# On a Particular Thin-shell Wormhole

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In this paper, using a black hole with scalar hair, we construct a scalar thin-shell wormhole (TSW) in 2+1 dimensions by applying the Visser's famous cut and paste technique. The surface stress, which is concentrated at the wormhole throat is determined by using the Darmois-Israel formalism. By employing the various gas models, we study the stability analysis of the TSW. The region of stability is changed by tuning the parameters of "$l$" and "$u$". It is observed that the obtained particular TSW originated from the black hole with scalar hair could be more stable with particular $l$ parameter, however it still needs exotic matter..

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## I. INTRODUCTION

In 1988 Morris and Thorne [1] devised the traversable wormholes, which are the solutions of the Einstein's equations of gravitation. They are cosmic shortcuts that connect two points of the Universe by a throat-like geometry. However, they violate one or more of the so-called energy conditions [weak energy condition (WEC), null energy condition (NEC), and strong energy condition (SEC)]; see for instance Refs. [2–4]. Because of this fact, most physicists agree that wormholes require exotic matter – a kind of antigravity – to keep their throat (the narrowest point) open [5]. Conversely, some of the physicists studying in this subject claimed that wormholes like the TSW can be supported by normal matter [6, 7].

Firstly, Visser [8] proposed the method of how the TSWs can be constructed via the Israel's junction conditions [9]. It is shown that the amount of exotic matter [10] around the throat can be minimized with a suitable choice of the geometry of the wormhole. Following the Visser's prescription, today there are many studies in the literature focused on the build of the TSWs described in the arbitrary (lower/higher) dimensions (see for instance [11–33]). In this paper, we consider the scalar hair black hole (SHBH) in 2+1 dimensions that is the solution to the Einstein-Maxwell theory with self-interacting scalar field described by the Liouville potential $V(\phi)$ [34]. Then, using the standard procedure of the cut and paste technique we construct TSW, and test its stability for the different physical gas states.

*Our main motivation in constructing a thinshell wormhole is to minimize the exotic matter, which is in general the main source for supporting the throat. In this paper, we focus on the stability of the SHBH spacetime in 2+1 dimensions inasmuch as this black hole depends on two variables and by fixing them, it is possible to reach a stable solutions.*

The paper is organized as follows: In Sec. 2, we first give a brief introduction about the SHBH described within 2+1 dimensional geometry. In Sec. 3, we firstly setup TSW, and then apply the various gas models to the equation of state (EoS) for testing its stability. The paper ends with our conclusions in Sect. 4.

## II. SHBH SPACETIME

In this section, we shall briefly overview the SHBH [34]. The following action describes the Einstein-Maxwell gravity that is minimally coupled to a scalar field $\phi$

$$S = \int d^3r\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - F^2 - V(\phi)\right), \tag{1}$$

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where $R$ denotes the Ricci scalar, $F = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, and $V(\phi)$ stands for the scalar ($\phi$) potential.SHBH is the solution to the action (1), which was found by [34] as follows

$$ds^2 = -f(r)dt^2 + \frac{4r^2dr^2}{f(r)} + r^2d\theta^2, \tag{2}$$

where the metric function is given by

$$f(r) = \frac{r^2}{l^2} - ur. \tag{3}$$

Here $u$ and $l$ are constants, and event horizon of the BH (2) is located at $r_h = u\ell^2$. It is clear that this BH possesses a non-asymptotically flat geometry. Metric (2) can alternatively be rewritten in the following form

$$ds^2 = -\frac{r}{\ell^2}(r - r_h)\,dt^2 + \frac{4r\ell^2dr^2}{(r - r_h)} + r^2d\theta^2. \tag{4}$$

The singularity located at $r = 0$ can be best seen by checking the Ricci and Kretschmann scalars:

$$R = -\frac{2r + r_h}{4r^3\ell^2}, \tag{5}$$

$$K = \frac{4r^2 - 4r_hr + 3r_h^2}{16r^6\ell^4}. \tag{6}$$

The scalar field and potential are respectively given by [34]

$$\phi = \frac{\ln r}{\sqrt{2}}, \tag{7}$$

$$V(\phi) = \frac{\lambda_1 + \lambda_2}{r^2}, \tag{8}$$

in which $\lambda_{1,2}$ are constants. The corresponding Hawking temperature of the BH (see, for example, [35]) is as follows

$$T_H = \frac{1}{4\pi}\left.\frac{\partial f}{\partial r}\right|_{r=r_h} = \frac{1}{8\pi\ell^2}, \tag{9}$$

which is constant. Having a radiation with constant temperature is the well-known isothermal process. It is worth noting that Hawking radiation of the linear dilaton black holes exhibits similar isothermal behavior [35**?** –42].

## III. STABILITY OF TSW

In this section, we take two identical copies of the SHBHs with ($r \geq a$):

$$M^{\pm} = (x|r \geq 0),$$

and the manifolds are bounded by hypersurfaces $M^+$ and $M^-$, to get the single manifold $M = M^+ + M^-$, we glue them together at the surface of the junction

$$\Sigma^{\pm} = (x|r = a).$$

where the boundaries $\Sigma$ are given . On the shell, the spacetime can be chosen to be

$$ds^2 = -d\tau^2 + a(\tau)^2d\theta^2, \tag{10}$$

where $\tau$ represents the proper time [21]. Setting coordinates $\xi^i = (\tau, \theta)$, the extrinsic curvature formula connecting the two sides of the shell is simply given by [15]

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm}\left(\frac{\partial^2x^{\gamma}}{\partial\xi^i\partial\xi^j} + \Gamma_{\alpha\beta}^{\gamma}\frac{\partial x^{\alpha}}{\partial\xi^i}\frac{\partial x^{\beta}}{\partial\xi^j}\right), \tag{11}$$

where the unit normals ($n^\gamma n_\gamma = 1$) are

$$n^{\pm}_{\gamma} = \pm \left| g^{\alpha\beta} \frac{\partial H}{\partial x^\alpha} \frac{\partial H}{\partial x^\beta} \right|^{-1/2} \frac{\partial H}{\partial x^\gamma}, \tag{12}$$

with $H(r) = r - a(\tau)$. Using the metric functions, the non zero components of $n^{\pm}_{\gamma}$ become

$$n_t = \mp 2a\dot{a}, \tag{13}$$

$$n_r = \pm 2\sqrt{\frac{al^2(4\dot{a}^2 l^2 a - l^2 u + a)}{(l^2 u - a)}}, \tag{14}$$

where the dot over a quantity denotes the derivative with respect to $\tau$. Then, the non-zero extrinsic curvature (11) components yield

$$K^{\pm}_{\tau\tau} = \mp \frac{\sqrt{-al^2}(8\dot{a}^2 l^2 a + 8\ddot{a} l^2 a^2 - l^2 u + 2a)}{4a^2 l^2 \sqrt{-4\dot{a}^2 l^2 a - l^2 u + a}}, \tag{15}$$

$$K^{\pm}_{\theta\theta} = \pm \frac{1}{2a^{\frac{3}{2}} l} \sqrt{4\dot{a}^2 l^2 a - l^2 u + a}. \tag{16}$$

Since $K_{ij}$ is not continuous around the shell ($H$) [15], we use the Lanczos equation [43–45]:

$$S_{ij} = -\frac{1}{8\pi} \left( [K_{ij}] - [K] g_{ij} \right). \tag{17}$$

where $K$ is the trace of $K_{ij}$, $[K_{ij}] = K^{+}_{ij} - K^{-}_{ij}$, and $S_{ij}$ is stress energy-momentum tensor at the junction which is given in general by [15, 17]

$$S^i_j = diag(\sigma, -p), \tag{18}$$

where $p$ is surface pressure, and $\sigma$ is surface energy density. Due to the circular symmetry, we have

$$K^i_j = \begin{pmatrix} K^{\tau}_{\tau} & 0 \\ 0 & K^{\theta}_{\theta} \end{pmatrix}. \tag{19}$$

Thus, from Eq.s (18) and (17) one obtains the surface pressure and surface energy density [15].

Using the cut and paste technique, we can demount the interior regions $r < a$ of the geometry (10), and links its exterior parts. However, there exists a bounce (deduced from the extrinsic curvature components at the surface $r = a$) that is related with the energy density and pressure:

$$\sigma = -\frac{1}{8\pi a^{\frac{3}{2}} l} \sqrt{4\dot{a}^2 l^2 a - l^2 u + a}, \tag{20}$$

$$p = \frac{1}{16\pi a^{\frac{3}{2}} l} \frac{\left(8\dot{a}^2 l^2 a + 8\ddot{a} l^2 a^2 - l^2 u + 2a\right)}{\sqrt{4\dot{a}^2 l^2 a - l^2 u + a}}. \tag{21}$$

Consequently, the energy and pressure quantities in a static case ($a = a_0$) become

$$\sigma_0 = -\frac{1}{8\pi a_0^{\frac{3}{2}} l} \sqrt{-l^2 u + a_0}, \tag{22}$$

$$p_0 = \frac{1}{16\pi a_0^{\frac{3}{2}} l} \frac{\left(-l^2 u + 2a_0\right)}{\sqrt{-l^2 u + a_0}}. \tag{23}$$

Once $\sigma \geq 0$ and $\sigma + p \geq 0$ hold, then WEC is satisfied. Besides, $\sigma + p \geq 0$ is the condition of NEC. Furthermore, SEC is conditional on $\sigma + p \geq 0$ and $\sigma + 2p \geq 0$. It is obvious from Eq. (24) that negative energy density violates the WEC, and consequently we are in need of the exotic matter for constructing TSW. We note that the total matter supporting the wormhole is given by [46]

$$\Omega_\sigma = \int_0^{2\pi} \left[\rho\sqrt{-g}\right]\Big|_{r=a_0} d\phi = 2\pi a_0 \sigma(a_0) = -\frac{1}{4a_0^{\frac{1}{2}}|l|}\sqrt{-l^2 u + a_0}. \tag{24}$$

Stability of such a wormhole is investigated through a linear perturbation in which the equation of state is given by

$$p = \psi(\sigma), \tag{25}$$

where $\psi(\sigma)$ is an arbitrary function of $\sigma$. The energy conservation equation is introduced as follows [15]

$$S^i_{j;i} = -T_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^j}n^\beta, \tag{26}$$

where $T_{\alpha\beta}$ is the bulk energy-momentum tensor. Eq. (28) can thus be rewritten in terms of the pressure and energy density:

$$\frac{d}{d\tau}(\sigma a) + \psi\frac{da}{d\tau} = -\dot{a}\sigma. \tag{27}$$

From above equation, one reads

$$\sigma' = -\frac{1}{a}(2\sigma + \psi), \tag{28}$$

and its second derivative yields

$$\sigma'' = \frac{2}{a^2}(\tilde{\psi} + 3)(\sigma + \frac{\psi}{2}). \tag{29}$$

where prime and tilde symbols denote derivative with respect to $a$ and $\sigma$, respectively. The equation of motion for the shell is in general given by

$$\dot{a}^2 + V = 0, \tag{30}$$

where the effective potential $V$ is found from Eq. (22) as

$$V = \frac{1}{4l^2} - \frac{u}{4a} - 16a^2\sigma^2\pi^2. \tag{31}$$

In fact, Eq. (32) is nothing but the equation of the oscillatory motion in which the stability around the equilibrium point $a = a_0$ is conditional on $V''(a_0) \geq 0$. Using Eqs. (30) and (31), we finally obtain

$$V'' = -\frac{1}{2a^3}\left[64\pi^2 a^5\left((\sigma\sigma')' + 4\sigma'\frac{\sigma}{a} + \frac{\sigma^2}{a^2}\right) + u\right]\Bigg|_{a=a_0}, \tag{32}$$

or equivalently,

$$V'' = \frac{1}{2a^3}\{-64\pi^2 a^3\left[(2\psi' + 3)\sigma^2 + \psi(\psi' + 3)\sigma + \psi^2\right] - u\}\Big|_{a=a_0}. \tag{33}$$

*The equation of motion of the throat, for a small perturbation becomes [47, 48, 50]*

$$\dot{a}^2 + \frac{V''(a_0)}{2}(a - a_0)^2 = 0.$$

*Noted that for the condition of $V''(a_0) \geq 0$ , TSW is stable where the motion of the throat is oscillatory with angular frequency $\omega = \sqrt{\frac{V''(a_0)}{2}}$.*

## IV. SOME MODELS OF EOS SUPPORTING TSW

In this section, we use particular gas models (linear barotropic gas (LBG) [49, 50], chaplygin gas (CG) [51, 52], generalized chaplygin gas (GCG) [53] and logarithmic gas (LogG) [21]) to explore the stability of TSW.

### A. Stability analysis of TSW via the LBG

The equation of state of LBG [49, 50] is given by

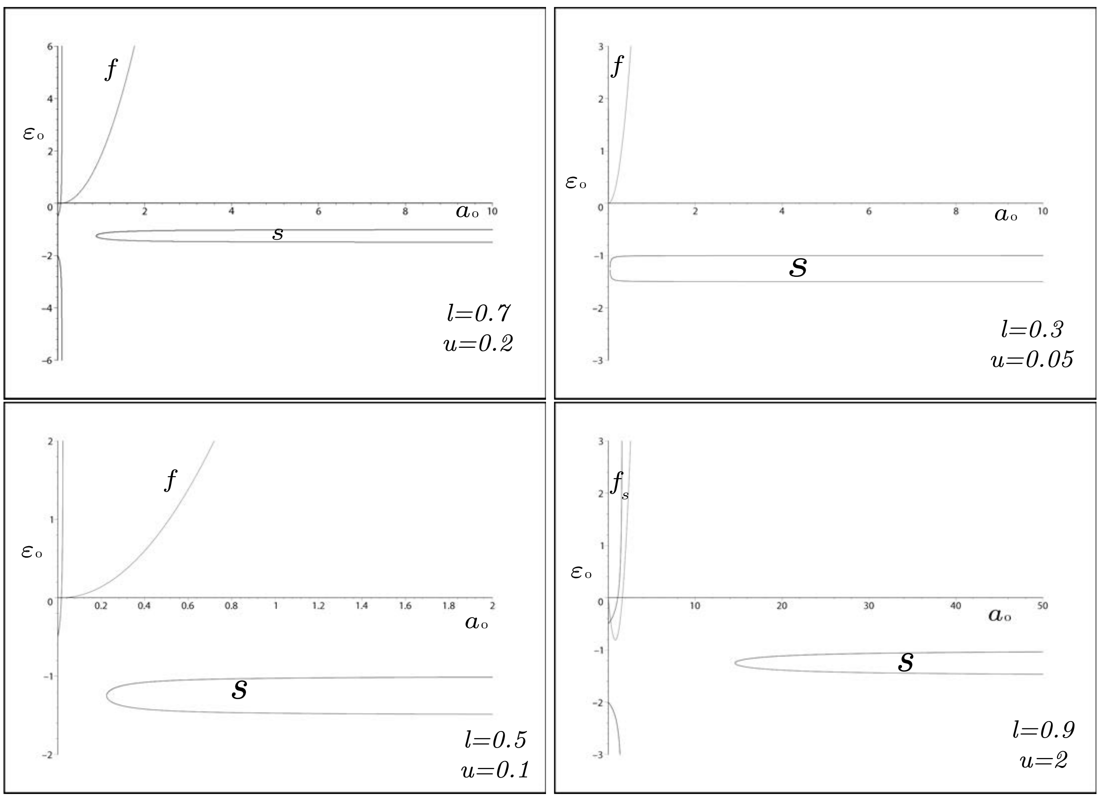
$$\psi = \varepsilon_0 \sigma, \tag{34}$$

and hence

$$\psi'(\sigma_0) = \varepsilon_0, \tag{35}$$

where $\varepsilon_0$ is a constant parameter. By changing the values of $l$ and $u$ in Eq. (35), we illustrate the stability regions for TSW, in terms of $\varepsilon_0$ and $a_0$, as depicted in Fig. 1.

FIG. 1: Stability Regions via the LBG




### B. Stability analysis of TSW via CG

The equation of state of CG that we considered is given by [51]

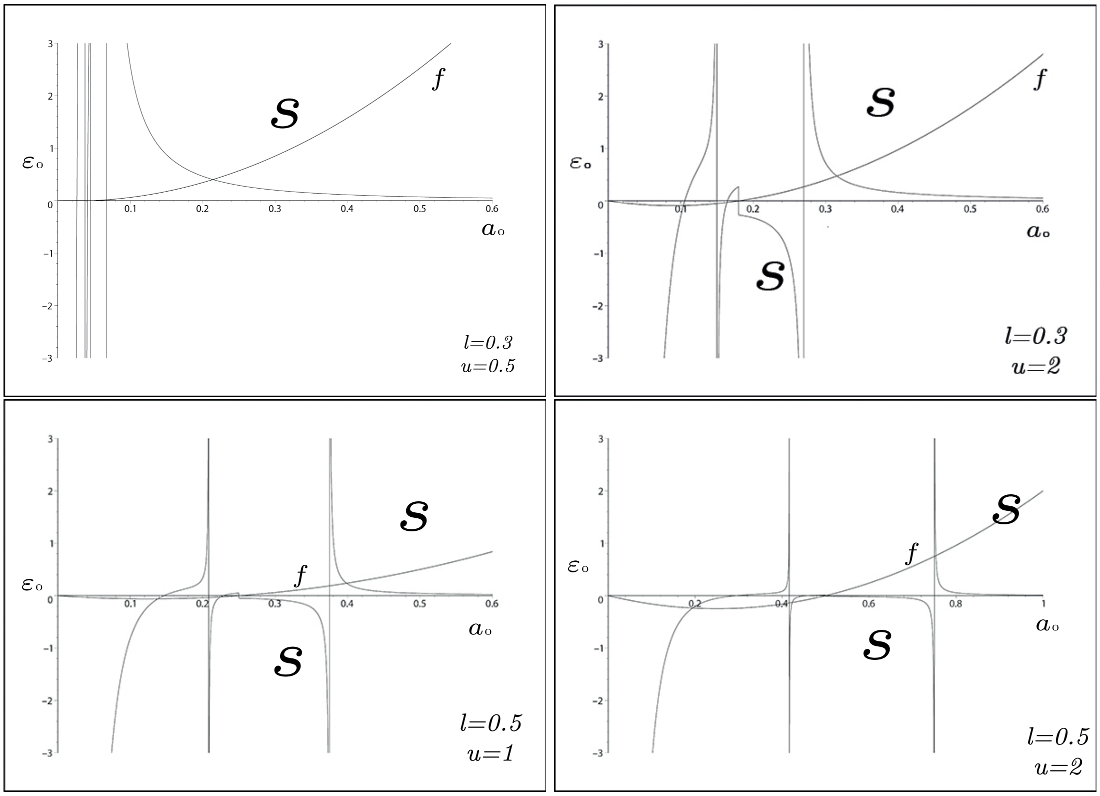
$$\psi = \varepsilon_0 \left(\frac{1}{\sigma} - \frac{1}{\sigma_0}\right) + p_0, \tag{36}$$

and one naturally finds

$$\psi'(\sigma_0) = \frac{-\varepsilon_0}{\sigma_0^2}. \tag{37}$$

After inserting Eq. (39) into Eq. (35), we plot the stability regions for TSW supported by CG in Fig. (2).

FIG. 2: Stability Regions via the CG




### C. Stability analysis of TSW via GCG

By using the equation of state of GCG [53]

$$\psi = p_0 \left(\frac{\sigma_0}{\sigma}\right)^{\varepsilon_0}, \tag{38}$$

and whence

$$\psi'(\sigma_0) = -\varepsilon_0 \frac{p_0}{\sigma_0}, \tag{39}$$

Substituting Eq. (41) in Eq. (35), one can illustrate the stability regions of TSW supported by GCG as seen in Fig. (3).

### D. Stability analysis of TSW via LogG

In our final example, the equation of state for LogG is selected as follows (see [21])

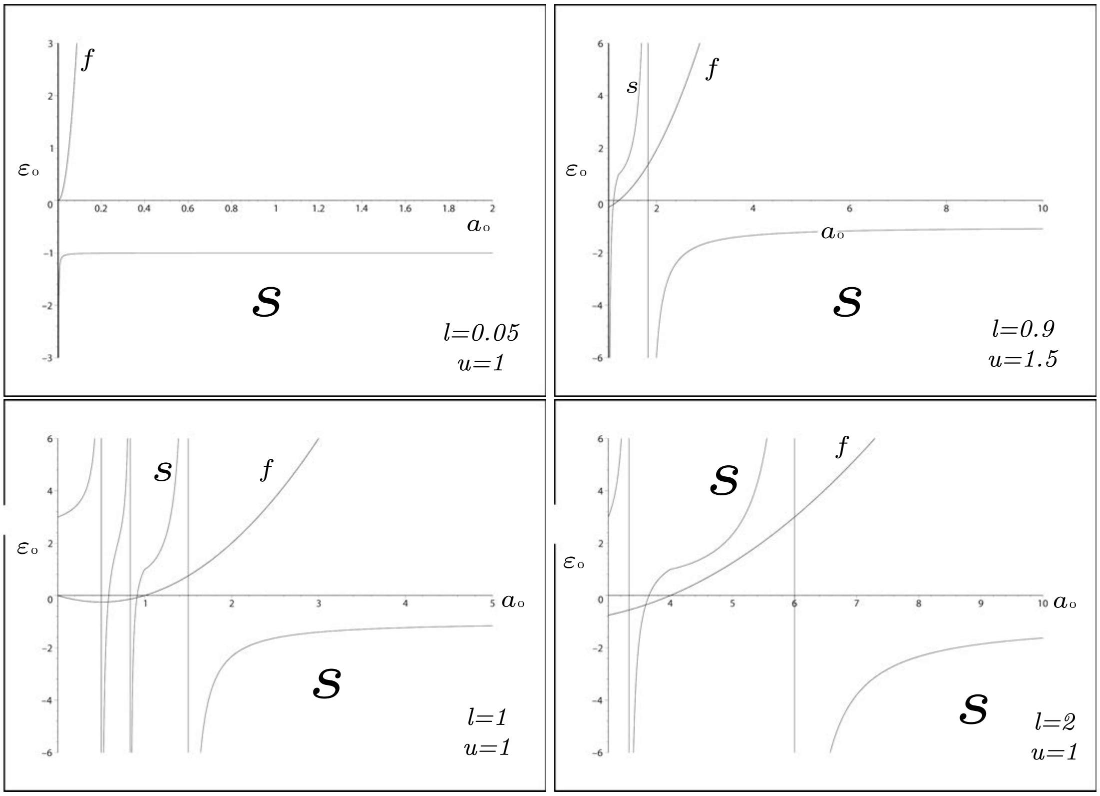
$$\psi = \varepsilon_0 \ln(\frac{\sigma}{\sigma_0}) + p_0, \tag{40}$$

which leads to

$$\psi'(\sigma_0) = \frac{\varepsilon_0}{\sigma_0}. \tag{41}$$

After inserting the above expression into Eq. (35), we show the stability regions of TSW supported by LogG in Fig. 4.

FIG. 3: Stability Regions via the GCG




## V. CONCLUSION

In this paper, we have constructed TSW by gluing two copies of SHBH via the cut and paste procedure. To this end, we have used the fact that the radius of throat must be greater than the event horizon of the metric given: ($a_0 > r_h$). We have adopted LBG, CG, GCG, and LogG gas equation of states to the exotic matter locating at the throat. Then, the stability analysis has become the study of checking positivity of the second derivative of an effective potential at the throat radius $a_0$: $V''(a_0) \geq 0$. In all cases, we have managed to find the stability regions in terms of the throat radius $a_0$ and constant parameter $\varepsilon_0$, which are associated with the EoS employed. The problem of the angular perturbation is out of scope for the present paper. That's why we have only worked on the linear perturbation. However, angular perturbation is in our agenda for the extension of this study. This is going to be studied in the near future.

One of the most trend topics in the theoretical physics is the relationship between ER and EPR, where ER refers to an Einstein-Rosen bridge (or wormhole) [54] and EPR, short for Einstein-Podolsky-Rosen [55], is another term for the entanglement [56, 57]. In our point of view, the yet another open problem here is that what is the link between TSW and EPR? Is it possible to solve exotic matter problem of the TSW by using the EPR? or vice versa. Also, are there any exotic forces between the EPR pairs? All of them are still open problems which are awaiting solutions, and they should be adequately explained. Our next project is to add a small piece of contribution to this big puzzle..

## VI. ACKNOWLEDGMENTS

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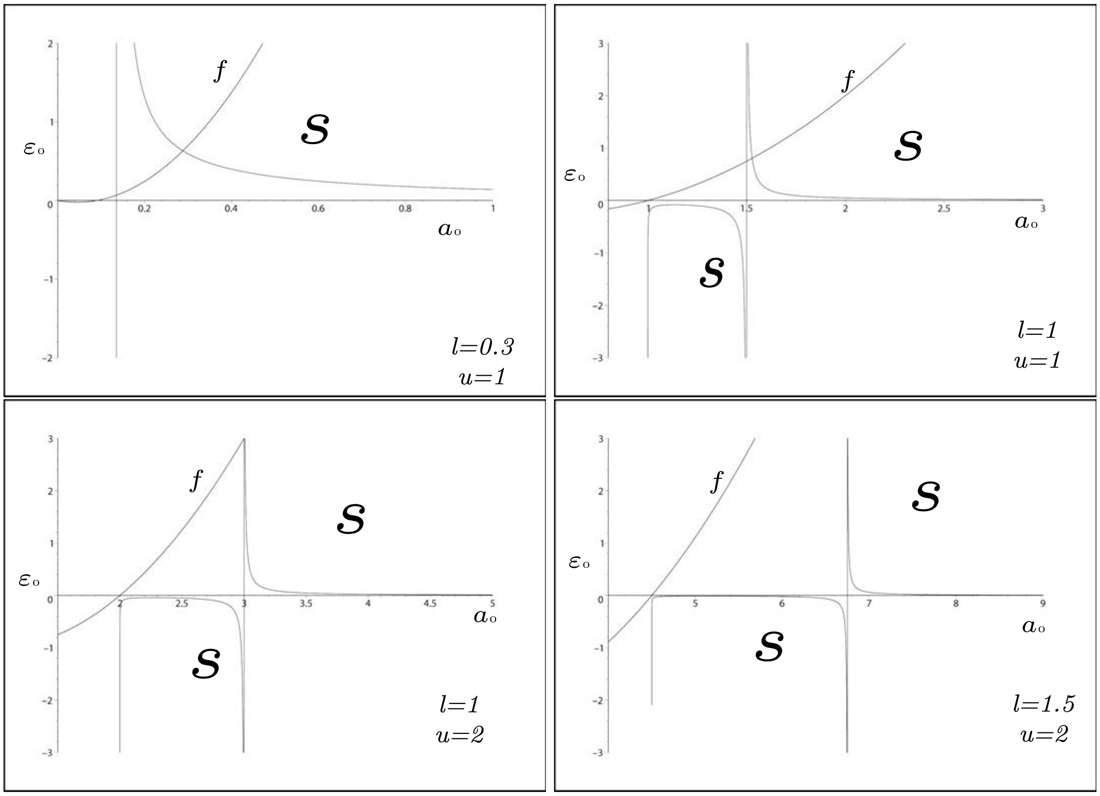
FIG. 4: Stability Regions via the LogG





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