Numerical Solutions of Fractional Differential Equations

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We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis of the degree of Master of Science in Mathematics.

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ABSTRACT

Fractional analysis has almost the same history as classical calculus. Fractional analysis did not attract enough attention for a long time. However, in recent decades, fractional analysis and fractional differential equations become very popular because of its powerful applications. A large number of new differential models that involve fractional calculus are developed. For most fractional differential equations we can not provide methods to compute the exact solutions analytically. Therefore it is necessary to revert to numerical methods.

The structure of this thesis is arranged in the following way. We begin by recalling some classical facts from calculus. Partically, we recall definition and some properties of gamma, beta and Mittag-Leffler function. Then, in Chapter 3, we introduce the fundamental concepts and definitions of fractional calculus. This includes, in particular, some basic results concerning Riemann–Liouville differentiation and integration, and basic properties of Caputo derivative. In Chapter 4 we discuss fractional variant of the classical second-order Adams–Bashforth–Moulton method. It has been introduced by K. Diethelm, A.D. Freed, and discussed in book by K. Diethelm.

Keywords: R-L Fractional Derivative, Caputo Fractional Derivative, Adams-Bashforth-Moulton Method, Fractional Differential Equations

ÖZ

Kesirli analiz, klasik kalkülüs ile hemen hemen aynı tarihe sahiptir. Kesirli analiz uzun bir süre dikkat çekmemesine rağmen son yıllarda güçlü uygulama alanları olduğu ortaya çıktıktan sonra kesirli diferansiyel denklemler ile birlikte en popüler çalışma alanları olmuştur. Bununla birlikte kesirli kalkülüsü de kapsayan çok sayıda diferansiyel model geliştirilmiştir. Birçok kesirli diferansiyel denklemlerin kesin çözümleri için analitik metodlar yetersiz kalmaktadır. Bu nedenle sayısal yöntemlere dönmek gerekmektedir.

Bu tezin yapısı şu şekilde düzenlenmiştir: Öncelikle klasik kalkülüsün bazı özellikleri hatırlatılacaktır. İkinci kısımda gamma, beta, mittag-leffler gibi bazı özel fonksiyonların tanım ve bazı özellikleri hatırlanacaktır. Daha sonra üçüncü bölümde kesirli analizin tanım ve temel kavramları tanıtılacaktır. Bu kısım Abel integral denkleminin çözüm koşullarını, Riemann-Liouville kesirli integral ve türevinin temel sonuçlarını ve Caputo kesirli türevinin tanım ve bazı temel özelliklerini içermektedir. Dördüncü bölümde ise ikinci dereceden klasik Adams-Bashford-Moulton metodunun kesirli varyantını tartışıp, hata analizini yapılacaktır. Bu method K. Diethelm ve A.D. Freed tarafından tanıtılmış ve K. Diethelm tarafından yazılan kitapta bahsedilmiştir.

Anahtar kelimeler: R-L Kesirli Turev, Caputo Kesirli Turev, Adams-Bashforth-Moulton Metodu, Kesirli Diferensiyel Denklemler.

To my family

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LIST OF SYMBOLS

$P_n(x)$ Taylor Polynomial,

$\mathscr{L}(f)$ Laplace Transform,

$\Gamma(z)$ Gamma Function,

$\beta(z,w)$ Beta Function,

$E_\alpha(z)$ one parameter Mittag-Leffler Function,

$(p)_k$ Pochammer Symbol,

${}^{RL}I^{\alpha}_{a+}f$ Left-sided R-L Fractional Integral,

${}^{RL}I^{\alpha}_{b-}f$ Right-sided R-L Fractional Integral,

${}^{RL}D^{\alpha}_{a+}f$ Left-sided R-L Fractional Derivative,

${}^{RL}D^{\alpha}_{b-}f$ Right-sided R-L Fractional Derivative,

${}^{C}D^{\alpha}_{a+}f$ Left-sided Caputo Fractional Derivative,

${}^{C}D^{\alpha}_{b-}f$ Right-sided Caputo Fractional Derivative.

$C[a,b]$ space of continuous functions on a domain $[a,b]$,

$C^n[a,b]$ space of $n-$times continuously differentiable functions on $[a,b]$,

$AC[a,b]$ space of absolutely continuous functions on a $[a,b]$,

$D(a,b)$	space of differentiable functions on a (a,b),
$L_1(a,b)$	space of integrable functions on (a,b),
$\|\cdot\|_\infty$	norm in $C[a,b]$.

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Chapter 1

PRELIMINARIES

The fraction

$$\frac{(f(d) - f(c))}{(d - c)}$$

defines the slope of the straight line joining $(c, f(c))$ and $(d, f(d))$, that is a chord of the graph of f. On the other hand $f'(x)$ defines the slope of the tangent to the curve at the point $(x, f(x))$. Therefore the MVT state that we can find a point lying between the end-points of the chord of a given smooth curve, s.t. the tangent at that point is parallel to the chord.

Theorem 1.0.1 ***(Mean Value Theorem) [5]*** *Let $f \in C[c,d]$, and $f \in D(c,d)$, where $c < d$. Then there exists some ξ in (c,d) s.t.*

$$f'(\xi) = \frac{f(d) - f(c)}{d - c}. \tag{1.0.1}$$

Theorem 1.0.2 ***(Fubini's Theorem)*** *: Assume that $\Phi_1 = [a,b], \Phi_2 = [c,d], -\infty \leq a < b \leq \infty, -\infty \leq c < d \leq \infty$, and let $f(x,y)$ be a measurable function defined on $\Phi_1 \times \Phi_2$. Then the following integrals coincide*

$$\int_{\Phi_1} dx \int_{\Phi_2} f(x,y)dy, \int_{\Phi_2} dy \int_{\Phi_1} f(x,y)dx, \iint_{\Phi_1 \times \Phi_2} f(x,y)dxdy,$$

if at least one of them is absolutely convergent. By using this theorem , we can inter-

change the order of integration in repeated integrals. The following special case of Fubini's Theorem is called Dirichlet Formula, which is,

$$\int_a^b dx \int_a^x f(x,y)dy = \int_a^b dy \int_y^b f(x,y)dx.$$

Here we assume that one of these integrals is absolutely convergent.

Theorem 1.0.3 ***(Taylor's theorem)*** *If $f \in C^n[c,d]$, and $f^{(n)} \in D(c,d)$, then $\exists \xi$ between c and d such that*

$$f(d) = f(c) + f'(c)(d-c) + \frac{f''(c)}{2!}(d-c)^2 + \ldots + \frac{f^n(c)}{n!}(d-c)^n + \frac{f^{n+1}(\xi)}{(n+1)!}(d-c)^{n+1}.$$

Definition 1.0.4 *(Divided difference)*

$$f[z,z_0] = \frac{f(z)-f(z_0)}{z-z_0}$$

$$f]z,z_0,z_1] = \frac{f[z_0,z_1]-f[z,z_0]}{z_1-z_0}.$$

Theorem 1.0.5 ***(Divided difference)*** *Let $z, z_0, z_1, \ldots, z_k \in [c,d]$, $f^{(k)} \in C[c,d]$,, and assume that $f^{(k+1)}$ exist on (c,d). Then $\exists\ \xi_y \in (c,d)$, so that*

$$f(z) - p_k(z) = (z-z_0)(z-z_1)\ldots(z-z_k)\frac{f^{(k+1)}\left(\xi_z\right)}{(n+1)!} \tag{1.0.2}$$

Proof. To prove the theorem, we repeatedly use Rolle's theorem. We define the function

$$H(z) = f(z) - p_k(z) - \frac{(z-z_0)...(z-z_k)}{(\alpha - z_0)...(\alpha - z_k)} \cdot (f(\alpha) - p_k(\alpha)) \tag{1.0.3}$$

where $\alpha \in [c,d] \setminus \{z_0, z_1, ..., z_k\}$. [1] Notice that H has at least $k+2$ zeros. Namely $\alpha, z_0, z_1, ..., z_k$. Then from theorem, we claim that H' must have at least $k+1$ zeros. By repeatedly applying this theorem, we claim that H'' has at least k zeros (if $k \geq 1$), $H^{(3)}$ has at least $k-1$ zeros (if $k \geq 2$), and at the end that $H^{(k+1)}$ has at least one zero, let's say at $z = \xi_\alpha$. Therefore, if we differentiate (1.0.3) $k+1$ times and inserting $z = \xi_\alpha$, we get the following

$$0 = f^{(k+1)}(\xi_\alpha) - \frac{(k+1)!\,(f(\alpha) - p_k(\alpha))}{(\alpha - z_0)...(\alpha - z_k)}$$

To finish the proof we solve the above equation for $f(\alpha) - p_k(\alpha)$, and then write z instead of α, and insert the obtained representation into (1.0.3). [1] ■

Now, let's obtain another error term for the interpolating polynomial. We start by using the following

$$f[z_0, z_1, ..., z_k] = \frac{f[z_1, z_2, ..., z_k] - f[z_0, z_1, ..., z_{k-1}]}{z_k - z_0}$$

to represent the divided difference $f[z, z_0, z_1, ..., z_k]$ of $f[z_0, z_1, ..., z_k]$ and $f[y, y_0, y_1, ..., y_{k-1}]$. Regulating this, we get

$$f[y, y_0, ..., y_{k-1}] = f[y_0, ..., y_k] + (y - y_k) f[y, y_0, ..., y_k] \tag{1.0.4}$$

Likewise, we have the following equation

$$f(z) = f(z_0) + (z - z_0)f[z, z_0]. \tag{1.0.5}$$

Next, replace $f[z, z_0]$, using (1.0.4) with $k = 1$ on the RHS of (1.0.5) , to give

$$f[z] = f[z_0] + (z - z_0)f[z_0, z_1] + (z - z_0)(z - z_1)f[z, z_0, z_1], \tag{1.0.6}$$

and notice that (1.0.6) can be written in the form

$$f(z) = p_1(z) + (z - z_0)(z - z_1)f[z, z_0, z_1].$$

Now let's replace $f[z, z_0, z_1]$ with $k = 2$ in (1.0.6), using (1.0.4). Continuing in this way, we finally get

$$f(z) = p_k(z) + (z - z_0)\cdots(z - z_0)f[z, z_0, z_1, \ldots, z_k]. \tag{1.0.7}$$

If comparing (1.0.7) and (1.0.2), we can easily see that if the conditions of Theorem 3. holds, then there exists a number ξ_z s.t.

$$f[z, z_0, z_1, \ldots, z_k] = \frac{f^{(k+1)}(\xi_z)}{(k+1)!}$$

Since this holds for any $z \in [c, d]$ and inside of which f holds the requirements. Now Let's write $k - 1$ instead of k, insert $z = z_k$, then get

$$f[z_0, z_1, ..., z_k] = \frac{f^{(k)}(\xi)}{k!} \tag{1.0.8}$$

where $\xi \in [c, d]$. [1]

Definition 1.0.6 *(Laplace Transform) Suppose that $f(t)$ is a piecewise continuous on $[0,\infty)$ and it is of exponential order α. Then the L-transform of the function $f(t)$ exists for all $k > \alpha$ and real numbers $c \geq 0$, which is given by:*

$$F(k) = \int_0^\infty e^{-kc} f(c) dc$$

Where k is a complex number:

$$k = \sigma + iw,$$

with $\sigma, w \in \mathbb{R}$.

Remark 1.0.7 *The Laplace transform has many properties that make it useful for analyzing LDE. The most important advantage is differentiation become multiplication. Due to this feature the L-transform k is also known as operator in the L domain: either derivative or (for k^{-1}) integration operator. The transform turns integral equation and differential equation to polynomial equation. These equations can be easily solved. Once these equations are solved, the use of the inverse L-transform reverts to the time domain.*

Definition 1.0.8 *A function $f(x)$ is called absolutely continuous on an interval Ω, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite set of pairwise nonintersecting intervals $[a_k, b_k] \subset \Omega, k = 1, 2, \ldots, n$, such that*

$$\sum_{k=1}^{n} (b_k - a_k) < \delta,$$

the inequality

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

holds. The space of these functions is denoted by $AC(\Omega)$.

Definition 1.0.9 *(The spaces* L_p *and* $L_p(p)$*) Let* $\Omega = [a,b]$, $-\infty \le a < b \le \infty$. *We denote by* $L_p = L_p(\Omega)$ *the set of all Lebesgue measurable functions* $f(x)$, *complex valued in general for which* $\int\limits_{\Omega} |f(x)|^p \, dx < \infty$, *where* $1 \le p < \infty$. *We set*

$$\|f\|_{L_p(\Omega)} = \left\{ \int\limits_{\Omega} |f(x)|^p \, dx \right\}^{1/p}.$$

If $p = \infty$ *the space* $L_p(\Omega)$ *is defined as the set of all measurable functions with a finite norm*

$$\|f\|_{L_\infty(\Omega)} = ess \sup_{x \in \Omega} |f(x)|,$$

where $ess\sup|f(x)|$ *is an essential maximum of the function* $|f(x)|$.

Chapter 2

SOME SPECIAL FUNCTIONS

In this chapter we provide the definitions and give certain features of some well-known functions such as the Beta, Gamma and Mittag-Leffler functions.

2.1 The Gamma Function

Definition 2.1.1 *(The Gamma function [2]) It is given by the Euler integral of the second kind*

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \quad (\mathrm{Re}(n) > 0) \tag{2.1.1}$$

where $t^{n-1} = e^{(n-1)\log(t)}$*.(2.1.1) is convergent for all* $n \in \mathbb{C}$ *with positive real part* $\mathrm{Re}(n) > 0$. *[3]*

We list some features of the Gamma function :

(i)

$$\Gamma(n+1) = n\Gamma(n) \quad (\mathrm{Re}(n) > 0); \tag{2.1.2}$$

using this relation, the Euler Gamma function is extended to the half-plane $\mathrm{Re}(n) \leq 0$ $(\mathrm{Re}(n) > -n; n \in \mathbb{N}; k \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\})$ by

$$\Gamma(n) = \frac{\Gamma(n+k)}{(n)_k}.$$

Here $(n)_k$ is the Pochammer symbol, defined for complex $n \in \mathbb{C}$ and non-negative integer, with $k \in \mathbb{N}$, by

$$(n)_0 = 1 \text{ and } (n)_k = n(n+1)\cdots(n+k-1).$$

(ii)

$$\Gamma(z+1) = (1)_z = z! \quad (z \in \mathbb{N}_0) \tag{1.3}$$

with (as usual) $0! = 1$.

(iii)

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(\pi n)} \quad (n \notin \mathbb{Z}_0;\ 0 < \mathrm{Re}(n) < 1, \tag{2.1.3}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{2.1.4}$$

(iv) Legendre duplication formula

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) \quad (n \in \mathbb{C}) \tag{2.1.5}$$

2.2 The Beta Function

Definition 2.2.1 *(Beta Function [2]) The Beta function β is given by the integral*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt \quad (\mathrm{Re}(z) > 0;\ \mathrm{Re}(w) > 0), \tag{2.2.1}$$

The Beta function and the Gamma function are related by [2]

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \quad (z,w \notin Z_0^- = \{0,-1,-2,...\}). \tag{2.2.2}$$

2.3 Binomial Coefficients

Definition 2.3.1 *(Binomial coefficients [2]) They are defined as*

$$\binom{\gamma}{0} = 1, \ \binom{\gamma}{n} = \frac{\gamma(\gamma-1)\cdots(\gamma-b+1)}{b!} = \frac{(-1)^b(-\gamma)_b}{b!}. \tag{2.3.1}$$

As a special case, when $\alpha = a$, $n \in \mathbb{N}_0 = \{0,1,\ldots\}$, with $a \geq b$, we have

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} \tag{2.3.2}$$

and

$$\binom{a}{b} = 0 \quad (a,b \in \mathbb{N}_0;\ 0 \leq a < b) \tag{2.3.3}$$

[3] If $\gamma \notin Z^- = \{-1,-2,-3,\ldots\}$, the formula (2.3.1) can be expressed via the Gamma function by the following

$$\binom{\gamma}{b} = \frac{\Gamma(\gamma+1)}{b!\Gamma(\gamma-b+1)} \quad (\gamma \in \mathbb{C};\ \gamma \notin \mathbb{Z}^-;\ b \in \mathbb{N}_0). \tag{2.3.4}$$

Next, we give the definitions and some well-known features of the Mittag-Leffler functions.

2.4 Mittag-Leffler Function

Definition 2.4.1 *(Mittag-Leffler Function [2]) Mittag-Leffler Function $E_\alpha(n)$ is given by*

$$E_\alpha(n) = \sum_{k=0}^{\infty} \frac{n^k}{\Gamma(\alpha k+1)} \quad (n \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0). \tag{2.4.1}$$

As a special case for $\alpha = 1, 2$

$$E_1(n) = e^n \text{ and } E_2(n) = \cosh(\sqrt{n}). \tag{2.4.2}$$

$E_{\alpha,\beta}(n)$, generalizing the one in (2.4.1), is given by

$$E_{\alpha,\beta}(n) = \sum_{k=0}^{\infty} \frac{n^k}{\Gamma(\alpha k+\beta)} \quad (n, \beta \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0). \tag{2.4.3}$$

As a special case, [3] when $\beta = 1, we have$

$$E_{\alpha,1}(n) = E_\alpha(n) \quad (n \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0) \tag{2.4.4}$$

and for $\beta = 1, we have$

$$E_{1,2}(n) = \frac{e^n - 1}{n}, \text{ and } E_{2,2}(n) = \frac{\sinh(\sqrt{n})}{\sqrt{n}}. \tag{2.4.5}$$

The $E_{\alpha,\beta}(n)$ has the integral expression

$$E_{\alpha,\beta}(n) = \frac{1}{2\pi} \int_C \frac{t^{\alpha-\beta} e^t}{t^\alpha - n} dt,$$

$|\arg(t)| \leq \pi$ on C. (2.4.6)

Here C is a loop with base point at $-\infty$ and encircles the circular disk $|t| \leq |n|^{1/\alpha}$ in the positive sense.

Chapter 3

FRACTIONAL DERIVATIVE

In this chapter we give the descriptions and some well-known features of Caputo and Riemann-Liouville types fractional derivatives and integrals.

3.1 R-L fractional integrals and derivatives

3.1.1 The Abel Integral Equation

Abel equation is given by

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{g(t)dt}{(x-t)^{1-\alpha}} = f(x),\ x>0,\ 0<\alpha<1. \tag{3.1.1}$$

The uniqueness of the solution of this equation can be shown as follows

Writing t instead of x and s instead of t in (3.1.1), we get

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s)ds}{(t-s)^{1-\alpha}} = f(t)$$

multiplying both sides of the equation by $(x-t)^{-\alpha}$, we obtain

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s)ds}{(t-s)^{1-\alpha}(x-t)^{\alpha}} = \frac{f(t)}{(x-t)^{\alpha}}$$

multiplying both sides by $\Gamma(\alpha)$ and integrating we have

$$\int_a^x \frac{1}{(x-t)^{\alpha}}dt\int_a^t \frac{g(s)ds}{(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)}{(x-t)^{\alpha}}$$

Now, interchanging the order of integration in the left-hand side by DirichletfFormula,

we get

$$\int_a^x g(s)ds \int_s^x \frac{dt}{(x-t)^{\alpha}(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \tag{3.1.2}$$

Putting $t = s + u(x-s)$ at the inner integral and using properties of Beta Function we have

$$\begin{aligned} \int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt &= \int_0^1 u^{\alpha-1}(1-u)du \\ &= B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha). \end{aligned}$$

if we substitute this result into (3.1.2) we get

$$\int_a^x g(s)ds = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f(t)}{(x-t)^{\alpha}} \tag{3.1.3}$$

If we differentiate (3.1.3) we have

$$g(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}. \tag{3.1.4}$$

Therefore, if (3.1.1) has a solution, this solution is necessarily given by (3.1.4). This means it is unique.

Similarly, the Abel equation in the form

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g(t)dt}{(t-x)^{1-\alpha}} = f(x),\ x \le b, \tag{3.1.5}$$

is considered and in place of (3.1.4) one obtains for $0 < \alpha < 1$ the below inversion formula

$$g(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(t)dt}{(t-x)^\alpha}. \tag{3.1.6}$$

3.1.2 On the Solvability of the Abel equation in the space of integrable functions

Let us start with investigating the conditions on $f(x)$ where the Abel equation is solvable. To formulate the main result of this section, we give the notation

$$f_{1-\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)}{(x-t)^\alpha}. \tag{3.1.7}$$

It's clear that

$$\int_a^b |f_{1-\alpha}(x)|dx \leq \frac{1}{\Gamma(2-\alpha)} \int_a^b |f(t)|(b-t)^{1-\alpha}dt, \tag{3.1.8}$$

so $f(x) \in L_1(a,b)$ implies that $f_{1-\alpha}(x) \in L_1(a,b)$ as well.

Theorem 3.1.1 *Abel Equation (3.1.1) with* $0 < \alpha < 1$ *is solvable in* $L_1(a,b)$ *iff*

$$f_{1-\alpha}(x) \in AC([a,b]) \text{ and } f_{1-\alpha}(a) = 0. \tag{3.1.9}$$

Proof. Necessity. Let (3.1.1) be solvable in $L_1(a,b)$. Then all considerations of the previous section are correct, the possibility of changing the order of integration in (3.1.2) being proved with the aid of Fubini Theorem. Thus (3.1.3) is valid. Hence we obtain (3.1.9).

Sufficiency. Since $f_{1-\alpha}(x) \in AC([a,b])$, we have $f'_{1-\alpha}(x) = \frac{d}{dx} f_{1-\alpha}(x) \in L_1(a,b)$. So the function given by (3.1.4) exists almost everywhere and belongs to $L_1(a,b)$. Now, let's show that it is indeed a solution of (3.1.1). For this purpose we put it into LHS of (3.1.1) and give the result by $g(x)$, i.e.

$$\frac{1}{\Gamma(\alpha)} \int_a^x \frac{f'_{1-\alpha}(t)}{(x-t)^{1-\alpha}} dt = g(x). \tag{3.1.10}$$

We should show that $g(x) = f(x)$, which proves the theorem. (3.1.10) is an equation of type (3.1.1) with respect to $f'_{1-\alpha}(x)$. It is certainly solvable since it is merely a notation. So by (3.1.4) we get

$$f'_{1-\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{g(t)dt}{(x-t)^{\alpha}}.$$

i.e. $f'_{1-\alpha}(x) = g'_{1-\alpha}(x)$. Functions $f_{1-\alpha}(x)$ and $g_{1-\alpha}(x)$ are absolutely continuous, the first by assumption, the second by virtue of (3.1.3) with $g(x)$ in the RHS. Hence $f_{1-\alpha}(x) - g_{1-\alpha}(x) = c$ note that the condition of absolute contunuity is essential in this reasoning : it can not be weakened to continuity, since it is known that there are continuous but not absolutely continuous functions different from constant and having the derivative equal to zero almost everywhere. We have $f_{1-\alpha}(a) = 0$ by conjecture, while $g_{1-\alpha}(a) = 0$ because (3.1.10) is a solvable equation. Hence $c = 0$, so

$$\int_a^x \frac{f(t) - g(t)}{(x-t)} dt = 0.$$

The second is an equation of the form (3.1.1). The uniqueness of its solution leds to the relation $f(t) = g(t) = 0$, which completes the proof. ■

The criterion of solvability for Abel's equation is given in below theorem in terms of the auxilary function $f_{1-\alpha}(x)$. The following lemma and corollary give a simple sufficient solvability condition in terms of the function $f(x)$ itself.

Lemma 3.1.2 *If $f(x) \in AC([a,b])$, then $f_{1-\alpha}(x) \in AC([a,b])$ and*

$$f_{1-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)}[f(a)(x-a)^{1-\alpha} + \int_a^x f'(t)(x-t)^{1-\alpha}dt]. \tag{3.1.11}$$

Proof. Substitute

$$f(t) = f(a) + \int_a^t f'(s)ds$$

into (3.1.7) we have

$$\begin{aligned} f_{1-\alpha}(x) &= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(a) + \int_a^t f'(s)ds}{(x-t)^\alpha} dt \\ &= \frac{1}{\Gamma(1-\alpha)} [\int_a^x \frac{f(a)}{(x-t)^\alpha} dt + \int_a^x \int_a^t \frac{f'(s)ds}{(x-t)^\alpha} dt \end{aligned} \tag{3.1.12}$$

For the fist term in the RHS if we substitute

$$\begin{aligned} u &= x - t \text{ and} \\ du &= -dt \end{aligned}$$

we get

$$\begin{aligned}
& \frac{1}{\Gamma(1-\alpha)} f(a)[-\int_{x-a}^{0} \frac{1}{u^{\alpha}} du] \\
= & \frac{1}{\Gamma(1-\alpha)} f(a) \left[-\frac{u^{1-\alpha}}{1-\alpha} \Big|_{x-a}^{0} \right] \\
= & \frac{1}{\Gamma(1-\alpha)} f(a) \left[\frac{(x-a)^{1-\alpha}}{1-\alpha} \right] \\
= & \frac{1}{(1-\alpha)\Gamma(1-\alpha)} f(a)(x-a)^{1-\alpha} \\
= & \frac{1}{\Gamma(2-\alpha)} f(a)(x-a)^{1-\alpha}
\end{aligned}$$

not putting this result into (3.1.12) we have

$$\begin{aligned}
f_{1-\alpha}(x) &= \frac{f(a)}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} \\
&\quad + \frac{1}{\Gamma(2-\alpha)} \int_a^x \frac{dt}{(x-t)^{\alpha}} \int_a^t f'(s) ds. \qquad (3.1.13)
\end{aligned}$$

The first term here is an absolutely continuous function because

$$(x-a)^{1-\alpha} = (1-\alpha) \int_a^x (t-a)^{-\alpha} dt.$$

Since

$$\int_a^x \frac{dt}{(x-t)^{\alpha}} \int_a^t f'(s) ds = \int_a^x \left(\int_a^t \frac{f'(s) ds}{(t-s)^{\alpha}} \right) dt \qquad (3.1.14)$$

which may be verified by direct interchange of order of integration in both parts of the equation, second term in (3.1.13) is also a primitive of a summable function hence it is absolutely continuous. The representation (3.1.12) follows from (3.1.13) after the interchange of the order of integration. This completes the proof. ■

Corollary 3.1.3 *If $f(x) \in AC([a,b])$, then Abel's equation (3.1.1) with $0 < \alpha < 1$ is solvable in $L_1(a,b)$ and its solution (3.1.4) can be represented in the form*

$$g(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(s)ds}{(x-s)^{\alpha}}\right]. \tag{3.1.15}$$

Indeed the solvability conditions (3.1.9) are satisfied owing to above Lemma and (3.1.13) and (3.1.14). Since $g(x) = \frac{d}{dx} f_{1-\alpha}(x)$ we observe that (3.1.15) can be obtained by differentiating (3.1.11), the differentiation itself under the sign of an integral being easily proved with the aid of (3.1.14).

We should also like to emphasize that we have simultaneously obtained a new form, (3.1.15), of Abel's integral equation inversion, which is applicable to absolutely continuous right-hand sides $f(x)$.

Similarly to above theorem one may show that (3.1.5) is solvable in $L_1(a,b)$ iff $\widetilde{f}_{1-\alpha}(x) \in AC([a,b])$ and $\widetilde{f}_{1-\alpha}(b) = 0$, where

$$\widetilde{f}_{1-\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{\alpha}},\ 0 < \alpha < 1.$$

The solution (3.1.6) of (3.1.5) with $f(x) \in AC([a,b])$ may be written down similarly to (3.1.15) as follows

$$g(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(b)}{(b-t)^{\alpha}} - \int_t^b \frac{f'(s)ds}{(s-t)^{\alpha}}\right]. \tag{3.1.16}$$

3.1.3 Definitions of R-L Fractional Integral and Derivatives and Some of Their Properties

Definition 3.1.4 *[2] Assume that $\Omega = [a,b] \subset \mathbb{R}$. The R-L fractional integrals ${}^{RL}I^{\gamma}_{a+}f$ and ${}^{RL}I^{\gamma}_{b-}f$ of order γ $(\mathrm{Re}(\gamma) > 0)$ are presented by*

$$\left({}^{RL}I^{\gamma}_{a+}f\right)(x) = \frac{1}{\Gamma(\gamma)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\gamma}} \ \ (x > a;$$

$\gamma \in \mathbb{C},\ \mathrm{Re}(\gamma) > 0)$(3.1.17)*and*

$$\left({}^{RL}I^{\gamma}_{b-}f\right)(x) = \frac{1}{\Gamma(\gamma)}\int_x^b \frac{f(t)dt}{(t-x)^{1-\gamma}} \ \ (x < b;$$

$\gamma \in \mathbb{C},\ \mathrm{Re}(\gamma) > 0)$(3.1.18) *respectively.*

Definition 3.1.5 *[2] The ${}^{RL}D^{\gamma}_{a+}y$ and ${}^{RL}D^{\gamma}_{b-}y$ of order $\gamma(\mathrm{Re}(\gamma) \geq 0)$ [3] are given by*

$$({}^{RL}D^{\gamma}_{a+}y)(x) = \left(\frac{d}{dx}\right)^n ({}^{RL}I^{n-\gamma}_{a+}y)(x) \quad (x > a), \tag{3.1.19}$$

and

$$({}^{RL}D^{\gamma}_{b-}y)(x) = \left(-\frac{d}{dx}\right)^n ({}^{RL}I^{n-\gamma}_{b-}y)(x) \quad (x < b), \tag{3.1.20}$$

where ($\gamma \in \mathbb{C}$), in the given order, with $n = -[-\mathrm{Re}(\gamma)]$, where $[\cdot]$ denotes the integral part of the argument, that is

$$n = \begin{cases} [\mathrm{Re}(\gamma)] + 1 & \text{for } \gamma \notin \mathbb{N}_0, \\ \gamma & \text{for } \gamma \in \mathbb{N}_0. \end{cases} \tag{3.1.21}$$

As a special case, when $\gamma = n \in \mathbb{N}_0$, then

$$({}^{RL}D^0_{a+}y)(x) = ({}^{RL}D^0_{b-}y)(x) = y(x), \tag{3.1.22}$$

$$({}^{RL}D^n_{a+}y)(x) = y^{(n)}(x), \quad ({}^{RL}D^\gamma_{b-}y)(x) = (-1)^n y^{(n)}(x), \tag{3.1.23}$$

$y^{(n)}(x)$ is the usual derivative of $y(x)$ of order n.

Proposition 3.1.6 *[2] We have*

$$({}^{RL}I^\gamma_{a+}(t-a)^\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}(x-a)^{\beta+\gamma}, \tag{3.1.24}$$

$$({}^{RL}D^\gamma_{a+}(t-a)^\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}(x-a)^{\beta-\gamma}, \tag{3.1.25}$$

$$({}^{RL}I^\gamma_{b-}(b-t)^\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}(b-x)^{\beta+\gamma}, \tag{3.1.26}$$

$$({}^{RL}D^\gamma_{b-}(b-t)^\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}(b-x)^{\beta-\gamma}. \tag{3.1.27}$$

Where $\gamma, \beta \in \mathbb{C}$, $\mathrm{Re}(\gamma) \geq 0$ *and* $\mathrm{Re}(\beta) > 0$.

For $0 < \mathrm{Re}(\gamma) < 1$ *this falls down to*

$$({}^{RL}D^\gamma_{a+}1)(x) = \frac{(x-a)^{-\gamma}}{\Gamma(1-\gamma)}, \quad ({}^{RL}D^\gamma_{b-}1)(x) = \frac{(b-x)^{-\gamma}}{\Gamma(1-\gamma)}, \tag{3.1.28}$$

and for $j = 1, 2, ..., n = -[-\operatorname{Re}(\gamma)]$, *we get*

$$({}^{RL}D^{\gamma}_{a+}(t-a)^{\gamma-j})(x) = 0, \;\; ({}^{RL}D^{\gamma}_{b-}(b-t)^{\gamma-j})(x) = 0 \tag{3.1.29}$$

From (3.1.29) we deduce that $(D^{\gamma}_{a+}y)(x) = 0$ is holds iff,

$$y(x) = \sum_{j=1}^{n} c_j (x-a)^{\gamma-j},$$

where $n = [\operatorname{Re}(\gamma)] + 1$ and $c_j \in \mathbb{R}$, $(j = 1, \ldots, n)$ are constant. As a special case, when $0 < \operatorname{Re}(\gamma) \leq 1$, $({}^{RL}D^{\gamma}_{a+}y)(x) = 0$ valid iff, $y(x) = c(x-a)^{\gamma-1}$ with any $c \in \mathbb{R}$. [2]

Similarly, the equality $({}^{RL}D^{\gamma}_{b-}y)(x) = 0$ is holds iff,

$$y(x) = \sum_{j=1}^{n} d_j (b-x)^{\gamma-j},$$

where $d_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are const. As a special case, when $0 < \operatorname{Re}(\gamma) \leq 1$, $({}^{RL}D^{\gamma}_{b-}y)(x) = 0$ valid iff, $y(x) = d(b-x)^{\gamma-1}$ with any $d \in \mathbb{R}$.

Proposition 3.1.7 (p.1) *The following results give us an another representation of the* ${}^{RL}D^{\gamma}_{a+}$ *and* ${}^{RL}D^{\gamma}_{b-}$, *for* $\operatorname{Re}(\gamma) \geq 0$, $n = [\operatorname{Re}(\gamma)] + 1$,

$$({}^{RL}D^{\gamma}_{a+}y)(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma} + \frac{1}{\Gamma(n-\gamma)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\gamma-n+1}}, \tag{3.1.30}$$

and

$$({}^{RL}D^{\gamma}_{b-}y)(x) = \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{\Gamma(1+k-\gamma)}(b-x)^{k-\gamma} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\gamma-n+1}}.$$

(3.1.31)

(p.2) *The semigroup property of the* ${}^{RL}I^{\gamma}_{a+}$ *and* ${}^{RL}I^{\gamma}_{b-}$ *provides that, if* $\mathrm{Re}(\gamma) > 0$ *and* $\mathrm{Re}(\beta) > 0$, *then the following equations*

$$\begin{aligned}({}^{RL}I^{\gamma}_{a+}\,{}^{RL}I^{\beta}_{a+}f)(x) &= ({}^{RL}I^{\gamma+\beta}_{a+}f)(x) \quad \textit{and} \\ ({}^{RL}I^{\gamma}_{b-}\,{}^{RL}I^{\beta}_{b-}f)(x) &= ({}^{RL}I^{\gamma+\beta}_{b-}f)(x),\end{aligned} \tag{3.1.32}$$

are hold almost everywhere since $f \in L_p(a,b)$, $p \geq 1$,.

Proof.

$$\begin{aligned} I^{\gamma}_{a+}I^{\beta}_{a+}f &= \frac{1}{\Gamma(\gamma)}\int_a^x \frac{1}{(x-t)^{1-\alpha}}(I^{\beta}_{a+}f)(t)dt \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_0^x\int_0^t \frac{f(u)}{(t-u)^{1-\beta}(x-t)^{1-\alpha}}dt \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_a^x\int_u^x \frac{f(u)dt}{(x-t)^{1-\gamma}(t-u)^{1-\beta}}du \end{aligned}$$

Now if we substitute

$$\begin{aligned} t &= u+s(x-u) \textit{ and} \\ dt &= (x-u)ds \end{aligned}$$

we get

$$\begin{aligned} &\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_a^x f(u)\int_0^1 \frac{(x-u)dsdu}{[(x-u)(1-s)]^{1-\gamma}[s(x-u)]^{1-\beta}} \\ =\;& \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_a^x \frac{f(u)du}{(x-u)^{1-\gamma-\beta}}\int_0^1 \frac{ds}{(1-s)^{1-\gamma}s^{1-\beta}} \end{aligned}$$

Here, if we consider on second integral, we can easily see that

$$\int_0^1 \frac{ds}{(1-s)^{1-\gamma} s^{1-\beta}} = B(\beta, \gamma)$$

and we know that

$$B(\beta, \gamma) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}$$

if we use this, we obtain the following desired result

$$\frac{1}{\Gamma(\gamma+\beta)} \int_a^x \frac{f(u)du}{(x-u)^{1-\gamma-\beta}} = I_{a+}^{\gamma+\beta} f$$

■

(p.3) *Likewise, we have the following index formulae*

$$({}^{RL}D_{a+}^{\gamma}\, {}^{RL}D_{a+}^{\beta} f)(x) = ({}^{RL}D_{a+}^{\gamma+\beta} f)(x) - \sum_{j=1}^{m} ({}^{RL}D_{a+}^{\beta-j} f)(a+) \frac{(x-a)^{-j-\gamma}}{\Gamma(1-j-\gamma)}, \tag{3.1.33}$$

since $\alpha, \beta > 0$, *s.t.* $n-1 < \gamma \leq n$, $m-1 < \beta \leq m$ $(n, m \in \mathbb{N})$ *and* $\gamma + \beta < n$.

For $f(x) \in L_p(a,b)$ $(1 \leq p \leq \infty)$, *the composition relations*

$$(D_{a+}^{\beta} I_{a+}^{\gamma} f)(x) = I_{a+}^{\gamma-\beta} f(x) \quad \text{and} \quad (D_{b-}^{\beta} I_{b-}^{\gamma} f)(x) = I_{b-}^{\gamma-\beta} f(x), \tag{3.1.34}$$

since $\mathrm{Re}(\gamma) > \mathrm{Re}(\gamma) > 0.$, *valid almost everywhere on* $[a,b]$ *between* D_{a+}^{β}*and*I_{a+}^{γ} *[3]. As a special case, when* $\beta = k \in \mathbb{N}$ *and* $\mathrm{Re}(\gamma) > k$, *then*

$$(D^k I_{a+}^{\gamma} f)(x) = I_{a+}^{\gamma-k} f(x) \text{ and } (D^k I_{b-}^{\gamma} f)(x) = (-1)^k I_{b-}^{\gamma-k} f(x). \tag{3.1.35}$$

This shows us that the fractional differentiation is an operation inverse to fractional integration from the left and

$$(D^{\gamma}_{a+}I^{\gamma}_{a+}f)(x) = f(x) \text{ and } (D^{\gamma}_{b-}I^{\gamma}_{b-}f)(x) = f(x), \tag{3.1.36}$$

since $\mathrm{Re}(\gamma) > 0$, *valid almost everywhere on* $[a,b]$. *[3]*

(p.4) *Besides,the following relation valid almost everywhere on* $[a,b]$

$$(I^{\gamma}_{a+}D^{\gamma}_{a+}f)(x) = f(x) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\gamma}(a)}{\Gamma(\gamma-j+1)}(x-a)^{\gamma-j} \tag{3.1.37}$$

if $\mathrm{Re}(\gamma) > 0$, $n = [\mathrm{Re}(\gamma)]+1$ *and* $f_{n-\gamma}(x) = (I^{n-\gamma}_{a+}f)(x)$. *Also, if* $g_{n-\gamma}(x) = (I^{n-\gamma}_{b-}g)(x)$, *then the formula*

$$(I^{\gamma}_{b-}D^{\gamma}_{b-}g)(x) = g(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j}g^{(n-j)}_{n-\gamma}(a)}{\Gamma(\gamma-j+1)}(b-x)^{\gamma-j}, \tag{3.1.38}$$

almost everywhere on $[a,b]$.

Assume that $\mathrm{Re}(\gamma) \geq 0$, $m \in \mathbb{N}$ and $D = d/dx$, then if the $(D^{\gamma}_{a+}y)(x)$ and $(D^{\gamma+m}_{a+}y)(x)$ exist, we have

$$(D^{m}D^{\gamma}_{b-}y)(x) = (D^{\gamma+m}_{a+}y)(x), \tag{3.1.39}$$

and, if the $(D^{\gamma}_{b-}y)(x)$ and $(D^{\gamma+m}_{b-}y)(x)$ exist, then

$$(D^{m}D^{\gamma}_{b-}y)(x) = (-1)^{m}(D^{\gamma+m}_{b-}y)(x). \tag{3.1.40}$$

If $\mathrm{Re}(\gamma) > 0$ and $n = [\mathrm{Re}(\gamma)] + 1$, we have

$$(\mathscr{L}D^{\gamma}_{0+}y)(s) = s^{\gamma}(\mathscr{L}y)(s) - \sum_{k=0}^{n-1} s^{n-k-1}D^{k}(I^{n-\gamma}_{0+}y)(0+) \tag{3.1.41}$$

for $(\mathrm{Re}(s) > q_0)$. [3]

Next, we have two formulas for fractional integration by parts :

a) For $\varphi(x) \in L_p(a,b)$ and $\psi(x) \in L_1(a,b)$, we have

$$\int_a^b \varphi(x)(I^{\gamma}_{a+}\psi)(x)dx = \int_a^b \psi(x)(I^{\gamma}_{b-}\varphi)(x)dx. \tag{3.1.42}$$

b) We have

$$\int_a^b f(x)(D^{\gamma}_{a+}g)(x)dx = \int_a^b g(x)(D^{\gamma}_{b-}f)(x)dx. \tag{3.1.43}$$

If $f(x) = (I^{\gamma}_{b-}h_1)(x)$ with some $h_1(x) \in L_p(a,b)$ and $g(x) = (I^{\gamma}_{a+}h_2)(x)$ with some $h_2(x) \in L_q(a,b)$,

Now let $\gamma > 0$, $p \geq 1$, $q \geq 1$, and $(1/p) + (1/q) \leq 1 + \gamma$ ($p \neq 1$ and $q \neq 1$ in the case when $(1/p) + (1/q) = 1 + \gamma$).

The extended fractional Leibniz formula for the R-L derivative, applied to sufficiently good function on $[a,b]$, gives

$$[D^{\gamma}_{a+}(fg)](x) = \sum_{j=0}^{\infty} \binom{\gamma}{j} (D^{\gamma-j}_{a+}f)(x)(D^{j}g)(x), \tag{3.1.44}$$

where $\gamma > 0$. Next, we give three special cases to demonstrate this feature.

a) Assume that $f(x) = x$ and $g(x)$ be a sufficiently good function. Then for $0 < \gamma < 1.$, we have

$$[D_{0+}^{\gamma}(fg)](x) = x(D_{0+}^{\gamma}g)(x) + (I_{0+}^{1-\gamma}g)(x). \tag{3.1.45}$$

b) Let $f(x) = x^{\gamma-1}$ and $g(x)$ be a sufficiently good function. Then for $0 < \gamma < 1.$, we have

$$[D_{0+}^{\gamma}(fg)](x) = \sum_{j=1}^{\infty} \binom{\gamma}{j} \frac{\Gamma(\gamma)}{\Gamma(j)} x^{j-1} g^{(j)}(x). \tag{3.1.46}$$

c) Let $p \in \mathbb{N}$, $f(x)$ be a sufficiently good function. Then for $\gamma > 0$,

$$(D_{0+}^{\gamma} t^p f)(x) = \sum_{j=0}^{p} \binom{\gamma}{j} (D^j x^p)(D_{0+}^{\gamma-j} f)(x). \tag{3.1.47}$$

Computing fractional R-L derivative of the composition of two sufficiently good function can be very intricated. The following formula

$$\begin{aligned}[D_{a+}^{\gamma}(f(g))](x) &= \frac{(x-a)^{-\gamma}}{\Gamma(1-\gamma)} f(g(x)) \\ &\quad + \sum_{j=1}^{\infty} \binom{\gamma}{j} \frac{j!(x-a)^{j-\gamma}}{\Gamma(j+1-\gamma)} \sum_{r=1}^{j} [D^i f(g)](x) \\ &\quad \cdot \sum \prod_{r=1}^{j} \frac{1}{a_r!} \left(\frac{(D^r g)(x)}{r!} \right)^{a_r} \end{aligned} \tag{3.1.48}$$

where $\sum_{r=1}^{j} r a_r = j$ and $\sum_{r=1}^{j} a_r = i$, displays the intricate structure very openly.

3.1.4 Fractional Integration and Differentiation as Reciprocal Operations

We already know that the ordinary differentiation and integration are reciprocal operations if the integration applied first as following

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

On the other hand, generally

$$\int_a^x f'(t)dt \neq f(x)$$

because of the constant $-f(a)$.

Similarly,

$$(\frac{d}{dx})^n I_{a+}^n f \equiv f,$$

but

$$I_{a+}^n f^{(n)} \neq f$$

because of a polynimial of the order $n-1$. Similarly, we should always have

$$D_{a+}^{\gamma} I_{a+}^{\gamma} f \equiv f,$$

but $I_{a+}^{\gamma} D_{a+}^{\gamma} f$ doesn't necessarily coincide with $f(x)$ because of the function $(x-a)^{\alpha-k}, k = 1,2,\ldots[\mathrm{Re}\,\alpha]+1$, can be arise, the linear combinations of which play the role of polynomials for fractional differentiation.

3.2 Caputo Fractional Derivatives

Definition 3.2.1 *[2] The Caputo and Riemann-Liouville fractional derivatives have close relations. Assume that $[a,b]\subset\mathbb{R}$. the left-sided and right-sided Caputo fractional derivatives of order α are defined by*

$$({}^{C}D^{\gamma}_{a+}y)(x)=\frac{1}{\Gamma(n-\gamma)}\int_a^x\frac{y^{(n)}(t)dt}{(x-t)^{\gamma-n+1}}=({}^{RL}I^{n-\gamma}_{a+}D^n y)(x), \tag{3.2.1}$$

$\alpha\in\mathbb{C}$ $(\mathrm{Re}(\gamma)\geq 0)$ *and*

$$({}^{C}D^{\gamma}_{b-}y)(x)=\frac{(-1)^n}{\Gamma(n-\gamma)}\int_x^b\frac{y^{(n)}(t)dt}{(t-x)^{\gamma-n+1}}=(-1)^n({}^{RL}I^{n-\gamma}_{b-}D^n y)(x), \tag{3.2.2}$$

$\gamma\in\mathbb{C}$ $(\mathrm{Re}(\gamma)\geq 0)$*respectively, where* $D=\frac{d}{dx}$ *and* $n=-[-\mathrm{Re}(\gamma)]$, *i.e.*

$$n=\begin{cases}[\mathrm{Re}(\gamma)]+1 & for\ \ \gamma\notin\mathbb{N}_0\\ \gamma & for\ \ \gamma\in\mathbb{N}_0\end{cases}$$

As a special case

$$({}^{C}D^{\gamma}_{a+}y)(x)=\frac{1}{\Gamma(1-\gamma)}\int_a^x\frac{y'(t)dt}{(x-t)^{\gamma}}=({}^{RL}I^{1-\gamma}_{a+}Dy)(x), \tag{3.2.3}$$

$0<\mathrm{Re}(\gamma)<1$.and

$$({}^{C}D^{\gamma}_{b-}y)(x)=-\frac{1}{\Gamma(1-\gamma)}\int_x^b\frac{y'(t)dt}{(t-x)^{\gamma}}=({}^{RL}I^{1-\gamma}_{b-}Dy)(x), \tag{3.2.4}$$

$0<\mathrm{Re}(\gamma)<1$.

Proposition 3.2.2 *[3] The relations between the Caputo and the R-L derivatives are* *defined as following*

$$({}^{C}D^{\gamma}_{a+}y)(x) = \left({}^{RL}D^{\gamma}_{a+}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x) \tag{3.2.5}$$

and

$$({}^{C}D^{\gamma}_{b-}y)(x) = \left({}^{RL}D^{\gamma}_{b-}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{k!}(b-t)^k\right]\right)(x), \tag{3.2.6}$$

respectively.

As a special case, (3.2.5) and (3.2.6) take the below forms

$$({}^{C}D^{\gamma}_{a+}y)(x) = ({}^{RL}D^{\gamma}_{a+}[y(t) - y(a)])(x), \tag{3.2.7}$$

$0 < \mathrm{Re}(\gamma) < 1$,

$$({}^{C}D^{\gamma}_{b-}y)(x) = ({}^{RL}D^{\gamma}_{b-}[y(t) - y(b)])(x), \tag{3.2.8}$$

$0 < \mathrm{Re}(\gamma) < 1$. If $\gamma = n \in \mathbb{N}_0$ and the classical derivative $y^{(n)}(x)$ exists, then $({}^{C}D^{n}_{a+}y)(x)$ coincides with $y^{(n)}(x)$, while $({}^{C}D^{n}_{b-}y)(x)$ coincides with $y^{(n)}(x)$ up to the const factor $(-1)^n$, i.e.,

$$({}^{C}D^{n}_{a+}y)(x) = y^{(n)}(x) \;\text{ and }\; ({}^{C}D^{n}_{b-}y)(x) = (-1)^n y^{(n)}(x) \;\; (n \in \mathbb{N}). \tag{3.2.9}$$

The $({}^{C}D^{\gamma}_{a+}y)(x)$ and $({}^{C}D^{\gamma}_{b-}y)(x)$ have some features similar to those defined in equations (3.1.25) and (3.1.27) for the R-L fractional derivatives. [2] If $\mathrm{Re}(\gamma) > 0$, $n =$

$-[-\mathrm{Re}(\gamma)]$ is defined by (3.1.21) and $\mathrm{Re}(\beta) > n-1$, then

$$({}^{C}D^{\gamma}_{a+}(t-a)^{\beta})(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}(x-a)^{\beta-\gamma} \tag{3.2.10}$$

and

$$({}^{C}D^{\gamma}_{b-}(b-t)^{\beta})(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}(b-x)^{\beta-\gamma}. \tag{3.2.11}$$

Besides, for $k = 0,1,...,n-1$, we get

$$({}^{C}D^{\gamma}_{a+}(t-a)^{k})(x) = 0 \quad \text{and} \quad ({}^{C}D^{\gamma}_{b-}(t-a)^{k})(x) = 0 \tag{3.2.12}$$

In particular,

$$({}^{C}D^{\gamma}_{a+}1)(x) = 0 \quad \text{and} \quad ({}^{C}D^{\gamma}_{b-}1)(x) = 0 \tag{3.2.13}$$

Besides, for any $\alpha \in \mathbb{R}$,

$$({}^{C}D^{\gamma}_{a+}e^{\lambda t})(x) \neq \lambda^{\gamma}e^{\lambda x}, \tag{3.2.14}$$

$\mathrm{Re}(\gamma) > 0$, $\lambda > 0$.

Let $y(x)$ be a appropriate function, for instance $y(x) \in C[a,b]$. Then the ${}^{C}D^{\gamma}_{a+}$ and ${}^{C}D^{\gamma}_{b-}$ provide operations inverse to the I^{γ}_{a+} and I^{γ}_{b-} from the left, that is

$$({}^{C}D^{\gamma}_{a+}I^{\gamma}_{a+}y)(x) = y(x) \quad \text{and} \quad ({}^{C}D^{\alpha\gamma}_{b-}I^{\gamma}_{b-}y)(x) = y(x), \tag{3.2.15}$$

since $\gamma \in \mathbb{N}$.

On the other hand,

$$({}^C D^{\gamma}_{a+} I^{\gamma}_{a+} y)(x) = y(x) - \frac{(I^{\gamma+1-n}_{a+} y)(a+)}{\Gamma(n-\gamma)}(x-a)^{n-\gamma}, \tag{3.2.16}$$

$\mathrm{Re}(\gamma) \in \mathbb{N}$, $\mathrm{Im}(\gamma) \neq 0$,and

$$({}^C D^{\gamma}_{b-} I^{\gamma}_{b-} y)(x) = y(x) - \frac{(I^{\gamma+1-n}_{b-} y)(b-)}{\Gamma(n-\gamma)}(b-x)^{n-\gamma}, \tag{3.2.17}$$

$\mathrm{Re}(\gamma) \in \mathbb{N}$, $\mathrm{Im}(\gamma) \neq 0$. However, if $\mathrm{Re}(\gamma) > 0$ and $n = -[-\mathrm{Re}(\gamma)]$ is defined by (3.1.21). Then, under good enough conditions for $y(x)$

$$(I^{\gamma}_{a+} {}^C D^{\gamma}_{a+} y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k \tag{3.2.18}$$

and

$$(I^{\gamma}_{b-} {}^C D^{\gamma}_{b-} y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!}(b-x)^k. \tag{3.2.19}$$

As a special case, if $0 < \mathrm{Re}(\alpha) \leq 1$, then

$$(I^{\gamma}_{a+} {}^C D^{\gamma}_{a+} y)(x) = y(x) - y(a) \text{ and } (I^{\gamma}_{b-} {}^C D^{\gamma}_{b-} y)(x) = y(x) - y(a). \tag{3.2.20}$$

Under good enough conditions, the L-transform of the Caputo fractional derivative ${}^C D^{\gamma}_{0+} y$ is given by

$$(\mathscr{L} {}^C D^{\gamma}_{0+} y)(s) = s^{\alpha} (\mathscr{L} y)(s) - \sum_{k=0}^{n-1} s^{\gamma-k-1} (D^k y)(0). \tag{3.2.21}$$

As a special case

$$(\mathscr{L}^C D_{0+}^{\gamma} y)(s) = s^{\gamma} (\mathscr{L} y)(s) - s^{\gamma-1} y(0), \tag{3.2.22}$$

if $0 < \gamma \leq 1$. We have identified the Caputo derivatives on $[a,b]$. (3.2.1) and (3.2.2) can be used to define the Caputo fractional derivatives on the whole axis $\mathbb{R}$. Therefore the corresponding Caputo fractional derivative of order $\alpha \in \mathbb{C}$ can be presented as below [2]

$$({}^C D_{+}^{\gamma} y)(x) = \frac{1}{\Gamma(n-\gamma)} \int_{-\infty}^{x} \frac{y^{(n)}(t)dt}{(x-t)^{\gamma+1-n}} \tag{3.2.23}$$

and

$$({}^C D_{-}^{\gamma} y)(x) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_{x}^{\infty} \frac{y^{(n)}(t)dt}{(t-x)^{\gamma+1-n}}, \tag{3.2.24}$$

with $x \in \mathbb{R}$.

The $({}^C D_{+}^{\gamma} y)(x)$ and $({}^C D_{-}^{\gamma} y)(x)$ have the following features

$$({}^C D_{+}^{\gamma} e^{\lambda t})(x) = \lambda^{\gamma} e^{\lambda x} \quad \text{and} \quad ({}^C D_{-}^{\gamma} e^{-\lambda t})(x) = \lambda^{\gamma} e^{-\lambda x}. \tag{3.2.25}$$

Chapter 4

AN ALGORITHM FOR SINGLE-TERM EQUATIONS

We can call this method "indirect" because, rather than discretizing the differential equation

$$D^{n}_{*0}y(x) = f(x, y(x))$$

with appropriate i.c. (initial conditions)

$$D^{k}y(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, \lceil n \rceil - 1$$

directly, it requires some preliminary analytical manipulation, namely an application of Lemma 6.2 in order to convert the IVP for the differential equation into an equivalent Volterra integral equation,

$$y(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} D^k y(0) + \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t, y(t))dt \quad \text{where } m = \lceil n \rceil \tag{4.0.1}$$

We will consider on fractional variant of the classical second-order Adams-Bashforth-Moulton method.

4.1 Classical Formulation

For motivating the construction of the method, firstly, we should shortly remember the basic the logic behind the classical Adams-Bashford-Moulton algorithm for first-order equations. Thus, ivp should be considered for the first-order differential equation;

$$Dy(x) = f(x, y(x)) \tag{4.1.1}$$

$$y(0) = y_0 \tag{4.1.2}$$

Let the f to be a unique solution exists on $[0, T]$. We will use the predictor-corrector technique and we let that we are working on a uniform grid $\{t_j = jh\}$ where $\{j = 0, 1, \ldots, N\}$ with some integer N and $h = \frac{T}{N}$.

The main idea in this method is, we let that the estimations $y_j \approx y(t_j)$ $(j = 1, 2, ..., k)$ are already calculated , that we try to get the approximation y_{k+1} by means of the

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(z, y(z))dz \tag{4.1.3}$$

This equation follows upon integration of (4.1.1) on the interval $[t_k, t_{k+1}]$. Surely, we know neither of the expressions on the RHS of (4.1.3) exactly, but we have y_k, which is an approximation for $y(t_k)$, and we can use it instead of $y(t_k)$.We write the following two-point trapezoidal quadrature formula instead of the integral

$$\int_a^b g(z)dz \approx \frac{b-a}{2}(g(a) + g(b)) \tag{4.1.4}$$

therefore defining an equation for the unknown approximation y_{k+1}, it being

$$y_{k+1} = y_k + \frac{t_{k+1} - t_k}{2}(f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))) \tag{4.1.5}$$

where again we have to write y_k instead of $y(t_k)$ and $y(t_{k+1})$ instead of y_{k+1}. This gives the following equation for the implicit one-step Adams-Moulton method

$$y_{k+1} = y_k + \frac{t_{k+1} - t_k}{2}(f(t_k, y_k) + f(t_{k+1}, y_{k+1})) \tag{4.1.6}$$

The unknown quantity y_{k+1} that appears on LHS and RHS is the problem with this equation, and because of the nonlinear property of the function f we can not solve for y_{k+1} directly in general.Therefore, we may use (4.1.6) in an iterative process, adding a preliminary estimation for y_{k+1}in the RHS in order to define a better estimation that we can use later.

Likewise, we obtain the preliminary approximation y^p_{k+1}, the so-called predictor via only writing the rectangle rule instead of trapezoidal quadrature formula

$$\int_a^b g(z)dz \approx (b-a)g(a) \tag{4.1.7}$$

defining the explicit (one-step Adams-Bashforth or forward Euler) method

$$y^p_{k+1} = y_k + hf(t_k, y_k) \tag{4.1.8}$$

It is well known that the process given by (4.1.8) and

$$y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, y^p_{k+1})) \tag{4.1.9}$$

known as the one-step Adams-Bashford-Moulton technique, is convergent of order 2, i.e.

$$\max_{j=1,2,\ldots,N} |y(t_j) - y_j)| = O(h^2) \tag{4.1.10}$$

In a actual implementation, we begin by evaluating the predictor in (4.1.8), then we evaluate $f(t_{k+1}, y_{k+1}^p)$, use this to calculate the corrector in (4.1.9), and finally evaluate $f(t_{k+1}, y_{k+1})$ that is why this method is type of the PECE (Predict,Evaluate,Correct,Evaluate).

4.2 Fractional Formulation

We now attempt to continue the necessary concept to the fractional-order problem with some unavoidable modifications. We now derive an equation similar to (4.1.3), which is (4.0.1). Because the integration starts at 0 instead of t_k, equation (4.0.1) looks a little bit different from (4.1.3). This but doesn't affects major problem in our try to generalize the Adams method.

Merely what we do is replacing the integral by using the product trapezoidal quadrature, i.e. we use the nodes t_j $(j = 0, 1, ..., k+1)$ and interpret the function $(t_{k+1} - \cdot)^{n-1}$ as a weight function for the integral. On the other hand, we apply the approximation

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} g(z) dz \approx \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \, \widetilde{g}_{k+1}(z) dz, \tag{4.2.1}$$

where $\widetilde{g}_{k+1}(z)$ is the piecewise linear interpolant for g with nodes and knots selected at the t_j, $j = 0, 1, ..., k+1$. We can write the integral on the RHS of (4.2.1) as

$$\int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \widetilde{g}_{k+1}(z) dz = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \tag{4.2.2}$$

where

$$a_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \Phi_{j,k+1}(z) dz \tag{4.2.3}$$

and

$$\Phi_{j,k+1}(z) = \begin{cases} \frac{(z-t_{j-1})}{(t_j - t_{j-1})} & \text{if} \quad t_{j-1} < z \leq t_j \\ \frac{(t_{j+1}-z)}{(t_{j+1}-t_j)} & \text{if} \quad t_j < z < t_{j+1} \\ 0 & \text{else} \end{cases} . \tag{4.2.4}$$

This is clear because the function $\Phi_{j,k+1}$ satisfy

$$\Phi_{j,k+1}(t_\mu) = \begin{cases} 0 & \text{if} \quad j \neq \mu \\ 1 & \text{if} \quad j = \mu \end{cases}$$

and that they are continuous and piecewise linear with breakpoints at the nodes t_μ, so they must be integrated exactly by our formula.

Let $j = 0$

$$a_{0,k+1} = \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \Phi_{0,k+1}(z) dz = \int_{t_0}^{t_1} (t_{k+1} - z)^{n-1} \frac{(t_1 - z)}{(t_1 - t_0)} dz$$

using integration by parts with

$$u = (t_1 - z)$$

$$du = -dz$$

and

$$dv = (t_{k+1} - z)^{n-1} dz$$
$$v = \tfrac{-(t_{k+1}-z)^n}{n}$$

we get

$$\begin{aligned}
& \frac{1}{(t_1 - t_0)} \left[-(t_1 - z) \frac{(t_{k+1} - z)^n}{n} \right]_{t_0}^{t_1} - \frac{1}{n} \int_{t_0}^{t_1} (t_{k+1} - z)^n dz \\
= \; & \frac{1}{t_1} \left(\frac{t_1 (t_{k+1})^n}{n} + \frac{1}{n} \left[\frac{(t_{k+1} - z)^{n+1}}{n+1} \right]_{t_0}^{t_1} \right) \\
= \; & \frac{t_{k+1}^n}{n} + \frac{(t_{k+1} - t_1)^{n+1} - (t_{k+1})^{n+1}}{t_1 . n . (n+1)} = \frac{(t_{k+1} - t_1)^{n+1} + t_{k+1}^n (t_1 + n t_1 - t_{k+1})}{t_1 . n . (n+1)} \\
= \; & a_{0,k+1} \qquad (4.2.5)
\end{aligned}$$

now let $j \in [1, k]$

$$\begin{aligned}
a_{j,k+1} &= \int_0^{t_{k+1}} (t_{k+1} - z)^{n-1} \Phi_{j,k+1}(z) dz \\
&= \int_{t_{j-1}}^{t_j} (t_{k+1} - z)^{n-1} \frac{(z - t_{j-1})}{(t_j - t_{j-1})} dz \qquad (4.2.6) \\
&\quad + \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{n-1} \frac{(t_{j+1} - z)}{(t_{j+1} - t_j)} dz \qquad (4.2.7)
\end{aligned}$$

for the integral on the (4.2.6) we have

$$\int_{t_{j-1}}^{t_j} (t_{k+1} - z)^{n-1} \frac{(z - t_{j-1})}{(t_j - t_{j-1})} dz$$

using integration by parts with

$$u = (z - t_{j-1})$$
$$du = dz$$

and

$$dv = (t_{k+1} - z)^{n-1} dz$$
$$v = \tfrac{-(t_{k+1}-z)^n}{n}$$

we get

$$\begin{aligned}
& \frac{1}{(t_j - t_{j-1})}\left[-(z - t_{j-1})\frac{(t_{k+1}-z)^n}{n}\right]_{t_{j-1}}^{t_j} - \frac{1}{n}\int_{t_{j-1}}^{t_j}(t_{k+1}-z)^n dz \\
= \; & \frac{1}{(t_j - t_{j-1})}\left(-(t_j - t_{j-1})\frac{(t_{k+1}-t_j)^n}{n} + \frac{1}{n}\left[\frac{-(t_{k+1}-z)^{n+1}}{n+1}\right]_{t_{j-1}}^{t_j}\right) \\
= \; & \frac{-(t_{k+1}-t_j)^n}{n} + \frac{-(t_{k+1}-t_j)^{n+1} + (t_{k+1}-t_{j-1})^{n+1}}{(t_j - t_{j-1}).n.(n+1)} \\
= \; & \frac{(t_{k+1}-t_{j-1})^{n+1} + (t_{k+1}-t_j)^n (t_{j-1} + n(t_{j-1}-t_j) - t_{k+1})}{(t_j - t_{j-1}).n.(n+1)}
\end{aligned}$$

and for the integral on the (4.2.7) we have

$$\int_{t_j}^{t_{j+1}} (t_{k+1}-z)^{n-1} \frac{(t_{j+1}-z)}{(t_{j+1}-t_j)} dz$$

using integration by parts with

$$u = (t_{j+1} - z)$$
$$du = -dz$$

and

$$dv = (t_{k+1} - z)^{n-1} dz$$
$$v = \tfrac{-(t_{k+1}-z)^n}{n}$$

we get

$$
\begin{aligned}
& \frac{1}{(t_{j+1}-t_j)}\left[-(t_{j+1}-z)\frac{(t_{k+1}-z)^n}{n}\right]_{t_j}^{t_{j+1}} - \frac{1}{n}\int_{t_j}^{t_{j+1}}(t_{k+1}-z)^n dz \\
= & \frac{1}{(t_{j+1}-t_j)}\left((t_{j+1}-t_j)\frac{(t_{k+1}-t_j)^n}{n} + \frac{1}{n}\left[\frac{-(t_{k+1}-z)^{n+1}}{n+1}\right]_{t_j}^{t_{j+1}}\right) \\
= & \frac{(t_{k+1}-t_j)^n}{n} + \frac{-(t_{k+1}-t_j)^{n+1}+(t_{k+1}-t_{j+1})^{n+1}}{(t_{j+1}-t_j).n.(n+1)} \\
= & \frac{(t_{k+1}-t_{j+1})^{n+1}-(t_{k+1}-t_j)^n(-t_{j+1}+n(t_j-t_{j+1})+t_{k+1})}{(t_{j+1}-t_j).n.(n+1)}
\end{aligned}
$$

Therefore, for the $j \in [1,k]$ we have

$$
\begin{aligned}
a_{j,k+1} \quad = \quad & \frac{(t_{k+1}-t_{j-1})^{n+1}+(t_{k+1}-t_j)^n(t_{j-1}+n(t_{j-1}-t_j)-t_{k+1})}{(t_j-t_{j-1}).n.(n+1)} \\
& + \frac{(t_{k+1}-t_{j+1})^{n+1}}{(t_{j+1}-t_j).n.(n+1)} \\
& - \frac{(t_{k+1}-t_j)^n(-t_{j+1}+n(t_j-t_{j+1})+t_{k+1})}{(t_{j+1}-t_j).n.(n+1)}
\end{aligned} \tag{4.2.8}
$$

now let $j = k+1$, then

$$
\begin{aligned}
a_{k+1,k+1} &= \int_0^{t_{k+1}}(t_{k+1}-z)^{n-1}\Phi_{k+1,k+1}(z)dz \\
&= \int_{t_k}^{t_{k+1}}(t_{k+1}-z)^{n-1}\frac{(z-t_k)}{(t_{k+1}-t_k)}dz
\end{aligned}
$$

using integration by parts with

$$u = (z-t_k)$$

$$du = dz$$

and

$$dv = (t_{k+1}-z)^{n-1}dz$$

$$v = \tfrac{-(t_{k+1}-z)^n}{n}$$

we get

$$\begin{aligned}
& \frac{1}{(t_{k+1}-t_k)}\left(\left[-(z-t_k)\frac{(t_{k+1}-z)^n}{n}\right]_{t_k}^{t_{k+1}}+\frac{1}{n}\int_{t_k}^{t_{k+1}}(t_{k+1}-z)^n dz\right) \\
= & \frac{-1}{(t_{k+1}-t_k)n}\left(\left[\frac{(t_{k+1}-z)^{n+1}}{n+1}\right]_{t_k}^{t_{k+1}}\right) \\
= & \frac{(t_{k+1}-t_k)^n}{n(n+1)} \\
= & a_{k+1,k+1} \qquad (4.2.9)
\end{aligned}$$

Therefore, from (4.2.5, 4.2.8 and 4.2.9) (taking $t_j = jh$ with some fixed h)

$$a_{j,k+1} = \begin{cases} \frac{h^n}{n(n+1)}(k^{n+1}-(k-n)(k+1)^n) & \text{if} \quad j=0 \\ \frac{h^n}{n(n+1)}\left[(k-j+2)^{n+1}+(k-j)^{n+1}-2(k-j+1)^{n+1}\right] & \text{if} \quad j\in[1,k] \\ \frac{h^n}{n(n+1)} & \text{if} \quad j=k+1 \end{cases} \qquad (4.2.10)$$

(4.2.10) gives the corrector formula as following

$$\begin{aligned}
y_{k+1} = & \sum_{j=0}^{m-1}\frac{t_{k+1}^j}{j!}y_0^{(j)} \\
& +\frac{1}{\Gamma(n)}\left(\sum_{j=0}^{k}a_{j,k+1}f(t_j,y_j)+a_{k+1,k+1}f(t_{k+1},y_{k+1}^p)\right) \qquad (4.2.11)
\end{aligned}$$

[4] The calculation of the predictor formula y_{k+1}^p is the remaining problem.The way that we are going to use for generalize the one-step Adams–Bashforth method is the following: We write product rectangle rule instead of the integral on the RHS of (4.0.1)

$$\int_0^{t_{k+1}}(t_{k+1}-z)^{n-1}g(z)dz \approx \sum_{j=0}^{k}b_{j,k+1}g(t_j) \qquad (4.2.12)$$

where now

$$b_{j,k+1} = \int_{t_j}^{t_{j+1}} (t_{k+1} - z)^{n-1} dz = \frac{(t_{k+1} - t_j)^n - (t_{k+1} - t_{j+1})^n}{n} \tag{4.2.13}$$

This expression for weights can be derived in a way similar to the method used in the derivation of (4.2.10).However, here we are dealing with a piecewise constant approximation and not a piecewise linear one, and hence we have to replace the "hat-shaped" functions Φ_{kj} by functions being of constant value 1 on $\left[t_j, t_{j+1}\right]$ and 0 on the remaining parts of the interval $[0, t_{k+1}]$. Again, in the equispaced case, we have the simpler expression

$$b_{j,k+1} = \frac{h^n}{n}((k+1-j)^n - (k-j)^n) \tag{4.2.14}$$

Thus, the predictor y_{k+1}^p is given by the fractional Adams-Bashforth method

$$y_{k+1}^p = \sum_{j=0}^{m-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j) \tag{4.2.15}$$

Therefore, the fractional Adams-Bashford-Moulton method, is completely defined now by (4.2.15) and (4.2.11) with the weight $a_{j,k+1}$ and $b_{j,k+1}$ being defined according to ((4.2.5, 4.2.8, 4.2.9)) and (4.2.13), respectively.

Chapter 5

ERROR ANALYSIS

For the error analysis of this algorithm, we will work on the case of an equispaced grid, i.e. from now on we assume that $t_j = jh = jT/N$ with some $N \in \mathbb{N}$.

What we need for our purposes is some information on the errors of the quadrature formulas that we have used in the derivation of the predictor and the corrector, respectively. We first give a statement on the product rectangle rule that we have used for the predictor.

Theorem 5.0.1 *(a) Assume that $z \in C^1[0,T]$. Then,*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq \frac{1}{n} \left\| z' \right\|_\infty t_{k+1}^n h$$

(b) Assume that $z(t) = t^p$ for some $p \in (0,1)$. Then,

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq C_{n,p}^{\mathrm{Re}} t_{k+1}^{n+p-1} h$$

where $C_{n,p}^{\mathrm{Re}}$ is a const. that depends on n and p.

Proof. By construction of the product rectangle formula, we find in both cases that the quadrature error has the representation

$$
\begin{aligned}
& \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \\
= \quad & \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1}(z(t)-z(t_j))dt \qquad (5.0.1)
\end{aligned}
$$

To prove statement (a), we apply the MVT of Differential Calculus to the second factor of the integrand on the RHS of (5.0.1) and derive

$$
\begin{aligned}
& \left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \\
= \quad & \left| \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1}(z(t)-z(t_j))dt \right| \\
= \quad & \left| \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1}(t-t_j)z'(\xi)dt \right|
\end{aligned}
$$

where $t_j < \xi < t$

$$
\begin{aligned}
\leq \quad & \|z'\|_\infty \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1}(t-jh)dt \\
= \quad & \|z'\|_\infty \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1}(t-t_{k+1}+t_{k+1}-jh)dt \\
= \quad & \|z'\|_\infty \sum_{j=0}^{k} \left[\int_{jh}^{(j+1)h} -(t_{k+1}-t)^{n} + (t_{k+1}-t)^{n-1}(k+1-j)hdt \right] \\
= \quad & \|z'\|_\infty \sum_{j=0}^{k} \left[\frac{(t_{k+1}-t)^{n+1}}{n+1} - (k+1-j)h\frac{(t_{k+1}-t)^{n}}{n} \right]_{t=jh}^{t=(j+1)h} \\
= \quad & \|z'\|_\infty \sum_{j=0}^{k} \frac{(k-j)^{n+1}h^{n+1}}{n+1} - \frac{(k+1-j)^{n+1}h^{n+1}}{n+1} \\
= \quad & -((k-j)+1)h\frac{(k-j)^{n}h^{n}}{n} + (k-j+1)h\frac{(k+1-j)^{n}}{n}h^{n}
\end{aligned}
$$

$$
\begin{aligned}
&= \|z'\|_\infty \sum_{j=0}^{k} h^{n+1}(\frac{1}{n(n+1)}(k+1-j)^{n+1} \\
&\quad -\frac{1}{n(n+1)}(k-j)^{n+1} - \frac{(k-j)^n}{n}) \\
&= \|z'\|_\infty \frac{h^{n+1}}{n} \sum_{j=0}^{k} \left(\frac{1}{n+1} \left[(k+1-j)^{n+1} - (k-j)^{n+1} \right] - (k-j)^n \right) \\
&= \|z'\|_\infty \frac{h^{n+1}}{n} \left(\frac{(k+1)^{n+1}}{1+n} - \sum_{j=0}^{k} j^n \right) \\
&= \|z'\|_\infty \frac{h^{n+1}}{n} \left(\int_0^{k+1} t^n dt - \sum_{j=0}^{k} j^n \right)
\end{aligned}
$$

Here the term in parentheses is simply the remainder of the standard rectangle quadrature formula, applied to the function t^n, and taken over the interval $[0, k+1]$. Since the integrand is monotonic, we may apply some standard results from quadrature theory to find that this term is bounded by the total variation of the integrand, viz. the quantity $(k+1)^n$. Thus

$$
\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \leq \|z'\|_\infty \frac{h^{n+1}}{n} (k+1)^n.
$$

Similarly, to prove (b), we use the monotonicity of z in (5.0.1) and derive

$$
\begin{aligned}
&\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} z(t)\, dt - \sum_{j=0}^{k} b_{j,k+1} z(t_j) \right| \\
&= \left| \sum_{j=0}^{k} \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1} (z(t) - z(t_j)) dt \right| \\
&\leq \sum_{j=0}^{k} \left| z(t_{j+1}) - z(t_j) \right| \int_{jh}^{(j+1)h} (t_{k+1}-t)^{n-1} dt \\
&= \frac{h^{n+p}}{n} \sum_{j=0}^{k} \left((j+1)^p - j^p \right) \left((k+1-j)^n (k-j)^n \right) \\
&\leq \frac{h^{n+p}}{n} \left((k+1)^n - k^n + (k+1)^p - k^p + pn \sum_{j=1}^{k-1} j^{p-1} (k-j+q)^{n-1} \right) \\
&\leq \frac{h^{n+p}}{n} \left(n(k+q)^{n-1} + pk^{p-1} + pn \sum j^{p-1} (k-j+q)^{n-1} \right)
\end{aligned}
$$

by additional applications of the Mean Value Theorem. Here $q = 0$ if $n \leq 1$, and $q = 1$ otherwise. In either case a brief asymptotic analysis using the Euler-MacLaurin formula yields that the term in parentheses is bounded from above by $C_{n,p}^{\mathrm{Re}}(k+1)^{p+n-1}$ where $C_{n,p}^{\mathrm{Re}}$ is a const depending on h and p. ■

Next we come to a corresponding result for the product trapezoidal formula that we have used for the corrector.

Theorem 5.0.2 *Suppose $z \in C^2[0,T]$,*

$$\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \leq C_n h^2 \tag{5.0.2}$$

where C_n only depends on n.

Proof.

$$\begin{aligned} & \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} z(t_j) \right| \\ = & \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} z(t) dt - \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} \widetilde{z}_{k+1}(t) dt \right| \\ = & \left| \int_0^{t_{k+1}} (t_{k+1} - t)^{n-1} (z(t) - \widetilde{z}_{k+1}(t)) dt \right| \end{aligned}$$

using divided difference formula (1.0.2)

$$\begin{aligned} \leq & \frac{\|z''\|_\infty}{2} \sum_{j=1}^{k+1} \int_{t_{j-1}}^{t_j} (t_{k+1} - t)^{n-1} (t_j - t)(t - t_{j-1}) dt \\ = & \frac{\|z''\|_\infty h^{n+2}}{2n(n+1)} \sum_{j=1}^{k+1} (k-j+2)^{n+1} + (k-j+1)^{n+1} \\ & + \frac{2}{n+2} \left[(k-j+2)^{n+2} - (k-j+1)^{n+2} \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\|z''\|_\infty h^{n+2}}{2n(n+1)} \sum_{j=1}^{k+1} \left((j+1)^{n+1} + j^{n+1} + \frac{2}{n+2}(1-(k+1)^{n+2}) \right) \\
&= -\frac{\|z''\|_\infty h^{n+2}}{2n(n+1)} \left(2\int_1^{k+1} t^{n+1} dt - \sum_{j=1}^{k+1} ((j+1)^{n+1} + j^{n+1}) \right) \\
&\le \begin{cases} \frac{\|z''\|_\infty h^{n+2}}{24} \sum_{j=1}^{k+1} j^{n-1} & \text{if} \quad n<1 \\ \frac{\|z''\|_\infty h^{n+2}}{24} \sum_{j=1}^{k+1} (j+1)^{n-1} & \text{if} \quad n \ge 1 \end{cases} \\
&\le \begin{cases} \frac{\|z''\|_\infty h^{n+2}}{24} \int_0^{k+1} t^{n-1} dt & \text{if} \quad n<1 \\ \frac{\|z''\|_\infty h^{n+2}}{24} \int_2^{k+2} t^{n-1} dt & \text{if} \quad n \ge 1 \end{cases} \\
&\le C_n h^2
\end{aligned}
$$

■

Lemma 5.0.3 *Assume that the solution y of the ivp is s.t.*

$$
\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_{*0}^n y(t) dt - \sum_{j=0}^{k} b_{j,k+1} D_{*0}^n y(t_j) \right| \le C_1 t_{k+1}^{\gamma_1} h^{\delta_1}
$$

and

$$
\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D_{*0}^n y(t) dt - \sum_{j=0}^{k+1} a_{j,k+1} D_{*0}^n y(t_j) \right| \le C_2 t_{k+1}^{\gamma_2} h^{\delta_2}
$$

with some $\gamma_1, \gamma_2 \ge 0$ *and* $\delta_1, \delta_2 > 0$. *Then, for some suitably chosen* $T > 0$, *we have*

$$
\max_{0 \le j \le N} \left| y(t_j) - y_j \right| = O(h^q)
$$

where $q = \min\{\delta_1 + n, \delta_2\}$ *and* $N = [T/h]$.

Proof. We will show that, for sufficiently small h,

$$\left|y(t_j) - y_j\right| \le Ch^q \tag{5.0.3}$$

for all $j \in \{0, 1, \ldots, N\}$, where C is a const. The proof will be based on mathematical induction. In view of the given initial condition, the induction basis $(j = 0)$ is presupposed. Now assume that (5.0.3) is correct for $j = 0, 1, \ldots, k$ for some $k \le N-1$. We must then prove that the inequality also valid for $j = k+1$. To do this, firstly, we look at the error of the predictor y^p_{k+1}. By construction of the predictor we find that

$$\begin{aligned}
\left|y(t_{k+1}) - y^p_{k+1}\right| &= \frac{1}{\Gamma(n)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} f(t, y(t))dt - \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j)\right| \\
&\le \frac{1}{\Gamma(n)}\left|\int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} D^n_{*0} y(t)dt - \sum_{j=0}^{k} b_{j,k+1} D^n_{*0} y(t_j)\right| \\
&\quad + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} \left|f(t_j, y(t_j)) - f(t_j, y_j)\right| \\
&\le \frac{C_1 t^{\gamma_1}_{k+1}}{\Gamma(n)} h^{\delta_1} + \frac{1}{\Gamma(n)} \sum_{j=0}^{k} b_{j,k+1} LCh^q \\
&\le \frac{C_1 T^{\gamma_1}}{\Gamma(n)} h^{\delta_1} + \frac{CLT^n}{\Gamma(n+1)} h^q
\end{aligned} \tag{5.0.4}$$

Here we have used the Lipschitz property of f, the assumption on the error of the rectangle formula, and the facts that, by construction of the quadrature formula underlying the predictor, $b_{j,k+1} > 0$ for all j and k and

$$\sum_{j=0}^{k} b_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1}-t)^{n-1} dt = \frac{1}{n} t^n_{k+1} \le \frac{1}{n} T^n.$$

On the basis of the bound (5.0.4) or the predictor error we begin the analysis of the corrector error.We recall the relation (4.2.10) which we shall use in particular for $j =$

$k+1$ and find, arguing in a similar way to above, that

$$\begin{aligned}
|y(t_{k+1})-y_{k+1}| &= \frac{1}{\Gamma(n)}\Big|\int_0^{t_{k+1}}(t_{k+1}-t)^{n-1}f(t,y(t))dt \\
&\quad -\sum_{j=0}^{k}a_{j,k+1}f(t_j,y_j)-a_{k+1,k+1}f(t_{k+1},y_{k+1}^{p})\Big| \\
&\leq \frac{1}{\Gamma(n)}\Big|\int_0^{t_{k+1}}(t_{k+1}-t)^{n-1}D_{*0}^{n}y(t)dt-\sum_{j=0}^{k+1}a_{j,k+1}D_{*0}^{n}y(t_j)\Big| \\
&\quad +\frac{1}{\Gamma(n)}\sum_{j=0}^{k}a_{j,k+1}\left|f(t_j,y(t_j))-f(t_j,y_j)\right| \\
&\quad +\frac{1}{\Gamma(n)}a_{k+1,k+1}\left|f(t_{k+1},y(t_{k+1}))-f(t_{k+1},y_{k+1}^{p})\right| \\
&\leq \frac{C_2t_{k+1}^{\gamma_2}}{\Gamma(n)}h^{\delta_2}+\frac{CL}{\Gamma(n)}h^{q}\sum_{j=0}^{k}a_{j,k+1} \\
&\quad +a_{k+1,k+1}\frac{L}{\Gamma(n)}\left(\frac{C_1T^{\gamma_1}}{\Gamma(n)}h^{\delta_1}+\frac{CLT^{n}}{\Gamma(n+1)}h^{q}\right) \\
&\leq \Big(\frac{C_2T^{\gamma_2}}{\Gamma(n)}+\frac{CLT^{n}}{\Gamma(n+1)} \\
&\quad +\frac{C_1LT^{\gamma_1}}{\Gamma(n)\Gamma(n+2)}+\frac{CL^2T^{n}}{\Gamma(n+1)\Gamma(n+2)}h^{n}\Big)h^{q}
\end{aligned}$$

in view of the nonnegativity of γ_1 and γ_2 and the relations $\delta_2 \leq q$ and $\delta_1+n \leq q$. By selecting T good enough small, we can make sure that the second summand in the parentheses is bounded by $C/2$. Having fixed this value for T, we can then make the sum of the remaining expressions in the parentheses smaller than $C/2$ too (for sufficiently small h) simply by choosing C sufficiently large. It is then obvious that the entire upper bound does not exceed Ch^q. ■

Theorem 5.0.4 *Let $0<n$ and assume $D_{*0}^{n}y\in C^2[0,T]$ for some good enough T. Then,*

$$\max_{0\leq j\leq N}\left|y(t_j)-y_j\right| = \begin{cases} O(h^2) & \text{if } n\geq 1 \\ O(h^{1+n}) & \text{if } n<1 \end{cases}$$

Before we come to the proof, we note one particular point: The order of convergence depends on n, and it is a non-decreasing function of n.This is due to the fact that we discretize the integral operator in (4.0.1) which behaves more smoothly (and hence can be approximated with a higher accuracy) as n increases. In contrast, socalled direct methods like the backward differentiation method use a different approach; as the name suggests they directly discretize the differential operator in the given ivp. The smoothness properties of such operators (and thus the ease with which they may be approximated) deteriorate as n increases, and so we find that the convergence order of the method from is a non-increasing function of n; in particular no convergence is achieved there for $n \geq 2$. It is a distinctive advantage of the Adams scheme presented here that it converges for all $n > 0$.

Remark 5.0.5 *We formally recover the error bound (4.1.10) if we set $c = 1$.*

Proof. In view of Theorem 1., we may apply Lemma 1. with $\gamma_1 = \gamma_2 = c > 0, \delta_1 = 1$ and $\delta_2 = 2$. Thus, defining

$$q = \min\{1+c, 2\} = \begin{cases} 2 & \text{if } c \geq 1, \\ 1+c & \text{if } c < 1, \end{cases}$$

we find an $O(h^q)$ error bound. ■

Conjecture 5.0.6 *Let $n > 0$ and assume that $D^n_{*0}y \in C^s[0,T]$ for some $s \geq 3$ and some suitable T. Then*

$$y(T) - y_{T/h} = \sum_{j=1}^{s_1} a_j h^{2j} + \sum_{j=1}^{s_2} d_j h^{j+n} + O(h^{s_3})$$

where s_1*,*s_2*, and* s_3 *are fixed constants depending only on* s *and holds* $s_3 > \max(2s_1, s_2 + n)$.

Note that the asymptotic expansion starts with an h^2 term and continues with h^{1+n} for $1 < n < 3$, whereas it starts with h^{1+n}, followed by h^2, for $0 < n < 1$.

Chapter 6

NUMERICAL EXAMPLE

Now we present a numerical example to illustrate the error bounds stated above. We only looked at example with $0 < \alpha < 2$ since the case $\alpha \geq 2$ does not seem to be of major practical interest.

Our example covers the case where the given function f (the RHS of the differential equation) is smooth. We look at the homogeneous linear equation

$$D_*^\alpha y(x) = -y(x),\ y(0) = 1,\ y'(0) = 0$$

(the second of the initial condition only for $\alpha > 1$ of course). It is well known that the exact solution is

$$y(x) = E_\alpha(-x^\alpha),$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

is the Mittag-Leffler function of order α.

In Table 1 we state some numerical results for this problem in the case $\alpha < 1$. The data given in the tables is the error of the Adams scheme at the point $x = 1$. We can see from the last line that the order of convergence is always close to $1 + \alpha$. In contrast,

Table 2 displays the case $\alpha > 1$; here the results confirm the $O(h^2)$ behaviour. This reflects the statement of equation which is

$$\max_{j=0,1,\ldots N} |y(t_j) - y_h(t_j)| = O(h^p)$$

where

$$p = \min(2, 1+\alpha)$$

and the quantities h and N are related according to $h = T/N$, and T is the upper bound of the interval on which we are looking for the solution.

Table 1. Errors for $\alpha < 1$.

h	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
1/10	−5.42(−3)	−1.86(−3)	−1.30(−3)	−9.91(−4)	−7.51(−4)
1/20	−1.22(−3)	−5.85(−4)	−3.93(−4)	−2.81(−4)	−1.91(−4)
1/40	−4.40(−4)	−1.97(−4)	−1.26(−4)	−8.28(−5)	−4.99(−5)
1/80	−1.68(−4)	−6.90(−5)	−4.18(−5)	−2.50(−5)	−1.32(−5)
1/160	−6.65(−5)	−2.49(−5)	−1.42(−5)	−7.63(−6)	−3.54(−6)
1/320	−2.68(−5)	−9.18(−6)	−4.86(−6)	−2.35(−6)	−9.48(−7)
EOC	1.31	1.44	1.54	1.70	1.90

Table 2. Errors for $\alpha > 1$.

h	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.85$
1/10	−5.61(−4)	−5.46(−4)	−4.40(−4)
1/20	−1.27(−4)	−1.28(−4)	−1.07(−4)
1/40	−2.90(−5)	−3.04(−5)	−2.65(−5)
1/80	−6.68(−6)	−7.33(−6)	−6.57(−6)
1/160	−1.55(−6)	−1.78(−6)	−1.63(−6)
1/320	−3.63(−7)	−4.37(−7)	−4.07(−7)
EOC	2.09	2.03	2.00

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