Information Loss Problem in Linear Dilaton Black Holes

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ABSTRACT

Using the Damour-Ruffini-Sannan and the Parikh-Wilczek methods, we analyze the Hawking radiation of uncharged massive particles for linear dilaton black holes with $N \geq 4$ dimensions. Contrary to the many studies in the literature in which the original Parikh-Wilczek's method are used, our results show that the obtained emission spectrum is precisely thermal. This implies that sole back-reaction effects do not retrieve the information from the linear dilaton black holes. On the other hand, when we recalculate the emission probability by taking into account the log-area quantum correction to the black hole entropy, it is seen that the radiation deviates from its pure thermal behavior. Besides, the quantum corrections give rise also to the statistical correlation between quanta emitted. The latter results yield that the information can leak out of the linear dilaton black holes together with preserving unitarity in quantum mechanics. In addition to these, we extend our study to the case in which quantum gravity $\hbar$ corrections in all orders in are considered. The obtained modified entropy and temperature are adjusted so finely that the scenario of fading Hawking radiation, in which both entropy and temperature vanish with zero mass, becomes possible. Finally, we highlight that, even in the case of fading Hawking radiation, the linear dilaton black holes could evaporate completely with conserving the total entropy − "no information loss".

Keywords: Linear dilaton black holes, Hawking radiation, information loss paradox, entropy conservation, quantum corrections.

ÖZ

Damour-Ruffini-Sannan ve Parikh-Wilczek yöntemleri kullanarak, $N \geq 4$ boyutlu lineer dilaton kara delikler için yüksüz kütleli parçacıkların Hawking radyasyonunu analiz ettik. Orjinal Parikh-Wilczek yönteminin kullanıldığı literatürdeki pek çok çalışmanın aksine elde edilen sonuçlar, emisyon spektrumunun tam ısıl olduğunu göstermektedir. Bu ise tek başına geri−reaksiyon etkisinin lineer dilaton kara deliklerinden bilgi çıkaramayacağını işaret etmektedir. Diğer taraftan, emisyon olasılığını, kara deliğin entropisine log-alan kuantum düzeltmesini dikkate alarak yeniden hesapladığımız zaman, radyasyonun saf ısıl davranışında sapma olduğu görüldü. Bunun yanında kuantum düzeltmeleri, yayılan kuantalar (kuantum parçacıkları) arasında istatistiksel bir ilişkinin de oluşmasına neden olmuştur. Son sonuçlar bilginin, kuantum mekaniğindeki üniterliği koruyarak lineer dilaton kara deliklerinden sızacağını göstermektedir. Bunlara ek olarak, çalışmamızı kuantum düzeltmelerini $\hbar$'ın tüm derecelerini içerecek şekilde genelledik. Elde edilen değiştirilmiş entropi ve sıcaklığa, sıfır kütle ile biten entropi ve sıcaklığa sahip sönümlü Hawking ışımasını mümkün kılacak şekilde ince bir ayar yaptık. Son olarak, sönümlü Hawking ışıması durumunda dahi lineer dilaton kara deliklerinin toplam entropiyi koruyarak − bilgi kayıpsız − tamamen buharlaşabileceğini vurguladık.

Anahtar Kelimeler: Entropi, lineer dilaton kara delik, Hawking ışıması, kuantum düzeltmeleri.

To My Family

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INTRODUCTION

In 1972, J.D. Bekenstein suggested that a black hole (BH) should have well-defined entropy [1]. From the point of view of information theory, it is natural to introduce the concept of BH entropy as the measure of information about a BH interior, which is inaccessible to an exterior observer. The exact theoretical model for how a BH could emit black body radiation was worked out by S.W. Hawking [2,3]. Hawking proved that a stationary BH can emit particles with a temperature proportional to the surface gravity from its event horizon. He indicated that vacuum fluctuations near the horizon cause the generation of particle-antiparticle pairs. The idea is that out of nothing, pair of particles is created, and exist for a short time, until getting annihilated. This pair of particles likes the electron-positron pairs; one has positive energy while the other has negative energy. If this pair of particles bumped up against the BH, Hawking released that the positive particle would have just enough energy to escape the BH where it materializes as a real particle, but the particle with negative energy would fall in. The particle that goes inside the BH eventually decreases the mass of the BH. However, the particle that goes off to the distant observer is known as the Hawking radiation.

Since the spectrum of such a radiation is pure thermal, it is understood that Hawking radiation has inconsistency with quantum theory. If one considers the information as a pure quantum state in the BH, according to the Hawking radiation the pure states should be converted into mixed ones. This problem came to be a non-unitary quantum evolution, and it gives rise to the information loss paradox of BH physics. and it gives rise to the information loss paradox of BH physics. Among them we mainly focus on Damour-Ruffini-Sannan (DRS) [4,5] and Parikh-Wilczek (PW) [6] methods to compute the Hawking radiation. DRS method is applicable to any Hawking temperature (T_H) problem in which the asymptotic behaviors of the wave equation near the event horizon are known. In this thesis, however, we will pay special attention to the PW method which corrects the Hawking's pure thermal radiation, at least for many well-known BHs like Schwarzschild, de Sitter, Kerr, Reissner-Nordström etc. [6-9]. Thus it has a therapeutic effect on the information loss paradox. The power of the PW method stems from the idea which considers the Hawking radiation as a quantum tunneling process. Because of this, it is also called as PW's tunneling formalism. To this end, it considers the outgoing particles as subsequently emitted spherical shells such that each shell has a small mass, compared with the mass of the BH, corresponding to its energy. Once the BH radiates, each shell decreases the total mass of the BH as the amount of its (shell's) own mass. This phenomenon is known as the self-gravitational (or back-reaction) effect. Then, it uses the null geodesics of the outgoing quanta together with the WKB approximation. As we mentioned above, the result for many well-known BHs is astonishing: the Hawking radiation is not pure thermal anymore. So, it is supposed by many others that PW's method is a general recipe for the information loss paradox. But as it will be shown explicitly in the following sections, contrary to the general belief, the original PW's method does not solve the information loss problem appeared in the linear dilaton black holes (LDBHs). Essentially, this explains why researchers are still in search for alternative approaches [10], at least in the framework of PW's tunneling method, since the complete quantum gravity (QG) is unknown yet. Among those studies, the fascinating one belongs to Zhang et al. [11]. They have explicitly shown that the amount of information that formerly was perceived to be lost is found to be hidden in the correlations (mutual information [12]) of Hawking radiation, and by virtue of the associated correlations it can be leaked out of the BH. This process, irrespective of the microscopic picture of the BH collapse, resolves the paradox of BH information loss. In this regard, [11] can be considered as the first study which gives the whole scenario of resolving the information paradox for the Schwarzschild BH. For a more recent account in the same line of thought applied to different types of BHs, including the case of quantum horizon, one may consult [10,13], which also revisits the BH information loss paradox. Meanwhile, it is worth noting that for the BHs considered in [11,13] the information-carrying correlations among their Hawking radiation emerge without reference to QG effects.

The considered LDBHs in this thesis are the solutions to Einstein-Maxwell-Dilaton (EMD), Einstein-Yang-Mills-Dilaton (EYMD) and Einstein-Yang-Mills-Born-Infeld-Dilaton (EYMBID) theories [14]. In Ref. [14], the Hawking radiation of the LDBHs is analyzed with another method: semi-classical radiation spectrum method. Meanwhile, the eponyms of the LDBHs are Clement and Gal'tsov [15]. These BHs are non-asymptotically flat (NAF) spacetimes and their event horizon hides the null singularity at the center. We first apply the DRS method in order to find T_H of the LDBHs and the tunneling rate of the chargeless particles crossing the event horizon. The resulting temperature obtained from DRS method is in agreement with the statistical Hawking temperature [16]. Afterward, we compute the emission rate of outgoing quanta. Similarly, we use the PW's method to derive the tunneling rate of the LDBHs. The obtained tunneling rate which also yields the Bekenstein-Hawking entropy does not attribute non-thermal radiation. Namely, the original form of the PW's tunneling formalism is inadequate while attempting to retrieve the information from a LDBH. So, as stated before, the sole PW's tunneling method cannot be a general recipe for resolving the BH information loss paradox.

The aim of this thesis is to show that tunneling probability of the emitted particles from LDBHs deviates from the pure thermal emission if the QG corrections are taken into account. To this end, we shall use the idea of Chen and Shao [17] who have modified the scenario of [11] by including the QG effects and the remnant, which is a minimal mass that remains at the end of the complete BH evaporation. For the subject of the BH remnant, one may refer to [18]. We show that in the LDBH case the crucial role of the QG corrections in finding the correlations between two sequential emissions becomes more apparent when compared with the Schwarzschild case [17]. Namely, in order to preserve the entropy conservation in a system of radiating LDBH plus its remnant, in conform with the Bekenstein's entropy bound (BEB) [19,20], QG effects must certainly be considered. We also model the remnant as an extreme LDBH spacetime with a point like horizon. By using the massless wave equation, we show that such a spacetime cannot radiate, which implies that its temperature must be zero.

As a further step, we consider the general form of the quantum corrected temperature given by Singleton, Vagenas, Zhu and Ren (SVZR) [21,22], and apply it to the LDBHs in order to derive specific entropy and temperature which vanish while mass of the BH ends; $S.T(M \rightarrow 0) \rightarrow 0$. Detailed calculations of these processes are given, and as a result we obtain this particular radiation that can be named as fading Hawking radiation [23]. According to our literature knowledge, such a radiation has not been obtained before. The behaviors of both the entropy and temperature of the LDBH with the quantum correction parameters coming from String Theory (ST) and Loop Quantum Gravity (LQG) are examined. We find that the results which have no any physical ambiguity are possible only in the LQG case. Moreover, it is highlighted that higher order QG corrections which are in conform with the back reaction effects provide the correlations between the emitted quanta. Finally, we show that the LDBHs could evaporate away completely with the entropy conservation which leads to the fact that information is not lost.

The thesis is organized as follows: In chapter 2, we make a brief review of the LDBHs in EMD, EYMD and EYMBID theories. Next, we apply the DRS and PW methods to the LDBHs to obtain the tunneling or emission rate of the chargeless particles crossing over the event horizon. By virtue of the tunneling rate, we obtain the difference of the Bekenstein-Hawking entropies. The result is interpreted in respect of information theory. Chapter 3 is devoted to entropy conservation of the LBBHs and their remnant structure. In this chapter, QG corrected entropy is used, and its role on the information conservation is emphasized. In chapter 4, a particular radiation which we call as "fading Hawking radiation" is thoroughly discussed by considering the QG $\hbar$ corrections in all orders. We draw our conclusions in chapter 5.

Throughout the thesis, the units $G = c = k_B = 1$ are used. Furthermore in chapters 2-3 $L_p = 1$, and in chapter 4 it is used as $L_p^2 = \hbar$.

Chapter 2

HAWKING RADIATION IN VARIOUS THEORIES FOR LINEAR DILATON BLACK HOLES[1]

2.1 $4D$-LDBHs, Calculation of Their Hawking Temperature and Tunneling Rate

The metric of the $4D$-LDBHs, which are static spherically symmetric solutions in various theories (EMD, EYMD and EYMBID) [14], is

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + A^2 r d\Omega^2 \tag{1}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$, the metric function $f = \tilde{\Sigma} r\left(1 - \frac{r_+}{r}\right)$, r_+ is the radius of the event horizon, and $\tilde{\Sigma}$ and A are constants.

One should consider the quasi-local mass definition M [24] for our metric (1), since the present form of the metric represents NAF geometry. In [14], the relationship between the mass M and the horizon r_+ is given as follows

$$r_+ = \frac{4M}{\tilde{\Sigma}\, A^2} \tag{2}$$

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1 This Chapter is mainly quoted from Ref. [25], which is *Pasaoglu, H., Sakalli, I. (2009). International Journal of Theoretical Physics. 48, 3517-3525.*

The coefficients $\tilde{\Sigma}$ and A take different values according to the concerned theory (EMD, EYMD or EYMBID). In the EMD theory [14,26,27], the coefficients $\tilde{\Sigma}$ and A are found as

$$\tilde{\Sigma} \rightarrow \tilde{\Sigma}_{EMD} = \frac{1}{\gamma^2} \; and \; A \rightarrow A_{EMD} = \gamma \tag{3}$$

where γ is a constant correlated to the electric charge of a BH. When inserting $\gamma \equiv r_0$, one can see that metric (1) matches with the solution given by Clément et al. [27]. Afterwards, if we consider the EYMD and EYMBID theories [28,29], the coefficients in the line-element (1) become

$$\tilde{\Sigma} \rightarrow \tilde{\Sigma}_{EYMD} = \frac{1}{2Q^2} \; and \; A \rightarrow A_{EYMD} = \sqrt{2}\, Q \tag{4}$$

$$\tilde{\Sigma} \rightarrow \tilde{\Sigma}_{EYMBID} = \frac{1}{Q_C^2}[1 - \sqrt{1 - \frac{Q_C^2}{Q^2}} \; and \; A \rightarrow A_{EYMBID}$$

$$= \sqrt{2}\, Q \left(1 - \frac{Q_C^2}{Q^2}\right)^{\frac{1}{4}} \tag{5}$$

where Q and Q_C are YM charge and the critical value of YM charge, respectively. According to EYMBID theory, the existence of the metric (1) depends strictly on the condition [29]

$$Q^2 > Q_C^2 = \frac{1}{4\tilde{\beta}^2} \tag{6}$$

where $\tilde{\beta}$ is the Born-Infeld parameter. It is needless to say that the constant $\tilde{\Sigma}$ in Eqs. (3), (4) and (5) should take positive values. This ensures the metric signature of the metric (1) as well.

When the definition in [16] is used for surface gravity, we get

$$\kappa = \lim_{r \to r_+} \frac{f'(r)}{2} = \frac{\tilde{\Sigma}}{2} \tag{7}$$

Positive surface gravity (7) shows that its direction is towards the singularity and therefore it is attractive. In other words, the matter can only fall into the BH.

Considering Eq. (1), we can use the covariant Klein-Gordon (KG) equation in curved spacetime for a massive test scalar field ϕ with mass μ, which is given by

$$\frac{1}{\sqrt{-detg}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-detg}\, g^{\mu\nu} \frac{\partial}{\partial x^{\nu}} \phi \right) - \mu^2 \phi = 0 \tag{8}$$

and by making the separation of variables as $\phi = Y(\theta, \varphi)\psi(t, r)$ the radical equation can be written as

$$-\frac{\partial^2 \psi}{\partial t^2} + f\left(\frac{f}{r} + \tilde{\Sigma}\right)\frac{\partial \psi}{\partial r} + f^2 \frac{\partial^2 \psi}{\partial r^2} - f\left(\mu^2 - \frac{l(l+1)}{r}\right)\psi = 0 \tag{9}$$

where l specifies the angular quantum number. In order to transform (9) to a standard wave equation at the horizon, conventionally one introduces the tortoise coordinate, which is obtained from

$$dr_* = \frac{dr}{f} \tag{10}$$

After making the straightforward calculation, we find an appropriate r_* as

$$r_* = \frac{1}{2\kappa}\ln(r - r_+) \tag{11}$$

Thus, the radical Eq. (9) can be rewritten as

$$\frac{\partial^2\psi}{\partial t^2} - \frac{f}{r}\frac{\partial\psi}{\partial r_*} - \frac{\partial^2\psi}{\partial {r_*}^2} + f\left[\mu^2 - \frac{l(l+1)}{r}\right]\psi = 0 \tag{12}$$

so, when $r \to r_+$, i.e., $f \to 0$, the radical Eq. (12) can be reduced to the standard form of the wave equation:

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial {r_*}^2} = 0 \tag{13}$$

Above form of the wave equation shows that there are waves which propagate near the horizon. The solutions of Eq. (13) give us the ingoing and outgoing waves at the surface of the BH horizon as

$$\psi_{out} = e^{-i\omega(t - r_*)} \tag{14}$$

$$\psi_{in} = e^{-i\omega(t + r_*)} \tag{15}$$

The metric form (1) attains singularity at the horizon, so we transform it to a new coordinate system which is non-singular at r_+. For this purpose, we introduce the Eddington-Finkelstein coordinate; $v = t + r_*$. Thus, the line-element (1) of the LDBHs becomes

$$ds^2 = -f dv^2 + 2dvdr + A^2 r d\Omega^2 \tag{16}$$

This yields the solutions of ingoing and outgoing waves at the event horizon, r_+ as follows

$$\psi_{out} = e^{-i\omega v} e^{2i\omega r_*} \tag{17}$$

$$\psi_{in} = e^{-i\omega v} \tag{18}$$

where ψ_{in} is the ingoing wave solution, which is analytic at the horizon. On the other hand, ψ_{out} which represents the outgoing wave solution is logarithmically singular at the horizon. To see this, we can rewrite the outgoing wave solution (17) as

$$\psi_{out} = e^{-i\omega v}(r - r_+)^{\frac{i\omega}{\kappa}} \qquad (r > r_+) \tag{19}$$

ψ_{out} can be analytically continued from the outside of the hole into the inside hole by the lower complex r-plane.

$$(r - r_+) \to (r_+ - r)e^{-i\pi} \tag{20}$$

Thus, we define the outgoing wave inside the horizon as

$$\tilde{\psi}_{out} = e^{-i\omega v}(r_+ - r)^{\frac{i\omega}{\kappa}} e^{\frac{\omega\pi}{\kappa}} \qquad (r < r_+) \tag{21}$$

Following the DRS method proposed in [4,5], we see that the thermal spectrum of the scalar particles radiating from the BH is given by

$$N_\omega^2 = \frac{\Gamma}{1-\Gamma} = \frac{1}{e^{\frac{\omega}{T}} - 1} \tag{22}$$

where Γ denotes the relative scattering probability (or the emission, tunneling rate) at the event horizon as

$$\Gamma = \left|\frac{\psi_{out}}{\tilde{\psi}_{out}}\right|^2 = e^{\frac{-2\pi\omega}{\kappa}} \tag{23}$$

Whence we can read the resulting temperature in Eq. (22) as

$$T = \frac{\kappa}{2\pi} \tag{24}$$

which is nothing but the statistical T_H. In Ref. [16], its computation is given by

$$T_H = \frac{f'(r_+)}{4\pi} \tag{25}$$

In brief, the DRS method is in agreement with the Hawking's original study [1,2].

2.2 Entropy of the LDBH

Another method to calculate the tunneling rate of the BH was developed by PW [6]. In the PW's study, the relationship between the entropy and the tunneling rate with the aid of the WKB approximation is laid bare. In short, this section is devoted to the application of the PW's method for the LDBHs.

In the seminal work [6], PW described the Hawking radiation as a tunneling process and used the WKB method. Their study is mainly based on the subjects of energy conservation and the self-gravitation effect. They also showed how the tunneling rate is exponentially related to the imaginary part of the particle action at stationary phase. In the PW model, it is described that an outgoing particle with positive energy ω which crosses the horizon outwards from initial radius of the horizon r_{in} to the final radius r_{out} has an imaginary part of the amplitude that is expressed in the WKB approximation as,

$$Im\,(I) = Im \int_{r_{in}}^{r_{out}} p_r dr = Im \int_{r_{in}}^{r_{out}} \int_{0}^{p_r} d\tilde{p}_r dr$$

$$= Im \int_{r_{in}}^{r_{out}} \int_{M}^{M-\omega} \frac{dr}{\dot{r}}\, dH \quad (26)$$

where p_r and H are momentum and Hamiltonian, respectively. Expression (26) is related to the emission rate of the tunneling particle by [30,31]

$$\Gamma \sim e^{-2Im(I)} \quad (27)$$

Remark: The above result is the consequence of the PW's method, which considers each emitted particle (with an energy ω) as a shell, fixes the total mass M, however it allows the hole mass to fluctuate. Thus, when the LDBH emits a particle, the horizon moves inwards and the mass of the BH changes from M to $M-\omega$. The Hamilton's equation of motion is in general written as $d\tilde{p}_r = \frac{dH}{\dot{r}}$. Introducing the total energy of the BH as $H = M - \omega$, i.e., $dH = -d\omega$, and substituting the value of $\dot{r}$, which is obtained from the null geodesic equation of the metric (16)

$$\dot{r} \equiv \frac{dr}{dv} = \frac{f}{2} \tag{28}$$

into Eq. (26), we obtain,

$$Im\,(I) = Im \int_0^{\omega} \int_{r_{in}}^{r_{out}} \frac{2dr}{\tilde{\Sigma}(r-r_+)}(-d\tilde{\omega}) \tag{29}$$

One can evaluate the r-integral by deforming the contour, where its semicircle centered at real axis pole r_+ . Thus we get

$$Im\,I = 2\pi \int_0^{\omega} \frac{d\omega'}{\tilde{\Sigma}} \tag{30}$$

So, the tunneling rate (27) becomes

$$\Gamma \sim exp(-4\pi \int_0^{\omega} \frac{d\omega'}{\tilde{\Sigma}}) = exp(\Delta S_{BH}) \tag{31}$$

where ΔS_{BH} represents the difference in Bekenstein-Hawking entropies of the LDBHs ($S_{BH} = \frac{A_h}{4} = \pi A^2 r_+$) before and after the emission of the particle. Namely,

$$\Delta S_{BH} = S(M-\omega) - S(M) = -\frac{2\pi\omega}{\kappa} \tag{32}$$

However, the foregoing result is not consistent with the results of the other works, see for instance [6-9,32-34]. Because Eq. (32) shows that the radiation spectrum still preserves its thermal character! In other words, the thermal spectrum does not suggest the underlying unitary theory of quantum mechanics, and therefore we understand that the conservation of information is violated. So, one should improve the original PW's method in order to satisfy the information conservation for the LDBHs as well. The next chapter will be based on this issue.

Chapter 3

ENTROPY CONSERVATION OF LDBHs IN QG CORRECTED HAWKING RADIATION[2]

3.1 4D-LDBHs' Tunneling Rate With QG Corrections

In chapter 2, we have discussed the $4D$-LDBHs entropy without adding the QG effects. In this section we will work on the same metric (1) by applying the QG corrections. For this purpose, we will mainly focus on the study of Chen and Shao [17], apply the steps given there to the LDBHs. To this end, we start with a minor modification on the typeface of the metric (1) as follows

$$ds^2 = -f(r)dt_L^2 + \frac{dr^2}{f(r)} + R^2 d\Omega_2^2 \tag{33}$$

where t_L denotes the LDBH time, $R^2 = A^2 r$ and $d\Omega_2^2 = d\theta^2 + sin^2\theta d\phi^2$. The metric function $f(r) = \tilde{\Sigma} r\left(1 - \frac{r_+}{r}\right)$ is already introduced in the previous chapter.

The curvature of metric (33) has coordinate singularities at the horizon, so in order to remove it non-singular at r_+, we pass to Painlevé-Gullstrand (PG) type coordinates with

$$dt = dt_L + \frac{\sqrt{1 - f(r)}}{f(r)} dr \tag{34}$$

Thus, the line element (33) transforms to

[2] This Chapter is mainly quoted from Ref. [35], which is *Sakalli, I., Halilsoy, M., Pasaoglu, H., (2011). International Journal of Theoretical Physics. 50, 3212-3224.*

$$ds^2 = -f(r)dt^2 + 2\sqrt{1-f(r)}dtdr + dr^2 + R^2 d\Omega^2 \tag{35}$$

The above metric has a number of advantages suitable for our present purpose. It is well known from the Schwarzschild case that the time t in the PG coordinates is linearly related to the proper time for a radially falling observer, [36].

Considering the test particle as a massless spherical shell, the radial null geodesics has a rather simple form as

$$\dot{r} = \frac{dr}{dt} = -\sqrt{1-f(r)} \pm 1 \tag{36}$$

where the choice of signs in equation (36) depends whether the rays are outgoing $(+)$ or ingoing $(-)$. In the PG coordinates, the strength of the gravitational field near a BH surface, which is known as the surface gravity, is one of the Christoffel components:

$$\kappa = \Gamma_{00}^{0} = \frac{1}{2} f'(r_+) \tag{37}$$

which becomes $\kappa = \frac{\tilde{\Sigma}}{2} = \frac{2M}{A^2 r_+}$ for the $4D$ LDBHs. The metric function *f(r)* is zero at the horizon, so we can expand it as

$$f(r) = f'(r_+)(r - r_+) + f''(r_+)O(r - r_+)^2 \tag{38}$$

As a result of vanishing term $O(r - \mathrm{r}_+)^2$ in (38), the radial outgoing null geodesic takes the following form

$$\dot{r} = \frac{dr}{dt} = \frac{1}{2} f'(r_+)(r - r_+) \tag{39}$$

Combining Eqs. (37) and (39), we obtain

$$\dot{r} = (r - r_+)\kappa = \frac{2}{A^2}\left(\frac{r}{r_+} - 1\right)M \tag{40}$$

When we assume that the whole system with fixed total mass M consists of a LDBH and a spherical shell of mass ω, which is emitted by the BH, Eq. (40) modifies to

$$\dot{r} = (r - r_+)\kappa = \frac{2}{A^2}\left(\frac{r}{r_+} - 1\right)(M - \omega) \tag{41}$$

where $(M - \omega)$ is the varying mass of the LDBH with $\omega \ll M$. This event is known as self-gravitational effect [30,31]. Following the PW's method [6] which was thoroughly employed in chapter 2, one can refer to Eq. (26) to obtain imaginary part of the particle's action. One gets the result as

$$Im\,(I) = -Im\int_{r_{in}}^{r_{out}}\int_0^{\omega} \frac{A^2 d\tilde{\omega}}{2\left(\frac{r}{r_+} - 1\right)(M - \tilde{\omega})} dr \tag{42}$$

After evaluating the r-integral by deforming a contour, where its semicircle is centered at the real axis pole r_+, we get

$$Im\,(I) = -\pi \int_0^{\omega} \frac{A^2 r_+ d\tilde{\omega}}{2(M - \tilde{\omega})} \tag{43}$$

The reason of the sign change in (43) is because of the shrinking of the horizon during the process of Hawking radiation i.e., the horizon tunnels inwards so, $r_{out} < r_{in}$.

The quantum surface gravity [37,38] of the LDBHs can be defined as,

$$\kappa_{QG} = \frac{2(M - \omega)}{A^2 r_+} \tag{44}$$

Therefore Eq. (43) turns out to be

$$Im\,(I) = -\pi \int_0^{\omega} \frac{d\tilde{\omega}}{\kappa_{QG}} \tag{45}$$

which changes the Hawking temperature as

$$T_H = \frac{\kappa_{QG}}{2\pi} \tag{46}$$

Accordingly, we can rewrite expression (45) as

$$Im\,(I) = -\frac{1}{2}\int_0^{\omega}\frac{d\widetilde{\omega}}{T_H} \;= -\frac{1}{2}\int_{S_{QG(M)}}^{S_{QG(M-\omega)}} dS$$

$$= -\frac{1}{2}\left[S_{QG(M-\omega)} - S_{QG(M)}\right] = -\frac{1}{2}\Delta S_{QG} \tag{47}$$

where S_{QG} is the QG corrected area entropy for the LDBH. In ST and LQG, the general definition of the S_{QG} is introduced with a logarithmic correction [39-42]

$$S_{QG} = \frac{A_h}{4} + \alpha ln A_h + O\left(\frac{1}{A_h}\right) \tag{48}$$

where α is the QG correction parameter, and it is a dimensionless constant. It takes different values according to the concerned theory. Since $A_h = 4\pi A^2 r_+ = \frac{16\pi M}{\tilde{\tilde{\Sigma}}}$, one can easily read the tunneling rate with QG corrections as

$$\Gamma \sim exp[-2Im(I)] = exp\left(\Delta S_{QG}\right) = \left(\frac{M-\omega}{M}\right)^{\alpha} exp\left(-4\pi\frac{\omega}{\tilde{\tilde{\Sigma}}}\right) \tag{49}$$

The additional term $\left(\frac{M-\omega}{M}\right)^{\alpha}$ in (49) compared with (31) comes from QG effects on the mass and the energy of the emitted particles of the LDBH. In chapter 2, the tunneling rate without QG corrections ($\alpha = 0$), brought us a contradiction with quantum mechanics. Its corresponding emission spectrum was not deviating from the pure thermal emission [25]. So while a LDBH radiates, keeping $\alpha \neq 0$ must be a precondition for unitarity in quantum mechanics as well as for the resolution of the information loss paradox. Meanwhile, it is observed that the value of the coefficient α depends on the power of r_+ in A_h. In the LDBHs its value reveals itself as 1 because of $A_h = 4\pi r_+$, while for the Schwarzschild BH [17] its value is 2 since its corresponding horizon area is $A_h = 4\pi r_+^2$.

According to the scenario of a radiating BH, which is employed by Chen and Shao [17], we assume that the quasilocal mass of a LDBH unites masses (energies) of n-particles $\omega_1, \omega_2, \dots, \omega_n$ together with a non-vanishing BH remnant (ω_c). Therefore, $M = \sum_{i=1}^{n} \omega_i + \omega_c$. The complete evaporation process corresponds to successively emitted quanta ($\omega_1, \omega_2, \dots, \omega_n$) from the BH. So the LDBH loses its mass M during its evaporation, such that at the final state one will only see its remnant; $M - \omega \to \omega_c$. We must emphasize that the existence of the BH remnant is crucial in the QG corrected emission rate (49). Because since LQG envisages a negative value for α [42], the case $M - \omega \to 0$ i.e., non-suppression of the BH emission, yields a diverging emission rate. One also quests for the case $\omega > M$, however, this is not allowed since our primary assumption is $M = \sum_{i=1}^{n} \omega_i + \omega_c$. Furthermore, such a case brings us an unphysical imaginary value (depending on the value of α) for the emission rate (49), which means that tunneling process does not occur. In short, the case $\omega > M$ should be excluded.

3.2 Statistical Correlation (Mutual Information) Between Two Successive Emissions

In this section we will discuss whether the emission probabilities of two successive emissions are statistically correlated [11,13,43] or not. Statistical correlation is a subject of the statistical physics which gives us a kind of quantitative measurement of how much a happened event tells about the probability of occurrence of another event. With the aid of the statistical correlation, we want to show that one can get a measurement of how much a successive emission tells about another successive emission. The statistical correlation is also known as mutual information or transinformation [44]. The existence of mutual information indicates that the information leaks out of a BH during its radiation. As time goes on, it will reduce the total information stored in a BH. So when the BH reaches the late stages of its evaporation, there would be enough space for the rest of information to be stored in the remnant. In brief, the existence of the mutual information gives support to the BEB [19,20].

As we stated before, in this section we will follow the method used in [17]. When we consider two successive emissions with energies ω_1 and ω_2, for the first emission of energy ω_1 from a LDBH mass M, the tunneling rate (49) becomes,

$$\Gamma(\omega_1) = \left(\frac{M-\omega_1}{M}\right)^{\alpha} exp\left(-4\pi\frac{\omega_1}{\tilde{\Sigma}}\right) \tag{50}$$

The conditional probability of a second emission with energy ω_2 after the first emission ω_1 becomes

$$\Gamma(\omega_2|\omega_1) = \left(\frac{M-\omega_1-\omega_2}{M-\omega_1}\right)^{\alpha} exp\left(-4\pi\frac{\omega_2}{\tilde{\Sigma}}\right) \tag{51}$$

On the other hand, direct condition on the second emission yields

$$\Gamma(\omega_2) = \left(\frac{M-\omega_2}{M}\right)^{\alpha} exp\left(-4\pi\frac{\omega_2}{\tilde{\Sigma}}\right) \tag{52}$$

which is the probability just for the second emission. The emission of the total energy is

$$\begin{aligned}\Gamma(\omega_1+\omega_2) &= \left(\frac{M-\omega_1-\omega_2}{M}\right)^{\alpha} exp\left(-4\pi\frac{\omega_1+\omega_2}{\tilde{\Sigma}}\right)\\ &= \Gamma(\omega_1)\Gamma(\omega_2|\omega_1)\end{aligned} \tag{53}$$

The statistical correlation [11] between two successive emissions is given by

$$\chi(\omega_1 + \omega_2; \omega_1, \omega_2) = ln\left[\frac{\Gamma(\omega_1 + \omega_2)}{\Gamma(\omega_1)\Gamma(\omega_2)}\right] \tag{54}$$

which is

$$\chi(\omega_1 + \omega_2; \omega_1, \omega_2) = \alpha ln\left(1 - \frac{\omega_1\omega_2}{(M - \omega_1)(M - \omega_2)}\right) \tag{55}$$

First of all, our result (55) shows that the subsequent emissions are statistically dependent, and thus correlations must exist between them. As explicitly shown in [11,13], the statistical correlation is equal to the mutual information between two sequential emissions. The reason of this equality comes from the fact that the mutual information is used in statistics as a measure of the information shared by two random variables [43]. If such two variables are designated by ω_1 and ω_2, then the mutual information is defined by

$$\begin{aligned} I(\omega_1 : \omega_2) &= S(\omega_1) + S(\omega_2) - S(\omega_{1,}\omega_2) \\ &= S(\omega_1) - S(\omega_1|\omega_2) = S(\omega_2) - S(\omega_2|\omega_1) \end{aligned} \tag{56}$$

where $S(\omega_1)$ and $S(\omega_2)$ are the entropies of ω_1 and ω_2, respectively. $S(\omega_1, \omega_2)$ is known as total (joint) entropy of ω_1 and ω_2. Besides, $S(\omega_1|\omega_2)$ is known as the conditional entropy of ω_1 and similarly $S(\omega_2|\omega_1)$ denotes the conditional entropy of ω_2. Conditional entropy describes the uncertainty in the specified event that remains after the other event is known. In terms of the mutual information, the conditional entropies of ω_1 and ω_2 tell us that a certain information needs to be transferred from ω_1 in order to determine ω_2 and vice versa. If we consider the event as an emission process of a particle with an emission rate $\Gamma(\omega)$, the uncertainty of the event (entropy) is found by $S(\omega) = -ln\Gamma(\omega)$ [11]. So, the conditional entropy; $S(\omega_1|\omega_2) = -ln\Gamma(\omega_1|\omega_2)$. After substituting those entropies in (56), one can easily see that the mutual information (56) exactly matches with the statistical correlation (54).

Remarkably, the most important point in (55) is that the obtained mutual information strictly depends on α. In the Schwarzschild BH [17], even in the case of $\alpha = 0$, the mutual information is non-zero. But, here once $\alpha = 0$ is set, the subsequent emissions become statistically independent, and thus information does not come out with the Hawking radiation. This result emphasizes the necessity of QG effects in the calculation of mutual information while the LDBHs radiate.

3.3 Entropy Conservation and BH Remnant

For the calculation of total entropy carried by Hawking radiation, one should consider the complete process of the BH evaporation. For this purpose, we use the emission of all particles with energies $\omega_1, \omega_2, \dots, \omega_n$, which are successively emitted from the LDBH. At the end of the evaporation, we should see only the BH remnant having energy ω_c such that $\omega_c = M - \sum_{i=1}^{n} \omega_i$.

In [11,13], it is shown that the chain rule of conditional entropies in quantum information theory [43] yields the total entropy S_R carried out by radiation

$$S_R = \sum_{i=1}^{n} S(\omega_i | \omega_1, \omega_2, \dots, \omega_{i-1})$$

$$= -\ln \left[\prod_{i=1}^{n} \Gamma\,(\omega_i | \omega_1, \omega_2, \dots, \omega_{i-1}) \right] \tag{57}$$

This expression states that the emitted particles extract entropies (or information) from the BH. Namely, the conditional entropies, part by part, transfer the entropy of the BH to S_R. Upon using the foregoing formula with Eq. (49), one finds

$$S_R = -ln\left\{\left(\frac{\omega_c}{M}\right)^{\alpha} exp\left[-\frac{4\pi}{\tilde{\Sigma}}(M-\omega_c)\right]\right\}$$

$$= \frac{4\pi}{\tilde{\Sigma}}M + \alpha\, ln\left(\frac{M}{\omega_c}\right) - \frac{4\pi}{\tilde{\Sigma}}\omega_c \qquad (58)$$

It is instructive to remark that if we require a physical result (avoiding divergence of S_R) with QG effects ($\alpha \neq 0$), the existence of remnant (ω_c) is of vital importance.

The common sense about the remnants is that they should have a Planck size length with zero temperature. Remnant formation is in accordance with the generalized uncertainty principle (GUP), which might cease the complete evaporation of the BH [45-47], and also with spacetime noncommutativity [48]. Beside these, thinking of the remnant as a non-radiate object having an infinitesimal surface area would not be absurd. From this point of view, in the next section we shall model the remnant as an extreme LDBH with a point-like horizon. It will be shown that such a BH cannot radiate and its temperature would vanish much like an extremal BH.

The entropy of the remnant S_c can be read from Eq. (58). To this end, Eq. (58) is rewritten as

$$S_R = \frac{4\pi}{\tilde{\Sigma}}M + \alpha\, ln\left(\frac{16\pi M}{\tilde{\Sigma}}\right) - \left[\alpha\, ln\left(\frac{16\pi\omega_c}{\tilde{\Sigma}}\right) + \frac{4\pi}{\tilde{\Sigma}}\omega_c\,\right]$$

$$= \left(\frac{A_h}{4} + \alpha ln A_h\right) - \left[\,\alpha \ln\left(\frac{16\pi\omega_c}{\tilde{\Sigma}}\right) + \frac{4\pi}{\tilde{\Sigma}}\omega_c\right]$$

$$= S_{QG} - S_c \qquad (59)$$

From here, one can easily see that the remnant's entropy S_c is

$$S_c = \alpha \ln\left(\frac{16\pi\omega_c}{\tilde{\Sigma}}\right) + \frac{4\pi}{\tilde{\Sigma}}\omega_c \tag{60}$$

In fact, Eq. (59) represents the conservation of entropy. Clearly, the total entropy of a radiating LDBH S_{QG} is equal to the entropy of its remnant S_c plus the entropy carried out by radiation S_R. Being in conform with [11,13,17], this interpretation implies that the information is not lost, and unitarity in quantum mechanics is restored during the Hawking radiation of the LDBH. Nevertheless, for a deeper analysis of the problem, we should emphasize that a complete QG theory is needed.

3.4 QG Corrected Entropy of the Remnant in Higher Dimensional LDBHs

The generic line element for higher dimensional ($N \geq 4$) static, spherically symmetric LDBHs in various theories can be found in [14]. In higher dimensions, the metric function $f(r)$ of the LDBHs and the spherical line-element of the metric (1) modify to

$$f(r) = \tilde{\Sigma} r\left[1 - \left(\frac{r_+}{r}\right)^{\frac{N-2}{2}}\right], \qquad d\Omega_{N-2}^2 = d\theta_1^2 + \sum_{i=2}^{N-3}\prod_{j=1}^{i-1} sin^2\theta_j d\theta_i^2 \tag{61}$$

where $0 \leq \theta_k \leq \pi$ with $k = 1..N-3$, and $0 \leq \theta_{N-2} \leq 2\pi$. The modified form of the physical constant $\tilde{\Sigma}$ in higher dimensions can also be seen in [14].

In this section, we aim to find the quantum corrected entropy of the remnant of the LDBHs in higher dimensions. The surface area of a higher dimensional LDBH is [32]

$$A_h = \frac{16\pi^{\frac{N-1}{2}}}{(N-2)\Gamma(\frac{N-1}{2})}\frac{M}{\tilde{\Sigma}} \tag{62}$$

By following the procedure given in section (3.1), one can find the dimensionful and QG corrected entropy S_{NQG}, and tunneling rate Γ_N of the higher dimensional LDBHs as

$$S_{NQG} = \frac{4\pi^{\frac{N-1}{2}}}{(N-2)\Gamma(\frac{N-1}{2})}\frac{M}{\tilde{\Sigma}} + \alpha \ln\left[\frac{16\pi^{\frac{N-1}{2}}}{(N-2)\Gamma\left(\frac{N-1}{2}\right)}\frac{M}{\tilde{\Sigma}}\right] \tag{63}$$

and

$$\Gamma_N \sim e^{-2Im(I)} = e^{\Delta S_{NQG}} = (1-\frac{\omega}{M})^{\alpha}\, exp\left[-\frac{4\pi^{\frac{N-1}{2}}}{(N-2)\Gamma\left(\frac{N-1}{2}\right)}\frac{\omega}{\tilde{\Sigma}}\right] \tag{64}$$

We notice that, higher dimensions do not change the statistical correlation computed for the two successive emissions. That is the correlation remains unchanged as in the $4D$ case (see Eq. (55)). If we proceed to extend the study of emission of n-particles with energies $\omega_1, \omega_2, \dots, \omega_n$, which are successively emitted from the higher dimensional LDBHs, a straightforward calculation leads us to obtain the dimensionful entropy carried out by radiation S_{NR} as

$$S_{NQG} = \frac{4\pi^{\frac{N-1}{2}}}{(N-2)\Gamma(\frac{N-1}{2})}\frac{M}{\tilde{\Sigma}} + \alpha \ln\left(\frac{M}{\omega_c}\right) - \frac{4\pi^{\frac{N-1}{2}}}{(N-2)\Gamma(\frac{N-1}{2})}\omega_c \tag{65}$$

This can be rearranged in the form

$$S_{NR} = S_{NQG} - S_{Nc} \tag{66}$$

where the dimensionful entropy of the remnant S_{Nc} is found to be

$$S_{Nc} = \alpha \ln\left[\frac{16\pi^{\frac{N-1}{2}}}{(N-2)\Gamma\left(\frac{N-1}{2}\right)}\frac{\omega_c}{\tilde{\Sigma}}\right] + \frac{4\pi^{\frac{N-1}{2}}}{(N-2)\Gamma(\frac{N-1}{2})}\omega_c \tag{67}$$

Eq. (66) is nothing but the conservation of entropy in the higher dimensional LDBHs. Thus, we conclude that even in the higher dimensional LDBHs information is not lost and unitarity in quantum mechanics remains intact.

Finally, as we stated in the previous section, we would like to model the remnant as an extreme LDBH with a point-like horizon. Our goal is to show that such a remnant cannot radiate and thus yields zero temperature, as expected.

In generic, we can use the metric functions (61) to describe the remnant in an arbitrary dimension. Thus, the metric of the remnant can be approximated by an extreme LDBH metric as

$$ds^2 = -\tilde{\Sigma} r dt^2 + \frac{dr^2}{\tilde{\Sigma} r} + R^2 d\Omega_{\mathrm{N}-2}^2 \tag{68}$$

One can find the statistical Hawking temperature of this metric as a finite temperature with $T_H = \frac{(N-2)}{8\pi}\tilde{\Sigma}$. But this result is not persuasive since we expect its temperature as zero. To this end, we proceed with a more precise computation of the temperature of the remnant from the study of wave scattering in such a spacetime. Metric (68) can be transformed into the vacuum metric [49]

$$ds^2 = \rho^2(-d\tau^2 + dx^2 + d\Omega_{N-2}^2) \tag{69}$$

by the transformation

$$r = e^{\beta x}, \quad t = \left(\frac{\beta}{\tilde{\Sigma}}\right)\tau, \qquad \rho = Ae^{\frac{\beta}{2}x} \tag{70}$$

where constant $\beta = A\sqrt{\tilde{\Sigma}}$ (for example, in $4D$-EMD theory $\beta = 1$ [14,15, 27]). Therefore metric (69) is conformal to the product of $M_2 \times S_{N-2}$ of a two-dimensional Minkowski spacetime with the $(N-2)$-sphere. The massless Klein-Gordon equation

$$\nabla^2 \Phi = 0 \tag{71}$$

with $\Phi = \rho^{-(\frac{N-2}{2})}\Psi$ can be reduced to

$$\rho^{-(\frac{N-2}{2})}\left\{\partial_{\tau\tau} - \partial_{xx} + \left[\frac{\beta(N-2)}{4}\right]^2 - \nabla_{N-2}^2\right\}\Psi = 0 \tag{72}$$

where ∇_{N-2}^2 is the $(N-2)$-dimensional Laplace-Beltrami operator with the eigenvalue $-l(l+N-3)$ [50]. The reduced Klein-Gordon equation can be rewritten in spherical harmonics with orbital quantum number l as

$$\nabla_2^2 \Psi_l + \mu^2 \Psi_l = 0 \tag{73}$$

where the effective mass μ can be found as

$$\mu = \left\{[\frac{\beta(N-2)}{4}]^2 + l(l+N-3)\right\}^{\frac{1}{2}} \tag{74}$$

In Eq. (73), ∇_2^2 is the d'Alembertian operator on M_2. Thus the problem of wave propagation of the remnant (69) reduces to the propagation of eigenmodes of a free Klein-Gordon field in two dimensions with effective mass μ. It is needless to say that this vacuum state of the LDBH admits quantum states, which can accommodate any amount of information as a remnant metric. However, according to the BEB [19,20] a remnant may not accommodate the huge information of the initial BH. The information capacity of a remnant has recently been discussed in [51] (and references therein), which certainly supports our scenario. In effect, the remnant cannot radiate and therefore contrary to the standard Hawking temperature calculation i.e., $T_N = \frac{(N-2)}{8\pi}\tilde{\Sigma}$, its temperature must vanish i.e., the model that we created for the remnant seems plausible.

Chapter 4

FADING HAWKING RADIATION[3]

4.1 Entropy and Temperature Expressions with QG Corrections to All Orders in $\hbar$ for LDBHs

In this chapter, before proceeding to the technical details, we first modify the unit of the Planck constant as $L_p^2 = \hbar$. Recall that it was scaled to one in the previous chapters. Thus, if one makes some elementary dimensional analysis, it can be seen that the units of M and A^2 in Eq.(2) become L_p, while $\tilde{\Sigma}$ has the unit of L_p^{-1} so that r_+ has the unit of L_p.

Recently, it has been shown that the temperature for a static and spherically symmetric BH with $\hbar$ corrections in all orders [21, 52] has the following form

$$T = \frac{\hbar\kappa}{2\pi}\left(1 + \sum_{j=1}^{\infty} \frac{\alpha_j \hbar^j}{r_+^{2j}}\right)^{-1} \tag{75}$$

where α_j's $-$ dimensionless constants – stand for the QG correction terms. In this expression $\frac{\hbar\kappa}{2\pi}$ is nothing but the well-known Hawking temperature, T_H. Here, we wish to highlight one of the important features of the LDBHs that their Hawking

[3] This Chapter is mainly quoted from Ref. [23], which is *Sakalli, I., Halilsoy, M., Pasaoglu, H., (2012). Astrophysics and Space Science. DOI: 10.1007/s10509-012-1028-3*

temperature, $T_H = \frac{\hbar\tilde{\Sigma}}{2\pi}$, is independent of their quasilocal mass M, and which is therefore a constant throughout the evaporation process i.e. an isothermal process.

In general, the first law of thermodynamics is about an expression for the entropy (S) as

$$S = \int \frac{dM}{T} \tag{76}$$

As we adopt the temperature with generic QG corrections from Eq. (75), the entropy with $\hbar$ corrections in all orders can be found by substituting Eq. (75) into Eq. (76), and by evaluating the integral. Thus, for the LDBHs one obtains the following modified entropy as a function of M

$$S(M) = \frac{M}{T_H}\left(1 - \sum_{j=1}^{\infty} \frac{\alpha_j}{2j-1} x^j\right) \tag{77}$$

where $x = \frac{\hbar\tilde{\Sigma}^2 \mathrm{A}^4}{16M^2}$ is a dimensionless quantity.

As we mentioned in the introduction, our ultimate aim is to find a specific condition by which it leads to a complete radiation of the LDBH with $S, T(M \to 0) \ \to 0$. This requirement implies conditions on the α_j's. It is remarkable to see that the only possibility which satisfies $S, T(M \to 0) \ \to 0$ is,

$$\alpha_j = \frac{(-1)^{j+1}(2j-1)}{j}\alpha_1 \tag{78}$$

Inserting this into the sum of (77), we find the modified LDBH entropy as

$$S(M) = \frac{M}{T_H}\left[1 + \alpha_1 ln\left(\frac{16M^2}{16M^2 + \hbar\tilde{\Sigma}^2 A^4}\right)\right] \tag{79}$$

Now, it can be easily checked that $S(M \to 0) \to 0$ and $S(M \to \infty) \to \infty$. Although the result of the sum in Eq. (79) stipulates that $M > \frac{\sqrt{\hbar\tilde{\Sigma}}A^2}{4}$, by making an analytical extension of the zeta function [21,53], one can redefine the sum via $\alpha_1 ln\left(\frac{16M^2}{16M^2+\hbar\tilde{\Sigma}^2 A^4}\right)$ such that it becomes valid also for $M < \frac{\sqrt{\hbar\tilde{\Sigma}}A^2}{4}$.

We plot $S(M)$ (79) versus M for the cases of semi-classical and QG $\hbar$ corrections in all orders, and display all graphs in Fig. 1. In all figures, we have used two different α_1 values such that $\alpha_1 = -\frac{1}{2}$ is taken as the representative of the LQG [42], while the choice $\alpha_1 = \frac{1}{2}$ stands for the ST [39,40]. Here, physically inadmissible case belongs to the ST's one in which the behavior of the entropy is not well-defined. Because, as seen in Fig. 1(b), just before the complete evaporation of the LDBH, the entropy first decreases to a negative value and then increases from below to become zero with $M = 0$.

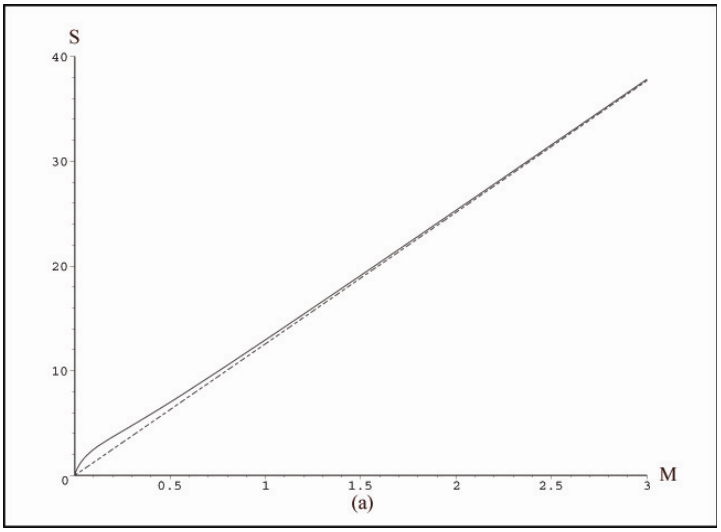

Figure 1(a). Entropy $S(M)$ in LQG. The relation is governed by (79). Here, $\alpha_1 = -\frac{1}{2}$. The two curves correspond to the semi-classical entropy (dashed line) and entropy with QG $\hbar$ corrections in all orders (solid line).

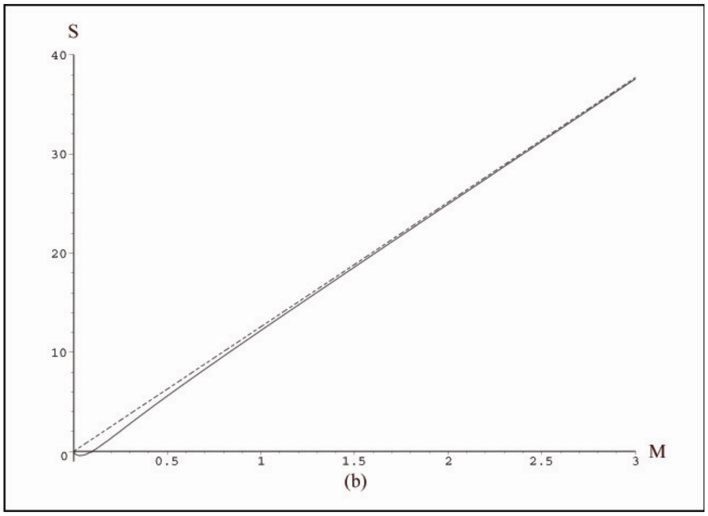

Figure 1(b). Entropy $S(M)$ in ST. The relation is governed by (79). Here, $\alpha_1 = \frac{1}{2}$. The two curves correspond to the semi-classical entropy (dashed line) and entropy with QG $\hbar$ corrections in all orders (solid line).

Furthermore, if we impose the same condition (78) in Eq. (75), a straightforward calculation of the sum shows that the temperature reads,

$$T(M) = \frac{T_H}{1 + \alpha_1 \left[\frac{2\hbar\tilde{\tilde{\Sigma}}^2 \mathrm{A}^4}{16M^2 + \hbar\tilde{\tilde{\Sigma}}^2 \mathrm{A}^4} + \ln\left(\frac{16M^2}{16M^2 + \hbar\tilde{\tilde{\Sigma}}^2 \mathrm{A}^4}\right)\right]} \tag{80}$$

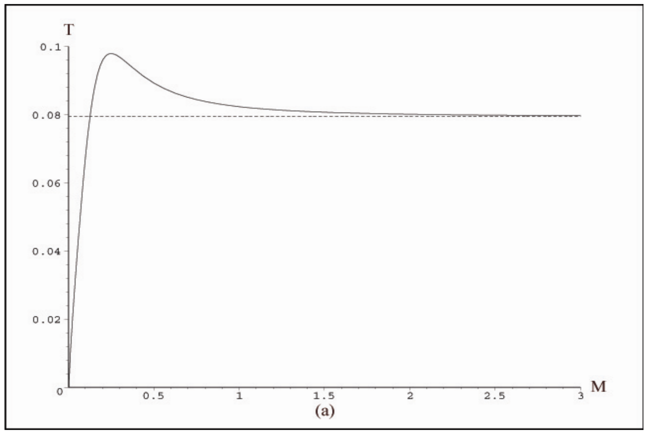

Figure 2(a). Temperature $T(M)$ in LQG. The relation is governed by (80). Here, $\alpha_1 = -\frac{1}{2}$. The two curves correspond to the semi-classical temperature (dashed line) and temperature with QG $\hbar$ corrections in all orders (solid line).

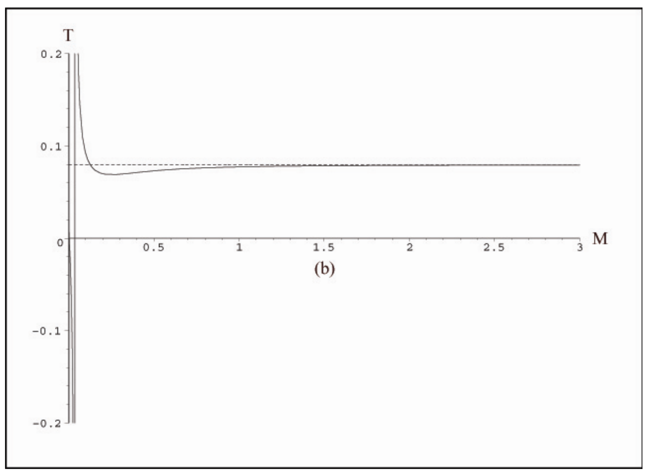

Figure 2(b). Temperature $T(M)$ in ST. The relation is governed by (87). Fig 2(b) stand for $\alpha_1 = \frac{1}{2}$. The two curves correspond to the semi-classical temperature (dashed line) and temperature with QG $\hbar$ corrections in all orders (solid line).

It is obvious that removing the QG corrections i.e., $\alpha_1 = 0$, leads T to the semi-classical result, T_H. Significantly, one can easily verify that $T(M \to 0) \to 0$ and $T(M \to \infty) \to T_H$. As it can be seen in Fig. 2(a), when $\alpha_1 < 0$ (the LQG case), the temperature does not take negative value, rather it remains always positive and goes to zero with $M \to 0$. On the other hand, for $\alpha_1 > 0$ (the ST case, see Fig. 2(b)), the temperature does not exhibit well-behaved behavior as obtained in the LQG case. Because it first diverges for some finite value of M, then becomes negative and approaches zero from below.

As a final remark for this section, our results suggest that the quantum corrected Hawking radiation of the LDBH should be considered with the LQG term $\alpha_1 < 0$ in order to avoid from any unphysical thermodynamical behavior. Because in the LQG case, both plots of $S(M)$ and $T(M)$ have physically acceptable thermodynamical behaviors and represent the deserved final; $S, T(M \to 0) \to 0$.

4.2 Entropy Conservation of LDBHs in QG Corrected Hawking Radiation

As it is seen in the previous chapter 2, in the WKB approximation, the tunneling rate for an outgoing positive energy particle with a field quantum of energy ω, which crosses the horizon from $r_{in}(M)$ to $r_{out}(M-\omega)$, is related to the entropy change ΔS

$$\Gamma \sim e^{\Delta S} = exp\,[S(M-\omega) - S(M)] \tag{81}$$

By using (79), ΔS becomes

$$\Delta S = \frac{1}{T_H}\left\{-\omega + \alpha_1 ln\left[\left(\frac{M-\omega}{\hat{Y}(\omega)}\right)^{M-\omega}\left(\frac{M}{\hat{Y}(0)}\right)^{-M}\right]\right\} \tag{82}$$

where

$$\hat{Y}(\omega) = \sqrt{(M-\omega)^2 + \frac{\hbar\tilde{\Sigma}^2 \mathrm{A}^4}{16}} \tag{83}$$

and after substituting (82) into (81), the tunneling rate with QG $\hbar$ corrections in all orders is found as

$$\Gamma(M;\omega) = exp\left(-\frac{\omega}{T_H}\right)\left[\left(\frac{M-\omega}{\hat{Y}(\omega)}\right)^{M-\omega}\left(\frac{M}{\hat{Y}(0)}\right)^{-M}\right]^{\frac{2\alpha_1}{T_H}} \tag{84}$$

In this expression, the term $exp\left(-\frac{\omega}{T_H}\right)$ arises due to the back reaction effects. The other term to the power $\frac{2\alpha_1}{T_H}$ shows the QG $\hbar$ corrections in all orders, and significantly it gives rise to a degeneracy in the pure thermal radiation. In the absence of the QG corrections ($\alpha_1 = 0$) the radiation of the LDBH is pure thermal since the rate (84) reduces to $exp\left(-\frac{\omega}{T_H}\right)$. The latter case was studied in detail in chapter 2, which is quoted from [25], in which it was stated that the Hawking radiation of the LDBH leads to the information loss paradox. The essential annoyance in the pure thermal radiation is that it never allows the information transfer, which can be possible with the correlations of the outgoing radiation. So it is prerequisite to keep the $\alpha_1 \neq 0$ in the tunneling rate (84) when the agenda is about obtaining a spectrum which is not pure thermal, and accordingly the correlations of the emitted quanta from the LDBH. Applying the definition of the statistical correlation (54), which is given in the chapter 2, for the present case one obtains it as

$$\chi(\omega_1+\omega_2;\omega_1,\omega_2) = \frac{2\alpha_1}{T_H} ln\left[\frac{\left(\frac{M-\omega_1-\omega_2}{\hat{Y}(\omega_{1+}\omega_2)}\right)^{M-\omega_1-\omega_2}}{\left(\frac{M-\omega_1}{\hat{Y}(\omega_1)}\right)^{M-\omega_1}\left(\frac{M-\omega_2}{\hat{Y}(\omega_2)}\right)^{M-\omega_2}}\right]\left(\frac{M}{\hat{Y}(0)}\right)^{M} \tag{85}$$

This result shows that successive emissions are statistically dependent if and only if the quantum correction parameter α_1 is non-zero. Since the amount of correlation is precisely equal to mutual information between two sequentially emitted quanta, one can deduce that the statistical correlation enables the information leakage from the LDBH during its evaporation process.

Now, one can assume that the quasilocal mass of a LDBH is a combination of n-particles with energies (masses) $\omega_1, \omega_2, \dots \omega_n$, $M = \sum_{j=1}^{\infty} \omega_j$ in which ω_j is the energy of the j^{th} emitted field quanta (particle). Namely, the whole radiation process constitutes of successively emitted quanta $(\omega_1, \omega_2, \dots \omega_n)$ from the BH, so that the LDBH loses its mass M during its evaporation, and at the final stage of the evaporation we find $S, T(M \to 0) \to 0$.

The probability of a radiation composed of correlated quanta is defined in the previous chapter (see Eq.(57)) as

$$P_{rad} = \Gamma(M;\omega_1) \times \Gamma(M-\omega_1;\omega_2) \times \dots \times \Gamma\left(M - \sum_{j=1}^{n-1} \omega_j\,;\omega_n\right) \tag{86}$$

where the probability of emission of each radiation of energy ω_j is given by

$$\Gamma(M;\omega_1) = exp\left(-\frac{\omega_1}{T_H}\right)\left\{\left[\frac{M-\omega_1}{Y(\omega_1)}\right]^{M-\omega_1}\left[\frac{M}{\hat{Y}(0)}\right]^{-M}\right\}^{\frac{2\alpha_1}{T_H}},$$

$$\Gamma(M-\omega_1;\omega_2) = exp\left(-\frac{\omega_2}{T_H}\right)\left\{\left[\frac{M-\omega_1-\omega_2}{Y(\omega_2)}\right]^{M-\omega_1-\omega_2}\left[\frac{M-\omega_1}{Y(\omega_1)}\right]^{-(M-\omega_1)}\right\}^{\frac{2\alpha_1}{T_H}},$$

…..,

$$\Gamma\left(M-\sum_{j=1}^{n-1}\omega_j\,;\omega_n\right)=$$

$$exp\left(-\frac{\omega_n}{T_H}\right)\left\{\left[\frac{M-\sum_{j=1}^{n}\omega_j}{Y(\omega_n)}\right]^{M-\sum_{j=1}^{n}\omega_j}\left[\frac{M-\sum_{j=1}^{n-1}\omega_j}{Y(\omega_{n-1})}\right]^{-(M-\sum_{j=1}^{n-1}\omega_j)}\right\}^{\frac{2\alpha_1}{T_H}},$$

$$=\ \exp\left(-\frac{\omega_n}{T_H}\right)\left[\frac{\omega_n}{Y(\omega_{n-1})}\right]^{-\frac{2\alpha_1}{T_H}\omega_n} \qquad (87)$$

in which

$$Y(\omega_k)=\sqrt{\left(M-\sum_{j=1}^{k}\omega_k\right)^2+\frac{\hbar\tilde{\Sigma}A^4}{16}} \qquad (88)$$

Here, $\Gamma\left(M-\omega_1-\omega_2-\cdots-\omega_{j-1}-\omega_j\right)$ is the conditional probability of an emission with energy ω_j following the emission before the energy $\omega_1+\omega_2+\cdots+\omega_{j-1}$.

We can now substitute Eq. (87) into Eq. (86), and calculate the total probability for the whole radiation, which turns out to be

$$P_{rad}=\ exp\left(-\frac{M}{T_H}\right)\left(\frac{M}{\hat{Y}(0)}\right)^{-\frac{2\alpha_1 M}{T_H}} \qquad (89)$$

According to the statistical mechanics, we recall that all microstates are equally likely for an isolated system. Since the radiation of a BH can be considered as an isolated system, the number of microstates Ω in the system is $1/P_{rad}$. Thus, one calculates the entropy of the radiation S_{rad} from the Boltzmann's definition as

$$S_{rad} = ln(\Omega) = ln\left(\frac{1}{P_{rad}}\right) = \frac{M}{T_H} + \frac{2\alpha_1 M}{T_H} ln\left(\frac{M}{\hat{Y}(0)}\right)$$

$$= \frac{M}{T_H}\left[1 + \alpha_1 ln\left(\frac{16M^2}{16M^2 + \hbar\tilde{\Sigma}^2 \mathrm{A}^4}\right)\right] \quad (90)$$

Clearly, the total entropy of the radiation S_{rad} is equal to the entropy of the initial LDBH $S(M)$ (79). We deduce therefore that the entropy is conserved; the entropy of the original LDBH (before radiation, initial state) is equal to the entropy of the radiation (after radiation, final state). From the microscopic point of view of the entropy, this result shows that the number of microstates of initial and final states is same. The latter remark implies also that under specific conditions it is possible to save the information during the Hawking radiation of the LDBHs. In this way, unitarity in quantum mechanics of the Hawking radiation is also restored.

Chapter 5

CONCLUSION

In this thesis, we considered the LDBHs, which are NAF spacetimes. One of the important thermodynamical aspects of them is that their Hawking temperature turns out to be a constant, i.e. independent of its event horizon. This is in analogy with a classical isothermal process where although the temperature remains constant this does not prevent absorption or emission of heat. For the deeper analysis, we effectively utilized two different methods (DRS and PW's tunneling models) to investigate the Hawking radiation for $4D$-LDBHs in the EMD, EYMD and EYMBID theories. Both methods yielded the same tunneling rate. In the framework of the original (without QG corrections) PW method, the inclusion of the back-reaction effects, which guarantees the conservation of energy during a particle tunneling the horizon, yields the tunneling rate of a BH in terms of difference of the Bekenstein-Hawking entropies ΔS_{BH} of the BH. Namely, the difference of the entropies corresponding to the entropies of the BH with mass M before and after the emission of a particle having energy ω. Contrary to the similar studies about the original PW model for the ordinary BHs like Schwarzschild, de Sitter, Kerr, Reissner-Nordström [6-9] etc., as it was shown in chapter 2, the obtained emission spectrum for the LDBHs did not exhibit any deviation from its thermal spectrum. This result violates the unitarity principle, which is a fundamental law in quantum mechanics. In fact, unitarity principle puts a restriction on the allowed evolution of quantum systems that ensures the sum of probabilities of all possible outcomes of any event is always 1. Correspondingly, it means the violation of the conservation of information in the LDBHs. Above all our result implies that the original form of the PW's method is inadequate while attempting to retrieve the information from a LDBH.

For a theoretical treatment of the original PW's method, we considered the QG corrections in chapter 3. Unlike the previous result obtained in chapter 2, the quantum corrected tunneling probability changed its pure thermal form. It contains an overall factor with power α, which pertains to QG effects, to its former expression (without QG correction). By using this new tunneling probability, the correlation between successively emitted quanta is found to be statistically dependent. It is also found that the associated correlation is independent of the dimension of the LDBH. Furthermore, this nontrivial correlation proves the mutual information, which means that in each emission an amount of information should leak out of the LDBH. Therefore, QG effects play crucial role in having non-zero statistical correlation and also in resolving the information paradox for the LDBH. On the other hand, it is seen that keeping $\alpha \neq 0$ and requiring non-divergent entropy carried out by radiation (S_R and S_{NR}) renders the existence of the BH remnant indispensable. When the complete evaporation process of the LDBH is considered, the conservation of entropy is obtained both for $4D$ and higher dimensional LDBHs. It is also shown that an extreme LDBH with a pointlike horizon can be used to describe the LDBH remnant. Using the massless Klein-Gordon equation, it is proved that such a remnant cannot radiate, as expected, and its Hawking temperature is zero. This latter result shows that a LDBH supplemented with QG effects does not radiate away all of its mass and leaves a remnant at the end or forms an extreme BH.

In chapter 4, we have used SVZR's analysis [21,22] in order to obtain a specific radiation which yields both zero temperature and entropy for the LDBH when its mass is radiated away, i.e. $S, T(M \rightarrow 0) \rightarrow 0$. According to this analysis, the complete evaporation of a BH is thought as a process in which both back reaction effects and QG $\hbar$ corrections in all orders are taken into consideration. For this purpose, we imposed a condition on α_j's which are the parameters of the QG $\hbar$ corrections in all orders. Unless the QG corrections are ignored, the choice of α_j's works finely in the LDBHs to end up with $S, T(M \rightarrow 0) \rightarrow 0$.

Upon using the specific form of the entropy (79), we derived the tunneling rate (84) with QG $\hbar$ corrections in all orders. Then, it is shown that this rate attributes to the correlations between the emitted quanta. On the other hand, existence of the correlations of the outgoing radiation allowed us to make calculations for the entropy conservation. Thus we proved that after a LDBH is completely exhausted due to its Hawking radiation, the entropy of the original LDBH is exactly equal to the entropy carried away by the outgoing radiation. The important aspect of this conservation is that it provides a probable decoding for the information loss paradox associated with the LDBHs. Another meaning of this conservation is that the process of the complete evaporation of the LDBH is unitary in regard to quantum mechanics. Because, it is precisely shown that the numbers of microstates before and after the complete evaporation are the same.

After analyzing the Figs. (1) and (2), which are about the scenario of $S, T(M)$ in the QG corrected Hawking radiation of the LDBH, it is seen that our specific choice of α_j's (78) with $\alpha_1 = \frac{1}{2}$ from ST led to unacceptable behavior for the entropy (79) in which it gets negative values for some M values. In addition to this, the behavior of the temperature (80) in the ST case is not well-behaved compared to the LQG case. However, we have no such unphysical thermodynamical behaviors in the LQG case ($\alpha_1 = -\frac{1}{2}$). So, for the scenario of S, $T(M \to 0)$, we conclude that only the QG correction term α_1 coming from the LQG should be taken into consideration.

In conclusion, we show in detail that in the QG corrected Hawking radiation of the LDBHs, the information is conserved, and unitarity in quantum mechanics is restored in the process of complete evaporation of the LDBHs. We also confirm that QG corrections with the back reaction effects (PW's original method) remain crucial for the information leakage. Therefore, it should be stressed that the present study is also supportive to the usage of higher order QG corrected Hawking radiation, which is first introduced by Banerjee and Majhi [52]. Finally, we point out that since the LDBHs are conformally related to the Brans-Dicke BHs [54], similar analysis might work for those BHs as well.

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