Position-dependent Mass Lagrangians: Nonlocal Transformations, Euler-Lagrange Invariance and Exact Solvability

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ABSTRACT

This thesis reviews some features of the PDM Lagrangians. We reach the mapping of the PDM equation into a unit-mass Lagrangian in the generalized coordinates by the insertion of a nonlocal point transformation. The invariance of the PDM Euler-Lagrange equations under specific conditions is proved. We analyze the dynamical equations obeyed by nonlinear oscillator systems with position-dependent mass. Four classes of such examples with PDM-nonlinear oscillators are specified. They include Mathews-Lakshmanan oscillators, a quadratic nonlinear oscillator, a Morse-type oscillator, and a nonlinear deformation of an isotonic oscillator. Attainment among them the mapping of an isotonic nonlinear oscillator into a PDM deformed isotonic oscillator.

Keywords: classical position-dependent mass, nonlocal point transformation, Euler-Lagrange equations invariance.

ÖZ

Bu tez, PDM Lagrangian'ların bazı özelliklerini gözden geçirmektedir. PDM denkleminin genelleştirilmiş koordinatlarda bir birim kütle Lagrangian'ı şeklinde eşlenebilmesine, lokal olmayan nokta dönüşümü ile ulaştık. Belirli koşullar atında PDM Euler-Lagrange denklemlerinin değişmezliği kanıtlanmıştır. Biz dinamik denklemleri analiz ederek konuma bağlı kitlelerle birlikte doğrusal olmayan osilatör sistemleri arasındaki uyumuna baktık. PDM doğrusal olmayan osilatörleri dört sınıfta örnekledik. Bunlar, Mathews-Lakshman osilatörleri, kuadratik doğrusal olmayan osilatör, Morse–tipi osilatör ve doğrusal olmayan deformasyonlu izotonik osilatörü içermektedir. Ayrıca, izotonik doğrusal olmayan osilatörün PDM defrome izotonik osilatörüne eşlenmesine de eriştik.

Anahtar kelimeler: Klasik pozisyon bağımlı kitle, yerel olmayan nokta dönüşümü, Euler-Lagrange denklemlerinin değişmezliği

DEDICATION

This thesis work is dedicated to:

- My dear father;
- My darling mother;
- My beloved wife;
- My son.

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INTRODUCTION

Recently, we have seen a new direction in the description of physical systems with position-dependent mass (PDM). Where the PDM equations provide a smooth dealing with issues as electronic characteristics of semiconductors, energy density many-body problems and quantum dots. This direction started from the last few decades in quantum mechanical systems with the von Roos PDM Hamiltonian [2]. The insertion of the PDM in the Lagrangian equation is very interesting because the Lagrange equation has a mass that is a function of position. This interest in mathematical and classical sides increased since Mathews and Lakshmanan worked on the equation of motion with PDM [1]. Despite the reality that the familiar Lagrange equations can be immediately applied to systems with mechanical properties of the mass that is explicitly dependent on time, where there is invariance with respect to that type of probability. This is different if the mass is dependent on position.

In the study of O. Mustafa [9], it has been illustrated that by the nonlocal point transformation from the coordinate q onto the coordinate x, where this transformation is invertible $\left(i.e., \frac{\partial x}{\partial q} \neq 0 \neq \frac{\partial q}{\partial x}\right)$, we protect the equality between the Euler-Lagrange equations and Newton's laws of motion. The insertion of the PDM into Euler-Lagrange equation would result to conserve the quasi-linear momentum $\Pi(x,\dot{x}) = \sqrt{m(x)}\dot{x}$ where ($\Pi(x,\dot{x}) = \Pi_0(x_0,\dot{x}_0)$ and $\dot{\Pi}(x,\dot{x}) = 0$), but the linear momentum $p(x,\dot{x}) = m(x)\dot{x}$ is not conserved, where $(p(x,\dot{x}) \neq p_0(x_0,\dot{x}_0)$ and $\dot{p}(x,\dot{x}) \neq 0$).

As a result for the Newton's equation of motion

$$F(x,\dot{x}) = \frac{dp(x,\dot{x})}{dt} = m(x)\ddot{x} + \frac{1}{2}m'(x)\dot{x}^2$$

would introduce a new PDM-byproducted reaction-type force into Newton's equation

$$F(x,\dot{x}) = F_{net}(x) + R_{PDM}(x),$$

where $R_{PDM}(x) = \frac{1}{2}m'(x)\dot{x}^2$; $m'(x) = \frac{dm(x)}{dx}$ and $\dot{x} = \frac{dx}{dt}$.

Finally, the total energy is remaining conservative.

1.1 Position-dependent mass Lagrangians

To illustrate the objective, let is assume a particle with mass $m(x)$ which is clearly dependent on a position x. The particle has a kinetic energy $T = \frac{1}{2}m(x)\dot{x}^2$, and it moves in a potential force field $V(x)$. Therefore, the Lagrange equation can be expressed simply as

$$L = \frac{1}{2}m(x)\dot{x}^2 - V(x). \tag{1.1}$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{1.2}$$

with equation (1.1), result that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m(x)\ddot{x} + m'(x)\dot{x}^2, \tag{1.3}$$

$$\frac{\partial L}{\partial x} = \frac{1}{2}m'(x)\dot{x}^2 - \frac{\partial V(x)}{\partial x}, \tag{1.4}$$

And

$$\ddot{x}+\frac{1}{2}\frac{m'(x)}{m(x)}\dot{x}^{2}+\frac{1}{m(x)}\frac{\partial V(x)}{\partial x}=0. \tag{1.5}$$

By nature, the equation (1.5) is invariant under some specified transformations. Furthermore, it is a second order differential equation of Liénard-type [15-17], it is quadratic in velocity, and it has interesting symmetry properties to make challenges in physics and mathematics.

Actually, the notion of PDM might constitute a deformity of the position dependent mass. It leads to deform the potential force field as well. To clear that, let the PDM $m(x)=m_0M(x)$ (m_0 is a constant with dimension of mass) moves in potential force field

$$V(x)=\frac{1}{2}m(x)\omega^{2}x^{2}.$$

We can write the harmonic oscillator field like

$$V(x)=\frac{1}{2}m_0M(x)\omega^{2}x^{2}\rightarrow V(u)=\frac{1}{2}m_0\omega^{2}u^{2};\quad u=\sqrt{M(x)}x.$$

Now, $V(u)$ can be considered as the mass-deformed potential corresponding to the constant-mass m_0.

The plan of the thesis is as following. Chapter 2 introduces a nonlocal point transformation for the PDM equation in generalized coordinates. We work on a constant unit-mass classical particle in a general coordinate (q,τ), where τ is a deformed-time, and the particle moves in a potential field $V(q)$. And we reach the condition that the Euler-Lagrange equation is invariant. Then, the oscillator-linearization is discussed in the structure of PDM for the Euler-Lagrange equation.

We start chapter 3 with applying the oscillator-linearization to a PDM-nonlinear oscillator. Starting with Mathews Lakshmanan PDM-nonlinear oscillators at three cases and plotting phase space trajectory using the Maple software. After that, we looked to PDM particles that have a quadratic and a Morse-type nonlinear oscillator equations of motion, and get their figures, respectively. Furthermore, we close this chapter with mapping of an isotonic nonlinear-oscillator by using our nonlinear transformation into a PDM isotonic deformed nonlinear-oscillator. The conclusions are given in Chapter 4.

Chapter 2

NONLOCAL POINT TRANSFORMATION AND EULER-LAGRANGE INVARIANCE

Let us start with unit-mass classical particle that moves in a system of the general coordinates (q, τ), where τ is a time-rescaled and q as a spatially function. The potential field is a force function on q, and the Lagrange equation is written as

$$L(q, \tilde{q}, \tau) = \frac{1}{2}\tilde{q}^2 - V(q);\ \tilde{q} = \frac{dq}{d\tau}\,. \tag{2.1}$$

The application of Euler-Lagrange equation,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \tilde{q}}\right) - \frac{\partial L}{\partial q} = 0, \tag{2.2}$$

Results

$$\frac{\partial L}{\partial \tilde{q}} = \tilde{q} \ \ \rightarrow \ \frac{d}{d\tau}(\tilde{q}) = \ \frac{d}{d\tau}\left(\frac{dq}{d\tau}\right) = \frac{d^2 q}{d\tau^2} \tag{2.3}$$

$$\frac{\partial L}{\partial q} = -\frac{\partial V(q)}{\partial q} \tag{2.4}$$

and

$$\frac{d^2 q}{d\tau^2} + \frac{\partial V(q)}{\partial q} = 0 \tag{2.5}$$

Now, to work out this nonlinear differential equation we need to define a point transformation to change it to a linear equation. That is, let us define

$$q \equiv q(x) = \int \sqrt{g(x)}dx,$$

$$\tau = \int f(x)\, dt \ \rightarrow \ \frac{d\tau}{dt} = f(x) \neq 0;\ \ x \equiv x(t) \tag{2.6}$$

Applying this transformations (2.6) on equation (2.5), with starting by take derivative of the equations

$$\frac{dq}{dx} = \sqrt{g(x)}, \qquad \frac{d\tau}{dt} = f(x), \qquad \frac{dx}{dt} = \dot{x}$$

$$\frac{dq}{d\tau} = \tilde{q} = \frac{\sqrt{g(x)}dx}{f(x)dt} = \frac{\sqrt{g(x)}}{f(x)}\dot{x}\ , \tag{2.7}$$

and

$$\frac{d^2q}{d\tau^2} = \frac{1}{f^2(x)}\left\{f(x)\left(\dot{x}\frac{d\sqrt{g(x)}}{d\tau} + \sqrt{g(x)}\frac{d\dot{x}}{d\tau}\right) - \dot{x}\sqrt{g(x)}\frac{df(x)}{d\tau}\right\}. \tag{2.8}$$

To find those derivatives we have to use the chain rule. To obtain

$$\frac{d^2q}{d\tau^2} = \frac{\sqrt{g(x)}}{f^2(x)}\left\{\ddot{x} + \frac{1}{2}\left(\frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)}\right)\dot{x}^2\right\} \tag{2.9}$$

$$\frac{\partial V(q)}{\partial q} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial q} = \frac{1}{\sqrt{g(x)}}\frac{\partial V(x)}{\partial x}\,. \tag{2.10}$$

Equations (2.9) and (2.10) into the equation (2.5) imply

$$\ddot{x} + \frac{1}{2}\left(\frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)}\right)\dot{x}^2 + \frac{f^2(x)}{g(x)}\frac{\partial V(x)}{\partial x} = 0. \tag{2.11}$$

To find the identification between equations (2.11) and (1.5), let us define

$$g(x) = m(x)f^2(x) \to \tilde{q} = \dot{x}\sqrt{m(x)}, \tag{2.12}$$

To obtain

$$\frac{dg(x)}{dx} = \frac{d}{dx}\left(m(x)f^2(x)\right) \to g'(x) = m'(x)f^2(x) + 2m(x)f(x)f'(x), \tag{2.13}$$

$$\frac{g'(x)}{g(x)} = \frac{m'(x)f^2(x)}{g(x)} + 2\frac{m(x)f(x)f'(x)}{g(x)},$$

and

$$\frac{m'(x)}{m(x)} = \frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)}. \tag{2.14}$$

Put equation (2.14) into the equation (2.11) to get the equation (1.5). This result means the position-dependent mass Euler-Lagrange equation (2.5) is invariance with the point transformation (2.6). So

$$L(q,\tilde{q},\tau)=\frac{1}{2}\tilde{q}^2-V(q)\leftrightarrow\left\{\begin{matrix} q(x)=\int\sqrt{g(x)}dx \\ g(x)=m(x)f^2(x) \\ \tau=\int f(x)\,dt \\ \tilde{q}=\dot{x}\sqrt{m(x)} \end{matrix}\right\}\leftrightarrow L(x,\dot{x},t)$$

$$=\tfrac{1}{2}m(x)\dot{x}^2-V(x). \tag{2.15}$$

Herein, we must pay attention to the transformation that we have used. It is only an especial part of the so called generalized Sundman transformation [19-23]

$$Z=F(x,t),\ \ dY=G(x,t)dt,\ \ F'(x,t)G(x,t)\neq 0;\ \ F'(x,t)=\tfrac{\partial F(x,t)}{\partial x}. \tag{2.16}$$

Corresponding with equation (2.6) we see that

$$Z=q(x),\quad G(x,t)=f(x),\qquad Y=\tau,\quad F(x,t)=\int\sqrt{m(x)}f(x)dx,$$

$$F'(x,t)G(x,t)=\sqrt{m(x)}f(x)^2\neq 0. \tag{2.17}$$

This relation is an illustration that our transformation could be applied to transform the nonlinear differential equation to a linear type in order to be able to solve it. That is, in fact, the spirit of the generalized Sundman transformation.

2.1 Linearizing Euler-Lagrange Equation with PDM

Again, let us assume a unit mass particle moving in some general coordinates and a deformed time, and the potential field is an oscillator force field. The Lagrange oscillator equation is given

$$L(q,\tilde{q},\tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2 q^2. \tag{2.18}$$

When we apply the Euler-Lagrange equation (2.2) we get

$$\frac{d}{d\tau}\left(\frac{dq}{d\tau}\right) = \frac{d^2q}{d\tau^2}, \quad \frac{\partial L}{\partial q} = -\omega^2 q \ , \tag{2.19}$$

Hence,

$$\frac{d^2q}{d\tau^2} + \omega^2 q = 0 \ . \tag{2.20}$$

The solution for this differential equation is known to be

$$q(\tau) = A cos(\omega\tau) + B sin(\omega\tau) \ , \tag{2.21}$$

Applying the boundary condition when $\mathrm{q}(0) = A$, say,

$$q(\tau) = A cos(\omega\tau + \varphi) \ . \tag{2.22}$$

φ is just a phase shift.

Substitute the equation (2.9) into the equation (2.20), to get

$$\frac{\sqrt{g(x)}}{f^2(x)}\left\{\ddot{x} + \frac{1}{2}\left(\frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)}\right)\dot{x}^2\right\} + \omega^2 q(x) = 0. \tag{2.23}$$

Using the result (2.14) to obtain the

$$\ddot{x} + \frac{m'(x)}{2m(x)}\dot{x}^2 + \frac{\omega^2 f(x)}{\sqrt{m(x)}} q(x) = 0 \ . \tag{2.24}$$

The correspondence between equation (1.5) and (2.24), leads to the condition

$$\frac{1}{m(x)}\frac{\partial V(x)}{\partial x} = \frac{\omega^2 f(x)}{\sqrt{m(x)}} q(x) \rightarrow \ q(x) = \frac{1}{\omega^2 f(x)\sqrt{m(x)}}\frac{\partial V(x)}{\partial x}. \tag{2.25}$$

Consequently, we can clearly see that this relation between the system in the generalized coordinate to the Cartesian coordinate system. To point out the quadratic Liénard-type equations of nonlinear oscillator under appropriate conditions admit Lagrangian characterization and show oscillatory action. In the next chapter, we explain that by some examples.

Chapter 3

ILLUSTRATIVE EXAMPLES

Consider a class of PDM-nonlinear oscillators. All these dynamical systems are straightforward applications of the classical Euler-Lagrange equation. Here, it is important to confirm that in all of the examples considered below we have used nonlocal point transformation in order to illustrate that they are exactly solvable and invariant.

3.1 Mathews-Lakshmanan Oscillators

As a first example, let us discuss a particle with mass that depends on position. This PDM-particle is moving in a nonlinear harmonic oscillator potential field (initially suggested by Mathews and Lakshmanan). We analyze three various cases.

3.1.1 Oscillator I:

Let us choose $f(x) = m(x)$, and the particle moving in harmonic oscillator field given by the Lagrangian

$$L(x, \dot{x}, t) = \frac{1}{2} m(x)\dot{x}^2 - \frac{1}{2} m(x)\omega^2 x^2. \tag{3.1}$$

Using equation (2.25) we get

$$q(x) = \frac{1}{f(x)} x\sqrt{m(x)} \left(1 + \frac{1}{2}\frac{m'(x)}{m(x)} x\right), \tag{3.2}$$

If

$$f(x) = m(x) = \left(1 + \frac{1}{2}\frac{m'(x)}{m(x)} x\right), \tag{3.3}$$

Then, one would obtain

$$q(x) = x\sqrt{m(x)}\ . \tag{3.4}$$

To find $m(x)$, we use

$$1 + \frac{1}{2}\frac{m'(x)}{m(x)}x = m(x) \rightarrow m'(x) = \frac{2}{x}m(x)(m(x) - 1), \tag{3.5}$$

Integrating the differential equation we get

$$\int \frac{dm(x)}{m(x)(m(x)-1)} = 2\int \frac{dx}{x}. \tag{3.6}$$

Let

$$\frac{1}{m(x)(m(x)-1)} = \frac{A}{m(x)-1} + \frac{B}{m(x)}, \tag{3.7}$$

to imply

$$Am(x) + Bm(x) - B = 1 \rightarrow m(x)(A + B) = 1 + B, \tag{3.8}$$

where A and B are two constants.

Choose $B = -1 \rightarrow m(x)(A - 1) = 0 \rightarrow m(x) \neq 0 \rightarrow A = 1.$

Then

$$\int \left(\frac{1}{m(x)-1} - \frac{1}{m(x)}\right) dm(x) = 2\ln(x) + \ln(c),$$

$$\frac{1-m(x)}{m(x)} = cx^2, \tag{3.9}$$

Therefore, the PDM-mass function with $c = \pm\lambda$ becomes

$$m(x) = \frac{1}{1\pm\lambda x^2}\ , \lambda \geq 0\ . \tag{3.10}$$

So, the nonlocal point transformation above shows

$$L(q, \tilde{q}, \tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2 q^2 \leftrightarrow \left\{\begin{matrix} q(x) = x\sqrt{m(x)} \\ g(x) = m^3(x) \\ d\tau = m(x)dt \\ \tilde{q} = \dot{x}\sqrt{m(x)} \end{matrix}\right\} \leftrightarrow L(x, \dot{x}, t)$$

$$= \frac{1}{2}\frac{(\dot{x}^2 - \omega^2 x^2)}{1\pm\lambda x^2}; \qquad \lambda \geq 0. \tag{3.11}$$

Apply the Euler-Lagrange equation (1.2)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m(x)\dot{x}) = m(x)\ddot{x} + m'(x)\dot{x}^2, \tag{3.12}$$

$$\frac{\partial L}{\partial x} = \frac{1}{2}m'(x)\dot{x}^2 - \frac{1}{2}m'(x)\omega^2 x^2 - m(x)\omega^2 x, \tag{3.13}$$

To get the dynamical equations of motion

$$\ddot{x} + \frac{1}{2}\frac{m'(x)}{m(x)}\dot{x}^2 + \omega^2 x\left(1 + \frac{1}{2}\frac{m'(x)}{m(x)}x\right) = 0, \tag{3.14}$$

Take into account equation (3.10) and substitute $m(x)$ and $m'(x)$ into the equation (3.14), to obtain

$$\ddot{x} \mp \frac{\lambda x}{1 \pm \lambda x^2}\dot{x}^2 + \frac{\omega^2 x}{1 \pm \lambda x^2} = 0\ . \tag{3.15}$$

This equation is the Mathews-Lakshmanan oscillator equation. To solve it we choose

$$F = \frac{dx}{dt} = \dot{x} \rightarrow \frac{dF}{dt} = \frac{d^2x}{dt^2} = \ddot{x}, \tag{3.16}$$

To get

$$\frac{2\tilde{\lambda}F}{(\tilde{\lambda}F^2 + \omega^2)}dF = \frac{-2\tilde{\lambda}x}{1 - \tilde{\lambda}x^2}dx; \quad \tilde{\lambda} = \mp\lambda. \tag{3.17}$$

Integration of both sides leads to

$$\ln(\tilde{\lambda}F^2 + \omega^2) = \ln(1 - \tilde{\lambda}x^2) + \ln(c), \tag{3.18}$$

$$\frac{dx}{\sqrt{1 - \frac{\tilde{c}\tilde{\lambda}x^2}{(\tilde{c}-1)}}} = \pm\omega\sqrt{\frac{(\tilde{c}-1)}{\tilde{\lambda}}}dt; \ \ \tilde{c} = \frac{c}{\omega^2}, \tag{3.19}$$

We get from the Integration

$$\sqrt{\frac{(\tilde{c}-1)}{\tilde{c}\tilde{\lambda}}}\cos^{-1}\left(\sqrt{\frac{\tilde{c}\tilde{\lambda}}{(\tilde{c}-1)}}x\right) = \pm\omega\sqrt{\frac{(\tilde{c}-1)}{\tilde{\lambda}}}t + c_1, \tag{3.20}$$

$$x = \sqrt{\frac{(\tilde{c}-1)}{\tilde{c}\tilde{\lambda}}}\cos(\omega t\sqrt{\tilde{c}} + c_2). \tag{3.21}$$

Let us define the Integration constants as

$$\sqrt{\frac{(\tilde{c}-1)}{\tilde{c}\tilde{\lambda}}} = A,\ \tilde{c}\omega^2 = \Omega^2,\ c_2 = \varphi, \tag{3.22}$$

$$\sqrt{\frac{(\tilde{c}-1)}{\tilde{c}\tilde{\lambda}}} = A \rightarrow \tilde{c} - A^2\tilde{c}\tilde{\lambda} - 1 = 0 \rightarrow \tilde{c} = \frac{1}{(1-A^2\tilde{\lambda})}, \tag{3.23}$$

Finally, the solution reduces to

$$x(t) = A\cos(\Omega t + \varphi), \qquad \Omega^2 = \frac{\omega^2}{(1 \pm A^2\lambda)}\,. \tag{3.24}$$

Moreover, the linear momentum is

$$p(x) = m(x)\dot{x}. \tag{3.25}$$

$$p(t) = -\frac{A\Omega\sin(\Omega t+\varphi)}{1 \pm \lambda x^2}\,. \tag{3.26}$$

In Fig.1 we plot $p(t)$ vs. $x(t)$ (i.e., the phase diagram) for different values of A, for $\lambda = +1$ (left) and $\lambda = -1$ (right), taking $\varphi = 0\ and\ \omega = 1$ [16,17].

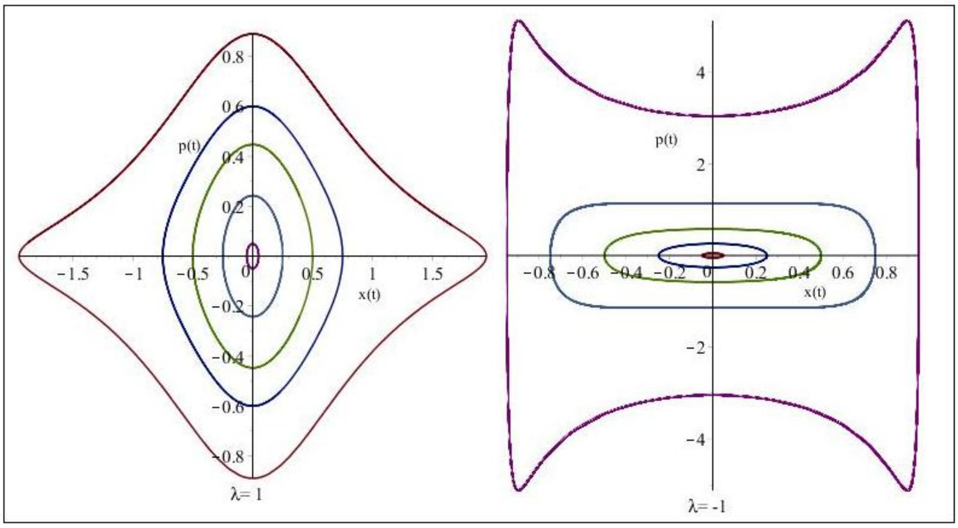

Figure 1: The phase trajectories in (*p(t) vs. x(t)*) plane for a Mathews-Lakshmanan oscillator.

In the figure on the left ($\lambda = +1$) there is no asymptote for arbitrary values of the position and momentum. The trajectories are soft curves which represent the deformity on the trajectory phase of the harmonic oscillator. They seem symmetry and have centralized circumferences at the origin. On the other hand, in the figure on the right ($\lambda = -1$) the behavior of the system is different, where there is a vertical asymptote in the momentum values at case ($A \rightarrow \pm 1$), it can be observed that the values of $p(t)$ quickly increase while an $x(t)$ tend to have limiting values.

3.1.2 Oscillator II:

Choose $f(x) = \frac{\beta}{2}\frac{m'(x)}{m(x)}$ where $\beta \neq 0$ is a constant, and the particle moves into a potential field $V(x) = \frac{1}{2}m(x)\omega^2\beta^2$. The Lagrangian is shown as

$$L(x, \dot{x}, t) = \frac{1}{2}m(x)\dot{x}^2 - \frac{1}{2}m(x)\omega^2\beta^2. \quad (3.27)$$

Using equation (2.25)

$$q(x) = \frac{1}{2}\frac{1}{\sqrt{m(x)}f(x)}m'(x)\beta^2, \quad (3.28)$$

If we choose

$$f(x) = \frac{1}{2}\frac{m'(x)}{m(x)}\beta\ , \quad (3.29)$$

and use the equation (2.6), we find

$$q(x) = \int \sqrt{m(x)}f(x)dx = \beta \int \frac{dm(x)}{2\sqrt{m(x)}} = \beta\sqrt{m(x)}. \quad (3.30)$$

This, in turn, suggests the nonlocal point transformation

$$L(q,\tilde{q},\tau)=\frac{1}{2}\tilde{q}^2-\frac{1}{2}\omega^2q^2\leftrightarrow\left\{\begin{matrix} q(x)=\beta\sqrt{m(x)} \\ g(x)=\dfrac{\beta}{4}\dfrac{m'(x)^2}{m(x)} \\ d\tau=\dfrac{\beta}{2}\dfrac{m'(x)}{m(x)}dt \\ \tilde{q}=\dot{x}\sqrt{m(x)} \end{matrix}\right\}L(x,\dot{x},t)$$

$$=\frac{1}{2}m(x)\dot{x}^2-\frac{1}{2}m(x)\omega^2\beta^2. \tag{3.31}$$

When we apply the Euler-Lagrange equation (1.2), we find

$$\ddot{x}+\frac{1}{2}\frac{m'(x)}{m(x)}\dot{x}^2+\frac{1}{2}\frac{m'(x)}{m(x)}\omega^2\beta^2=0\ . \tag{3.32}$$

We now compare equations (3.14) and (3.32) to solve for $m(x)$

$$\omega^2x\left(1+\frac{1}{2}\frac{m'(x)}{m(x)}x\right)=\frac{1}{2}\frac{m'(x)}{m(x)}\omega^2\beta^2 \tag{3.33}$$

$$x=\frac{1}{2}\frac{m'(x)}{m(x)}(\beta^2-x^2)\rightarrow\frac{2x}{(\beta^2-x^2)}=\frac{m'(x)}{m(x)} \tag{3.34}$$

$$\int\frac{dm(x)}{m(x)}=\int\frac{2xdx}{(\beta^2-x^2)}\rightarrow\ln(m(x))=-\ln(\beta^2-x^2)+\ln(c) \tag{3.35}$$

$$m(x)=\frac{c}{\beta^2-x^2} \tag{3.36}$$

Take $c=\beta^2$ and $\lambda=\mp\frac{1}{\beta^2}$, to get

$$m(x)=\frac{1}{1\pm\lambda x^2}. \tag{3.37}$$

Equation (3.37) into the equation (3.32) implies

$$\ddot{x}\mp\frac{\lambda x}{1\pm\lambda x^2}\dot{x}^2+\frac{\omega^2x}{1\pm\lambda x^2}=0;\qquad\lambda=\mp\frac{1}{\beta^2}\ ,\lambda\geq0. \tag{3.38}$$

This equation has the same form as that of (3.15) and has the exact solution

$$x(t)=A\cos(\Omega\mathrm{t}+\varphi)\,,\Omega^2=\frac{\omega^2}{(1\pm A^2\lambda)}\,. \tag{3.39}$$

However, to obtain the oscillator equations of Mathews-Lakshmanan we can directly use a group of four potential fields

$$V(x) = \begin{cases} \frac{-1}{2\lambda} m(x)\omega^2; \ \ for\ m(x) = 1/(1+\lambda x^2) \\ \frac{+1}{2\lambda} m(x)\omega^2; \ \ for\ m(x) = 1/(1-\lambda x^2) \\ \frac{1}{2} m(x)\omega^2 x^2; \ \ for\ m(x) = 1/(1\pm\lambda x^2) \end{cases} \tag{3.40}$$

in (1.5), (one potential at a particular time). That is almost completely associated to the nature of the specified PDM equation.

3.1.3 A Shifted Oscillator Case

Now, let us choose $f(x) = m(x)$, and the particle moves in a shifted harmonic oscillator force field given by the Lagrangian

$$L(x, \dot{x}, t) = \frac{1}{2} m(x)\dot{x}^2 - \frac{1}{2} m(x)\omega^2 (x+\zeta)^2. \tag{3.41}$$

Using equation (2.25) yields

$$q(x) = \frac{1}{f(x)} (x+\zeta)\sqrt{m(x)} \left(1 + \frac{1}{2}\frac{m'(x)}{m(x)}(x+\zeta)\right). \tag{3.42}$$

By choosing

$$f(x) = \left(1 + \frac{1}{2}\frac{m'(x)}{m(x)}(x+\zeta)\right), \tag{3.43}$$

we obtain

$$q(x) = (x+\zeta)\sqrt{m(x)}\,. \tag{3.44}$$

To find $m(x)$ we set $f(x) = m(x)$ and use the equation (3.43)

$$f(x) = 1 + \frac{1}{2}\frac{m'(x)}{m(x)}(x+\zeta) = m(x) \rightarrow m'(x) = \frac{2}{(x+\zeta)} m(x)(m(x)-1) \tag{3.45}$$

$$\int \frac{dm(x)}{m(x)(m(x)-1)} = 2\int \frac{dx}{x} \tag{3.46}$$

$$m(x) = \frac{1}{1 \pm \lambda(x+\zeta)^2}\ , \lambda \geq 0\ . \tag{3.47}$$

Then, the nonlocal point transformation comes as

$$L(q, \tilde{q}, \tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2 q^2 \leftrightarrow \left\{\begin{matrix} q(x) = (x+\zeta)\sqrt{m(x)} \\ \tilde{q} = \dot{x}\sqrt{m(x)} \\ d\tau = m(x)dt \\ g(x) = m^3(x) \end{matrix}\right\} \leftrightarrow L(x, \dot{x}, t)$$

$$= \frac{1}{2}\frac{\left(\dot{x}^2 - \omega^2(x+\zeta)^2\right)}{1 \pm \lambda(x+\zeta)^2}\ . \tag{3.48}$$

By apply the Euler-Lagrange equation (1.2) one obtains

$$\ddot{x} \mp \frac{\lambda(x+\zeta)}{1 \pm \lambda(x+\zeta)^2}\dot{x}^2 + \frac{\omega^2(x+\zeta)}{1 \pm \lambda(x+\zeta)^2} = 0\ . \tag{3.49}$$

With an exact solution

$$x(t) = A\cos(\Omega \mathrm{t} + \varphi) - \zeta; \quad \Omega^2 = \frac{\omega^2}{(1 \pm A^2\lambda)}. \tag{3.50}$$

Here, it is easy to notice that the case of shifted Mathews-Lakshmanan is perfectly symmetric to the first case, and we can get same results if the PDM in (3.47) moves in the potential field $V(x) = \frac{1}{2}m(x)\omega^2\beta^2$. So, we can note that the shifted oscillator equations of Mathews-Lakshmanan (3.49) are acquired by taking the group of four potential force fields

$$V(x) = \begin{cases} \frac{-1}{2\lambda}m(x)\omega^2; & for\ m(x) = 1/(1+\lambda(x+\zeta)^2) \\ \frac{+1}{2\lambda}m(x)\omega^2; & for\ m(x) = 1/(1-\lambda(x+\zeta)^2) \\ \frac{1}{2}m(x)\omega^2(x+\zeta)^2; & for\ m(x) = 1/(1 \pm \lambda(x+\zeta)^2) \end{cases} \tag{3.51}$$

in (1.5) (one at a specific time).

3.2 A nonlinear quadratic PDM oscillator

Let us suggest $f(x) = 1$. Define a PDM particle moves in a harmonic oscillator system with a potential field $V(x) = -\frac{\alpha^2}{2\lambda^2} m(x)(1+2\lambda x)(1+\lambda x)^2$. Then the Lagrangian is

$$L(x, \dot{x}, t) = \frac{1}{2} m(x)\dot{x}^2 + \frac{\alpha^2}{2\lambda^2} m(x)(1+2\lambda x)(1+\lambda x)^2. \tag{3.52}$$

We have to introduce

$$q(x) = x m(x)^{1/4} = \int \sqrt{m(x)} dx \tag{3.53}$$

$$\frac{dq(x)}{dx} = m(x)^{1/4} + \frac{1}{4} \frac{m'(x)}{m(x)^{3/4}} x = \sqrt{m(x)} \;\rightarrow\; 1 + \frac{1}{4} \frac{m'(x)}{m(x)} x = m(x)^{1/4} \tag{3.54}$$

In a straightforward manner we get

$$m(x) = \frac{1}{(1+\lambda x)^4}. \tag{3.55}$$

This produces the nonlocal point transformation

$$L(q, \tilde{q}, \tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2 q^2 \leftrightarrow \left\{ \begin{array}{c} q(x) = x m(x)^{1/4} \\ g(x) = m(x) \\ d\tau = dt \\ \tilde{q} = \dot{x}\sqrt{m(x)} \end{array} \right\} \leftrightarrow L(x, \dot{x}, t)$$

$$= \frac{1}{2} m(x) \left(\dot{x}^2 + \frac{\alpha^2}{\lambda^2} (1+2\lambda x)(1+\lambda x)^2 \right). \tag{3.56}$$

Using the equations (3.55) and (3.56) equation (1.2) we obtain

$$\ddot{x} - \frac{2\lambda}{1+\lambda x}\dot{x}^2 + \alpha^2 x(1+\lambda x) = 0;\; \alpha^2 = \omega^2. \tag{3.57}$$

With the solution [16,17]

$$x(t) = \frac{A\cos(\alpha t+\varphi)}{1-\lambda A\cos(\alpha t+\varphi)};\quad 0 \leq A < \frac{1}{\lambda}. \tag{3.58}$$

Then, the momentum $p(x) = m(x)\dot{x}$ becomes

$$p(t) = -A\alpha\sin(\alpha t+\varphi)\,(1-\lambda A\cos(\alpha t+\varphi))^2. \tag{3.59}$$

Plot $p(t)$ vs. $x(t)$ for different values of A and choose λ *and* $\alpha = 1$ [17].

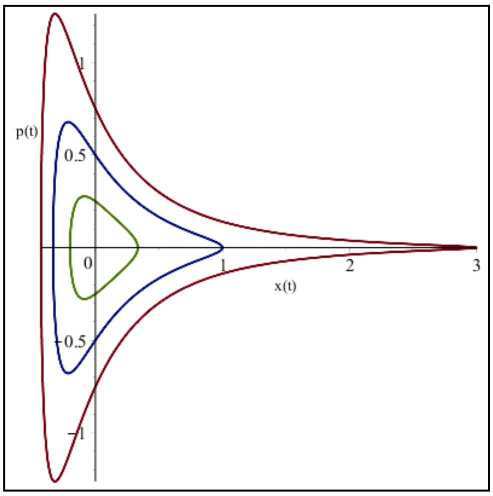

Figure 2: The phase trajectories in (*p(t) vs. x(t)*) plane for a quadratic nonlinear PDM oscillator.

In the figure 2, note that the trajectories are bounded in the left half $(x(t),\ p(t))$ plane by the condition $0 \leq A < {}^{1}/_{\lambda}$, beyond which the trajectories have a horizontal asymptote at $(x \to +\infty)$. Also, we see that the trajectories are symmetry on x and centered at the origin.

3.3 A Morse-oscillator

Let us defined a position-dependent mass particle for $f(x) = \eta$, moving in a potential field $V(x) = -\frac{1}{2}m(x)\alpha^2(1-e^{-\eta x})^2$, working like a Morse oscillator. Then, the Lagrangian is

$$L(x,\dot{x},t) = \tfrac{1}{2}m(x)\dot{x}^2 + \tfrac{1}{2}m(x)\alpha^2(1-e^{-\eta x})^2. \tag{3.60}$$

To work on the nonlocal point transformation we insert

$$q(x) = \sqrt{m(x)} - 1 = \int f(x)\sqrt{m(x)}dx \tag{3.61}$$

$$\frac{dq(x)}{dx} = \frac{1}{2}\frac{m'(x)}{\sqrt{m(x)}} = \eta\sqrt{m(x)} \;\rightarrow\; \frac{m'(x)}{m(x)} = 2\eta \tag{3.62}$$

$$\int \frac{dm(x)}{m(x)} = 2\int \eta dx \rightarrow \ln\big(m(x)\big) = 2\eta x + \mathrm{c}, \tag{3.63}$$

With the choice $\mathrm{c} = 0$ we get

$$m(x) = \exp(2\eta x). \tag{3.64}$$

We define η as a ratio between the deformed-time and the exact time $\eta = {}^{\tau}/_{t}$. After this processes, we cast the nonlocal point transformation as

$$L(q,\tilde{q},\tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2 q^2 \;\leftrightarrow\; \left\{\begin{matrix} q(x) = \sqrt{m(x)} - 1 \\ g(x) = \eta^2\, m(x) \\ d\tau = \eta\; dt \\ \tilde{q} = \dot{x}\sqrt{m(x)} \end{matrix}\right\} \leftrightarrow\; L(x,\dot{x},t)$$

$$= \tfrac{1}{2}m(x)(\dot{x}^2 - \alpha^2(1-e^{-\eta x})^2). \tag{3.65}$$

And the equation of a Morse-oscillator can be obtained using equation (1.2) with the PDM-mass of (3.64)

$$\ddot{x} + \frac{1}{2}\frac{m'(x)}{m(x)}\dot{x}^2 - \alpha^2(1-e^{-\eta x})\left(\frac{1}{2}\frac{m'(x)}{m(x)}(1-e^{-\eta x}) + \eta e^{-\eta x}\right) = 0, \tag{3.66}$$

To obtain

$$\ddot{x} + \eta\dot{x}^2 + \frac{\alpha^2}{\eta}(1-e^{-\eta x}) = 0; \quad \alpha^2 = \eta^2\omega^2, \tag{3.67}$$

with an exact solution [16,17]

$$x(t) = \frac{1}{\eta} \ln(1 + A\cos(\alpha t + \varphi))\, ;\; 0 \leq A < 1. \tag{3.68}$$

Hence, the momentum becomes

$$p(t) = -\frac{\alpha}{\eta} A\sin(\alpha t + \varphi)\,(1 + A\cos(\alpha t + \varphi)). \tag{3.69}$$

Plot $p(t)$ vs. $x(t)$ for different values of A and choose $\lambda, \eta, \omega\,, \alpha\; = 1\; and\; \varphi = 0$ [17].

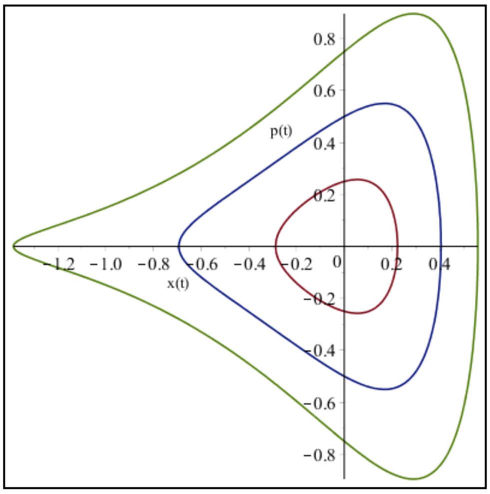

Figure 3: The phase trajectories in (*p(t) vs. x(t)*) plane for a Morse-type oscillator.

In the Morse-type figure 3, we observe that it has the same behavior of the previous example but the trajectories bounded in the right half $(x(t), p(t))$ plane by the condition $0 \leq A < 1$, and the asymptote at $(x \to -\infty)$.

3.4 Nonlinear Deformation of an Isotonic Oscillator

Let us look on a classical unit mass particle moves in general coordinate with time deformed, and the potential force field is an isotonic oscillator. The Lagrangian for isotonic oscillators' equation is

$$L(q, \tilde{q}, \tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2 q^2 - \frac{\beta}{q^2}\,. \tag{3.70}$$

Applying the Euler-Lagrange equation (2.2)

$$\frac{d}{d\tau}\left(\frac{dq}{d\tau}\right) = \frac{d^2q}{d\tau^2}, \quad \frac{\partial\mathcal{L}}{\partial q} = -\omega^2 q + \frac{2\beta}{q^3}, \tag{3.71}$$

one would get

$$\frac{d^2q}{d\tau^2} + \omega^2 q - \frac{2\beta}{q^3} = 0, \tag{3.72}$$

Equation (3.72) is called as the Ermakov-Pinney equation [26]. We can write its general solution [24,25] as

$$q(\tau) = \frac{1}{\omega A}\sqrt{(\omega^2A^4 - 2\beta)sin^2(\omega\tau + \delta) + 2\beta}. \tag{3.73}$$

We will use the same condition of the nonlocal transformation that we have used in example (3.1.1). Applying the condition $f(x) = m(x)$ and the equations (2.7) and (3.10) on the equation (3.70) we get the nonlocal transformation

$$L(q,\tilde{q},\tau) = \frac{1}{2}\tilde{q}^2 - \frac{1}{2}\omega^2q^2 - \frac{\beta}{q^2} \leftrightarrow \left\{\begin{matrix} q(x) = x\sqrt{m(x)} \\ \tilde{q} = \dot{x}\sqrt{m(x)} \\ d\tau = m(x)dt \\ g(x) = m^3(x) \end{matrix}\right\} \leftrightarrow L(x,\dot{x},t)$$

$$= \frac{1}{2}\frac{(\dot{x}^2 - \omega^2x^2)}{(1\pm\lambda x^2)} - \frac{\beta(1\pm\lambda x^2)}{x^2}. \tag{3.74}$$

Applying the Euler-Lagrange equation (1.2) one obtains

$$\ddot{x} \mp \frac{\lambda x}{1\pm\lambda x^2}\dot{x}^2 + \frac{\omega^2 x}{1\pm\lambda x^2} - \frac{2\beta}{x^3}(1 \pm \lambda x^2) = 0;\ \alpha = \mp\omega^2 . \tag{3.75}$$

The general solution is shown to be [24,25]

$$x(t) = \frac{1}{\Omega A}\sqrt{(\Omega^2A^4 - 2\beta)sin^2(\Omega t + \delta) + 2\beta};\ \ \omega^2 = (1 \pm \lambda A^2)\left(\Omega^2 \pm \frac{2\lambda\beta}{A^2}\right). \tag{3.76}$$

Using the equation (3.25) to find the momentum

$$p(t) = \frac{(\Omega^2 A^4 - 2\beta)\sin(2(\Omega t + \delta))}{2A\sqrt{(\Omega^2 A^4 - 2\beta)sin^2(\Omega t + \delta) + 2\beta}\left(1 \pm \frac{\lambda}{\Omega^2 A^2}((\Omega^2 A^4 - 2\beta)sin^2(\Omega t + \delta) + 2\beta)\right)}. \qquad (3.77)$$

The plotting of $p(t)$ vs. $x(t)$ for various values of A , $\lambda = -1$, $\beta = \omega = 1$ and $\delta = 0$ [10] give

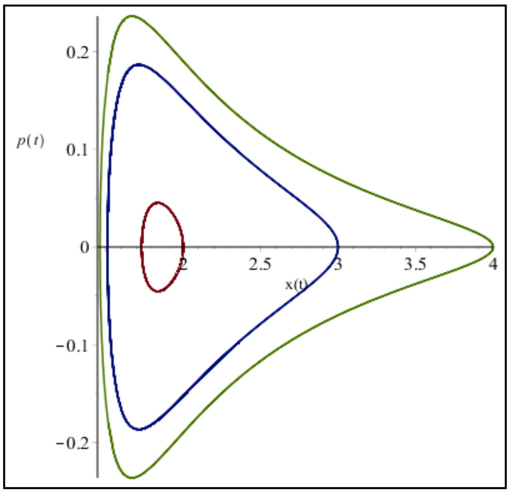

Figure 4: The phase trajectories in (p(t) vs. x(t)) plane for Nonlinear Deformation of an Isotonic Oscillator.

In this figure, we can see that the trajectories are not centered at the origin because we have a point singularity at $A = 0$. It can be noted also that when $(x \to +\infty)$ the momentum p tends to its limiting values.

Chapter 4

CONCLUSION

The PDM Lagrangian is discussed in this thesis carefully. We started by introducing a nonlocal point transformation in the generalized coordinates for PDM Lagrangians which is mapped to a unit-mass Lagrangian. Additionally, Euler-Lagrange equations are derived, and determination the appropriate conditions for its invariance. Several functions of the nonlinear harmonic oscillator are solved to find the linearization equations for the Euler-Lagrange equation with PDM. The examples illustrated comprised of the Mathews Lakshmanan PDM-nonlinear oscillators, a quadratic nonlinear oscillator, a Morse-type oscillator, and PDM-deformed isotonic oscillator.

With reference to the Mathews-Lakshmanan nonlinear oscillators (3.1.1) and (3.1.2), we see that the PDM-function (3.37) moved in a group of potential fields (3.40) comprised from four potential each one at a time, which have accurately the same dynamical properties in the equations of motion (3.15) and (3.38), as well as the trajectories are completely similar. The total energy $E = T + V$ is also the same

$$E = \frac{\omega^2}{2}\frac{A^2}{(1 \pm \lambda A^2)}, \tag{4.1}$$

this is due to the characteristics of the PDM-function used. The shifted Mathews-Lakshmanan nonlinear oscillators (3.1.3) has the same properties, paths but with total energies E,

$$E = \frac{\omega^2}{2}\frac{(A-\zeta)^2}{(1 \pm \lambda (A-\zeta)^2)}. \tag{4.2}$$

In addition to, our nonlocal point transformation (2.6) used to linearize the PDM nonlinear oscillators. Also, we noticed that the transformation (2.6) has mapped the nonlinear isotonic oscillator of unit-mass constant into a nonlinear deformation isotonic oscillator with PDM. Where, $L(q, \tilde{q}, \tau)$ plays like a reference Lagrangian and $L(x, \dot{x}, t)$ like a purpose Lagrangian in (3.74).

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