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Differential equations for the extended 2D Bernoulli and Euler polynomials

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Abstract

In this paper, we introduce the extended $2D$ Bernoulli polynomials by

$$\frac{t^{\alpha}}{(e^t-1)^{\alpha}}c^{xt+yt^j}=\sum_{n=0}^{\infty}B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}$$

and the extended $2D$ Euler polynomials by

$$\frac{2^{\alpha}}{(e^t+1)^{\alpha}}c^{xt+yt^j}=\sum_{n=0}^{\infty}E_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!},$$

where $c>1$. By using the concepts of the monomiality principle and factorization method, we obtain the differential, integro-differential and partial differential equations for these polynomials. Note that the above mentioned differential equations for the extended $2D$ Bernoulli polynomials reduce to the results obtained in (Bretti and Ricci in Taiwanese J. Math. 8(3): 415–428, 2004), in the special case $c=e$, $\alpha=1$. On the other hand, all the results for the second family are believed to be new, even in the case $c=e$, $\alpha=1$. Finally, we give some open problems related with the extensions of the above mentioned polynomials.

MSC: Primary 11B68; secondary 33C05

Keywords: $2D$ Bernoulli polynomials; $2D$ Euler polynomials; $2D$ Appell Polynomials; Hermite-Kampé de Fériet (or Gould-Hopper) polynomials; differential equations; generalized heat equation

1 Introduction

A polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ is quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$, independent of n, such that

$$\hat{P}\bigl(P_n(x)\bigr)=nP_{n-1}(x)\quad\text{and}\quad\hat{M}\bigl(P_n(x)\bigr)=P_{n+1}(x).$$

Here, $\hat{M}$ and $\hat{P}$ play the role of multiplicative and derivative operators, respectively. Owing to the fact that every polynomial set is quasi-monomial [1], by using the monomiality principle, new results were obtained for Hermite, Laguerre, Legendre and Appell polynomials in [2–6].

In this paper, we consider Appell polynomials. Before proceeding, we recall some basic definitions and properties of the polynomial families that we discuss throughout the paper.

The celebrated Appell polynomials can be defined by the following generating relation:

$$G_A(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}, \tag{1}$$

where

$$A(t) = \sum_{k=0}^{\infty} R_k \frac{t^k}{k!}, \qquad A(0) \neq 0$$

is an analytic function at $t = 0$ and $R_k := R_k(0)$. Assuming that

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}$$

it is easy to see that for any $A(t)$ the derivatives of $R_n(x)$ satisfy

$$R_n'(x) = nR_{n-1}(x).$$

Letting $\Phi_n := \frac{1}{n}D_x$, where $D_x := \frac{d}{dx}$, it is straightforward that

$$(\Phi_1\Phi_2\cdots\Phi_{n-1}\Phi_n)R_n(x) = R_0(x).$$

On the other hand, it is shown in [2] that, if

$$\Psi_n := (x + \alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}$$

then $\Psi_n(R_n(x)) = R_{n+1}(x)$. Hence, we have the following relation:

$$(\Phi_{n+1}\Psi_n)R_n(x) = R_n(x). \tag{2}$$

Since Φ_n and Ψ_n are differential realizations, equation (2) gives the differential equation that is satisfied by Appell polynomials [2]. In [2], M.X. He and Paolo E. Ricci obtained the differential equations of the Appell polynomials via the factorization method. Moreover, they found differential equations satisfied by Bernoulli and Euler polynomials as a special case. Afterward, Da-Qian Lu found differential equations for generalized Bernoulli polynomials in [7].

Bernoulli polynomials are defined by the following generating relation:

$$G(x,t) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}; \quad |t| < 2\pi$$

and Bernoulli numbers $B_n := B_n(0)$ can be obtained by the generating relation

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Particularly,

$$B_0 = 1, \qquad B_1 = \frac{-1}{2}, \qquad B_2 = \frac{1}{6} \tag{3}$$

and $B_{2k+1} = 0$ for ($k = 1, 2, \ldots$). Bernoulli numbers play an important role in many mathematical formulas. For instance,

- MacLaurin expansion of the trigonometric and hyperbolic tangent and cotangent functions,
- the sums of powers of natural numbers,
- the residual term of the Euler-Maclaurin quadrature formula [8].

Bernoulli polynomials, first studied by Euler [9], are employed in the integral representation of differentiable periodic functions, and play an important role in the approximation of such functions by means of polynomials [10].

First, the three Bernoulli polynomials are

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2}, \qquad B_2(x) = x^2 - x + \frac{1}{6}. \tag{4}$$

The following properties are straightforward:

$$B_n(0) = B_n(1) = B_n, \quad n \neq 1,$$
$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$
$$B_n'(x) = nB_{n-1}(x).$$

Taking $A(t) = \frac{2}{e^t+1}$ in (1), we meet with the well-known Euler polynomials. More precisely, the Euler polynomials are defined via the generating relation

$$G_E(x,t) = \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi.$$

On the other hand, the Euler numbers E_n are defined by the following relation:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Moreover,

$$E_n\left(\frac{1}{2}\right) = 2^{-n}E_n$$

and (see in [2, 11])

$$e_k = -\frac{1}{2^k}\sum_{h=0}^{k}\binom{k}{h}E_{k-h}.$$

Note that some extensions of these and related polynomials were given in [12–16].

Recently, Gabriella Bretti and Paolo E. Ricci defined the two-dimensional Bernoulli polynomials $B_n^{(j)}(x,y)$ ($j \in \mathbb{N}_2 := \{2,3,4,\ldots\}$) via the generating relation

$$G^{(j)}(x,y;t) = \frac{t}{e^t - 1} e^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(j)}(x,y) \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{5}$$

The two-dimensional Euler polynomials $E_n^{(j)}(x,y)$ are defined as

$$G^{(j)}(x,y;t) = \frac{2}{e^t + 1} e^{xt+yt^j} = \sum_{n=0}^{\infty} E_n^{(j)}(x,y) \frac{t^n}{n!}, \quad |t| < \pi. \tag{6}$$

They obtained explicit forms of the polynomials $B_n^{(j)}(x,y)$ by means of Hermite-Kampé de Fériet (or Gould-Hopper) polynomials $H_n^{(j)}(x,y)$, where these polynomials are defined by

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x,y) \frac{t^n}{n!}.$$

Furthermore, Gabriella Bretti and Paolo E. Ricci gave a recurrence relation, shift operators, differential, integro-differential and partial differential equations for two-dimensional Bernoulli polynomials in [10]. We gather these results in the following theorem:

Theorem 1 [10] Gabriella Bretti and Paolo E. Ricci *For $n \in \mathbb{N}$, the recurrence relation of the 2D Bernoulli polynomials is given by*

$$\begin{aligned} &B_0^{(j)}(x,y) = 1, \\ &B_{n+1}^{(j)}(x,y) = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_{n-k+1} B_k^{(j)}(x,y) + \left(x - \frac{1}{2}\right) B_n^{(j)}(x,y) \\ &\qquad + jy \frac{n!}{(n-j+1)!} B_{n-j+1}^{(j)}(x,y). \end{aligned} \tag{7}$$

Shift operators are given by

$$\begin{aligned} &L_n^- := \frac{1}{n} D_x, \\ &L_n^+ := \left(x - \frac{1}{2}\right) - \sum_{k=0}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{n-k} + jy D_x^{j-1}, \\ &\mathcal{L}_n^- := \frac{1}{n} D_x^{-(j-1)} D_y, \\ &\mathcal{L}_n^+ := \left(x - \frac{1}{2}\right) + jy D_x^{-(j-1)^2} D_y^{j-1} - \sum_{k=0}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{-(j-1)(n-k)} D_y^{n-k}. \end{aligned}$$

Differential, integro-differential and partial differential equations are

$$\begin{aligned} &\left[\frac{B_n}{n!} D_x^n + \cdots + \frac{B_{j+1}}{(j+1)!} D_x^{j+1} + \left(\frac{B_j}{j!} - jy\right) D_x^j \right. \\ &\quad \left. + \frac{B_{j-1}}{(j-1)!} D_x^{j-1} + \cdots + \left(\frac{1}{2} - x\right) D_x + n\right] B_n^{(j)}(x,y) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned}&\Bigg[\left(x-\frac{1}{2}\right)D_y+jD_x^{-(j-1)^2}D_y^{j-1}+jyD_x^{-(j-1)^2}D_y^{j}\\&\quad-\sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1}-(n+1)D_x^{j-1}\Bigg]B_n^{(j)}(x,y)=0,\end{aligned}\tag{9}$$

$$\begin{aligned}&\Bigg[\left(x-\frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y+jD_x^{(j-1)(n-j)}\left(D_y^{j-1}+yD_y^{j}\right)\\&\quad-\sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)D_x^{(j-1)n}\Bigg]B_n^{(j)}(x,y)=0,\quad n\geq j\end{aligned}\tag{10}$$

respectively.

From here and throughout the paper,

$$D_x:=\frac{d}{dx},\qquad D_y:=\frac{d}{dy},\qquad D_x^{-1}f(x):=\int_0^x f(\xi)\,d\xi.$$

Note that Gabriella Bretti and Paolo E. Ricci investigated the case $j=2$ separately.

In this paper, we consider the 2D extension of the Bernoulli and Euler polynomials. To obtain the explicit forms of them, we take into consideration of the extended Hermite-Kampé de Fériet (or Gould-Hopper) polynomials. Let us define the extended Hermite-Kampé de Fériet (or Gould-Hopper) polynomials by the following generating relation:

$$c^{xt+yt^j}=\sum_{n=0}^{\infty}P_n^{(j,c)}(x,y)\frac{t^n}{n!},\qquad c>1.\tag{11}$$

It is clear that $P_n^{(j,c)}(x,y)$ is explicitly given by

$$P_n^{(j,c)}(x,y)=n!\sum_{s=0}^{[\frac{n}{j}]}\frac{x^{n-js}y^s}{(n-js)!s!}(\ln c)^{n+s-js},\tag{12}$$

where $j\geq 2$ is an integer. Note that $c=e$, gives $P_n^{(j,c)}(x,y)=H_n^{(j)}(x,y)$ where

$$H_n^{(j)}(x,y)=n!\sum_{s=0}^{[\frac{n}{j}]}\frac{x^{n-js}y^s}{(n-js)!s!}$$

are Hermite-Kampé de Fériet (or Gould-Hopper) polynomials.

It is meaningful to mention that the polynomials $P_n^{(j,c)}(x,y)$ are very important in solving the generalized heat equation:

$$\begin{aligned}&(\ln c)^{1-j}\frac{\partial^j}{\partial x^j}F(x,y,c)=\frac{\partial}{\partial y}F(x,y,c),\\&\quad F(x,0,c)=x^n(\ln c)^n.\end{aligned}\tag{13}$$

Moreover, other generalizations which include $P_n^{(j,c)}(x,y)$ polynomials can be defined by

$$c^{(x_1 t + x_2 t^2 + \cdots + x_r t^r)} = \sum_{n=0}^{\infty} P_n^{(c,r)}(x_1, \ldots, x_r)\frac{t^n}{n!}. \tag{14}$$

Gabriella Bretti and Paolo E. Ricci gave the explicit form of 2D Bernoulli polynomials by

$$B_n^{(\alpha,j)}(x,y) = \sum_{h=0}^{n} \binom{n}{h} B_{n-h} H_h^{(j)}(x,y).$$

On the other hand, generalized Bernoulli and Euler polynomials were defined by H. M. Srivastava, Mridula Garg and Sangeeta Choudhary in [17] as follows.

Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda;a;b;c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating relation:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \quad \left(\left|t \ln\left(\frac{b}{a}\right) + \ln\lambda\right| < 2\pi; 1^{\alpha} := 1; x \in \mathbb{R}\right). \tag{15}$$

Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Euler polynomials $E_n^{(\alpha)}(x;\lambda;a;b;c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating relation:

$$\left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!} \quad \left(\left|t \ln\left(\frac{b}{a}\right) + \ln\lambda\right| < \pi; 1^{\alpha} := 1; x \in \mathbb{R}\right). \tag{16}$$

These definitions motivate us to consider the following extended $2D$ Bernoulli and Euler polynomials:

Definition 2 The extended $2D$ Bernoulli polynomials of order α is defined as

$$\frac{t^{\alpha}}{(e^t - 1)^{\alpha}} c^{xt + yt^j} = \sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}, \tag{17}$$

where $(j \in \mathbb{N}_2 := \{2, 3, 4, \ldots\})$ and $c > 1$.

In the case $c = e$ in (17), we call the polynomials $B_n^{(\alpha,j)}(x,y) := B_n^{(\alpha,j)}(x,y,e)$, as the generalized $2D$ Bernoulli polynomials. Note that the generalized Bernoulli numbers are defined by

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}\frac{t^n}{n!}. \tag{18}$$

Definition 3 The extended $2D$ Euler polynomials of order α is defined as

$$\frac{2^{\alpha}}{(e^t + 1)^{\alpha}} c^{xt + yt^j} = \sum_{n=0}^{\infty} E_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}, \tag{19}$$

where $(j \in \mathbb{N}_2 := \{2, 3, 4, \ldots\})$ and $c > 1$.

Note that in the case $c = e$ in (19), we call the polynomials $E_n^{(\alpha,j)}(x,y) := E_n^{(\alpha,j)}(x,y,e)$, as the generalized $2D$ Euler polynomials.

In the following section, we obtain the explicit forms of the $2D$ extension of Bernoulli polynomials, by means of Hermite-Kampé de Fériet (or Gould Hopper) polynomials and Bernoulli numbers. Moreover, we obtain differential, integro-differential, partial differential equations and shift operators for the extended $2D$ Bernoulli polynomials by using the factorization method, introduced in [18]. We list the results for the extended $2D$ Bernoulli polynomials. In Section 3, we deal with finding the recurrence relation, differential, integro-differential and partial differential equations for the extended $2D$ Euler polynomials. Finally, in Section 4, we present some open problems that will be investigated in the future.

2 2*D* extension of generalized Bernoulli polynomials and their differential equations

We begin by the following theorem that gives the explicit form of extended $2D$ Bernoulli polynomials via Hermite-Kampé de Fériet (or Gould Hopper) polynomials:

Theorem 4 *The relationship between* $P_n^{(j,c)}(x,y)$ *and* $B_n^{(\alpha,j)}(x,y,c)$ *is given by*

$$B_n^{(\alpha,j)}(x,y,c) = \sum_{k=0}^{n} \binom{n}{k} P_k^{(j,c)}(x,y) B_{n-k}^{\alpha}, \quad c > 1, \tag{20}$$

where B_k *denotes the Bernoulli numbers.*

Proof Since

$$\sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y,c) \frac{t^n}{n!} = \frac{t^{\alpha}}{(e^t - 1)^{\alpha}} c^{xt+yt^j}$$

the result is obtained by using (11) and (18) and then applying the Cauchy product of the series. □

Corollary 5 *For* $\alpha = 1$, $c = e$, *we obtain Theorem* 4.1 *of* [10].

In the following theorem, the recurrence relation, shift operators and differential, integro-differential, partial differential equations are obtained for extended $2D$ Bernoulli polynomials.

Theorem 6 *The extended* $2D$ *Bernoulli polynomials satisfy the following recurrence relation*:

$$\begin{aligned}
&B_0^{(\alpha,j)}(x,y,c) = 1, \qquad B_{-k}^{(\alpha,j)}(x,y,c) := 0, \\
&B_{n+1}^{(\alpha,j)}(x,y,c) = \left(x \ln c - \frac{\alpha}{2}\right) B_n^{(\alpha,j)}(x,y,c) + yj \frac{n!}{(n-j+1)!} (\ln c) B_{n-j+1}^{(\alpha,j)}(x,y,c) \\
&\qquad - \frac{\alpha}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k^{(\alpha,j)}(x,y,c) B_{n+1-k},
\end{aligned} \tag{21}$$

where B_n *denotes the Bernoulli numbers and* $n \in N$.

Shift operators are given by

$$\begin{aligned}
L_n^- &:= \frac{1}{n\ln c}D_x,\\
L_n^+ &:= x\ln c - \frac{\alpha}{2} + yj(\ln c)^{(2-j)}D_x^{(j-1)}\\
&\quad - \alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(k-n)}D_x^{n-k},\\
\mathcal{L}_n^- &:= \frac{(\ln c)^{j-2}}{n}D_x^{1-j}D_y,\\
\mathcal{L}_n^+ &:= \left(x\ln c - \frac{\alpha}{2}\right) + yj(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^{j-1}\\
&\quad - \alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(n-k)(j-2)}D_x^{-(j-1)(n-k)}D_y^{n-k},
\end{aligned}$$

where $n \geq 1$, $j \geq 2$ *is an integer and* $c > 1$.

The differential, integro-differential and partial differential equations for the extended $2D$ *Bernoulli polynomials are given by*

$$\begin{aligned}
&\Bigg[\left(x - \frac{\alpha}{2\ln c}\right)D_x + yj(\ln c)^{1-j}D_x^j\\
&\quad - \alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{k-n-1}D_x^{n+1-k} - n\Bigg]B_n^{(\alpha,j)}(x,y,c) = 0, \qquad (22)\\
&\Bigg[\left(x\ln c - \frac{\alpha}{2}\right)D_y + j(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^{j-1} + yj(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^j\\
&\quad - \alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(j-2)(n-k)}D_x^{-(j-1)(n-k)}D_y^{n-k+1}\\
&\quad - (n+1)(\ln c)^{2-j}D_x^{j-1}\Bigg]B_n^{(\alpha,j)}(x,y,c) = 0, \qquad (23)\\
&\Bigg[\left(x\ln c - \frac{\alpha}{2}\right)D_x^{(j-1)(n-1)}D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y\\
&\quad + j(\ln c)^{(j-1)(j-2)+1}D_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y)\\
&\quad - \alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(j-2)(n-k)}D_x^{(j-1)(k-1)}D_y^{n-k+1} - (n+1)(\ln c)^{2-j}D_x^{n(j-1)}\Bigg]\\
&\quad \times B_n^{(\alpha,j)}(x,y,c) = 0, \quad n \geq j, \qquad (24)
\end{aligned}$$

respectively.

Note that the partial differential equation (24) does not contain anti-derivatives for $n \geq j$.

Proof Taking derivative on both sides of the generating relation

$$\frac{t^{\alpha}}{(e^t-1)^{\alpha}}c^{xt+yt^j}=\sum_{n=0}^{\infty}B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}$$

with respect to t, then using some series manipulations and (3), we get the recurrence relation

$$\begin{aligned}B_{n+1}^{(\alpha,j)}(x,y,c)&=\left(x\ln c-\frac{\alpha}{2}\right)B_n^{(\alpha,j)}(x,y,c)+yj\frac{n!}{(n-j+1)!}(\ln c)B_{n-j+1}^{(\alpha,j)}(x,y,c)\\&\quad-\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(\alpha,j)}(x,y,c)B_{n+1-k}.\end{aligned}$$

Differentiating generating equation (17) with respect to x and equating coefficients of t^n, we obtain

$$D_xB_n^{(\alpha,j)}(x,y,c)=n\ln cB_{n-1}^{(\alpha,j)}(x,y,c).$$

Hence, the operator $L_n^-:=\frac{1}{n\ln c}D_x$ satisfies the following relation:

$$L_n^-B_n^{(\alpha,j)}(x,y,c)=B_{n-1}^{(\alpha,j)}(x,y,c).$$

Since, we have the relations

$$\begin{aligned}B_k^{(\alpha,j)}(x,y,c)&=\left[L_{k+1}^-L_{k+2}^-\cdots L_n^-\right]B_n^{(\alpha,j)}(x,y,c)\\&=\frac{k!}{n!}(\ln c)^{k-n}D_x^{n-k}B_n^{(\alpha,j)}(x,y,c),\end{aligned}\tag{25}$$

$$\begin{aligned}B_{n-j+1}^{(\alpha,j)}(x,y,c)&=\left[L_{n-j+2}^-L_{n-j+3}^-\cdots L_n^-\right]B_n^{(\alpha,j)}(x,y,c)\\&=\frac{(n-j+1)!}{n!}(\ln c)^{1-j}D_x^{j-1}B_n^{(\alpha,j)}(x,y,c),\end{aligned}\tag{26}$$

writing (25) and (26) in the recurrence relation, we get L_n^+

$$L_n^+:=x\ln c-\frac{\alpha}{2}+yj(\ln c)^{(2-j)}D_x^{(j-1)}-\alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(k-n)}D_x^{n-k}.$$

By applying the factorization method (see [18, 19]),

$$L_{n+1}^-L_n^+B_n^{(\alpha,j)}(x,y,c)=B_n^{(\alpha,j)}(x,y,c)$$

we get differential equation

$$\begin{aligned}&\Bigg[\left(x-\frac{\alpha}{2\ln c}\right)D_x+yj(\ln c)^{1-j}D_x^j\\&\quad-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{k-n-1}D_x^{n+1-k}-n\Bigg]B_n^{(a,j)}(x,y,c)=0.\end{aligned}$$

To obtain the integro-differential equation

$$\begin{aligned}
&\Bigg[\left(x\ln c-\frac{\alpha}{2}\right)D_y+j(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^{j-1}+yj(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^{j}\\
&\quad-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(j-2)(n-k)}D_x^{-(j-1)(n-k)}D_y^{n-k+1}-(n+1)(\ln c)^{2-j}D_x^{j-1}\Bigg]\\
&\quad\times B_n^{(a,j)}(x,y,c)=0,
\end{aligned}$$

we take derivative with respect to y in the generating relation (17) to obtain

$$(\ln c)B_{n-j}^{(a,j)}(x,y,c)n(n-1)\cdots(n-j+1)=\frac{\partial B_n^{(a,j)}(x,y,c)}{\partial y}.$$

From this fact, we write $\mathcal{L}_n^-$ as follows:

$$\mathcal{L}_n^-:=\frac{(\ln c)^{j-2}}{n}D_x^{1-j}D_y.$$

By using this lowering operator in (21), we get

$$\begin{aligned}
\mathcal{L}_n^+:=&\left(x\ln c-\frac{\alpha}{2}\right)+yj(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^{j-1}\\
&-\alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(n-k)(j-2)}D_x^{-(j-1)(n-k)}D_y^{n-k}.
\end{aligned}$$

Using the factorization relation

$$\mathcal{L}_{n+1}^-\mathcal{L}_n^+B_n^{(\alpha,j)}(x,y,c)=B_n^{(\alpha,j)}(x,y,c),$$

we get the integro-differential equation (23).

Differentiating both sides of (23) with respect to x, $(j-1)(n-1)$ times, we obtain the partial differential equation

$$\begin{aligned}
&\Bigg[\left(x\ln c-\frac{\alpha}{2}\right)D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y\\
&\quad+j(\ln c)^{(j-1)(j-2)+1}D_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y)\\
&\quad-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(j-2)(n-k)}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)(\ln c)^{2-j}D_x^{n(j-1)}\Bigg]\\
&\quad\times B_n^{(a,j)}(x,y,c)=0.
\end{aligned}$$

□

Since the case $c=e$ reduces to the generalized $2D$ Bernoulli polynomials, it is important to state this result for this case.

Corollary 7 *For the generalized* 2*D Bernoulli polynomials*, *the recurrence relation is given by*

$$\begin{aligned} B_{n+1}^{(\alpha,j)}(x,y) &= \left(x-\frac{\alpha}{2}\right)B_n^{(\alpha,j)}(x,y) + yj\frac{n!}{(n-j+1)!}B_{n-j+1}^{(\alpha,j)}(x,y) \\ &\quad - \frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(\alpha,j)}(x,y)B_{n+1-k}. \end{aligned} \tag{27}$$

Shift operators are given by

$$\begin{aligned} L_n^- &:= \frac{1}{n}D_x, \\ L_n^+ &:= x-\frac{\alpha}{2}+yjD_x^{(j-1)}-\alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{n-k}, \\ \mathcal{L}_n^- &:= \frac{1}{n}D_x^{1-j}D_y, \\ \mathcal{L}_n^+ &:= \left(x-\frac{\alpha}{2}\right)+yjD_x^{-(j-1)^2}D_y^{j-1} \\ &\quad -\alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{-(j-1)(n-k)}D_y^{n-k}. \end{aligned}$$

Differential, *integro-differential and partial differential equations are*

$$\begin{aligned} &\Bigg[\left(x-\frac{\alpha}{2}\right)D_x+yjD_x^j \\ &\quad -\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{n+1-k}-n\Bigg]B_n^{(a,j)}(x,y)=0, \end{aligned} \tag{28}$$

$$\begin{aligned} &\Bigg[\left(x-\frac{\alpha}{2}\right)D_y+jD_x^{-(j-1)^2}D_y^{j-1}+yjD_x^{-(j-1)^2}D_y^j \\ &\quad -\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1}-(n+1)D_x^{j-1}\Bigg]B_n^{(a,j)}(x,y)=0, \end{aligned} \tag{29}$$

$$\begin{aligned} &\Bigg[\left(x-\frac{\alpha}{2}\right)D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y \\ &\quad +jD_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y) \\ &\quad -\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)D_x^{n(j-1)}\Bigg]B_n^{(a,j)}(x,y)=0, \quad n\geq j. \end{aligned} \tag{30}$$

Remark 8 Taking $\alpha = 1$ in the above Corollary, one can get Theorem 1.1 of [10].

The differential equation of one variable Bernoulli polynomials was obtained in [2]. On the other hand, the differential equation of the generalized Bernoulli polynomials was given in [7]. For this reason, and for the sake of completeness, we list the recurrence relation, shift operators, differential equations for the two dimensional generalized Bernoulli polynomials in the case $c = e, \alpha = 1, j = 2$ in the following corollary. (Note that the following corollary was recorded in [10].)

Corollary 9 *Recurrence relation of the* 2*D Bernoulli polynomials is written as*

$$\begin{aligned} B_0^{(2)}(x,y) &= 1, \\ B_{n+1}^{(2)}(x,y) &= \left(x - \frac{1}{2}\right) B_n^{(2)}(x,y) \\ &\quad + 2nyB_{n-1}^{(2)}(x,y) - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k^{(2)}(x,y) B_{n+1-k}. \end{aligned}$$

Shift operators are given by

$$\begin{aligned} L_n^- &:= \frac{1}{n} D_x, \\ L_n^+ &:= x - \frac{1}{2} + 2yD_x - \sum_{k=0}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{n-k}, \\ \mathcal{L}_n^- &:= \frac{1}{n} D_x^{-1} D_y, \\ \mathcal{L}_n^+ &:= \left(x - \frac{1}{2}\right) + 2yD_x^{-1}D_y - \sum_{k=0}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{-(n-k)} D_y^{n-k}. \end{aligned}$$

Differential equation is

$$\left[\left(x - \frac{1}{2}\right) D_x + 2yD_x^2 - \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} D_x^{n+1-k} - n\right] B_n^{(2)}(x,y) = 0,$$

integro-differential equation is given by

$$\begin{aligned} &\left[\left(x - \frac{1}{2}\right) D_y + 2D_x^{-1} D_y + 2yD_x^{-1} D_y^2 \right. \\ &\quad \left. - \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} D_x^{-(n-k)} D_y^{n-k+1} - (n+1) D_x \right] B_n^{(2)}(x,y) = 0, \end{aligned}$$

partial differential equation is written as

$$\begin{aligned} &\left[\left(x - \frac{1}{2}\right) D_x^{(n-1)} D_y + (n-1) D_x^{(n-2)} D_y + 2D_x^{(n-2)} D_y (1 + yD_y) \right. \\ &\quad \left. - \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} D_x^{(k-1)} D_y^{n-k+1} - (n+1) D_x^n \right] B_n^{(2)}(x,y) = 0, \quad n \geq 2. \end{aligned}$$

3 Euler polynomials

In this section, we study the Euler polynomials and the equations satisfied by extended $2D$ Euler polynomials. Since the extended $2D$ Euler differential equations have not been studied before, the results are new even in the cases $c = e$, $\alpha = 1$. The proof is very similar as in the previous section, therefore, we only exhibit the results.

Theorem 10 *The recurrence relation of the extended* $2D$ *Euler polynomials is given by*

$$\begin{aligned} E_{n+1}^{(\alpha,j)}(x,y,c) &= \left(x\ln c - \frac{\alpha}{2}\right)E_n^{(\alpha,j)}(x,y,c) + yjE_{n-j+1}^{(\alpha,j)}(x,y,c)\frac{n!}{(n-j+1)!}(\ln c) \\ &\quad + \frac{\alpha}{2}\sum_{k=0}^{n-1}\binom{n}{k}e_{n-k}E_k^{(\alpha,j)}(x,y,c). \end{aligned} \tag{31}$$

Shift operators are given by

$$\begin{aligned} L_n^- &:= \frac{1}{n\ln c}D_x, \\ L_n^+ &:= x\ln c - \frac{\alpha}{2} + yj(\ln c)^{2-j}D_x^{j-1} + \frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{e_{n-k}}{(n-k)!}(\ln c)^{k-n}D_x^{n-k}, \\ \mathcal{L}_n^- &:= \frac{(\ln c)^{j-2}}{n}D_x^{1-j}D_y, \\ \mathcal{L}_n^+ &:= \left(x\ln c - \frac{\alpha}{2}\right) + yj(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^{j-1} \\ &\quad + \frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{e_{n-k}}{(n-k)!}(\ln c)^{(n-k)(j-2)}D_x^{-(n-k)(j-1)}D_y^{n-k}. \end{aligned}$$

Differential, integro-differential and partial differential equations are as follows, respectively:

$$\begin{aligned} &\Bigg[\left(x - \frac{\alpha}{2\ln c}\right)D_x + yj(\ln c)^{1-j}D_x^j \\ &\quad + \frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}(\ln c)^{k-n-1}D_x^{n-k+1} - n\Bigg]E_n^{(\alpha,j)}(x,y,c) = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} &\Bigg[\left(x\ln c - \frac{\alpha}{2}\right)D_y + (\ln c)^{(j-1)(j-2)+1}jD_x^{-(j-1)^2}D_y^{j-1} + yj(\ln c)^{(j-1)(j-2)+1}D_x^{-(j-1)^2}D_y^j \\ &\quad + \frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}(\ln c)^{(j-2)(n-k)}D_x^{-(j-1)(n-k)}D_y^{n-k+1} - (n+1)(\ln c)^{2-j}D_x^{j-1}\Bigg] \\ &\quad \times E_n^{(\alpha,j)}(x,y,c) = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &\Bigg[\left(x\ln c - \frac{\alpha}{2}\right)D_x^{(j-1)(n-1)}D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y \\ &\quad + (\ln c)^{(j-1)(j-2)+1}jD_x^{(j-1)(n-j)}\left(D_y^{j-1} + yD_y^j\right) \end{aligned}$$

$$+\frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}(\ln c)^{(j-2)(n-k)}D_x^{(j-1)(k-1)}D_y^{n+1-k}-(n+1)(\ln c)^{2-j}D_x^{(j-1)n}\Bigg]$$
$$\times E_n^{(\alpha,j)}(x,y,c)=0,\quad n\geq j. \tag{34}$$

Since the case $c=e$ reduces to the generalized $2D$ Euler polynomials, we thus have the following corollary.

Corollary 11 *For the generalized* $2D$ *Euler polynomials, we have the recurrence*:

$$E_{n+1}^{(\alpha,j)}(x,y)=\left(x-\frac{\alpha}{2}\right)E_n^{(\alpha,j)}(x,y)+yjE_{n-j+1}^{(\alpha,j)}(x,y)\frac{n!}{(n-j+1)!}$$
$$+\frac{\alpha}{2}\sum_{k=0}^{n-1}\binom{n}{k}e_{n-k}E_k^{(\alpha,j)}(x,y).$$

Shift operators:

$$L_n^-:=\frac{1}{n}D_x,$$
$$L_n^+:=x-\frac{\alpha}{2}+yjD_x^{j-1}+\frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{n-k},$$
$$\mathcal{L}_n^-:=\frac{1}{n}D_x^{1-j}D_y,$$
$$\mathcal{L}_n^+:=\left(x-\frac{\alpha}{2}\right)+yjD_x^{-(j-1)^2}D_y^{j-1}+\frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{-(n-k)(j-1)}D_y^{n-k}.$$

Differential, integro-differential and partial differential equations:

$$\Bigg[\left(x-\frac{\alpha}{2}\right)D_x+yjD_x^j$$
$$+\frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{n-k+1}-n\Bigg]E_n^{(\alpha,j)}(x,y)=0,$$
$$\Bigg[\left(x-\frac{\alpha}{2}\right)D_y+jD_x^{-(j-1)^2}D_y^{j-1}+yjD_x^{-(j-1)^2}D_y^j$$
$$+\frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1}-(n+1)D_x^{j-1}\Bigg]E_n^{(\alpha,j)}(x,y)=0,$$
$$\Bigg[\left(x-\frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y+jD_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y)$$
$$+\frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)D_x^{(j-1)n}\Bigg]E_n^{(\alpha,j)}(x,y)=0;\quad n\geq j.$$

Note that as it is noticed before, even the case $\alpha=1$ has not been studied before. The interested reader can obtain this case as a consequence of the above corollary.

4 Concluding remarks

As it is mentioned in the Introduction section, generalized Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda;a;b;c)$ and generalized Euler polynomials $E_n^{(\alpha)}(x;\lambda;a;b;c)$ of order $\alpha \in \mathbb{C}$ were constructed by the following generating relations, respectively [17]:

$$\frac{t^\alpha}{(\lambda b^t - a^t)^\alpha} c^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!},$$
$$\frac{2^\alpha}{(\lambda b^t + a^t)^\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!},$$

where $a,b,c \in \mathbb{R}^+$ $(a \neq b)$ $n \in \mathbb{N}_0$. Using factorization method, differential equations can be obtained for these polynomials.

On the other hand, introducing the two variable polynomial families

$$\frac{t^\alpha}{(\lambda b^t - a^t)^\alpha} c^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y;\lambda;a,b,c)\frac{t^n}{n!},$$

and

$$\frac{2^\alpha}{(\lambda b^t + a^t)^\alpha} c^{xt+yt^j} = \sum_{n=0}^{\infty} E_n^{(\alpha,j)}(x,j;\lambda;a,b,c)\frac{t^n}{n!},$$

where $a,b,c \in \mathbb{R}^+$ $(a \neq b)$ $n \in \mathbb{N}_0$, following the lines of proof given in Section 2, the differential, integro-differential and partial differential equations can be given for these families. Future works are left to the interested readers.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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