On Continued Fractions

Mahmoud Jafari Shah Belaghi

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
January 2013
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Elvan Yılmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Agamiza Bashirov
Supervisor

Examining Committee

1. Prof. Dr. Adiguzel Dosiyev
2. Prof. Dr. Agamiza Bashirov
3. Prof. Dr. Alexey Lukashov
4. Prof. Dr. Oktay Veliev
5. Assoc. Prof. Dr. Mehmet Ali Özarslan

ABSTRACT

In this thesis we concern two problems related to continued fractions.

Euler's differential method: we apply Euler's differential method, which was not used by mathematicians for a long time, to derive a new formula for a certain kind continued fraction depending on a parameter. This formula is in the form of the ratio of two integrals. In case of integer values of the parameter, the formula reduces to the ratio of two finite sums. Asymptotic behavior of this continued fraction is investigated numerically and it is shown that it increases in the same rate as the root function.

Bauer-Muir transform: we define a transformation of a certain kind of continued fractions to the same kind of continued fractions. This transformation is obtained by multiple application of the Bauer-Muir transform and then using the limiting process. It is shown that a double application of this transformation is the identity transformation. The obtained result is applied to some classic continued fractions due to Euler and Ramanujan. As a result a new transformation was found which in some special cases infinite continued fraction can be transformed to finite continued fraction.

Keywords: Continued fractions, Euler’s differential method, Bauer-Muir transform.

ÖZ

Bu tezde sürekli kesirlerle alakalı iki konu çalışıldı.

Euler'in differensiyel metodu: Matematikçilerin uzun zamandır kullanmadığı Euler diferensiyel metodunu kullanarak, bir parametreye bağlı sürekli kesirler için yeni bir formül bulundu. Bu formül iki integralin oranı formundadır. Parametrelerin tam sayı olduğu durumlarda bu formül iki sonlu toplamın oranı şeklinde değişir. Bu sürekli kesirlerin asimptotik davranışları üzerinde yapılan sayısal çalışmalar sonunda, kök fonksiyonu ile aynı oranda büyüdükleri görüldü.

Bauer-Muir dönüşümü: Belirli bir türden olan sürekli kesirleri yine aynı türe çeviren bir dönüşüm tanımlandı. Bu dönüşüm, birçok kez Bauer-Muir dönüşümü ve daha sonra limit işlemleri uygulanarak bulundu. Dönüşümün iki kez uygulandığı durumlarda birim dönüşüm elde edildiği görüldü. Elde edilen dönüşüm Euler ve Ramanujan'ın sürekli kesirlerine uygulandı. Sonuç olarak, belirli parametreler için sonsuz sürekli kesirleri sonlu sürekli kesirlere çeviren bir dönüşüm bulundu.

Anahtar Kelimeler: Sürekli kesirlerle, Euler diferensiyel metodunu, Bauer-Muir dönüşümü.

ACKNOWLEDGMENTS

I would like to express my appreciation to my supervisor Prof. Dr. Agamirza Bashirov. I benefited a lot from your his supervision, critical comments and advice.

I would also like to thank Prof. Dr. Sergey Khrushchev for providing me the freedom to work on my own. This helped me to hone my skills to work independently. He constantly inspired me throughout the whole period and his support was inevitable for the success of this dissertation.

This is the time to thank all members of the Department of Mathematics for providing a friendly atmosphere.

Last, but not least, I would like to thank my wife Armina for her understanding and love during the past few years. Her support and encouragement was in the end what made this dissertation possible.

Mahmoud J. S. Belaghi

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

1.1 Historical Background

Continued fractions belong to classical research areas of mathematics. Elements of continued fraction were used in Euclid's Elements (300B.C.). Indian mathematician Aryabhata (475−550) used continued fractions to solve linear equations. Specific examples of continued fractions were considered by Italian mathematicians Rafael Bombelli and Pietro Cataldi in the 16th century. An intensive study of continued fractions started since John Wallis and Lord Brouncker. Many great mathematicians such as Leonhard Euler, Karl Jacobi, Oscar Perron, Charles Hermit, Karl Friderich Gauss, Augustin Cauchy, Thomas Stieltjes etc. investigated continued fractions. We refer to the books Cuyt et al [6], Jones and Thron [15], Khrushchev [16], Lorentzen and Waadeland [18], Olds [23], Wall [30], etc. for the present state of the theory.

In this section we will see how a continued fraction appears. Continued fraction is used when there is repeated division and also can be used for solving equations.

- **Continued fractions appear by "repeated divisions".** Take for instance, $\frac{243}{88}$, we can write $\frac{243}{88}$ as: $\frac{243}{88} = 2 + \frac{67}{88}$. By inverting the fraction $\frac{67}{88}$, we will have

$$\frac{243}{88} = 2 + \frac{1}{\frac{88}{67}}.$$

Repeating the above process for $\frac{88}{67}$, give us $\frac{88}{67} = 1 + \frac{21}{67} = 1 + \frac{1}{67/21}$. Therefore we can write $\frac{243}{88}$ in the different form

$$\frac{243}{88} = 2 + \frac{1}{1 + \frac{1}{\frac{67}{21}}}.$$

Again we can rewrite $\frac{67}{21}$ as $\frac{67}{21} = 3 + \frac{1}{21/4}$. Hence

$$\frac{243}{88} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{\frac{21}{4}}}}.$$

Since $\frac{21}{4} = 5 + \frac{1}{4}$, we will have

$$\frac{243}{88} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{4}}}}.$$

Since 4 is an integer number, this process stops.

- **Continued fractions appear by solving equations.** Let us try to solve the equation $x^2 - 2x - 3 = 0$, (3 is the only positive solution). We can write $x^2 - 2x - 3 = 0$ as $x^2 = 2x + 3$ and dividing the both sides by x, we get

$$x = 2 + \frac{3}{x} \quad \text{or, since } x = 3, \qquad 3 = 2 + \frac{3}{x}.$$

Since $x = 2 + 3/x$, by substituting $2 + 3/x$ into the denominator of $3 =$

$2+3/x$, we obtain

$$3 = 2 + \cfrac{3}{2 + \frac{3}{x}}.$$

By iterating, we have

$$3 = 2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \ddots + \frac{3}{x}}}}.$$

Repeating this process infinity many times, we get

$$3 = 2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \ddots}}}}.$$

Following example (see [16], page 18) show us how we can write the **golden ratio** ($\Phi = \frac{1+\sqrt{5}}{2}$) as continued fractions form.

Example 1.1.1 *Consider the equation $x^2 - x - 1 = 0$. Since $\frac{1+\sqrt{5}}{2}$ is the only positive solution, we can write $\Phi^2 - \Phi - 1 = 0$.*

By rewriting the equation, we will get $\Phi = 1 + \frac{1}{\Phi}$ and by substituting Φ into the denominator of right-hand side. By iterating this process infinity many times, we can write

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}.$$

Remark 1.1.2 *Many false rumours have been spread around about the golden ratio[20].*

1.2 Definition of Continued Fractions.

Let $\{x_k\}_{k=0}^{n}$ be a decreasing finite sequence of all positive integers, such that

$$\begin{aligned} x_0 &= b_0x_1 + x_2, \\ x_1 &= b_1x_2 + x_3, \\ x_2 &= b_2x_3 + x_4, \\ &\vdots \\ x_{n-2} &= b_{n-2}x_{n-1} + x_n, \\ x_{n-1} &= b_{n-1}x_n, \end{aligned} \tag{1.2.1}$$

where $b_j \in \mathbb{N},\ j = 0,\ 1,\ \ldots$. Eliminating x_k from (1.2.1) we obtain

$$\frac{x_{k-1}}{x_k} = b_{k-1} + \frac{1}{\frac{x_k}{x_{k+1}}}, \quad k = 1,\ 2,\ \ldots,$$

which transform x_0/x_1 into a **finite simple continued fraction**

$$\frac{x_0}{x_1} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots + \cfrac{1}{b_{n-1}}}}.$$

The following notation was proposed by Rogers [28] which is written in line form:

$$\frac{x_0}{x_1} = b_0 + \frac{1}{b_1}{}_{+}\frac{1}{b_2}{}_{+\ldots+}\frac{1}{b_{n-1}}.$$

By multiplying nonzero coefficients a_j to the x_{j+1} on the right-hand side of (1.2.1) and letting the number of equations be infinite, we obtain

$$\begin{aligned} x_0 &= b_0 x_1 + a_1 x_2 \\ x_1 &= b_1 x_2 + a_2 x_3 \\ x_2 &= b_2 x_3 + a_3 x_4 \\ &\vdots \end{aligned} \tag{1.2.2}$$

Eliminating x_k, from (1.2.2) we obtain

$$\frac{x_{k-1}}{x_k} = b_{k-1} + \frac{a_k}{\frac{x_k}{x_{k+1}}}, \quad k = 1,\, 2,\, \ldots,$$

which transform x_0/x_1 into a general **continued fraction**

$$\frac{x_0}{x_1} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}},$$

and by using the Rogers' notation we can write it as

$$\frac{x_0}{x_1} = b_0 + \frac{a_1}{b_1}{}_{+}\frac{a_2}{b_2}{}_{+}\frac{a_3}{b_3}{}_{+\ldots}.$$

In general, we can give a definition of continued fraction (see [16], page11) which is a fraction as follows

$$b_0 + K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{1.2.3}$$

where b_k's and a_k's are real numbers. Here K stands for **Kettenbruch**, the German word for "continued fraction". This is probably the most compact and convenient way to express continued fractions. The numbers a_k and b_k are called the **k-th partial numerators** and **denominators** of (1.2.3), respectively.

More precisely, for every $n \in \mathbb{N}$, we can stop the process in (1.2.3) at the term a_n/b_n and perform all algebraic operations without cancellations. Then

$$c_n = \frac{P_n}{Q_n} \equiv b_0 + K_{k=1}^{n}\left(\frac{a_k}{b_k}\right) \tag{1.2.4}$$

is called the **n-th convergent** to the continued fraction (1.2.3). The continued fraction converges if the limit $\lim_{n\to\infty} c_n$, exists and is finite.

Simple (or regular) continued fraction is a continued fraction where all a_k's are 1 and b_k's are positive integers.

Positive continued fraction is a continued fraction where a_n, b_n's are positive real numbers for all $n \geq 1$.

Nonnegative continued fraction is a continued fraction where $b_n > 0$, $a_n \geq 0$ for all $n \geq 1$.

1.3 Convergents and Recurrence Relations.

Consider the continued fraction

$$b_0 + K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right), \tag{1.3.1}$$

with convergents $\{P_n/Q_n\}_{n\geq 0}$. Therefore the sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ satisfy the Euler-Wallis formulas (see Euler [8] and Wallis [31])

$$\begin{cases} P_n = b_n P_{n-1} + a_n P_{n-2} \\ \\ Q_n = b_n Q_{n-1} + a_n Q_{n-2} \quad , \end{cases} \tag{1.3.2}$$

where $P_{-1} = 1$, $P_0 = b_0$, $Q_{-1} = 0$ and $Q_0 = 1$. The zero-th and first convergents of the continued fraction (1.3.1) are $c_0 = b_0 = \frac{P_0}{Q_0}$ and $c_1 = b_0 + \frac{a_1}{b_1} = \frac{b_1 b_0 + a_1}{b_1} = \frac{P_1}{Q_1}$, respectively.

Theorem 1.3.1 *[see[16], page 12] Let $\{\frac{P_n}{Q_n}\}_{n\geq 1}$ be a sequence of convergents of the following continued fraction*

$$b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\frac{a_3}{b_3 +\cdots}\frac{a_n}{\xi},$$

where ξ is any positive real number. Then we have

$$b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\frac{a_3}{b_3 +\cdots}\frac{a_n}{\xi} = \frac{\xi P_{n-1} + a_n P_{n-2}}{\xi Q_{n-1} + a_n Q_{n-2}}, \quad n = 1, 2, 3, \ldots. \tag{1.3.3}$$

The equation (1.3.3) can be proven by induction. For the case $n = 1$;

$$b_0 + \frac{a_1}{x} = \frac{b_0 x + a_1}{x} = \frac{x P_0 + a_1 P_{-1}}{x Q_0 + a_1 Q_{-1}},$$

since $P_{-1} = 1$, $P_0 = b_0$, $Q_{-1} = 0$ and $Q_0 = 1$. Assume that (1.3.3) is true for the case n.

For the case $n+1$, we have

$$b_0 + \frac{a_1}{b_1} \, {}_{+} \frac{a_2}{b_2} \, {}_{+} \frac{a_3}{b_3} \, {}_{+ \dots +} \frac{a_n}{b_n} \, {}_{+} \frac{a_{n+1}}{x} = b_0 + \frac{a_1}{b_1} \, {}_{+} \frac{a_2}{b_2} \, {}_{+} \frac{a_3}{b_3} \, {}_{+ \dots +} \frac{a_n}{y}$$

where $y := b_n + \frac{a_{n+1}}{x} = \frac{xb_n + a_{n+1}}{x}$.

Then by induction hypothesis, we have

$$\begin{aligned}
b_0 + \frac{a_1}{b_1} \, {}_{+} \frac{a_2}{b_2} \, {}_{+} \frac{a_3}{b_3} \, {}_{+ \dots} \frac{a_n}{b_n} \, {}_{+} \frac{a_{n+1}}{x} &= \frac{yP_{n-1} + a_n P_{n-2}}{yQ_{n-1} + a_n Q_{n-2}} \\
&= \frac{\left(\frac{xb_n + a_{n+1}}{x}\right) P_{n-1} + a_n P_{n-2}}{\left(\frac{xb_n + a_{n+1}}{x}\right) Q_{n-1} + a_n Q_{n-2}} \\
&= \frac{x(b_n P_{n-1} + a_n P_{n-2}) + a_{n+1} P_{n-1}}{x(b_n Q_{n-1} + a_n Q_{n-2}) + a_{n+1} Q_{n-1}} \\
&= \frac{xP_n + a_{n+1} P_{n-1}}{xQ_n + a_{n+1} Q_{n-1}}.
\end{aligned}$$

Theorem 1.3.2 *The following identities hold (see [16], page* 14*):*

$$\begin{cases} P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 a_2 \cdots a_n & \text{for } n \geq 1, \\ \\ P_n Q_{n-2} - P_{n-2} Q_n = (-1)^n a_1 a_2 \cdots a_{n-1} b_n & \text{for } n \geq 2. \end{cases} \tag{1.3.4}$$

First equality in (1.3.4) can be proven by induction. For the case $n = 1$, we obtain

$$P_1 Q_0 - P_0 Q_1 = (b_1 b_0 + a_1).1 - b_0 b_1 = a_1.$$

Assume that the equality holds for the case n. We have to show that it also holds for the case $n+1$. By Euler-Wallis Formulas (1.3.2), we obtain

$$\begin{aligned} P_{n+1}Q_n - P_nQ_{n+1} &= Q_n(b_{n+1}P_n + a_{n+1}P_{n-1}) - P_n(b_{n+1}Q_n + a_{n+1}Q_{n-1}) \\ &= -a_{n+1}(P_nQ_{n-1} - P_{n-1}Q_n) \\ &= -a_{n+1}.(-1)^{n-1}a_1a_2\cdots a_n \\ &= (-1)^n a_1a_2\cdots a_na_{n+1}. \end{aligned}$$

Similarly, we can prove the second equality in (1.3.4).

Corollary 1.3.3 *[13] Consider the simple continued fraction*

$$b_0 + K_{k=1}^{\infty}\left(\frac{1}{b_k}\right),$$

with convergents $\{\frac{P_n}{Q_n}\}_{n\geq 0}$. *The* P_n *and* Q_n *are relatively prime for all* $n \geq 0$.

This corollary can be proved by use of Euler-Wallis formula (1.3.2) for simple continued fraction. Indeed, we have

$$P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}.$$

If there exist any integer number to divide both P_n and Q_n, then it must divide $P_nQ_{n-1} - P_{n-1}Q_n$ also, therefore it must divide $(-1)^{n-1} = \pm 1$.

Theorem 1.3.4 *(see [16], page 14) Let* $\{\frac{P_n}{Q_n}\}_{n\geq 0}$ *be a sequence of convergents of* $b_0 + K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right)$*, then* P_n *and* Q_n *satisfy*

$$\begin{aligned}\frac{P_n}{P_{n-1}} &= b_n + \frac{a_n}{b_{n-1}} {}_{+} \frac{a_{n-1}}{b_{n-2}} {}_{+\ldots+} \frac{a_1}{b_0},\\ \frac{Q_n}{Q_{n-1}} &= b_n + \frac{a_n}{b_{n-1}} {}_{+} \frac{a_{n-1}}{b_{n-2}} {}_{+\ldots+} \frac{a_2}{b_1}.\end{aligned}$$

This theorem can be proved by applying Euler-Wallis formulas (1.3.2) to the left-hand sides of the equations, iteratively.

1.4 Transformation of Continued Fractions.

It this section we will see how one continued fraction can be transformed to another one. The following example shows this transformation.

Example 1.4.1 *Consider the finite continued fraction*

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}},$$

where b_k*'s and* a_k*'s are real numbers. Assume that* ρ_1, ρ_2, ρ_3 *are nonzero real numbers. Then multiplying the numerator and denominator of the first fraction by* ρ_1*, we obtain*

$$\xi = b_0 + \frac{\rho_1 a_1}{\rho_1 b_1 + \frac{\rho_1 a_2}{b_2 + \frac{a_3}{b_3}}}.$$

Multiplying the numerator and denominator of the second fraction by ρ_2 gives

$$\xi = b_0 + \frac{\rho_1 a_1}{\rho_1 b_1 + \frac{\rho_1 \rho_2 a_2}{\rho_2 b_2 + \frac{\rho_2 a_3}{b3}}}.$$

Finally, multiplying the numerator and denominator of the last fraction by ρ_3 gives

$$\xi = b_0 + \frac{\rho_1 a_1}{\rho_1 b_1 + \frac{\rho_1 \rho_2 a_2}{\rho_2 b_2 + \frac{\rho_2 \rho_3 a_3}{\rho_3 b_3}}}.$$

In summary,

$$b_0 + \frac{a_1}{b_1}{}_{+}\frac{a_2}{b_2}{}_{+}\frac{a_3}{b_3} = b_0 + \frac{\rho_1 a_1}{\rho_1 b_1}{}_{+}\frac{\rho_1 \rho_2 a_2}{\rho_2 b_2}{}_{+}\frac{\rho_2 \rho_3 a_3}{\rho_3 b_3}.$$

In general, Two continued fractions are said to be ***equivalent*** *if and only if they have the same sequence of convergents. This example leads us to the following theorem (see [15], page* 31*).*

Theorem 1.4.2 *(**Equivalence Transform**). For any real numbers $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ and nonzero constants $\{\rho_k\}_{k=1}^{\infty}$, the following equation holds:*

$$b_0 + \frac{a_1}{b_1}{}_{+}\frac{a_2}{b_2}{}_{+}\frac{a_3}{b_3}{}_{+\cdots}\frac{a_n}{b_n}{}_{+\cdots} = b_0 + \frac{\rho_1 a_1}{\rho_1 b_1}{}_{+}\frac{\rho_1 \rho_2 a_2}{\rho_2 b_2}{}_{+}\frac{\rho_2 \rho_3 a_3}{\rho_3 b_3}{}_{+\cdots}\frac{\rho_{n-1} \rho_n a_n}{\rho_n b_n}{}_{+\cdots}.$$

By using Euler-Wallis formulas (1.3.2) one can prove the theorem by simple induction.

1.5 Convergence Theorems

Consider the following continued fraction

$$b_0 + K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right).$$

By using the Theorem 1.3.2, which is

$$\begin{cases} P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}a_1a_2\cdots a_n \\ P_nQ_{n-2} - P_{n-2}Q_n = (-1)^{n}a_1a_2\cdots a_{n-1}b_n, \end{cases}$$

we will prove the following monotonicity properties of the c_n's, which is an important part in the study of continued fractions (see [16], page 14).

Theorem 1.5.1 *Convergents of positive continued fractions, satisfy the following inequalities:*

$$c_0 < c_2 < c_4 < \cdots < c_{2n} < \cdots < c_{2n-1} < \cdots < c_5 < c_3 < c_1.$$

That is, the sequence of even and odd convergents are strictly increasing and decreasing, respectively.

By using Theorem 1.3.2, we can expand the following equation for $c_n - c_{n-2}$,

$$
\begin{aligned}
c_n - c_{n-2} &= \frac{P_n}{Q_n} - \frac{P_{n-2}}{Q_{n-2}} \\
&= \frac{P_n Q_{n-2} - P_{n-2} Q_n}{Q_n Q_{n-2}} \\
&= \frac{(-1)^n a_1 a_2 \cdots a_{n-1} b_n}{Q_n Q_{n-2}}.
\end{aligned}
$$

By substituting $2n$ instead of n in the above equation for $c_n - c_{n-2}$, we obtain

$$
c_{2n} - c_{2n-2} = \frac{(-1)^{2n} a_1 a_2 \cdots a_{2n-1} b_{2n}}{Q_{2n} Q_{2n-2}} = \frac{a_1 a_2 \cdots a_{2n-1} b_{2n}}{Q_{2n} Q_{2n-2}} > 0.
$$

This shows that $c_{2n-2} < c_{2n}$ for all $n \geq 1$ and hence, $c_0 < c_2 < c_4 < \cdots$.

Similarly, by substituting $2n-1$ instead of n in the above equation for $c_n - c_{n-2}$, we will have $c_{2n} < c_{2n-1}$ for all $n \geq 1$ and hence $c_1 > c_3 > c_5 > \cdots$.

The easiest way to prove the convergence is to apply a simple old theorem by Pringsheim [27].

Theorem 1.5.2 *Let $b_n > 0$, $a_n > 0$ for all $n \geq 1$, and*

$$
\sum_{n=1}^{\infty} \frac{b_n b_{n+1}}{a_{n+1}} = \infty.
$$

Then the continued fraction $b_0 + K_{k=1}^{\infty} \left(\frac{a_k}{b_k} \right)$ converges.

This theorem can be proved by use of Euler-Wallis formulas (1.3.2). Indeed, we have

$$Q_{n-1} = b_{n-1}Q_{n-2} + a_{n-1}Q_{n-3} \geq b_{n-1}Q_{n-2},$$

since $a_{n-1}Q_{n-3} \geq 0$. By using Euler-Wallis formulas (1.3.2), for $n \geq 2$ we can write

$$\begin{aligned} Q_n &= b_nQ_{n-1} + a_nQ_{n-2} \\ &\geq b_n(b_{n-1}Q_{n-2}) + a_nQ_{n-2} \\ &= Q_{n-2}(b_nb_{n-1} + a_n). \end{aligned}$$

Therefore, we have

$$Q_n \geq Q_{n-2}(b_nb_{n-1} + a_n), \quad \text{for all } n \geq 2.$$

Applying this formula n-times, we find that for any $n \geq 1$,

$$\begin{aligned} Q_{2n} &\geq Q_{2n-2}(b_{2n}b_{2n-1} + a_{2n}) \\ &\geq Q_{2n-4}(b_{2n-2}b_{2n-3} + a_{2n-2})(b_{2n}b_{2n-1} + a_{2n}) \\ &\geq \vdots \\ &\geq Q_0(b_2b_1 + a_2)(b_4b_3 + a_4)\cdots(b_{2n}b_{2n-1} + a_{2n}). \end{aligned}$$

Similarly, we can find that for any $n \geq 2$,

$$Q_{2n-1} \geq Q_1(b_3b_2 + a_3)(b_5b_4 + a_5)\cdots(b_{2n-1}b_{2n-2} + a_{2n-1}).$$

Therefore, for any $n \geq 2$, we have

$$\begin{aligned} Q_{2n}Q_{2n-1} &\geq Q_0Q_1(b_2b_1 + a_2)(b_3b_2 + a_3)\cdots(b_{2n-1}b_{2n-2} + a_{2n-1})(b_{2n}b_{2n-1} + a_{2n}) \\ &= Q_0Q_1a_2a_3\cdots a_{2n}\Big(1 + \frac{b_2b_1}{a_2}\Big)\Big(1 + \frac{b_3b_2}{a_3}\Big)\cdots\Big(1 + \frac{b_{2n}b_{2n-1}}{a_{2n}}\Big). \end{aligned}$$

By easy simplification we can write that

$$\frac{a_1a_2\cdots a_{2n}}{Q_{2n}Q_{2n-1}} \leq \frac{a_1}{Q_0Q_1}.\frac{1}{\prod_{k=1}^{2n-1}\Big(1 + \frac{b_kb_{k+1}}{a_{k+1}}\Big)}.$$

Now recall that a series $\sum_{k=1}^{\infty}\alpha_k$ of positive real numbers converges if and only if the infinite product $\prod_{k=1}^{\infty}(1+\alpha_k)$ converges. Therefore, $\prod_{k=1}^{\infty}\Big(1 + \frac{b_kb_{k+1}}{a_{k+1}}\Big) = \infty$, since $\sum_{k=1}^{\infty}\frac{b_kb_{k+1}}{a_{k+1}} = \infty$ is given. so the right-hand side of inequality vanishes as $n \to \infty$.

Corollary 1.5.3 *(see [16], page* 20*) Simple continued fractions always converge.*

Consider a simple continued fraction, therefore

$$\sum_{n=1}^{\infty}\frac{b_nb_{n+1}}{a_{n+1}} = \sum_{n=1}^{\infty}b_nb_{n+1} = \infty,$$

since all the b_n's are positive integers and a_n's $= 1$.

Theorem 1.5.4 *[19] Let* $b_0 + K_{k=1}^{\infty}(\frac{a_k}{b_k})$ *be a positive continued fraction such that* $\sum_{n=1}^{\infty}\frac{b_nb_{n+1}}{a_{n+1}} = \infty$*. Let* ξ_n *is also defined as*

$$\xi_0 = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+\cdots}\frac{a_n}{\xi_n},$$

which is a positive infinitely often. Then ξ_0 and $b_0+K_{k=1}^{\infty}(\frac{a_k}{b_k})$ are equivalent.

By Theorem 1.5.2, the continued fraction $b_0+K_{k=1}^{\infty}(\frac{a_k}{b_k})$ converges, since $\sum_{n=1}^{\infty}\frac{b_n b_{n+1}}{a_{n+1}}=\infty$. Let $\{c_n\}_{n\geq 0}$ be a sequence of convergents of $b_0+K_{k=1}^{\infty}(\frac{a_k}{b_k})$. Let $\varepsilon>0$ and by using Theorem 1.5.2, observe that there exist an N such that

$$\forall\ n>N \quad \Longrightarrow \quad |c_n-c_{n-1}|=\frac{a_1a_2\cdots a_n}{Q_nQ_{n-1}}<\varepsilon.$$

Fix $n>N$ and write out ξ_o to the n-th term:

$$\xi_0=b_0+\frac{a_1}{b_1}{}_{+}\frac{a_2}{b_2}{}_{+}\frac{a_3}{b_3}{}_{+\cdots+}\frac{a_{n-1}}{b_{n-1}}{}_{+}\frac{a_n}{\xi_n}.$$

Let $\{c_k^{'}=\frac{P_k^{'}}{Q_k^{'}}\}$ be a sequence of convergents for this finite continued fraction. Therefore on can see that $Q_k=Q_k^{'}$ and $P_k=P_k^{'}$ for all $k\leq n-1$ and $\xi_0=c_n^{'}$. Then, by using Theorem 1.3.2, we can write

$$|\xi_0-c_{n-1}|=\left|c_n^{'}-c_{n-1}^{'}\right|=\frac{a_1a_2\cdots a_n}{Q_n^{'}Q_{n-1}^{'}}=\frac{a_1a_2\cdots a_n}{Q_n^{'}Q_{n-1}}.$$

By the Euler-Wallis formulas (1.3.2), we have

$$Q_n^{'}=\xi_nQ_{n-1}^{'}+a_nQ_{n-2}^{'}=\left(b_n+\frac{a_{n+1}}{\xi_{n+1}}\right)Q_{n-1}+a_nQ_{n-2}>b_nQ_{n-1}+a_nQ_{n-2}=Q_n.$$

Hence,

$$|\xi_0-c_{n-1}|\leq\frac{a_1a_2\cdots a_n}{Q_n^{'}Q_{n-1}}<\frac{a_1a_2\cdots a_n}{Q_nQ_{n-1}}<\varepsilon.$$

Therefore $\xi_0 = \lim c_{n-1}$, Since $\varepsilon > 0$ was arbitrary.

1.6 Continued Fractions and Some Series

Convergents of some continued fractions coincide with partial sums of series. This phenomenon was first studied in detail by Euler [7].

Theorem 1.6.1 *[5] The sequence $\{d_n\}_{n\geq 0}$ is the sequence of convergents to a continued fraction $q_0 + K_{n=1}^{\infty}(\frac{p_n}{q_n})$ if and only if $d_0 \neq \infty, d_n \neq d_{n-1}$, $n = 1, 2, 3, \ldots$.*

Let $d_n = \frac{P_n}{Q_n}$, $n = 0, 1, 2, \ldots$, be a sequence of convegents to a continued fraction, then $d_0 = q_0 \neq \infty$ and by Theorem 1.3.2 we have

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} p_1 p_2 \cdots p_n \neq 0.$$

Therefore Q_{n-1} and Q_n cannot both vanish. Similarly, if $Q_n = 0$ then $P_n \neq 0$. This shows that $d_n \neq d_{n-1}$.

To prove the converse we assume that the numerators of the convergents are d_n and the denominators of the convergents all equals 1. Let all the d_n be finite. Then the Euler-Wallis formulas (1.3.2) takes the form

$$\begin{aligned} d_n &= q_n d_{n-1} + p_n d_{n-2}, \\ 1 &= q_n + p_n. \end{aligned}$$

The determinant of this linear system in two unknowns p_n and q_n is $d_{n-1} - d_{n-2} \neq 0$. It follows that

$$p_n = \frac{d_{n-1} - d_n}{d_{n-1} - d_{n-2}}, \qquad q_n = \frac{d_n - d_{n-2}}{d_{n-1} - d_{n-2}}, \qquad n = 2, 3, \ldots . \tag{1.6.1}$$

The initial values are $q_0 = d_0$, $p_1 = d_1 - d_0$, $q_1 = 1$. If, say, $d_n = \infty$ then by the assumption both d_{n-1} and d_{n+1} are finite. We put $P_n = 1$, $Q_n = 0$ and by Euler-Wallis formulas (1.3.2) obtain the system

$$\begin{aligned} 1 &= q_n d_{n-1} + p_n d_{n-2}, \\ 0 &= q_n + p_n. \end{aligned}$$

The second equation shows that $q_n = -p_n$, and

$$p_n = \frac{1}{d_{n-1} - d_{n-2}}, \qquad q_n = \frac{1}{d_{n-1} - d_{n-2}}, \qquad n = 2, 3, \cdots$$

follows from the first.

Following theorems are given by Euler (see [7] and [11]) which are about relation between convergents of continued fraction and partial sums of a given series.

Theorem 1.6.2 *[11] Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $\alpha_k \neq 0$*

and $\alpha_k \neq \alpha_{k-1}$ *for all* k*, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_k} = \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 +} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 + \dots}.$$

The theorem can be proved by induction. One can show that the equality holds for the case $n = 1$. Assuming that it holds for the case n, we can prove that it holds for the case $n + 1$. we have

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} = \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + \frac{(-1)^{n-1}}{\alpha_n} + \frac{(-1)^n}{\alpha_{n+1}}.$$

By writing the last two terms together, we obtain sum of n terms.

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} &= \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + (-1)^{n-1}\Big(\frac{1}{\alpha_n - \alpha_{n+1}}\Big) \\ &= \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + (-1)^{n-1} \frac{1}{\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1} - \alpha_n}}.\end{aligned}$$

Now, we can apply the induction hypothesis and obtain

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} &= \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 +} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 + \dots} \frac{\alpha_{n-1}^2}{\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1} - \alpha_n} - \alpha_{n-1}} \\ &\quad \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 +} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 + \dots} \frac{\alpha_{n-1}^2}{\frac{\alpha_n(\alpha_{n+1} - \alpha_n) + \alpha_n^2}{\alpha_{n+1} - \alpha_n} - \alpha_{n-1}} \\ &\quad \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 +} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 + \dots} \frac{\alpha_{n-1}^2}{\alpha_n - \alpha_{n-1} + \frac{\alpha_n^2}{\alpha_{n+1} - \alpha_n}} \\ &\quad \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 +} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 + \dots} \frac{\alpha_{n-1}^2}{\alpha_n - \alpha_{n-1} +} \frac{\alpha_n^2}{\alpha_{n+1} - \alpha_n}.\end{aligned}$$

Theorem 1.6.3 *[11] Let* $\{\alpha_k\}_{k=1}^{\infty}$ *be a sequence of real numbers such that* $\alpha_k \neq$

$0, 1$. *The following equality holds*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} = \frac{1}{\alpha_1}{}_{+}\frac{\alpha_1}{\alpha_2 - 1}{}_{+}\frac{\alpha_2}{\alpha_3 - 1}{}_{+\cdots}\frac{\alpha_{n-1}}{\alpha_n - 1}{}_{+\cdots}.$$

The theorem can be proved by induction. One can show that the equality holds for the case $n = 1$. Assuming that it holds for the case n, we can prove that it holds for the case $n + 1$. We have

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} \quad = \quad \frac{1}{\alpha_1} - \frac{1}{\alpha_1 \alpha_2} + \cdots + \frac{(-1)^{n-1}}{\alpha_1 \cdots \alpha_n} + \frac{(-1)^n}{\alpha_1 \cdots \alpha_{n+1}}.$$

By writing the last two terms together, we obtain sum of n terms.

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} \quad &= \quad \frac{1}{\alpha_1} - \frac{1}{\alpha_1 \alpha_2} + \cdots + \frac{(-1)^{n-1}}{\alpha_1 \cdots \alpha_{n-1}} \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n \alpha_{n+1}} \right) \\
&= \quad \frac{1}{\alpha_1} - \frac{1}{\alpha_1 \alpha_2} + \cdots + \frac{(-1)^{n-1}}{\alpha_1 \cdots \alpha_{n-1}} \left(\frac{1}{\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1} - 1}} \right).
\end{aligned}$$

By applying the induction hypothesis, we obtain

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} = \frac{1}{\alpha_1}{}_{+}\frac{\alpha_1}{\alpha_2 - 1}{}_{+}\frac{\alpha_2}{\alpha_3 - 1}{}_{+\cdots}\frac{\alpha_{n-1}}{\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1} - 1} - 1}{}_{+\cdots}.$$

Since

$$\begin{aligned}
\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1} - 1} - 1 \quad &= \quad \frac{\alpha_n(\alpha_{n+1} - 1) + \alpha_n}{\alpha_{n+1} - \alpha_n} - 1 \\
&= \quad \alpha_n - 1 + \frac{\alpha_n}{\alpha_{n+1} - 1},
\end{aligned}$$

we have

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} = \frac{1}{\alpha_1} \underset{+}{} \frac{\alpha_1}{\alpha_2 - 1} \underset{+}{} \frac{\alpha_2}{\alpha_3 - 1} \underset{+\dots}{} \frac{\alpha_{n-1}}{\alpha_n - 1 + \frac{\alpha_n}{\alpha_{n+1}-1}} \underset{+\dots}{} .$$

Theorem 1.6.4 *[7] Let $\{c_n\}_{n\geq 0}$ be a sequence of nonzero numbers, then*

$$\sum_{k=0}^{n} c_k = \frac{c_0}{1} \underset{-}{} \frac{c_1/c_0}{1 + c_1/c_0} \underset{-}{} \frac{c_2/c_1}{1 + c_2/c_1} \underset{-\dots-}{} \frac{c_n/c_{n-1}}{1 + c_n/c_{n-1}}. \tag{1.6.2}$$

Apply Theorem 1.3.2 to $d_n = \sum_{k=0}^{n} c_k$, $n \geq 0$. Since $c_n \neq 0$, we have $d_n \neq d_{n-1}$ for $n = 1, 2, \ldots$. Next, $d_0 = c_0 \neq \infty$. Since $d_n \neq \infty$, formula (1.6.1) shows that

$$\begin{aligned} p_n &= \frac{d_n - d_{n-1}}{d_{n-2} - d_{n-1}} = -\frac{c_n}{c_{n-1}} \\ q_n &= \frac{d_n - d_{n-2}}{d_{n-1} - d_{n-2}} = \frac{c_n + c_{n-1}}{c_{n-1}} \qquad n = 2, 3, \ldots. \end{aligned}$$

Since $q_0 = d_0 = c_0$, $p_1 = d_1 - d_0 = c_1$, $q_1 = 1$, Theorem 1.6.1 shows that

$$\sum_{k=0}^{n} c_k = c_0 + \frac{c_1}{1} \underset{-}{} \frac{c_2/c_1}{1 + c_2/c_1} \underset{-}{} \frac{c_3/c_2}{1 + c_3/c_2} \underset{-\dots-}{} \frac{c_n/c_{n-1}}{1 + c_n/c_{n-1}}. \tag{1.6.3}$$

The application to (1.6.3) of the elementary identity

$$c_0 + \frac{c_1}{1 + w} = \frac{c_0}{1} \underset{-}{} \frac{c_1/c_0}{1 + c_1/c_0 + w}$$

proves (1.6.2).

If we put

$$\rho_0 = c_0, \qquad c_k = \rho_1\rho_2\cdots\rho_k, \quad \text{for } k = 1,\, 2,\, \ldots\,,$$

then (1.6.3) and (1.6.2) turn into the following formula

$$\sum_{k=0}^{n} \rho_0\rho_1\cdots\rho_k = \frac{\rho_0}{1}{}_{-}\frac{\rho_1}{1+\rho_1}{}_{-\,\ldots\,-}\frac{\rho_n}{1+\rho_n}.$$

Following example is the application of Theorem1.6.2,

Example 1.6.5 *[17]* ***Continued fraction for*** π*: Consider the following telescoping series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Big(\frac{1}{n}+\frac{1}{n+1}\Big) = \Big(\frac{1}{1}+\frac{1}{2}\Big) - \Big(\frac{1}{2}+\frac{1}{3}\Big) + \Big(\frac{1}{3}+\frac{1}{4}\Big) - \cdots = 1,$$

and

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = 1 - \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}.$$

By multiplying the second series by 4*, we get*

$$\pi = 4 - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1},$$

and using the first telescoping series, we obtain

$$\begin{aligned}
\pi &= 3+1-4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1} \\
&= 3+\sum_{n=1}^{\infty}(-1)^{n-1}\Big(\frac{1}{n}+\frac{1}{n+1}\Big)-4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1} \\
&= 3+\sum_{n=1}^{\infty}(-1)^{n-1}\Big(\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\Big) \\
&= 3+4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)}.
\end{aligned}$$

Now, We apply Theorem 1.6.2 with $\alpha_n = 2n(2n{+}1)(2n{+}2)$*. We can write the* k*-th partial denominator of continued fraction in Theorem 1.6.2 as* $\alpha_n - \alpha_{n-1} = 24n^2$*. Therefore by applying Theorem 1.6.2, we obtain*

$$4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)} = 4\Big(\frac{1}{24.(1^2)}\,{}_{+}\,\frac{(2.3.4)^2}{24.(2^2)}\,{}_{+}\,\frac{(4.5.6)^2}{24.(3^2)}\,{}_{+\cdots}\Big).$$

Hence,

$$\pi = 3+\frac{1}{6}\,{}_{+}\,\frac{(2.3.4)^2}{24.(2^2)}\,{}_{+}\,\frac{(4.5.6)^2}{24.(3^2)}\,{}_{+\cdots+}\,\frac{\Big(2(n-1)(2n-1)(2n)\Big)^2}{24.(n^2)}\,{}_{+\cdots}.$$

Using the equivalence transformation rule from Theorem 1.4.2:

$$b_0+\frac{a_1}{b_1}\,{}_{+}\,\frac{a_2}{b_2}\,{}_{+}\,\frac{a_3}{b_3}\,{}_{+\cdots+}\,\frac{a_n}{b_n}\,{}_{+\cdots} = b_0+\frac{\rho_1 a_1}{\rho_1 b_1}\,{}_{+}\,\frac{\rho_1\rho_2 a_2}{\rho_2 b_2}\,{}_{+}\,\frac{\rho_2\rho_3 a_3}{\rho_3 b_3}\,{}_{+\cdots+}\,\frac{\rho_{n-1}\rho_n a_n}{\rho_n b_n}\,{}_{+\cdots}.$$

By setting $\rho_1 = 1$ *and* $\rho_n = \frac{1}{4n^2}$ *for* $n \geq 2$*, we see that*

$$\frac{\rho_{n-1}\rho_n a_n}{\rho_n b_n} = \frac{\frac{1}{4(n-1)^2}\cdot\frac{1}{4n^2}\cdot\Big(2(n-1)(2n-1)(2n)\Big)^2}{\frac{1}{4n^2}.24.n^2} = \frac{(2n-1)^2}{6}.$$

Thus,

$$\pi = 3 + \frac{1^2}{6 +} \frac{3^2}{6 +} \frac{5^2}{6 +} \frac{7^2}{6 + \cdots +} \frac{(2n-1)^2}{6 + \cdots},$$

or

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}.$$

The following example is the application of Theorem1.6.3.

Example 1.6.6 *(see [16], page* 161*)* ***Continued fractions for*** e*: By using the power series for exponential function we obtain*

$$\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1} + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \cdots,$$

so

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots.$$

Now, we apply Theorem 1.6.3 with $\alpha_k = k$ *to get*

$$1 - \frac{1}{e} = \frac{1}{1 +} \frac{1}{1 +} \frac{2}{2 +} \frac{3}{3 + \cdots}.$$

By inverting both sides, we obtain after simple calculations,

$$\frac{1}{e-1} = \frac{1}{1+}\frac{2}{2+}\frac{3}{3+\cdots}.$$

Inverting again and adding 1 *to both sides, we obtain*

$$e = 2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+\cdots}. \tag{1.6.4}$$

1.7 Irrationality

In this section we are going to see how continued fraction represent irrational number.

Theorem 1.7.1 *[22] Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of positive rational numbers such that for all sufficiently large n, a_n and b_n be positive integers and $a_n \leq b_n$, and also $\sum_{n=1}^{\infty} \frac{b_n b_{n+1}}{a_{n+1}} = \infty$. Then the real number*

$$\xi = b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\frac{a_3}{b_3 +\cdots} \quad \textit{is irrational.}$$

By using Theorem 1.5.2, the continued fraction defining ξ converges, since $\sum_{n=1}^{\infty} \frac{b_n b_{n+1}}{a_{n+1}} = \infty$. Suppose that $m > 0$ and $0 < a_n \leq b_n$ for all $n \geq m+1$. Consider the following continued fraction

$$\eta = b_m + \frac{a_{m+1}}{b_{m+1} +}\frac{a_{m+2}}{b_{m+2} +}\frac{a_{m+3}}{b_{m+3} +\cdots}.$$

By using Theorem 1.5.2, the continued fraction η converges, then $\eta > b_m > 0$ and we have

$$\xi = b_0 + \frac{a_1}{b_1} {}_+ \frac{a_2}{b_2} {}_+ \frac{a_3}{b_3} {}_{+ \cdots +} \frac{a_m}{\eta}.$$

By using Theorem 1.3.1, we can write that

$$\xi = b_0 + \frac{a_1}{b_1} {}_+ \frac{a_2}{b_2} {}_+ \frac{a_3}{b_3} {}_{+ \cdots +} \frac{a_m}{\eta} = \frac{\eta P_m + a_m P_{m-1}}{\eta Q_m + a_m Q_{m-1}}.$$

Also, we can write that

$$\xi = \frac{\eta P_m + a_m P_{m-1}}{\eta Q_m + a_m Q_{m-1}} \quad \Longleftrightarrow \quad \eta = \frac{\xi a_m Q_{m-1} - a_m P_{m-1}}{P_m - \xi Q_m}.$$

Note that$\xi \neq P_m/Q_m$, since $\eta > b_m$. It is clear that both ξ and η are rational or irrational, Since all the b_n, a_n's are rational. Therefore, we have to show that η is irrational. Assume, by way of contradiction, that ξ is rational. Let us define positive continued fraction

$$\xi_n := \frac{a_n}{b_n} {}_+ \frac{a_{n+1}}{b_{n+1}} {}_+ \frac{a_{n+2}}{b_{n+2}} {}_{+ \cdots}, \quad \text{for all } n \in \mathbb{N}.$$

Then we can write it as

$$\xi_n = \frac{a_n}{b_n + \xi_{n+1}} \quad \Longrightarrow \quad \xi_{n+1} = \frac{a_n}{\xi_n} - b_n. \tag{1.7.1}$$

The continued fraction ξ_n is a positive continued fraction ($\xi_n > 0$), since $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ are sequences of positive rational numbers, then we have

$$\xi_n = \frac{a_n}{b_n + \xi_{n+1}} < \frac{a_n}{b_n} \leq 1,$$

which implies that $0 < \xi_n < 1$ for all n. By assumption of contradictory, ξ_1 is rational. Assuming that ξ_n is rational, We have to show that ξ_{n+1} is also rational.

Since $0 < \xi_n < 1$ for all n, then we can write $\xi_n = s_n/t_n$ where $t_n > s_n > 0$ for all n (t_n and s_n are relatively prime). By using (1.7.1) we have

$$\frac{s_{n+1}}{t_{n+1}} = \xi_{n+1} = \frac{a_n}{\xi_n} - b_n = \frac{a_n t_n}{s_n} - b_n = \frac{a_n t_n - b_n s_n}{s_n}.$$

Hence,

$$s_n s_{n+1} = (a_n t_n - b_n s_n) t_{n+1}.$$

Thus, $t_{n+1} \,|\, s_n s_{n+1}$. Therefore t_{n+1} must divide s_n, since t_{n+1} and s_{n+1} are relatively prime. In particular, $t_{n+1} < s_n$. So $t_{n+1} < t_n$, since, $s_n < t_n$. Hence, $\{t_n\}_{n \geq 1}$ should satisfy

$$0 < \ldots < t_{n+1} < t_n < \ldots < t_3 < t_2 < t_1,$$

which is contradiction to the assumption.

Example 1.7.2 *[8] Consider continued fraction for e (1.6.4), which is*

$$e = 2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+\cdots}.$$

Since $a_n \leq b_n$ for all n, then e is irrational.

Corollary 1.7.3 *(see [16], page 21) Any infinite simple continued fraction represents an irrational number.*

This corollary can be proved by using Theorem 1.7.1. In fact, simple continued fraction is a continued fraction where $a_n = 1$ and b_n are positive integer number for all n, then for all $n \geq 1$, $0 < a_n \leq b_n$ holds.

Remark 1.7.4 *Any finite simple continued fraction represents a rational number and any infinite simple continued fraction represents an irrational number.*

1.8 Summary of the Results

The rest of thesis has been divided into two chapters. In the second chapter, we apply Euler's differential method, which was not used by mathematicians for a long time, to derive a new formula for a certain kind continued fraction depending on a parameter. This formula is in the form of the ratio of two integrals. In case of integer values of the parameter, the formula reduces to the ratio of two finite sums. Asymptotic behavior of this continued fraction is investigated numerically and it is shown that it increases in the same rate as the root function. The results of this chapter are based on [3]. In the third chapter, we define a transformation of a certain kind of continued fractions to the same kind of continued fractions. This transformation is obtained by multiple application of the Bauer-Muir transform and then using the limiting process. It is shown that a double application of this transformation is the identity transformation. The obtained result is applied to some classic continued fractions due to Euler and Ramanujan. The results of this chapter are published in [2].

Chapter 2

EULER'S DIFFERENTIAL METHOD

In [10] Euler considered a generalization of (1.2.2):

$$
\begin{aligned}
c_0x_0 &= b_0x_1 + a_1x_2 \\
c_1x_1 &= b_1x_2 + a_2x_3 \\
c_2x_2 &= b_2x_2 + a_3x_4 \\
&\vdots \\
c_nx_n &= b_nx_{n+1} + a_{n+1}x_{n+2} \\
&\vdots
\end{aligned}
\tag{2.0.1}
$$

Assume that sequences $\{x_n\}$ in (2.0.1) are nonzero. Now we can rewrite (2.0.1) as a following continued fraction form

$$\frac{c_0x_0}{x_1} = b_0 + \frac{a_1c_1}{b_1} {}_{+} \frac{a_2c_2}{b_2} {}_{+} \cdots {}_{+} \frac{a_nc_n}{b_n + a_{n+1}\frac{x_{n+2}}{x_{n+1}}}. \tag{2.0.2}$$

Example 2.0.1 *Consider the function*

$$S(x) = x^n(\alpha - \beta x - \gamma x^2)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and all be positive. With simple calculation we can easily find $x_1 = 0$ and $x_2 = \xi = (\sqrt{\beta^2 + 4\alpha\gamma} - \beta)/2\gamma > 0$ are two real roots of $S(x)$.

After differentiating of $S(x)$, we obtain

$$dS \;=\; n\alpha x^{n-1}dx - (n+1)\beta x^n dx - (n+2)\gamma x^{n+1}dx.$$

Integrating both sides and simplifying, we obtain

$$n\alpha \int_0^\xi x^{n-1}dx = (n+1)\beta \int_0^\xi x^n dx + (n+2)\gamma \int_0^\xi x^{n+1}dx. \tag{2.0.3}$$

Let us to define

$$x_n := \int_0^\xi x^n dx = \frac{\xi^{n+1}}{n+1}.$$

By replacing x_n in (2.0.3), we can write

$$n\alpha x_{n-1} = (n+1)\beta x_n + (n+2)\gamma x_{n+1}. \tag{2.0.4}$$

Comparing (2.0.1) and (2.0.4), we can choose

$$c_n = (n+1)\alpha, \qquad b_n = (n+2)\beta, \qquad a_n = (n+2)\gamma.$$

Then by using Theorem 1.5.4 and (2.0.2), we can get

$$\frac{2\alpha}{\xi} = 2\beta + \frac{2\times 3\alpha\gamma}{3\beta\ +}\,\frac{3\times 4\alpha\gamma}{4\beta\ +}\,\frac{4\times 5\alpha\gamma}{5\beta\ +}\cdots. \tag{2.0.5}$$

We can write the left side of equation (2.0.5) as below

$$\frac{2\alpha}{\xi} = \frac{4\alpha\gamma}{\sqrt{\beta^2+4\alpha\gamma}-\beta} = \beta + \sqrt{\beta^2+4\alpha\gamma},$$

since $\xi = (\sqrt{\beta^2+4\alpha\gamma}-\beta)/2\gamma$. Let us take $x = \frac{\alpha\gamma}{\beta^2}$, and substitute it in (2.0.5),

$$\beta + \sqrt{\beta^2+4x\beta^2} = 2\beta + \frac{2\times 3x\beta^2}{3\beta\ +}\,\frac{3\times 4x\beta^2}{4\beta\ +}\,\frac{4\times 5x\beta^2}{5\beta\ +}\cdots. \tag{2.0.6}$$

After simplification, we obtain

$$\sqrt{1+4x} = 1 + \frac{2x}{1\ +}\,\frac{x}{1\ +}\,\frac{x}{1\ +}\,\frac{x}{1\ +}\,\frac{x}{1\ +}\cdots, \qquad x>0.$$

More examples can be found in [16].

2.1 Euler's Differential Method

Euler transformed the above computation into the following theorem.

Theorem 2.1.1 *[9] Let R and P be two real-valued functions on the interval $[0\,,\,1]$, which are positive on $(0\,,\,1)$, let n be a nonnegative integer, let a, b and c be any real numbers, and let α, β and γ be any positive numbers. If*

$$(a+n\alpha)\int_0^1 PR^n dx = (b+n\beta)\int_0^1 PR^{n+1}dx + (c+n\gamma)\int_0^1 PR^{n+2}dx \tag{2.1.1}$$

then

$$\frac{\int_0^1 PR\,dx}{\int_0^1 P\,dx} = \frac{a}{b_+}\,\frac{(a+\alpha)c}{b+\beta}\,_+\,\frac{(a+2\alpha)(c+\gamma)}{b+2\beta}\,_+\,\frac{(a+3\alpha)(c+2\gamma)}{b+3\beta}\,_+\cdots. \tag{2.1.2}$$

Moreover, (2.1.1) *holds if R and P satisfy the differential relations*

$$\begin{cases} R\,dS + S\,dR = (bR + cR^2 - a)P\,dx \\ \\ S\,dR = (\beta R + \gamma R^2 - \alpha)P\,dx \end{cases} \tag{2.1.3}$$

on $(0\,,\,1)$ for some function S on $[0\,,\,1]$ such that $R^{n+1}S$ vanishes at 0 and 1.

Proof. The proof of this theorem can be found in Khrushchev[16], page 184. ■

In [12] Euler considered a continued fraction of the form

$$K(s) = \frac{s}{1_+}\,\frac{s+1}{2}\,_+\,\frac{s+2}{3}\,_+\,\frac{s+3}{4}\,_+\cdots. \tag{2.1.4}$$

In the above notation,$K(s) = K_{k=1}^{\infty}\left[\frac{k+s-1}{k}\right]$. Clearly, at $s = 0, -1, -2, \dots$ the right hand side of (2.1.4) is a finite continued fraction and, hence, $K(s)$ can be calculated directly. But at $s = 1, 2, \dots$ it is an infinite continued fraction and straightforward calculation of $K(s)$ becomes complicated. For this, by using Theorem 2.1.1, Euler transferred the continued fraction in (2.1.4) to the following continued fraction

$$K(s) = \frac{s}{2_+}\,\frac{s-2}{3\ \ _+}\,\frac{s+1}{4\ \ _+}\,\frac{s-3}{5\ \ _+}\,\frac{s+2}{6\ \ _+}\,\frac{s-4}{7\ \ _+}\,\frac{s+3}{8\ \ _+} \dots, \tag{2.1.5}$$

which is finite at all integer values of s except $s = 1$. Based on this, he calculated $K(2) = 1$, $K(3) = \frac{4}{3}$ etc. and obtained that $K(s)$ is rational for all integer values of s except $s = 1$. It is remarkable that $K(1) = (e-1)^{-1}$ is irrational, the fact again proved by Euler (see [16], page 162).

Example 2.1.2 *Following formula was obtained by Stieltjes [29].*

$$\frac{1}{s + K_{n=1}^{\infty}\left[\frac{n(n+1)}{s}\right]} = \int_0^{\infty} \frac{e^{-sx}}{cosh^2 x}\,dx = s\int_0^{\infty} e^{-sx} tanh\, x\, dx \tag{2.1.6}$$

By using the Euler's differential method, we can find a very simple proof of Stieltjes's formula.

In Theorem 2.1.1, let us take

$$a = 1, \qquad b = s, \qquad c = 1$$

$$\alpha = 1, \qquad \beta = 0, \qquad \gamma = 1$$

and choose $R(x) = x$ *and* $P(x) = (\frac{1-x}{1+x})^{s/2}$. *Therefore, it suffices to verify the condition in* (2.1.3) *of Theorem 2.1.1 for a suitable choice of* S. *Let* $S(x) = (\frac{1-x}{1+x})^{s/2}$. *Then* $R^n(x)S(x) = x^n(\frac{1-x}{1+x})^{s/2}$, *implying* $R^n(0)S(0) = R^n(1)S(1) = 0$ *for* $n \geq 0$. *For our choice of functions* P, R *and* S, *the equations in* (2.1.3) *have the form*

$$\begin{cases} x\,dS + S\,dx = (sx + x^2 - 1)P\,dx, \\ S\,dx = (x^2 - 1)P\,dx. \end{cases}$$

Therefore we can write

$$\int_0^1 P\,dx = \int_0^1 (\frac{1-x}{1+x})^{s/2} \frac{dx}{1-x^2} = \frac{1}{2}\int_0^1 t^{\frac{s}{2}-1} dt = \frac{1}{s},$$

the result obtained by substituting $x = \frac{1-t}{1+t}$. *Similarly,*

$$\begin{aligned} \int_0^1 RP\,dx &= \frac{1}{2}\int_0^1 t^{\frac{s}{2}-1}(\frac{1-t}{1+t})\,dt = \int_0^\infty e^{-sx} tanh\,x\,dx \\ &= -\frac{1}{s}\int_0^\infty tanh\,x\,de^{-sx} = \frac{1}{s}\int_0^\infty \frac{e^{-sx}}{cosh^2 x}\,dx, \end{aligned}$$

the first integral obtained by substituting $t = e^{-x}$, *and the second integral obtained from the first by using the integration by parts.*

2.2 Application of Euler's Differential Method

In this section we discover new application of Euler's differential method . The object of our study is the function

$$f(t) = K_{k=1}^{\infty}\left[\frac{k+t}{k}\right], \ -1 < t < \infty. \tag{2.2.1}$$

2.2.1 The Analyticity

In this subsection we prove that f is analytic.

At first, note that the mth convergent $c_m(t) = K_{k=1}^{m}\left[\frac{k+t}{k}\right]$ of $K_{k=1}^{\infty}\left[\frac{k+t}{k}\right]$ is a rational function of t. Hence, we can represent it in the form

$$c_m(t) = \frac{P_m(t)}{Q_m(t)}$$

for some polynomials P_m and Q_m. One can also observe that Q_m is a positive function on $(-1, \infty)$. Consequently, c_m is an analytic function on $(-1, \infty)$ for all $m = 1, 2, \ldots$.

Lemma 2.2.1 *The functions*

$$g_m(t) = \frac{Q_{m+1}(t)}{Q_m(t)}, \ m = 1, 2, \ldots,$$

are positive and strictly increasing on $(-1, \infty)$.

Proof. The positivity of g_m follows from the positivity of Q_m for all $m = 1, 2, \ldots$. To prove that g_m is strictly increasing, we will verify the condition $g_m'(t) > 0$ for

$m = 1, 2, \ldots$. By Theorem 1.3.4

$$g_m(t) = m + 1 + \frac{m+1+t}{m} \,\frac{m+t}{+m-1} {}_{+\cdots+} \frac{2+t}{1} = m + 1 + \frac{m+1+t}{g_{m-1}(t)}.$$

Hence, for $m = 1$,

$$g_1'(t) = \left(2 + \frac{2+t}{1}\right)' = 1 > 0,$$

and for $m = 2$,

$$g_2'(t) = \left(3 + \frac{3+t}{g_1(t)}\right)' = \frac{1}{(4+t)^2} > 0.$$

Assume that $g_n'(t) > 0$ for all $n = 1, \ldots, m-1$. Then

$$\begin{aligned}
g_m'(t) &= \left(m + 1 + \frac{m+1+t}{g_{m-1}(t)}\right)' = \frac{g_{m-1}(t) - (m+1+t)g_{m-1}'(t)}{g_{m-1}(t)^2} \\
&= g_{m-1}(t)^{-2}\left(m + \frac{m+t}{g_{m-2}(t)} - (m+1+t)\frac{g_{m-2}(t) - (m+t)g_{m-2}'(t)}{g_{m-2}(t)^2}\right) \\
&= g_{m-1}(t)^{-2}\left(m - \frac{1}{g_{m-2}(t)} + \frac{(m+1+t)(m+t)g_{m-2}'(t)}{g_{m-2}(t)^2}\right).
\end{aligned}$$

Since $g_{m-2}(t) > m-1$, we obtain $g_m'(t) > 0$. By induction, $g_m'(t) > 0$ for all $m = 1, 2, \ldots$. ■

Lemma 2.2.2 *The functions*

$$h_m(t) = |c_{m+1}(t) - c_m(t)|, \ m = 1, 2, \ldots,$$

are positive and strictly increasing on $(-1, \infty)$.

Proof. If $m = 1$, then

$$h_1(t) = |c_2(t) - c_1(t)| = \frac{(1+t)(2+t)}{4+t}.$$

One can verify that $h_1(t) > 0$ and $h_1'(t) > 0$. Assume that h_m is positive and strictly increasing. By Theorem 1.3.2

$$\begin{aligned} h_{m+1}(t) &= \frac{(1+t)(2+t)\cdots(m+1+t)}{Q_{m+1}(t)Q_m(t)} \\ &= \frac{(1+t)(2+t)\cdots(m+t)}{Q_m(t)Q_{m-1}(t)} \cdot \frac{(m+1+t)Q_{m-1}(t)}{Q_{m+1}(t)}. \end{aligned}$$

Hence, h_{m+1} is positive. Moreover, from the Euler-Wallis formula (1.3.2)

$$Q_{m+1}(t) = (m+1)Q_m(t) + (m+1+t)Q_{m-1}(t),$$

we obtain

$$h_{m+1}(t) = h_m(t) \cdot \frac{Q_{m+1}(t) - (m+1)Q_m(t)}{Q_{m+1}(t)} = h_m(t)\left(1 - \frac{m+1}{g_m(t)}\right).$$

Thus, by Lemma 2.2.1, h_{m+1} equals to the product of two positive strictly increasing functions. Hence, h_{m+1} is strictly increasing. By induction, h_m is strictly increasing for all $m = 1, 2, \ldots$. ■

Theorem 2.2.3 *The function f, defined by* (2.2.1), *is analytic on the interval* $(-1, \infty)$.

Proof. At first, note that the continued fraction $K_{k=1}^{\infty}\left[\frac{k+t}{k}\right]$ converges at every

$t > -1$. This follows from the theorem of Pringsheim (Theorem 1.5.2) since

$$\sum_{k=1}^{\infty} \frac{(k-1)k}{k+t} = +\infty$$

for $t > -1$. Next, let us show that this convergence is uniform on every compact subinterval $[a, b]$ of $(-1, \infty)$. Since the sequence $\{c_m(b)\}$ converges, it is a Cauchy sequence. Hence, for given $\varepsilon > 0$, there is N such that for all $m > N$,

$$|c_{m+1}(b) - c_m(b)| < \varepsilon.$$

By Lemma 2.2.2, for all $m > N$,

$$\max_{t \in [a,b]} |c_{m+1}(t) - c_m(t)| = |c_{m+1}(b) - c_m(b)| < \varepsilon.$$

By Theorem 1.5.1

$$c_2(b) < c_4(b) < \cdots < c_{2k}(b) < \cdots < c_{2k+1}(b) < \cdots < c_3(b) < c_1(b).$$

Hence, for all $n > m > N$,

$$\max_{t \in [a,b]} |c_n(t) - c_m(t)| \leq \max_{t \in [a,b]} |c_{m+1}(t) - c_m(t)| = |c_{m+1}(b) - c_m(b)| < \varepsilon.$$

This means that the sequence of functions $\{c_m\}$ is uniformly Cauchy on $[a, b]$. Hence, it converges uniformly on $[a, b]$. Finally, since all terms of the sequence $\{c_m\}$ are analytic functions on $[a, b]$, the limit function f is also analytic on $[a, b]$. From the analyticity on every compact subinterval of $(-1, \infty)$, we obtain that f

is analytic on $(-1, \infty)$. ■

2.2.2 The Representation by Integrals

In this subsection we represent the function f, defined by (2.2.1), as a ratio of two integrals. We will use the following.

Lemma 2.2.4 *The integral*

$$\int_0^1 (1-x)^{b-1} x^{a-1} e^x \, dx$$

is well-defined for $a > 0$ and $b > 0$. Otherwise, it diverges to ∞.

Proof. Since the exponential function is bounded positive on $[0, 1]$, the lemma follows from comparison of the above integral with the Euler's integral

$$\int_0^1 (1-x)^{b-1} x^{a-1} \, dx,$$

which is well-defined for $a > 0$ and $b > 0$, and diverges to ∞ whenever $a \leq 0$ or $b \leq 0$. ■

Lemma 2.2.5 *For $-1 < t < 1$,*

$$f(t) = \frac{\int_0^1 (1-x)^{-t} x^{t+1} e^x \, dx}{\int_0^1 (1-x)^{-t} x^t e^x \, dx}, \tag{2.2.2}$$

where f is defined by (2.2.1).

Proof. It is easily seen that (2.2.2) is same as (2.1.2) for the selection $a = t+1$, $b = c = \alpha = \beta = 1$, $\gamma = 0$, $R(x) = x$ and $P(x) = x^t(1-x)^{-t}e^x$. Therefore, it suffices to verify the condition in (2.1.3) of Theorem 2.1.1 for a suitable choice of S. Let $S(x) = -x^t(1-x)^{1-t}e^x$. Then $R^{n+1}(x)S(x) = -x^{t+n+1}(1-x)^{1-t}e^x$, implying $R^{n+1}(0)S(0) = R^{n+1}(1)S(1) = 0$. For our choice of functions P, R and S, the equations in (2.1.3) have the form

$$\begin{cases} x\,dS + S\,dx = (x + x^2 - t - 1)P\,dx, \\ S\,dx = (x-1)P\,dx, \end{cases}$$

which can be verified easily. ■

Corollary 2.2.6 $K(1) = (e-1)^{-1}$, *where* $K(s)$ *is defined by* (2.1.4).

Proof. Let $t = 0$ in Lemma 2.2.5. Then

$$K(1) = f(0) = \frac{\int_0^1 xe^x\,dx}{\int_0^1 e^x\,dx} = \frac{1}{e-1},$$

proving the corollary. ■ By Lemma 2.2.4, the integrals in (2.2.2) are divergent for $t \geq 1$, producing $\frac{\infty}{\infty}$ in the right hand side. The next lemma extend (2.2.2) to the interval $(-1, \infty)$.

Lemma 2.2.7 *For* $0 < t < 2$,

$$f(t) = \frac{\int_0^1 (1-x)^{1-t}(x^{t+1}e^x)'\,dx}{\int_0^1 (1-x)^{1-t}(x^te^x)'\,dx}, \tag{2.2.3}$$

where f is defined by (2.2.1).

Proof. For $0 < t < 1$, formula (2.2.3) can be deduced by the application of the integration by parts formula to the integrals in (2.2.2). The right hand side of (2.2.3) is analytic on $(0, 2)$. By Theorem 2.2.3, the left hand side of (2.2.3) is an analytic on $(-1, \infty)$. Hence, by the uniqueness theorem of analytic functions, (2.2.2) holds for $0 < t < 2$. ■ Formula (2.2.3) is still not valid for $t > 2$ since its right hand side produces $\frac{\infty}{\infty}$. It is not valid for $-1 < t \leq 0$ as well for the same reason. Next, we generalize Lemmas 2.2.5 and 2.2.7 in the following form.

Theorem 2.2.8 *For $p - 1 < t < p + 1$, where $p = 0, 1, 2, \ldots,$*

$$f(t) = \frac{\int_0^1 (1-x)^{p-t} \frac{d^p}{dx^p}(x^{t+1}e^x)\,dx}{\int_0^1 (1-x)^{p-t} \frac{d^p}{dx^p}(x^t e^x)\,dx}. \tag{2.2.4}$$

Proof. This follows by multiple application of the procedure used in the proof of Lemma 2.2.7. ■

2.2.3 The Representation by Finite Sums

Now we are interested in integer values of t.

Theorem 2.2.9 *For $p = 1, 2, \ldots,$*

$$f(p) = (p+1)\frac{\sum_{k=0}^{p-1} \frac{a_{p,k}}{p-k+1}}{\sum_{k=0}^{p-1} a_{p,k}}, \tag{2.2.5}$$

where

$$a_{p,k} = \begin{pmatrix} p \\ \\ k \end{pmatrix} \cdot \frac{1}{(p-k-1)!}.$$

Proof. Substituting $t = p$ in (2.2.4) and integrating, we obtain

$$f(p) = \frac{\frac{d^{p-1}}{dx^{p-1}}(x^{p+1}e^x)\Big|_{x=1}}{\frac{d^{p-1}}{dx^{p-1}}(x^p e^x)\Big|_{x=1}}. \tag{2.2.6}$$

By Leibnitz's formula for the higher order derivatives of product function,

$$\begin{aligned}
\frac{d^{p-1}}{dx^{p-1}}(x^{p+1}e^x)\Big|_{x=1} &= \sum_{k=0}^{p-1} \begin{pmatrix} p-1 \\ \\ k \end{pmatrix} e^x (x^{p+1})^{(k)}\Big|_{x=1} \\
&= e\sum_{k=0}^{p-1} \frac{(p-1)!}{k!(p-k-1)!} \cdot \frac{(p+1)!}{(p-k+1)!} \\
&= e\sum_{k=0}^{p-1} \frac{(p-1)!}{k!(p-k-1)!} \cdot \frac{(p+1)p!}{(p-k+1)(p-k)!} \\
&= e(p+1)\sum_{k=0}^{p-1} \begin{pmatrix} p \\ \\ k \end{pmatrix} \frac{(p-1)!}{(p-k+1)(p-k-1)!} \\
&= e(p+1)(p-1)!\sum_{k=0}^{p-1} \frac{a_{p,k}}{p-k+1}.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\frac{d^{p-1}}{dx^{p-1}}(x^p e^x)\bigg|_{x=1} &= \sum_{k=0}^{p-1} \begin{pmatrix} p-1 \\ \\ k \end{pmatrix} e^x (x^p)^{(k)}\bigg|_{x=1} \\
&= e\sum_{k=0}^{p-1} \frac{(p-1)!}{k!(p-k-1)!} \cdot \frac{p!}{(p-k)!} \\
&= e\sum_{k=0}^{p-1} \begin{pmatrix} p \\ \\ k \end{pmatrix} \frac{(p-1)!}{(p-k-1)!} \\
&= e(p-1)!\sum_{k=0}^{p-1} a_{p,k}.
\end{aligned}$$

Substituting the calculated expressions in (2.2.6) produces the formula in (2.2.5).

■ Using Theorem 2.2.9, one can easily recalculate $K(2) = f(1) = 1$, $K(3) = f(2) = \frac{4}{3}$, etc.

2.2.4 Concluding Remarks

The Euler's differential method, that was forgotten for a long time, is applied to a continued fraction depending a parameter. A new formula, different from the Euler's one, proved for this continued fraction. In case of integer values of the parameter, this formula takes a simple form.

The function f, defined by (2.2.1), has a very interesting behavior. If

$$\sigma(t) = \sqrt{t} - f(t),\ 0 < t < \infty,$$

then σ is a slowly increasing function. The values of σ, calculated by use of Wolfram Mathematica Software, are presented in Table 1 for t changing from

Table 2.1. The values of $\sigma(t)$ for $t = 10^4$ to 10^{15}

t	$\sigma(t)$		t	$\sigma(t)$		t	$\sigma(t)$
$1 \cdot 10^4$	0.247814		$11 \cdot 10^4$	0.249341		10^6	0.249781
$2 \cdot 10^4$	0.248454		$12 \cdot 10^4$	0.249369		10^7	0.249931
$3 \cdot 10^4$	0.248738		$13 \cdot 10^4$	0.249393		10^8	0.249978
$4 \cdot 10^4$	0.248907		$14 \cdot 10^4$	0.249415		10^9	0.249993
$5 \cdot 10^4$	0.249022		$15 \cdot 10^4$	0.249435		10^{10}	0.249998
$6 \cdot 10^4$	0.249107		$16 \cdot 10^4$	0.249453		10^{11}	0.249999
$7 \cdot 10^4$	0.249173		$17 \cdot 10^4$	0.249470		10^{12}	0.250000
$8 \cdot 10^4$	0.249227		$18 \cdot 10^4$	0.249484		10^{13}	0.250000
$9 \cdot 10^4$	0.249271		$19 \cdot 10^4$	0.249498		10^{14}	0.250000
$10 \cdot 10^4$	0.249308		$20 \cdot 10^4$	0.249511		10^{15}	0.250000

$t = 10^4$ to $t = 10^{15}$ with different steps. This table shows that the difference between $\sqrt{t}$ and $f(t)$ continuously increases but the increased value become tiny in comparison to the change of t. From $t = 10^{12}$ to $t = 10^{15}$ the program calculates the value 0.25. This allows to conjecture whether σ has a horizontal asymptote. If yes, then it becomes interesting to prove whether $\sigma_\infty = \lim_{t\to\infty} \sigma(t) = 0.25$. Theorem 2.2.9 may be used for this purpose since σ_∞ can be evaluated by giving t integer values p in the limiting process. The formula in (2.2.5) is heavily based on factorials. Therefore, the Stirling's approximation formula for factorials may be efficient to solve this conjecture.

Chapter 3

BAUER-MUIR TRANSFORM

Consider the continued fraction

$$b_0 + K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right). \tag{3.0.1}$$

Its Bauer–Muir transform with respect to a sequence of real numbers $\{x_n\}$ is a continued fraction

$$b_0 + x_0 + \frac{\phi_1}{b_1 + x_1 +} \frac{a_1\phi_2/\phi_1}{b_2 + x_2 - x_0\phi_2/\phi_1 + \cdots +} \frac{a_{n-1}\phi_n/\phi_{n-1}}{b_n + x_n - x_{n-2}\phi_n/\phi_{n-1} + \cdots}, \tag{3.0.2}$$

where

$$\phi_n = a_n - x_{n-1}(b_n + x_n),\ n = 1, 2, \ldots,$$

are assumed to be nonzero. This transform was introduced in Bauer [1] and Muir [21]. Its importance is predefined by the following theorem, the proof of which can be found in Khrushchev (see [16], page 230).

3.1 The Value of the Bauer-Muir Transform

Theorem 3.1.1 *Assume that the continued fraction in* (3.0.1) *has positive elements and* $x_n \geq 0$ *starting from some* n. *If the continued fraction converges, then its Bauer–Muir transform also converges to the same value.*

Recently, Jacobson [14] has proved that the Bauer-Muir transform is useful also for negative elements of the continued fraction in (3.0.1). But for our purposes this transform will be used for positive elements.

Using this transform, Bauer showed that the Brouncker's continued fraction

$$b(s) = s + K_{n=1}^{\infty}\left[\frac{(2n-1)^2}{2s}\right]$$

equals to $(s+1)^2/b(s+2)$ and provided an easy proof of the Brouncker's theorem, stating that $b(s)b(s+2) = (s+1)^2$.

More significant application of the Bauer-Muir transform belongs to Perron [24, 25], who proved the Ramanujan's formula

$$\frac{1}{s^2-1_+}\,\frac{4\cdot 1^2}{1\quad_+}\,\frac{4\cdot 1^2}{s^2-1_+}\,\frac{4\cdot 2^2}{1\quad_+}\,\frac{4\cdot 2^2}{s^2-1_{+\cdots}} = \int_0^{\infty}\frac{2te^{-st}}{e^t+e^{-t}}\,dt.$$

Some other applications can be found in [18].

In [12] Euler considered a continued fraction of the form

$$K(s) = K_{n=1}^{\infty}\left[\frac{n+s-1}{n}\right] = \frac{s}{1_+}\,\frac{s+1}{2\quad_+}\,\frac{s+2}{3\quad_+}\,\frac{s+3}{4\quad_{+\cdots}}. \tag{3.1.1}$$

Using differential method, Euler transferred this continued fraction to

$$K(s) = \frac{s}{2_+}\,\frac{s-2}{3\quad_+}\,\frac{s+1}{4\quad_+}\,\frac{s-3}{5\quad_+}\,\frac{s+2}{6\quad_+}\,\frac{s-4}{7\quad_+}\,\frac{s+3}{8\quad_{+\cdots}}, \tag{3.1.2}$$

which is finite at all integer values of s except $s = 1$. Based on this, he calculated

$K(2) = 1$, $K(3) = \frac{4}{3}$ etc. and obtained that $K(s)$ is rational for all integer values of s except $s = 1$. Note that $K(1) = (e-1)^{-1}$ is irrational, the fact again proved by Euler (see [16], page 162).

3.2 Application of Bauer-Muir Transform

In this chapter we apply the Bauer-Muir transform to the continued fraction in the more general form

$$K_{n=1}^{\infty}\left[\frac{an+b}{dn+c}\right], \tag{3.2.1}$$

which equals to (3.1.1) when $a = d = 1$, $b = s-1$ and $c = 0$. By this, we transfer (3.2.1) to a continued fraction which becomes finite at certain values of parameters allowing to calculate its values. In particular, for Euler's continued fraction $K(s)$, we find another finite representation, reducing the number of calculations.

Our aim is studying continued fractions of the form $K_{n=1}^{\infty}\left[\frac{\alpha n+\beta}{\varepsilon n+\delta}\right]$ with $\alpha, \beta, \delta, \varepsilon \in \mathbb{R}$ and $\varepsilon \neq 0$. Applying the equivalent transform (Theorem 1.4.2), one can show that

$$K_{n=1}^{\infty}\left[\frac{\alpha n+\beta}{\varepsilon n+\delta}\right] = \varepsilon K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right], \tag{3.2.2}$$

where $a = \alpha/\varepsilon^2$, $b = \beta/\varepsilon^2$ and $c = \delta/\varepsilon$. Therefore, the problem reduces to study of $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ for $a, b, c \in \mathbb{R}$.

In the sequel, we are interested in the continued fraction $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ for two sets of parameters (a, b, c). The first one is

$$D = \{(a, b, c) \in \mathbb{R}^3 : 0 \leq a \leq 1,\ a + b \geq 1,\ c \geq 0\}, \tag{3.2.3}$$

and the second one is

$$R = \{(a, b, c) \in \mathbb{R}^3 : -1 \leq a \leq 0,\ c + 2a \geq 0,\ b \geq a^2 + ac + 1\}. \tag{3.2.4}$$

3.2.1 Convergence Theorems

In this section, we state convergence theorems for sets of parameters (3.2.3) and (3.2.4). For this, at first we modify the Tietze's criterion (see [26], page 56) for continued fractions.

Theorem 3.2.1 *The continued fraction $K_{n=1}^{\infty}\left[\frac{a_n}{b_n}\right]$ converges if the following inequalities hold:*

$$\begin{cases} a_n > 0,\ b_n > 1 + \varepsilon, & \text{if } n < N, \\ a_n < 0,\ b_n \geq 1,\ a_n + b_n \geq 1 + \varepsilon & \text{if } n \geq N, \end{cases} \tag{3.2.5}$$

where N is a natural number and ε is a small positive value.

Proof. Note that in condition (3.2.5), n takes natural values. If $N = 1$, then no n satisfies $n < N$. Hence, only the second line in (3.2.5) takes place. Also, one can observe that the inequality $a_n + b_n \geq 1 + \varepsilon$ is assumed explicitly or implicitly for all n.

Let $\frac{P_n}{Q_n}$ be nth convergent of $K_{n=1}^{\infty}\left[\frac{a_n}{b_n}\right]$. Conventionally, we assume that $Q_0 = P_{-1} = 1$ and $Q_{-1} = P_0 = 0$. At the first step, let us prove that

$$Q_n > (1+\varepsilon)Q_{n-1}, \quad n \geq 1. \tag{3.2.6}$$

For $n = 1$, we have

$$Q_1 = b_1 > 1 + \varepsilon = (1+\varepsilon)Q_0.$$

If $n = 2$, then

$$Q_2 = b_2 Q_1 + a_2 Q_0 > b_2 Q_1 > (1+\varepsilon)Q_1.$$

In a similar way, by using Euler-Wallis formula (1.3.2), we can continue up to $N-1$ and obtain

$$Q_{N-1} = b_{N-1}Q_{N-2} + a_{N-1}Q_{N-3} > b_{N-1}Q_{N-2} > (1+\varepsilon)Q_{N-2}.$$

Starting $n = N$ our arguments change since we pass from the inequalities in the first line to the inequalities in the second line of (3.2.5). For $n = N$, we can write

$$\begin{aligned} Q_N &= b_N Q_{N-1} + a_N Q_{N-2} > b_N Q_{N-1} + a_N Q_{N-1} \\ &= (b_N + a_N)Q_{N-1} \geq (1+\varepsilon)Q_{N-1}. \end{aligned}$$

Here we used the inequality $Q_{N-1} > Q_{N-2}$ combined with $a_N < 0$. For $n > N$, the arguments of the case $n = N$ are valid. So, by induction we can conclude that (3.2.6) holds for all $n \geq N$, and hence, for all $n = 1, 2 \ldots$.

In the second step, let us prove that

$$\prod_{i=1}^{n}(b_i - 1 - \varepsilon) < Q_n - Q_{n-1}, \;\; n = 1, 2, \ldots. \tag{3.2.7}$$

If $n = 1$, then this inequality holds trivially in the form $b_1 - 1 - \varepsilon < b_1 - 1$. Assume it holds for n. Then

$$\begin{aligned}
Q_{n+1} - Q_n &= b_{n+1}Q_n + a_{n+1}Q_{n-1} - Q_n \\
&> b_{n+1}Q_n + (1 - b_{n+1} + \varepsilon)Q_{n-1} - (1 + \varepsilon)Q_n \\
&= (Q_n - Q_{n-1})(b_{n+1} - 1 - \varepsilon) > \prod_{i=1}^{n+1}(b_i - 1 - \varepsilon).
\end{aligned}$$

By induction, (3.2.7) holds for all $n = 1, 2, \ldots$. This implies

$$\prod_{i=1}^{n}(b_i - 1 - \varepsilon) < Q_n, \;\; n = 1, 2, \ldots. \tag{3.2.8}$$

In the third step, for $n \geq N$, by Theorem 1.3.2

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 \cdots a_n,$$

we can write

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1} a_1 \cdots a_n}{Q_n Q_{n-1}} = \frac{(-1)^N a_1 \cdots a_{N-1} |a_N| \cdots |a_n|}{Q_n Q_{n-1}}.$$

Hence, the sequence of convergents $\frac{P_n}{Q_n}$ is monotone starting from N. It is decreasing if N is odd and it is increasing if N is even. Furthermore, for $n \geq N$,

$$\frac{P_n}{Q_n} - \frac{P_{N-1}}{Q_{N-1}} = \sum_{i=N}^{n} \left(\frac{P_i}{Q_i} - \frac{P_{i-1}}{Q_{i-1}} \right) = (-1)^N a_1 \cdots a_{N-1} \sum_{i=N}^{n} \frac{|a_N| \cdots |a_i|}{Q_i Q_{i-1}}.$$

Hence, from (3.2.5) and (3.2.8),

$$\begin{aligned} \left| \frac{P_n}{Q_n} - \frac{P_{N-1}}{Q_{N-1}} \right| &= a_1 \cdots a_{N-1} \sum_{i=N}^{n} \frac{|a_N| \cdots |a_i|}{Q_i Q_{i-1}} \\ &\leq a_1 \cdots a_{N-1} \sum_{i=N}^{n} \frac{\prod_{j=N}^{i} (b_j - 1 - \varepsilon)}{Q_i Q_{i-1}} \\ &< a_1 \cdots a_{N-1} \sum_{i=N}^{n} \frac{\prod_{j=N}^{i} (b_j - 1 - \varepsilon)}{\prod_{j=1}^{i} (b_j - 1 - \varepsilon) Q_{i-1}} \\ &< \frac{a_1 \cdots a_{N-1}}{\prod_{j=1}^{N-1} (b_j - 1 - \varepsilon)} \sum_{i=N}^{n} \frac{1}{Q_{i-1}}. \end{aligned}$$

One can observe that in the case $N = 1$, we simply have

$$a_1 \cdots a_{N-1} = \prod_{j=1}^{N-1} (b_j - 1 - \varepsilon) = 1.$$

So, $\frac{P_n}{Q_n}$ is bounded if the series $\sum_{i=N}^{\infty} \frac{1}{Q_{i-1}}$ converges. To prove the latter, we use (3.2.6). From

$$Q_{i-1} > (1+\varepsilon) Q_{i-2} > \cdots > (1+\varepsilon)^{i-N} Q_{N-1} > (1+\varepsilon)^{i-N},$$

the series $\sum_{i=N}^{\infty} \frac{1}{Q_{i-1}}$ is majorized by convergent geometric series. Thus $\frac{P_n}{Q_n}$ converges. ■

Theorem 3.2.2 *The continued fraction* $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ *converges if* $(a,b,c) \in D \cup R$.

Proof. For $(a,b,c) \in D$, we have $an+b>0$, $n+c>0$ for all $n=1,2,\ldots$ and

$$\sum_{n=1}^{\infty} \frac{(n-1+c)(n+c)}{an+b} = \infty.$$

Hence, by Theorem 1.5.2, the continued fraction $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ converges.

For $(a,b,c) \in R$, first note that

$$D \cap R = \{(a,b,c) : a=0,\ b \geq 1,\ c \geq 0\}.$$

Therefore, we have to prove the convergence of $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ under the conditions

$$-1 \leq a < 0,\ c+2a \geq 0,\ b \geq a^2+ac+1.$$

We will prove this by verifying the conditions of Theorem 3.2.1. Since $a<0$, we have $a_n = an+b<0$ starting some N. Next,

$$b_n = n+c \geq n-2a = n+2|a| > 1+\varepsilon$$

if we let $\varepsilon = -a$. Finally,

$$\begin{aligned} a_n+b_n = (a+1)n+b+c &\geq (a+1)n+a^2+ac+1+c \\ &= (a+1)(n+c)+a^2+1 > (a+1)(1+\varepsilon)+a^2+1 \\ &= (a+1)(1-a)+a^2+1 = 2 \geq 1+\varepsilon. \end{aligned}$$

Thus, $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ converges on R. ■

3.2.2 Main Result

Theorem 3.2.3 *For $(a, b, c) \in D$,*

$$K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right] = a + K_{n=1}^{\infty}\left[\frac{An+B}{n+C}\right], \tag{3.2.9}$$

where

$$A = -a,\ \ B = b + a - a(a+c),\ \ C = c + 2a. \tag{3.2.10}$$

Proof. One can verify that $(a, b, c) \in D$ implies $(A, B, C) \in R$. Hence, both continued fractions in (3.2.9) are convergent by Theorem 3.2.2. It remains to show the equality holds in (3.2.9). For this, we will use Theorem 3.1.1.

Let $a_n = an + b,\ b_n = n + c,\ x_n = a$, observing that $\{x_n\}$ is a constant sequence. Calculate $\phi_n = b - a(a+c)$, observing that $\{\phi_n\}$ is also a constant sequence. Then by Theorem 3.1.1, the Bauer-Muir transform of $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ for this selection of parameters converges and

$$\begin{aligned} K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right] &= a + \frac{b-a(a+c)}{1+c+a} {}_{+}\frac{a+b}{2+c}{}_{+}\frac{2a+b}{3+c}{}_{+\cdots+}\frac{na+b}{n+1+c}{}_{+\cdots} \\ &= a + \frac{b-a(a+c)}{1+c+a+K_{n=1}^{\infty}\left[\frac{an+b}{n+1+c}\right]}. \end{aligned} \tag{3.2.11}$$

Apply the same to $K_{n=1}^{\infty}\left[\frac{an+b}{n+1+c}\right]$ selecting $a_n = an+b,\ b_n = n+1+c,\ x_n = a$ and calculating $\phi_n = b - a(a+c+1)$. Then

$$\begin{aligned} K_{n=1}^{\infty}\left[\frac{an+b}{n+1+c}\right] &= a + \frac{b-a(a+c+1)}{2+c+a}\,{}_{+}\frac{a+b}{3+c}{}_{+\cdots+}\frac{na+b}{n+2+c}{}_{+\cdots} \\ &= a + \frac{b-a(a+c+1)}{2+c+a+K_{n=1}^{\infty}\left[\frac{an+b}{n+2+c}\right]}. \end{aligned}$$

Substituting this continued fraction in (3.2.11), we obtain

$$K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right] = a + \frac{b-a(a+c)}{1+c+2a+\frac{b-a(a+c+1)}{2+c+a+K_{n=1}^{\infty}\left[\frac{an+b}{n+2+c}\right]}}.$$

Applying the same to $K_{n=1}^{\infty}\left[\frac{an+b}{n+2+c}\right]$ and repeating this procedure N times, we obtain

$$K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right] = a + \frac{b-a(a+c)}{1+c+2a}{}_{+\cdots+}\frac{b-a(a+c+N-1)}{N+c+a+K_{n=1}^{\infty}\left[\frac{an+b}{n+N+c}\right]},$$

or, by using (3.2.10),

$$K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right] = a + \frac{A+B}{1+C}{}_{+\cdots+}\frac{AN+B}{N+C}\,{}_{+}\frac{-a+K_{n=1}^{\infty}\left[\frac{an+b}{n+N+c}\right]}{1}. \tag{3.2.12}$$

Next, we deduce the equality (3.2.9) from (3.2.12). At first, introduce the following notation. Let

$$\alpha_N = K_{n=1}^{\infty}\left[\frac{an+b}{n+N+c}\right],\ N = 0, 1, \ldots,$$

and let $\frac{P_N}{Q_N}$ be Nth convergent of $K_{n=1}^{\infty}\left[\frac{An+B}{n+C}\right]$. Since $(a, b, N+c) \in D$, α_N is finite for all N and, therefore, by Theorem 1.5.1, it is located between the first two convergents of $K_{n=1}^{\infty}\left[\frac{an+b}{n+N+c}\right]$, i.e.,

$$\frac{a+b}{N+1+c+\frac{2a+b}{N+2+c}} \leq \alpha_N \leq \frac{a+b}{N+1+c}.$$

Hence, $\alpha_N \to 0$ as $N \to \infty$. By Theorem 1.3.1

$$\alpha_0 = \frac{P_N + (\alpha_N - a)P_{N-1}}{Q_N + (\alpha_N - a)Q_{N-1}}.$$

Then

$$\left|\alpha_0 - \frac{P_N}{Q_N}\right| = \left|\frac{P_N + (\alpha_N - a)P_{N-1}}{Q_N + (\alpha_N - a)Q_{N-1}} - \frac{P_N}{Q_N}\right| = \frac{|\alpha_N - a|\left|\frac{P_{N-1}}{Q_{N-1}} - \frac{P_N}{Q_N}\right|}{\left|\frac{Q_N}{Q_{N-1}} + \alpha_N - a\right|}.$$

Here $\alpha_N \to 0$ and, by Theorem 3.2.2,

$$\left|\frac{P_{N-1}}{Q_{N-1}} - \frac{P_N}{Q_N}\right| \to 0 \text{ as } N \to \infty.$$

Hence,

$$|\alpha_N - a|\left|\frac{P_{N-1}}{Q_{N-1}} - \frac{P_N}{Q_N}\right| \to 0 \text{ as } N \to \infty. \tag{3.2.13}$$

Also, by (3.2.6),

$$\frac{Q_N}{Q_{N-1}} \geq 1 + \varepsilon,$$

which implies

$$\left|\frac{Q_N}{Q_{N-1}} + \alpha_N - a\right| \geq \left|\frac{Q_N}{Q_{N-1}}\right| - |\alpha_N - a|$$
$$\geq 1 + \varepsilon - |\alpha_N - a| \geq 1 + \varepsilon - |\alpha_N| - |a| \geq \varepsilon - |\alpha_N|$$

since $0 \leq a \leq 1$. From $\lim_{N\to\infty} \alpha_N = 0$, for large values of N, we have $|\alpha_N| < \varepsilon/2$. Therefore,

$$\left|\frac{Q_N}{Q_{N-1}} + \alpha_N - a\right| > \frac{\varepsilon}{2} \tag{3.2.14}$$

for large values of N. (3.2.13) and (3.2.14) yield $\alpha_0 = \lim_{N\to\infty} \frac{P_N}{Q_N}$. Theorem is proved. ■

Corollary 3.2.4 *For $a, b, c, d \in \mathbb{R}$ with $d \neq 0$ and $\left(\frac{a}{d^2}, \frac{b}{d^2}, \frac{c}{d}\right) \in D$,*

$$K_{n=1}^{\infty}\left[\frac{an+b}{dn+c}\right] = \frac{a}{d} + K_{n=1}^{\infty}\left[\frac{a'n+b'}{dn+c'}\right],$$

where

$$a' = -a,\ b' = b + a - \frac{a}{d}\left(\frac{a}{d} + c\right),\ c' = c + \frac{2a}{d}.$$

Proof. By (3.2.2) and Theorem 3.2.3, we have

$$\begin{aligned}
K_{n=1}^{\infty}\left[\frac{an+b}{dn+c}\right] &= dK_{n=1}^{\infty}\left[\frac{\frac{a}{d^2}n+\frac{b}{d^2}}{n+\frac{c}{d}}\right] \\
&= \frac{a}{d} + dK_{n=1}^{\infty}\left[\frac{-\frac{a}{d^2}n+\frac{a+b}{d^2}-\frac{a}{d^2}\left(\frac{a}{d^2}+\frac{c}{d}\right)}{n+\frac{c}{d}+\frac{2a}{d^2}}\right] \\
&= \frac{a}{d} + K_{n=1}^{\infty}\left[\frac{-an+a+b-a\left(\frac{a}{d^2}+\frac{c}{d}\right)}{dn+c+\frac{2a}{d}}\right] \\
&= \frac{a}{d} + K_{n=1}^{\infty}\left[\frac{a'n+b'}{dn+c'}\right].
\end{aligned}$$

This proves the corollary. ■

Corollary 3.2.5 *For every real $s \geq 1$,*

$$K(s) = K_{n=1}^{\infty}\left[\frac{n+s-1}{n}\right] = 1 + \frac{s-2}{3}{}_{+}\,\frac{s-3}{4}{}_{+}\,\frac{s-4}{5}{}_{+\cdots}.$$

Proof. Just let $a = 1$, $c = 0$ and $b = s - 1$ in Theorem 3.2.3. ■

Corollary 3.2.6 *For every $s = 2, 3, \ldots,$*

$$K(s) = K_{n=1}^{\infty}\left[\frac{n+s-1}{n}\right] = 1 + K_{n=1}^{s-2}\left[\frac{s-1-n}{n+2}\right],$$

where $K_{n=1}^{0}\left[\frac{1-n}{n+2}\right] = 0$ in the case $s = 2$.

Proof. This follows from the fact that the integer values of s makes the continued fraction in Corollary 3.2.5 finite. ■ Corollary 3.2.6 provides a fast formula for calculation of $K(s)$ at integer values of s in comparison with the formula in (3.1.2).

For example, for $s = 4$, the formula in (3.1.2) produces

$$K(4) = \frac{4}{2 + \frac{2}{3 + \frac{5}{4 + \frac{1}{5 + \frac{6}{6}}}}} = \frac{21}{13},$$

that is 5 step fractional calculation. While Corollary 3.2.6 reduces the number of steps to 2 as follows

$$K(4) = 1 + \frac{2}{3 + \frac{1}{4}} = \frac{21}{13}.$$

Generally, for $s = 2, 3, \ldots$, Corollary 3.2.6 gives a formula consisting of $s - 2$ fractions, while the number of fractions in (3.1.2) is $2s - 3$.

Theorem 3.2.3 allows to obtain some of Ramanujan's formulas. Letting $a = 1$, $b = x$ and $c = x - 1$ in Theorem 3.2.3, we obtain the Ramanujan's formula (see [4], page 112)

$$K_{n=1}^{\infty}\left[\frac{n + x}{n + x - 1}\right] = 1.$$

Letting $a = 1$, $b = m - 1$ and $c = m - \alpha - 1$ in Theorem 3.2.3, we obtain another Ramanujan's formula (see [4], page 118)

$$K_{n=1}^{\infty}\left[\frac{n + m - 1}{n + m - \alpha - 1}\right] = 1 + K_{n=1}^{\infty}\left[\frac{\alpha - n}{n + m - \alpha + 1}\right].$$

Also, letting $a = \alpha$, $b = x$, $c = x - \alpha - 1$ and $d = \alpha$ in Corollary 3.2.4, we obtain one more Ramanujan's formula (see [4], page 115)

$$K_{n=1}^{\infty}\Big[\frac{\alpha n + x}{\alpha n + x - \alpha - 1}\Big] = 1 + \frac{\alpha}{x+1}.$$

Theorem 3.2.3 suggests also a mapping T, that assigns $(A, B, C) \in \mathbb{R}^3$ to every $(a, b, c) \in \mathbb{R}^3$ by the formulas in (3.2.10). This mapping has the following properties:

- The mapping T is an involution, i.e., $T(T(a, b, c)) = (a, b, c)$. This can be verified by straightforward calculations.
- The mapping T is a bijection since it is an involution.
- The image of D under the mapping T equals to $T(D) = R$. Indeed, let $(a, b, c) \in D$. From $(A, B, C) = T(a, b, c)$, we have

  $$A = -a, \;\; B = b + a - a(a + c), \;\; C = c + 2a.$$

  From $0 \le a \le 1$, we get $-1 \le A \le 0$. Also, $c \ge 0$ implies $C + 2A \ge 0$. Finally, $a + b \ge 1$ produces $B \ge A^2 + AC + 1$.
- The image of R under the mapping T equals to $T(R) = D$ since T is an involution.
- $T(0, b, c) = (0, b, c)$, i.e., in the case $a = 0$ the right and left hand sides of the equality in Theorem 3.2.3 are the same continued fraction.

- The continued fractions $K_{n=1}^{\infty}\left[\frac{an+b}{n+c}\right]$ and $a+K_{n=1}^{\infty}\left[\frac{An+B}{n+C}\right]$ converge for $(a, b, c) \in D$ and $(A, B, C) \in R$, respectively. In the case of $(A, B, C) = T(a, b, c)$ they converge to the same value.

REFERENCES

[1] Bauer G., Von einem Kettenbruch Eulers und einem Theorem von Wallis. Abh. der kgl. Bayerischen Akad. der Wissenschaften zu München, zweite Klasse 11, 1872.

[2] Belaghi M.J.S., Khrushchev S. and Bashirov A.E., On Bauer–Muir Transform of Continued Fractions, Int. J. Number Theory, Vol. 09, 321–332, 2013.

[3] Belaghi M.J.S. and Bahsirov A.E., On application of Euler's differential method to a continued fraction depending on parameter, (Submitted for publication).

[4] Berndt B.C., Ramanujan's Notebook's, Part II, Springer, New York, 1989.

[5] Bernoulli D., Disquisitiones ulteriores de indole fractionum continuarum, Novi. Comm. Acad. Sc. Imp. St Petersburg 20, 1775.

[6] Cuyt A., Brevik Peterson V., Verdonk B., Waadeland H. and Jones W.B., Handbook of Continued Fractions for Special Functions, Springer, 2008.

[7] Euler L., De fractionibus continuus, Dissertatio, Commentarii Academiae Scientiarum Imperialis Petropolitanae IX for 1737, 98–137 (presented on February 7, 1737); Opera Omnia, ser. 1, vol. 14, 187–216, 1744; E071. English translation: Mathematical Systems Theory (1985) 4 (18).

[8] Euler L., Introductio in Analysin Infinitorum (Lausanne) 1748; E101.

[9] Euler L., De fractionibus continuus, observationes, Commentarii Academiae Scientiarum Imperialis Petropolitanae XI for 1739, 32–81 (presented on 22 January 1739); Opera Omnia, ser. I, vol. 14, pp. 291–349, 1750; E123.

[10] Euler L., De formatione fractionum continuarum, Acta Academiae Scientarum Imperialis Petropolitinae 3, 3–29 (presented to the St Petersburg Academy on 4 September 1775); Opera Omnia, ser. 1, vol. 15, pp. 314–337, 1782; E522.

[11] Euler L., De transformatione serierum in fractiones continuas, ubi simul haec theoria non mediocriter amplificatur, Opuscula Analytica 2, 138–177; Opera Omnia, ser. 1, vol. 15, pp. 661–700, 1785; E593.

[12] Euler L., Observationes circa fractiones continuas in hac forma contentas $S = n/(1+(n+1)/(2+(n+2)/(3+(n+3)/(4+etc.))))$, Mmoires de l'Acadmie des Sciences de St Petersbourg, 4, 52-74 (presented on 18 November 1779); Opera Omnia, ser. 1, vol. 16, pp. 139–161, 1813; E742.

[13] Huygens Ch., Descriptio Automati Planetarium (The Hague), 1698.

[14] Jacobson L., On the Bauer-Muir transformation for continued fractions and its applications, Journal of Mathematical Analysis and Applications, Vol.152, Issue2, pp.496-514, 1990.

[15] Jones W. B. and Thron W. J., Continued Fractions. Analytic Theory and Applications (Addison-Wesley, Reading), 1980.

[16] Khrushchev S.V., Orthogonal Polynomials and Continued Fractions from Euler's Point of View, Encyclopedia of Mathematics and Its Applications, Vol.~122, Cambridge University Press, Cambridge, 2008.

[17] Lange L. J., An elegant continued fraction for π, Amer. Math. Monthly 106, no. 5, 456-458, 1999.

[18] Lorentzen L., Waadeland H., Continued Fraction with Applications, Studies in Computational Mathematics 3, Vol.122, North-Holland, Amsterdam, 1992.

[19] Markoff A. A., Izbrannye Trudy po Teorii Neoreryvnyukh Drobei i Teorii Funkcii naimenee uklonyayushikhsya ot nulya (GITTL, Moscow), 1948.

[20] Markowsky G., Misconceptions about the golden ratio, Two Year College Math. J. 23, no. 1, 2-19, 1992.

[21] Muir Th., A theorem in continuants and Extension of a theorem in continuants, with an important application. The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science 3 (5), 137, 360, 1877.

[22] Nathan J. A., The irrationality of e^x for nonzero rational x, Amer. Math. Monthly 105, no. 8, 762-763, 1998.

[23] Olds C.D., Continued Fractions, Mathematical Association of America, Washington, 1963.

[24] Perron O., Über die Preeceschen Kettenbrüche, Sitzungsberichte der ($kgl.$) Bayerischen Akademie der Wissenschaften zu München, matematische-physikalisch ($naturwissenschaftlicheKlasse$), 1953.

[25] Perron O., Die Lehre von der Kettenbrüchen, Band I: Elementare Kettenbrüche, Teubner, Stuttgart, 1954.

[26] Perron O., Die Lehre von der Kettenbrüchen, Band II: Analytisch-funktionentheoretische Kettenbrüche, (Teubner, Stuttgart), 1957.

[27] Pringsheim A., Über ein Convergenz-Kriterium für die Kettenbrüche mit positiven Gliedern, Sitzungsber. der math.-phys Klasse der Kgl. Bayer. Akad. Wiss., München 29, 261-268, 1899.

[28] Rogers L. J., On the representation of certain asymptotic series as convergent continued fractions. Proc. London Math. Soc., second series, 4 (2), 72–98, 1907.

[29] Stieltjes T. J., Sur quelques intgrales dfinies et leur dveloppement en fractions continues, Q. J. Pure Appl. Math. 24, 370–382, 1890; OEuvres compltes, vol. 2 (Gronigen, 1918), pp. 378–391.

[30] Wall H.S., Analytic Theory of Continued Fractions, Chelsea Publishing Company, New York, 1967.

[31] Wallis J., Aritheoremetica Infinitorum (London), 1656.