Approximation by Kantorovich Type Operators

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ABSTRACT

In this thesis, new type q-Bernstein - Kantorovich polynomials $(q > 0)$ and complex q-Szász-Kantorovich operators $(q > 1)$ are introduced. In additon, The exact order of approximation, quantitative Voronovskaja-type theorems, simultaneous approximation properties for complex q-Bernstein - Kantorovich polynomials $(q > 0)$, complex Szász-Kantorovich and complex q-Szász-Kantorovich operators $(q > 1)$ are studied.

Keywords : q-Bernstein - Kantorovich polynomials, q-Szász-Kantorovich operator, complex Szász-Kantorovich operator, q-Szász-Kantorovich operator.

ÖZET

Bu tezde, yeni tip karmaşık q-Bernstein - polinomları $(q > 0)$ ve karmaşık q-Szasz-Kantorovich operatörleri $(q > 1)$ tanımlanmıştır. Buna ek olarak, karmaşık q-Bernstein-Kantorovich polinomlarının $(q > 0)$, karmaşık Szász-Kantorovich operatörünün ve karmaşık q-Szász-Kantorovich operatörünün $(q > 1)$ yakınsaklık oranları, yakınsaklık özellikleri ve Voronovskaja tipi teoremler incelenmiştir.

Anahtar Kelimeler: q-Bernstein - Kantorovich polinomları, q-Szasz-Kantorovich operatörü, karmaşık Szász-Kantorovich operatörü, karmaşık q-Szász-Kantorovich operatörü.

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NOTATIONS and SYMBOLS

$\mathbb{N}$	the set of natural numbers,
$:=$	is the sign indicating equal by definition .
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{C}$	the set of complex numbers,
$\mathbb{R}$	the set of real number,
$\mathbb{R}_+$	the set of positive real numbers,
(a, b)	an open interval,
$[a, b]$	a closed interval
$\\|P_n\\|_r$	$:= \max\{P_n(z); \|z\| \leq r\}$
$\mathbb{C}_2([0,\infty))$	$:=\left\{f \in C([0,\infty)) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite}\right\}$
$\mathbb{M}_R$	$:= \{z \in \mathbb{C} : \|z\| \leq R\}$ with $R > 1$
$\\|.\\|_{\mathbb{C}[0,+\infty)}$	uniform norm on $\mathbb{C}[0, +\infty)$ the space of all real valued bounded functions on $[0,+\infty)$,
$H(\mathbb{M}_R)$	space of all analytic function on $\mathbb{M}_R$.
$\\|f\\|_\infty$	$:= \sup\{\|f(x)\| : x \in X\}$

$\Delta f(x_j)$ is the forward difference defined as

$$\Delta f(x_j) = f(x_{j+1}) - f(x_j) = f(x_j + h) - f(x_j)),$$

with step size h

$\Delta^0 f(x_j) = f(x_j), \Delta^r f(x_j) = \Delta(\Delta^{r-1} f(x_j))$

$\Delta_h^k f(x_j)$ is the finite difference of order $k \in \mathbb{N}$,

with step size $h \in \mathbb{R} \setminus \{0\}$ and starting point

$x \in X$. Its formula is given by

$$\Delta_h^k f(x_j) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh),$$

$\mathbb{C}[a, b]$ the set of all real-valued and continuous functions

defined on the compact interval $[a, b]$

Chapter 1

INTRODUCTION

The first constructive (and simple) proof of Weierstrass approximation theorem was given by S. N. Bernstein [40]. He gave an alternative proof to the Weierstrass Approximation Theorem. He introduced the following polynomial

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \qquad f : [0,1] \to \mathbb{R}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$.

In 1997, George M. Phillips [41] suggested the q analogue of the Bernstein polynomials as follows:

$$B_{n,q}(f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-k-1} \left(1 - q^j x\right), \qquad f \in C[0;1].$$

If one replaces $x \in [0,1]$ by $z \in \mathbb{C}$, in the expression of $B_{n,q}(f)(x)$ where f is supposed to be analytic function, then we get the following complex $q-$Bernstein polynomals

$$B_{n,q}(f)(z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} \left(1 - q^j z\right).$$

The analogue of the Bernstein polynomials on an unbounded interval is Szász-Mirakjan operator. If $f : [0,\infty) \to \mathbb{R}$ and $\forall\, n \in \mathbb{N}$ the Szász-Mirakjan operators (Szász [18], Mirakjan [19])

$S_n : C_2([0,\infty)) \to C([0,\infty))$ are defined by

$$S_n(f)(x) = e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}f(j/n),\ x \in [0,\infty)$$

where $C_2([0,+\infty)) := \left\{ f \in C([0,\infty)) : \lim_{x\to\infty}\frac{f(x)}{1+x^2} \right\}$ exists and is finite.

The following complex Szász-Mirakjan operator is obtained from real version, simply replacing the real variable x by the complex $z \in \mathbb{C}$,

$$S_n(f)(z) = e^{-nz}\sum_{j=0}^{\infty}\frac{(nz)^j}{j!}f(j/n).$$

In this thesis, we studied approximation properties of complex q-Bernstein-Kantorovich, complex Szász-Kantorovich and complex q-Szász-Kantorovich operators.

This thesis was organized as follows

In Chapter 2, the following studied.

- Some basic definitions and properties related to q-integers,
- Some auxilary results in complex analysis are mentioned,
- main definitions, some elementary properties and approximation properties of Bernstein operators, Szász-Mirakjan operators and their q analogues of real variable as well as complex variable.

In Chapter 3, we introduced the following complex q-Bernstein-Kantorovich operators ($q > 0$)

$$K_{n,q}(f;z) = \sum_{k=0}^{n} p_{n,k}(q;z) \int_0^1 f\left(\frac{q[k]_q + t}{[n+1]_q}\right) dt \tag{1.0.1}$$

where $z \in \mathbb{C}$ and $p_{n,k}(q;z) = \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} \left(1 - q^j z\right)$.Notice that in the case $q = 1$, these operators coincide with the classical Kantorovich operators. For $0 < q \leq 1$, the operator $K_{n,q} : C[0,1] \to C[0,1]$ is positive and for $q > 1$ it is not positive. The problems studied in this thesis in the case $q = 1$ were investigated in [29] and [13]. Our study on the operator (1.0.1) is listed below;

- Quantitative estimates of the convergence for complex q-Bernstein-Kantorovich-type operators attached to an analytic function in a disk of radius $R > 1$ and center 0,
- Voronovskaja type result in compact disks, for complex q-Bernstein-Kantorovich operators (1.0.1) attached to an analytic function in $\mathbb{M}_R$, $R > 1$ and center 0,
- The order of approximation for complex q-Bernstein-Kantorovich operators (1.0.1).

The approximation properties for the following complex Szász-Mirakjan operators

$$S_n(f)(x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f(j/n), \ x \in [0, \infty)$$

were studied by S. Gal [22] and Mahmudov [29].

For the convergence of $S_n(f;x)$ to $f(x)$, usually f is supposed to be of exponential growth, that is $|f(x)| \leq C \exp(Bx)$, for all $x \in [0, \infty)$, with $C, B > 0$, (see Favard [17]). Also, concerning quantitative estimates in approximation of $f(x)$ by $S_n(f;x)$, in [20], it is proved that under some additional assumptions on f, we actually have $|S_n(f;x) - f(x)| \leq \frac{C}{n}$, for all $x \in [0, \infty)$,

$n \in \mathbb{N}$. In [21] Gal, under the condition that $f : [0, \infty) \to \mathbb{C}$ of exponential growth, obtained quantitative estimates in closed disks with center in origin. Unlike the convergence results in [22], all the results in the present thesis are obtained in the absence of the exponential-type growth conditions for analytic f in the disk. The approximation properties of the q-Szász-Mirakjan operators are studied in [39].

In Chapter 4, we introduce the following complex Szász-Kantorovich operators

$$K_n(f;z) = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \int_0^1 f\left(\frac{j+t}{n+1}\right) dt. \tag{1.0.2}$$

If f is bounded on $[0, \infty)$ then it is clear that $K_n(f;z)$ are well defined for all $z \in \mathbb{C}$. In this chapter,

- we investigate the quantitative estimates of the convergence for complex Szász-Kantorovich operators (1.0.2) attached to an analytic function in a disk of radius $R > 1$ and center 0,
- we prove Voronovskaja-type theorem and saturation of convergence for complex Szász-Kantorovich operators (1.0.2).

The approximation properties of q-Szasz-Mirakjan operators in compact disks were studied for $q = 1$ in Gal [23] (see also Gal [13], pp. 114-120) and for $q > 1$ in Mahmudov [27]. Also, it is worth noting that the approximation properties for other complex Bersntein-type operators were collected by the book Gal [13].

In Chapter 5 we introduce and study approximation properties of the following complex q-Szász-Kantorovich operators in the case $q > 1$,

$$K_{n,q}(f;z) = \sum_{j=0}^{\infty} e_q\left(-[n]_q q^{-j} z\right) \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} \int_0^1 f\left(\frac{q[j]_q + t}{[n+1]_q}\right) dt. \tag{1.0.3}$$

If f is bounded on $[0, +\infty)$ then it is clear that $K_{n,q}(f; z)$ is well-defined for all $z \in \mathbb{C}$.

In this chapter, the following results were obtained:

- The upper estimates in approximation by $K_{n,q}(f; z)$(1.0.3) and by its derivatives,
- The quantitative and qualitative Voronovskaja-type results in compact disks for $K_{n,q}(f; z)$ (1.0.3),
- The exact estimate in the approximation by the complex q-Szász-Kantorovich operators (1.0.3).

Chapter 2

PRELIMINARY and AUXILIARY RESULTS

In this Chapter, some basic results of Quantum Calculus, Complex analysis and Approximation Theory are collected. These results can be found in standard books on q-Calculus, Complex Analysis and Approximation Theory, see examples, [1], [2], [5], [13], and [41].

2.1. Elements of $q-$Calculus

In this section we will give some definitions related to $q-$integer.

Definition 2.1.1 [1] *For each integer $k \geq 0$, the q-integer $[k]_q$is defined by*

$$[k]_q = 1 + q^2 + ... + q^{k-1} := \begin{cases} \dfrac{1-q^k}{1-q}, & if \quad q \in R^+ \backslash \{1\}, \\ k, & if \quad q = 1 \end{cases}$$

Note that, $[0]_q = 0$.

Definition 2.1.2 [1] *For each integer $k \geq 0$, the q-factorial $[k]_q!$ is defined by*

$$[k]_q! := \begin{cases} [k]_q\, [k-1]_q \ldots [1]_q\,, & if \quad k = 1, 2, 3, ... \\ 1, & if \quad k = 0. \end{cases}$$

Definition 2.1.3 [1] *For integers $0 \leq j \leq k$, the q-binomial coefficient is defined by*

$$\left[\begin{array}{c} k \\ j \end{array} \right]_q \frac{[k]!}{[j]_q!\,[k-j]_q!} := \left[\begin{array}{c} k \\ k-j \end{array} \right]_q .$$

Definition 2.1.4 [1] *The q-analogue of* $(x-a)^n$ *is a polynomial of the form*

$$(x-a)_q^n := \begin{cases} 1 & if \;\; n = 0 \\ (x-a)(x-qa)\left(x-q^2a\right)\ldots\left(x-q^{n-1}a\right) & if \;\; n \geq 1. \end{cases}$$

Definition 2.1.5 [1] *For fixed* $1 \neq q > 0$*, we denote the q-derivative* $D_q f(x)$ *of* f *by*

$$D_q f(x) := \begin{cases} \frac{f(qx)-f(x)}{(q-1)x}, & x \neq 0 \\ \\ f'(0), & x = 0 \end{cases}$$

Example 2.1.6 [1] *Compute the q-derivative of* $f(x) = x^k$*, where n is a positive integer. By definition*

$$D_q x^n = \frac{(qx)^k - x^k}{(q-1)x} = \frac{q^k-1}{q-1}x^{k-1} = [k]_q\, x^{k-1}.$$

Proposition 2.1.7 [1] *For any integer n,*

$$D_q (x-a)_q^n = [n]\,(x-a)_q^{n-1}\,.$$

Lemma 2.1.8 [1] *For any integer* $k > 0$ *and a be a number. Gauss's Binomial Formula defined as,*

$$(x+a)_q^k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j(j-1)/2} a^j x^{k-j}.$$

Lemma 2.1.9 [1] *For a nonnegative integer n, we have*

$$\frac{1}{(1-x)_q^k} = 1 + \sum_{j=1}^{\infty} \frac{[k]_q\,[k+1]_q \ldots [k+j-1]_q}{[j]_q!}.$$

In addition, for $|q| < 1$, we have

$$\lim_{k\to\infty} [k]_q = \lim_{k\to\infty} \frac{1-q^k}{1-q} = \frac{1}{1-q}$$

and

$$\begin{aligned} \lim_{k\to\infty} \begin{bmatrix} k \\ j \end{bmatrix}_q &= \lim_{k\to\infty} \frac{\left(1-q^k\right)\left(1-q^{k-1}\right)\ldots\left(1-q^{k-j+1}\right)}{(1-q)\,(1-q^2)\ldots(1-q^j)}. \\ &= \frac{1}{(1-q)\,(1-q^2)\ldots(1-q^j)} \end{aligned} \tag{2.1.2}$$

If we apply the formulas (2.1.1) and (2.1.2) to Gauss's and Heine's Binomial Formulas, we obtain, the following two identities of formal power series in x ($|q| < 1$),as $k \to \infty$.

$$(1+x)_q^\infty = \sum_{j=0}^{\infty} q^{j(j-1)} \frac{x^j}{(1-q)\,(1-q^2)\ldots(1-q^j)} \tag{2.1.3}$$

$$\begin{aligned} \frac{1}{(1-x)_q^\infty} &= \sum_{j=0}^{\infty} \frac{x^j}{(1-q)\,(1-q^2)\ldots(1-q^j)} \\ &= \sum_{j=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q^2}{1-q}\right)\ldots\left(\frac{1-q^j}{1-q}\right)} \\ &= \sum_{j=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^j}{[j]_q!}. \end{aligned} \tag{2.1.4}$$

which resembles Taylor's expansion of the classical exponential function:

$$e^x = \sum_{j=0}^{\infty} \frac{(x)^j}{j!}$$

The series (2.1.3) and (2.1.4) are called Euler's first and Euler's second identities, or E1 and E2. E1 and E2 are obtained by Gauss and Heine.

Definition 2.1.10 [1] *A q-analogue of the classical exponential function e^x is*

$$e_q(x) = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}.$$

Exponential function on q based can also be expressed in terms of infinite product as follows

$$e_q(x) = \begin{cases} \prod_{j=0}^{\infty} \left(1 + (q-1)\frac{x}{q^{j+1}}\right) & \text{if } |q| > 1 \\ \prod_{j=0}^{\infty} \frac{1}{(1-(1-q)q^n x)} & \text{if } 0 < |q| < 1. \end{cases}$$

2.2. Bernstein Polynomials

This section contains some theorems which are related to Bernstein polynomials. see [2]. Given a function f defined on the closed interval [0, 1], we define the Bernstein polynomial

$$B_n(f; x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \tag{2.2.1}$$

for any integer $n > 0$. $B_n(f; x)$ is a polynomial in x is of degree $\leq n$.

For all $n \geq 1$, Bernstein polynomials have the following property,

$$B_n(f; 0) = f(0) \text{ and } B_n(f; 1) = f(1)$$

which is called end point interpolation property.

In addition, the following identities will be useful for us

$$\begin{aligned} B_n(1; x) &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \\ &= (x + (1-x))^n = 1. \end{aligned}$$

so that the Bernstein polynomial for the constant function 1 is also 1. Since

$$\frac{r}{k}\binom{k}{r} = \binom{k-1}{r-1}$$

For $1 \leq k \leq n$, Bernstein polynomial for the function t is

$$\begin{aligned} B_n(t; x) &= \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} \\ &= x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} \end{aligned}$$

Putting $l = k - 1$

$$= x\sum_{l=0}^{n-1}\begin{pmatrix} n-1 \\ l \end{pmatrix} x^l (1-x)^{n-1-l} = x$$

Finally, for a function t^2

$$\begin{aligned} B_n\left(t^2; x\right) &= \sum_{k=0}^{n}\frac{k^2}{n^2}\begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} \\ &= x^2 + \frac{1}{n}x(1-x). \end{aligned}$$

Theorem 2.2.1 [2] *Given a function* $f \in C[0,1]$ *and any* $\varepsilon > 0$, *there exists an integer N such that*

$$|f(x) - B_n(f; x)| < \varepsilon, \qquad 0 \leq x \leq 1$$

for all $n \geq N$.

Theorem 2.2.2 [2] *Let* $f(x)$ *be a bounded function on* $[0,1]$. *For any* $x \in [0,1]$ *at which* $f''(x)$ *exists,then*

$$\lim_{n\to\infty} n\left(B_n(f; x) - f(x)\right) = \frac{1}{2}x(1-x) f''(x).$$

2.3. q-Bernstein Polynomials

In this section, we are given some general information about the q-Bernstein polynomials, see [2] and [41].

Generalization of Bernstein polynomials based on the q-integers, which were proposed by Phillips [41], as given below:

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-k-1} \left(1 - q^j x\right) \tag{2.3.1}$$

where $n > 0$. Here on empty product is taken to be equal to 1.

Note that, also $B_{n,q}(f;x)$ can be written in the form $B_n(f;q,x)$. When $q = 1$, we recover the classical Bernstein polynomials.

For all $q > 0$, q-Bernstein polynomials have the following property,

$$B_n(f;q,0) = f(0) \text{ and } B_n(f;q,1) = f(1)$$

is called end point interpolation property. It is known that the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This difference is caused by the fact that, for $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0,1]$ while for $q > 1$, the positivity fails.

Theorem 2.3.1 [42] (Il'inskii and Ostrovska). *Given $q \in (0,1)$ and $f \in C[0,1]$, there exists a continuous function $B_{\infty,q}(f;x)$ such that*

$$B_{n,q}(f;x) \to B_{\infty,q}(f;x) \text{ for } x \in [0,1] \text{ as } n \to \infty.$$

where

$$B_{\infty,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1 - q^k\right) \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty} (1 - q^s x), & \text{if } x \in [0,1) \\ f(1), & \text{if } x = 1. \end{cases}$$

Theorem 2.3.2 [2] *The generalized Bernstein polynomial may be expressed in the form*

$$B_n(f;q,x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \triangle_q^k f_0 x^k, \tag{2.3.2}$$

where

$$\triangle_q^k f_j = \triangle_q^{k-1} f_{j+1} - q^{k-1} \triangle_q^{k-1} f_j, \quad r \geq 1,$$

with $\triangle_q^0 f_j = f_j = f([j]/[n])$.

In particular, we need to evaluate $B_n(f;q,x)$ for $f = 1, x, x^2$ in order to justify applying the Bohman-Korovkin Theorem on the uniform convergence of monotone operators. Due to the above Theorem 2.3.2, for $f(x) = 1$;

$$B_n(1;q,x) = 1. \tag{2.3.3}$$

for $f(x) = x$; we have

$$\Delta_q^0 f_0 = f_0 = 0, \Delta_q^1 f_0 = f_1 - f_0 = 1/[n]_q,$$

and it follows from Theorem 2.3.2 that

$$B_n(x;q,x) = x. \tag{2.3.4}$$

For $f(x) = x^2$; we have

$$\Delta_q^0 f_0 = f_0 = 0, \Delta_q^1 f_0 = 1/[n]_q^2$$

and

$$\Delta_q^2 f_0 = f_2 - (1+q) f_1 + q f_0 = \frac{q(1+q)}{[n]_q^2}.$$

Thus from Theorem 2.3.2

$$\begin{aligned} B_n\left(x^2; q, x\right) &= x \begin{bmatrix} n \\ 1 \end{bmatrix} \frac{1}{[n]_q^2} + x^2 \begin{bmatrix} n \\ 2 \end{bmatrix} \frac{q(1+q)}{[n]_q^2} \\ &= \frac{1}{[n]_q} x + \frac{[n]_q [n-1]_q q (1+q) x (1-x)}{(1+q)[n]_q^2} \\ &= x^2 + \frac{x(1-x)}{[n]_q}. \end{aligned} \tag{2.3.5}$$

Theorem 2.3.3 [2] *Let (q_n) denote a sequence such that $q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$. Then, for any $f \in C[0,1]$, $B_n(f; q_n, x)$ converges uniformly to $f(x)$ on $[0,1]$, where $B_n(f; q_n, x)$ is defined by (2.3.2) with $q = q_n$.*

2.4. Auxilary Results in Complex Analysis

In this section we give some known results and methods in Complex Analysis which we use in our study (See [3] and [4]).

Let $\mathbb{M}_R := \{z \in \mathbb{C} : |z| < R\}$ with $R > 1$ and assume that F is a segment included in $\mathbb{M}_R$ and the compact subset considered will be the closed disks $\overline{\mathbb{M}}_r = \{z \in \mathbb{C} : |z| \leq r\}$ with $1 \leq r < R$.

In addtion, $H(\mathbb{M}_R)$ is the space of all analytic functions on $\mathbb{M}_R$. For $f \in H(\mathbb{M}_R)$ we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$.

Theorem 2.4.1 (Cauchy) [3] *Let $r > 0$ and $f : \overline{\mathbb{M}_r} \to \mathbb{C}$ be analytic in $\mathbb{M}_r$ and continuous in $\overline{\mathbb{M}_r}$. Then, for any $l \in \{0, 1, 2, ...\}$ and all $|z| < r$ we have*

$$f^{(l)}(z) = \frac{l!}{2\pi i} \int_{\Gamma} \frac{f(u)}{(u-z)^{l+1}} du,$$

where $\Gamma = \{z \in \mathbb{C} : |z| = r\}$ and $i^2 = -1$.

Theorem 2.4.2 (Weierstrass) [3] *Let $G \subset \mathbb{C}$ be an open set. If the sequence $(f_n)_{n\in\mathbb{N}}$ of analytic functions on G converges to the analytic function f, uniformly in each compact in G, then for any $l \in \mathbb{N}$, the sequence of lth derivatives $\left(f_n^{(l)}\right)_{n\in\mathbb{N}}$ converges to $f^{(l)}$ uniformly on compact in G.*

Indeed, note that by the above Cauchy's formula we can write as

$$f_n^{(l)}(z) - f^{(l)}(z) = \frac{l!}{2\pi i} \int_{\Gamma} \frac{f_n(u) - f(u)}{(u-z)^{l+1}} du,$$

from which by passing to modulus the theorem easily follows.

Finally, we state a basic result very useful in the proofs of the approximation results and called Bernstein's inequality for complex polynomials in compact disks.

Theorem 2.4.3 [4] *Let $P_n(z) = \sum_{k=0}^{n} a_k z^k$ be with $a_k \in \mathbb{C}$, for all $k \in \{0, 1, 2, ..., \}$ and for $r > 0$ denote $\|P_n\|_r = \max\{P_n(z); |z| \le r\}$. Then*

(i) For all $|z| \le 1$ we have $\left|P_n'(z)\right| \le n \|P_n\|_1$;

(ii) If $r > 0$ then for all $|z| \le r$ we have $\left|P_n'(z)\right| \le \frac{n}{r} \|P_n\|_r$.

2.5. Bernstein Polynomials on Compact Disks

This section contains some theorems which are related to complex Bernstein polynomials, see [13].

If in the expression of $B_n(f;x)$ one replaces $x \in [0,1]$ by z in some regions in $\mathbb{C}$ (containing $[0,1]$) where f is suppused to be analytic, then we obtain the following complex Bernstein polynomials;

$$B_n(f;z) = \sum_{k=0}^{n} p_{n,k}(z) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}, \quad z \in \mathbb{C}.$$

Theorem 2.5.1 [13]

(i) *(Bernstein)* [5] *For the open* $G \subset \mathbb{C}$, *such that* $\overline{\mathbb{M}_1} \subset G$ and $f : G \to \mathbb{C}$ *is analytic in* G, *the complex Bernstein polynomials*

$$B_n(f;z) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} z^k (1-z)^{n-k} f(k/n),$$

uniformly convergence to f *in* $\overline{\mathbb{M}_1}$. *Here* $\mathbb{M}_1$ *denotes an open unit disk.*

(ii) *(Tonne)* [6] *If* $f(z) = \sum_{k=0}^{n} c_k z^k$ *is analytic in an open disk* M_1, $f(1)$ *is a complex number and there exist* $M > 0$ *and* $m \in \mathbb{N}$ *such that* $|c_k| \leq M(k+1)^m$, *for all* $k = 0, 1, 2, ...$ *then* $B_n(f;z)$ *converges uniformly (as* $n \to \infty$*) to* f *on each closed subset of* M_1.

(iii) *(Kantorovich)* [5] *If* f *is analytic in the interior of an ellipse of foci* 0 *and* 1, *then* $B_n(f;z)$

converges uniformly to $f(z)$ in any closed set contained in the interior of ellipse.

The following upper quantitative estimates results were obtained by Sorin Gal [7], [8] and [9].

Theorem 2.5.2 [7] *Suppose that $R > 1$ and $f : \mathbb{M}_R \rightarrow \mathbb{C}$ is analytic in $\mathbb{M}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{M}_R$.*

(i) Let $1 \leq r < R$ be arbitrary fixed. For all $|z| \leq r$ and *$n \in \mathbb{N}$, we have*

$$|B_n(f; z) - f(z)| \leq \frac{D_r(f)}{n},$$

where $0 < D_r(f) = \frac{3r(1+r)}{2} \sum_{j=2}^{\infty} j(j-1)\left|c_j\right| r^{j-2} < \infty$. (ii) For the simultaneous approximation by complex Bernstein polynomials, we have: if $1 \leq r < r_1 < R$ are arbitrary fixed, then for all $|z| \leq r$ and *$n, l \in \mathbb{N}$,*

$$\left|B_n^{(l)}(f)(z) - f^{(l)}(z)\right| \leq \frac{C_{r_1}(f)\, l! r_1}{n (r_1 - r)^{l+1}},$$

where $D_{r_1}(f)$ is given as at the above item(i).

The next theorem gives the Voronovskaja-type results in compact disks for $B_n(f; z)$.

Theorem 2.5.3 [8] *Let $R > 1$ and suppose that $f : \mathbb{M}_R \rightarrow \mathbb{C}$ is analytic in $\mathbb{M}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{M}_R$.*

(i) The following Voronoskaja-type result in the closed unit disk holds

$$\left|B_n(f; z) - f(z) - \frac{z - z^2}{2n} f''(z)\right| \leq \frac{10M(f)\left|z - z^2\right|}{2n^2}.$$

for all $n \in \mathbb{N}, z \in \overline{\mathbb{M}}_1$, *where* $0 \leq M(f) = \sum_{k=3}^{\infty} k(k-1)(k-2)^2 |c_k| < \infty$.

(ii) Let $r \in [1, R)$. Then for all $n \in \mathbb{N}, |z| \leq r$, we have

$$\left| B_n(f; z) - f(z) - \frac{z - z^2}{2n} f''(z) \right| \leq \frac{5M_r(f)(1+r)^2}{2n^2}.$$

where $M_r(f) = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 r^{k-2} < \infty$.

Also, S. Gal proved that the order of approximation for complex Bernstein polynomials in Theorem 2.5.2 (i) and (ii) are exactly 1/n.

Theorem 2.5.4 [9] *Let* $R > 1, \mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{M}_R \to \mathbb{C}$ *is analytic in* $\mathbb{M}_R$, *that is we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{M}_R$. *If* f *is not a polynomial of degree* ≤ 1, *then for any* $r \in [1, R)$ *we have*

$$\|B_n(f) - f\|_r \geq \frac{D_r(f)}{n}, \qquad n \in \mathbb{N},$$

where $\|f\|_r = \max\{f(z); |z| \leq r\}$ *and the constant* $D_r(f)$ *depends only on* f *and* r.

Corollary 2.5.5 [9] *Let* $R > 1, \mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{M}_R \to \mathbb{C}$ *is analytic in* $\mathbb{M}_R$. *If* f *is not a polynomial of degree* ≤ 1, *then for any* $r \in [1, R)$ *we have*

$$\|B_n(f) - f\|_r \sim \frac{1}{n}, \ n \in \mathbb{N},$$

where the constant in the equivalence depend on f *and* r.

In the case of simultaneous approximation presented in the following theorem.

Theorem 2.5.6 [9] *Let* $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $R > 1$ *and let us suppose that* $f : \mathbb{M}_R \to \mathbb{C}$ *is analytic in* $\mathbb{M}_R$,*i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{M}_R$. *Also, let* $1 \leq r < r_1 < R$ *and* $l \in \mathbb{N}$ *be fixed. If* f *is not polynomial of degree* $\leq \max\{1, l-1\}$, *then we have*

$$\left\| B_n^{(l)}(f) - f^{(l)} \right\|_r \sim \frac{1}{n},$$

where the constant in the equivalence depend on f, r, r_1 *and* p.

2.6. Complex q-Bernstein Polynomials

In this section we give the approximation and shape properties of the complex q-Bernstein polynomials. For $f : [0,1] \to \mathbb{C}$, the complex q-Bernstein polynomials are defined simply replacing x by z in the Phillips [41] definition in (2.31), that is

$$B_n(f; q, z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} \left(1 - q^j z\right), n \in \mathbb{N}, z \in \mathbb{C}.$$

Here the empty product is equal to be 1. Also, note that for $q = 1$, we obtain the classical complex Bernstein polynomials.

S. Ostrovska investigated the convergence properties for q-Bernstein polynomials in the case $q > 1$ and she has obtained the following results.

Theorem 2.6.1 [10] *Let* $q \in (1, \infty)$, *and let* f *be a function analytic in an* ε*-neighborhood of* $[0,1]$. *Then for any compact set* $K \subset D_\varepsilon := \{z : |z| < \varepsilon\}$,

$$B_n(f; q, z) \rightrightarrows f(z) \text{ for } z \in K \text{ as } n \to \infty.$$

The expression $f_n(x) \rightrightarrows f(x)$ means uniform convergence of a sequence $\{f_n(x)\}$ to $f(x)$.

Theorem 2.6.2 [10] *If f is a function analytic in a disk $\mathbb{M}_R$, $R > 1$, then for any compact set $K \subset D_{R-1}$,*

$$B_n(f; q, z) \rightrightarrows f(z) \text{ for } z \in K \text{ as } n \to \infty$$

Theorem 2.6.3 [10] *If f is an entire function, then for any compact set $K \subset \mathbb{C}$,*

$$B_n(f; q, z) \rightrightarrows f(z) \text{ for } z \in K \text{ as } n \to \infty.$$

For $q > 1$, S. Ostrovska proved that $B_n(t^m; q, z)$ converges to z^m essentially faster than the classical Bernstein polynomial.

Theorem 2.6.4 (Ostrovska [12] and Gal [11]) *Let $q > 0$, $R > 1$, $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{M}_R \to \mathbb{C}$ is analytic in $\mathbb{M}_R$.That is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{M}_R$. Then for the complex q-Bernstein polynomials we have the estimate*

$$|B_n(f; q, z) - f(z)| \leq \frac{Y_{r,q}(f)}{[n]_q}, \text{ for all } n \in \mathbb{N},$$

valid for all $n \in \mathbb{N}$ and $|z| \leq r$, with $1 \leq r < R$, where

$$0 < Y_{r,q}(f) = 2\sum_{k=2}^{\infty} (k-1)[k-1]_q |c_k| r^k.$$

Morever,

$$Y_{r,q}(f) \leq 2\sum_{k=2}^{\infty}(k-1)|c_k| r^k := M_r(f) < \infty,$$

for all $r \in [1,R)$ *and* $q \in (0,1]$, *while if* $q > 1$, *then* $Y_{r,q}(f) < \infty$, *for all* $q < R$ *and* $r \in \left[1, \frac{R}{q}\right)$.

Remark 2.6.5 [13]

1) Let $0 < q \leq 1$ *be fixed. Since* $[n]_q \to (1-q)^{-1}$ *as* $n \to \infty$ *in the estimate in Theorem 2.6.4, we do not obtain convergence of* $B_n(f;q,z)$ *to* $f(z)$*. But this situation can be improved by choosing* $0 < q = q_n < 1$ *with* $q_n \nearrow 1$ *as* $n \to \infty$*. Since in this case* $[n]_{q_n} \to \infty$ *as* $n \to \infty$*; from Theorem 2.6.4 we get uniform convergence in* $\mathbb{M}_R$ *.*
2) If $q > 1$*, since the estimate* $\frac{1}{[n]_q} \leq \frac{1}{n}$ *then by Theorem 2.6.4, it follows that* $r \geq 1$ *with* $rq < R$*, we have* $B_n(f;q,z) \to f(z)$ *as* $n \to \infty$*, uniformly for* $|z| \leq r$*.*
For $0 < q < 1$, Voronovskaja-type results for the for complex q-Bernstein polynomials are given by the following theorem.

Theorem 2.6.6 [11] *Let* $0 < q < 1$, $R > 1$, $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{M}_R \to \mathbb{C}$ *is analytic in* $\mathbb{M}_R$*, that is, we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in \mathbb{M}_R$ *.*

(i) The following estimate holds:

$$\left| B_n(f;q,z) - f(z) - \frac{z-z^2}{2[n]_q} f''(z) \right| \leq \frac{9M(f)\,|z(1-z)|}{2[n]_q^2}.$$

for all $n \in \mathbb{N}$, $z \in \overline{\mathbb{M}}_1$*, where* $0 < M(f) = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 < \infty$.

(ii) Let $r \in [1,R)$. Then

$$\left|B_n(f;q,z) - f(z) - \frac{z-z^2}{2[n]_q} f''(z)\right| \le \frac{9K_r(f)(1+r)}{2[n]_q^2}.$$

for all $n \in \mathbb{N}$, $|z| \le r$, where $K_r(f) = \sum_{k=3}^{\infty} |c_k|\, k\,(k-1)\,(k-2)^2\, r^k < \infty$.

Remark 2.6.7 [13] *In the hypothesis on f in Theorem 2.6.6 by choosing $0 < q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$, it follows that*

$$\lim_{n\to\infty} [n]_{q_n} \left[B_{n,}(f;q_n,z) - f(z)\right] = \frac{\left(z-z^2\right) f''(z)}{2},$$

uniformly in any compact disks included in the open disks of center 0 and radius R.

In the following theorems, Gal obtained the exact order in approximation by complex q-Bernstein polynomials and their derivatives on compact disks.

Theorem 2.6.8 [11] *Let $0 < q_n \le 1$ with $\lim_{n\to\infty} q_n = 1$, $R > 1$, $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{M}_R \to \mathbb{C}$ is analytic in $\mathbb{M}_R$. That is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{M}_R$. If f is not a polynomial of degree ≤ 1, then for any $r \in [1,R)$ we have*

$$\left\|B_{n,q_n}(f) - f\right\|_r \ge \frac{D_r(f)}{[n]_{q_n}}, n \in \mathbb{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \le r\}$ and the constant $D_r(f) > 0$ depends on f, r and on the seqeuence $(q_n)_{n\in\mathbb{N}}$ but it is independent of n.

Corollary 2.6.9 [11] *Let $0 < q_n \le 1$ with $\lim_{n\to\infty} q_n = 1$, $R > 1$, $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{M}_R \to \mathbb{C}$ is analytic in $\mathbb{M}_R$. If f is not a polynomial of degree ≤ 1, then for* *any $r \in [1, R)$ we have*

$$\left\|B_{n,q_n}(f) - f\right\|_r \sim \frac{1}{[n]_{q_n}}, n \in \mathbb{N},$$

where the constant in the equivalence depend on on f, r and on the seqeuence $(q_n)_{n\in\mathbb{N}}$ but are independent of n.

Remark 2.6.10 [13] *Theorem 2.6.8 and Corollary 2.6.9 in the case when $q_n = 1$ for all $n \in \mathbb{N}$ were obtained by Theorem 2.5.4 and Corollary 2.5.5.*

Theorem 2.6.11 [13] *Let $0 < q_n \leq 1$ be with $\lim\limits_{n\to\infty} q_n = 1$, $R > 1$, $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{M}_R \to \mathbb{C}$ is analytic in $\mathbb{M}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{M}_R$. Also, let $1 \leq r < r_1 < R$ and $l \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq \max\{1, p-1\}$, then we have*

$$\left\|B^{(l)}_{n,q_n}(f) - f^{(l)}\right\|_r \sim \frac{1}{[n]_{q_n}}$$

where the constant in the equivalence depend on f, r, r_1, p and on the sequence $(q_n)_n$, but are independent of n.

2.7. Szász-Mirakjan Operators

Let $\mathbb{N}$ be a set of positive integer number and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $f : [0, \infty) \to \mathbb{R}$ and $\forall\, n \in \mathbb{N}$ the Szász-Mirakjan operators (Szász [18], Mirakjan [19] $S_n : C_2([0, \infty)) \to C([0, \infty))$ given by

$$S_n(f)(x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f(j/n), \ x \in [0, \infty)$$

where $C_2([0,+\infty)) := \left\{f \in C([0,\infty)) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite}\right\}$. $S_n(f)(x)$ can be also written in the form $S_n(f;x)$. In the convergence of $S_n(f;x)$ to $f(x)$, $f(x)$ is to be exponential growth, that is $|f(x)| \leq Ce^{Bx}$, for all $x \in [0,\infty)$, with $C, B > 0$. Also, Totik [20] was studied quantitative estimates of this converges and he proved that the following inequality;

$$|S_n(f;x) - f(x)| \leq \frac{C_r}{n} \text{ for all } x \in \mathbb{R}_+,\ n \in \mathbb{N}$$

The complex Szász-Mirakjan operators is obtained from real version, simply replacing x by complex one z in the real version. That is

$$S_n(f;z) = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} f(j/n) \tag{2.7.1}$$

S. Gal proved that approximation and Voronovskaja theorems with quantitative estimates for complex Szász-Mirakjan operators attached to analytic functions in a disks of radius $R > 1$ and center 0.

Theorem 2.7.1 [21] *Let* $\mathbb{M}_R = \{z \in \mathbb{C}; |z| < R\}$ *with* $1 < R < +\infty$ *and suppuse that* $f : [R,\infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *is continuous in* $[R,+\infty) \cup \overline{\mathbb{M}}_R$, *analytic in* $\mathbb{M}_R$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{M}_R$, *and that there exists* $M, C, B > 0$ *and* $A \in \left(\frac{1}{R}, 1\right)$, *with the property* $|c_k| \leq M\frac{A^k}{k!}$ *for all* $k = 0, 1, ...,$ *(which implies* $|f(z)| \leq Me^{B|z|}$ *for all* $z \in \mathbb{M}_R$ *) and* $|f(x)| \leq Ce^{Bx}$, *for all* $x \in [R,+\infty)$.

(*i*) *Let* $1 \leq r < \frac{1}{A}$ *be arbitrary fixed. For all* $|z| \leq r$ *and* $n \in \mathbb{N}$, *we have*

$$|S_n(f;z) - f(z)| \leq \frac{D_{r,A}}{n},$$

where $D_{r,A} = \frac{M}{2r} \sum_{k=2}^{\infty} |c_k| (k+1)(rA)^k < \infty$.

(ii) *If* $1 \leq r < r_1 < \frac{1}{A}$ *are arbitrary fixed, then for all* $|z| \leq r$ *and* $n, l \in \mathbb{N}$,

$$\left|S_n^{(l)}(f;z) - f^{(l)}(z)\right| \leq \frac{l! r_1 D_{r_1,A}}{n(r_1 - r)^{l+1}},$$

where $D_{r_1,A}$ *is given as at the above point* (i).

Voronovskaja-type formula with quantitative estimate for the complex Szász-Mirakjan operators is given by the following theorem.

Theorem 2.7.2 [21] *Suppose that the hypothesis on the function* f *and the constant* R, M, C, B, A *in the statement of Theorem 2.7.1 hold and let* $1 \leq r < \frac{1}{A}$ *be arbitrary fixed.*

(i) *The following upper estimate in the Voronovskaja-type formula holds*

$$\left|S_n(f;z) - f(z) - \frac{z}{2n} f''(z)\right| \leq \frac{3MA|z|}{r^2 n^2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1},$$

for all $n \in \mathbb{N}, |z| \leq r$.

(ii) *We have the following equivalence in the Voronovskaja's formula*

$$\left\|S_n(f) - f - \frac{e_1}{2n} f''\right\|_r \sim \frac{1}{n^2},$$

where the constant in the equivalence depend on f *and* r *but are independent of* n.

In the next theorem, Sorin Gal proved that the order of the approximation is exactly $1/n$ in theorem 2.7.1 .

Corollary 2.7.3 [21] *In the hypothesis of Theorem 2.7.1, if f is not a polynomial of degree≤ 1 in the case (i) and if f is not a polynomial of degree $\leq l$, $(l \geq 1)$ in the case (ii), then $\frac{1}{n}$ is in fact the exact order of approximation.*

The exponential-type growth condition on the function f was ignored by S. Gal. Then, he was obtained the following results.

Theorem 2.7.4 [22] *For $2 < R < +\infty$ let $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{M}_R$. That is $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{M}_R$.*

(i) *Let $1 \leq r < \frac{R}{2}$ then for all $|z| \leq r$ and $n \in \mathbb{N}$ it follows*

$$|S_n(f; z) - f(z)| \leq \frac{D_{r,f}}{n},$$

with $D_{r,f} = 6\sum\limits_{k=2}^{\infty} |c_k| (k-1)(2r)^{k-1} < \infty$.

(ii) *If $1 \leq r < r_1 < \frac{R}{2}$ then for all $|z| \leq r$ and $n, l \in \mathbb{N}$ it follows*

$$\left|S_n^{(l)}(f)(z) - f^{(l)}(z)\right| \leq \frac{l! r_1 D_{r_1,f}}{n (r_1 - r)^{l+1}},$$

where $D_{r_1,f}$ is as above.

Theorem 2.7.5 [22] *For $2 < R < +\infty$ let $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{M}_R$, that is $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{M}_R$. Also, let $1 \leq r < \frac{R}{2}$.*

(i) *For all $|z| \leq r$ and $n \in \mathbb{N}$ it follows*

$$\left|S_n(f)(z) - f(z) - \frac{z}{2n} f''(z)\right| \leq Y_{r,,f} . \frac{|z|}{n^2},$$

with $Y_{r,f} = 26\sum_{k=3}^{\infty} |c_k| (k-1)^2 (k-2) (2r)^{k-3} < \infty$.

(ii) *For all* $n \in \mathbb{N}$ *we have*

$$\left\| S_n(f) - f - \frac{e_1}{2n} f'' \right\|_r \sim \frac{1}{n^2},$$

where the constant in the equivalence depend on f *and* r *but are independent of* n.

In additon, Gal proved that the order of approximation in Theorem 2.7.4 is exactly $\frac{1}{n}$.

Theorem 2.7.6 [22] *Let* $2 < R < +\infty$, $1 \le r < \frac{R}{2}$ *and* $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *be bounded on* $[0, +\infty)$ *and analytic in* $\mathbb{M}_R$, *that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{M}_R$. *If* f *is not a polynomial of degree* ≤ 1, *then the estimate*

$$\|S_n(f) - f\|_r \ge \frac{D_r(f)}{n}, \quad n \in \mathbb{N},$$

holds, where the constant $D_r(f)$ *depends on* f *and* r *but is independent of* n.

Theorem 2.7.7 [22] *Let* $2 < R < +\infty$ *and* $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *be bounded on* $[0, +\infty)$ *and analytic in* $\mathbb{M}_R$, *that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{M}_R$. *If* $1 \le r < \frac{R}{2}$ *is arbitrary fixed and if* f *is not polynomial of degree*≤ 1, *then the estimate*

$$\|S_n(f) - f\|_r \sim \frac{1}{n}, n \in \mathbb{N},$$

holds, where the constant in the equivalence depend only on f *and* r.

Theorem 2.7.8 *[22] Let* $2 < R < +\infty$ *and* $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *be bounded on* $[0, +\infty)$ *and* *analytic in* $\mathbb{M}_R$*, that is* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in \mathbb{M}_R$ *.*

If $1 \le r < r_1 < \frac{R}{2}$*,* $l \in \mathbb{N}$ *and if* f *is not a polynomial of degree*$\le l$*, then we have*

$$\left\| S_n^{(l)}(f)(z) - f^{(l)}(z) \right\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f, r, r_1 *and* p*.*

2.8. Complex q-Szász-Mirakjan Operators

$q-$Szász-Mirakjan operators were defined and their approximation properties were investigated in [23] and [25]. In [23], $q-$ Szász-Mirakjan operators defined as follows

$$S_{n,q} = E\left(-\frac{[n]_q x}{b_n}\right) \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{[n]_q^k x^k}{[k]_q! b_n^k},$$

where $0 \le x < \frac{b_n}{(1-q)[n]}$, $f \in C[0,\infty)$ and $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$.

Mahmudov [24] has obtained new $q-$Szász-Mirakjan operators as the following

$$S_n(f; q, x) = \frac{1}{\prod_{j=0}^{\infty} \left(1 + (1-q) q^j [n]_q x\right)} \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right) q^{\frac{k(k-1)}{2}} \frac{[n]_q^k x^k}{[k]_q!}, \qquad (2.8.1)$$

where $x \in [0,\infty)$, $0 < q < 1$ and $f \in C[0,\infty)$.

Mahmudov [24] investigated convergence properties of this operators (2.8.1). Also Mahmudov [24], obtained the inequalities for the weighted approximation error of $q-$ Szász-Mirakjan operators. In addition, Mahmudov [24] discussed Voronovskaja-type formula for $q-$ Szász-Mirakjan operators (2.8.1). In [26], Mahmudov introduced the following $q-$Szász operators in the case $q > 1$.

$$M_n(f;q,x) := \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n]_q}\right) \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left(-[n]_q q^{-k} x\right) \tag{2.8.2}$$

Mahmudov [26] proved that the rate of approximation by the $q-$ Szász-Mirakjan operators (2.8.2) ($q > 1$) is q^{-n} versus $\frac{1}{n}$ for the classical Szász-Mirakjan operators (2.7.1).Also, Mahmudov [27] constructed the following complex generalized Szász-Mirakjan operators based on the q- integer, in the case $q > 1$

$$S_n(f;q,z) := \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n]_q}\right) \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]_q^k z^k}{[k]_q!} e_q\left(-[n]_q q^{-k} z\right) \tag{2.8.3}$$

and he investigated approximation properties of complex q- Szász-Mirakjan operators(2.8.3) in compact disks. Firstly, Mahmudov [27] obtained the following quantitative estimates of the convergence for complex q-Szász-Mirakjan operators attached to an analytic function in a disk of radius $R > 2$ and center 0.

Theorem 2.8.1 [27] *Let* $1 < q < \frac{R}{2} < \infty$ *and suppose that* $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *is continuous and bounded in* $[R, \infty) \cup \overline{\mathbb{M}}_R$ *and analytic in* $\mathbb{M}_R$*. Let* $1 \leq r < \frac{R}{2q}$ *be arbitrary fixed. For all* $|z| \leq r$ *and* $n \in \mathbb{N}$*, we have*

$$\left|S_{n,q}(f;z) - f(z)\right| \leq \frac{D_{r,A}}{[n]_q},$$

where $D_{r,A} = 2\sum_{m=2}^{\infty} |c_m| (m-1)(2qr)^{m-1} < \infty$.

Theorem 2.8.1 shows that, the rate of approximation q-Szász-Mirakjan operators ($q > 1$) is of order q^{-n} versus $\frac{1}{n}$ for the classical Szász-Mirakjan operators, see [21]. Secondly Mahmudov

[27] gives Voronovskaja type result in compact disks, for complex q-Szász-Mirakjan operators attached to an analytic in $\mathbb{M}_R$, $R > 2$ and center 0.

Theorem 2.8.2 [27] *Let* $1 < q < \frac{R}{2} < \infty$ *and suppose that* $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *is continuous and bounded in* $[R, \infty] \cup \overline{\mathbb{M}}_R$ *and analytic in* $\mathbb{M}_R$. *Let* $1 \leq r < \frac{R}{2q}$ *be arbitrary fixed. The following Voronovskaja-type result holds. For all* $|z| \leq r$ *and* $n \in \mathbb{N}$, *we have*

$$\left| S_n(f; q, z) - f(z) - \frac{1}{[n]_q} L_q(f; z) \right| \leq \frac{4|z|}{[n]_q^2} \sum_{m=2}^{\infty} |c_m| (m-1)(m-2)(2qr)^{m-3}$$

where

$$L_q(f; z) = \begin{cases} \frac{D_q f(z) - f'(z)}{q-1}, & \text{if } q > 1 \\ \frac{f''(z) z}{2}, & \text{if } q = 1 \end{cases}$$

Thirdly, Mahmudov [27] proved that the order of approximation in Theorem 2.8.1 is exactly q^{-n} versus $\frac{1}{n}$ for the classical Szász-Mirakjan operators (see [21]).

Theorem 2.8.3 [27] *Let* $1 < q < \frac{R}{2}, 1 \leq r < \frac{R}{2q}$ *and* $f : [R, \infty) \cup \overline{\mathbb{M}}_R \to \mathbb{C}$ *be bounded on* $[0, \infty)$ *and analytic in* $\mathbb{M}_R$. *If* f *is not a polynomial of degree* ≤ 1, *the estimate*

$$\left\| S_{n,q}(f) - f \right\|_r \geq D_{r,q}(f), \quad n \in \mathbb{N},$$

holds, where the constant $D_{r,q}(f)$ *depends on* f, q *and* r *but is independent of* n.

Theorem 2.8.4 [27] *Let* $1 < q < \frac{R}{2}, 1 \leq r < \frac{R}{2q}$. *If a function is analytic in the disks* $\mathbb{M}_R$, *then* $|S_n(f; q, z) - f(z)| = \circ(q^{-n})$ *for infinite number of points having an accumulation point on*

$\mathbb{M}_{R/2q}$ *if and only if* f *is linear.*

The next theorem shows that $L_q(f; z)$, $q \geq 1$, is continuous about the parameter q for $f \in H(\mathbb{M}_R)$, $R > 2$.

Theorem 2.8.5 [27] *Let* $R > 2$ *and* $f \in H(\mathbb{M}_R)$. *Then for any* r, $0 < r < R$,

$$\lim_{q \to 1^+} L_q(f; z) = L_1(f; z)$$

uniformly on $\mathbb{M}_R$.

Chapter 3

APPROXIMATION THEOREMS FOR COMPLEX q-BERNSTEIN KANTOROVICH OPERATORS

In this chapter, we introduce complex Bernstein-Kantorovich operators based on the q-integers and investigate their approximation properties. Morever, Voronovskaja type results and quantitative estimates of the convergence for the complex q-Bernstein-Kantorovich operators attached to disc $\mathbb{M}_R$ are obtained.

3.1. Construction and Auxilary Results

Let $\mathbb{M}_R$ be a disc $\mathbb{M}_R := \{z \in \mathbb{C} : |z| < R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{M}_R)$ the space of all analytic functions on $\mathbb{M}_R$. For $f \in H(\mathbb{M}_R)$ we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$. Firstly, Ostrovska studied convergence properties of complex q-Bernstein polynomials, proposed by Phillips [28], defined by

$$B_n(f; q, x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k \prod_{j=0}^{n-k-1} \left(1 - q^j x\right) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q; x)$$

Later many author studied approximation properties of q-Bernstein and q-Bernstein type operator. See [13] and references their in. It is known that the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This difference is caused by the fact that, for $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0, 1]$ while for $q > 1$, the positivity fails. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than that for $0 < q < 1$.

We introduce new type complex Bernstein-Kantorovich operators based on the q-integer, in the case $q > 0$.

Definition 3.1.1 *For $f \in H(\mathbb{M}_R)$, $q > 0$ and $n \in \mathbb{N}$, we define the following q-Bernstein-*

Kantorovich operators

$$K_{n,q}(f;z)=\sum_{k=0}^{n}p_{n,k}(q;z)\int_{0}^{1}f\left(\frac{q[k]_q+t}{[n+1]_q}\right)dt \tag{3.1.1}$$

where $z\in\mathbb{C}$ and $p_{n,k}(q;z)=\begin{bmatrix} n \\ k \end{bmatrix}_q z^k\prod_{j=0}^{n-k-1}\left(1-q^jz\right)$.

Notice that in the case $q=1$ these operators coincide with the classical Bernstein-Kantorovich operators. For $0<q\leq 1$ the operator $K_{n,q}:C[0,1]\to C[0,1]$ is positive and for $q>1$ it is not positive.

To investigate approximation properties of q-Bernstein-Kantorovich operators, we need to several lemmas. First lemma gives formula for $K_{n,q}(e_m;z)$. Using this formula we can easily calculate the value of $K_{n,q}(e_m;z)$.

Lemma 3.1.2 *Let $q>0$. For all $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $z\in\mathbb{C}$ we have*

$$K_{n,q}(e_m;z)=\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}B_{n,q}\left(e_j;z\right), \tag{3.1.2}$$

where $e_m(z)=z^m$.

Proof. Using the definition of $K_{n,q}(f;z)$, for $f(z)=e_m(z)=z^m$ one can obtain.

$$\begin{aligned}
K_{n,q}(e_m;z) &= \sum_{k=0}^{n}p_{n,k}(q;z)\sum_{j=0}^{m}\int_0^1\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[k]_q^jt^{m-j}}{[n+1]_q^m}dt \\
&= \sum_{k=0}^{n}p_{n,k}(q;z)\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[k]_q^j}{[n+1]_q^m}\int_0^1 t^{m-j}dt \\
&= \sum_{k=0}^{n}p_{n,k}(q;z)\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[k]_q^j}{[n+1]_q^m(m-j+1)} \\
&= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}\sum_{k=0}^{n}\frac{[k]_q^j}{[n]_q^j}p_{n,k}(q;z)
\end{aligned}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q} \right)^j p_{n,k}(q;z)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \underbrace{\sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q} \right) p_{n,k}(q;z)}_{B_{n,q}(e_j;z)}$$

$$K_{n,q}(e_m;z) = \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} B_{n,q}\left(e_j;z\right).$$

■

The second lemma says that $\left|K_{n,q}(e_m;z)\right|$ is bounded by r^m in the disc of radius $r \geq 1$.

Lemma 3.1.3 *For all* $z \in \mathbb{M}_r$, $r \geq 1$ *we have*

$$\left|K_{n,q}(e_m;z)\right| \leq r^m, \quad n,\ m \in \mathbb{N}.$$

Proof. Indeed, using the inequality $\left|B_{n,q}\left(e_j;z\right)\right| \leq r^j$ (see S. Ostrovska [10]) we get

$$\left|K_{n,q}(e_m;z)\right| \leq \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \underbrace{\left|B_{n,q}\left(e_j;z\right)\right|}_{\leq r^m}$$

$$\leq \frac{1}{[n+1]_q^m} \sum_{j=0}^{m} \binom{m}{j} q^j [n]_q^j r^m$$

$$= \left(\frac{1+q[n]_q}{[n+1]_q} \right)^m r^m = r^m.$$

■

The third lemma gives recurrence formula for $K_{n,q}(e_{m+1};z)$.

Lemma 3.1.4 *For all $n, m \in \mathbb{N}$, $z \in \mathbb{C}$, $1 \neq q > 0$, we have*

$$K_{n,q}(e_{m+1};z) = \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m;z) + zK_{n,q}(e_m;z) + \frac{1}{[n+1]_q^{m+1}} \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{q^j [n]_q^j}{(m-j+2)} \cdot \left(\frac{(m+1)q[n]_q - j[n+1]_q}{(m+1)q[n]_q} \right) B_{n,q}(e_j;z). \tag{3.1.3}$$

Proof. We know that (see Gal [13])

$$\frac{z(1-z)}{[n]_q} D_q B_{n,q}(e_j;z) = B_{n,q}(e_{j+1};z) - zB_{n,q}(e_j;z).$$

Taking the derivative of the formula (3.1.2), using the above formula and multiplying obtained identity $\frac{z(1-z)}{[n]_q}$ we get

$$D_q K_{n,q}(e_m;z) = \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} D_q B_{n,q}(e_j;z)$$

$$\frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m;z) = \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \underbrace{\frac{z(1-z)}{[n]_q} D_q B_{n,q}(e_j;z)}_{B_{n,q}(e_{j+1};z) - zB_{n,q}(e_j;z).}$$

$$\frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m;z) = \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \cdot \left(B_{n,q}(e_{j+1};z) - zB_{n,q}(e_j;z) \right)$$

$$\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)=\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}B_{n,q}\left(e_{j+1};z\right)$$

$$-z\underbrace{\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}B_{n,q}\left(e_j;z\right)}_{K_{n,q}(e_m;z)}$$

$$\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)=\sum_{j=1}^{m+1}\begin{pmatrix} m \\ j-1 \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^m(m-j+2)}B_{n,q}\left(e_j;z\right)$$

$$-zK_{n,q}(e_m;z)$$

and

$$0=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)-\sum_{j=1}^{m+1}\begin{pmatrix} m \\ j-1 \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^m(m-j+2)}B_{n,q}\left(e_j;z\right)$$

İf we add $K_{n,q}(e_{m+1};z)=\sum_{j=0}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^{m+1}(m-j+2)}B_{n,q}\left(e_j;z\right)$ on both sides of the above equation, we obtain

$$K_{n,q}(e_{m+1};z)=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)$$

$$+\sum_{j=0}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^{m+1}(m-j+2)}B_{n,q}\left(e_j;z\right)$$

$$-\sum_{j=1}^{m+1}\begin{pmatrix} m \\ j-1 \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^m(m-j+2)}B_{n,q}\left(e_j;z\right)$$

$$K_{n,q}(e_{m+1};z)=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)+\frac{1}{[n+1]_q^{m+1}(m+2)}$$
$$+\sum_{j=1}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^{m+1}(m-j+2)}B_{n,q}\left(e_j;z\right)$$
$$-\sum_{j=1}^{m+1}\begin{pmatrix} m \\ j-1 \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^{m}(m-j+2)}B_{n,q}\left(e_j;z\right)$$

Using the identity

$$\begin{pmatrix} m \\ j-1 \end{pmatrix}=\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{j}{(m+1)},$$

We may obtain the desired formula (3.1.3)

$$K_{n,q}(e_{m+1};z)=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)+\frac{1}{[n+1]_q^{m+1}(m+2)}$$
$$+\sum_{j=1}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^{m+1}(m-j+2)}B_{n,q}\left(e_j;z\right)$$
$$-\sum_{j=1}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{j}{(m+1)}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^{m}(m-j+2)}B_{n,q}\left(e_j;z\right)$$
$$K_{n,q}(e_{m+1};z)=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)+\frac{1}{[n+1]_q^{m+1}(m+2)}$$
$$+\sum_{j=1}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^{m}(m-j+2)}\left(\frac{q[n]_q}{[n+1]_q}-\frac{j}{(m+1)}\right)$$

$$K_{n,q}(e_{m+1};z)=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)+\frac{1}{[n+1]_q^{m+1}(m+2)}$$
$$+\sum_{j=1}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^m(m-j+2)}\frac{(m+1)q[n]_q-j[n+1]_q}{(m+1)[n+1]_q}B_{n,q}\left(e_j;z\right)$$
$$K_{n,q}(e_{m+1};z)=\frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)$$
$$+\sum_{j=0}^{m+1}\begin{pmatrix} m+1 \\ j \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^m(m-j+2)}\frac{(m+1)q[n]_q-j[n+1]_q}{(m+1)[n+1]_q}B_{n,q}\left(e_j;z\right).$$

■

3.2. Convergence Properties of $K_{n,q}$

We start with the following quantitative estimates of the convergence for complex q-Bernstein-Kantorovich operators attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 3.2.1 *Let $f \in H(\mathbb{M}_R)$.*

(i) *Let $0 < q \leq 1$ and $1 \leq r < R$. For all $z \in \mathbb{M}_r$ and $n \in \mathbb{N}$, we have*

$$\left|K_{n,q}(f;z)-f(z)\right| \leq \frac{3+q^{-1}}{2[n]_q}\sum_{m=1}^{\infty}|a_m|\,m(m+1)\,r^m \tag{3.2.1}$$

(ii) *Let $1 < q < R < \infty$ and $1 \leq r < \frac{R}{q}$. For all $z \in \mathbb{M}_r$ and $n \in \mathbb{N}$, we have*

$$\left|K_{n,q}(f;z)-f(z)\right| \leq \frac{2}{[n]_q}\sum_{m=1}^{\infty}|a_m|\,m(m+1)\,q^m r^m \tag{3.2.2}$$

Proof. (i) The use of the above recurrence we obtain the following relationship

$$\begin{aligned} K_{n,q}(e_m;z)-e_m(z) &= \frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_{m-1};z) \\ &+zK_{n,q}(e_{m-1};z)+\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[n]_q^j}{(m-j+1)} \\ &\cdot\left(\frac{mq[n]_q-j[n+1]_q}{mq[n]_q}\right)B_{n,q}\left(e_j;z\right)-z^m \end{aligned} \tag{3.2.3}$$

We can easily estimate the sum in the above formula as follows.

$$\begin{aligned} &\left|\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}q^j[n]_q^j\frac{1}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mq[n]_q}\right)B_{n,q}\left(e_j;z\right)\right| \\ \leq\ & \frac{1}{[n+1]_q^m}\left(\sum_{j=0}^{m-1}\begin{pmatrix} m-1 \\ j \end{pmatrix}\frac{m}{m-j}\frac{q^j[n]_q^j}{m-j+1}\left|1-\frac{j}{m}-\frac{j}{mq[n]_q}\right|\right)\left|B_n\left(e_j;z\right)\right| \\ &+\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^m}r^m \\ \leq\ & \frac{m\left(q[n]_q+1\right)^{m-1}+q^{m-1}[n]_q^{m-1}}{[n+1]_q^m}r^m\leq\frac{m}{1+q[n]_q}r^m\leq\frac{m}{q[n]_q}r^m \end{aligned}$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$\left|P_m'(z)\right|\leq\frac{m}{qr}\|P_m\|_{qr},\quad \text{for all}\ \ |z|\leq qr,\ \ r\geq 1,$$

(where $\|P_m\|_{qr}=\max\{|P_m(z)|:|z|\leq qr\}$).

$$\left|D_q(P_m;z)\right|=\left|\frac{P_m(qz)-P_m(z)}{qz-z}\right|\leq\left\|P_m'\right\|_{qr}\leq\frac{m}{qr}\|P_m\|_{qr},$$

for all $|z| \leq r$, where $P_m(z)$ is a complex polynomial of degree $\leq m$. From the above recurrence formula (3.2.3) we get

$$\begin{aligned}
&\left|K_{n,q}(e_m;z) - e_m(z)\right| \\
&\leq \frac{|z|\,|1-z|}{[n]_q}\left|D_q K_{n,q}(e_{m-1};z)\right| + |z|\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| \\
&+ \frac{mq^{-1}}{[n]_q} r^m
\end{aligned}$$

$$\begin{aligned}
&\left|K_{n,q}(e_m;z) - e_m(z)\right| \\
&\leq \frac{r(1+r)}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})\right\|_{qr} + r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| \\
&+ \frac{mq^{-1}}{[n]_q} r^m \\
&\leq r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + \frac{2m}{[n]_q} q^{m-2} r^{m-2} \\
&+ \frac{mq^{-1}}{[n]_q} r^m \\
&\leq r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + \frac{\left(3+q^{-1}\right)m}{[n]_q} r^m
\end{aligned} \tag{3.2.4}$$

By writing the last inequality for $m = 1, 2, \ldots$,step by step the following we easily obtain the following;

$$\begin{aligned}
&\left|K_n(e_m;z) - e_m(z)\right| \\
\leq\ & \frac{\left(3+q^{-1}\right)m}{[n]_q} r^m + r\frac{\left(3+q^{-1}\right)(m-1)}{[n]_q} r^{m-1} + \\
& r^2\frac{\left(3+q^{-1}\right)(m-2)}{[n]_q} r^{m-2} + \ldots + r^{m-1}\frac{\left(3+q^{-1}\right)}{[n]_q} r \\
=\ & \frac{\left(3+q^{-1}\right)}{[n]_q} r^m (m + m - 1 + \ldots + 1)
\end{aligned}$$

$$|K_n(e_m;z)-e_m(z)| \leq \frac{\left(3+q^{-1}\right)m(m+1)}{2[n]_q}r^m. \tag{3.2.5}$$

Since $K_{n,q}(f;z)$ is analytic in $\mathbb{M}_R$, we can write

$$K_{n,q}(f;z)=\sum_{m=0}^{\infty}a_m K_{n,q}(e_m;z), \quad z\in\mathbb{M}_R,$$

which together with (3.2.5) immediately implies for all $|z|\leq r$,

$$\begin{aligned}\left|K_{n,q}(f;z)-f(z)\right| &\leq \sum_{m=0}^{\infty}|a_m|\left|K_{n,q}(e_m;z)-e_m(z)\right| \\ &\leq \frac{\left(3+q^{-1}\right)}{2[n]_q}\sum_{m=1}^{\infty}|c_m|\,m(m+1)\,r^m\end{aligned}$$

(ii) To prove Theorem 3.2.1 (i) we again use the formula (3.2.3) and the following estimations are obtained.

$$\begin{aligned}&\left|\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\begin{pmatrix} m \\ j\end{pmatrix}q^j[n]_q^j\frac{1}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mq[n]_q}\right)B_{n,q}\left(e_j;z\right)\right| \\ \leq\ & \frac{1}{[n+1]_q^m}\left(\sum_{j=0}^{m-1}\begin{pmatrix} m-1 \\ j\end{pmatrix}\frac{m}{m-j}\frac{q^j[n]_q^j}{m-j+1}\left|1-\frac{j}{m}-\frac{j}{mq[n]_q}\right|\right)\left|B_n\left(e_j;z\right)\right| \\ &+\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^m}r^m \\ \leq\ & \frac{2m\left(q[n]_q+1\right)^{m-1}+q^{m-1}[n]_q^{m-1}}{[n+1]_q^m}r^m \leq \frac{2m+1}{[n+1]_q}r^m\end{aligned}$$

From the above recurrence formula (3.2.3), we get

$$\left|K_{n,q}(e_m;z)-e_m(z)\right| \leq \frac{|z|\,|1-z|}{[n]_q}\left|D_qK_{n,q}(e_{m-1};z)\right|+|z|\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right| + \frac{2m+1}{[n+1]_q}r^m$$

$$\begin{aligned}
&\left|K_{n,q}(e_m;z)-e_m(z)\right| \\
&\leq \frac{r(1+r)}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})\right\|_{qr}+r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right| \\
&+\frac{2m+1}{[n+1]_q}r^m \\
&\leq r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right|+\frac{2(m-1)}{[n]_q}q^{m-1}r^m \\
&+\frac{2m+1}{[n+1]_q}r^m \\
&\leq r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right|+\frac{4m}{[n]_q}q^m r^m.
\end{aligned}$$

By writing the last inequality for $m=1,2,...$, we easily obtain, step by step the following

$$\begin{aligned}
&\left|K_n(e_m;z)-e_m(z)\right| \\
\leq\ & \frac{4m}{[n]_q}q^m r^m + r\frac{4(m-1)}{[n]_q}q^{m-1}r^{m-1} \\
& +r^2\frac{4(m-2)}{[n]_q}q^{m-2}r^{m-2}+...+r^{m-1}\frac{4}{[n]_q}qr \\
=\ & \frac{4}{[n]_q}q^m r^m(m+m-1+...+1) \\
\leq\ & \frac{2m(m+1)}{[n]_q}q^m r^m.
\end{aligned} \tag{3.2.6}$$

Since $K_{n,q}(f;z)$ is analytic in $\mathbb{M}_R$, we can write

$$K_{n,q}(f;z)=\sum_{m=0}^{\infty}a_mK_{n,q}(e_m;z),\quad z\in\mathbb{M}_R,$$

which together with (3.2.6) immediately implies for all $|z| \leq r$

$$\begin{aligned}\left|K_{n,q}(f;z) - f(z)\right| &\leq \sum_{m=0}^{\infty} |a_m| \left|K_{n,q}(e_m;z) - e_m(z)\right| \\ &\leq \frac{2}{[n]_q} \sum_{m=1}^{\infty} |c_m| \, m(m+1)(qr)^m .\end{aligned}$$

■

Remark 3.2.2 *(i) Since $[n]_q \to (1-q)^{-1}$ as $n \to \infty$ in the estimate in Theorem (3.2.1)(i) we do not obtain convergence of $K_{n,q}(f;z)$ to $f(z)$. But this situation can be improved by choosing $0 < q = q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$. Since in this case $[n]_{q_n} \to \infty$ as $n \to \infty$, from Theorem (3.2.1)(i) we get uniform convergence in $\mathbb{M}_r$.*

(ii)Theorem (3.2.1)(ii) says that for functions analytic in M_R, $R > q$, the rate of approximation by the q-Bernstein-Kantorovich operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Kantorovich operators.

3.3. Voronovskaja Type Results

Let $f \in H(\mathbb{M}_R)$. Let us define

$$L_q(f;z) = \begin{cases} \dfrac{1-2z}{2} f'(z) + \dfrac{(1-z)\left(D_q f(z) - f'(z)\right)}{1-q^{-1}}, & \text{if } |z| < R/q,\ R > q > 1 \\ \dfrac{1-2z}{2} f'(z) + \dfrac{z(1-z)}{2} f''(z), & \text{if } |z| < R,\ 0 < q \leq 1 \end{cases} \tag{3.3.1}$$

It is not difficult to show that

$$
\begin{aligned}
&L_q(f;z) \\
&= q(1-z)\sum_{m=1}^{\infty} a_m \frac{[m]_q - m}{q-1} z^{m-1} + \frac{1-2z}{2}\sum_{m=1}^{\infty} a_m m z^{m-1} \\
&= q\sum_{m=1}^{\infty} a_m \left([1]_q + ... + [m-1]_q\right) z^{m-1}(1-z) + \frac{1-2z}{2}\sum_{m=1}^{\infty} a_m m z^{m-1}, \quad q > 1.
\end{aligned}
$$

Here we used the identity

$$
\frac{[m]_q - m}{q-1} = [1]_q + ... + [m-1]_q
$$

In order to prove quantitative Voronovskaja type result we need the following polynomials. We consider the cases $0 < q < 1$ and $q > 1$ separately.

If $0 < q \leq 1$

$$
E_{n,m}(z) = K_{n,q}(e_m; z) - e_m(z) - \frac{1-2z}{2[n+1]_q} m z^{m-1} - \frac{z(1-z)}{2[n+1]_q} m(m-1) z^{m-2},
$$

If $q > 1$

$$
E_{n,m}(z) = K_{n,q}(e_m; z) - e_m(z) - \frac{1-2z}{2[n+1]_q} m z^{m-1} - \sum_{j=1}^{m-1} [j]_q \frac{q z^{m-1}(1-z)}{[n+1]_q},
$$

Here it is assumed that $\sum_{j=1}^{0} [j]_q = 0$.

Lemma 3.3.1 *Let $n, m \in \mathbb{N}$.*

(a) If $0 < q < 1$, we have the following recurrence formula;

$$\begin{aligned} & E_{n,m}(z) \\ = \ & \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\ & -\left(\frac{m-1}{[n+1]_q} - \frac{[m-1]_q}{[n]_q}\right) z^{m-1}(1-z) - \frac{1-2z}{2[n+1]_q} z^{m-1} \\ & + \frac{1}{[n+1]_q^m} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \frac{q^j [n]_q^j}{(m-j+1)} \\ & \cdot \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}\left(e_j; z\right). \end{aligned} \tag{3.3.2}$$

(b) If $q > 1$, we have

$$\begin{aligned} & E_{n,m}(z) \\ = \ & \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\ & + \frac{[m-1]_q}{[n]_q [n+1]_q} z^{m-1}(1-z) - \frac{1-2z}{2[n+1]_q} z^{m-1} \\ & + \frac{1}{[n+1]_q^m} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}\left(e_j; z\right). \end{aligned} \tag{3.3.3}$$

Proof. *(a)* It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z) = E_{n,1}(z) = 0$.

Using the formula (3.1.3), we get

$$\begin{aligned} K_{n,q}(e_m; z) = \ & \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_{m-1}; z) + zK_{n,q}(e_{m-1}; z) \\ & + \frac{1}{[n+1]_q^{m-1}} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \frac{q^{j-1} [n]_q^{j-1}}{(m-j+1)} \\ & \cdot \left(\frac{mq[n]_q - j[n+1]_q}{m[n+1]_q}\right) B_{n,q}\left(e_j; z\right). \end{aligned}$$

Then

$$\begin{aligned} E_{n,m}(z) &= \frac{z(1-z)}{[n]_q}\left\{D_q\left\{K_{n,q}(e_{m-1};z)-z^{m-1}\right\}+[m-1]_q z^{m-2}\right\} \\ &+zE_{n,m-1}(z)+z^m+\frac{1-2z}{2[n+1]_q}(m-1)z^{m-1} \\ &+\frac{z(1-z)}{2[n+1]_q}(m-1)(m-2)z^{m-2} \\ &+\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{[n+1]_q^{m-1}(m-j+1)}\left(\frac{mq[n]_q-j[n+1]_q}{m[n+1]_q}\right)B_{n,q}\left(e_j;z\right) \\ &-z^m-\frac{1-2z}{2[n+1]_q}mz^{m-1}-\frac{z(1-z)}{2[n+1]_q}m(m-1)z^{m-2}. \end{aligned}$$

A simple calculation leads to the following relationship

$$\begin{aligned} E_{n,m}(z) &= \frac{z(1-z)}{[n]_q}D_q\left\{K_{n,q}(e_{m-1};z)-z^{m-1}\right\}+zE_{n,m-1}(z) \\ &+z^{m-1}(1-z)\left\{\frac{[m-1]_q}{[n]_q}-\frac{(m-1)}{[n+1]_q}\right\}-\frac{1-2z}{2[n+1]_q}z^{m-1} \\ &+\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)} \\ &.\left(\frac{mq[n]_q-j[n+1]_q}{mq[n]_q}\right)B_{n,q}\left(e_j;z\right), \end{aligned}$$

which is the desired recurrence formula.

(b) It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z)=E_{n,1}(z)=0$.

Using the formula (3.1.3), we get

$$\begin{aligned} K_{n,q}(e_m;z) &= \frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_{m-1};z)+zK_{n,q}(e_{m-1};z) \\ &+\frac{1}{[n+1]_q^{m-1}}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^{j-1}[n]_q^{j-1}}{(m-j+1)} \\ &.\left(\frac{mqj[n]_q-j[n+1]_q}{m[n+1]_q}\right)B_{n,q}\left(e_j;z\right). \end{aligned}$$

Then

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q\left\{K_{n,q}(e_{m-1};z) - z^{m-1}\right\} + \frac{[m-1]_q z^{m-1}(1-z)}{[n]_q} \\
&+ zE_{n,m-1}(z) + z^m + \frac{1-2z}{2[n+1]_q}(m-1)z^{m-1} + \sum_{j=1}^{m-2}[j]_q \frac{qz^{m-2}z(1-z)}{[n+1]_q} \\
&+ \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \left(\frac{mq[n]_q - j[n+1]_q}{mq[n]_q}\right) B_{n,q}\left(e_j;z\right) \\
&- z^m - \frac{1-2z}{2[n+1]_q} m z^{m-1} - \sum_{j=1}^{m-1}[j]_q \frac{qz^{m-1}(1-z)}{[n+1]_q}
\end{aligned}
$$

A simple calculation leads to the following relationship

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q\left\{K_{n,q}(e_{m-1};z) - e_{m-1}\right\} + zE_{n,m-1}(z) \\
&+ \frac{z^{m-1}(1-z)[m-1]_q}{[n]_q[n+1]_q} - \frac{1-2z}{2[n+1]_q} z^{m-1} \\
&+ \frac{1}{[n+1]_q^m} \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix} \frac{q^j [n]_q^j}{(m-j+1)} \left(\frac{mq[n]_q - j[n+1]_q}{mq[n]_q}\right) B_{n,q}\left(e_j;z\right),
\end{aligned}
$$

which is the desired recurrence formula. ■

Remark 3.3.2 *Lemma 3.1.4 and 3.3.1 are true in the case $q = 1$. In the formulae we have to replace $q-$derivative by the ordinary derivative.*

The next theorem gives Voronovskaja type result in compact disks, for complex q-Bernstein-Kantorovich operators attached to an analytic function in $\mathbb{M}_R$, $R > 1$ and center 0.

Theorem 3.3.3 *Let $f \in H(\mathbb{M}_R)$.*

(i) *Let* $0 < q \leq 1$ *and* $1 \leq r < R$. *For all* $z \in \mathbb{M}_r$ *and* $n \in \mathbb{N}$ *we have*

$$\left| K_{n,q}(f;z) - f(z) - \frac{1-2z}{2[n+1]_q} f'(z) - \frac{z(1-z)}{2[n+1]_q} f''(z) \right|$$

$$\leq \frac{12+2q^{-1}}{[n]_q^2} \sum_{m=2}^{\infty} |a_m| \, m(m-1)^2 r^m. \qquad (3.3.4)$$

(ii) Let $1 < q < R < \infty$ and $1 \leq r < \frac{R}{q^2}$. For all $z \in \mathbb{M}_r$ and $n \in \mathbb{N}$, we have

$$\left| K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q} L_q(f;z) \right|$$

$$\leq \frac{14}{[n]_q^2} \sum_{m=2}^{\infty} |a_m| \, m(m-1)^2 q^{2m} r^m$$

Proof. (i) For $0 < q < 1$, as previously $E_{n,m}(z)$ describe Lemma (3.3.1) as the following

$$\begin{aligned} E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\ &- \left(\frac{m-1}{[n+1]_q} - \frac{[m-1]_q}{[n]_q}\right) z^{m-1}(1-z) - \frac{1-2z}{2[n+1]_q} z^{m-1} \\ &+ \frac{1}{[n+1]_q^m} \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j;z) \\ E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\ &- \frac{1}{2[n+1]_q} z^{m-1} + \frac{z^m}{[n+1]_q} - \left(\frac{m-1}{[n+1]_q} - \frac{[m-1]_q}{[n]_q}\right) z^{m-1}(1-z) \\ &- \frac{1}{[n+1]_q^m} q^{m-1} [n]_q^{m-1} B_{n,q}(e_m;z) - \frac{(m-1)}{2[n+1]_q^m} q^{m-2} [n]_q^{m-2} \\ &+ \frac{1}{2[n+1]_q^m} q^{m-1} [n]_q^{m-1} B_{n,q}(e_{m-1};z) \\ &+ \frac{1}{[n+1]_q^m} \sum_{j=0}^{m-2} \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j;z) \end{aligned}$$

After simple calculations we get

$$\begin{aligned} E_{n,m}(z) &= \underbrace{\frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)}_{I_1} + \underbrace{zE_{n,m-1}(z)}_{I_2} \\ & \underbrace{-\left(\frac{m-1}{[n+1]_q}-\frac{[m-1]_q}{[n]_q}\right) z^{m-1}(1-z)}_{I_3} \\ & \underbrace{+\frac{1}{[n+1]_q}\left(z^m - B_{n,q}(e_m;z)\right)}_{I_4} \\ & \underbrace{+\frac{1}{[n+1]_q}\left(1-\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right) B_{n,q}(e_m;z)}_{I_5} \\ & \underbrace{+\frac{1}{2[n+1]_q}\left(\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}-1\right) B_{n,q}(e_{m-1};z)}_{I_6} \\ & \underbrace{+\frac{1}{2[n+1]_q}\left(z^{m-1}-B_{n,q}(e_{m-1};z)\right)}_{I_7} \\ & \underbrace{-\frac{(m-1)}{[n+1]_q} q^{m-2}[n]_q^{m-2} B_{n,q}(e_{m-1};z)}_{I_8} \\ & \underbrace{+\frac{1}{[n+1]_q^m}\sum_{j=0}^{m-2}\left(\begin{array}{c} m \\ j \end{array}\right)\frac{q^j[n]_q^j}{(m-j+1)}\frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}\left(e_j;z\right)}_{I_9} \end{aligned}$$

$$E_{n,m}(z) = \sum_{k=1}^{9} I_k. \tag{3.3.5}$$

Firstly, we estimate I_3, I_8. It is clear that

$$|I_3| \le \frac{(m-1)(m-2)}{2[n]_q^2} r^{m-1}(1+r),$$

and

$$|I_8| \leq \frac{(m-1)}{2\,[n+1]_q^2}\left|B_{n,q}\left(e_{m-1};z\right)\right| \leq \frac{(m-1)}{2\,[n+1]_q^2}r^{m-1} \tag{3.3.6}$$

Secondly, using the known inequality

$$1-\prod_{k=1}^{m}x_k \leq \sum_{k=1}^{m}\left(1-x_k\right),\ 0 \leq x_k \leq 1$$

to estimate I_5, I_6, I_9

$$\begin{aligned}
|I_5| &\leq \frac{1}{[n+1]_q}\left(1-\frac{q^{m-1}\,[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|B_{n,q}\left(e_m;z\right)\right| \\
&\leq \frac{m-1}{[n+1]_q^2}r^m
\end{aligned}$$

$$\begin{aligned}
|I_6| &\leq \frac{1}{2\,[n+1]_q}\left(1-\frac{q^{m-1}\,[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|B_{n,q}\left(e_{m-1};z\right)\right| \\
&\leq \frac{m-1}{2\,[n+1]_q^2}r^{m-1} \\
|I_9| &\leq \frac{1}{[n+1]_q^m}\sum_{j=0}^{m-2}\begin{pmatrix} m-2 \\ j \end{pmatrix}\frac{m\,(m-1)}{(m-j)\,(m-j-1)} \\
&\cdot\left(\frac{q^j\,[n]_q^j}{(m-j+1)}\right)\left(1-\frac{j}{m}-\frac{j}{mq\,[n]_q}\right)r^j \\
&\leq \frac{(1+q^{-1})m\,(m-1)\,[n+1]_q^{m-2}}{[n+1]_q^m}r^m \\
&= \frac{(1+q^{-1})m\,(m-1)}{[n+1]_q^2}r^m
\end{aligned} \tag{3.3.7}$$

Finally, we estimate I_4, I_7. We use [13]

$$\begin{aligned} |I_4| + |I_7| &\leq \frac{1}{[n+1]_q} \left| z^m - B_{n,q}(e_m; z) \right| \\ &+ \frac{1}{2[n+1]_q} \left| B_n(e_{m-1}; z) - z^{m-1} \right| \\ &\leq \frac{2[m-1]_q (m-1)}{[n]_q [n+1]_q} r^m \\ &+ \frac{[m-2]_q (m-2)}{[n]_q [n+1]_q} r^{m-1} \end{aligned} \tag{3.3.8}$$

Using (3.2.5), (3.3.6), (3.3.7) and (3.3.8) in (3.3.5) finally we have ($m \geq 3$)

$$\begin{aligned} & \left| E_{n,m}(z) \right| \\ \leq \; & \frac{r(1+r)}{[n]_q} \left| D_q \left(K_{n,q}(e_{m-1}; z) - e_{m-1}(z) \right) \right| + r \left| E_{n,m-1}(z) \right| + \\ & + \frac{(m-1)(m-2)}{2[n]_q^2} r^{m-1} (1+r) + \frac{2[m-1]_q (m-1)}{[n]_q [n+1]_q} r^m \\ & + \frac{m-1}{[n+1]_q^2} r^m + \frac{m-1}{2[n+1]_q^2} r^{m-1} + \frac{[m-2]_q (m-2)}{[n]_q [n+1]_q} r^{m-1} + \\ & + \frac{(m-1)}{2[n+1]_q^2} r^{m-1} + \frac{(1+q^{-1}) m (m-1)}{[n+1]_q^2} r^m \end{aligned}$$

$$\begin{aligned} \leq \; & \frac{r(1+r)}{[n]_q} \left| D_q \left(K_{n,q}(e_{m-1}; z) - e_{m-1}(z) \right) \right| + r \left| E_{n,m-1}(z) \right| \\ & + \frac{(m-1)(m-2)}{[n]_q^2} r^{m-1} (1+r) + \frac{2(m-1)^2}{[n]_q [n+1]_q} r^m + \frac{m-1}{[n+1]_q^2} r^m \\ & + \frac{m-1}{2[n+1]_q^2} r^{m-1} + \frac{(m-2)^2}{[n]_q [n+1]_q} r^{m-1} + \frac{(m-1)}{2[n+1]_q^2} r^{m-1} \\ & + \frac{(1+q^{-1}) m (m-1)}{[n+1]_q^2} r^m \end{aligned}$$

$$\begin{aligned}
\left|E_{n,m}(z)\right| &\leq \frac{r(1+r)}{[n]_q}\left|D_q\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)\right|+r\left|E_{n,m-1}(z)\right| \\
&+\frac{2(m-1)^2}{[n]_q^2}r^m+\frac{2(m-1)^2}{[n]_q^2}r^m+\frac{(m-1)}{[n]_q^2}r^m+\frac{(m-1)^2}{[n]_q^2}r^m \\
&+\frac{(m-1)^2}{[n]_q^2}r^m+\frac{(m-1)^2}{[n]_q^2}r^m+\frac{(1+q^{-1})m(m-1)}{[n]_q^2}r^m \\
\left|E_{n,m}(z)\right| &\leq r\left|E_{n,m-1}(z)\right|+\frac{r(1+r)}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})-e_{m-1}\right\|_{qr} \\
&+\frac{8m(m-1)^2}{[n]_q^2}r^m+\frac{(1+q^{-1})m(m-1)^2}{[n]_q^2}r^m \\
\left|E_{n,m}(z)\right| &\leq r\left|E_{n,m-1}(z)\right|+\frac{\left(3+q^{-1}\right)m(m-1)^2}{[n]_q^2}r^m \\
&+\frac{8m(m-1)^2}{[n]_q^2}r^m+\frac{(1+q^{-1})m(m-1)^2}{[n]_q^2}r^m \\
\left|E_{n,m}(z)\right| &\leq r\left|E_{n,m-1}(z)\right|+\frac{\left(12+2q^{-1}\right)m(m-1)^2}{[n]_q^2}r^m
\end{aligned}$$

As a consequence, we get

$$\left|E_{n,m}(z)\right| \leq \frac{\left(12+2q^{-1}\right)m(m-1)^2}{[n]_q^2}q^{2m}r^m.$$

This inequality combined with

$$\left|K_{n,q}(f;z)-f(z)-\frac{1-2z}{2[n+1]_q}f'(z)-\frac{z(1-z)}{2[n+1]_q}f''(z)\right| \leq \sum_{m=1}^{\infty}|a_m|\left|E_{n,m}(z)\right|$$

immediately implies the required estimate in statement.

Note that since $f^{(3)} = \sum_{m=3}^{\infty} a_m m(m-1)(m-2)z^{m-3}$ and the series is absolutely convergent for all $|z| < R$, it easily follows the finiteness of the involved constants in the statement.

(b) For $q > 1$, a simple calculation and the use of the recurrence formula (3.1.3) lead us to the following relationship;

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\
&+ \frac{[m-1]_q}{[n]_q [n+1]_q} z^{m-1}(1-z) + \frac{1}{[n+1]_q}\left(z^m - B_{n,q}(e_m;z)\right) \\
&+ \frac{1}{[n+1]_q}\left(1 - \frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right) B_{n,q}(e_m;z) \\
&+ \frac{1}{2[n+1]_q}\left(\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}} - 1\right) B_{n,q}(e_{m-1};z) \\
&+ \frac{1}{2[n+1]_q}\left(B_{n,q}(e_{m-1};z) - z^{m-1}\right) - \frac{(m-1)q^{m-2}[n]_q^{m-2}}{2[n+1]_q^m} B_{n,q}(e_{m-1};z) \\
&+ \frac{1}{[n+1]_q^m} \sum_{j=0}^{m-2} \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}\left(e_j;z\right) \\
: &= \sum_{k=1}^{9} I_k. \qquad (3.3.9)
\end{aligned}
$$

Firstly, we estimate I_3, I_8. It is clear that

$$
\begin{aligned}
|I_3| &\leq \frac{[m-1]_q}{[n]_q [n+1]_q} r^{m-1}(1+r) \qquad (3.3.10) \\
|I_8| &\leq \frac{(m-1)}{2[n+1]_q^2}\left|B_{n,q}(e_{m-1};z)\right| \leq \frac{(m-1)}{2[n+1]_q^2} r^{m-1}
\end{aligned}
$$

Secondly, using the known inequality

$$
1 - \prod_{k=1}^{m} x_k \leq \sum_{k=1}^{m}(1-x_k),\ 0 \leq x_k \leq 1,
$$

to estimate I_5, I_6, I_9.

$$\begin{aligned}
|I_5| &\leq \frac{1}{[n+1]_q}\left(1-\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|B_{n,q}(e_m;z)\right| \leq \frac{m-1}{[n+1]_q^2}r^m \\
|I_6| &\leq \frac{1}{2[n+1]_q}\left(1-\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|B_{n,q}(e_{m-1};z)\right| \leq \frac{m-1}{2[n+1]_q^2}r^{m-1} \\
|I_9| &\leq \frac{1}{[n+1]_q^m}\sum_{j=0}^{m-2}\binom{m-2}{j}\frac{m(m-1)}{(m-j)(m-j-1)} \\
&\quad \cdot\left(\frac{q^j[n]_q^j}{(m-j+1)}\right)\left(1-\frac{j}{m}-\frac{j}{mq[n]_q}\right)r^j \\
&\leq \frac{2m(m-1)[n+1]_q^{m-2}}{[n+1]_q^m}r^m = \frac{2m(m-1)}{[n+1]_q^2}r^m
\end{aligned} \tag{3.3.11}$$

Finally, we estimate I_4, I_7. We use [13]

$$\begin{aligned}
|I_4|+|I_7| &\leq \frac{1}{[n+1]_q}\left|z^m - B_{n,q}(e_m;z)\right| + \frac{1}{2[n+1]_q}\left|B_n(e_{m-1};z)-z^{m-1}\right| \\
&\leq \frac{2[m-1]_q(m-1)}{[n]_q[n+1]_q}r^m + \frac{[m-2]_q(m-2)}{[n]_q[n+1]_q}r^{m-1}
\end{aligned} \tag{3.3.12}$$

Using (3.2.6), (3.3.10), (3.3.11) and (3.3.12) in (3.3.9) finally we have ($m \geq 3$)

$$\begin{aligned}
\left|E_{n,m}(z)\right| &\leq \frac{r(1+r)}{[n]_q}\left|D_q\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)\right| + r\left|E_{n,m-1}(z)\right| \\
&\quad + \frac{[m-1]_q}{[n]_q[n+1]_q}r^{m-1}(1+r) + \frac{2[m-1]_q(m-1)}{[n]_q[n+1]_q}r^m \\
&\quad + \frac{m-1}{[n+1]_q^2}r^m + \frac{m-1}{2[n+1]_q^2}r^{m-1} + \frac{[m-2]_q(m-2)}{[n]_q[n+1]_q}r^{m-1} \\
&\quad + \frac{(m-1)}{2[n+1]_q^2}r^{m-1} + \frac{2m(m-1)}{[n+1]_q^2}r^m \\
\left|E_{n,m}(z)\right| &\leq \frac{r(1+r)}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})-e_{m-1}\right\|_{qr} + r\left|E_{n,m-1}(z)\right| \\
&\quad + \frac{10m[m-1]_q}{[n]_q^2}r^m
\end{aligned}$$

$$\left|E_{n,m}(z)\right| \leq \frac{(m-1)(1+r)}{[n]_q}\frac{2(m-1)m}{[n]_q}q^{2(m-1)}r^{m-1} + r\left|E_{n,m-1}(z)\right|$$
$$+ \frac{10m[m-1]_q}{[n]_q^2}r^m$$
$$\left|E_{n,m}(z)\right| \leq r\left|E_{n,m-1}(z)\right| + \frac{4m(m-1)^2}{[n]_q^2}q^{2m}r^m + \frac{10m(m-1)}{[n]_q^2}q^m r^m$$
$$\left|E_{n,m}(z)\right| \leq r\left|E_{n,m-1}(z)\right| + \frac{14m(m-1)^2}{[n]_q^2}q^{2m}r^m.$$

As a consequence, we get

$$\left|E_{n,m}(z)\right| \leq \frac{14m(m-1)^2}{[n]_q^2}q^{2m}r^m$$

This inequality combined with

$$\left|K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q}L_q(f,z)\right| \leq \sum_{m=1}^{\infty}|a_m|\left|E_{n,m}(z)\right|$$

immediately implies the required estimate in statement. Note that since $f^{(3)} = \sum_{m=3}^{\infty} a_m m(m-1)(m-2)z^{m-3}$ and the series is absolutely convergent for all $|z| < R$, it easily follows the finiteness of the involved constants in the statement. ■

Remark 3.3.4 *(i) In the hypothesis on f in Theorem 3.3.3, (i) choosing* $0 < q_n < 1$ *with* $q_n \nearrow 1$ *as* $n \to \infty$*, it follows that*

$$\lim_{n\to\infty}[n+1]_{q_n}\left[K_{n,q_n}(f;z) - f(z)\right] = \frac{1-2z}{2}f'(z) + \frac{z(1-z)}{2}f''(z)$$

uniformly in any compact disk included in the open disk $\mathbb{M}_R$*.*

(ii) Theorem 3.3.3 (ii) gives explicit formulas of Voronovskaja-type for the q-Bernstein-Kantorovich

polynomials for $q > 1$.

(iii) Obviously that the best order of approximation that can be obtained from the estimate Theorem 3.3.3(i) is $O\left(1/[n]_{q_n}^2\right)$ and $O\left(1/n^2\right)$ for $q = 1$, while the order given by Theorem 3.3.3 (ii) is $O\left(1/q^{2n}\right)$, $q > 1$, which is essentially better.

Next theorem shows that $L_q(f;z)$, $q \geq 1$, is continuous about the parameter q for $f \in H(\mathbb{M}_R)$, $R > 1$.

Theorem 3.3.5 *Let $R > 1$ and $f \in H(\mathbb{M}_R)$. Then for any r, $0 < r < R$,*

$$\lim_{q \to 1+} L_q(f;z) = L_1(f;z)$$

uniformly on $\mathbb{M}_r$.

As an application of Theorem 3.3.3, we present the order of approximation for complex q-Bernstein-Kantorovich operators.

Theorem 3.3.6 *Let $1 < q < R$, $1 \leq r < \frac{R}{q^2}$ (or $0 < q \leq 1$, $1 \leq r < R$) and $f \in H(\mathbb{M}_R)$. If f is not a constant function then the estimate*

$$\left\| K_{n,q}(f) - f \right\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N},$$

holds, where the constant $C_{r,q}(f)$ depends on f, q and r but is independent of n.

Proof. For all $z \in \mathbb{M}_R$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} & K_{n,q}(f;z) - f(z) \\ = & \frac{1}{[n+1]_q}\left\{L_q(f;z) + [n+1]_q\left(K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q}L_q(f;z)\right)\right\}. \end{aligned}$$

We apply

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$$

to get

$$\begin{aligned} & \left\|K_{n,q}(f) - f\right\|_r \\ \geq & \frac{1}{[n+1]_q}\left\{\left\|L_q(f;z)\right\|_r - [n+1]_q\left\|K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q}L_q(f;z)\right\|_r\right\}. \end{aligned}$$

Because by hypothesis f is not a constant in $\mathbb{M}_R$, it follows $\left\|L_q(f;z)\right\|_r > 0$. Indeed, assuming the contrary it follows that $L_q(f;z) = 0$ for all $z \in \overline{\mathbb{M}}_R$, that is

$$\sum_{m=1}^{\infty} a_m\left(\frac{1}{2} - z\right)mz^{m-1} + \sum_{m=1}^{\infty} a_m \sum_{j=1}^{m-1}[j]_q z^{m-1}(1-z) = 0,$$

$$\frac{1}{2}a_1 + a_1 + \sum_{m=1}^{\infty}\left(\frac{1}{2}(m+1)a_{m+1} - a_m + a_{m+1}\sum_{j=1}^{m}[j]_q - a_m\sum_{j=1}^{m-1}[j]_q\right)z^m = 0$$

for all $z \in \overline{\mathbb{M}}_R \setminus \{0\}$. Thus $a_m = 0$, $m = 1, 2, 3, ...$ Thus, f is constant, which is contradiction with the hypothesis. ■

Now, by Theorem 3.3.3 we have

$$\begin{aligned}
& [n+1]_q \left| K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q} L_q(f;z) \right| \\
\leq \; & \frac{[n+1]_q}{[n]_q} \frac{14}{[n]_q} \sum_{m=2}^{\infty} |a_m| m^2 (m-1)^2 q^{2m} r^m \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

Consequently, there exists n_1 (depending only on f and r) such that for all, $n \geq n_1$ we have

$$\begin{aligned}
& \left\| L_q(f;z) \right\|_r - [n+1]_q \left\| K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q} L_q(f;z) \right\|_r \\
\geq \; & \frac{1}{2} \left\| L_q(f;z) \right\|_r ,
\end{aligned}$$

which implies

$$\left\| K_{n,q}(f) - f \right\|_r \geq \frac{1}{[n+1]_q} \frac{1}{2} \left\| L_q(f;z) \right\|_r , \quad \text{for all } n \geq n_1.$$

For $1 \leq n \leq n_1 - 1$, we have

$$\begin{aligned}
\left\| K_{n,q}(f) - f \right\|_r & \geq \frac{1}{[n+1]_q} \left([n+1]_q \left\| K_{n,q}(f) - f \right\|_r \right) \\
& = \frac{1}{[n+1]_q} M_{r,n}(f) > 0,
\end{aligned}$$

which finally implies that

$$\left\| K_{n,q}(f) - f \right\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f) ,$$

for all n, with $C_{r,q}(f) = \min \left\{ M_{r,1}(f), ..., M_{r,n_1-1}(f), \frac{1}{2} \left\| L_q(f;z) \right\|_r \right\}$.

Chapter 4

APPROXIMATION THEOREMS FOR COMPLEX SZÁSZ–KANTOROVICH OPERATORS

In this Chapter, we investigate the order of approximation and quantitative estimates of the convergence for the new type complex Szász-Kantorovich operators. Morever, Voronovskaja-type results with quantitative estimates for the new type complex Szász-Kantorovich operators attached to analytic functions on compact disks.

4.1. Construction and Auxilary Results

In this section, we introduce new type complex Szász–Kantorovich operators. For $f \in H(\mathbb{M}_R)$, we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$. New type complex Szász-Kantorovich operators defined as follows;

Definition 4.1.1 *For $f \in H(\mathbb{M}_R)$, $z \in \mathbb{C}$ and $n \in \mathbb{N}$, we define the following Szász–Kantorovich operators*

$$K_n(f;z) = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \int_0^1 f\left(\frac{j+t}{n+1}\right) dt \tag{4.1.1}$$

where $j \in \mathbb{N}$.

If f is bounded on $[0,\infty)$ then it is clear that $K_n(f;z)$ is well defined for all $z \in \mathbb{C}$. To investigate approximation properties of Szász-Kantorovich operators, we need to several lemmas. First lemma gives formula for $K_{n,q}(e_m;z)$. Using this formula, we can easily calculate the value of $K_{n,q}(e_m;z)$.

Lemma 4.1.2 *For all $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have*

$$K_n(e_m; z) = \frac{1}{(n+1)^m} \sum_{k=0}^{m} \binom{m}{k} \frac{n^k}{m-k+1} S_n(e_k; z) \tag{4.1.2}$$

where $e_m = z^m$.

Proof. The recurrence formula can be derived by direct computation:

$$\begin{aligned} K_n(e_m; z) &= e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \int_0^1 \left(\frac{j+t}{n+1} \right)^m dt \\ &= e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \sum_{k=0}^{m} \binom{m}{k} \int_0^1 \frac{j^k t^{m-k}}{(n+1)^m} dt \\ &= e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \sum_{k=0}^{m} \frac{j^k}{(n+1)^m} \int_0^1 t^{m-k} dt \end{aligned}$$

Then

$$\begin{aligned} K_n(e_m; z) &= e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \sum_{k=0}^{m} \binom{m}{k} \frac{j^k}{(n+1)^m (m-k+1)} dt \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{n^k}{(n+1)^m (m-k+1)} e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \left(\frac{j}{n} \right)^k \\ &= \frac{1}{(n+1)^m} \sum_{k=0}^{m} \binom{m}{k} \frac{n^k}{(m-k+1)} S_n(e_k; z). \end{aligned}$$

■

The second lemma says that $\left| K_{n,q}(e_m; z) \right|$ is bounded by $(2r)^m$ in the disc of radius $r \geq 1$.

Lemma 4.1.3 *For all $z \in \mathbb{C}$ we have*

$$|K_n(e_m; z)| \leq (2r)^m, \quad m \in \mathbb{N}. \tag{4.1.3}$$

Proof. Indeed, using the inequality $\left|S_n\left(e_j; z\right)\right| \leq (2r)^j$ from [13] p. 115, we get

$$\begin{aligned}
|K_n(e_m; z)| &\leq \sum_{j=0}^{m} \binom{m}{j} \frac{n^j}{(m-j+1)} \left|S_n(e_j; z)\right| \\
&\leq |S_n(e_m; z)| \\
&\leq (2r)^m
\end{aligned}$$

■

Lemma 4.1.4 *We have*

$$\begin{aligned}
K_n(e_0; z) = 1, \quad K_n(e_1; z) &= \frac{1}{2(n+1)} + \frac{n}{n+1} z, \\
K_n(e_2; z) &= \frac{1}{3(n+1)^2} + \frac{2n}{(n+1)^2} z + \frac{n^2}{(n+1)^2} z^2, \\
K_n\left((e_1 - z e_0)^2; z\right) &= \frac{1}{3(n+1)^2} + \frac{n-1}{(n+1)^2} z + \frac{1}{(n+1)^2} z^2.
\end{aligned}$$

Proof.

$$
\begin{aligned}
K_n(e_0;z) &= \frac{1}{(n+1)^0}\sum_{k=0}^{0}\binom{0}{k}\frac{n^k}{0-k+1}S_n(e_k;z)\\
&= S_n(e_0;z) = 1,\\
K_n(e_1;z) &= \frac{1}{(n+1)}\sum_{k=0}^{1}\binom{1}{k}\frac{n^k}{2-k}S_n(e_k;z)\\
&= \frac{1}{2(n+1)}S_n(e_0;z) + \frac{n}{n+1}S_n(e_1;z)\\
&= \frac{1}{2(n+1)} + \frac{n}{n+1}z,\\
K_n(e_2;z) &= \frac{1}{(n+1)^2}\sum_{k=0}^{2}\binom{2}{k}\frac{n^k}{3-k}S_n(e_k;z)\\
&= \frac{1}{3(n+1)^2}S_n(e_0;z) + \frac{2n}{(n+1)^2}S_n(e_1;z)\\
&\quad + \frac{n^2}{(n+1)^2}S_n(e_2;z)\\
&= \frac{1}{3(n+1)^2} + \frac{2n}{(n+1)^2}z + \frac{n^2}{(n+1)^2}z^2,
\end{aligned}
$$

$$
\begin{aligned}
K_n\left((e_1-ze_0)^2;z\right) &= K_n(e_2;z) - 2zK_n(e_1;z) + z^2K_n(e_0;z)\\
&= \frac{1}{3(n+1)^2} + \frac{2n}{(n+1)^2}z + \frac{n^2}{(n+1)^2}z^2\\
&\quad -2z\left(\frac{1}{2(n+1)} + \frac{n}{n+1}z\right) + z^2\\
&= \frac{1}{3(n+1)^2} + \frac{n-1}{(n+1)^2}z + \frac{1}{(n+1)^2}z^2
\end{aligned}
$$

■

The third lemma gives recurrence formula for $K_n(e_{m+1};z)$.

Lemma 4.1.5 *For all $n, m \in \mathbb{N}$ and $z \in \mathbb{C}$, we have*

$$K_n(e_{m+1};z) = \frac{z}{n}K_n^{'}(e_m;z) + zK_n(e_m;z) + \frac{1}{(n+1)^m}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{n^j}{(m-j+2)}\left(\frac{1}{n+1} - \frac{j}{(m+1)n}\right)S_n\left(e_j;z\right) \tag{4.1.4}$$

Proof. We know that, (see [13] p. 115)

$$S_n^{'}\left(e_j;z\right) = -nS_n\left(e_j;z\right) + \frac{n}{z}S_n\left(e_{j+1};z\right) \tag{4.1.5}$$

Taking the derivative of the formula (4.1.2) and using the above formula (4.1.5), we have

$$\begin{aligned} K_n^{'}(e_m;z) &= \frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\underbrace{S_n^{'}\left(e_j;z\right)}_{-nS_n\left(e_j;z\right)+\frac{n}{z}S_n\left(e_{j+1};z\right)} \\ &= \frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(-nS_n\left(e_j;z\right) + \frac{n}{z}S_n\left(e_{j+1};z\right)\right) \end{aligned}$$

$$\begin{aligned} K_n^{'}(e_m;z) &= \frac{n}{z}\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}S_n\left(e_{j+1};z\right) \\ &\quad -n\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}S_n\left(e_j;z\right) \end{aligned}$$

$$\frac{z}{n}K_n'(e_m;z) = \frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m}{j-1}\frac{n^{j-1}}{(m-j+2)}S_n\left(e_j;z\right) - zK_n(e_m;z).$$

$$0 = \frac{z}{n}K_n'(e_m;z) + zK_n(e_m;z) - \frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m}{j-1}\frac{n^{j-1}}{(m-j+2)}S_n\left(e_j;z\right) \quad (4.1.6)$$

It follows that

$$K_n(e_{m+1};z) = \frac{z}{n}K_n'(e_m;z) + zK_n(e_m;z) + \frac{1}{(n+1)^{m+1}}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{n^{j}}{(m-j+2)}S_n\left(e_j;z\right) - \frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m}{j-1}\frac{n^{j-1}}{(m-j+2)}S_n\left(e_j;z\right)$$

$$K_n(e_{m+1};z) = \frac{z}{n}K_n'(e_m;z) + zK_n(e_m;z) + \frac{1}{n^{m+1}(m+2)} + \frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m+1}{j}\frac{n^{j}}{(m-j+2)}.\left\{\frac{1}{n+1} - \frac{j}{(m+1)\,n}\right\}S_n\left(e_j;z\right).$$

$$K_n(e_{m+1};z) = \frac{z}{n}K_n'(e_m;z) + zK_n(e_m;z) + \frac{1}{(n+1)^m}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{n^{j}}{(m-j+2)}.\left\{\frac{1}{n+1} - \frac{j}{(m+1)\,n}\right\}S_n\left(e_j;z\right).$$

Here we used the identity

$$\begin{pmatrix} m \\ j-1 \end{pmatrix} = \begin{pmatrix} m+1 \\ j \end{pmatrix} \frac{j}{(m+1)}$$

■ Define

$$E_{n,m}(z) := K_n(e_m; z) - e_m(z) - \frac{\left(m^2 - 2mz\right) z^{m-1}}{2(n+1)} \tag{4.1.7}$$

Lemma 4.1.6 *Let $n, m \in \mathbb{N}$, we have the following recurrence formula*

$$\begin{aligned} E_{n,m}(z) &= \frac{z}{n}\left(K_n(e_{m-1}; z) - e_{m-1}(z)\right)' + zE_{n,m-1}(z) + \frac{m-1}{n(n+1)} z^{m-1} \\ &- \frac{1}{2(n+1)} z^{m-1} + \frac{1}{n+1} z^m + \frac{1}{(n+1)^m} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \frac{n^j}{(m-j+1)} \\ &. \left\{1 - \frac{j}{m} - \frac{j}{mn}\right\} S_n\left(e_j; z\right) \end{aligned} \tag{4.1.8}$$

Proof. It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z) = 0$.

Using the formula (4.1.4), we get

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{n}K_n'(e_{m-1};z) + zK_n(e_{m-1};z) \\
&+ \frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{n^j}{(m-j+1)}.\left(\frac{1}{n+1} - \frac{j}{mn}\right)S_n\left(e_j;z\right) \\
&- z^m - \frac{\left(m^2-2mz\right)}{2(n+1)}z^{m-1} \\
E_{n,m}(z) &= \frac{z}{n}\left\{(K_n(e_{m-1};z) - e_{m-1}(z))' + \left(z^{m-1}\right)'\right\} \\
&+ zE_{n,m-1}(z) + z^m + \frac{\left((m-1)^2 - 2(m-1)z\right)z^{m-1}}{2(n+1)} \\
&+ \frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{n^j}{(m-j+1)}.\left(\frac{1}{n+1} - \frac{j}{mn}\right)S_n\left(e_j;z\right) \\
&- z^m - \frac{\left(m^2-2mz\right)}{2(n+1)}z^{m-1}
\end{aligned}
$$

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{n}(K_n(e_{m-1};z) - e_{m-1}(z))' + \frac{m-1}{n}z^{m-1} + \\
&+ zE_{n,m-1} + \frac{\left((m-1)^2 - 2(m-1)z\right)z^{m-1}}{2(n+1)} - \frac{\left(m^2-2mz\right)}{2(n+1)}z^{m-1} \\
&+ \frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{n^j}{(m-j+1)}\left(\frac{1}{n+1} - \frac{j}{mn}\right)S_n\left(e_j;z\right) \\
E_{n,m}(z) &= \frac{z}{n}(K_n(e_{m-1};z) - e_{m-1}(z))' + zE_{n,m-1} + \frac{m-1}{n}z^{m-1} \\
&+ \frac{\left((m-1)^2 - 2z(m-1) - m^2 + 2mz\right)}{2(n+1)}z^{m-1} \\
E_{n,m}(z) &= \frac{z}{n}(K_n(e_{m-1};z) - e_{m-1}(z))' + zE_{n,m-1} + \frac{m-1}{n}z^{m-1} \\
&+ \frac{(m-1)^2 - m^2}{2(n+1)}z^{m-1} + \frac{z^m}{n+1}
\end{aligned}
$$

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{n}\left(K_n\left(e_{m-1};z\right)-e_{m-1}(z)\right)'+zE_{n,m-1} \\
&+\frac{(2n+2)(m-1)+(m-1)^2 n-m^2 n}{2n(n+1)}z^{m-1} \\
&+\frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(\frac{1}{n+1}-\frac{j}{mn}\right)S_n\left(e_j;z\right)
\end{aligned}
$$

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{n}\left(K_n\left(e_{m-1};z\right)-e_{m-1}(z)\right)'+zE_{n,m-1}+\frac{2m-n-2}{2n(n+1)}z^{m-1} \\
&+\frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(\frac{1}{n+1}-\frac{j}{mn}\right)S_n\left(e_j;z\right) \\
&+\frac{1}{n+1}z^m
\end{aligned}
$$

$$
\begin{aligned}
E_{n,m}(z) = {} & \frac{z}{n}\left(K_n\left(e_{m-1};z\right)-e_{m-1}(z)\right)'+zE_{n,m-1}+\frac{m-1}{n(n+1)}z^{m-1} \\
&-\frac{1}{2(n+1)}z^{m-1}+\frac{1}{n+1}z^m \\
&+\frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(\frac{1}{n+1}-\frac{j}{mn}\right)S_n\left(e_j;z\right)
\end{aligned}
$$

which is the desired recurrence formula. ∎

4.2. Convergence Properties Of K_n

In this section, we investigate the quantitative estimates of the convergence for complex Szász-Kantorovich-type operators attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 4.2.1 *Let $f \in H(\mathbb{M}_R)$ and $f : [R, \infty) \cup \overline{\mathbb{M}_R} \to \mathbb{C}$ be bounded on $[0, \infty)$. If $1 \le r < \frac{R}{2}$,* *then for all* $|z| \le r$ *and* $n \in \mathbb{N}$ *we have*

$$|K_n(f;z) - f(z)| \le \frac{3}{2n}\sum_{m=1}^{\infty}|c_m|\, m(m+1)(2r)^m \tag{4.2.1}$$

Proof. Using the recurrence formula (4.1.4) we obtain the following relationship:

$$\begin{aligned} K_n(e_m;z) - e_m(z) &= \frac{z}{n}K_n'(e_{m-1};z) + z\left(K_n(e_{m-1};z) - e_{m-1}(z)\right) \\ &+ \frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)} \\ &.\left\{1 - \frac{j}{m} - \frac{j}{mn}\right\}S_n\left(e_j;z\right). \end{aligned} \tag{4.2.2}$$

We can easily estimate the sum in the above formula as follows:

$$\begin{aligned} &\left|\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left\{1 - \frac{j}{m} - \frac{j}{mn}\right\}S_n\left(e_j;z\right)\right| \\ \le\ & \frac{1}{(n+1)^m}\left\{\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{m}{m-j}\frac{n^j}{m-j+1}\left|1 - \frac{j}{m} - \frac{j}{mn}\right|\right\}\left|S_n\left(e_j;z\right)\right| \\ &+ \frac{n^{m-1}}{(n+1)^m}S_n(e_m;z) \\ \le\ & \frac{2m(n+1)^{m-1} + n^{m-1}}{(n+1)^m}(2r)^m \le \frac{2m+1}{n+1}(2r)^m \end{aligned}$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$\left|P_m'(z)\right| \le \frac{m}{r}\|P_m\|_r, \quad \text{for all} \quad |z| \le r, \quad r \ge 1,$$

where $P_m(z)$ is a complex polynomial of degree $\leq m$. From the above recurrence formula (4.2.2), we get

$$\begin{aligned}
&\left|K_{n,q}(e_m;z)-e_m(z)\right| \\
&\leq \frac{|z|}{n}\left|K_n'(e_{m-1};z)\right|+|z|\left|K_n(e_{m-1};z)-e_{m-1}(z)\right| \\
&+\frac{(2m+1)}{n+1}(2r)^m \\
&\leq \frac{r}{n}\frac{m-1}{r}\|K_n(e_{m-1})\|_r+r\left|K_n(e_{m-1};z)-e_{m-1}(z)\right| \\
&+\frac{(2m+1)}{n+1}(2r)^m \\
&\leq \frac{(m-1)}{n}(2r)^{m-1}+\frac{(2m+1)}{n+1}(2r)^m \\
&+r\left|K_n(e_{m-1};z)-e_{m-1}(z)\right| \\
&\leq r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right|+\frac{3m}{n}(2r)^m.
\end{aligned}$$

Writing the last inequality for $k=1,2,...$, we easily obtain

$$\begin{aligned}
\left|K_n(e_m;z)-e_m(z)\right| &\leq \frac{(2r)^m}{n}3m+r\frac{(2r)^{m-1}}{n}3(m-1) \\
&\quad +r^2\frac{(2r)^{m-2}}{n}+...+r^{m-1}\frac{(2r)}{n}3 \\
&= \frac{3(2r)^m}{n}(m+m-1+...+1) \\
&\leq \frac{3m(m+1)}{2n}(2r)^m. \qquad (4.2.3)
\end{aligned}$$

Since $K_n(f:z)$ is analytic in $\mathbb{M}_R$, we can write

$$K_n(f;z)=\sum_{m=0}^{\infty}a_mK_n(e_m;z), \qquad z\in\mathbb{M}_R$$

which together with estimate (4.2.3) immediately implies for all $|z| \le r$,

$$|K_n(f;z) - f(z)| \le \sum_{m=0}^{\infty} |a_m| \, |K_n(e_m;z) - e_m(z)| \le \frac{3}{2n}\sum_{m=1}^{\infty} |a_m| \, m(m+1)\,(2r)^m .$$

■

4.3. Voronovskaja Type Results of K_n

The following theorem gives Voronovskaja type results in compact disks for complex Szász-Kantorovich operators attached to an analytic function in $\mathbb{M}_R$, $R > 1$.

Theorem 4.3.1 *Let $f \in H(\mathbb{M}_R)$ and $f : [R,\infty) \cup \overline{\mathbb{M}_R} \to \mathbb{C}$ be bounded on $[0,\infty)$. If $1 \le r < \frac{R}{2}$ then for all $|z| \le r$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned} & \left| K_{n,q}(f;z) - f(z) - \frac{1-2z}{2(n+1)} f'(z) - \frac{z(1-z)}{2(n+1)} f''(z) \right| \\ \le \; & \frac{10}{n^2} \sum_{m=2}^{\infty} |a_m| \, m(m-1)^3 \, (2r)^m . \end{aligned}$$

Proof. A simple calculation and applying the recurrence formula (4.1.4) leads us to the following relationship

$$\begin{aligned} E_{n,m}(z) \; = \; & \frac{z}{n}\left(K_n(e_{m-1};z) - e_{m-1}(z)\right)' + zE_{n,m-1}(z) + \frac{m-1}{n(n+1)} z^{m-1} \\ & -\frac{1}{2(n+1)} z^{m-1} + \frac{1}{n+1} z^m - \frac{1}{(n+1)^m} n^{m-1} S_n(e_m;z) \\ & -\frac{n^{m-2}}{2(n+1)^m}(m-1) S_n(e_m;z) + \frac{n^{m-1}}{2(n+1)^m} S_n(e_{m-1};z) \\ & +\frac{1}{(n+1)^m} \sum_{j=0}^{m-2} \binom{m}{j} \frac{n^j}{(m-j+1)} \left\{1 - \frac{j}{m} - \frac{j}{mn}\right\} S_n\left(e_j;z\right) \end{aligned}$$

Then

$$\begin{aligned}
E_{n,m}(z) &= \frac{z}{n}\left(K_n\left(e_{m-1};z\right)-e_{m-1}(z)\right)' + zE_{n,m-1}(z)\\
&+ \frac{m-1}{n(n+1)}z^{m-1} + \frac{1}{n+1}\left(z^m - S_n\left(e_m;z\right)\right) + \frac{1}{n+1}\left(1-\frac{n^{m-1}}{(n+1)^{m-1}}\right)S_n\left(e_m;z\right)\\
&- \frac{n^{m-2}}{2(n+1)^m}(m-1)S_n\left(e_{m-1};z\right) + \frac{1}{2(n+1)}\left(S_n\left(e_{m-1};z\right)-z^{m-1}\right)\\
&- \frac{1}{2(n+1)}\left(1-\frac{n^{m-1}}{(n+1)^{m-1}}\right)S_n\left(e_{m-1};z\right)\\
&+ \frac{1}{(n+1)^m}\sum_{j=0}^{m-2}\binom{m}{j}\frac{n^j}{(m-j+1)}\left\{1-\frac{j}{m}-\frac{j}{mn}\right\}S_n\left(e_j;z\right)\\
&:= \sum_{k=1}^{9} I_k.
\end{aligned}$$

From the proof, we use Theorem 1.8.4 of [13],

$$\left|z^m - S_n\left(e_m;z\right)\right| \le \frac{6(m-1)}{n}(2r)^{m-1}$$

It follows that

$$\begin{aligned}
|I_4| &\le \frac{1}{n+1}\left|z^m - S_n\left(e_m;z\right)\right|\\
&\le \frac{6(m-1)(2r)^{m-1}}{n(n+1)},
\end{aligned}$$

and

$$\begin{aligned}
|I_7| &\le \frac{1}{2(n+1)}\left|z^{m-1} - S_n\left(e_{m-1};z\right)\right|\\
&\le \frac{3(m-2)(2r)^{m-2}}{n(n+1)}.
\end{aligned}$$

Applying the inequality

$$1-\prod_{j=1}^{k} x_j \leq \sum_{j=1}^{k}\left(1-x_j\right), \quad 0 \leq x_j \leq 1, \quad j=1,...,k$$

we have

$$\begin{aligned}|I_5| &\leq \frac{1}{n+1}\left(1-\frac{n^{m-1}}{(n+1)^{m-1}}\right)|S_n(e_m;z)| \\ &\leq \frac{m-1}{(n+1)^2}(2r)^m,\end{aligned}$$

and

$$\begin{aligned}|I_8| &\leq \frac{1}{2(n+1)}\left(1-\frac{n^{m-1}}{(n+1)^{m-1}}\right)|S_n(e_{m-1};z)| \\ &\leq \frac{m-1}{2(n+1)^2}(2r)^{m-1}\end{aligned}$$

For I_9, we have

$$\begin{aligned}|I_9| &\leq \frac{1}{(n+1)^m}\sum_{j=0}^{m-2}\left(\begin{array}{c} m-2 \\ j \end{array}\right)\frac{m(m-1)}{(m-j)(m-j-1)}\frac{n^j}{(m-j+1)}\left|1-\frac{j}{m}-\frac{j}{mn}\right|\left|S_n\left(e_j;z\right)\right| \\ &\leq \frac{2m(m-1)(n+1)^{m-2}}{(n+1)^m}(2r)^{m-2} \\ &\leq \frac{2m(m-1)}{(n+1)^2}(2r)^{m-2}\end{aligned}$$

Thus

$$
\begin{aligned}
&\left|E_{n,m}(z)\right| \\
&\leq \frac{r}{n}\left|(K_n(e_{m-1};z)-e_{m-1}(z))'\right| + r\left|E_{n,m-1}(z)\right| \\
&+ \frac{m-1}{n(n+1)}r^{m-1} + \frac{6(m-1)}{n(n+1)}(2r)^{m-1} + \frac{m-1}{(n+1)^2}(2r)^m \\
&+ \frac{m-1}{2(n+1)^2}(2r)^{m-1} + \frac{3(m-2)(2r)^{m-2}}{n(n+1)} \\
&+ \frac{m-1}{2(n+1)^2}(2r)^{m-1} + \frac{2m(m-1)}{(n+1)^2}(2r)^{m-2}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|E_{n,m}(z)\right| \\
&\leq \frac{r}{n}\frac{m-1}{r}\frac{3m(m-1)}{2n}(2r)^m + r\left|E_{n,m-1}(z)\right| \\
&+ \frac{8m(m-1)}{(n+1)^2}(2r)^m \\
&\leq r\left|E_{n,m-1}(z)\right| + \frac{10m(m-1)^2}{n^2}(2r)^m
\end{aligned}
$$

$$
\leq r\left|E_{n,m-1}(z)\right| + \frac{10m(m-1)^2}{n^2}(2r)^m
$$

As a consequence, we get

$$
\left|E_{n,m}(z)\right| \leq \frac{10m(m-1)^3}{n^2}(2r)^m .
$$

Note that since $f^{(4)} = \sum_{m=4}^{\infty} a_m m(m-1)(m-2)(m-3)z^{m-4}$ and the series is absolutely convergent for all $|z| < R$, it easily follows the finiteness of the involved constants in the statement.

As an application of Theorem 4.3.1, we present the order of approximation for complex q-Kantorovich operators. ■

Theorem 4.3.2 *Let* $f \in H(\mathbb{M}_R)$ *and* $f : [R, \infty) \cup \overline{\mathbb{M}_R} \to \mathbb{C}$ *be bounded on* $[0, \infty)$*. If* $1 \leq r < \frac{R}{2}$ *and if* f *is not a constant function then the estimate*

$$\|K_n(f) - f\|_r \geq \frac{1}{n} C_r(f), \quad n \in \mathbb{N}$$

holds, where the constant $C_r(f)$ *depends on* f *and* r *but it is independent of* n.

Proof. For all $z \in \mathbb{M}_R$ and $n \in \mathbb{N}$ we get

$$K_n(f; z) - f(z) = \frac{1}{n} \left\{ \begin{array}{c} \frac{(1-2z)}{2} f'(z) + \frac{z}{2} f''(z) \\ +n\left(K_n(f; z) - f(z) - \frac{(1-2z)}{2n} f'(z) - \frac{z}{2n} f''(z)\right) \end{array} \right\}$$

We apply

$$\|F + G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$$

to get

$$\|K_n(f) - f\|_r \geq \frac{1}{n} \left\{ \begin{array}{c} \left\|\frac{(1-2z)}{2} f'(z) + \frac{z}{2} f''(z)\right\|_r \\ -\left\|n\left(K_n(f; z) - f(z) - \frac{(1-2z)}{2n} f'(z) - \frac{z}{2n} f''(z)\right)\right\|_r \end{array} \right\}.$$

Taking into account that by hypothesis f is not a polynomial of degree 0 in $\mathbb{M}_R$, we get $\|e_1(1 - e_1)f'' - e_1 f'\|_r > 0$.

Indeed, supposing the contrary it follows that $(1-2z)f'(z)+zf''(z)=0$ for all $|z|\le r$, that is $(zf'(z))'-2zf'(z)=0$ for all $|z|\le r$. The last equality is equivalent to $zf'(z)=Ce^{2z}$ for all $|z|\le r$ with $z\neq 0$. Therefore we get $f'(z)=C\frac{e^{2z}}{z}$, for all $|z|\le r$ with $z\neq 0$. But since f is analytic in $\overline{\mathbb{M}_r}$, we necessarily have $C=0$, which implies $f'(z)=0$ and $f(z)=c$ for all $z\in\overline{\mathbb{M}_r}$, a contradiction with the hypothesis.

Now, by Theorem 4.3.1 we have

$$\begin{aligned} & n\left|K_n(f;z)-f(z)-\frac{(1-2z)}{2n}f'(z)-\frac{z}{2n}f''(z)\right| \\ \le\ & \frac{10}{n}\sum_{m=2}^{\infty}|a_m|\,m(m-1)^3(2r)^m\to 0 \quad\text{as}\quad n\to\infty. \end{aligned}$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n\ge n_1$ we have

$$\begin{aligned} & \left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r-\left\|n\left(K_n(f;z)-f(z)-\frac{(1-2z)}{2n}f'(z)-\frac{z}{2n}f''(z)\right)\right\|_r \\ \ge\ & \frac{1}{2}\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r, \end{aligned}$$

which implies

$$\|K_n(f)-f\|_r\ge\frac{1}{2n}\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r,\quad\text{for all } n\ge n_1.$$

For $1\le n\le n_1-1$ we have

$$\|K_n(f)-f\|_r\ge\frac{1}{n}\left(n\,\|K_n(f)-f\|_r\right)=\frac{1}{n}M_{r,n}(f)>0,$$

which finally implies that

$$\|K_n(f) - f\|_r \geq \frac{1}{n} C_r(f),$$

for all n, with $C_r(f) = \min\left\{M_{r,1}(f), ..., M_{r,n_1-1}(f), \frac{1}{2}\left\|\frac{(1-2z)}{2} f'(z) + \frac{z}{2} f''(z)\right\|_r\right\}$. ■

Chapter 5

APPROXIMATION THEOREMS FOR COMPLEX SZÁSZ–KANTOROVICH OPERATORS IN COMPACT DISKS, $q > 1$

In this chapter, we introduce the new type complex $q-$Szász–Kantorovich operators ($q > 1$) and investigate qualitative and quantitative Voronovskaja-type results and the exact order of approximation for the new complex $q-$Szász–Kantorovich operators attached to analytic functions on compact disks.

5.1. Construction and Auxilary results

In this section, we introduce new type complex $q-$Szász–Kantorovich operators, $q > 1$ and some auxiliary results for this operators. The proof of the main results stated in the next section require the following auxiliary lemmas.

The complex $q-$Szász–Kantorovich operators in the case $q > 1$, defined as

Definition 5.1.1 *For $f \in H(\mathbb{M}_R)$, $q > 1$ and $n \in \mathbb{N}$, we define the following complex $q-$Szász–Kantorovich operators*

$$K_{n,q}(f;z) = \sum_{j=0}^{\infty} e_q\left(-[n]_q q^{-j} z\right) \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} \int_0^1 f\left(\frac{q[j]_q + t}{[n+1]}\right) dt. \tag{5.1.1}$$

where $j \in \mathbb{N}$ and $z \in \mathbb{C}$.

If f is bounded on $[0, +\infty)$ then it is clear that $K_{n,q}(f;z)$ is well-defined for all $z \in \mathbb{C}$.
Lemma 5.1.2 *Let $q > 1$. For all $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have*

$$K_{n,q}(e_m;z) = \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \binom{m}{k} \frac{q^k [n]_q^k}{(m-k+1)} S_{n,q}(e_k;z) \tag{5.1.2}$$

where $e_m(z) = z^m$.

Proof. The recurrence formula can be derived by direct computation. Indeed,

$$
\begin{aligned}
K_{n,q}(e_m;z) &= \sum_{j=0}^{\infty} e_q\left(-[n]_q q^j z\right) \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} \cdot \int_0^1 f\left(\frac{q[j]_q + t}{[n+1]_q}\right) dt \\
&= \sum_{j=0}^{\infty} e_q\left(-[n]_q q^j z\right) \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} \\
&\cdot \sum_{k=0}^{m} \binom{m}{k} \int_0^1 \frac{q^k [j]_q^k t^{m-k}}{[n+1]_q^m} dt \\
&= \sum_{j=0}^{\infty} e_q\left(-[n]_q q^j z\right) \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} \\
&\cdot \sum_{k=0}^{m} \binom{m}{k} \frac{q^k [j]_q^k}{[n+1]_q^m} \int_0^1 t^{m-k} dt
\end{aligned}
$$

$$
\begin{aligned}
K_{n,q}(e_m;z) &= \sum_{j=0}^{\infty} e_q\left(-[n]_q q^j z\right) \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} \\
&\cdot \sum_{k=0}^{m} \binom{m}{k} \frac{q^k [j]_q^k}{[n+1]_q^m (m-k+1)}
\end{aligned}
$$

$$
\begin{aligned}
K_{n,q}(e_m;z) &= \sum_{k=0}^{m} \binom{m}{k} \frac{q^k [n]_q^k}{[n+1]_q^m (m-k+1)} \\
&\underbrace{\cdot \sum_{j=0}^{\infty} \frac{[j]_q^k}{[n]_q^k} \frac{\left([n]_q z\right)^j}{[j]_q!} \frac{1}{q^{\frac{j(j-1)}{2}}} e_q\left(-[n]_q q^j z\right)}_{S_{n,q}(e_k;z)}
\end{aligned}
$$

$$
K_{n,q}(e_m;z) = \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \binom{m}{k} \frac{q^k [n]_q^k}{(m-k+1)} S_{n,q}(e_k;z).
$$

This proves the lemma. ■

Also we have:

Lemma 5.1.3 *For all $z \in \mathbb{C}$ we have*

$$\left|K_{n,q}\left(e_m;z\right)\right| \leq (2qr)^m, \quad m \in \mathbb{N}. \tag{5.1.3}$$

Proof. Indeed, using the inequality $\left|S_{n,q}\left(e_j;z\right)\right| \leq (2qr)^j$ (see relation (6), p. 1788 in Mahmudov [27]), we get

$$\begin{aligned}\left|K_{n,q}\left(e_m;z\right)\right| &\leq \frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{q^j\,[n]_q^j}{(m-j+1)}\left|S_{n,q}(e_j;z)\right| \\ &\leq \left|S_{n,q}(e_m;z)\right| \leq (2qr)^m\end{aligned}$$

which proves the lemma. ■ The next result is immediate.

Lemma 5.1.4 *We have*

$$\begin{aligned}K_{n,q}\left(e_0;z\right) = 1, \quad K_{n,q}\left(e_1;z\right) &= \frac{1}{2\,[n+1]_q} + \frac{q\,[n]_q}{[n+1]_q}z, \\ K_{n,q}\left(e_2;z\right) &= \frac{1}{3\,[n+1]_q^2} + \frac{2q\,[n]_q}{[n+1]_q^2}z + \frac{q^2\,[n]_q^2}{[n+1]_q^2}z^2, \\ K_{n,q}\left(\left(e_1 - ze_0\right)^2;z\right) &= \frac{1}{3\,[n+1]_q^2} + \frac{2q\,[n]_q - 1}{[n+1]_q^2}z + \frac{1}{[n+1]_q^2}z^2.\end{aligned}$$

Also, $K_{n,q}(e_m;z)$ is a polynomial of degree m in z.

The following recurrence formulas in Lemma 5.1.5 and Lemma 5.1.6 hold.

Lemma 5.1.5 *For all* $n, m \in \mathbb{N}$, $z \in \mathbb{C}$ *and* $q > 1$, *we have*

$$K_{n,q}(e_m; z) = \frac{z}{[n]_q} D_q K_{n,q}(e_{m-1}; z) + z K_{n,q}(e_{m-1}; z)$$
$$+ \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k+1}$$
$$. \left\{ 1 - \frac{k}{m} - \frac{k}{mq[n]_q} \right\} S_{n,q}(e_k; z). \tag{5.1.4}$$

Proof. We know that (see Mahmudov [27], p. 1788, the first relation just after relation (6) there)

$$S_{n,q}(e_{k+1}; z) = \frac{z}{[n]_q} D_q S_{n,q}(e_k; z) + z S_{n,q}(e_k; z).$$

Taking the derivative of the formula (5.1.4) and using the above formula we have

$$\begin{aligned}
\frac{z}{[n]_q} D_q K_{n,q}(e_{m-1}; z) &= \frac{1}{[n+1]_q^{m-1}} . \sum_{k=1}^{m-1} \begin{pmatrix} m-1 \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k} \\
&\quad . \{ S_n(e_{k+1}; z) - z S_n(e_k; z) \} \\
&= \frac{1}{[n+1]_q^{m-1}} \sum_{k=1}^{m-1} \begin{pmatrix} m-1 \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k} S_n(e_{k+1}; z) \\
&\quad - z \underbrace{\frac{1}{[n+1]_q^{m-1}} \sum_{k=1}^{m-1} \begin{pmatrix} m-1 \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k} S_n(e_k; z)}_{K_{n,q}(e_{m-1}; z)}
\end{aligned}$$

$$\begin{aligned}
\frac{z}{[n]_q} D_q K_n(e_{m-1}; z) &= \frac{1}{[n+1]^{m-1}} \sum_{k=1}^{m} \begin{pmatrix} m-1 \\ k \end{pmatrix} \frac{q^{k-1} [n]_q^{k-1}}{m-k+1} S_{n,q}(e_k; z) \\
&\quad - z K_{n,q}(e_{m-1}; z)
\end{aligned}$$

It follows that

$$
\begin{aligned}
K_{n,q}\left(e_m;z\right) &= \frac{z}{[n]_q} D_q K_{n,q}\left(e_{m-1};z\right) + zK_{n,q}\left(e_{m-1};z\right) \\
&+ \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k+1} S_{n,q}\left(e_k;z\right) \\
&- \frac{1}{[n+1]^{m-1}} \sum_{k=1}^{m} \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} \frac{q^{k-1} [n]_q^{k-1}}{m-k+1} S_{n,q}\left(e_k;z\right) \\
&= \frac{z}{[n]_q} D_q K_{n,q}\left(e_{m-1};z\right) + zK_{n,q}\left(e_{m-1};z\right) \\
&+ \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k+1} \left\{1 - \frac{k\,[n+1]_q}{mq\,[n]_q}\right\} S_{n,q}\left(e_k;z\right)
\end{aligned}
$$

Here we used the identity $\binom{m-1}{k-1} = \binom{m}{k} \cdot \frac{k}{m}$. ■

Define

$$
\begin{aligned}
E_{n,m}(z) \quad : &= K_{n,q}\left(e_m;z\right) - e_m(z) - \frac{1-2z}{2\,[n]_q} m z^{m-1} \\
&- \sum_{k=1}^{m-1} [k]_q \frac{q z^{m-1}}{[n]_q}.
\end{aligned} \tag{5.1.5}
$$

Lemma 5.1.6 *Let $n, m \in \mathbb{N}$ and $q > 1$. We have the following recurrence formula*

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{[n]_q} D_q \left(K_n\left(e_{m-1};z\right) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\
&- \frac{(1-2z)}{2\,[n]_q} z^{m-1} + \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k+1} \\
&. \left\{1 - \frac{k}{m} - \frac{k}{mq\,[n]_q}\right\} S_{n,q}\left(e_k;z\right).
\end{aligned} \tag{5.1.6}
$$

Proof. It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z) = 0$.

Using the formula (5.1.4), we get

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{[n]_q}\left\{D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) + D_q\left(z^{m-1}\right)\right\} \\
&+ zE_{n,m-1}(z) + \frac{(1-2z)}{2[n]_q}(m-1)z^{m-1} + z^m \\
&+ \sum_{k=1}^{m-2}[k]_q\frac{qz^{m-1}}{[n]_q} - \frac{(1-2z)mz^{m-1}}{2[n]_q} \\
&+ \frac{1}{[n+1]_q^m}\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\frac{q^k[n]_q^k}{m-k+1}.\left\{1 - \frac{k}{m} - \frac{k}{mq[n]_q}\right\}S_{n,q}(e_k;z) \\
&- z^m - \sum_{k=1}^{m-1}[k]_q\frac{qz^{m-1}}{[n]_q}
\end{aligned}
$$

Then

$$
\begin{aligned}
E_{n,m}(z) &= \frac{z}{[n]_q}D_q(K_n(e_{m-1};z) - e_{m-1}(z)) + \frac{[m-1]_q}{[n]_q}z^{m-1} \\
&+ zE_{n,m-1} + \frac{(m-1)(1-2z)z^{m-1}}{2[n]_q} + \sum_{k=1}^{m-2}[k]_q\frac{qz^{m-1}}{[n]_q} \\
&- \frac{(1-2z)mz^{m-1}}{2[n]_q} - \sum_{k=1}^{m-1}[k]_q\frac{qz^{m-1}}{[n]_q} + \frac{1}{[n+1]_q^m} \\
&.\sum_{k=0}^{m}\binom{m}{k}\frac{q^k[n]_q^k}{m-k+1}\left\{1 - \frac{k}{m} - \frac{k}{mq[n]_q}\right\}S_{n,q}(e_k;z)
\end{aligned}
$$

which is the desired recurrence formula. ∎

5.2. Convergence Properties of $K_{n,q}$

Upper estimates in approximation by $K_{n,q}(f;z)$ and by its derivatives can be stated as follows.

Theorem 5.2.1 *Let* $1 < q < \frac{R}{2}$ *and suppose that* $f : \overline{\mathbb{M}_R} \bigcup [R, +\infty) \rightarrow \mathbb{C}$ *is continuous and bounded in* $\overline{\mathbb{M}_R} \bigcup [R, +\infty)$ *and analytic in* $\mathbb{M}_R$*, namely* $f(z) = \sum_{m=0}^{\infty} a_m z^m$ *for all* $z \in \mathbb{M}_R$.

(i) If $1 \leq r < \frac{R}{2q}$*, then for all* $|z| \leq r$ *and* $n \in \mathbb{N}$*, we have*

$$\left|K_{n,q}(f;z) - f(z)\right| \leq \frac{3}{2[n]_q} \sum_{m=1}^{\infty} |a_m| \, m(m+1)(2qr)^m \tag{5.2.1}$$

(ii) If $1 \leq r < r_1 < \frac{R}{2q}$ *then for all* $|z| \leq r$ *and* $n, p \in \mathbb{N}$*, we have*

$$\left|K_{n,q}^{(p)}(f;z) - f^{(p)}(z)\right| \leq \frac{p! C_{r_1}(f)}{[n]_q (r_1 - r)^{p+1}} \tag{5.2.2}$$

where

$$C_{r_1}(f) = \frac{3}{2} \sum_{m=1}^{\infty} |a_m| \, m(m+1)(2qr_1)^m < \infty.$$

Proof. (i) From the use of the recurrence formula in Lemma 5.1.5, we obtain the following relationship

$$\begin{aligned} K_{n,q}(e_m;z) - e_m(z) &= \frac{z}{[n]} D_q\left(K_{n,q}(e_{m-1};z)\right) \\ &+ z\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) \\ &+ \frac{1}{[n+1]_q^m} \sum_{k=0}^{m} \binom{m}{k} \frac{q^k [n]_q^k}{m-k+1} \\ &. \left\{1 - \frac{k}{m} - \frac{k}{mq[n]_q}\right\} S_{n,q}(e_k;z). \end{aligned} \tag{5.2.3}$$

We can easily estimate the sum in the above formula as follows:

$$\begin{aligned}
&\left|\frac{1}{[n+1]_q^m}\sum_{k=0}^{m}\binom{m}{k}\frac{q^k[n]_q^k}{m-k+1}\left\{1-\frac{k}{m}-\frac{k}{mq[n]_q}\right\}S_{n,q}(e_k;z)\right| \\
&\leq \frac{1}{[n+1]_q^m}\left\{\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{m}{m-k}\frac{q^k[n]_q^k}{m-k+1}\left|1-\frac{k}{m}-\frac{k}{qm[n]_q}\right|\right\}\left|S_{n,q}(e_k;z)\right| \\
&+\frac{[n]_q^{m-1}q^{m-1}}{[n+1]_q^m}S_{n,q}(e_m;z) \\
&\leq \frac{\left(1+q^{-1}\right)m[n+1]_q^{m-1}+[n]_q^{m-1}q^{m-1}}{[n+1]_q^m}(2qr)^m \\
&\leq \frac{2m+1}{[n+1]_q}(2qr)^m \qquad (5.2.4)
\end{aligned}$$

It is known that by a linear transformation, the Bernstein's inequality (for polynomials) in the closed unit disk becomes

$$\left|P'_m(z)\right| \leq \frac{m}{qr}\|P_m\|_{qr}, \quad \text{for all} \quad |z| \leq qr, \quad r \geq 1,$$

Where $\left(\|P_m(z)\|_{qr} = \max\{|P_m(z)| : |z| \le qr\}\right)$ is a complex polynomial of degree $\le m$. From the above recurrence formula (5.2.3), we get

$$\left|K_{n,q}(e_m;z) - e_m(z)\right| \le \frac{|z|}{[n]_q}\left|D_q K_{n,q}(e_{m-1};z)\right| + |z|\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + \frac{(2m+1)}{[n+1]_q}(2qr)^m$$

$$\left|K_{n,q}(e_m;z) - e_m(z)\right| \le \frac{r}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})\right\|_{qr} + r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + \frac{(2m+1)}{[n+1]_q}(2qr)^m$$

$$\left|K_{n,q}(e_m;z) - e_m(z)\right| \le r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + \frac{(m-1)}{[n]_q}\frac{(2qr)^{m-1}}{q} + \frac{(2m+1)}{[n+1]_q}(2qr)^m$$

$$\left|K_{n,q}(e_m;z) - e_m(z)\right| \le r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + (m-1+2m+1)\frac{(2qr)^m}{[n]_q}$$

$$\left|K_{n,q}(e_m;z) - e_m(z)\right| \le r\left|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right| + 3m\frac{(2qr)^m}{[n]_q}$$

By writing the last inequality for $k = 1, 2, ...$, step by step we easily obtain, the following

$$|K_n(e_m;z) - e_m(z)| \le m(2+q)\frac{(2qr)^m}{[n]_q} + r(m-1)(2+q)\frac{(2qr)^{m-1}}{[n]_q}$$

$$+ r^2(m-2)(2+q)\frac{(2qr)^{m-2}}{[n]_q} + ... + r^{m-1}\frac{(2qr)}{[n]_q}(2+q)$$

$$= \frac{(2qr)^m}{[n]_q}(2+q)(m+m-1+...+1) \le \frac{3m(m+1)}{2[n]_q}(2qr)^m \qquad (5.2.5)$$

Since $K_{n,q}(f;z)$ is analytic in $\mathbb{M}_R$, we can write

$$K_{n,q}(f;z) = \sum_{m=0}^{\infty} a_m K_{n,q}(e_m;z), \quad z \in \mathbb{M}_r.$$

Indeed, for this purpose, for any $m \in \mathbb{N}$ let us define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \le r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

From the hypothesis on f it is clear that for any $m \in \mathbb{N}$ it follows $|f_m(x)| \le C_{m,r}$, for all $x \in [0, +\infty)$. This implies that for each fixed $m, n \in \mathbb{N}$ and z,

$$|K_{n,q}(f_m)(z)| \le C_{m,r} \sum_{j=0}^{\infty} |e_q(-[n]_q q^{-j} z)| \frac{\left([n]_q|z|\right)^j}{[j]_q!} \cdot \frac{1}{q^{j(j-1)/2}} < \infty$$

since the last series is convergent (for fixed n and z). Therefore $K_{n,q}(f_m)(z)$ is well-defined.

Now, denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \le r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, \infty),$$

It is clear that each $f_{m,k}$ is bounded on $[0, \infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. Since from the linearity of $K_{n,q}$ we have

$$K_{n,q}(f_m)(z) = \sum_{k=0}^{m} c_k K_{n,q}(e_k)(z), \text{ for all } |z| \le r,$$

It suffices to prove that $\lim_{m\to\infty} K_{n,q}(f_m)(z) = K_{n,q}(f)(z)$ for any fixed $n \in \mathbb{N}$ and $|z| \le r$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \le \|f_m - f\|_r$ and from the inequality

$$\begin{aligned} |K_{n,q}(f_m)(z) - K_{n,q}(f)(z)| &\leq M_{r,n} \cdot \|f_m - f\|_{B[0,\infty)} \\ &\leq M_{r,n}\|f_m - f\|_r \end{aligned}$$

valid for all $|z| \leq r$. Here $\| \cdot \|_{B[0,+\infty)}$ denotes the uniform norm on $C[0, +\infty)$-the space of all real-valued bounded functions on $[0, +\infty)$.

In conclusion, together with (5.2.5) immediately implies for all $|z| \leq r$

$$\begin{aligned} \left|K_{n,q}(f;z) - f(z)\right| &\leq \sum_{m=0}^{\infty} |a_m| \left|K_{n,q}(e_m;z) - e_m(z)\right| \\ &\leq \frac{3}{2[n]_q} \sum_{m=1}^{\infty} |a_m| \, m(m+1)(2qr)^m \end{aligned}$$

(ii) Denote by γ the circle of radius $r_1 > r$ and center 0. For any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$ and by the Cauchy's formula, for all $|z| \leq r$ and $n \in \mathbb{N}$, it follows

$$\begin{aligned} |K_{n,q}^{(p)}(f;z) - f^{(p)}(z)| &= \frac{p!}{2\pi} \left| \int_{\gamma} \frac{K_{n,q}(f;v) - f(v)}{(v-z)^{p+1}} dv \right| \\ &\leq \frac{C_{r_1}(f)}{[n]_q} \cdot \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}} \\ &= \frac{C_{r_1}(f)}{[n]_q} \cdot \frac{p! r_1}{(r_1 - r)^{p+1}} \end{aligned}$$

■

5.3. Voronovskaja Type Results of $K_{n,q}$

For $R > q > 1$ and $|z| \leq \frac{R}{q}$, we also define

$$L_q(f;z) := \frac{1-2z}{2} f^{'}(z) + \frac{D_q f(z) - f^{'}(z)}{1-q^{-1}}.$$

It is not difficult show that

$$\begin{aligned} L_q(f;z) &= q \sum_{m=2}^{\infty} a_m \frac{[m]_q - m}{q-1} z^{m-1} + \frac{1-2z}{2} \sum_{m=1}^{\infty} a_m m z^{m-1} \\ &= q \sum_{m=2}^{\infty} a_m \left([1]_q + ... + [m-1]_q\right) z^{m-1} \\ &+ \frac{1-2z}{2} \sum_{m=1}^{\infty} a_m m z^{m-1} \\ L_q(f;z) &= \sum_{m=1}^{\infty} a_m \cdot V_m^{(q)}(z) \end{aligned}$$

where

$$V_m^{(q)}(z) = q([1]_q + ... + [m-1]_q) z^{m-1} + \frac{m z^{m-1}}{2}(1-2z)$$

with the convention that $[1]_q + ... + [m-1]_q = 0$ for $m = 1$.

Theorem 5.3.1 *Suppose that $f : \overline{\mathbb{M}_R} \bigcup [R, +\infty) \to \mathbb{C}$ is continuous and bounded in $\overline{\mathbb{M}_R} \bigcup [R, +\infty)$ and analytic in $\mathbb{M}_R$, namely $f(z) = \sum_{m=0}^{\infty} a_m z^m$ for all $z \in \mathbb{M}_R$.*

(i) If $1 < 2q^2 < R$ and $1 \leq r < \frac{R}{2q^2}$, then for all $z \in \mathbb{M}_r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} &\left| K_{n,q}(f;z) - f(z) - \frac{1}{[n]_q} L_q(f;z) \right| \\ \leq \; & \frac{9}{[n]_q^2} \sum_{m=2}^{\infty} |a_m| \, m (m-1)^3 \left(2q^2 r\right)^m \end{aligned}$$

(ii) Let $1 < q < R$. For any $1 < r < \frac{R}{q}$, we have

$$\lim_{n\to\infty} [n]_q (K_{n,q}(f;z) - f(z)) = L_q(f;z)$$

uniformly in $\overline{\mathbb{M}_r}$, where

$$L_q(f;z) = \sum_{m=1}^{\infty} a_m \cdot V_m^{(q)}(z),\ z \in \mathbb{M}_R,$$

$$V_m^{(q)}(z) = q([1]_q + ... + [m-1]_q)z^{m-1} + \frac{mz^{m-1}}{2}(1-2z),$$

with the convention that $[1]_q + ... + [m-1]_q = 0$ for $m = 1$.

Proof. (i) A simple calculation and the use of the recurrence formula (5.1.4) lead us to the following relationship

$$\begin{aligned}
E_{n,m}(z) &= \frac{z}{[n]_q} D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) \\
&+ \frac{1}{2q[n]_q}\left(S_{n,q}(e_{m-1};z) - z^{m-1}\right) - \frac{1}{2q[n]_q}\left(1 - \frac{q^m [n]_q^m}{[n+1]_q^m}\right) S_{n,q}(e_{m-1};z) \\
&+ \frac{1}{q[n]_q}\left(z^m - S_{n,q}(e_m;z)\right) + \frac{1}{q[n]_q}\left(1 - \frac{q^m [n]_q^m}{[n+1]_q^m}\right) S_{n,q}(e_m;z) \\
&- \frac{(m-1)q^{m-2}[n]_q^{m-2}}{2[n+1]_q^m} S_{n,q}(e_{m-1};z) \\
&+ \frac{1}{[n+1]_q^m} \sum_{k=0}^{m-2} \begin{pmatrix} m \\ k \end{pmatrix} \frac{q^k [n]_q^k}{m-k+1} . \left\{1 - \frac{k}{m} - \frac{k}{mq[n]_q}\right\} S_{n,q}(e_k;z) \\
&:= \sum_{k=1}^{8} I_k.
\end{aligned}$$

By Mahmudov [27], p. 1789, relation (8), we have

$$\left|z^m - S_{n,q}(e_m;z)\right| \leq \frac{2(m-1)}{[n]_q}(2qr)^{m-1}.$$

It follows that

$$\begin{aligned}|I_3| &\leq \frac{1}{2q[n]_q}\left|S_{n,q}(e_{m-1};z) - z^{m-1}\right| \\ &\leq \frac{1}{2q[n]_q}\cdot\frac{2(m-1)}{[n]_q}(2qr)^{m-2} \\ &\leq \frac{(m-1)(2qr)^{m-1}}{[n]_q^2}\end{aligned}$$

Applying the inequality

$$1 - \prod_{j=1}^{k} x_j \leq \sum_{j=1}^{k}\left(1 - x_j\right), \quad 0 \leq x_j \leq 1, \quad j = 1, ..., k$$

we have

$$\begin{aligned}|I_4| &\leq \frac{1}{2q[n]_q}\left(1 - \frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|S_{n,q}(e_{m-1};z)\right| \\ &\leq \frac{1}{2q[n]_q}\left(1 - \frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right) \\ &\leq \frac{(m-1)}{[n]_q^2}(2qr)^{m-1}\end{aligned}$$

and

$$
\begin{aligned}
|I_5| &\leq \frac{1}{q\,[n]_q}\left|z^m - S_{n,q}\left(e_m;z\right)\right| \\
&\leq \frac{2\left(m-1\right)}{[n]_q^2}\left(2qr\right)^{m-1}
\end{aligned}
$$

$$
\begin{aligned}
|I_6| &\leq \frac{1}{q\,[n]_q}\left(1-\frac{q^{m-1}\,[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|S_{n,q}\left(e_m;z\right)\right| \\
&\leq \frac{\left(m-1\right)}{[n]_q^2}\left(2qr\right)^{m-1}
\end{aligned}
$$

and

$$
\begin{aligned}
|I_7| &\leq \frac{\left(m-1\right)q^{m-2}\,[n]_q^{m-2}}{2\,[n+1]_q^m}\left|S_{n,q}\left(e_{m-1};z\right)\right| \\
&\leq \frac{\left(m-1\right)}{2\,[n]_q^2}\left(2qr\right)^{m-1}
\end{aligned}
$$

Hence

$$
|I_3| + |I_6| + |I_8| + |I_5| + |I_7| \leq \frac{11\left(m-1\right)}{2\,[n]_q^2}\left(2qr\right)^{m-1}
$$

For I_8, we have

$$
\begin{aligned}
|I_8| \leq \frac{1}{[n+1]_q^m}\sum_{k=0}^{m-2}\left(\begin{array}{c} m-2 \\ k \end{array}\right)\frac{m\left(m-1\right)}{\left(m-k\right)\left(m-k-1\right)}\frac{q^k\,[n]_q^k}{\left(m-k+1\right)} \\
.\left\{1-\frac{k}{m}-\frac{k}{qm\,[n]_q}\right\}\left(2qr\right)^m \\
\leq \frac{2m\left(m-1\right)[n+1]_q^{m-2}}{[n+1]_q^m}\left(2qr\right)^m \leq \frac{2m\left(m-1\right)}{[n]_q^2}\left(2qr\right)^m
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|E_{n,m}(z)\right| \\
&\leq \frac{r}{[n]_q}|D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right)| + r\left|E_{n,m-1}(z)\right| \\
&+ \frac{11(m-1)}{2[n]_q^2}(2qr)^{m-1} + \frac{2m(m-1)}{[n]_q^2}(2qr)^m \\
&\leq \frac{r}{[n]_q} \cdot \frac{m-1}{qr}\left\|K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right\|_{qr} \\
&+ r\left|E_{n,m-1}(z)\right| + \frac{11(m-1)}{2[n]_q^2}(2qr)^{m-1} + \frac{2m(m-1)}{[n]_q^2}(2qr)^m \\
&\leq r\left|E_{n,m-1}(z)\right| + \frac{3m(m-1)^2}{2[n]_q^2}\left(2q^2r\right)^m \\
&+ \frac{15m(m-1)}{2[n]_q^2}(2qr)^m \\
&\leq r\left|E_{n,m-1}(z)\right| + \frac{9m(m-1)^2}{[n]_q^2}\left(2q^2r\right)^m
\end{aligned}
$$

which implies

$$\leq r\left|E_{n,m-1}(z)\right| + \frac{9m(m-1)^2}{[n]_q^2}\left(2q^2r\right)^m$$

As a consequence, we get

$$\left|E_{n,m}(z)\right| \leq \frac{9m(m-1)^3}{[n]_q^2}\left(2q^2r\right)^m$$

Note that since $f^{(4)}(z) = \sum_{m=4}^{\infty} a_m m(m-1)(m-2)(m-3)z^{m-4}$ and the series is absolutely convergent for all $|z| < R$, it easily follows the finiteness of the involved constants in the statement.

(ii) Let $1 \le r < \frac{R}{2q}$. Since $\frac{R}{2q^{1+t}} \nearrow \frac{R}{2q}$ as $t \searrow 0$, evidently that given $1 < r < \frac{R}{2q}$, there exists a $t \in (0,1)$, such that $2q^{1+t}r < R$. Because f is analytic in $\mathbb{M}_R$, this implies that

$$\sum_{m=1}^{\infty} |a_m| m^4 q^{(1+t)m} (2r)^m = \sum_{m=1}^{\infty} |a_m| \cdot m^4 (2q^{1+t} r)^m < \infty$$

for all $z \in \overline{\mathbb{M}_r}$.

Also, the analyticity of f in $\mathbb{M}_R$ implies the convergence of the series $\sum_{m=1}^{\infty} |a_m|(m+1)^2(2qr)^m$, which means that for arbitrary $\varepsilon > 0$, there exists n_0, such that $\sum_{m=n_0+1}^{\infty} |a_m| \cdot (m+1)^2(2qr)^m < \varepsilon$.

Now, by using the formula for $L_q(f;z)$ just before the statement of Theorem 5.2.1 and the estimate for $|E_{n,m}(z)|$ in the above point (i), for all $z \in \overline{\mathbb{M}_r}$ and $n > n_0$, we get

$$\begin{aligned}
& |[n]_q(K_{n,q}(f;z) - f(z)) - L_q(f;z)| \\
\le & \sum_{m=1}^{n_0} |a_m| \cdot |[n]_q(K_{n,q}(e_m;z) - e_m(z)) - V_m^{(q)}(z)| \\
& + \sum_{m=n_0+1}^{\infty} |a_m| \cdot \left([n]_q |K_{n,q}(e_m;z) - z^m| + |V_m^{(q)}(z)|\right) \\
\le & \sum_{m=1}^{n_0} |a_m| \cdot \frac{9m(m-1)^3}{[n]_q} \cdot (2q^2 r)^m \\
& + \sum_{m=n_0+1}^{\infty} |a_m| \cdot \left([n]_q |K_{n,q}(e_m;z) - z^m| + |V_m^{(q)}(z)|\right)
\end{aligned}$$

Then

$$\begin{aligned}
& |[n]_q(K_{n,q}(f;z) - f(z)) - L_q(f;z)| \\
\le & q^2 \sum_{m=1}^{n_0} |a_m| \cdot \frac{9m(m-1)^3 [m]_q^2}{[n]_q} \cdot (2r)^m \\
& + \sum_{m=n_0+1}^{\infty} |a_m| \cdot \left([n]_q |K_{n,q}(e_m;z) - z^m| + |V_m^{(q)}(z)|\right)
\end{aligned}$$

Note that for the last inequality we used the obvious inequality $q^{2m} \leq q^2[m]_q^2$.

But by the proof of Theorem 5.2.1, (i), relationship (5.2.5), we have

$$|K_{n,q}(e_m; z) - z^m| \leq \frac{3m(m+1)}{2[n]_q} \cdot (2qr)^m$$

for all $z \in \overline{\mathbb{M}_r}$.

Also, since $[1]_q + ... + [m-1]_q \leq (m-1)[m-1]_q$, it is immediate that

$$\begin{aligned} |V_m^{(q)}(z)| &\leq qr^{m-1}(m-1)[m-1]_q + \frac{1+2r}{2}mr^{m-1} \\ &\leq (m-1)^2(qr)^{m-1} + \frac{3}{2}mr^m \end{aligned}$$

Therefore, we easily obtain

$$\begin{aligned} &\sum_{m=n_0+1}^{\infty} |a_m| \cdot \left([n]_q|K_{n,q}(e_m; z) - z^m| + |V_m^{(q)}(z)|\right) \\ \leq &\sum_{m=n_0+1}^{\infty} |a_m| \cdot \left[\frac{3m(m+1)}{2}(2qr)^m + (m-1)^2(qr)^{m-1} + \frac{3}{2}mr^m\right] \\ \leq &\sum_{m=n_0+1}^{\infty} |a_m| \cdot \left[\frac{3m(m+1)}{2}(2qr)^m + (m-1)^2 + \frac{3}{2}m\right](2qr)^m \\ \leq &\frac{5}{2}\sum_{m=n_0+1}^{\infty} |a_m|(m+1)^2(2qr)^m, \end{aligned}$$

valid for all $z \in \overline{\mathbb{M}_r}$. Concluding, for all $z \in \overline{\mathbb{M}_r}$ and $n > n_0$, we have

$$\begin{aligned}
& |[n]_q(K_{n,q}(f;z) - f(z)) - L_q(f;z)| \\
\leq\ & q^2 \sum_{m=1}^{n_0} |a_m| \cdot \frac{9m(m-1)^3[m]_q^2}{[n]_q} \cdot (2r)^m \\
& + \frac{5}{2} \sum_{m=n_0+1}^{\infty} |a_m| \cdot (m+1)^2 (2qr)^m \\
\leq\ & 9q^2 \sum_{m=1}^{n_0} |a_m| \cdot \frac{m^4[m]_q^2}{[n]_q} \cdot (2r)^m \\
& + \frac{5}{2} \sum_{m=n_0+1}^{\infty} |a_m| \cdot (m+1)^2 (2qr)^m \\
=\ & \frac{9q^2}{[n]_q^t} \sum_{m=1}^{n_0} |a_m| \cdot m^4 \frac{[m]_q^2}{[n]_q^{1-t}} \cdot (2r)^m \\
& + \frac{5}{2} \sum_{m=n_0+1}^{\infty} |a_m| \cdot (m+1)^2 (2qr)^m \\
\leq\ & \frac{9q^2}{[n]_q^t} \sum_{m=1}^{n_0} |a_m| \cdot m^4 \frac{[m]_q^2}{[m]_q^{1-t}} \cdot (2r)^m \\
& + \frac{5}{2} \sum_{m=n_0+1}^{\infty} |a_m| \cdot (m+1)^2 (2qr)^m
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{9q^2}{[n]_q^t} \sum_{m=1}^{n_0} |a_m| \cdot m^4 [m]_q^{1+t} \cdot (2r)^m \\
& + \frac{5}{2} \sum_{m=n_0+1}^{\infty} |a_m| \cdot (m+1)^2 (2qr)^m
\end{aligned}$$

$$|[n]_q(K_{n,q}(f;z) - f(z)) - L_q(f;z)| \leq \frac{9q^2}{[n]_q^t (q-1)^{1+t}} \cdot \sum_{m=1}^{\infty} |a_m| \cdot m^4 q^{(1+t)m} \cdot (2r)^m + \frac{5}{2}\varepsilon.$$

Now, since $\frac{9q^2}{(q-1)^{1+t}[n]_q^t} \to 0$ as $n \to \infty$ and

$$\sum_{m=1}^{\infty} |a_m| \cdot m^4 q^{(1+t)m} (2r)^m = \sum_{m=1}^{\infty} |a_m| \, m^4 . (2q^{1+t} r)^m < \infty,$$

For the given $\varepsilon > 0$, there exists n_1, such that $\frac{9q^2}{(q-1)^{1+t}[n]_q^t} \cdot \sum_{m=1}^{\infty} |a_m| \cdot m^4 q^{(1+t)m} \cdot (2r)^m < \varepsilon/2$ for all $n > n_1$.

Note that n_1 depends on t, q and ε while n_0 depends on q, r and ε, but both are independent of z.

As a final conclusion, for all $n > \max\{n_0, n_1\}$ and $z \in \overline{\mathbb{M}_r}$, we get

$$|[n]_q(K_{n,q}(f;z) - f(z)) - L_q(f;z)| \leq 3\varepsilon$$

which shows that

$$\lim_{n\to\infty} [n]_q(K_{n,q}(f;z) - f(z)) = L_q(f;z), \text{ uniformly in } \overline{\mathbb{M}_r}.$$

The theorem is proved. ■

As an application of Theorem 5.3.1, (ii) and of Theorem5.2.1, (i), we get the following exact estimate in the approximation by the complex q-Száz-Kantorovich operators.

Theorem 5.3.2 *Suppose that* $f : \overline{\mathbb{M}_R} \bigcup [R, +\infty) \to \mathbb{C}$ *is continuous and bounded in* $\overline{\mathbb{M}_R} \bigcup [R, +\infty)$ *and analytic in* $\mathbb{M}_R$*. Let* $1 < q < \frac{R}{2}$ *and* $1 \leq r < \frac{R}{2q}$.

If f *is not a constant function in* $\mathbb{M}_R$ *then*

$$\left\|K_{n,q}(f) - f\right\|_r \sim \frac{1}{[n]_q}$$

where the constants in the equivalence depend on f, q *and* r*, but are independent of* n *;*

Proof. Suppose that f given by $f(z) = \sum_{m=0}^{\infty} a_m z^m$ for all $z \in \mathbb{M}_R$, is such that the approximation order in approximation by $K_{n,q}(f)$ is better than $\frac{1}{[n]_q}$, that is $\|K_{n,q}(f) - f\|_r \leq M \frac{s_n}{[n]_q}$, for all $n \in \mathbb{N}$, where $s_n \to 0$ as $n \to \infty$. This would imply that $\lim_{n\to\infty} [n]_q \|K_{n,q}(f) - f\|_r = 0$.

Then, by Theorem 5.3.1, (ii), would immediately follow that $L_q(f; z) = 0$ for all $z \in \overline{\mathbb{M}_r}$, where $L_q(f; z)$ is defined in the statement of Theorem 5.3.1, (ii).

But $L_q(f; z) = 0$ for all $z \in \overline{\mathbb{M}_r}$ implies

$$\frac{1}{2}\sum_{m=1}^{\infty} m a_m z^{m-1} - \sum_{m=1}^{\infty} m a_m z^m + q \sum_{m=2}^{\infty} a_m \sum_{j=1}^{m-1} [j]_q \, z^{m-1} = 0$$

$$\frac{1}{2}\sum_{m=0}^{\infty} (m+1)\, a_{m+1} z^m - \sum_{m=1}^{\infty} m a_m z^m + q \sum_{m=1}^{\infty} a_{m+1} \sum_{j=1}^{m} [j]_q \, z^m = 0$$

$$\frac{1}{2}a_1 + \sum_{m=1}^{\infty}\left(\frac{1}{2}(m+1)\, a_{m+1} - m a_m + q a_{m+1}\sum_{j=1}^{m} [j]_q\right) z^m = 0$$

$$a_1 = 0,$$

$$\frac{1}{2}(m+1)\, a_{m+1} + q a_{m+1}\sum_{j=1}^{m} [j]_q = m a_m, \quad m \geq 1$$

$$a_{m+1} = \frac{m a_m}{\frac{1}{2}(m+1) + q\sum_{j=1}^{m} [j]_q}$$

for all $z \in \overline{\mathbb{M}_R} \setminus \{0\}$. Thus we get $a_m = 0, m = 1, 2, 3, ...$, which implies that f is constant, a contradiction with the hypothesis.

In conclusion, if f is not a constant function, then the approximation order cannot be better than $\frac{1}{[n]_q}$, which combined with Theorem 5.2.1, (i), implies that the approximation order is exactly $\frac{1}{[n]_q}$, which proves the theorem. ■

Remark 5.3.3 *By the obvious inequalities* $\frac{q-1}{q^n} \leq \frac{1}{[n]_q} \leq \frac{q}{q^n}$, *for all* $n \in \mathbb{N}$ *and* $q > 1$, *it follows* *that the approximation order in Theorem 5.2.1, (i) and in Theorem 5.3.2 is geometrical, namely* $\frac{1}{q^n}$ *with* $q > 1$.

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