Perihelion Precession in the Solar System

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ABSTRACT

In this thesis, we study about perihelion precession in the solar system, one of the most interesting aspects of astrophysics that include both aspects of General Relativity and classical mechanics. The phenomenon, by which perihelion of elliptical orbital path of a planet appears to rotate around a central body (which our central body is the Sun) is known as the precession of the orbital path. The special situation of Mercury arises as it is the smallest and the closest to the Sun amongst eight planets in the solar system and since the precession of Mercury's orbital path is much greater than other planets so it has attracted much attention in comparison to others. This natural phenomenon was realized by astronomers many years ago where they could not explain many strange observatory data.

This thesis deals with the derivation of the equation of motion and the corresponding approximate solution leading to the perihelion advance formula. Therefore, our preliminary aim is to find solutions for equations of motion and derive a general formula by considering the General Relativity concepts and Classical Mechanics.

Keywords: Perihelion, Advance of Perihelion, Aphelion, Perihelion precession

ÖZ

Bu tez, Güneş Sistemi'nde perihelionun ilerlemesi ile ilgilidir. Araştırma konumuz hem genel görelilik, hem de klasik mekanik bakış açılarını içerdiğinden ötürü; astrofiziğin en ilginç konularından biri olarak tanımlanabilir. Bir gezegenin eliptik orbital yörüngesine ait prehelionunun merkezi cisim etrafında dönmesi, yol yörüngesinin presesyonu olarak bilinir. Merkür, Güneş Sistemi'ndeki sekiz gezegenin en küçüğü ve Güneş'e en yakın olanıdır ve diğer gezegenlere göre daha fazla dikkat çekmektedir. Bunun nedenlerinden biri, Merkür'ün yol yörüngesinin presesyonunun diğerlerine kıyasla daha büyük olmasıdır.

Bu tez, hareket denkleminin derivasyonu ve perihelion ilerleyişi formulü ile sonuçlanan yaklaşık çözümler ile ilgilidir. Bu yüzden temel amacımız, genel görelilik ve klasik mekaniği hesaba katarak çözümler bulmak ve genel bir formül elde etmektir.

Anahtar Kelimeler: Perihelion, Perihelion ilerlemesi, Aphelion, Perihelion ilerlemesi

DEDICATION

To My Family

ACKNOWLEDGMENT

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Chapter 1

INTRODUCTION

One of the first phenomenon which was elucidated by Einstein's General Theory of Relativity was anomalous precession of the perihelion of Mercury. This theory became successful because Einstein provided a numerical value for perihelion precession of Mercury that it had an excellent similarity with observation value. He changed the apprehension of astronomers and physicist about the concept of space and time, and led to a different way of viewing the problems.

According to the meaning, precession is a change in the orientation of rotatioanl planet around the Sun or a central body as it illustrates in the Fig (1.1), the semi major axis rotate around the central body. The figure shows four elliptical orbits which they are shifting with respect to each others, this shifting or advance called the advance of planet's perihelion or perihelion precession of planet. Furthermore the aphelion which is opposite point of perihelion and it is the farthest distance between planets and the Sun, it advances at the same angular rate as the perihelion is shown in the figure (1.1).

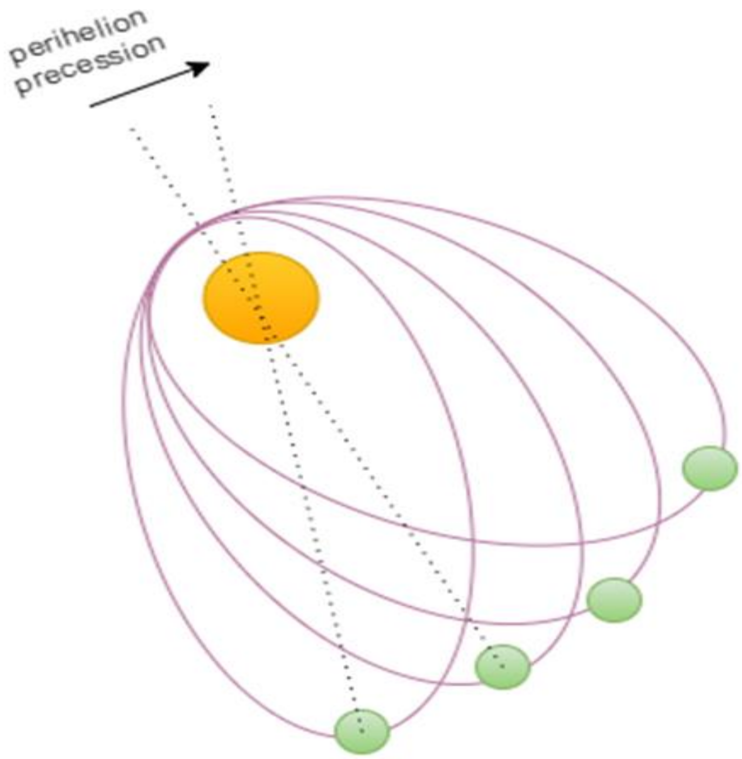

Figure 1.1: Exaggerated view of the perihelion precession of a planet

The perihelion precession for the first time was reported in 1859, by a French mathematician Urbain Jean Joseph Le Verrier (1811-1877), which motivated the astronomers and theorists to study about the solar system more and more. The unusual orbital motion of Mercury turned his attention to discern advance of perihelion of Mercury [8]. He intromted this phenomenon to an unknown planet which he named Vulcan and it was never found. He obtained his results by using Newtonian mechanics that his value of precession of the perihelion was 38 arcsecond per century.

But there was something wrong with the value that was obtained by Verrier because advertently determined in 1895 by an Canadian–American astronomer and mathematithion Simon Newcomb (1835-1909) [10]. The theory of Newcomb confirmed the Varrier's finding about the advance of Mercury's perihelion. He also followed the Newtonian method with just making some small changes in the planetary masses. He could obtain an amazing value for Mercury's advance, it was 42.95 arcseconds per century which was incredibly close to the real value.

There is an important point which according to Newton's law the planets (or at that time the Mercury) can not have advance when one considers only the gravitational force between the planets and the Sun. On the other hand the 90% of the mass of the solar system belonged to the Sun, so it shows that the masses of other planets in comparison with Sun are negligible, and since the planets with small masses move in the static gravitational field of the Sun, also the planet's static gravitational potential is neglegible.

Later on this natural phenomenon was eventually explained in 1915 by Albert Einstein's General theory of Relativity that could give axceptable answers for some questions.

Einstein published a paper in 25^{th} of November 1915 based on vacuum field equations Actually the derivation of him in this paper mathematically was interesting because he obtained the equation of motion from the vacuum field equation without considering Schwarzschild metrics. Einstein used an approximation to the spherically symmetric metric for finding the solution for vacuum field equation, he used this instead of Cartesian coordinate system and this approximate metric can he expressed in Polar coordinatea

$$(d\tau)^2 = \left(1 - \frac{2m}{r}\right)(dt)^2 - \left(1 + \frac{2m}{r}\right)(dr)^2 - r^2(d\theta)^2 - r^2 sin(\theta)^2(d\varphi)^2 \qquad (1.1)$$

where m is mass of the central body and r is distnce between planet ant the Sun. The relation between Einstein's approximation for coefficient of $(dr)^2$and the real one which expressed very soon by Schwarzchild is

$$(g_{rr})_E = x(g_{rr})_S \tag{1.2}$$

Which

$$x = 1 - (\frac{2m}{r})^2. \tag{1.3}$$

Einstein by estimating of Christoffel symbol and on the other hand by using his approximate metric for spherical symmetry, defined the geodesic equations of motion as

$$x(\frac{du}{d\varphi})^2 = \frac{2A}{B^2} + \frac{\alpha}{B^2}u - u^2 + \alpha u^3 \tag{1.4}$$

where $u = 1/r$, φ is the angular coordinate in the orbital plane, and A and B are the constants of integration such that A is proportional angular momentum and B is related to energy. The exact value of x according to Schwarzschild metric is 1, but according to Einstein's approximation it is $(1 - \alpha^2 u^2)$.and after some calculation finally he realized that it must be one. By integrating from Eq. (1.4), the angular difference $\Delta\varphi$,was obtained. He calculated the angular differece by just accurate and necessary values and considered two points as limitation, from aphelion point to perihelion point.

After solving the polynomial, which is exactly Eq. (1.4), he found the arc length from the perihelion to aphelion and equivalently from aphelion to the next perihelion. So for finding the total $\Delta\varphi$ for one orbit from one perihelion to the next, this value will be twice and for finding precession per orbit this amount should be subtracted by 2π. So the result was obtained as

$$2\Delta\varphi - 2\pi = \frac{6\pi m}{L} \tag{1.5}$$

Where L is semi-latus rectom of the elliptical orbit (for Mercury is 55.4430 million km) and m is the Sun's mass in geometrical units (1.475 km). By substituting the amounts in Eq. (1.5) the result will give 0.1034 arc seconds per revolution.and Mercury has 414.9378 revolutions per century so we have 42.9195 arc seconds per century, which was close to the observed amount. Furthermore Einstein's result applies to any eccentricity, not just for circular orbit.

In 1907 he started to work on his gravitational theory that he hoped to lead him for finding perihelion precession of the Mercury. After eight years finally he could obtain it.

After some month (in 22 December) that Einstein published his paper, a German Physicist and astronomer Karl Schwarzschild (1873-1916), could obtaine the exact solution for the Einstain's field equation of General Relativity for non-rotating gravitational fields. At first he changed the first order approximation of Einstein and found an exact solution for it. Next he introduced only one line element which satisfied four conditions of Einstein. On the other hand Schwarzschild considered a spherical symmetry , with considering a body exactly in the origin of the coordinate by assuming the isotropy of space and a static solution, (a static solution means there is no dependence on time) then we can say there is spherical symmetry around the center. So his line element showed the spherical coordinate in the best way as

$$(ds)^2 = \left(1 - \frac{\beta}{R}\right) dt^2 - \left(1 - \frac{\beta}{R}\right)^{-1} dr^2 - R^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (1.6)$$

where $\beta = 2GM/c^2$, $(c^2 = 1)$ and G is Newton's gravitational constant. But the equation of orbit of Schwarzschild remained same as Einstein's equation. Unfortunately he died in the following year (in 11 may 1916) during the First world war and the Astroied 837 Schwarzchilda was named in his honor.

Actually Mercury is not the only planet in the solar system that has precession and it can be seen even for the nearly circular orbit or small eccentricity as the Earth or Venus, at first it seems difficult to find the precession of this kinds of orbits with small eccentricity but modern measurement techniques and computerized analysis of the values make it possible and more accurate.

There have been several studies in this issue for finding more accurate value, and we are interested to explore more in this thesis, we are aiming to provide an exact solution for the second and higher order corrections with all steps explained in Chapter 2.

This will start by the geodesic equations obtained from the Schwarzschild gravitational metric. We assume that the motion of the planets is a time-like geodesic in Schwarzschild metric rotating around the Sun. According to the computations for finding the equation for perihelion precession we follow all the steps by considering both important aspects of physics, the General Relativity theory and classical mechanics.

At the end of the second chapter, according to some data and the perihelion preccesion equation we prepere a table that represents the results for eight planets in the solar system (Mercury,Venus, Earth, Mars, Jupiter, Saturn, Uranuse and Neptune). There is a conversion in our table that gives us two different values for advance of perihelion that we express one of them as $(rad/orbit)$ and the other one as $(sec/century)$, the relation between these two conversion is represented by

$$(\Delta\delta)_{\frac{s}{c}} = \left(\frac{100\ yr}{sidereal\ period\ years}\right)\left(\frac{360 \times 60 \times 60}{2\pi}\right)(\Delta\delta)_{\frac{r}{o}} \tag{1.7}$$

The sidereal period is the orbital period of each planet in a year, for example for the Mercury the orbital period is 87.969 day and each year has 365 days, 5 hours, 48 minutes and 46 seconds, the division of these two numbers will give us the values of sidereal period per year. for the Mercury. If we follow the same rule we will obtain for each planet as in Table (1.1).

Table 1.1: Sidereal Period of The Planets in The Solar System

Planet	Mercurt	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Nepton
Sidereal period (year)	0.2408	0.6152	1.00	1.8809	11.865	11.865	83.744	165.95

Chapter 2

EQUATION OF MOTION OF THE PLANETS IN THE SOLAR SYSTEM

Sun is almost spherically symmetric and compared to the position of the planets, its radius is very small. Hence, one may consider the spacetime around the Sun to be in form of the solution of the vacuum Einstein's equations which is very well known to be the Schwarzschild spacetime with the line element

$$ds^2 = -\left(1 - \frac{2M_\odot}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M_\odot}{r}\right)} + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{2.1}$$

in which $M_\odot$ is the mass of Sun, c is the speed of light and $G = 1$ is the Newton's gravitational constant. For motion of the planets in the Solar System, we assume that the effect of the planets on the spacetime, individually, is negligible and therefore each planet moves as a test particle. Thus, the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\mathrm{m}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{2.2}$$

can be used for the motion of each planet with its mass m. Let us note that

$$g_{\alpha\beta} = diag\left[-\left(1 - \frac{2M_\odot}{r}\right), \frac{1}{\left(1 - \frac{2M_\odot}{r}\right)}, r^2, r^2 sin^2\theta\right] \tag{2.3}$$

is the metric tensor for the Schwarzschild spacetime.

Applying the metric tensor in the Lagrangian, one finds

$$\mathcal{L} = \frac{m}{2}\left(-\left(1 - \frac{2M_{\odot}}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M_{\odot}}{r}\right)} + r^2\left(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2\right) \right) \quad (2.4)$$

where dot stands for taking derivative with respect to the proper time τ measured by an observer located on the particle.and consequently the Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^{\alpha}}\right) - \frac{\partial\mathcal{L}}{\partial x^{\alpha}} = 0 \quad (2.5)$$

give the basic equations of motion. Before we give the explicit form of the equations, we note that the spacetime is spherically symmetric and as a result, the angular momentum of the test particle in a preferable direction (say ʐ) remains constant. Therefore, from the beginning we know that the motion happens to be in a 2-dimensional plane which by setting the proper system of coordinates one can choose $\theta = \frac{\pi}{2}$ at which the equatorial plane is. Based on this fact, the three Euler–Lagrange equations are given by

$$\frac{d}{d\tau}\left[\left(1 - \frac{2M_{\odot}}{r}\right)\dot{t}\right] = 0 \quad (2.6)$$

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0 \quad (2.7)$$

and

$$\frac{d}{d\tau}(r^2\dot{\varphi}) = 0. \quad (2.8)$$

The first and the third equations imply

$$\left(1 - \frac{2M_{\odot}}{r}\right)\dot{t} = E \tag{2.9}$$

and

$$r^2\dot{\varphi} = \ell \tag{2.10}$$

in which E and ℓ denote two integration constants related to the energy and angular momentum of the test particle. As the planets are moving on a timelike worldline, their four-velocity must satisfy

$$U^{\mu}U_{\mu} = -1 \tag{2.11}$$

in which $U^{\mu} = \frac{\partial x^{\mu}}{\partial \tau}$. Explicitly, Eq. (2.11) reads

$$-\left(1 - \frac{2M_{\odot}}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M_{\odot}}{r}\right)} + r^2\dot{\varphi}^2 = -1 \tag{2.12}$$

where we set $\theta = \frac{\pi}{2}$ .From Eq. (2.9) and Eq. (2.10) one finds $\dot{t}$ and $\dot{\varphi}$ which upon a substitution in Eq. (2.12) we find the proper equation for the radial coordinate i.e

$$\dot{r}^2 + \left(1 - \frac{2M_{\odot}}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = E^2. \tag{2.13}$$

To proceed further, we use the chain rule to find a differential equation for r with respect to φ. Therefore, the latter equation becomes

$$\left(\frac{dr}{d\varphi}\dot{\varphi}\right)^2 + \left(1 - \frac{2M_{\odot}}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = E^2 \tag{2.14}$$

or in more convenient form it reads

$$\left(\frac{dr}{d\varphi}\right)^2 \frac{\ell^2}{r^4} + \left(1 - \frac{2M_\odot}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = E^2. \tag{2.15}$$

As in the Kepler problem, we introduce a new variable $u = \frac{1}{r}$ and rewrite the last differential equation given by

$$\left(\frac{du}{d\varphi}\right)^2 + (1 - 2M_\odot u)\left(\frac{1}{\ell^2} + u^2\right) = \frac{E^2}{l^2}. \tag{2.16}$$

This is the master first order differential equation to be solved for $u(\varphi)$. To get closer to this differential equation, let us take its derivative with respect to φ which after a rearrangement yields

$$\frac{d^2u}{d\varphi^2} + u = \frac{M_\odot}{\ell^2} + 3M_\odot u^2. \tag{2.17}$$

A comparison with the Newtonian equation of motion of planets reveals that $3M_\odot u^2$ is the additional term to the Classical Mechanics due to General Relativity (GR). The solution without GR correction is very straight forward and is given by

$$u = \frac{1}{r} = \frac{M_\odot}{\ell^2}(1 + ecos(\varphi - \varphi_0)) \tag{2.18}$$

in which φ_0 is an arbitrary phase and e stands for the eccentricity of the elliptic orbit of the planet. The initial phase φ_0 is an arbitrary constant which can be set to zero without loss of generality, as we are allowed to rotate our system of coordinate about the symmetry axis. Unfortunately, the master Eq.(2.16) with the GR correction cannot be solved analytically. However, the nature of the additional term suggests that it is a very small correction to the classical motion. Therefore, an appropriate approximation method may give results being significantly acceptable. Hence, we shall consider the GR term as a small perturbation to the classical path of the planets. To do so, we consider $\frac{3M^2_{\odot}}{\ell^2} = \lambda \ll 1$ and hence we expand the orbit of the planets in terms of λ,i.e.,

$$u = \sum_{n=0}^{\infty} \lambda^n u_n \tag{2.19}$$

and

$$u'' = \sum_{n=0}^{\infty} \lambda^n u''_n \tag{2.20}$$

and the prime stands for taking derivative with respect to φ, in which u_0 is the orbit without the GR correction, i.e.,

$$u_0 = \frac{M_{\odot}}{\ell^2}(1 + ecos\varphi), \tag{2.21}$$

Plugging this into the master Eq. (2.16), one finds

$$\sum_{n=0}^{\infty} \lambda^n u''_n + \sum_{n=0}^{\infty} \lambda^n u_n = \frac{M_{\odot}}{\ell^2} + \frac{\ell^2}{M_{\odot}} \lambda \left(\sum_{n=0}^{\infty} \lambda^n u_n \right)^2. \tag{2.22}$$

In the zeroth order, one evaluate the Newtonian equation of motion given by

$$u''_0 + u_0 = \frac{M_{\odot}}{\ell^2} \tag{2.23}$$

whose solution has already been given in Eq. (2.21). In general, the n^{th} order equation (with $n \geq 1$) is obtained to be

$$u_n'' + u_n = \frac{\ell^2}{M_\odot} \sum_{i=0}^{n-1} u_i u_{n-1-i}. \tag{2.24}$$

For instance the first order equation becomes

$$u_1'' + u_1 = \frac{\ell^2}{M_\odot} u^2{}_0 \tag{2.25}$$

which is another second order differential equation and it is worthwhile to mention that it is nonhomogeneous due to the presence of $u^2{}_0$ at the right hand side. Some of the higher order corrections which can be extracted from the master Eq. (2.16) s are as follows. For $n = 2,3$ and 4 Eq. (2.24) admits

$$u_2'' + u_2 = \frac{\ell^2}{M_\odot} (2u_0 u_1) \tag{2.26}$$

$$u_3'' + u_3 = \frac{\ell^2}{M_\odot} (2u_0 u_2 + u^2{}_1) \tag{2.27}$$

and

$$u_4'' + u_4 = \frac{\ell^2}{M_\odot} (2\, u_0 u_3 + 2u_1 u_2) \tag{2.28}$$

respectively.

Our next step is to solve the equation of the first order correction which explicitly reads

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \frac{M_{\odot}}{\ell^2}(1 + ecos\varphi)^2 \tag{2.29}$$

This is a nonhomogeneous ordinary differential equation of second order with a constant coefficient whose solution involves two distinct parts. The first part is the solution to its homogenous form, whereas the second part is the particular solution. Both solutions will be discussed in the sequel.

First the homogenous equation which is given by

$$\frac{d^2u_1}{d\varphi^2} + u_1 = 0 \tag{2.30}$$

and its solution simply

$$u_{1h} = A_1 sin\varphi + B_1 cos\varphi \tag{2.31}$$

in which both A_1 and $B_1$are integration constants. The particular solution of Eq. (2.29) can be estimated by an expantion of the right-hand-side as

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \mu\left(1 + \frac{e^2}{2} + 2ecos\varphi + \frac{e^2}{2}cos2\varphi\right) \tag{2.32}$$

in which we set $\mu = \frac{M_{\odot}}{\ell^2}$ and $cos^2\varphi = (1 + cos2\varphi)/2$. Using the standard method of solving the nonhomogeneous second order differential equation with constant coefficients, one considers the ansatz

$$u_{1p} = A + (Bsin\varphi + Ccos\varphi)\varphi + Dsin2\varphi + Ecos2\varphi \tag{2.33}$$

in which all constants will be found by matching the left and the right side of Eq. (2.32). We apply this ansatz in Eq. (2.30) to get

$$A + 2B\cos\varphi - 2C\sin\varphi - 3D\sin 2\varphi - 3E\cos 2\varphi$$

$$= \mu\left(1 + \frac{e^2}{2} + 2e\cos\varphi + \frac{e^2}{2}\cos 2\varphi\right) \quad (2.34)$$

which after matching the two sides we find $A = \mu(1 + \frac{e^2}{2}), B = \mu e, C = 0, D = 0$ and $E = -\mu\frac{e^2}{6}$. Consequently, the particular solution becomes

$$u_{1p} = \left[1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6}\ \cos 2\varphi\right]. \quad (2.35)$$

Finally, the full solution is the sum of the homogenous and particular solutions which reads

$$u_1 = A_1\sin\varphi + B_1\cos\varphi + \left(\frac{M_\odot}{\ell^2}\right)\left[1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6}\cos 2\varphi\right]. \quad (2.36)$$

We note that the homogeneous solution can be written as

$$u_{1h} = \mathcal{A}\cos(\varphi - \varphi_0) \quad (2.37)$$

and for the same reason as for u_0, one can set the initial phase φ_0 to be zero.

Up to the first order correction, the orbit of a planet around the Sun is expressed by

$$u = \mu\left(1 + e\cos\varphi + \lambda\left(1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6}\cos 2\varphi\right)\right) \quad (2.38)$$

where we have absorbed the $\mathcal{A}\cos\varphi$ term into the other similar term in u_0. In other words, the homogeneous solution is not the solution we are really looking for but instead, the particular solution is the correction to be taken into account. Hence, for our next step, we consider

$$u_0 = \mu(1 + ecos\varphi) \tag{2.39}$$

and

$$u_1 = \mu(1 + \frac{2e^2}{3} + e\varphi sin\varphi - \frac{e^2}{3}cos^2\varphi) \tag{2.40}$$

For the second order correction, we have to solve the particular solution of Eq. (2.26). After we plug in the explicit forms of u_0 and u_1 , Eq. (2.26) reads

$$u_2'' + u_2 = \mu(1 + ecos\varphi)\left(1 + \frac{2e^2}{3} + e\varphi sin\varphi - \frac{e^2}{3}cos^2\varphi\right). \tag{2.41}$$

Using the standard method of solving the particular solution of second order nonhomogeneous differential equation, we obtain

$$u_2 = \mu\left\{-\frac{1}{2}e\varphi^2 cos\varphi - \frac{1}{12}e\varphi(-5e^2 + 8ecos\varphi - 18)sin\varphi \right.$$
$$\left. + \frac{1}{12}e^2cos^2\varphi(ecos\varphi \ - 8) + 2 \ + \frac{4e^2}{3}\right\}. \tag{2.42}$$

Since we are interested in the second and third order corrections, in the next step we find the particular solution for $n = 3$ equation which is given by Eq. (2.27). Without going through the details of the procedure, we give the final solution as

$$u_3 = \mu\left\{-\frac{1}{6}e\varphi^3 sin\varphi + \frac{1}{12}e\varphi^2(8ecos^2\varphi - 4e - (5e^2 + 18)cos\varphi)\right.$$

$$+\frac{1}{36}e\varphi sin\varphi((9ecos\varphi - 84 - 10e^2)ecos\varphi + 126 + 45e^2)$$

$$-\frac{1}{54}e^3cos^3\varphi(ecos\varphi - 18) - \frac{2}{27}(4e^2 + 27)e^2cos^2\varphi + 5$$

$$\left.+\frac{e^2(13e^2 + 108)}{27}\right\}. \tag{2.43}$$

This can be continued to any order of corrections in principle, but in the case of the Solar System, one may not need to go more than the first order.

In our solar system, λ is very small and for a good approximation one can use only the first order approximation i.e.,

$$u \simeq \mu\left(1 + ecos\varphi + \lambda\left(1 + \frac{e^2}{2} + e\varphi sin\varphi - \frac{e^2}{6}cos2\varphi\right)\right) \tag{2.44}$$

although $\lambda \ll 1$, the term including $e\varphi sin\varphi$ with large φ becomes significant and as a consequence, we can simplify this expression even further as

$$u \simeq \mu(1 + e(cos\varphi + \lambda\varphi sin\varphi)). \tag{2.45}$$

Let us note that while φ is increasing, we may still consider $\lambda \ll \lambda\varphi \ll 1$ which implies $\lambda\varphi \simeq sin(\lambda\varphi)$ and $cos(\lambda\varphi) \simeq 1$. Applying these into Eq. (2.45), one obtains

$$u \simeq \mu(1 + e(cos(\lambda\varphi)cos\varphi + sin(\lambda\varphi)sin\varphi)). \tag{2.46}$$

After using $cos(a)cos(b) + sin(a)sin(b) = cos(a - b)$, the latter becomes

$$u \simeq \mu(1 + e\cos((1-\lambda)\varphi)). \tag{2.47}$$

This relation clearly suggests that the period of the motion is not $2\boldsymbol{\pi}$ any more and instead it is given by

$$(1-\lambda)\Delta \simeq 2\pi \tag{2.48}$$

in which Δ is the period of the motion. This, results

$$\Delta \simeq \frac{2\pi}{1-\lambda} \tag{2.49}$$

and since $\lambda \ll 1$, one may apply $\frac{1}{1-\lambda} = \sum_{k=0}^{\infty} \lambda^k$ which in first order it yields

$$\Delta \simeq 2\pi(1+\lambda) \tag{2.50}$$

This expression shows a perihelion precession per orbit for the planet under study due to the GR term equal to $\delta\Delta = \Delta - \Delta_0 \simeq 2\pi\lambda$ in which $\Delta_0 = 2\pi$ is the period of the planet's orbit predicted by Newton's gravity. As λ in natural units was given by $\lambda = \frac{3M_\odot^2}{\ell^2}$ where both $M_\odot$ and ℓ are in natural units one has to convert λ into geometrized units which is given as

$$\lambda = \left(\frac{3M_\odot^2}{\ell^2}\right)_{natural\ units} = \left(\frac{3M_\odot^2 G^2}{\ell^2 c^2}\right)_{geometrized\ units}. \tag{2.51}$$

Let us add that to convert mass and angular momentum per unit mass from natural units into geometrized units, we must use the proper coefficients. In this case $(M_\odot)_{NU} = \frac{G}{c^2}(M_\odot)_{GU}$ and $(L)_{NU} = \frac{G}{c^3}(L)_{GU}$ which amounts to $(\ell)_{NU} = \frac{1}{c}(\ell)_{GU}$. Finally, in SI units we find

$$\delta\Delta \simeq \frac{6\pi M_{\odot}^2 G^2}{\ell^2 c^2}. \tag{2.52}$$

Herein, $M_{\odot}$ is the mass of sun in kg, G is the Newton's gravitational constant, c is the speed of light in m/s and $\ell = \frac{L}{m}$ in which L is the angular momentum of the planet and m is the mass of the planet. Therefore, more precisely one finds

$$\delta\Delta \simeq \frac{6\pi M_{\odot}^2 m^2 G^2}{L^2 c^2}. \tag{2.53}$$

Next, we go back to the classical Newtonian gravity and the well-known Kepler's law. First law states that the planets orbit the Sun on an ellipse with the semi-major and semi- minor; a and b respectively and we must keep in mind that Sun is located on one of the foci of the ellipse. Second law states that a line from the Sun to the planets sweeps out an equal area in equal time. Finally, the third law implies that the square of the period of the planet is proportional to the cube of the semi-major axis. According to the second and the third laws

$$T^2 = \frac{4\pi^2(1-e^2)a^4}{\ell^2} \tag{2.54}$$

And

$$T^2 = \frac{4\pi^2 a^3}{G(M_{\odot}+m)} \tag{2.55}$$

in which in our Solar System for all planets $m \ll M_{\odot}$ it can be approximated as

$$T^2 \simeq \frac{4\pi^2 a^3}{GM_{\odot}}. \tag{2.56}$$

From Eq. (2.54) we find $\ell^2 = \frac{4\pi^2(1-e^2)a^4}{T^2}$ and from Eq. (2.56) we find $GM_{\odot} = \frac{4\pi^2 a^3}{T^2}$

which after substitution into Eq. (2.53) one finds

$$\delta\Delta = \frac{24\pi^3 a^2}{T^2(1-e^2)c^2} \tag{2.57}$$

In Table 1, we provide perihelion precession $\delta\Delta$ for all planets in our solar system.

Table 2.1:Perihelion precession of the solar system due to the effect of general relativity.

Planets	$a(m) \times 10^{11}$	$T(d)$	e	$\delta\Delta(\frac{rad}{orbit}) \times 10^{-6}$	$\delta\Delta(\frac{sec}{century})$
Mercury	0.579091757	87.969	0.20563069	0.5018545204	42.980
Venus	1.082089255	224.701	0.00677323	0.2571130671	8.6247
Earth	1.495978871	365.256	0.01671022	0.1859498484	3.8374
Mars	2.279366372	686.98	0.09341233	0.1230815591	1.3504
Jupiter	7.784120267	4332.589	0.04839266	0.03581036194	0.0623
Saturn	14.26725413	10759.22	0.0541506	0.01954946842	0.0136
Uranus	28.7097222	30685.4	0.04716771	0.00982264895	0.0024
Neptune	44.9825291	60189	0.00858587	0.00618284201	0.0008

Chapter 3

CONCLUSION

In this thesis we have studied "Perihelion Precession In the Solar System". In the chapter one (introduction) we had a review from the first spark of this subject to the last correction that was done on the equations for finding the best amounts. We explained that it was born by the Newtonian's law at the first time and some astronomers and mathematicians tried to render methods which had the closest result with the observed value. After propounding General Rilativity theory by Albert Einstein, he could change the traditional physics worldview and one of the phenomenon that was solved by this theory was perihelion precession.

In the second chapter we have started with Schwarzschild spacetime equation and we assumed that each planet moves as a test particle for the motion of the planets in solar system The best way for derivation of the equation for the perihelion precession of orbits in general relativity theory involves the solution of the Euler-Lagrangian equations where the line element is given in Schwarzschild coordinates. For finding the basic equation of motion from the Eular-Lagrange equations and by choosing $\theta = \frac{\pi}{2}$, the result obtained were three Eular-Lagrange equations which we obtained the two conserved quantities energy and angular momentum of the test particle. To proceed further, we used the chain rule to find a differential equation for r with respect to φ, where through a change of variables we substituted $u = \frac{1}{r}$, and rewrited the last differential equation that named it as the master equation of first order.

These reduce to one equation is similar to the classical Kepler problem with an additional and small term which is $3M_{\odot}u^2$, due to the general relativity. For finding the solution by considering $\frac{3M_{\odot}^2}{\ell^2} = \lambda \ll 1$ and expanding the orbit of planets in terms of λ. In this way the zeroth order obtained without the GR correction. For finding the higher order we obtained a general solution as Eq. (2.24). After considering the first solution we used the standard method of solving the particular solution of nonhomogeneous differential equation.

As regards $\lambda \ll 1$ and using other simplifications and mathematical methods the perihelion precession due to thr GR was obtained. Since the λ depends on mass and angular momentum that both of them are in natural units according to Eq. (2.50) were converted to geometrized units

We compared the classical Kepler orbit with an orbit in Schwarzschild space, related by the invariance of Kepler's second law which states that a line that is from sun to the planet swept out the equal area in the equal time and the third law states the square of the period of planets is proportional to the qcube of semi major axis (a), we found the period as T. After using some mathematical method we obtained the perihelion precession. This treatment more clearly demonstrates the action of the effect of perturbation in spacetime due to the presence of a gravitating body. Contary that the Eq. (2.57) was shown, may this thought comes in our mind that this approximation is influenced by the eccentricity, but the eccentricity appears in this equation just for changing the semi-latus rectum L, (as we show it in Einstein's result) of the ellipse to the semi-major axis, which in our equation we know it as 'a'. The geometrical relation between these two concept is $L = a(1 - e^2)$.

According to data's table of the planets in the solar system [14] the table (2.1) was prepared with respect to Eq. (2.57) we calculated the perihelion precession in $(rad/orbit)$ and in $(sec/century)$ by considering Table (1.1).

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