GRAVITATIONAL LENSING EFFECT ON THE HAWKING RADIATION OF DYONIC BLACK HOLES

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In this paper, we analyze the Hawking radiation (HR) of a non-asymptotically flat (NAF) dyonic black hole (dBH) in four-dimensional (4D) Einstein-Maxwell-Dilaton (EMD) gravity by using one of the semiclassical approaches which is the so-called Hamilton-Jacobi (HJ) method. We particularly motivate on the isotropic coordinate system (ICS) of the dBH in order to highlight the ambiguity to be appeared in the derivation of the Hawking temperature (T_H) via the HJ method. Besides, it will be shown that the ICS allows us to write the metric of the dBH in form of the Fermat metric, which renders possible of identification of the refractive index (n) of the dBH. It is unraveled that the value of n and therefore the gravitational lensing effect is decisive on the the tunneling rate of the HR. We also uncloak how one can resolve the discrepancy about the T_H of the dBH in spite of that lensing effect.

I. INTRODUCTION

In classical point of view, black holes (BHs) are such objects that due to their huge attractive forces even the light cannot escape from them. But, if we take the quantum effects into account, they could be gray i.e., not entirely black. That surprising claim was made by Hawking [1, 2] over forty years ago. Hawking thought that the quantum mechanics should allow the particles to tunnel through the event horizon of the BH. After making marvelous calculations, Hawking theoretically showed that BHs should emit a steady flux of thermal radiation with a temperature $T_H = \frac{\kappa}{2\pi}$ in which κ is the gravitational field strength at the event horizon. On this basis, a number of new methods have been proposed (a reader may consult [3, 4] for the topical review).

Among those methods, the universal consent one is the tunneling method which is devised by Kraus and Wilczek (KW) [5, 6]. KW proved that the HR can be thought as a dynamical model in which while BH shrinks, the particles radiate. In this dynamical model, both energy conservation and self-gravitational effects which were not considered in the original derivation of HR [1, 2] are taken into account. In fact, they treated the each radiating particle as a self-gravitating thin spherical shell with energy ω. Hence, by using the null geodesics of the associated particles, it became possible to get the action (I) (and henceforth its imaginary part, ImI) of the tunneling particle, which yields the tunneling rate as $\Gamma \approx e^{-2ImI}$ and admits the T_H due the well-known quantum mechanical result, $\Gamma \approx e^{-\omega/T}$. Later on, KW's method was improved by Parikh and Wilczek (PW) [7] in order to show that the spectrum of the HR is not pure thermal, which implies unitarity of the underlying quantum process, and the resolution of the information loss paradox [8, 9].

Among the other methods, the HJ method which uses the relativistic HJ equation attracts great interest. This method firstly devised by Angheben et al. [10] (and references therein). They indeed built up an alternative method for calculating ImI. The HJ method within the WKB approximation ignores the self-gravitational effect and energy conservation. Generally, the HJ method is employed by substituting a suitable ansatz into the relativistic HJ equation. For the separability, the chosen ansatz should involve the symmetries of the spacetime. The final radial equation is solved by an integration which is along the classically forbidden trajectory starting from inside of the BH and extends to the observation point. But during this calculation, the integral always possesses a pole located at the horizon. This pole can be circumvented by applying the method of complex path analysis [11–13].

According to us, most of the BHs in the macrocosm should be portrayed with NAF geometries. The mainstay of our thoughts is the Friedmann-Robertson-Walker [14], which is assumed to be one of the best theoretical models in describing our universe and, as is known, its geometry is NAF. For this reason, we focus on the commonly acceptable NAF BHs in order to compute their T_H by employing the HJ method. In the same line of thought, we consider the dBHs which are originally found by Yazadjiev [15]. The dBH solutions considered here have two horizons hiding a curvature singularity at the origin. As mentioned in [15], they may serve as backgrounds for non-supersymmetric

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holography and lead to possible extensions of AdS/CFT correspondence [16]. On the other hand, when the standard HJ method is applied to the dBH which can be expressed in the ICS, we encounter with a discrepancy between its computed horizon's temperature and its standard T_H. This discrepancy problem arising during the use of the ICS has become popular anew. Recently, Schwarzschild [17] and the linear dilaton BHs (from now on we abbreviate it as LDBHs) [18] within the ICS have been thoroughly studied. Inspiring from [18], we shall also calculate the n of the medium of the dBH. Then, we highlight the effect of n on the tunneling rate, and consequently on the T_H. The latter remark could be associated with the gravitational lensing effect on the HR as in [19].

The structure of the paper is as follows. In Sec. II, we introduce and review the dBH. We give its some geometrical and thermodynamical features. For the sake of example, we follow the systematic HJ method for deriving the T_H of the dBH in its naive coordinates. In Sec. III, we firstly transform the naive coordinates of the dBH into the ICS. Then, the effect of the n on the calculation of the ImI is explicitly represented. The obtained horizon temperature is the half of the T_H. At the last stage of this section, a detailed analytical analysis is performed in order to resolve the differences in the temperatures. We draw our conclusions in Sec. IV.

The paper uses the signature $(-,+,+,+)$ and units where $c = G = \hbar = k_B = 1$

II.   DBH GEOMETRY AND HJ METHOD

In this section, we shall first describe the dBH spacetime which is the solution to the 4D EMD gravity. Its line-element represents a static and spherically symmetric solution in the low-energy limit of the string theory in which gravity is coupled to the electromagnetic field and dilaton. These solutions posses both electric and magnetic charges, and they admit NAF geometry. Then, we will apply the HJ method to the dBH metric given in its naive coordinates in order to show that it concludes with the T_H.

In the EMD theory [20], the 4D action is given by

$$S = \int d^4x \sqrt{-g}(\Re - 2g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu}), \tag{1}$$

where $\Re$ denotes the scalar curvature with respect to the spacetime metric $g_{\mu\nu}$, φ is the scalar dilaton field with a coupling constant α and $F_{\mu\nu}$ is the electromagnetic field. The dBH solution of the above action is given by [15]. Here, we shall take $\alpha = 1$ as it has been considered very recently in [21]. The dBH spacetime is described by the following line-element

$$ds^2 = -fdt^2 + f^{-1}dr^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

in which the metric functions are

$$f = \frac{(r - r_+)(r - r_-)}{r_0 r},$$
$$R^2 = r_0 r. \tag{3}$$

The constants r_+ and r_- represent inner and outer horizons, respectively. While r_+ and r_- determine the value of the magnetic charge Q_m, another constant r_0 governs the electric charge Q_e. Those relationships are given by

$$r_0 = \sqrt{2}Q_e, \qquad r_+r_- = 2Q_m^2, \tag{4}$$

When the Q_m is vanished by setting the inner horizon to zero i.e., $r_- = 0$, the metric functions (3) describe the pure electrical LDBH spacetime. The eponyms of the LDBH are Clément and Gal'tsov [22]. One of the most interesting features of the LDBHs is that while performing the HR, they undergo an isothermal process. Namely, their HR is such a special process that the energy (mass) transfer out of them typically happens at such a slow rate that thermal equilibrium is always maintained. Today, there are numerous studies on the LDBHs (see for instance [18, 23–29]).

Due to the NAF structure of the dBH, one should follow Brown and York's quasi-local mass (M) formalism [30]. Thus, one derives a relationship between the horizons and the M as follows

$$M = \frac{1}{4}(r_+ + r_-), \tag{5}$$

The T_H is expressed in terms of the surface gravity κ [31] as follows

$$T_H = \frac{\kappa}{2\pi} = \left.\frac{\partial_r f}{4\pi}\right|_{r=r_+}, \tag{6}$$

After substituting the metric function f (3) into the above equation, T_H of the dBH yields

$$T_H = \frac{r_+ - r_-}{4\pi r_0 r_+}, \tag{7}$$

In the extremal BH solution i.e., $r_+ = r_-$, it can be easily seen that we obtain zero temperature, which is a well-known issue.

Here, we focus on the problem of a scalar particle which crosses the event horizon from inside to outside. While it acts this classically forbidden motion, we ignore the back-reaction and self-gravitational effects. Within the semi-classical framework, the action I of the particle in question should satisfy the relativistic HJ equation which is given by

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0, \tag{8}$$

where m is the mass of the scalar particle, and $g^{\mu\nu}$ represents contravariant form of the metric tensors of Eq. (2). By considering Eqs.(2), (3) and (8), we get

$$\frac{-1}{f}(\partial_t I)^2 + f(\partial_r I)^2 + \frac{1}{R^2}\left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi I)^2\right] + m^2 = 0, \tag{9}$$

The technique of separation of variables is the commonly used method for the HJ equation. To this end, the I is chosen as follows

$$I = -Et + W(r) + J(x^k), \tag{10}$$

So, one finds

$$\partial_t I = -E, \qquad \partial_r I = \partial_r W(r), \qquad \partial_k I = J_k, \tag{11}$$

where J_k's are constants in which $k = 1, 2$ label angular coordinates θ and φ, respectively. The norm of the timelike Killing vector ∂_t becomes (negative) unity at a particular location:

$$r \equiv R_0 = \frac{1}{2}\left[r_+ + r_- + r_0\left(1 + \sqrt{\frac{(r_+ - r_-)^2}{r_0^2} + 2\frac{(r_+ + r_-)}{r_0} + 1}\right)\right], \tag{12}$$

It means that when a detector of an observer is located at R_0 which is outside the horizon, the energy of the particle measured by the observer corresponds to E. After substituting Eq. (11) into Eq. (9) and solving it for $W(r)$, we find out that

$$W(r) = \pm\int \frac{\sqrt{E^2 - \frac{f}{R^2}[J_\theta^2 + \frac{J_\varphi^2}{\sin^2(\theta)} + (mR)^2]}}{f}dr, \tag{13}$$

The positive and negative signs appeared in the above equation is due to the quadratic form of Eq. (9). Solution of Eq. (13) with "+" signature corresponds to outgoing scalar particles and the other solution i.e., the solution with "−" signature refers to the ingoing particles. Evaluating the above integral around the pole at the horizon (following the Feyman's prescription given by [11–13, 32]), one reaches to

$$W(r) \cong W_{(\pm)} = \pm W_h + \chi, \tag{14}$$

where $W_h = \frac{i\pi E r_+ r_0}{r_+ - r_-}$ and χ is a complex integration constant. Thus, we can deduce that ImI arises from the pole at the horizon and the complex constant χ. Thence, we can determine the probabilities of ingoing and outgoing particles while crossing r_+ as

$$P_{out} = e^{-2ImI} = \exp\left[-2ImW_{(+)}\right], \tag{15}$$

$$P_{in} = e^{-2ImI} = \exp\left[-2ImW_{(-)}\right], \tag{16}$$

In the classical point of view, a BH absorbs any ingoing particles passing its horizon. In other words, there is no reflection for the ingoing waves which corresponds to $P_{in} = 1$. This is enabled by setting $W_{(-)} = 0$ which amounts to $Im\chi = \frac{\pi E r_+ r_0}{r_+ - r_-}$. This choice signifies also that the ImI for a tunneling particle solely depends on $W_{(+)}$. Namely, we get

$$ImI = ImW_{(+)} = 2\pi E r_+ r_0 r_+ - r_-, \tag{17}$$

Therefore, the tunneling rate for the dBH can be obtained as

$$\Gamma = P_{out} = \exp\left(-\frac{4\pi E r_+ r_0}{r_+ - r_-}\right), \tag{18}$$

and according to [7]

$$\Gamma = e^{-\beta E}, \tag{19}$$

in which β denotes the Boltzmann factor and $T = \frac{1}{\beta}$, one can easily read the horizon temperature of the dBH as

$$\check{T}_H = \frac{r_+ - r_-}{4\pi r_+ r_0}. \tag{20}$$

This is equal to the T_H obtained in Eq. (7).

III.   ISOTROPIC COORDINATES OF THE DBH AND LENSING EFFECT ON ITS HR

The universe is homogeneous and isotropic on large scales. Isotropy of the universe is well known from experiments such as distribution of radio galaxies at Very Large Array in Mexico, Cosmic X-Ray Background and Cosmic Microwave Background at Planck and WMAP satellites [33]. In this section, we are deal with the ICS. Isotropic coordinates are used in general to kill the horizon and make the time direction a Killing vector. Furthermore, time slices appear as Euclidean with a conformal factor. By using this transformation, the index of refraction, n, of light rays around the BH can be calculated.

The dBH solution in isotropic coordinates [34] can be found by the following transformation

$$r = r_- + \frac{1}{4\rho}(\rho + \rho_h)^2 \tag{21}$$

where $\rho_h = r_+ - r_-$ defines the horizon in the ICS. By making this transformation, the metric (2) becomes

$$ds^2 = -Fdt^2 + G(d\rho^2 + \rho^2 d\Omega^2), \tag{22}$$

with

$$F = \frac{1}{4\rho r_0}\left[\frac{\left(\rho^2 - \rho_h^2\right)^2}{\left(\rho + \rho_h\right)^2 + 4r_-\rho}\right], \tag{23}$$

$$G = \frac{r_0}{4\rho^3} \left[4\rho r_+ + (\rho - \rho_h)^2 \right], \tag{24}$$

Once we rewrite the metric (22) as follows

$$ds^2 = F(-dt^2 + \Delta), \tag{25}$$

we obtain the dBH in the form of the Fermat metric [35]

$$\Delta = n^2(d\rho^2 + \rho^2 d\Omega^2), \tag{26}$$

in which, as mentioned before, n is called the index of refraction or the refractive index. For the dBH medium, we compute it as

$$n = \frac{r_0 \sqrt{[(\rho - \rho_h)^2 + 4\rho r_+][(\rho + \rho_h)^2 + 4\rho r_-]}}{\rho(\rho^2 - \rho_h^2)}, \tag{27}$$

When we use the HJ equation (8) on the ICS metric (22), we get

$$\frac{-1}{F}(\partial_t I)^2 + \frac{1}{G}(\partial_\rho I)^2 + \frac{1}{G\rho^2} \left[(\partial_\theta I)^2 + \frac{1}{\sin^2 \theta}(\partial_\varphi I)^2 \right] + m^2 = 0, \tag{28}$$

It is possible to obtain the solution to the above equation in the following form

$$I = -Et + W_{iso}(\rho) + J(x^i), \tag{29}$$

Thus we get an integral solution for $W_{iso}(\rho)$ as

$$W_{iso}(\rho) = \pm \int n \sqrt{E^2 - \frac{F}{G\rho^2} \left(J_\theta^2 + \frac{J_\varphi^2}{\sin^2 \theta} \right) - m^2 F} d\rho, \tag{30}$$

Around the horizon the above solution reduces to

$$W_{iso}(\rho) \cong W_{iso(\pm)} = \pm E \int n d\rho, \tag{31}$$

One can easily observe that the n of the dBH is decisive on the $W_{iso(\pm)}$. After making a straightforward calculation covering the Feyman's prescription, we obtain

$$\begin{aligned} W_{iso(\pm)} &= \pm i2\pi E r_0 \frac{\sqrt{r_+[(\rho_h + r_-)]}}{\rho_h} + \mu \\ &= \pm i2\pi E \frac{r_0 r_+}{r_+ - r_-} + \mu, \end{aligned} \tag{32}$$

where μ is another complex integration constant. Similar to the procedure followed in the previous section i.e., setting $P_{in} = 1$ which yields $Im\mu = 2\pi E r_0 \frac{r_+}{r_+ - r_-}$, we derive the imaginary part of the action I of the tunneling particle as follows

$$ImI = ImW_{iso(+)} = 4\pi E \frac{r_0 r_+}{r_+ - r_-}, \tag{33}$$

Thus, by considering the tunneling rate expresions (18) and (19) one obtains the horizon temperature of the dBH as

$$\check{T}_H = \frac{r_+ - r_-}{8\pi r_+ r_0}. \tag{34}$$

As it can be seen above, the result is the half of the standard Hawking temperature; $\check{T}_H = \frac{1}{2}T_H$. Therefore, we deduce that the present result infers that ICS leads to an apparent temperature of the dBH that it is less than its T_H. This phenomenon resembles in the case that the apparent depth h of a fish swimming at a depth d below the surface of a pool is less than the true depth d. Namely, $h < d$. This illusion is due to the discrepancy of the refractive indexes between the mediums. In particular, such events take place when $n_{observer} < n_{object}$, as happened at here. Because, it can be checked from Eq. (27) that the n value of the medium of an observer who is placed at the outer region ($\rho > \rho_h$) is less than the n for the medium of the near the horizon ($\rho \approx \rho_h$). Since the value of $W_{iso(\pm)}$ (31) acts as a decision-maker on the value of the horizon temperature $\check{T}_H$ of the dBH, one can deduce that the n (27), and consequently the gravitational lensing effect, plays an important role on the observation of the true T_H.

On the other hand, we admittedly know that coordinate transformation of the naive coordinates to the isotropic coordinates should not alter the true temperature of the BH. Since the appearances are deceptive, one should make deeper analysis to get the real. This problem has been recently discussed by Chatterjee and Mitra [17] for the Schwarzschild BH and by Sakalli and Mirekhtiary [18] for the LDBH. They have proven that during the evaluation of the integral (31) around the horizon, the path across the horizon involves a change of $\pi/2$ instead of π. By getting inspired from their studies, here we shall follow the same strategy to overcome that factor-2 problem. For this purpose, we rewrite Eq. (21) as follows

$$\begin{aligned} r &= r_- + \frac{1}{4\rho}\left(\rho + \rho_h\right)^2, \\ &= r_+ + \frac{1}{4\rho}\left(\rho - \rho_h\right)^2, \end{aligned} \tag{35}$$

which means that

$$\frac{dr}{r - r_+} = -\frac{d\rho}{\rho} + \frac{2d\rho}{\rho - \rho_h}. \tag{36}$$

The first term at the right hand side (RHS) of the above equation does not admit any imaginary part at the horizon. Only the second term of the RHS of the Eq. (36) admits the imaginary result. On the other hand, in order to match the RHS and the left hand side results any imaginary contribution coming from $\frac{d\rho}{\rho - \rho_h}$ must be half of the $\frac{dr}{r - r_+}$. The latter remark produces a factor $i\pi/2$ for the integral (31) and therefore it results in $ImW_{iso(+)} = 2\pi E \frac{r_0 r_+}{r_+ - r_-}$ as obtained in the previous section. Thus, we read the horizon temperature as $\check{T}_H = \frac{r_+ - r_-}{4\pi r_+ r_0}$ which is nothing but the T_H.

IV. CONCLUSION

In this study, the HR of the dBH which has double horizons is studied via the HJ method. We have first applied the associated HJ method to the dBH metric expressed with the naive coordinates. Thus, we have explicitly shown how the HJ method produces the T_H. On the other hand, the most interesting part of the present paper belongs to the Sec. III in which the dBH spacetime is transformed into the ICS. In this system, with the aid of the Fermat metric we have managed to determine the n of the dBH. In particular, it is proven that the n plays a decisive role on the tunneling rate. Unlike to the naive coordinates, in the ICS the integration around the pole which appears at the horizon has led the factor-2 problem in the horizon temperature. For fixing this discrepancy, we inspired from recent studies [17] and [18] which have demonstrated how the proper regularization of singular integrals give the true horizon temperature, i.e., T_H, in the ICS. As a result, it has been clarified that the path across the horizon entails the value of $\frac{i\pi}{2}$ on the integration instead of $i\pi$. Our results are in accordance with the work of [18] if one gets the

limit $r_- \to 0$.

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