Markov Chains and Markov Processes

Segun Ogunbayo

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Approval of the Institute of Graduate Studies and Research

Prof. Dr. Cem Tanova
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Mathematics

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics

Prof. Dr. Sonuç Zorlu Oğurlu
Supervisor

Examining Committee

1. Prof. Dr. Agamirza Bashirov
2. Prof. Dr. Sonuç Zorlu Oğurlu
3. Asst. Prof. Dr. Yücel Tandoğdu

ABSTRACT

Markov chain, which was named after Andrew Markov is a mathematical system that transfers a state to another state**.** Many real world systems contain uncertainty. This study helps us to understand the basic idea of a Markov chain and how is been useful in our daily lives. For some times there had been suspense on distinct predictions and future existences. Also in different games there had been different expectations or results involved. That is the reason why we need Markov chains to predict our expectation for the future. In this thesis we specifically talk about Markov Chains and how it has been processed, the gaming tactics which gives us a clue in a game that requires expectation. Also, we gave some applications of Markov chains such as Random walk, Games of chance, Queuing chain etc.

Keywords: Stochastic Process, Conditional Expectation, Markov chain, Random Walk, Birth and Death Chains

ÖZ

Andrew Markov'dan sonra adlandırılan Markov zinciri durumlar arası geçişleri çalışan matematiksel bir modeldir. Gerçek hayatta birçok olay belirsizlik içerir. Bu çalışma Markov zincirinin temel fikrini anlamaya yardımcı olmayı ve günlük yaşamdaki kullanımını belirtmeyi amaçlamaktadır. Farklı oyunlarda farklı beklentiler veya sonuçlar yer almaktadır. Gelecek için yapılacak tahminlerde Markov zincirleri önem taşımaktadır. Bu tezde özellikle Markov Zincirlerinin tanım ve özellikleri, oyun taktikleri, ayrıca Rastgele yürüyüş, şans oyunu, kuyruk zinciri gibi Markov zincirlerinin bazı uygulamaları çalışılmıştır.

Anahtar Kelimeler: Stokastik Süreç, Koşullu Beklenti, Markov Zinciri, Rasgele Yürüyüş, Doğum ve Ölüm Zincirleri

To My Family

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TABLE OF CONTENTS

LIST OF TABLES

Chapter 1

INTRODUCTION

According to Alexander Volfovsky, August 17, 2007 in a deterministic world, it is good to know that occasionally randomness can still occur. A stochastic process is the exact opposite of a deterministic one, and is a random process that can have several outcomes as time advances. This means that if we know an initial state for the process and the function by which it is den, we can tell of likely outcomes of the process. One of the most generally discussed stochastic processes is the Markov chain.

Markov Chains which also refers to Markov processes are defined as cycles of states which transition from one to another, and have a certain probability for each transition. They are used as a statistical model to represent and predict real world events. It can be refers to stochastic process or random variable having Markov property.

Most of our study of probability has concerted on independent trials processes. The results of these trials processes have their source from the theory of probability and statistics.

We have observed that when a series of experiments forms an independent trials process, the possible results for each experiment are the same and it occur with the same probability. Further, the existence of outcomes of the preceding experiments does not have any effects on our expectations for the outcomes of the next experiment.

In modern probability theory, Kwang Ho Jo said that the study of chance processes gives an ideal of understanding the previous outcomes of a given experiments always influenced expectations for future experiments. In principle, when we notice a sequence of chance experiments, all of the previous outcomes could generate impact on our predictions for the next experiment. For example, if Water Company charges 60 to 70tl per month for waters bill then, all the previous bills could generate impact on our predictions for the next month charges.

According to Guy Leonard Kouemou EADS Deutschland GmbH in 1906, Andrey Andreyevich Markov a Russian mathematician created the first theoretic results for stochastic processes by use of the term called chain. He went further by generating the type of chain process. In this process, the outcome generated from a given experiment determined the result of the next experiment. This type of process is referred to Markov chain. In the literature, different classes of Markov processes are taken as Markov chains. Mostly, the term is used for a process with a discrete set of times, while the time parameter is usually discrete and the state space of a Markov chain does not have any generally agreed-on limitations.

However, many applications of Markov chains employed countable infinite state spaces, which have more statistical analysis. Besides time and state-space parameters, there are many other variations, extensions and generalizations. Most of our study focuses on the discrete-time, discrete state-space case etcA change in the state of the system is referred to transitions while the probabilities assigned to different state changes are called transition probabilities. The process is described by a state space, a transition matrix studying the probabilities of a specific transitions, and an initial state or initial distribution through the state space. Without any doubt, we accept all likely states and transitions have been included in the definition of the process, so there is always a succeeding state, and the process does not lay off. This will be discussed further in Chapter three.

This study has four chapters in which are ordered as follows. Chapter 1 is the review part of our study. Basic definitions and related concepts are presented in Chapter 2. Markov chain definition with its states, examples and applications are given in Chapter 3. Finally, Chapter 4 consists of Conclusion.

Chapter 2

REVIEW OF PROBABILITY THEORY

In this part we shall think through some notations and basic part of probability theory. Topics to be revised are;

(1) Probability space and σ-fields

(2) Random variables and their distributions

(3) Conditional probability and independence

(4) Stochastic process

(5) Conditional expectation

2.1 Probability space and σ-fields

The probability space will be explained by using the system language of measure theory.

2.1.1 Definition of Sample space($\mathbf{\Omega}$)

This is the set of all possible result of a given random experiments e.g betting on players to score randomly, the results can either be winning or losing.

2.1.2 Definition of Event space($\mathcal{F}$)

This is a collection of all possible events under a given consideration. Hence, every set belonging to $\mathcal{F}$ is called an events. It can also be define as subsets of Ω

For example, if sample space Ω contains N elements, then the number of possible events will be $\sum_{k=1}^{N}\binom{N}{K}$.

2.1.3 Definition of Probability measure(P)

Probability measure Pis defined as the function P: $\mathcal{F} \rightarrow [0,1]$ such that the following axioms are satisfied

(1)$P(\Omega) = 1$ Infers that there is always an outcome from Ω on every trial carry out.

(2)For two events E_1 and E_2 which are disjoint set i.e. $E_y \cap E_z = \emptyset$ for all $y \neq z$ then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Therefore, a probability space is a triplet $(\Omega, \mathcal{F}, \mathrm{P})$ in which the three component are used to determine the outcome of a given experiment.

2.1.4 Definition of σ-fields

We defined σ-fields $\mathcal{F}$ on Ω if it satisfies the following condition.

(a) $\Omega \in \mathcal{F}$

(b) if an event $T \in \mathcal{F}$ then $T^{|} \in \mathcal{F}$ (closed under complements)

(c) if $T_i \in \mathcal{F}$ for $i = 1,2, \dots.$, then $\bigcup_i T_i \in \mathcal{F}$ (closed under countable union)

Note that the σ-fields $\mathcal{F}$ always containt least Ω and ∅ which is called the trivial σ-field $\mathcal{F}_o$.

Example 2.1: Let $\Omega = \{1 \dots \dots .4\}$, then the following are σ-field on Ω:

$$\mathcal{F}_1 = \{\emptyset, \{1\}, \{2,3,4\}, \Omega\}.$$

$$\mathcal{F}_2 = \{\emptyset, \{1,3\}, \{2,4\}, \Omega\}.$$

2.1.5 Definition of Borel set

We defined $B(R)$ where R is the set of real numbers as the smallest σ-field covering all interval in R. It can also be defined as the smallest σ- algebra which can be derived from an open and closed sets done by countable unions and complementations.

Note: The pair $(\Omega, \mathcal{F})$ is said to be measurable space and any events fitting to $\mathcal{F}$ are said to be $\mathcal{F}$-measurable. This implies that the events help to decide on whether they happened or not, given the information of $\mathcal{F}$. In other words, if one knows the information of $\mathcal{F}$, then one is able to state which events of $\mathcal{F}$ (= subsets of Ω).

2.2 Random variables and Their Distributions

2.2.1 Definition of $\mathcal{F}$-measurable

If $\mathcal{F}$ is a σ-field of subsent of Ω, then a function $Z\colon \Omega \longrightarrow \mathrm{R}$ is $\mathcal{F}$-measurable if $(Z \epsilon B) \epsilon \mathcal{F}$ for all Borel set ϵ $B(R)$. If $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space then the function Z is called a random variable.

2.2.2 Definition of Smallest σ-field generated Z

The smallest σ-field generated by random variable $Z\colon \Omega \longrightarrow \mathrm{R}$ consist of all sets of the form $(Z \epsilon B)$, where B is the borel set in R.

2.2.3 Definition of Distribution function of *Z*

Every random variable $Z\colon \Omega \longrightarrow \mathrm{R}$ result to a probability measure $P_z(B) = P(Z \in B)$ on R which is defined on the σ-field of Borel sets $B(R)$. Therefore we call P_z the distribution of Z. Also the function $F_z\colon R \longrightarrow [0,1]$ defined by $F_z(x) = P(Z \leq x)$ is called the distribution function of Z. The distribution function have the following properties.

The distribution function F_z is non-decreasing right continuous and

$$\lim_{x \longrightarrow -\infty} F_z(x) = 0, \; \lim_{x \longrightarrow \infty} F_z(x) = 1$$

2.2.4 Definition of Borel measurable function

A function $f\colon R \rightarrow R$ is said to be a Borel measurable function if F is a random variable on (R, B).

2.2.5 Definition of Density of Z

Assuming there exist a integrable function $f: R \rightarrow R$ such that for any close set $a, b \subset R$, $P(Z \in [a,b]) = F(a) - F(b) = \int_a^b f(x)dx$, then Z is said to be a random variable with absolutely continuous distribution and f is called the density of Z.

2.2.6 Definition of Discrete Distribution

If there is a finite sequence of distinct real numbers $x_{1,}x_{2,}$ .... such that for any Borel set $B \subset R,\ P(Z \in B) = \sum_{x_i \in B} P(Z = x_i)$ then Z is said to have discrete distribution with value $x_{1,}x_{2,}$ .....

Example 2.2: Assuming that Z has a continuous distribution with density f_z, show that $\frac{d}{dx} F_Z(x) = f_Z(x)$ if f is continuous at x.

Solution: Since Z has a densityf_z then the distribution function $F_z(x)$ can be written as $F_Z(x) = P(Z \leq x) = \int_{-\infty}^{x} f_Z(y)dy$.

Therefore, if f_z is continuous at x, then F_z is differentiable at x and

$\frac{d}{dx} F_Z(x) = \frac{d}{dx} \int_{-\infty}^{x} f_Z(y)dy = f_Z(x)$.

2.2.7 Joint Distribution of Numerous Random Variable $Z_{1}, \dots, Z_n$

This is said to be a probability measure $P_{Z_1 \cdots Z_n}$ on R^n such that

$P_{Z_1 \cdots Z_n}(B) = \{(Z_{1,} \dots . Z_n) \in B\}$ for every Borel set B in R^n. Suppose there is Borel function $F_{Z_1 \cdots Z_n} : R^n \rightarrow R$ such that

$P\{(Z_{1,} \dots . Z_n) \in B\} = \int_B \quad f_{Z_1 \cdots Z_n} \quad (x_{1 \dots,,} x_n) d_{x_1} \dots . d_{x_n}$ for any Borel set B in R^n, then $f_{Z_1 \cdots Z_n}$ is called the joint density of $Z_1 \dots, Z_n$. R^n.

2.2.2 Definition of Indicator function

For any event $A \epsilon \mathcal{F}$, the function $I_A(r) = \begin{cases} 1, & r \in A \\ 0, & r \notin A \end{cases}$ is a random variable and we call such random variable an indicator function.

The following are the properties of the Indicator Random Variable:

(a) $I_{\emptyset}(r) = 0$ and $I_{\Omega}(r) = 1$.

(b) $I_{A^c}(r) = 1 - I_A(r)$.

(c) $I_A(r) \leq I_B(r)$ if and only if $A \subseteq B$.

(d) $I_{\cap_i A_i}(r) = \prod_i I_{A_i}(r)$.

(e) If A_i are disjoint then $I_{\cup_i A_i}(r) = \sum_i I_{A_i}(r)$.

2.3 Conditional Probability and Independence

2.3.1 Conditional Probability

Assuming events $D, B \in \mathcal{F}$ such that $P(B) \neq 0$ then the conditional probability of an event D given event B can be expressed by

$$P(D|B) = \frac{P(B \cap D)}{P(B)}$$

Example 2.3: If 60% of my classmate like chicken kebab and 45% like chicken kebab and ham kebab. What is the percentage of those who like chicken also like ham kebab?

Solution

$$P(\text{ram kebab} \mid \text{chicken kebab}) = \frac{P(\text{chicken kebab and ham kebab })}{P(\text{chicken kebab})}$$

$$= \frac{0.45}{0.6} = 0.75$$

Therefore, 0.75 is the percentage of those who like chicken also ham kebab.

2.3.2 Definition of Independence of an events

An events D,$B \in \mathcal{F}$ are said to independent if the existence of D does not affect the probability of B. This implies $P(D \cap B) = P(D)P(B)$ or events D and B are said to be independent if $P(D|B) = P(D)$ which is the same as $P(B|D) = P(B)$.

In general we conclude that an events $D_1 \ldots D_n \in \mathcal{F}$ are independent if

$P\left(D_{i_1} \cap D_{i_2} \cap \ldots D_{i_k}\right) = P\left(D_{i_1}\right)P\left(D_{i_2}\right) \ldots P(D_{i_k})$.

Example 2.4: Consider the experiment of rolling a 3 on a die and spinning a tail on a coin. Rolling the 3 does not affect the probability of spinning the tail. If the events are independent, then the probability that boths events will occur is the product of the probabilities of each occurring i.e. $P(D \cap C) = 0.5$.

2.3.3 Definition of Independence of Two Random Variables

Two random variables J and Q are said to be independent if for any Borel sets $D, B \in B(R)$ then the two events $(J \in D)$ and $q \in B$ are independent. In general we conclude that random variable $J_{1,\ldots,}J_n$are independent if for any Borel sets

$D_{1,\ldots,}D_n \in B(R)$ then the events $(J_1 \in D_1)$ and $(J_n \in D_n)$ are also independent.

2.34 Definition of Independence of Two σ-Fields

Two σ-fields $\mathcal{H}, G \subseteq \mathcal{F}$ are independent if $P(D \cap B) = P(D)P(B)$ such that for all $D \in \mathcal{H}$ and $B \in G$.

2.35 Definition of Independence of Finite number of σ-Fields

A finite number of σ-fields $\mathcal{H}_1, \ldots, \mathcal{H}_n$ contained in $\mathcal{F}$ is said to be independent for any n events if $D_1 \in \mathcal{H}_1, \ldots., D_n \in \mathcal{H}_n$ are all independent. In general, we say an infinite or finite family of σ-fields is said to be independent if any finite number of them are independent.

2.4 Stochastic Process

This section is essential for the understanding of stochastic process.

2.4.1 Stochastic Process

Assuming T is a subset of $(-\infty, \infty)$. A family of random variables $\{D_t\}_{t\in T}$ defined on Ω is called a stochastic process. Here we represent $(-\infty, \infty)$ as the infinite past to infinite future respectively in which are called Time.

Types of stochastic processes

(a) Discrete time process: A stochastic process is called a discrete time process if and only if T is continuous and Ω is discrete. This implies that, as T is continuous,Ω takes a discrete set of values.

Example 2.5: If $D(t)$ represent the number of costumer received in kebab shop in the interval of $(0, t)$ then $\{D(t)\}$ is a discrete time process since $\Omega = \{0,1,2,3 \ldots \ldots\}$

(b) Continuous time process: A stochastic process is said to be continuous if and only if both T and Ω are continuous or if T is an interval which has a positive length.

Example 2.6

If $D(t)$ represent the maximum temperature at a place in the interval $(0, t)$, then we say that $D(t)$ is continuous.

2.4.2 Range of Random Variable

The range of random variable or possible value in stochastic process is referred to state spaces of the process.

2.4.2 Transition Process

A change between any given state spaces in stochastic process is called transition process.

Example 2.7

Let $D_n: n = \{0, 1, 2, 3 \dots\}$ where the state space of D_n is $\{0, 1, 2, 3, 4, 5, 6\}$ which signify the six types of transactions submitted to a data service where time n relates to the number of transactions submitted.

2.4.3 Sample Path

A sample path is described as time ordered which show what happened to a process in one instant. This can be either continuous or discrete.

2.4.4 Filtration

We defined a filtration as the increase in the family of σ-fields that is if a sequence of σ-fields $\mathcal{F}_1, \mathcal{F}_2$, ... on Ω such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ then we call it filtration.

Example 2.8

Let $D = \{the\ outcome\ of\ the\ first\ four\ tosses\ produce\ at\ least\ two\ tails\}$ at discrete time $n = 4$. Whenever the coin has been tossed four times, it is likely to determine if D has occurred or not. It implies that $D \in \mathcal{F}_4$. Nevertheless, at $n = 3$ it is not always possible to determine if D has occurred or not. Assuming the outcome of the first three tossed are heads, heads, tails, then the event D is unsure. This implies that $D \notin \mathcal{F}_4$. Assuming that we are able to get two tails at first three tossed, then we say that D has already occurred no matter the outcome of the fourth toss. This does not mean that $D \in \mathcal{F}_3$.

2.4.5 Definition of Sequence of Random Variables

We defined a sequence $p_{1,}p_{2,\dots\dots\dots}$ of random variables to be martingale with regard to a filtration $\mathcal{F}_{1,}\mathcal{F}_2, \dots \dots ..$ if the following properties are satisfied:

(a) $J_1 J_{2,\dots}$ are adapted to a filtration $\mathcal{F}_{1,}\mathcal{F}_2, \dots$

(b) J_n is integrable for each $n = 1, 2, 3 \dots$

(c) $E(J_{n+1} \mid \mathcal{F}_n) = J$ for each $n = 1, 2, 3 \dots$

2.5 Conditional Expectation

Recall that the conditional probability of D given B

$$P(D|B) = \frac{P(B \cap D)}{P(B)}.$$

Clearly, $P(D|B) = P(D)$ if and only if $D \; and \; B$ are independent. Given that $P(B) > 0$, then the conditional distribution function of a random variable where $x \in R$ is

$$F_X(x|B) = \frac{P((X \leq x) \cap B)}{P(B)}.$$

Therefore the expectation

$$E(X|B) = \frac{E(X \cap B)}{P(B)}.$$

Is called the conditional expectation of *X* given *B*.

2.6 Conditioning on an Event

For any given integrable random variable p and any event $B \in \mathcal{F}$ such that $P(B) \neq 0$, the conditional expectation of p given B is defined as

$$E(p|B) = \frac{1}{P(B)} \int_B pdP.$$

Example 2.9: Assuming three coins 15J, 25J and 60J are flipped. The outcome of those coins that land tails up are added to get the total amount of p. Find the expected total amount of p if and only if two coins have landed tails up.

Solution: Let B represent two coins that have landed tails up. We will find $E(J|B)$. Obviously, $B = \{TTH, THT, HTT\}$ where T represent tails, H represent heads and each having total probability of $\frac{1}{8}$

i.e.$\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$. Therefore, the corresponding values of J are

$$J(TTH) = 15 + 25 = 40 .$$

$$J(THT) = 15 + 60 = 75.$$

$$J(HTT) = 25 + 60 = 80 .$$

Then

$$E(J|B) = \frac{1}{P(B)} \int_B JdP = \frac{1}{\frac{3}{8}} \left(\frac{40}{8} + \frac{75}{8} + \frac{85}{8} \right) = 66\frac{2}{3}.$$

2.7 Conditioning on an Arbitrary Random Variable

Assuming p is an integrable random variable and τ is an arbitrary random variable, then the conditional expectation of p given τ is assumed to be a random variable $E(J|\tau)$ if it satisfies the following properties below

(a) $E(J|\tau)$ is $\sigma(\tau)$-measurable.

(b) For any $D \in \sigma(\tau)$.

$$\int_D E(J|\tau)dP = \int_D pdP.$$

2.8 General Properties of Conditional Expectation

Let $x, y \in R,\ p, \zeta \in\ \Omega,\ \mathcal{F},$ P and G, H are sub σ-algebra on Ω then

(1) $E(xp + y\zeta|\text{G}) = xE(p|\text{G}) + yE(\zeta|\text{G})$ for all $x, y \in R$ (linearity property).

(2) $E(E(p|\text{G})) = E(p)$.

(3) If $p \geq 0$, then $E(p|\text{G})\ \geq 0$ (Positivity property).

(4) $E(E(p|\text{G})|H) = E(p|\text{G})$ if and only if H⊂G.

(5) $E(p|\text{G}) =\ E(p)$ if and only if p is an independent of G.

Chapter 3

MARKOV CHAINS

3.1 Definition

A Markov chain is a family of stochastic processes in which the process is a discrete time. The discrete time process is always characterized by the set called the State space of the system where X_n denotes the state of the system at time n = 0, 1, 2...

Many systems have the property that given the present state, the past states have no influence on the future. This property is called the Markov property and the system having this property is called a Markov chain. Since the system have Markov property that is, a process is$\{X_n\}_n^{\infty} = 0$ called a Markov if

$P\left(X_{n+1} \epsilon A \middle| X_{0,}X_{1,\dots\dots,} \; X_{n,}\right) = P(X_{n+1}\epsilon A|X_n).$

The Conditional probabilities $P(X_{n+1} = y|X_n = x)$ are called the Transition Probabilities of the chain. A Markov chain is said to have Stationary Transition Probabilities if $P(X_{n+1} = y|X_n = x)$ is independent of n. Note that in all states of Markov chain, it is possible to go from any state in more than one step to every other state and you can only return to a state in an even number of steps.

3.2 Markov Chains Having Two States

For an example, consider a Markov chain having two chain states. Assume that a Generator at the start of any particular day is either broken down or in operating condition. Let X_n be random variable denoting the state of the Generator at time n and let $\pi_0(0)$ be the probability that the generator is broken down initial. Then the following are the stationary transition probabilities:

$$P(X_{n+1} = 1|X_n = \ 0) = \varkappa \quad (1)$$

$$P(X_{n+1} = 0|X_n \ = 1) \ = q \quad (2)$$

Where $\varkappa$ is the probability that it will successfully repaired and in operating condition at the start of the $(n+1)^{st}$ day when the generator is broken at the start of nth day. Also q is the probability that it will fail causing it to be broken down at the start of the $(n+1)^{st}$ day when the generator is in operating condition at the start of the nth day. Since there are only two states which are 0 and 1, it follows that

$$P(X_{n+1} = 0 \ |X_n = 0) = 1 - \varkappa \quad (3)$$

$$P(X_{n+1} = 1 \ |X_n \ = 1) \ = \ 1 - q \quad (4)$$

And $\pi_0(1) = P(X_0 = 1) = 1 - \pi_0(0)$ are called the initial distribution.

By applying matrix transition to (3) and (4) we have

$$P = \begin{pmatrix} 1-\varkappa & \varkappa \\ q & 1-q \end{pmatrix}$$ Where sum of any row of the matrix is1.

Given the initial distribution and transition probabilities, we can find distribution of all X_n which are $P(X_n = 0)$ and $P(X_n = 1)$.

We observe that

$$P(X_{n+1} = 0) = P(X_n = 0, X_{n+1} = 0) + P(X_n = 1, X_{n+1} = 0) \quad (5)$$

By applying multiplicative rule to equation (5) we get

$$P(X_{n+1} = 0|X_n = 0) \ P(X_n = 0) + P(X_{n+1} = 0|X_n = 1) \ P(X_n = 1) \quad (6)$$

By applying transition function which has been stated above we have

$$P(0,0)\, P(X_n = 0) + P(1,0)P(X_n = 1).$$

Recall that

$$P(X_n = 1) = 1 - P(X_n = 0).$$

Then we have

$$(1 - \varkappa)P(X_n = 0) + q(1 - P(X_n = 0)).$$

$$= \quad (1 - \varkappa)P(X_n = 0) + q - q\, P(X_n = 0).$$

$$= \quad P(X_{n+1} = 0) = (1 - \varkappa - q)P(X_n = 0) + q. \qquad (7)$$

Then for $n = 0$, substitute for n in equation (7) we have

$$P(X_1 = 0) = (1 - \varkappa - q)P(X_0 = 0) + q \qquad (8)$$

Since $\pi_0(0) = P(X_0 = 0)$ substitute it into equation (8) we have

$$P(X_1 = 0) = (1 - \varkappa - q)\pi_0(0) + q \qquad (9)$$

Therefore for state 1 we have

$$P(X_1 = 1) = 1 - \ P(X_1 = 0)$$

From equation (7) when $n = 1$ we have

$$P(X_2 = 0) = (1 - \varkappa - q)P(X_1 = 0) + q \qquad (10)$$

By substituting equation (9) into (10) we have

$$= \quad (1 - \varkappa - q)\big((1 - \varkappa - q)\pi_0(0) + q\big) + q$$

$$= \quad (1 - \varkappa - q)^2\pi_0(0) + (1 - \varkappa - q)q + q$$

By factorization we get $(1 - \varkappa - q)^2\pi_0(0) + q[1 + (1 - \varkappa - q)]$ (11)

Then for n times, apply induction we have

$$P(X_n = 0) = (1 - \varkappa - q)^n\pi_0(0) + q[1 + (1 - \varkappa - q)(1 - \varkappa - q)^{n-1}]$$

$$= \quad (1 - \varkappa - q)^n\pi_0(0) + \sum_{j=0}^{n-1}(1 - \varkappa - q)^j \qquad (12)$$

Since the sequence in (12) is a geometric sequence then we can rewrite it as

$$(1 - \varkappa - q)^n\pi_0(0) + q[\tfrac{1-(1-\varkappa-q)^n}{1-(1-\varkappa-q)}] \qquad (13)$$

Hence by simplify (13) we have

$$P(X_n = 0) = \frac{q}{\varkappa+q} + (1 - \varkappa - q)^n(\pi_0(0) - \frac{q}{\varkappa+q}) \qquad (14)$$

For

$$P(X_n = 1) = 1 - P(X_n = 0) \qquad (15)$$

Substitute (14) into (15) we have

$$1 - [\frac{q}{\varkappa+q} + (1 - \varkappa - q)^n \left(\pi_0(0) - \frac{q}{\varkappa+q}\right)].$$

$$= \quad \frac{\varkappa}{\varkappa+q} + (1 - \varkappa - q)^n \left(1 - \pi_0(1) - \frac{q}{\varkappa+q}\right).$$

$$= \quad \frac{\varkappa}{\varkappa+q} + (1 - \varkappa - q)^n \left(\pi_0(1) - \frac{\varkappa}{\varkappa+q}\right) \qquad (16)$$

Assuming that $\varkappa$ and q are neither equal to 0 or 1 then, $0 < \varkappa + q < 2$ .This implies that $|1 - \varkappa - q| < 1$.

In this case, will can find the limit of $P(X_n = 0)$ and $P(X_n = 1)$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} P(X_n = 0) = \frac{q}{\varkappa+q} \text{ and} \lim_{n\to\infty} P(X_n = 1) = \frac{\varkappa}{\varkappa+q}.$$

Also, since it is not specified whether the $X_n, n \geq 0$ then we can assume that it Satisfy Markov Property and compute for Joint distribution of $X_0, X_1, X_2 \dots, X_n$. For example take n = 2 and assume that X_0, X_1 and X_2 each equal to 1 or 0.Then by applying multiplicative rule, let

$$X_0 = x_0 \text{ and } X_1 = x_1 \text{ be } A \text{ and } X_2 = x_2 \text{ be } B.$$

Then we have $P(A \cap B) = P(A)P(B|A)$ which implies

$$P(X_0 = x_0, X_1 = x_1, X_2 = x_2).$$

$$= P(X_0 = x_0, X_1 = x_1)P(X_2 = x_2|X_0 = x_0, X_1 = x_1) \qquad (17)$$

Apply Multiplicative rule to and Markov property to (17) we get

$$P(X_0 = x_0)\, P(X_1 = x_1|X_0 = x_0)\, P(X_2 = x_2|X_1 = x_1) \qquad (18)$$

Recall that $P(X_0 = x_0) = \pi_0(x_0)$ therefore substitute it into equation (18) we get the Joint Distribution table of X_0, X_1 and X_2 which are

Table 1: Joint distribution

X_0	X_1	X_2	$P(X_0 = x_0, X_1 = x_1, X_2 = x_2)$
1	1	1	$(1-\pi_0(0))(1-q)^2$
1	1	0	$(1-\pi_0(0))(1-q)q$
1	0	1	$(1-\pi_0(0))\varkappa q$
1	0	0	$(1-\pi_0(0))q(1-\varkappa)$
0	1	1	$\pi_0(0)\varkappa(1-q)$
0	1	0	$\pi_0(0)\varkappa q$
0	0	1	$\pi_0(0)(1-\varkappa)\varkappa$
0	0	0	$\pi_0(0)(1-\varkappa)^2$

The function $P(x, y) = P(X_1 = y|X_0 = x)$ where $x, y \in$S is called the Transition function of the Chain such that $P(x, y) \geq 0$, where $x, y \in$S and $\sum_y P(x, y) = 1$ where $x, y \in$S .Here $P(x, y)$ is the probability the chain is in state y at step *n*+1 provided that it was in state X at time *n.*

The function $\pi_0(x) = P(X_0 = x)$, $x \in$S is called the initial distribution of the chain such that $\pi_0(x) \geq 0, x \in$ S and $\sum_x \pi_0(x) = 1$.

The Joint distribution of $X_0, X_1, X_2 \dots, X_n$ can simply expressed in term of initial distribution and transition function.

For $P(X_0 = x_0, X_1 = x_1) = P(X_0 = x_0)P(X_1 = x_1|X_0 = x_0)$

$$= \pi_0(x_0)\, P(x_0, x_1).$$

Also $P(X_0 = x_0, X_1 = x_1, X_2 = x_2) = \pi_0(x_0)\, P(x_0, x_1)P(x_1, x_2)$.

Since $X_n, x \geq 0$ which has stationary transition probabilities and satisfies Markov property. By induction it is easily seen that $P(X_0, X_1, X_2 \dots, X_n)$

$$= \pi_0(x_0)\, P(x_0, x_1)P(x_1, x_2) \dots \dots P(X_{n-1}, X_n) \qquad (19)$$

3.3 Examples of Markov Chains

3.3.1 Random walk

Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ ... be independent integer valued random variables and let X_0 integer valued random variables that is independent of the $\mathcal{E}_i's$, and set

$X_n = X_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_n$. This set of sequence is called random walk.It is a Markov whose state space is the integers and whose transition function is

$$P(x, y) = f(y - x) \qquad (20)$$

To verify (20), let π_0 denote the distribution of X_0. Then $P(X_0 = x_0 \dots . X_n = x_n)$

$$= P(X_0 = x_0, \mathcal{E}_1 = x_1 - x_0, \dots ., \mathcal{E}_n = X_n - X_{n-1}).$$

$$= P(X_0 = x_0)P(\mathcal{E}_1 = x_1 - x_0)P(\mathcal{E}_2 = x_2 - x_1)\dots P(\mathcal{E}_n = X_n - X_{n-1}).$$

$$= \pi_0(x_0)\, f(\mathcal{E}_1 = x_1 - x_0)f(\mathcal{E}_2 = x_2 - x_1)\dots f(\mathcal{E}_n = X_n - X_{n-1}).$$

$$= \pi_0(x_0)\, P(x_1 x_0)\dots P(X_{n-1}, X_n).$$

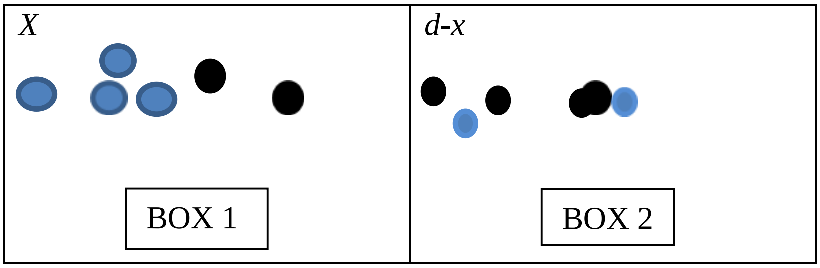
Thus (19) holds.

As a special case, consider a simple random walk in which $f(1) = \varkappa$, $f(-1) = q$ and $f(0) = r$, where $\varkappa + q + r = 1$, then the transition function is given by

$$f(y - x) = P(x, y) = \begin{cases} \varkappa, & y = x + 1, \\ q, & y = x - 1, \\ r, & y = x, \\ 0, & \text{elsewhere.} \end{cases}$$

3.3.2 Ehrenfest chain

This is a simple model of the exchange of heat or gas molecules between two isolated bodies.



Let X_n denote the number of molecules (or balls) in box 1 after the n^{th} trial. (Trials are independent). The $X_n, n \geq 0$ is a Markov chain S ={0,1,2 … … , d}. the transition function of this Markov chain is given by

$$P(x,y) = \begin{cases} {d-x}/{d} & y = x+1 \ (from\ box\ 2\ to\ box\ 1) \\ {x}/{d} & y = x-1 \quad (from\ box\ 1\ to\ box\ 2) \end{cases}$$

A state m of a Markov chain is called an absorbing state if $P(m,m) = 1$ or equivalently if $P(a,y) = 0$ for $y \neq m$

3.3.3 Gambler's Ruin Chain

Let p be the probability of winning 1 unit at any bet and q be the probability of losing 1 unit at any bet. If the gamblers capital ever reach zero he is ruined and his capital remains zero therefore, (absorbing state.)

Let X_n denote the gamblers capital at time n. this is a Markov chain in which zero is an absorbing states and for $x \geq 1$.

$$P(x,y) = \begin{cases} p, & y = x+1 \\ q, & y = x-1 \\ 0, & elsewhere \end{cases}$$

Such a chain is called a Gambler's Ruin Chain on S ={0,1,2, … . . }.

If S′ $= \{0,1,2,\ldots,d\}$, in this case 0 and d are both absorbing states holds for

$x = 1,2,\ldots d$.

3.3.4 Birth and Death Chain

The transition of a Birth and Death chain on S $=\{0,1,2,\ldots\ldots\}$ or on S $=\{0,1,2,\ldots\ldots d\}$ is given by

$$P(x,y) = \begin{cases} q_x, & y = x-1 \ (corresponding\ to\ death) \\ r_x, & y = x \\ p_x, & y = x+1 \ (corresponding\ to\ birth) \\ 0, & elsewhere \end{cases}$$

$where\ p_x + q_x + r_x = 1$.

The Ehrenfest chain and Gambler's ruin chain are the examples of Birth and Death chains**.**

3.3.5 Queuing Chain

Consider a service facility such as checkout at supermarket. Let $\mathcal{E}_n$ denote the number of new customers arriving during the n^{th} period. We assume that $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ ...are independent integer valued random variables and exactly one customer will be served during any given period. Let X_0 denote the number of customers present initially and for $n \geq 1$, let X_n denote the number of customers present at the end of the n^{th} period.

If $X_n = 0$ then $X_{n+1} = \mathcal{E}_{n+1}$ and if $X_n \geq 1$ then $X_{n+1} = X_n + \mathcal{E}_{n+1} - 1$.

$X_n, n \geq 0$ is a Markov chain on S $=\{0,1,2,\ldots\ldots\}$ with

$P(0,y) = f(y)$ and $P(x,y) = f(y-x+1),\ x \geq 1$.

3.4 Computation with Transition Functions

Let $X_n, n \geq 0$ be a Markov chain on S having transition function *P*. In this section we will show how various conditional probabilities can be expressed in terms of P. We will also define the *n*- step transition function of the Markov chain. We begin with the formula

$$P(X_{n+1} = x_{n+1} \dots, X_{n+m} = x_{n+m} | X_0 = x_0, \dots, X_n = x_n) . \quad (21)$$

By definition of conditional probability

$$P(x_{n,} x_{n+1}) P(x_{n+1}, x_{n+2}) \dots P(x_{n+m-1}, x_{n+m}). \quad (22)$$

Also (21) can be written as

$$P(X_{n+1} = y_1, X_{n+2} = y_2, \dots, X_{n+m} = y_m | X_0 = x_0, .., X_n = x_n). \quad (23)$$

$$= P(x, y_1) P(y_1, y_2) \dots \dots .. P(y_{m-1}, y_m). \quad (24)$$

Note that, $P_B(.) = P(.|B)$ where $(.) \epsilon$ S .

If $A_1 \cap A_2 = \phi$, $P_B(.) = (A_1 \cap A_2|B) = P(A_1|B) + P((A_2|B)$.

But $(A|B_1 \cup B_2) \neq P(A_1|B) + P(A_2|B)$.

Lemma 1 (Paul G. Hoel).

I. If D_i are disjoint and $P(C|D_i) = P$ for all *i*, then $P\left(C \middle| \ _i \cup D_i\right) = P$.

II. If C_i are disjoint , then $P\left(C \middle| \ _i \cup D_i\right) = \sum_i P(C_i|D)$.

Let $A_0, A_1, \dots, A_{n-1}$ be subset of S. It follows from (24) and lemma (I) that

$$P(X_{n+1} = y_{1,} \dots X_{n+m} = y_m | X_0 \in A_0, \dots, X_{n-1} \in A_{n-1}, X_n = x) \quad (25)$$

$$= P(x, y_1) P(y_1, y_2) \dots P(y_{m-1}, y_m).$$

Let $B_1 \dots B_m$ be subsets of S. It follows from (25) and lemma (II) that

$$P(X_{n+1} \in B_1, \dots, X_{n+m} \in B_m | X_0 \in A_1, \dots, X_{n-1} \in A_{n-1}, X_n = x)$$

$$= \sum_{y_1 \in B_1} \sum_{y_2 \in B_2} \dots \sum_{y_m} P(x, y_1) P(y_1, y_2) \dots P\ (y_{m-1}, y_m).$$

The *m*-step transition function $P^m(x, y)$, which gives the probability of going from *x* to *y* in *m*-step is defined by

$$P^m(x, y) = \sum_{y_1} \dots \sum_{y_{m-1}} P(x, y_1) P(y_1 , y_2) \dots P(y_{m-1}, y_m),$$

For $m \geq 2$,

$P^1(x,y) = P(x,y)$ and $P^0(x,y) = \begin{cases} 1, & x = y \\ 0, & elsewhere \end{cases}$

Furthermore, $P(X_{n+m} = y|X_n = x) = P^m(x,y)$ and for n+m step probability we have

$$P^{n+m}(x,y) = \sum_{z\in S} P^n(x,z)P^m\,(z,y) \qquad (26)$$

$P(X_n = y) = \sum_{x,y} \pi_0(x)P^n(x,y)$, distribution of X_n while

$P(X_{n+1} = y) = \sum_{x,y} P(X_n = x)P(x,y)$, is the recursion between distribution of X_n and X_{n+1}.

Note: $P_x(A) = P(A|X_0 = x)$.

$$P_x(X_1 \neq a, X_2 \neq a, X_3 = a) = P(X_0 = x, X_1 \neq a, X_2 \neq a, X_3 \neq a).$$

Starting at X, the chain will be in a at time 3.

3.4.1 Hitting Times

Let $A \subset S$. The hitting time T_A of A is defined by $T_A = min\{n > 0: X_n \epsilon A\}$.

If $X_n \epsilon A$ for some $n > 0$ and by $T_A = \infty$ if $X_n \notin A$ for all $n > 0$.

Hitting times play an important role in the theory of Markov chains. T_a denotes the hitting time of a point $a\epsilon$S.

An important equation involving hitting times is given by

$$P^n(x,y) = \sum_{m=1} P_x(T_y = m)\, P^{n-m}(y,y) \qquad n \geq 1 \qquad (27)$$

Let us verify equation (27). To do this, note that the events $(T_y = m, X_n = y)$ where $1 \leq m \leq n$ are disjoint and $(X_n = y) = \cup_{m-1}^{n}(T_y = m, X_n = y)$.

We have in effect decomposed the event ($X_n = y$) according to the hitting time of y.

We see from this decomposition that

$$P^n(x,y) = P_x(X_n = y)$$

$$= \sum_{m=1}^{n} P_x\,(T_y = m, X_n = y)$$

$$= \sum_{m=1}^{n} P_x\,(T_y = m) P\big(X_n = y \big| X_0 = x, T_y = m\,\big)$$

$$= \sum_{m=1}^{n} P_x\,\big(T_y = m\big) P\big(X_n = y \big| X_0 = x, X_1 \neq y, \dots X_{m-1} \neq y, X_y = y\,\big)$$

$$= \sum_{m=1} P_x(T_y = m)\; P^{n-m}(y,y)\;.$$

Example1: Show that if a is an absorbing state, then

$P^n(x,a) = P_x(T_a \leq n)\;,\; n \geq 1$

If a is an absorbing state, then $P^{n-m}(a,a) = 1$ for $1 \leq m \leq n$ then equation (27) implies that

$$P^n(x,a) = \sum_{m=1}^{n} P_x\,(T_a = m) P^{n-m}(a,a)$$

$$= \sum_{m=1}^{n} P_x\,(T_a = m) P_x(T_a \leq n).$$

Observe that

$$P_x\big(T_y = 1\big) = P(x,y)$$

$$P_x\big(T_y = 2\big) = \sum_{z\neq y}^{n} P_x\,(X_1 = z, X_2 = y)$$

$$= \sum_{z\neq y}^{n} P\,(x,z) P(z,y).$$

And

$$P_x\big(T_y = n+1\big) = \sum_{z\neq y} P(x,z) P_z\,(T_n = y), \qquad n \geq 1\;.$$

3.4.2 Transition Matrix

Suppose that S is finite, say $S = \{0,1,2, \ldots \ldots \ldots, d\}$ then

$$\begin{matrix} 0 \\ 1 \\ \\ d \end{matrix} \begin{bmatrix} P(0,0) \ldots \ldots \ldots \ldots P(0,d) \\ P(1,0) \ldots \ldots \ldots \ldots P(1,d) \\ \ldots \ldots \\ P(d,0) \ldots \ldots \ldots \ldots P(d,d) \end{bmatrix}$$

for *i*, *j* = 0, 1. . . *d*, where

$$\sum_{y=0}^{d} P(x,y) = 1, \text{ for all } x \in S.$$

Example 2: the transition matrix of the Gamblers ruin chain on {0,1,2,3} is

$$\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & \varkappa & 0 \\ 0 & q & 0 & \varkappa \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \varkappa + q = 1$$

P is one-step transition matrix similarly, P^n is *n*-step transition matrix

Then (26) with *m* = *n* = 1 becomes

$$P^2(x,y) = \sum_z P(x,z)P(z,y) \tag{28}$$

$$P^{\,n+1}(x,y) = \sum_z P^n(x,z)P(z,y) \tag{29}$$

It follows from (29) by induction that the *n*-step transition matrix P^n is the n^{th} power of P and the initial distribution π_0 is

$$\pi_0 = (\pi_0(0), \pi_0(1), \ldots \pi_0(d)$$

and for we have π_n

$$\pi_n = (P(X_n = 0), \ldots, P(X_n = d))$$

Also $\pi_n = \pi_0 P^n$ and $\pi_{n+1} = \pi_n P$.

Example 3: Consider two state Markov having one-step transition matrix

$$P = \begin{pmatrix} 1-\varkappa & \varkappa \\ q & 1-q \end{pmatrix}$$

Where $\varkappa + q > 0$. Find P^n.

Firstly let $\pi_0(0) = 1$ in (14) then

$$P^n(0,0) = P_0(X_n = 0) = \frac{q}{\varkappa+q} + (1-\varkappa-q)^n - \frac{\varkappa}{\varkappa+q}$$

Also if we set $\pi_0(1) = 0$ in (16) then

$$P^n(0,1) = P_0(X_n = 1) = \frac{\varkappa}{\varkappa+q} - (1-\varkappa-q)^n \frac{\varkappa}{\varkappa+q}$$

Similarly, for $P^n(1,0)$ and $P^n(1,1)$ we have

$$P^n(1,0) = P_1(X_n = 0) = \frac{q}{\varkappa+q} - (1-\varkappa-q)^n \frac{q}{\varkappa+q}$$

$$P^n(1,1) = P_1(X_n = 1) = \frac{\varkappa}{\varkappa+q} + (1-\varkappa-q)^n \frac{q}{\varkappa+q}$$

It follows that

$$P^n = \frac{1}{\varkappa+q}\begin{bmatrix} q & \varkappa \\ q & \varkappa \end{bmatrix} + \frac{(1-\varkappa-q)^n}{\varkappa+q}\begin{bmatrix} \varkappa & -\varkappa \\ -q & q \end{bmatrix}.$$

3.5 Classification of States

Let $X_n, n \geq 0$ be Markov having state space S and transition function P then set

$$\zeta_{xy} = P_x(T_y < \infty).$$

Then ζ_{xy} denote that the probability that a markov chain starting at x will visited state y in finite time.

ζ_{yy} Denote that the probability that a Markov chain starting at *y* will ever return to *y*.

A state *y* is called recurrent state if $\zeta_{yy} = 1$, and Transient if $\zeta_{yy} < 1$.

If *y* is recurrent state then a Markov chain starting at *y* returns to *y* with probability 1 but if *y* is a transient state then a Markov chain starting at *y* has a positive probability $1 - \zeta_{yy}$ that never return to *y*.

Therefore, $1-\zeta_{yy} = P_y(T_y = \infty) > 0$ implies probability of no return to y

If y is an absorbing state, then $P_y(T_y = 1) = P(y,y) = 1$ and hence $\zeta_{yy} = 1$, thus an absorbing state is necessarily recurrent.

Let $1_y(z), z \in S$, denote the indicator function of the $\{y\}$ defined by

$$1_y(z) = \begin{cases} 1, & z = y \\ 0, & z \neq y \end{cases}$$

Let $N(y)$ denote the number of times $n \geq 1$ that the chain is in state y.

Since $1_y(X_n) = 1$ if the chain is in state y at time n and $1_y(X_n) = 0$ otherwise, we see that

$$N(y) = \sum_{n=1}^{\infty} 1_y(X_n)$$

implies number of visits to y. Therefore the

$$P_x(N(y) \geq 1) = P_x(T_y < \infty) = \zeta_{xy}$$

$$P_x(N(y) \geq 2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_x(T_y = m)P_y(T_y = n)$$

$$= \left(\sum_{m=1}^{\infty} P_x(T_y = m)\right)\left(\sum_{n=1}^{\infty} P_y(T_y = n)\right)$$

$$= \zeta_{xy}, \zeta_{yy}.$$

Similarly we conclude that

$$P_x(N(y) \geq m) = \zeta_{xy}\,\zeta_{yy}{}^{m-1}, \qquad m \geq 1. \tag{30}$$

Since $P_x(N(y) = m) = P_x(N(y) \geq m) - P_x(N(y) \geq m+1)$.

By (30) we have

$$\zeta_{xy}\,\zeta_{yy}{}^{m-1} - \zeta_{xy}\,\zeta_{yy}{}^{m}$$

$$= \zeta_{xy}\,\zeta_{yy}{}^{m-1}(1-\zeta_{yy}), \qquad m \geq 1\,.$$

Also

$$P_x(N(y) = 0) = 1 - P_x(N(y) \geq 1)$$

$$= 1 - \zeta_{xy}.$$

We use the notation $E_x(.) = E(.|X_0 = x)$ as the expectation of random variables defined in term of Markov chain starting at *x*. for example,

$$E_x(I_y(X_n)) = 1P_x(X_n = y) + 0P_x(X_n \neq y) = P^n(x,y).$$

$$= E_x\big(N(y)\big) = E_x\left(\sum_{n=1}^{\infty} I_y(X_n)\right)$$

$$= E_x\left(\sum_{n=1}^{\infty} E_x(I_y(X_n)\right)$$

$$= \sum_{n=1}^{\infty} P^n\,(x,y).$$

Set

$$G(x,y) = E_x\big(N(y)\big) = \sum_{n=1}^{\infty} P^n\,(x,y).$$

Then $G(x,y)$ represent the expected number of visits to *y* starting at *x*.

The following theorem describes the fundamental difference between a Transient and Recurrent state.

Theorem1: (i) Let *y* be a transient state then

$$P_x(N(y) < \infty) = 1$$

And

$$G(x,y) = \frac{\zeta_{xy}}{1-\zeta_y}\ ,\text{where } x\epsilon\zeta,$$

Which is finite for all $x\epsilon\zeta$.

(ii)Let *y* be a recurrent state then

$$P_x(N(y) < \infty) = 1$$

And

$$G(x,y) = \infty$$

Also

$$P_x(N(y) = \infty) = P_x(T_y < \infty) = \ \zeta_{xy},\ x\epsilon\zeta.$$

If $\zeta_{xy} = 0$ then $G(x,y) = 0$, while if $\zeta_{xy} > 0$ then $G(x,y) = \infty$.

Proof. **(i)** If *y* is in transient state then by definition $0 \le \zeta_{xy} < 1$, then it follows from (30) that

$$P_x(N(y) = \infty) = \lim_{m\to\infty} P_x(N(y) \ge m)$$

$$\lim_{m\to\infty} \zeta_{xy}\zeta_{yy}^{m-1} = 0$$

Here $P_x(N(y) = \infty) = 1$

Since $G(x,y) = E_x\big(N(y)\big) = \sum_{m=1}^{\infty} mP_x(\,N(y) = m)$

$$= \sum_{m=1}^{\infty} m\zeta_{xy}\zeta_{yy}^{m-1}(\,1 - \zeta_{yy}) \qquad (31)$$

$$= \ \zeta_{xy}\big(1 - \ \zeta_{yy}\big)\sum_{m-1}^{\infty} m\,\zeta_{yy}^{m-1} \qquad (32)$$

Apply power series to (32), let $\zeta_{yy} = t$ then differentiate to have

$$= \sum_{m=1}^{\infty} \frac{d}{dt}(t^m) = \frac{d}{dt}\sum_{m=1}^{\infty} t^m.$$

When $m = 0$ we have

$$= \frac{d}{dt}\left(\frac{1}{1-t} - 1\right)$$

$$= \frac{d}{dt}\left(\frac{t}{1-t}\right) = \frac{1}{(1-t)^2}$$

We conclude that

$$G(x, y) = \frac{\zeta_{xy}}{1 - \zeta_{yy}}$$

This completes the proof of (i).

Now let y be a recurrent state then $P_x(N(y) = \infty) = \lim_{m\to\infty} P_x(N(y) \geq m)$

$$\lim_{m\to\infty} \zeta_{xy} = \zeta_{xy}.$$

In particular, $P_y(N(y) = \infty) = 1$.

If a nonnegative random variables has positive of being infinite, then

$$G(y, y) = E_y(N(y)) = \infty.$$

If $\zeta_{xy} = 0$, then $P_x(T_y = m) = 0$ for all finite positive integers m, so (27) implies that $P^n(T_y = m) = 0, \quad n \geq 1$; thus $G(x, y) = 0$.

If $\zeta_{xy} > 0$, then $P_x(N(y) = \infty) = \zeta_{xy} > 0$ and hence $G(x, y) = E_x(N(y)) = \infty$

3.6.3 Absorption Probabilities

Let y be transient state, since

$$\sum_{n=1}^{\infty} P^n (x,y) = G(x,y) < \infty, \qquad x \in \zeta .$$

Then

$$\lim_{n\to\infty} P^n(x,y) = 0, \qquad x \in \zeta .$$

3.5.1 Transient and Recurrent Chain

A Markov chain is called a transient chain if all of its states are transient and a recurrent chain if all of its states are recurrent.

3.6 Decomposition of the State Space

3.6.1 Definition

A non-empty set C of states is said to be closed if no inside of C leads to any state outside of C, i.e., if $\zeta_{xy} = 0,\ where\ \ x \in C$ And $y \notin C$. Equivalently, C is closed if and only if $P^n(x,y) = 0 \quad x \in C, y \notin C, n \geq 1$

If $\mathrm{P}(x,y) = 0 \quad x \in C, y \notin C$, then C is closed. If C is closed, then a Markov chain starting in C will with probability one stay in C for all time. If a is an absorbing state, then $\{a\}$ is closed.

3.6.2 Irreducible of a Close Set

A close set C is called irreducible if x leads to y for all choice of x and y in C.

Corollary 1: Let C be an irreducible closed set of recurrent states. Then $\zeta_{xy} = 1$, $P_x(N(y) = \infty) = 1$ and $G(x,y) = \infty$ for all choices of x and y in C.

An irreducible Markov chain is a chain whose state space is irreducible, that is, a chain in which every state leads back to itself and also to every other state.

Theorem 2: Let C be a finite irreducible closed set of state. Then every state in C is recurrent.

Assuming a Markov chain have a finite number states, the theorem implies that for a chain to be irreducible it must be recurrent. In a situation where the chain cannot be irreducible then we tried to determined which states are recurrent and which are transient.

Example 4: Consider a finite Markov chain having transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 1/5 & 2/5 & 1/5 & 0 & 1/5 \\ 0 & 0 & 0 & 1/6 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/4 & 0 & 3/4 \end{bmatrix} \end{array}$$

Determine which states are recurrent and which are transient.

Solution: the following matrix shows which state leads to which other states.

For example $P^2(1,3) = P(1,2)P(2,3) > 0$ and $P^2(2,0) = P(2,1)P(1,0) > 0$

0 is an absorbing state, hence also a recurrent state. Also $\{3,4,5\}$ is an irreducible closed set. By theorem (2), $3,4, and$ 5 are recurrent states. State 1 and 2 both leads to zero, but neither can be reached from zero. By Theorem (2) both 1 and 2 must be transient.

Let S_T denote the collection of transient states in S, and let S_R denote the collection of recurrent states.

Hence $S_T = \{1,2\}$ and $S_R = \{0,3,4,5\}$. the set S_T can be decomposed into disjoint irreducible closed set $C_1 = \{0\}$ and $C_2 = \{3,4,5\}$ irreducible hence $S_R = C_1 \cup C_2$.

3.6.3 Absorption Probabilities

Let *C* be closed irreducible recurrent set and $\zeta_c(x) = P_x(T_c < \infty)$ be the starting in *x* absorbing probability. A chain starting at *x* is absorbed by the set *C*.

Clearly $\zeta_c(x) = 1$, if $x \in C_1$ and $\zeta_c(x) = 0$ if $\zeta_c(x) = 0$ if $x \in C_i$ where $i \neq 1$ implies that *x* recurrent not in C_1.

What if $x \in \zeta_T$ then we can find $\zeta_c(x)$ to be

$$= \sum_{y \in C} P(x, y) + \sum_{y \in \zeta_T} P(x, y)\, \zeta_c(y), \qquad x \in \zeta_T .$$

This equation holds whether ζ_T is finite or infinite.

Theorem 3: Suppose the set ζ_T of transient states is finite and *C* be an irreducible closed set of recurrent states. The system of equations.

$$f(x) = \sum_{y \in C} P(x, y) + \sum_{y \in \zeta_T} P(x, y) f(y) \qquad x \in \zeta_T \qquad (33)$$

Has the unique solution $f(x) = \zeta_c(x) \qquad x \in \zeta_T \qquad (34)$

Example 5: Consider the Markov chain discussed in the previous example.

Find $\zeta_{10} = \zeta_{(0)}(1)$ and $\zeta_{20} = \zeta_{(0)}(2)$

Solution: by apply equation (33) with transition matrix in example 4 we have

$$\zeta_{10} = P(1,0) + P(1,1)\, \zeta_{10} + P(1,2)\, \zeta_{20}$$

$$= \frac{1}{4} + \frac{1}{2}\, \zeta_{10} + \frac{1}{4}\, \zeta_{20} \qquad (35)$$

And $$\zeta_{20} = P(2,1)\,\zeta_{10} + P(2,2)\,\zeta_{20} \tag{36}$$

$$= \frac{1}{5}\zeta_{10} + \frac{2}{5}\,\zeta_{20}\,. \tag{37}$$

Solving (35) and (37), we get $\zeta_{10} = \frac{3}{5}$ and $\zeta_{20} = \frac{1}{5}$.

By similar methods, we conclude that

$$\zeta_{\{3,4,5\}}(1) = \frac{2}{5} \text{ and } \zeta_{\{3,4,5\}}(2) = \frac{4}{5}$$

Alternatively, since $\sum \zeta_{c_i}(x) = 1 \quad x \in \zeta_T$

$\zeta_{\{3,4,5\}}(1) = 1 - \zeta_{\{0\}}(1) = 1 - \frac{3}{5} = \frac{2}{5}$,

And

$$\zeta_{\{3,4,5\}}(2) = 1 - \zeta_{\{3,4,5\}}(2) = 1 - \frac{1}{5} = \frac{4}{5}.$$

Note: Once a Markov chain starting at a transient state *x* enter an irreducible closed set *C* of recurrent states. It visits every state in *C*. thus

$$\zeta_{xy} = \zeta_c(x)\,, x \in \zeta_T, y \in C.$$

From this relation it follows that

$$\zeta_{13} = \zeta_{14} = \zeta_{15} = \zeta_{\{3,4,5\}}(1) = \frac{2}{5},$$

And

$$\zeta_{23} = \zeta_{24} = \zeta_{15} = \zeta_{\{3,4,5\}}(2) = \frac{4}{5}\,.$$

3.7 Birth and Death Chains

For an irreducible Markov chain either every state is recurrent or every state is transient, so that as irreducible Markov chain is either a recurrent chain or a transient chain. An irreducible Markov chain having only finitely many states is necessarily recurrent. In the case state space is infinite, it is not so easy to identify ζ_R and ζ_T.

But for the birth and death chain, we are able to do so. Consider a Birth and Death chain on the nonnegative integers or on the finite set $\{0,1, \dots \dots, d\}$, $d < \infty$.

The transition function is of the form

$$P(x,y) = \begin{cases} q_{x,} & y = x-1 \\ r_{x,} & y = x \\ P_{x,} & y = x+1 \end{cases}$$

Where $P_x + r_x + q_x = 1$ for $x \in \zeta, q_0 = 0, Prd = 0$ if $d \geq \infty$ then

$P_x > 0$ And $q_x > 0$ for $0 < x < d$.

Set $u(x) = P_x(T_a < T_b)$ where $a < x < b$ and $a, b \in \zeta$

Assume that $u(a) = 1$ and $u(b) = 0$ and if the birth and death chain start at y then by taking one step it goes from $y-1, y,$ or $y+1$ with respective probabilities p_y, r_y or q_y . It follows that

$$u(y) = q_y u(y-1) + r_y u(y) + p_y u(y+1), \qquad a < y < b \tag{38}$$

Since $r_y = 1 - p_y - q_y$, we write (38) as

$$u(y+1) - u(y) = \frac{q_y}{p_y}(u(y) - u(y-1)), \qquad a < y < b \tag{39}$$

$$\text{Set } \gamma_0 = 1 \text{ and } \gamma_y = q_1 \dots . q_y / p_1 \dots p_y , \qquad 0 < y < d \tag{40}$$

From (39), we see that

$$u(y+1) - u(y) = \frac{\gamma_y}{\gamma_{y-1}}(u(y) - u(y-1))$$

$$= \left(\frac{\gamma_y}{\gamma_{y-1}}\right)\left(\frac{\gamma_{y-1}}{\gamma_{y-2}}\right)[\, u(y-1) - u(y-2)]$$

$$= \left(\frac{\gamma_y}{\gamma_{y-1}}\right)\left(\frac{\gamma_{y-1}}{\gamma_{y-2}}\right) \dots \dots \dots \dots . \frac{\gamma_{a+1}}{\gamma_y}[\, u(a+1) - u(a)]$$

$$= \frac{\gamma_y}{\gamma_a}[\, u(a+1) - u(a)].$$

Consequently

$$u(y) - u(y+1) = \frac{\gamma_y}{\gamma_a}[\, u(a) - u(a+1)], \qquad a \leq y < b \tag{41}$$

Hence by summing (41) on $y = a, \dots \dots \dots .., b-1$ and recall that $u(a) = 1$ and $u(b) = 0$, then we conclude that

$$\frac{u(a)-u(a+1)}{\gamma_y} = \frac{1}{\sum_{y=a}^{b-1} \gamma_y} \tag{42}$$

By summing (42) on $y = x, x+1, \dots, b-1$ $\quad a < x < b$

We obtain

$$u(x) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} \qquad a < x < y$$

Therefore from definition of $u(x)$, it follows that

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} \qquad a < x < b \tag{43}$$

Subtracting (43) from 1 we have

$$P_x(T_b < T_a) = 1 - \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} \qquad a < x < b$$

$$P_x(T_b < T_a) = \frac{\sum_{y=a}^{x-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} \qquad a < x < b \tag{44}$$

Example 5: A gambler playing roulette makes a series of one dollar bets. He has respective probabilities $\frac{9}{10}$ and $\frac{10}{19}$ of winning and losing each bet. The gambler decided to quit playing as soon as his net winning reach 25 dollars or his net losses reach 10 dollars.

(a) Find the probability that when he quit playing he will have won 25 dollars

(b) Find his expected loss.

Solution

Since his respective probabilities of winning and losing each bet are $\frac{9}{10}$ and $\frac{10}{19}$ respectively and he also decided to quit playing as soon as his net winning reach 25 dollars or his net losses reach 10 dollar therefore, Let X_n denote the capital of the gambler at time *n* with $X_0 = \$10$

X_n form a death-birth chain on $\{0, \dots \dots \dots, 35\}$ with birth and death rates

$$P_x = \frac{9}{10}, \qquad 0 < x < 35$$

$$q_x = \frac{10}{19}, \qquad 0 < x < 35$$

Where 0 and 35 are absorbing states.

To solve (a) applied equation (44) we have

$$P_{10}(T_{35} < T_0) = \frac{\sum_{y=0}^{9} \gamma^y}{\sum_{y=0}^{34} \gamma^y} \qquad \text{Where } \gamma^y = \left(\frac{10}{19} \times \frac{19}{9}\right) = \frac{10}{9}.$$

Therefore we have

$$P_{10}(T_{35} < T_0) = \frac{\sum_{y=0}^{9} \left(\frac{10}{9}\right)^{10}}{\sum_{y=0}^{34} \left(\frac{10}{9}\right)^{35}}$$

$$= \frac{\left(\frac{10}{9}\right)^{10} - 1}{\left(\frac{10}{9}\right)^{35} - 1} = 0.047$$

Thus the gambler has probability 0.047 of winning 25 dollars.

Then for (b), his expected loss is $10 - 35(0.047) = \$8.36$.

In the reminder of this part, we consider a Birth and Death chain on the nonnegative integers which is irreducible that is $P_x > 0$ for $x > 0$ and $q_x > 0$ for $x \geq 1$. We will determine when such a chain is recurrent and when it is transient.

Let consider a special case of equation (43)

$$P_1(T_0 < T_n) = 1 - \frac{1}{\sum_{y=0}^{n-1} \gamma_y} \qquad n > 1 \tag{45}$$

Let the process start in state 1 so,

$$1 \leq T_2 < T_3 \dots \tag{46}$$

It follows from (46) that $\{T_0 < T_n\}, \quad n > 1$ forms an expanding sequence of events. Assuming that $A_n = \{T_0 < T_n\}$, then the expanding sequence will be $A_n \subset A_{n+1}$ given that $X_0 = 1$

Then

$$\lim_{n\to\infty} P_1\{T_0 < T_n\} = \lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P(\cup_n A_n)$$

Since A_n is an expanding sequence,

$$\lim_{n\to\infty} P_1\{T_0 < T_n\} = P_1(T_0 < T_n) \quad \text{for some } n. \tag{47}$$

Then (46) implies that $T_n \geq n$ and thus $T_n \to \infty$ as $n \to \infty$ hence, the even $\{T_0 < T_n\}$ for some $n > 1$ occurs if and only if the event$\{T_0 < \infty\}$ occurs.

We rewrite (47) as

$$\lim_{n\to\infty} P_1(T_0 < T_n) = P_1(T_0 < \infty) \tag{48}$$

Hence by (45) and (48) we have

$$P_1(T_0 < \infty) = 1 - \frac{1}{\sum_{y=0}^{\infty} \gamma_y} \tag{49}$$

We now show that the chain is recurrent $\Leftrightarrow \sum_{y=0}^{\infty} \gamma_y = \infty$ (irreducible)

(a) If the chain is recurrent, then $P_1(T_0 < \infty) = 1$and hence (49) implies

$$\sum_{y=0}^{\infty} \gamma_y = \infty$$

(b) If $\sum_{y=0}^{\infty} \gamma_y = \infty$, show that the chain is recurrent.

Since $P(0, y) = 0$ for $y \geq 2$ hence,

$P_0(T_0 < \infty) =$P$(0,0) + P(0,1)\ P_1(T_0 < \infty) = 1.$

0 is a recurrent state, thus whole chain is recurrent, since the chain is irreducible.

In conclusion an irreducible birth and death chain on $\{0, 1, 2, \dots \dots.\}$ is recurrent if and only if

$$\sum_{x=1}^{\infty} \frac{q_1 \cdots q_x}{p_{1\dots} p_x} = \infty \,.$$

Example 6: Consider the birth and death chain on $\{0, 1, 2, \dots \dots.\}$ defined by

$P_x = \frac{x+2}{2(x+1)}$ and $q_x = \frac{x}{2(x+1)}$ $x \geq 0$.

Determine whether this chain is recurrent or transient.

Solution

Since $\frac{q_x}{p_x} = \frac{x}{x+2}$, it follows that

$$\gamma_y = \sum_{x=1}^{\infty} \frac{q_1 \dots . q_x}{p_{1\dots} p_x} = \frac{1, 2 \dots x}{3, 4 \dots (x+2)} = \frac{2}{(x+1)(x+2)}$$

$$= 2\left(\frac{1}{x+1} - \frac{1}{x+2}\right).$$

Thus

$$\sum_{x=1}^{\infty} \gamma_x = 2 \sum_{x=1}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots\right)$$

$$= 2\,\frac{1}{2} = 1.$$

We conclude that the chain is transient.

Example 7: Consider a Markov chain on the nonnegative integers such that starting from x, the chain goes to state $x+1$ with probability P, where $0 < p < 1$ and goes to state 0 with probability$(1-p)$.

(a) Show that this chain is irreducible.

(b)Find $P_0(T_0 = n)$,$n \geq 1$.

(c) Show that the chain is recurrent.

Solution

(a) Since every state leads back to itself and also to every other state, $\left(\gamma_{yx} > 0\right)$, the chain is irreducible.

(b)By applying

$$P_x\left(T_y = n\right) = \sum_{z \neq y} P(x,z)\, P_z(T_y = n-1)$$

$P_0(T_0 = n) = \sum_{z \neq 0} P(0,z)\ P_z(T_0 = n-1)$.

If $n \geq 1$, for $n = 1$ $\quad P_0(T_0 = 1) = P(0,0) = 1 - \varkappa$.

For $n = 2$

$$P_0(T_0 = 2) = \sum_{z \neq 0} P(0,z)\ P_z(T_0 = 1)$$

$$= P(0,1) = \ P_1(T_0 = 1) = P(1 - \varkappa).$$

For $n = 3$

$$P_0(T_0 = 3) = \sum_{z \neq 0} P(0,z)\ P_z(T_0 = 2)$$

$$= P(0,1) = \ P_1(T_0 = 2)$$

$$= P(0,1)\, P(1,2)\, P(2,0)$$

$$= P^2(1 - \varkappa).$$

By induction, for *n* we have

$$P_0(T_0 = 3) = \sum_{z \neq 0} P(0, z)\ P_z(T_0 = n - 1)$$

$$= P(0,1)\ P_1(T_0 = n - 1)$$

$$= P^{n-1}(1 - \varkappa)\ .$$

(c) A state *y* is recurrent if $\gamma_{yy} = 1$.

Try *0* state.

$$\gamma_{00} = P_0(T_0 < \infty) = \sum_{n=1}^{\infty} P_0(T_0 = n)$$

$$= \sum_{n=1}^{\infty} P^n(1 - \varkappa) = (1 - \varkappa) \sum_{n=1}^{\infty} P^{n-1} = 1 - \varkappa\left(\frac{1}{1 - \varkappa}\right) = 1.$$

Since $\gamma_{00} = 1$, *0* is recurrent, and since the chain is irreducible, then every state is recurrent, thus the chain is recurrent.

Example 8: Consider the Markov Chain on $\{0,1, \ldots ,5\}$ having transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 7/8 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 0 & 1/4 & 0 \\ 0 & 1/5 & 0 & 1/5 & 1/5 & 2/5 \end{bmatrix} \end{array}$$

(a) Determine which states are transient and which are recurrent

(b) Find $\gamma_{\{0,1\}}(x)$, where $x = 0, \ldots ,5$

Solution

(a)C_1 = the recurrent states are 0,1,2,4.

$C_2 =$ the transient states are 3 and 5 .

(b)Recall that from (33)

$$f(x) = \sum_{y \in C} P(x,y) + \sum_{y \in \zeta_T} P(x,y) f(y) \;\; where \;\; x = 0,1, \dots ,5$$

$C_1 = \{0,1\}$ is recurrent

When $x = 0$ then

$$\gamma_{\{0,1\}}(0) = \sum_{y \in \{0,1\}} P(0,y) + \sum_{y \in \{3,5\}} P(0,y) \gamma_{\{0,1\}}(y)$$

$$= P(0,0) + P(0,1) = 1.$$

When $x = 1$ we have

$$\gamma_{\{0,1\}}(1) = \sum_{y \in \{0,1\}} P(1,y) + \sum_{y \in \{3,5\}} P(1,y) \gamma_{\{0,1\}}(y)$$

$$= P(1,0) + P(1,1) = 1.$$

Therefore $\gamma_{\{0,1\}}(0) = \gamma_{\{0,1\}}(1) = 1.$

When $x = 2$ then

$$\gamma_{\{0,1\}}(2) = \sum_{y \in \{0,1\}} P(2,y) + \sum_{y \in \{3,5\}} P(2,y) \gamma_{\{0,1\}}(y)$$

$$= P(2,0) + P(2,1) = 0.$$

When $x = 3$ then we have

$$\gamma_{\{0,1\}}(3) = \sum_{y \in \{0,1\}} P(3,y) + \sum_{y \in \{3,5\}} P(3,y) \gamma_{\{0,1\}}(y)$$

$$= P(3,0) + P(3,1) + P(3,3)\gamma_{\{0,1\}}(3) + P(3,5)\gamma_{\{0,1\}}(5) \, .$$

Therefore

$$\gamma_{\{0,1\}}(3) = {}^{1}/_{4} + {}^{1}/_{4} + {}^{1}/_{4}\,\gamma_{\{0,1\}}(5).$$

When $x = 4$ then

$$\gamma_{\{0,1\}}(4) = \sum_{y\in\{0,1\}} P(4,y) + \sum_{y\in\{3,5\}} P(4,y)\gamma_{\{0,1\}}(y).$$

$$= P(4,0) + P(4,1) + P(4,3)\gamma_{\{0,1\}}(3) + P(4,5)\gamma_{\{0,1\}}(5) = 0\ .$$

When $x = 5$ then

$$\gamma_{\{0,1\}}(5) = \sum_{y\in\{0,1\}} P(5,y) + \sum_{y\in\{3,5\}} P(5,y)\gamma_{\{0,1\}}(y).$$

$$\gamma_{\{0,1\}}(5) = {}^{1}/_{5} + {}^{1}/_{5}\,\gamma_{\{0,1\}}(3) + {}^{2}/_{5}\,\gamma_{\{0,1\}}(5)\ .$$

By collecting like terms in the above equation we have,

$${}^{3}/_{5}\,\gamma_{\{0,1\}}(5) = {}^{1}/_{5} + {}^{1}/_{5}\,\gamma_{\{0,1\}}(3).$$

Chapter 4

CONCLUSION

The Markov chain is very important stochastic model in probability theory. With the good understanding of Markov chains, it can be practically applied in different stages and areas of life. For example, if we make an attempt of taking a risk to gambling of which we cannot determine the future outcome, then the proper understanding of Markov chain is applicable.

There are other areas where Markov chain can be applied. For an example, a Markov chain model is formulated to solve a problem on the "Genetics of Inbreeding". Assuming two individuals are randomly mated then in the next generation, two of their offspring of opposite sex are randomly mated. The process of brother and sister mating or inbreeding continues each year. This process can be formulated as a finite discrete time Markov chain.

Another example is a new state of our wardrobe which depends on the present launched brands of clothes, if a cloth is torn out or old then it gets removed from the wardrobe.

The Markov chain is an example of a stochastic process which is applied to our daily lives. By acquiring and understanding the concept, we have and know more about the expectation and possible outcomes of future predictions. Therefore the knowledge of the Markov chain cannot be ignored.

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