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Singular integral equation involving a multivariable analog of Mittag-Leffler function

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Abstract

Motivated by the recent work of the second author (Özarslan in Appl. Math. Comput. 229:350-358, 2014), we present, in this paper, some fractional calculus formulas for a mild generalization of the multivariable Mittag-Leffler function, a Schläfli's type contour integral representation, some multilinear and mixed multilateral generating functions; and, finally, we consider a singular integral equation with the function $E_{(\rho_r),\lambda}^{(\gamma_r),(1)}(x_1,\ldots,x_r)$ in the kernel and we provide its solution.

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1 Introduction

The celebrated Mittag-Leffler function [1, 2] is defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{1.1}$$

$$\bigl(\alpha\in\mathbb{C};\Re(\alpha)>0;z\in\mathbb{C}\bigr),$$

where $\mathbb{C}$ denotes the set of complex numbers.

The Mittag-Leffler function arises naturally in the solution of fractional integral equations [3]. A generalization of the Mittag-Leffler function $E_\alpha(z)$ has been investigated by Wiman [4]. He studied the following function:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.2}$$

$$\bigl(\alpha,\beta\in\mathbb{C};\Re(\alpha)>0;\Re(\beta)>0;z\in\mathbb{C}\bigr).$$

Other generalizations of the Mittag-Leffler functions were given in [5, 6]. Let us recall the one given by Srivastava and Tomovski [6]:

$$E_{\alpha,\beta}^{\gamma,K}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_{Kn}}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \tag{1.3}$$

$$\bigl(\alpha,\beta,\gamma\in\mathbb{C};\Re(\alpha)>\max\{0,\Re(K)-1\};\Re(K)>0;\Re(\beta)>0;z\in\mathbb{C}\bigr),$$

where $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa=n\in\mathbb{N};\lambda\in\mathbb{C}),\\ 1 & (\kappa=0;\lambda\in\mathbb{C}\setminus\{0\}),\end{cases} \tag{1.4}$$

where $\mathbb{N}$ denotes the set of positive integers.

Multivariable analog of the Mittag-Leffler function has been introduced and investigated by Saxena *et al.* [7, p.536, Eq. (1.14)] in the following form:

$$\begin{aligned}E_{(\rho_r),\lambda}^{(\gamma_r)}(z_1,\ldots,z_r)&=E_{(\rho_1,\ldots,\rho_r),\lambda}^{(\gamma_1,\ldots,\gamma_r)}(z_1,\ldots,z_r)\\&=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots(\gamma_r)_{k_r}}{\Gamma(k_1\rho_1+\cdots+k_r\rho_r+\lambda)}\frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!}\end{aligned} \tag{1.5}$$

$$\bigl(\lambda,z_j,\gamma_j,\rho_j\in\mathbb{C};\Re(\rho_j)>0;j=1,2,\ldots,r\bigr).$$

This function is, in fact, a special case of the generalized Lauricella series in several variables, introduced by Srivastava and Daoust [8] and Srivastava and Karlsson [9].

A mild generalization of the multivariable analog of the Mittag-Leffler function, which will play an important role in this paper, has been given by Saxena *et al.* [7, p.547, Eq. (7.1)]:

$$E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1,\ldots,z_r)=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1l_1}\cdots(\gamma_r)_{k_rl_r}}{\Gamma(k_1\rho_1+\cdots+k_r\rho_r+\lambda)}\frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} \tag{1.6}$$

$$\bigl(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-;\gamma_j,\rho_j,l_j\in\mathbb{C};\Re(\rho_j)>0;\Re(l_j)>0;j=1,2,\ldots,r\bigr).$$

Recently, the second author in [10] introduced a class of polynomials suggested by the multivariate Laguerre polynomials in the following form:

$$\begin{aligned}&Z_{n_1,\ldots,n_r}^{(\alpha)}(x_1,\ldots,x_r;\rho_1,\ldots,\rho_r)\\&\quad=\frac{\Gamma(\rho_1n_1+\cdots+\rho_rn_r+\alpha+1)}{n_1!\cdots n_r!}\sum_{k_1,\ldots,k_r}^{n_1,\ldots,n_r}\frac{(-n_1)_{k_1}\cdots(-n_r)_{k_r}x_1^{\rho_1k_1}\cdots x_r^{\rho_rk_r}}{\Gamma(\rho_1k_1+\cdots+\rho_rk_r+\alpha+1)k_1!\cdots k_r!}\end{aligned} \tag{1.7}$$

$$\bigl(\alpha,\rho_1,\ldots,\rho_r\in\mathbb{C};\Re(\rho_j)>0\ (j=1,2,\ldots,r)\bigr).$$

It is easy to see that the following relation between the class of polynomials given by (1.7) and the generalized multivariable Mittag-Leffler function (1.6) exists:

$$\begin{aligned}&Z_{n_1,\ldots,n_r}^{(\alpha)}(x_1,\ldots,x_r;\rho_1,\ldots,\rho_r)\\&\quad=\frac{\Gamma(\rho_1n_1+\cdots+\rho_rn_r+\alpha+1)}{n_1!\cdots n_r!}E_{(\rho_1,\ldots,\rho_r),\alpha+1}^{(-n_1,\ldots,-n_r),(1,\ldots,1)}\bigl(x_1^{\rho_1},\ldots,x_r^{\rho_r}\bigr).\end{aligned} \tag{1.8}$$

Note that by further specializing the several parameters involved, we can obtain many well-known classes of polynomials such as the Laguerre polynomials of r variables defined by Erdélyi [11] and the Konhauser polynomials [12].

Another interesting generalization of the polynomials $Z^{(\alpha)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)$ is given by

$$\begin{aligned}
&Z^{(\alpha;N_1,\dots,N_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)\\
&\quad=\frac{\Gamma(\rho_1 n_1+\cdots+\rho_r n_r+\alpha+1)}{n_1!\cdots n_r!}\sum_{k_1,\dots,k_r=0}^{[\frac{n_1}{N_1}],\dots,[\frac{n_r}{N_r}]}\frac{(-n_1)_{N_1k_1}\cdots(-n_r)_{N_rk_r}x_1^{\rho_1k_1}\cdots x_r^{\rho_rk_r}}{\Gamma(\rho_1k_1+\cdots+\rho_rk_r+\alpha+1)k_1!\cdots k_r!} \qquad (1.9)\\
&\big(\alpha,\rho_1,\dots,\rho_r\in\mathbb{C},\Re(\rho_i)>0,N_i\in\mathbb{N}\ (i=1,\dots,r)\big).
\end{aligned}$$

Obviously, setting $N_i = 1$ $(i = 1,\dots,r)$ leads to (1.8).

In this paper, we obtain a Schläfli's type contour integral representation for the multivariable polynomials given in (1.9). Next, we give some multilinear and mixed multilateral generating functions. We also recall the fractional order integral of the generalized multivariable Mittag-Leffler function. Finally, we consider a singular integral equation with $E^{(\gamma_r),(1)}_{(\rho_r),\lambda}(x_1,\dots,x_r)$ in the kernel and we give its solution. Throughout this paper, the variables $x_1,\dots,x_r$ are assumed to be real variables.

2 Schläfli's type contour integral representation of $Z^{(\alpha;N_1,\dots,N_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)$

Let us define the following polynomials set:

$$\begin{aligned}
&P^{(N_1,\dots,N_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)\\
&\quad:=\sum_{k_1,\dots,k_r=0}^{[\frac{n_1}{N_1}],\dots,[\frac{n_r}{N_r}]}(-n_1)_{N_1k_1}\cdots(-n_r)_{N_rk_r}\frac{x_1^{\rho_1k_1}\cdots x_r^{\rho_rk_r}}{k_1!\cdots k_r!} \qquad (2.1)\\
&\big(\rho_j\in\mathbb{C};\Re(\rho_j)>0;N_j\in\mathbb{N}\ (j=1,\dots,r)\big).
\end{aligned}$$

The Schläfli's type contour integral representation of $Z^{(\alpha;N_1,\dots,N_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)$ in terms of $P^{(N_1,\dots,N_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)$ is given in the next theorem.

Theorem 2.1 *Let* $\alpha,\rho_j\in\mathbb{C}$ *with* $\Re(\rho_j)>0$ $(j=1,\dots,r)$ *and let* $N_j\in\mathbb{N}$ $(j=1,\dots,r)$. *Then the following integral representation holds true*:

$$\begin{aligned}
&Z^{(\alpha;N_1,\dots,N_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r;\rho_1,\dots,\rho_r)\\
&\quad=\frac{\Gamma(\rho_1 n_1+\cdots+\rho_r n_r+\alpha+1)}{n_1!\cdots n_r!}\frac{1}{2\pi i}\\
&\qquad\times\int_{-\infty}^{(0+)}P^{(N_1,\dots,N_r)}_{n_1,\dots,n_r}\left(\frac{x_1}{t},\dots,\frac{x_r}{t};\rho_1,\dots,\rho_r\right)t^{-\alpha-1}\mathrm{e}^t\,\mathrm{d}t. \qquad (2.2)
\end{aligned}$$

Proof We have

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} P_{n_1,\ldots,n_r}^{(N_1,\ldots,N_r)}\left(\frac{x_1}{t},\ldots,\frac{x_r}{t};\rho_1,\ldots,\rho_r\right)t^{-\alpha-1}\mathrm{e}^t\,\mathrm{d}t$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \sum_{k_1,\ldots,k_r=0}^{[\frac{n_1}{N_1}],\ldots,[\frac{n_r}{N_r}]} \frac{(-n_1)_{N_1k_1}\cdots(-n_r)_{N_rk_r}}{k_1!\cdots k_r!} x_1^{\rho_1k_1}\cdots x_r^{\rho_rk_r}\frac{\mathrm{e}^t}{t^{\rho_1k_1+\cdots+\rho_rk_r+\alpha+1}}\,\mathrm{d}t. \tag{2.3}$$

With the help of Hankel's formula [13]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-z}\mathrm{e}^t\,\mathrm{d}t, \tag{2.4}$$

we find from (2.3) and (2.4) the result asserted by Theorem 2.1. □

3 Multilinear and multilateral generating functions

We begin this section by proving a linear generating function for the polynomials $Z_{n_1,\ldots,n_j}^{(\alpha;N_1,\ldots,N_j)}(x_1,\ldots,x_j;\rho_1,\ldots,\rho_j)$ by means of the mild generalization of the multivariate analog of Mittag-Leffler functions.

Theorem 3.1 *We have*

$$\sum_{n_1,\ldots,n_j=0}^{\infty} \frac{(\gamma_1)_{n_1}\cdots(\gamma_j)_{n_j}Z_{n_1,\ldots,n_j}^{(\alpha;N_1,\ldots,N_j)}(x_1,\ldots,x_j;\rho_1,\ldots,\rho_j)}{\Gamma(\rho_1n_1+\cdots+\rho_jn_j+\alpha+1)}t_1^{n_1}\cdots t_j^{n_j}$$

$$= \prod_{i=1}^{j}(1-t_i)^{-\gamma_i}E_{\rho_1,\ldots,\rho_j,\alpha+1}^{(\gamma_j),(N_j)}\left(\frac{x_1^{\rho_1}(-t_1)^{N_1}}{(1-t_1)^{N_1}},\ldots,\frac{x_j^{\rho_j}(-t_j)^{N_j}}{(1-t_j)^{N_j}}\right),$$

where $|t_i| < 1$ $(i = 1,\ldots,j)$.

Proof Direct calculations yield

$$\sum_{n_1,\ldots,n_j=0}^{\infty} \frac{(\gamma_1)_{n_1}\cdots(\gamma_j)_{n_j}Z_{n_1,\ldots,n_j}^{(\alpha;N_1,\ldots,N_j)}(x_1,\ldots,x_j;\rho_1,\ldots,\rho_j)}{\Gamma(\rho_1n_1+\cdots+\rho_jn_j+\alpha+1)}t_1^{n_1}\cdots t_j^{n_j}$$

$$= \sum_{n_1,\ldots,n_j=0}^{\infty}\sum_{k_1,\ldots,k_j=0}^{[\frac{n_1}{N_1}],\ldots,[\frac{n_j}{N_j}]} \frac{(-1)^{N_1k_1+\cdots+N_jk_j}(\gamma_1)_{n_1}\cdots(\gamma_j)_{n_j}x_1^{\rho_1k_1}\cdots x_j^{\rho_jk_j}t_1^{n_1}\cdots t_j^{n_j}}{\Gamma(\rho_1k_1+\cdots+\rho_jk_j+\alpha+1)(n_1-N_1k_1)!\cdots(n_j-N_kk_j)!k_1!\cdots k_j!}$$

$$= \sum_{n_1,\ldots,n_j=0}^{\infty}\sum_{k_1,\ldots,k_j=0}^{\infty} \frac{(-1)^{N_1k_1+\cdots+N_jk_j}(\gamma_1)_{n_1+N_1k_1}\cdots(\gamma_j)_{n_j+N_jk_j}x_1^{\rho_1k_1}\cdots x_j^{\rho_jk_j}t_1^{n_1+N_1k_1}\cdots t_j^{n_j+N_jk_j}}{\Gamma(\rho_1k_1+\cdots+\rho_jk_j+\alpha+1)n_1!\cdots n_j!k_1!\cdots k_j!}$$

$$= \sum_{k_1,\ldots,k_j=0}^{\infty} \frac{(\gamma_1)_{N_1k_1}\cdots(\gamma_j)_{N_jk_j}(x_1^{\rho_1}(-t_1)^{N_1})^{k_1}\cdots(x_j^{\rho_j}(-t_j)^{N_j})^{k_j}}{\Gamma(\rho_1k_1+\cdots+\rho_jk_j+\alpha+1)k_1!\cdots k_j!}$$

$$\times \sum_{n_1,\ldots,n_j=0}^{\infty} \frac{(\gamma_1+N_1k_1)_{n_1}\cdots(\gamma_j+N_jk_j)_{n_j}}{n_1!\cdots n_j!}t_1^{n_1}\cdots t_j^{n_j}$$

$$= \prod_{i=1}^{j}(1-t_i)^{-\gamma_i}E_{\rho_1,\ldots,\rho_j,\alpha+1}^{(\gamma_j),(N_j)}\left(\frac{x_1^{\rho_1}(-t_1)^{N_1}}{(1-t_1)^{N_1}},\ldots,\frac{x_j^{\rho_j}(-t_j)^{N_j}}{(1-t_j)^{N_j}}\right),$$

where we have interchanged the order of summations which is guaranteed because of the uniform convergence of the series under the conditions $|t_i| < 1$ $(i = 1, \ldots, j)$. □

Now let $(\gamma) := (\gamma_1, \ldots \gamma_j)$, $(\lambda) := (\lambda_1, \ldots, \lambda_j)$, $(\eta) := (\eta_1, \ldots, \eta_j)$, $(\psi) := (\psi_1, \ldots, \psi_j)$, $(\rho) := (\rho_1, \ldots, \rho_j)$, $(N) := (N_1, \ldots, N_j)$ be complex j-tuples. By making use of the above theorem we have the following.

Theorem 3.2 *Corresponding to an identically non-vanishing function* $\Omega_{(\eta)}(\xi_1, \ldots, \xi_s)$ *of complex variables* $\xi_1, \ldots, \xi_s$ $(s \in \mathbb{N})$, *let*

$$\begin{aligned}
&\Lambda_{(\eta),(\psi)}(\xi_1, \ldots, \xi_s; \varsigma_1, \ldots, \varsigma_j) \\
&\quad := \sum_{k_1,\ldots,k_j=0}^{\infty} a_{k_1,\ldots,k_j} \Omega_{\eta_1+\psi_1 k_1,\ldots,\eta_j+\psi_j k_j}(\xi_1, \ldots, \xi_s) \varsigma_1^{k_1} \cdots \varsigma_j^{k_j} \quad (a_{k_1,\ldots,k_j} \neq 0).
\end{aligned} \tag{3.1}$$

Suppose also that

$$\begin{aligned}
&\Theta_{n_1,\ldots,n_j;q_1,\ldots,q_j}^{(\gamma),(\lambda),(\eta),(\psi),\alpha,(N)}\bigl(\xi_1, \ldots, \xi_s; x_1, \ldots, x_j; (\rho); \varsigma_1, \ldots, \varsigma_j\bigr) \\
&\quad = \sum_{k_1,\ldots,k_j=0}^{[\frac{n_1}{q_1}],\ldots,[\frac{n_j}{k_j}]} a_{k_1,\ldots,k_j} \Omega_{\eta_1+\psi_1 k_1,\ldots,\eta_j+\psi_j k_j}(\xi_1, \ldots, \xi_s) \\
&\qquad \times \frac{(\gamma_1 + \lambda_1 k_1)_{n_1-q_1k_1} \cdots (\gamma_j + \lambda_j k_j)_{n_j-q_jk_j} Z_{n_1-q_1k_1,\ldots,n_j-q_jk_j}^{(\alpha;N_1,\ldots,N_j)}(x_1, \ldots, x_j; \rho_1, \ldots, \rho_j)}{\Gamma(\rho_1(n_1 - q_1k_1) + \cdots + \rho_j(n_j - q_jk_j) + \alpha + 1)} \\
&\qquad \times \varsigma_1^{k_1} \cdots \varsigma_j^{k_j} \quad (q_1, \ldots, q_j \in \mathbb{N}).
\end{aligned} \tag{3.2}$$

Then

$$\begin{aligned}
&\sum_{n_1,\ldots,n_j=0}^{\infty} \Theta_{n_1,\ldots,n_j;q_1,\ldots,q_j}^{(\gamma),(\lambda),(\eta),(\psi),\alpha,(N)}\left(\xi_1, \ldots, \xi_s; x_1, \ldots, x_j; (\rho); \frac{\varsigma_1}{t_1^{q_1}}, \ldots, \frac{\varsigma_j}{t_j^{q_j}}\right) t_1^{n_1} \cdots t_j^{n_j} \\
&\quad = \prod_{i=1}^{j} (1 - t_i)^{-\gamma_i} \Lambda_{(\eta),(\psi)}\left(\xi_1, \ldots, \xi_s; \frac{\varsigma_1}{(1 - t_1)^{\lambda_1}}, \ldots, \frac{\varsigma_j}{(1 - t_j)^{\lambda_j}}\right) \\
&\qquad \times E_{\rho_1,\ldots,\rho_j,\alpha+1}^{(\gamma_j),(N_j)}\left(\frac{x_1^{\rho_1}(-t_1)^{N_1}}{(1 - t_1)^{N_1}}, \ldots, \frac{x_j^{\rho_j}(-t_j)^{N_j}}{(1 - t_j)^{N_j}}\right),
\end{aligned} \tag{3.3}$$

provided that each member of (3.3) *exists and* $|t_i| < 1$ $(i = 1, \ldots, j)$.

Proof Following similar lines to [10], the proof is completed. □

4 Fractional integrals and derivatives

In this section, we first recall the definitions of the Riemann-Liouville fractional integrals and derivatives. Next, we give the fractional integral and derivative of the generalized multivariable Mittag-Leffler function $E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(x_1, \ldots, x_r)$ where x_j are real variables for $j = 1, \ldots, r$.

Definition 4.1 Let $\Omega = [a,b]$ be a finite interval of the real axis. The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ is defined by

$$ {}_xI_{a^+}^{\alpha}[f] = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}} \quad (x > a). \tag{4.1}$$

It is well known [14, p.71] that

$$ {}_xI_{0^+}^{\alpha}\left[x^p\right] = \frac{\Gamma(1+p)}{\Gamma(1+p+\alpha)} x^{p+\alpha} \quad \bigl(\Re(\alpha) > 0; \Re(p) > -1\bigr). \tag{4.2}$$

Definition 4.2 Let $\Omega = [a,b]$ be a finite interval of the real axis. The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ is defined by

$$ {}_xD_{a^+}^{\alpha}[f] = \left(\frac{d}{dx}\right)^n {}_xI_{a^+}^{n-\alpha}[f] \quad \bigl(n = \left[\Re(\alpha)\right] + 1; x > a\bigr), \tag{4.3}$$

where $[\Re(\alpha)]$ denotes the integral part of $\Re(\alpha)$.

Using (4.2), we see easily that

$$ {}_xD_{0^+}^{\alpha}\left[x^p\right] = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} x^{p-\alpha} \quad \bigl(\Re(\alpha) \geq 0; \Re(p) > -1\bigr). \tag{4.4}$$

Now, let us give two fractional calculus formulas obtained by Jaimini and Gupta [15, p.145, Eqs. (1) and (2)] involving the generalized multivariable Mittag-Leffler function.

Theorem 4.3 *Let* $\alpha, \lambda, \rho_j, \gamma_j, l_j, \omega_j \in \mathbb{C}$ *such that* $\Re(\alpha) > 0$; $\Re(\lambda) > 0$; $\Re(\rho_j) > 0$; $\Re(l_j) > 0$ $(j = 1, \ldots, r)$. *Then the following fractional calculus formulas*:

$$ {}_xI_{0^+}^{\alpha}\left[x^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\left(\omega_1 x^{\rho_1}, \ldots, \omega_r x^{\rho_r}\right)\right] = x^{\lambda+\alpha-1} E_{(\rho_r),\lambda+\alpha}^{(\gamma_r),(l_r)}\left(\omega_1 x^{\rho_1}, \ldots, \omega_r x^{\rho_r}\right) \tag{4.5}$$

and

$$ {}_xD_{0^+}^{\alpha}\left[x^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\left(\omega_1 x^{\rho_1}, \ldots, \omega_r x^{\rho_r}\right)\right] = x^{\lambda-\alpha-1} E_{(\rho_r),\lambda-\alpha}^{(\gamma_r),(l_r)}\left(\omega_1 x^{\rho_1}, \ldots, \omega_r x^{\rho_r}\right) \tag{4.6}$$

hold true.

Setting $\lambda = \lambda + 1$, $l_j = \omega_j = 1$ $(j = 1, \ldots, r)$, replacing $\gamma_1, \ldots, \gamma_r$, respectively, by $-n_1, \ldots, -n_r$, where n_j $(j = 1, \ldots, r)$ are positive integers in (4.5) and (4.6), and making use of (1.8) yield the following special cases given by Özarslan [10, p.353, Theorem 6 and Theorem 8]:

$$\begin{aligned} &{}_xI_{0^+}^{\alpha}\left[x^{\lambda} Z_{n_1,\ldots,n_r}^{(\lambda)}(x, \ldots, x; \rho_1, \ldots, \rho_r)\right] \\ &\quad = \frac{\Gamma(\rho_1 n_1 + \cdots + \rho_r n_r + \lambda + 1)}{\Gamma(\rho_1 n_1 + \cdots + \rho_r n_r + \lambda + \alpha + 1)} x^{\lambda+\alpha} Z_{n_1,\ldots,n_r}^{(\lambda+\alpha)}(x, \ldots, x; \rho_1, \ldots, \rho_r) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &{}_xD_{0^+}^{\alpha}\left[x^{\lambda} Z_{n_1,\ldots,n_r}^{(\lambda)}(x, \ldots, x; \rho_1, \ldots, \rho_r)\right] \\ &\quad = \frac{\Gamma(\rho_1 n_1 + \cdots + \rho_r n_r + \lambda + 1)}{\Gamma(\rho_1 n_1 + \cdots + \rho_r n_r + \lambda - \alpha + 1)} x^{\lambda-\alpha} Z_{n_1,\ldots,n_r}^{(\lambda+\alpha)}(x, \ldots, x; \rho_1, \ldots, \rho_r). \end{aligned} \tag{4.8}$$

Further special cases of (4.5) and (4.6) can be obtained by suitably specializing the coefficients involved. For instance, if we set $l_j = 1$ ($j = 1, \ldots, r$), then (4.5) and (4.6) reduce to two results obtained by Saxena *et al.* [7].

We end this section by giving a recurrence relation for the generalized multivariable Mittag-Leffler function $E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r)$.

Theorem 4.4 *Let* $\lambda, \rho_j, \gamma_j, l_j \in \mathbb{C}$ *such that* $\Re(\lambda) > 0$; $\Re(\rho_j) > 0$; $\Re(l_j) > 0$ $(j = 1, \ldots, r)$. *Then the following recurrence relation holds true*:

$$E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r) = \lambda E_{(\rho_r),\lambda+1}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r) + \sum_{i=1}^{r} \rho_i z_i \frac{\partial}{\partial z_i} E_{(\rho_r),\lambda+1}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r). \tag{4.9}$$

Proof From (1.6), we have

$$\begin{aligned} &\lambda E_{(\rho_r),\lambda+1}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r) + \sum_{i=1}^{r} \rho_i z_i \frac{\partial}{\partial z_i} E_{(\rho_r),\lambda+1}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r) \\ &\quad = \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(k_1\rho_1 + \cdots + k_r\rho_r + \lambda + 1)} [\lambda + \rho_1 k_1 + \cdots + \rho_r k_r] \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1! \cdots k_r!} \\ &\quad = E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1, \ldots, z_r). \end{aligned} \tag{4.10}$$

□

5 Singular integral equation

In this section, we solve a singular integral equation with the generalized multivariable Mittag-Leffler function in the kernel. To do so, we first find the Laplace transform of the function $E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}((\mu x)^{\rho_1}, \ldots, (\mu x)^{\rho_r})$ and we compute an integral involving the product of two generalized multivariable Mittag-Leffler functions.

We denote the Laplace transform of a function f [16, p.218] by

$$\mathbb{L}\big[f(t)\big](p) = \widetilde{f}(p) = \int_0^{\infty} \mathrm{e}^{-pt} f(t)\,dt \quad \big(\Re(p) > 0\big). \tag{5.1}$$

Lemma 5.1 *Let* $p, \lambda, \mu, \rho_j, \gamma_j, l_j \in \mathbb{C}$ *such that* $\Re(p) > 0$; $\Re(\mu) > 0$; $\Re(\lambda) > 0$; $\Re(\rho_j) > 0$; $\Re(l_j) > 0$ $(j = 1, \ldots, r)$, *we have*

$$\begin{aligned} &\mathbb{L}\big[t^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\big((\mu t)^{\rho_1}, \ldots, (\mu t)^{\rho_r}\big)\big](p) \\ &\quad = \frac{1}{p^{\lambda}} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{k_1! \cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1 + \cdots + \rho_r k_r}. \end{aligned} \tag{5.2}$$

Proof Using (5.1), we get

$$\begin{aligned}
&\mathbb{L}\bigl[t^{\lambda-1}E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\bigr](p)\\
&\quad=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1l_1}\cdots(\gamma_r)_{k_rl_r}}{\Gamma(k_1\rho_1+\cdots+k_r\rho_r+\lambda)}\frac{\mu^{\rho_1k_1+\cdots+\rho_rk_r}}{k_1!\cdots k_r!}\cdot\int_0^{\infty}\mathrm{e}^{-pt}t^{\rho_1k_1+\cdots+\rho_rk_r+\lambda-1}\,dt\\
&\quad=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1l_1}\cdots(\gamma_r)_{k_rl_r}}{\Gamma(k_1\rho_1+\cdots+k_r\rho_r+\lambda)}\frac{\mu^{\rho_1k_1+\cdots+\rho_rk_r}}{k_1!\cdots k_r!}\cdot\frac{\Gamma(k_1\rho_1+\cdots+k_r\rho_r+\lambda)}{p^{k_1\rho_1+\cdots+k_r\rho_r+\lambda}}\\
&\quad=\frac{1}{p^{\lambda}}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1l_1}\cdots(\gamma_r)_{k_rl_r}}{k_1!\cdots k_r!}\left(\frac{\mu}{p}\right)^{\rho_1k_1+\cdots+\rho_rk_r},
\end{aligned}\tag{5.3}$$

where we used the well-known formula [16, p.218, Eq. (3)]

$$\int_0^{\infty}\mathrm{e}^{-pt}t^{\lambda-1}\,dt=\frac{\Gamma(\lambda)}{p^{\lambda}}\quad\bigl(\min\bigl\{\Re(\lambda),\Re(p)\bigr\}>0\bigr).\tag{5.4}$$

□

Theorem 5.2 *Let* $p,\lambda,\mu,\nu,\rho_j,\gamma_j,l_j,\sigma_j,m_j\in\mathbb{C}$ *such that* $\Re(p)>0$; $\Re(\mu)>0$; $\Re(\nu)>0$; $\Re(\lambda)>0$; $\Re(\rho_j)>0$; $\Re(\sigma_j)>0$; $\Re(l_j)>0$; $\Re(m_j)>0$ $(j=1,\ldots,r)$, *we have*

$$\begin{aligned}
&\int_0^x(x-t)^{\lambda-1}E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\bigl(\bigl(\mu[x-t]\bigr)^{\rho_1},\ldots,\bigl(\mu[x-t]\bigr)^{\rho_r}\bigr)\\
&\qquad\times t^{\nu-1}E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\,dt\\
&\quad=t^{\lambda+\nu-1}E_{\rho_1,\ldots,\rho_r,\rho_1,\ldots,\rho_r,\lambda+\nu}^{\gamma_1,\ldots,\gamma_r,\sigma_1,\ldots,\sigma_1,l_1,\ldots,l_r,m_1,\ldots,m_r}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r},(\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr).
\end{aligned}\tag{5.5}$$

Proof With the help of the convolution theorem for the Laplace transform (see [17])

$$\mathbb{L}\left[\int_0^x f(x-t)g(t)\,dt\right](p)=\mathbb{L}\bigl[f(x)\bigr](p)\mathbb{L}\bigl[g(x)\bigr](p),\tag{5.6}$$

we have

$$\begin{aligned}
&\mathbb{L}\left[\int_0^x(x-t)^{\lambda-1}E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\bigl(\bigl(\mu[x-t]\bigr)^{\rho_1},\ldots,\bigl(\mu[x-t]\bigr)^{\rho_r}\bigr)\right.\\
&\qquad\left.\times t^{\nu-1}E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\,dt\right](p)\\
&\quad=\mathbb{L}\bigl[t^{\lambda-1}E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\bigr](p)\\
&\qquad\cdot\mathbb{L}\bigl[t^{\nu-1}E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\bigr](p).
\end{aligned}\tag{5.7}$$

From Lemma 5.1, we have

$$\begin{aligned}
&\mathbb{L}\left[\int_0^x(x-t)^{\lambda-1}E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}\bigl(\bigl(\mu[x-t]\bigr)^{\rho_1},\ldots,\bigl(\mu[x-t]\bigr)^{\rho_r}\bigr)\right.\\
&\qquad\left.\times t^{\nu-1}E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\,dt\right](p)
\end{aligned}$$

$$= \frac{1}{p^{\lambda}} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1}\cdots(\gamma_r)_{k_r l_r}}{k_1!\cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1+\cdots+\rho_r k_r}$$

$$\cdot \frac{1}{p^{\mu}} \sum_{i_1,\ldots,i_r=0}^{\infty} \frac{(\sigma_1)_{i_1 m_1}\cdots(\sigma_r)_{i_r m_r}}{i_1!\cdots i_r!} \left(\frac{\mu}{p}\right)^{\rho_1 i_1+\cdots+\rho_r i_r}$$

$$= \frac{1}{p^{\lambda+\nu}} \sum_{k_1,\ldots,k_r,i_1,\ldots,i_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1}\cdots(\gamma_r)_{k_r l_r}\cdot(\sigma_1)_{i_1 m_1}\cdots(\sigma_r)_{i_r m_r}}{k_1!\cdots k_r!\cdot i_1!\cdots i_r!} \left(\frac{\mu}{p}\right)^{\rho_1(k_1+i_1)+\cdots+\rho_r(k_r+i_r)}$$

$$= \mathbb{L}\bigl[t^{\lambda+\nu-1} E_{\rho_1,\ldots,\rho_r,\rho_1,\ldots,\rho_r,\lambda+\nu}^{\gamma_1,\ldots,\gamma_r,\sigma_1,\ldots,\sigma_1,l_1,\ldots,l_r,m_1,\ldots,m_r}\bigl((\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r},(\mu t)^{\rho_1},\ldots,(\mu t)^{\rho_r}\bigr)\bigr](p). \tag{5.8}$$

Finally, taking the inverse Laplace transform on both sides of (5.8), the result follows. □

Now, let us consider the following convolution equation involving the generalized multivariable Mittag-Leffler in the kernel:

$$\int_0^x (x-t)^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(1)}\bigl(\bigl(\mu[x-t]\bigr)^{\rho_1},\ldots,\bigl(\mu[x-t]\bigr)^{\rho_r}\bigr)\cdot\phi(t)\,dt = \psi(x), \tag{5.9}$$

where $\Re(\alpha) > -1$.

Theorem 5.3 *The singular integral equation* (5.9) *admits a locally integrable solution*

$$\phi(x) = \int_0^x (x-t)^{\omega-\lambda-1} E_{(\rho_r),\omega-\lambda}^{(-\gamma_r),(1)}\bigl(\bigl(\mu[x-t]\bigr)^{\rho_1},\ldots,\bigl(\mu[x-t]\bigr)^{\rho_r}\bigr)\cdot {}_tI_{0^+}^{-\omega}\psi(t)\,dt, \tag{5.10}$$

provided that ${}_tI_{0^+}^{-\omega}\psi(t)$ *exists for* $\Re(\omega) > \Re(\alpha+1)$ *and is locally integrable for* $0 < t < \delta < \infty$.

Proof Applying the Laplace transform on both sides of (5.9), using the convolution theorem as well as Lemma 5.1, we find

$$\frac{1}{p^{\lambda}} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\gamma_1)_{k_1}\cdots(\gamma_r)_{k_r}}{k_1!\cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1+\cdots+\rho_r k_r} \cdot \mathbb{L}\bigl[\phi(t)\bigr](p) = \mathbb{L}\bigl[\psi(t)\bigr](p), \tag{5.11}$$

which under the assumptions that $|\frac{\mu}{p}| < 1$ can be rewritten as

$$\frac{1}{p^{\lambda}} \prod_{j=1}^{r} \left(1 - \left(\frac{\mu}{p}\right)^{\rho_j}\right)^{-\gamma_j} \cdot \mathbb{L}\bigl[\phi(t)\bigr](p) = \mathbb{L}\bigl[\psi(t)\bigr](p). \tag{5.12}$$

Therefore, we have

$$\mathbb{L}\bigl[\phi(t)\bigr](p) = \left\{\prod_{j=1}^{r}\left(1-\left(\frac{\mu}{p}\right)^{\rho_j}\right)^{\gamma_j} p^{-\omega+\lambda}\right\}\cdot\bigl\{p^{\omega}\mathbb{L}\bigl[\psi(t)\bigr](p)\bigr\}. \tag{5.13}$$

Taking the inverse Laplace transform on both sides of (5.13) and with the help of the following property [5, p.217, Eq. (3.8)]:

$$p^{\mu}\mathbb{L}\bigl[f(t)\bigr](p) = \mathbb{L}\bigl[{}_tI_{0^+}^{-\mu}f(t)\bigr](p) \quad \bigl(\mu, p \in \mathbb{C}; \Re(p) > 0\bigr), \tag{5.14}$$

which holds for suitable f, we thus obtain

$$\phi(x) = \int_0^x (x-t)^{\omega-\lambda-1} E_{(\rho_r),\omega-\lambda}^{(-\gamma_r),(1)}\bigl(\bigl(\mu[x-t]\bigr)^{\rho_1},\ldots,\bigl(\mu[x-t]\bigr)^{\rho_r}\bigr)\cdot {}_tI_{0^+}^{-\omega}\psi(t)\,dt. \tag{5.15}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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References

1. Mitag-Leffler, GM: Sur la nouvelle function $E_\alpha(x)$. C. R. Acad. Sci. Paris **137**, 554-558 (1903)
2. Mitag-Leffler, GM: Sur la représentation analytique d'une fonction monogène (cinquième note). Acta Math. **29**, 101-181 (1905)
3. Saxena, RK, Mathai, AM, Haubold, HJ: On fractional kinetic equations. Astrophys. Space Sci. **282**(1), 281-287 (2002)
4. Wiman, A: Über den fundamental Satz in der Theorien der Funktionen $E_\alpha(x)$. Acta Math. **29**, 191-201 (1905)
5. Prabhakar, TR: A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. **19**, 7-15 (1971)
6. Srivastava, HM, Tomovski, Z: Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. Appl. Math. Comput. **211**(1), 198-210 (2009)
7. Saxena, RK, Kalla, SL, Saxena, R: Multivariable analogue of generalized Mittag-Leffler function. Integral Transforms Spec. Funct. **22**, 533-548 (2011)
8. Srivastava, HM, Daoust, MC: Certain generalized Neumann expansion associated with Kampé de Fériet function. Proc. K. Ned. Akad. Wet., Ser. A, Indag. Math. **31**, 449-457 (1969)
9. Srivastava, HM, Karlsson, PW: Multiple Gaussian Hypergeometric Series. Ellis Horwood, Chichester (1985)
10. Özarslan, MA: On a singular integral equation including a set of multivariate polynomials suggested by Laguerre polynomials. Appl. Math. Comput. **229**, 350-358 (2014)
11. Erdélyi, A: Beitrag zur theorie der konfluenten hypergeometrischen funktionen von mehreren veränderlichen. Sitzungsber. Akad. Wiss. Wien, Math.-Naturw. Kl., Abt. IIa **146**, 431-467 (1937)
12. Konhauser, JDE: Biorthogonal polynomials suggested by the Laguerre polynomials. Pac. J. Math. **21**, 303-314 (1967)
13. Erdélyi, A, Magnus, W, Oberhettinger, F, Tricomi, F: Higher Transcendental Functions, vols. 1-3. McGraw-Hill, New York (1953)
14. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematical Studies, vol. 204. Elsevier, Amsterdam (2006)
15. Jaimini, BB, Gupta, J: On certain fractional differential equations involving generalized multivariable Mittag-Leffler function. Note Mat. **32**(2), 141-156 (2012)
16. Srivastava, HM, Manocha, HL: A Treatise on Generating Functions. Ellis Horwood, Chichester (1984)
17. Titchmarsh, EC: Introduction to the Theory of Fourier Integrals, 3rd edn. Chelsea, New York (1986). The first edition in Oxford University Press, Oxford (1937)

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