High Order Accurate Approximation of the First and Pure Second Derivatives of the Laplace Equation on a Rectangle and a Rectangular Parallelepiped

Hamid Mir Mohammad Sadeghi

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Applied Mathematics and Computer Science

Eastern Mediterranean University
June 2016
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Cem Tanova
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Doctor of Philosophy in Applied Mathematics and Computer Science.

Prof. Dr. Nazim Mahmudov
Acting Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis of the degree of Doctor of Philosophy in Applied Mathematics and Computer Science.

Prof. Dr. Adıgüzel Dosiyev
Supervisor

Examining Committee

1. Prof. Dr. Allaberen Ashyralyev
2. Prof. Dr. Adıgüzel Dosiyev
3. Prof. Dr. Mahir Rasulov
4. Assoc. Prof. Dr. Derviş Subaşı
5. Asst. Prof. Dr. Suzan C. Buranay

ABSTRACT

In this thesis, we discuss the approximation of the first and pure second order derivatives for the solution of the Dirichlet problem for Laplace's equation on a rectangular domain and in a rectangular parallelepiped. In the case when the domain is a rectangle, the boundary values on the sides of the rectangle are supposed to have sixth derivatives satisfying the Hölder condition. On the vertices, besides the continuity, the compatibility conditions, which result from the Laplace equation, for the second and fourth derivatives of the boundary functions, given on the adjacent sides, are also satisfied. Under these conditions a uniform approximation of order $O\left(h^4\right)$ (h is the grid size), is obtained for the solution of the Dirichlet problem on a square grid, its first and pure second derivatives, by a simple difference schemes.

In the case a rectangular parallelepiped, we propose and justify difference schemes for the first and pure second derivatives approximation of the solution of the Dirichlet problem for 3D Laplace's equtation.The boundary values on the faces of the parallelepiped are assumed to have the sixth derivatives satisfying the Hölder condition. They are continuous on the edges, and their second and fourth order derivatives satisfy the compatibility conditions which results from the Laplace equation. It is proved that the solutions of the proposed difference schemes converge uniformly on the cubic grid with order $O(h^4)$, where h is the grid step. For both cases numerical experiments are demonstrated to support the analysis made.

Keywords: Finite difference method, approximation of derivatives, uniform error, Laplace equation.

ÖZ

Bu tezde, Laplace Denkleminin dikdörtgensel bölgede ve dikdörtgenler prizması üzerinde Dirichlet probleminin çözümü için birinci mertebeden ve pür ikinci mertebeden türevlerinin yaklaşımı tartışılır. Tanım bölgesinin dikdörtgen olduğu durumda dikdörtgenin kenarlarında verilen sınır fonksiyonlarının altıncı türevlerinin Hölder şartını sağladıkları kabul edildi. Köşelerde süreklilik şartının yanında Laplace denkleminden sonuçlanan köşelerin komşu kenarlarında verilen sınır değer fonksiyonlarının ikinci ve dördüncü türevleri icin uyumluluk şartları da sağlandı. Bu şartlar altında Dirichlet probleminin kare ızgara üzerinde çözümü için ve çözümün birinci ve pür ikinci türevleri için $O(h^4)$ (h adım uzunluğu) düzgün yaklaşımı sade bir fark şeması ile elde edildi.

İkinci durumda tanım bölgesi dikdörtgenler prizması olduğunda Laplace denkleminin Dirichlet probleminin çözümünün birinci ve pür ikinci türevlerinin yaklaşımı için fark şemaları önerilir ve sağlanır. Prizmanın yüzeylerinde verilen sınır değerlerinin altıncı türevlerinin Hölder koşulunu sağladığı kabul edildi. Köşelerde süreklidirler ve onların ikinci ve dördüncü mertebeden türevleri Laplace denklemlerinden sonuçlanan uyumluluk koşulunu sağlar. Önerilen fark şemalarının çözümünün küp ızgaralar üzerinde h ızgara uzunluğu olduğunda $O(h^4)$ mertebesinden düzgün yakınsadığı ispatlandı. Her iki durum için sayısal örnekler yapılan analizleri desteklemek için verildi.

Anahtar Kelimeler: Sonlu fark metodu, türevlerin yaklasımı, düzgün hata, Laplace denklemi.

DEDICATION

To My Beloved Family

ACKNOWLEDGMENT

First of all I would like to express my profound gratitude to my supervisor Prof. Dr. Adıgüzel Dosiyev. He supported me throughout my studies and research with his patience and immense knowledge. This dissertation would have never been accomplished without his incredible patience and his guidance.

Worthy of extreme respect are also Prof. Dr. Allaberen Ashyralyev, Prof. Dr. Mahir Rasulov, Asst. Prof. Dr. Suzan C. Buranay and Assoc. Prof. Dr. Derviş Subaşı, who have served as my committee members and have offered valuable comments and suggestions during my oral defense. They made my defense an unforgettable and enjoyable time for me.

I must offer special appreciation to my dear friend, Dr. Emine Çeliker, who reviewed and corrected my thesis. This thesis owes its English form to her.

I would like to express great respect to my father Prof. Dr. Javad Mir-Mohammad-Sadeghi who encouraged me to endeavor towards my aim. My special thanks also goes to my mother Ashraf Tadayon for her prayers and sacrifices that she has done on my behalf.

I would like to present my sincere thankfulness to my dear brother Ali Mir-Mohammad-Sadeghi, who passed away in 1997 for his great role in my life. He constantly provided emotional support and took care of me in many aspects. He was truly brother when needed. Also I would like to thank to my brother Dr. Amir Mir-Mohammad-Sadeghi for supporting and encouraging me throughout the time of my studies.

Last but not least, I owe my beloved wife Shaghayegh Parchami who devoted to me her love throughout my studies. I am indebted to her support and concern in all moments of my degree.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

Pierre Simon Marquis de Laplace (1749-1827) identified arguably same of the most well known partial differential equations. These equations are widely employed in a number of topics in applied sciences in order to illustrate equilibrium or steady-state problems. One of the most important elliptic equations is Laplace's equation which has been employed to model as many problems as real-life situations. Laplace's equation can be employed in the formulation of problems relevant to the theory of gravitation, electrostatics, dielectrics and problems arising in magneto statics, in the field of interest to mathematical physics. Further it is applied in engineering, when dealing with problems related to the torsion of prismatic elastic solids, analysis of steady heat conduction in solid bodies, the irrotational flow of incompressible fluid, and so on (see [1]-[33]).

Undoubtedly, the derivative of the solution can be just as important as of finding the solution itself. For instance, the fundamental problem of fracture mechanics is the fracture problem of the stress intensity factor, which it comes from the derivative of the intensity function, and in electrostatics problems the electric field can be obtained from the first derivative of electrostatics potential function.

Another torsion example of the Dirichlet problem for of Poisson's equation is the torsion problem for a rectangular prism.

The problem of the torsion of any prismatic frame whose section is the region D, bounded by the contour L is reduced to the following boundary value problem using the theory of Saint-Venant. The solution of the Poisson equation

$$\Delta u = -2, \tag{1.0.1}$$

that is reduced to zero on the contour L:

$$u = 0 \text{ on } L.$$

The elements of tangential stress are

$$\tau_{zx} = G\vartheta \frac{\partial u}{\partial y}, \quad \tau_{zy} = -G\vartheta \frac{\partial u}{\partial x},$$

and the torsional moment is shown by

$$M = G\vartheta \int\int_D u dx dy.$$

The angle of twist per unit length and the modulus of shear are indicated by ϑ and G, respectively.

Now, the solution of the torsion problem is given for a rectangle of sides a and b. The solution of equation (1.0.1) decreasing to zero on the contour should be found. We attempt to find the exact solution, u_0, of equation (1.0.1) to decrease the problem to the solution of the Laplace equation.

Let u_0 be represented in the form:

$$u_0 = Ax^2 + By^2$$

where $A = -1$ and $B = 0$. Furthermore, an arbitrary linear function can be added to the solution obtained. Hence, u_0 is obtained as follows

$$u_0 = -x^2 + ax$$

Since u_0 decreases to zero on the sides $x = 0$ and $x = a$. If we introduce the unknown function $u_1 = u - u_0$ which satisfies the equation $\Delta u_1 = 0$; then the boundary conditions for that are (see [34])

$$\begin{aligned} u_1 &= -\left(ax - x^2\right) \text{ for } y = \pm\frac{b}{2} \\ u_1 &= 0 \qquad\quad , x = 0, x = a. \end{aligned}$$

As the operation of differentiation is ill-conditioned, to find a highly accurate approximation for the derivatives of the solution of a differential equation becomes problematic, especially when smoothness is restricted. In many studies, finding the nonsmooth solution of elliptic equations in the classical finite difference scheme are considered (see [35]-[48] and references therein). In [56] (for two dimension), [41] (for n dimension), for the solution of the finite difference problem on a square grids, the uniform error $O(h^2)$ is acquired. The minimum requirements on the smoothness of the boundary functions are used to solve Dirichlet problem for Laplace's equation in the bounded domain Ω. From these requirements it follows that the Hölder condition is satisfied by the second order derivatives of the exact solution on Ω, i.e., $u \in C^{2,\lambda}(\Omega), 0 < \lambda < 1$. In addition, taking into account results in [37] and [62] follows that $u \in C^{2,\lambda}(\bar{\Pi})$, thus the uniform error on the rectangular domain Π is $O(h^k), k = 2,4,6$, for the finite difference solution of the mixed boundary value problem (for the proof see [37], when $k = 2$, and [62], when $k = 4,6$).

A highly accurate method is one of the powerful tools to reduce the number of unknowns, which is the main problem in the numerical solution of differential equations, to get reasonable results. This becomes more valuable in 3D problems when we are looking for the derivatives of the unknown solution by the finite difference or finite element methods for a small discretization parameter h.

E.A. Volkov proved in [56] that to acquire a second-order approximation, the smoothness requirement on the boundary functions can be lowered to $C^{2,\lambda}, 0 < \lambda < 1$, when the domain is rectangular.

However, approximating the boundary value problem of Laplace's equation when the harmonic functions $u(x,y) = r^{\frac{1}{\alpha}} \cos\frac{\theta}{\alpha}, v(x,y) = r^{\frac{1}{\alpha}} \sin\frac{\theta}{\alpha}$ are the exact solution, in a domain with an interior angle of $\alpha\pi, \frac{1}{2} < \alpha \leq 2$, is problematic as these functions do not belong to $C^{2,\lambda}, 0 < \lambda < 1$. E. A. Volkov demonstrated that in the presence of angular singularities, for the numerical solution of the Dirichlet problem for Laplace's equation with the use of the 5-point scheme in square grids, the order of approximation of $O(h^{\frac{1}{\alpha}})$ is obtained on a bounded domain with an interior angle of $\alpha\pi, \frac{1}{2} < \alpha \leq 2, \alpha \neq 1$. Similarly, $O(h^{\frac{1}{2}\alpha})$ is obtained for the mixed boundary-value problem. Hence, the approximation is significantly worse than $O(h^2)$.

In [46], A.A.Dosiyev introduced a highly accurate difference-analytical method. The uniform error $O(h^6)$ is attained for the solution of the mixed boundary value problem for Laplace's equation on graduated polygons. Further the error of approximation is order $O(h^6/r_j^{p-\lambda_j})$ for p-order derivatives in a finite neighborhood of reentrant angles. The mesh step is denoted by h, the distance between current point and vertex containing the corner singularity is indicated by r_j, $\lambda_j = \frac{1}{a\alpha_j}$, and $a = 1$ or 2 depending on the type of the boundary condition. Moreover, the value of the interior angle at the investigated vertex is represented by $\alpha_j\pi$.

In [62], A.A.Dosiyev investigated the mixed boundary value problem for Laplace equation on a rectangular domain R. If the exact solution u of the problem is in $\tilde{C}^{6,\lambda}(\tilde{R})$, then the uniform error will be $O(h^6)$, where $\tilde{C}^{6,\lambda}$, is wider than $C^{6,\lambda}$.

Smoother (in $C^{8,0}$) set of solutions than $\tilde{C}^{6,\lambda}$ are obtained by many authors for $O(h^6)$ order of error estimations in the maximum norm. Hackbusch [49] acquired the same order of estimation for Dirichlet problem if $u \in C^{7,1}(\tilde{R})$. Also, Volkov [50] investigated mixed boundary value problem when $u \in \tilde{C}^{8,\lambda}(\tilde{R})$.

In [51], it was proved that the higher order difference derivatives uniformly converge to the corresponding derivatives of the solution of the Laplace equation in any strictly interior subdomain, with the same order of h as which the difference solution converges on the given domain. In [52], by using the difference solution of the Dirichlet problem for the Laplace equation on a rectangle, the uniform convergence of its first and pure second divided difference over the whole grid domain to the corresponding derivatives of the exact solution with the rate $O(h^2)$ is proved. In [54], the difference schemes on a rectangular parallelepiped were constructed, where the approximate solution of the Dirichlet problem for the Laplace equation and its first and second derivatives were obtained. Under the assumptions that the boundary functions belong to $C^{\{4,\lambda\}}, 0 < \lambda < 1$, on the faces, are continuous on the edges, and their second-order derivatives satisfy the compatibility condition, the solution to their difference schemes converge uniformly on the grid with a rate of $O\left(h^2\right)$. In [53] for the 3D Laplace equation the convergence of order $O\left(h^2\right)$ of the difference derivatives to the corresponding first order derivatives of the exact solution is proved. It was assumed that the boundary functions have third derivatives on the faces satisfying the Hölder condition. Furthermore, it is assumed that they are continuous on the edges, and their second derivatives satisfy the compatibility condition that is implied by the Laplace equation.

In this thesis, the use of the square grid has been investigated for the solution of the first and second pure derivatives of the Laplace equation on a rectangle and also on a rectangular parallelepiped and high-order accuracy of the approximate solution is justified. In the two dimensional case (rectangular domain), we consider the classical $9-point$ finite difference approximation of the problem to find the approximate solution of Laplace's equation and also of the first and second pure derivatives of Laplace equation. In the three dimensional case (in a rectangular parallelepiped), we used the $27-point$ scheme to find a similar solution to the problems two dimensional case.

In Chapter 2, we consider the Dirichlet problem for the Laplace equation on a rectangle, when the boundary values belong to $C^{6,\lambda}, 0 < \lambda < 1$, on the sides of the rectangle and as a whole are continuous on the vertices. Also the $2\tau, \tau = 1,2$, order derivatives satisfy the compatibility conditions on the vertices which result from the Laplace equation. Under these conditions, we construct the difference problems, the solutions of which converge to the first and pure second derivatives of the exact solution with the order $O(h^4)$.

In Chapter 3, we consider the Dirichlet problem for the Laplace equation in a rectangular parallelepiped. It is assumed that the boundary functions on the faces have sixth order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy some compatibility conditions on the edges. Three different schemes are constructed on a cubic grid with mesh size h, whose solutions separately approximate the solution of the Dirichlet problem for Laplace's equation with the order $O(h^6 |\ln h|)$, its first and pure second derivatives with the order $O(h^4)$.

In Chapter 4, the theoretical results in Chapter 2 and 3 are demonstrated by numerical experiments. We illustrated the higher order accurate approximation of the first and second pure derivatives of the Laplace equation on a rectangle and also in a rectangular parallelepiped.

Concluding remarks are given in Chapter 4.4.2.2.

Chapter 2

A FOURTH ORDER ACCURATE APPROXIMATION OF THE FIRST AND PURE SECOND DERIVATIVES OF THE LAPLACE EQUATION ON A RECTANGLE

2.1 The Dirichlet Problem on Rectangular Domains

Let $\Pi = \{(x,y) : 0 < x < a, 0 < y < b\}$ be a rectangle and a/b is a rational number. The sides are denoted by $\gamma_j(\gamma_j')$, $j = 1,2,3,4$, including (excluding), the ends. These sides are enumerated counterclockwise which γ_1 is the left side of Π $(\gamma_0 \equiv \gamma_4,\ \gamma_5 \equiv \gamma_1)$, hence, the boundary of Π is defined by $\gamma = \cup_{j=1}^{4} \gamma_j$. The arclength along γ is denoted by s, and s_j is the value of s at the beginning of γ_j. If f has k-th derivatives on D satisfying a Hölder condition, we say that $f \in C^{k,\lambda}(D)$, where exponent $\lambda \in (0,1)$.

We consider the following boundary value problem

$$\Delta u = 0 \text{ on } \Pi, \ \ u = \varphi_j(s) \text{ on } \gamma_j, \ j = 1,2,3,4, \tag{2.1.1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j are given functions of s. Assume that

$$\varphi_j \in C^{6,\lambda}(\gamma_j), \ 0 < \lambda < 1, \ j = 1,2,3,4, \tag{2.1.2}$$

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \ q = 0,1,2. \tag{2.1.3}$$

Lemma 2.1.1 *The solution u of problem* (2.1.1) *is from* $C^{5,\lambda}(\overline{\Pi})$,

The proof of Lemma 2.1.1 follows from Theorem 3.1 in [40].

Lemma 2.1.2 *The inequality is true*

$$\max_{0\le p\le 3}\sup_{(x,y)\in\Pi}\left|\frac{\partial^6 u}{\partial x^{2p}\partial y^{6-2p}}\right|<\infty, \tag{2.1.4}$$

where u is the solution of problem (2.1.1).

Proof. From Lemma 2.1.1 follows that the functions $\frac{\partial^4 u}{\partial x^4}$ and $\frac{\partial^4 u}{\partial y^4}$ are continuous on $\overline{\Pi}$. We put $w=\frac{\partial^4 u}{\partial x^4}$. The function w is harmonic in Π, and is the solution of the problem

$$\Delta w=0 \text{ on } \Pi,\ \ w=\Phi_j \text{ on } \gamma_j,\ j=1,2,3,4,$$

where

$$\Phi_\tau=\frac{\partial^4\varphi_\tau}{\partial y^4},\ \ \tau=1,3$$
$$\Phi_\nu=\frac{\partial^4\varphi_\nu}{\partial x^4},\ \ \nu=2,4.$$

By considering the conditions (2.1.2) and (2.1.3) follows that

$$\Phi_j\in C^{2,\lambda}(\gamma_j),\ 0<\lambda<1,\ \ \Phi_j(s_j)=\Phi_{j-1}(s_j),\ j=1,2,3,4.$$

Hence, on the basis of Theorem 6.1 in [55], we have

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^2 w}{\partial x^2}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^6 u}{\partial x^6}\right|<\infty, \tag{2.1.5}$$

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^2 w}{\partial y^2}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^6 u}{\partial x^4\partial y^2}\right|<\infty. \tag{2.1.6}$$

Similarly, it is proved that

$$\sup_{(x,y)\in\Pi}\left\{\left|\frac{\partial^6 u}{\partial y^6}\right|,\left|\frac{\partial^6 u}{\partial y^4\partial x^2}\right|\right\}<\infty. \tag{2.1.7}$$

when $w=\frac{\partial^4 u}{\partial y^4}$. The function w is harmonic in Π, and is the solution of the problem

$$\Delta w=0 \text{ on } \Pi,\ \ w=\Phi_j \text{ on } \gamma_j,\ j=1,2,3,4,$$

where

$$\Phi_\tau = \frac{\partial^4 \varphi_\tau}{\partial y^4}, \quad \tau = 1,3$$

$$\Phi_\nu = \frac{\partial^4 \varphi_\nu}{\partial x^4}, \quad \nu = 2,4.$$

by considering the conditions (2.1.2) and (2.1.3) follows that

$$\Phi_j \in C^{2,\lambda}(\gamma_j),\ 0 < \lambda < 1,\ \ \Phi_j(s_j) = \Phi_{j-1}(s_j),\ j = 1,2,3,4.$$

$$\sup_{(x,y)\in\Pi} \left| \frac{\partial^2 w}{\partial x^2} \right| = \sup_{(x,y)\in\Pi} \left| \frac{\partial^6 u}{\partial x^2 \partial y^4} \right| < \infty, \tag{2.1.8}$$

$$\sup_{(x,y)\in\Pi} \left| \frac{\partial^2 w}{\partial y^2} \right| = \sup_{(x,y)\in\Pi} \left| \frac{\partial^6 u}{\partial y^6} \right| < \infty. \tag{2.1.9}$$

From (2.1.5) – (2.1.9), estimation (2.1.4) follows. ■

Lemma 2.1.3 *Let* $\rho(x,y)$ *be the distance from a current point of the open rectangle* Π *to its boundary and let* $\partial/\partial l \equiv \alpha\partial/\partial x + \beta\partial/\partial y$, $\alpha^2+\beta^2 = 1$. *Then the next inequality holds*

$$\left| \frac{\partial^8 u}{\partial l^8} \right| \le c\rho^{-2}, \tag{2.1.10}$$

where c is a constant independent of the direction of the derivative $\partial/\partial l$, *u is a solution of problem* (2.1.1).

Proof. According to Lemma 2.1.2, we have

$$\max_{0\le p\le 3} \sup_{(x,y)\in\Pi} \left| \frac{\partial^6 u}{\partial x^{2p}\partial y^{6-2p}} \right| \le c < \infty.$$

Since any eighth order derivative can be obtained by two times differentiating some of the derivatives $\partial^6/\partial x^{2p}\partial y^{6-2p}$, $0 \le p \le 3$, on the basis of estimations (29) and (30) from [56], we obtain

$$\max_{\nu+\mu=8} \left| \frac{\partial^8 u}{\partial x^\nu \partial y^\mu} \right| \le c_1\rho^{-2}(x,y) < \infty. \tag{2.1.11}$$

From (2.1.11), inequality (2.1.10) follows. ■

Let $h > 0$, and $\min\{a/h, b/h\} \geq 6$ whereas, a/h and b/h be integers. A square net on Π is assigned by Π^h, with step h, created by the lines $x, y = 0, h, 2h, \ldots$. The set of nodes on the interior of γ_j is denoted by γ_j^h, and let

$$\gamma^h = \cup_{j=1}^4 \gamma_j^h,\ \dot{\gamma}_j = \gamma_{j-1} \cap \gamma_j,\ \overline{\gamma}^h = \cup_{j=1}^4 (\gamma_j^h \cup \dot{\gamma}_j),\ \overline{\Pi}^h = \Pi^h \cup \overline{\gamma}^h.$$

The averaging operator B be defined by following

$$\begin{aligned} Bu(x,y) &= (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/5 \\ &\quad + (u(x+h,y+h) + u(x+h,y-h) \\ &\quad + u(x-h,y+h) + u(x-h,y-h))/20. \end{aligned} \tag{2.1.12}$$

The classical 9-point finite difference approximation of problem (2.1.1) is considered as follows:

$$u_h = Bu_h \text{ on } \Pi^h,\ \ u_h = \varphi_j \text{ on } \gamma_j^h \cup \dot{\gamma}_j,\ j = 1,2,3,4. \tag{2.1.13}$$

By the maximum principle, problem (2.1.13) has a unique solution.

In what follows and for simplicity, we will denote by $c, c_1, c_2, \ldots$ constants which are independent of h and the nearest factor, identical notation will be used for various constants.

Let Π^{1h} be the set of nodes of the grid Π^h that are at a distance h from γ, and let $\Pi^{2h} = \Pi^h \backslash \Pi^{1h}$.

Proposition 2.1.4 *The equation holds*

$$Bp_7(x_0, y_0) = u(x_0, y_0) \tag{2.1.14}$$

where $p_7(x_0, y_0)$ *is the seventh order Taylor's polynomial at* (x_0, y_0) *and* u *is a harmonic* *function.*

Proof. Taking into account that the function u is harmonic, by exhaustive calculations, we have

$$Bp_7(x_0,y_0) = (p_7(x_0+h,y_0) + p_7(x_0-h,y_0) + p_7(x_0,y_0+h) + p_7(x_0,y_0-h))/5+$$

$$(p_7(x_0+h,y_0+h) + p_7(x_0+h,y_0-h) + p_7(x_0-h,y_0+h) + p_7(x_0-h,y_0-h))/20$$

$$= (u(x_0,y_0) + \frac{\partial u(x_0,y_0)}{\partial x}h + \frac{\partial^2 u(x_0,y_0)}{\partial x^2}\frac{h^2}{2!} + \frac{\partial^3 u(x_0,y_0)}{\partial x^3}\frac{h^3}{3!} + \frac{\partial^4 u(x_0,y_0)}{\partial x^4}\frac{h^4}{4!} +$$
$$\frac{\partial^5 u(x_0,y_0)}{\partial x^5}\frac{h^5}{5!} + \frac{\partial^6 u(x_0,y_0)}{\partial x^6}\frac{h^6}{6!} + \frac{\partial^7 u(x_0,y_0)}{\partial x^7}\frac{h^7}{7!} + u(x_0,y_0) - \frac{\partial u(x_0,y_0)}{\partial x}h +$$
$$\frac{\partial^2 u(x_0,y_0)}{\partial x^2}\frac{h^2}{2!} - \frac{\partial^3 u(x_0,y_0)}{\partial x^3}\frac{h^3}{3!} + \frac{\partial^4 u(x_0,y_0)}{\partial x^4}\frac{h^4}{4!} - \frac{\partial^5 u(x_0,y_0)}{\partial x^5}\frac{h^5}{5!} +$$
$$\frac{\partial^6 u(x_0,y_0)}{\partial x^6}\frac{h^6}{6!} - \frac{\partial^7 u(x_0,y_0)}{\partial x^7}\frac{h^7}{7!} + u(x_0,y_0) + \frac{\partial u(x_0,y_0)}{\partial y}h + \frac{\partial^2 u(x_0,y_0)}{\partial y^2}\frac{h^2}{2!} +$$
$$\frac{\partial^3 u(x_0,y_0)}{\partial y^3}\frac{h^3}{3!} + \frac{\partial^4 u(x_0,y_0)}{\partial y^4}\frac{h^4}{4!} + \frac{\partial^5 u(x_0,y_0)}{\partial y^5}\frac{h^5}{5!} + \frac{\partial^6 u(x_0,y_0)}{\partial y^6}\frac{h^6}{6!} +$$
$$\frac{\partial^7 u(x_0,y_0)}{\partial y^7}\frac{h^7}{7!} + u(x_0,y_0) - \frac{\partial u(x_0,y_0)}{\partial y}h + \frac{\partial^2 u(x_0,y_0)}{\partial y^2}\frac{h^2}{2!} - \frac{\partial^3 u(x_0,y_0)}{\partial y^3}\frac{h^3}{3!} +$$
$$\left.\frac{\partial^4 u(x_0,y_0)}{\partial y^4}\frac{h^4}{4!} - \frac{\partial^5 u(x_0,y_0)}{\partial y^5}\frac{h^5}{5!} + \frac{\partial^6 u(x_0,y_0)}{\partial y^6}\frac{h^6}{6!} - \frac{\partial^7 u(x_0,y_0)}{\partial y^7}\frac{h^7}{7!}\right)/5+$$

$$(u(x_0,y_0) + \left(\frac{\partial u(x_0,y_0)}{\partial x} + \frac{\partial u(x_0,y_0)}{\partial y}\right)h + \left(\frac{\partial^2 u(x_0,y_0)}{\partial x^2} + 2\frac{\partial^2 u(x_0,y_0)}{\partial x \partial y} +\right.$$

$$\left.\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\right)\frac{h^2}{2!} + \left(\frac{\partial^3 u(x_0,y_0)}{\partial x^3} + 3\frac{\partial^3 u(x_0,y_0)}{\partial x^2 \partial y} + 3\frac{\partial^3 u(x_0,y_0)}{\partial x \partial y^2} +\right.$$

$$\left.\frac{\partial^3 u(x_0,y_0)}{\partial y^3}\right)\frac{h^3}{3!} + \left(\frac{\partial^4 u(x_0,y_0)}{\partial x^4} + 4\frac{\partial^4 u(x_0,y_0)}{\partial x^3 \partial y} + 6\frac{\partial^4 u(x_0,y_0)}{\partial x^2 \partial y^2} +\right.$$

$$\left.4\frac{\partial^4 u(x_0,y_0)}{\partial x \partial y^3} + \frac{\partial^4 u(x_0,y_0)}{\partial y^4}\right)\frac{h^4}{4!} + \left(\frac{\partial^5 u(x_0,y_0)}{\partial x^5} + 5\frac{\partial^5 u(x_0,y_0)}{\partial x^4 \partial y} +\right.$$

$$\left.10\frac{\partial^5 u(x_0,y_0)}{\partial x^3 \partial y^2} + 10\frac{\partial^5 u(x_0,y_0)}{\partial x^2 \partial y^3} + 5\frac{\partial^5 u(x_0,y_0)}{\partial x \partial y^4} + \frac{\partial^5 u(x_0,y_0)}{\partial y^5}\right)\frac{h^5}{5!}+$$

$$\left(\frac{\partial^6 u(x_0,y_0)}{\partial x^6} + 6\frac{\partial^6 u(x_0,y_0)}{\partial x^5 \partial y} + 15\frac{\partial^6 u(x_0,y_0)}{\partial x^4 \partial y^2} + 20\frac{\partial^6 u(x_0,y_0)}{\partial x^3 \partial y^3} +\right.$$

$$\left.15\frac{\partial^6 u(x_0,y_0)}{\partial x^2 \partial y^4} + 6\frac{\partial^6 u(x_0,y_0)}{\partial x \partial y^5} + \frac{\partial^6 u(x_0,y_0)}{\partial y^6}\right)\frac{h^6}{6!} + \left(\frac{\partial^7 u(x_0,y_0)}{\partial x^7} +\right.$$

$$7\frac{\partial^7 u(x_0,y_0)}{\partial x^6 \partial y} + 21\frac{\partial^7 u(x_0,y_0)}{\partial x^5 \partial y^2} + 35\frac{\partial^7 u(x_0,y_0)}{\partial x^4 \partial y^3} + 35\frac{\partial^7 u(x_0,y_0)}{\partial x^3 \partial y^4} +$$

$$21\frac{\partial^7 u(x_0,y_0)}{\partial x^2\partial y^5}+7\frac{\partial^7 u(x_0,y_0)}{\partial x\partial y^6}+\frac{\partial^7 u(x_0,y_0)}{\partial y^7}\Bigg)\frac{h^7}{7!}+u(x_0,y_0)+$$

$$\Bigg(\frac{\partial u(x_0,y_0)}{\partial x}-\frac{\partial u(x_0,y_0)}{\partial y}\Bigg)h+\Bigg(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}-2\frac{\partial^2 u(x_0,y_0)}{\partial x\partial y}+$$

$$\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Bigg)\frac{h^2}{2!}+\Bigg(\frac{\partial^3 u(x_0,y_0)}{\partial x^3}-3\frac{\partial^3 u(x_0,y_0)}{\partial x^2\partial y}+3\frac{\partial^3 u(x_0,y_0)}{\partial x\partial y^2}-$$

$$\frac{\partial^3 u(x_0,y_0)}{\partial y^3}\Bigg)\frac{h^3}{3!}+\Bigg(\frac{\partial^4 u(x_0,y_0)}{\partial x^4}-4\frac{\partial^4 u(x_0,y_0)}{\partial x^3\partial y}+6\frac{\partial^4 u(x_0,y_0)}{\partial x^2\partial y^2}-$$

$$4\frac{\partial^4 u(x_0,y_0)}{\partial x\partial y^3}+\frac{\partial^4 u(x_0,y_0)}{\partial y^4}\Bigg)\frac{h^4}{4!}+\Bigg(\frac{\partial^5 u(x_0,y_0)}{\partial x^5}-5\frac{\partial^5 u(x_0,y_0)}{\partial x^4\partial y}+$$

$$10\frac{\partial^5 u(x_0,y_0)}{\partial x^3\partial y^2}-10\frac{\partial^5 u(x_0,y_0)}{\partial x^2\partial y^3}+5\frac{\partial^5 u(x_0,y_0)}{\partial x\partial y^4}-\frac{\partial^5 u(x_0,y_0)}{\partial y^5}\Bigg)\frac{h^5}{5!}+$$

$$\Bigg(\frac{\partial^6 u(x_0,y_0)}{\partial x^6}-6\frac{\partial^6 u(x_0,y_0)}{\partial x^5\partial y}+15\frac{\partial^6 u(x_0,y_0)}{\partial x^4\partial y^2}-20\frac{\partial^6 u(x_0,y_0)}{\partial x^3\partial y^3}+$$

$$15\frac{\partial^6 u(x_0,y_0)}{\partial x^2\partial y^4}-6\frac{\partial^6 u(x_0,y_0)}{\partial x\partial y^5}+\frac{\partial^6 u(x_0,y_0)}{\partial y^6}\Bigg)\frac{h^6}{6!}+\Bigg(\frac{\partial^7 u(x_0,y_0)}{\partial x^7}-$$

$$7\frac{\partial^7 u(x_0,y_0)}{\partial x^6\partial y}+21\frac{\partial^7 u(x_0,y_0)}{\partial x^5\partial y^2}-35\frac{\partial^7 u(x_0,y_0)}{\partial x^4\partial y^3}+35\frac{\partial^7 u(x_0,y_0)}{\partial x^3\partial y^4}-$$

$$21\frac{\partial^7 u(x_0,y_0)}{\partial x^2\partial y^5}+7\frac{\partial^7 u(x_0,y_0)}{\partial x\partial y^6}-\frac{\partial^7 u(x_0,y_0)}{\partial y^7}\Bigg)\frac{h^7}{7!}+u(x_0,y_0)+$$

$$\Bigg(-\frac{\partial u(x_0,y_0)}{\partial x}+\frac{\partial u(x_0,y_0)}{\partial y}\Bigg)h+\Bigg(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}-2\frac{\partial^2 u(x_0,y_0)}{\partial x\partial y}+$$

$$\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Bigg)\frac{h^2}{2!}+\Bigg(-\frac{\partial^3 u(x_0,y_0)}{\partial x^3}+3\frac{\partial^3 u(x_0,y_0)}{\partial x^2\partial y}-3\frac{\partial^3 u(x_0,y_0)}{\partial x\partial y^2}+$$

$$\frac{\partial^3 u(x_0,y_0)}{\partial y^3}\Bigg)\frac{h^3}{3!}+\Bigg(\frac{\partial^4 u(x_0,y_0)}{\partial x^4}-4\frac{\partial^4 u(x_0,y_0)}{\partial x^3\partial y}+6\frac{\partial^4 u(x_0,y_0)}{\partial x^2\partial y^2}-$$

$$10\frac{\partial^5 u(x_0,y_0)}{\partial x^3\partial y^2}+10\frac{\partial^5 u(x_0,y_0)}{\partial x^2\partial y^3}-5\frac{\partial^5 u(x_0,y_0)}{\partial x\partial y^4}+\frac{\partial^5 u(x_0,y_0)}{\partial y^5}\Bigg)\frac{h^5}{5!}+$$

$$\Bigg(\frac{\partial^6 u(x_0,y_0)}{\partial x^6}-6\frac{\partial^6 u(x_0,y_0)}{\partial x^5\partial y}+15\frac{\partial^6 u(x_0,y_0)}{\partial x^4\partial y^2}-20\frac{\partial^6 u(x_0,y_0)}{\partial x^3\partial y^3}+$$

$$15\frac{\partial^6 u(x_0,y_0)}{\partial x^2\partial y^4}-6\frac{\partial^6 u(x_0,y_0)}{\partial x\partial y^5}+\frac{\partial^6 u(x_0,y_0)}{\partial y^6}\Bigg)\frac{h^6}{6!}+\Bigg(-\frac{\partial^7 u(x_0,y_0)}{\partial x^7}+$$

$$7\frac{\partial^7 u(x_0,y_0)}{\partial x^6\partial y}-21\frac{\partial^7 u(x_0,y_0)}{\partial x^5\partial y^2}+35\frac{\partial^7 u(x_0,y_0)}{\partial x^4\partial y^3}-35\frac{\partial^7 u(x_0,y_0)}{\partial x^3\partial y^4}+$$

$$21\frac{\partial^7 u(x_0,y_0)}{\partial x^2\partial y^5}-7\frac{\partial^7 u(x_0,y_0)}{\partial x\partial y^6}+\frac{\partial^7 u(x_0,y_0)}{\partial y^7}\Bigg)\frac{h^7}{7!}+u(x_0,y_0)+$$

$$\Bigg(-\frac{\partial u(x_0,y_0)}{\partial x}-\frac{\partial u(x_0,y_0)}{\partial y}\Bigg)h+\Bigg(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+2\frac{\partial^2 u(x_0,y_0)}{\partial x\partial y}+$$

$$\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{h^2}{2!}+\Big(-\frac{\partial^3 u(x_0,y_0)}{\partial x^3}-3\frac{\partial^3 u(x_0,y_0)}{\partial x^2\partial y}-3\frac{\partial^3 u(x_0,y_0)}{\partial x\partial y^2}-$$

$$\frac{\partial^3 u(x_0,y_0)}{\partial y^3}\Big)\frac{h^3}{3!}+\Big(\frac{\partial^4 u(x_0,y_0)}{\partial x^4}+4\frac{\partial^4 u(x_0,y_0)}{\partial x^3\partial y}+6\frac{\partial^4 u(x_0,y_0)}{\partial x^2\partial y^2}+$$

$$4\frac{\partial^4 u(x_0,y_0)}{\partial x\partial y^3}+\frac{\partial^4 u(x_0,y_0)}{\partial y^4}\Big)\frac{h^4}{4!}+\Big(-\frac{\partial^5 u(x_0,y_0)}{\partial x^5}-5\frac{\partial^5 u(x_0,y_0)}{\partial x^4\partial y}-$$

$$10\frac{\partial^5 u(x_0,y_0)}{\partial x^3\partial y^2}-10\frac{\partial^5 u(x_0,y_0)}{\partial x^2\partial y^3}-5\frac{\partial^5 u(x_0,y_0)}{\partial x\partial y^4}-\frac{\partial^5 u(x_0,y_0)}{\partial y^5}\Big)\frac{h^5}{5!}+$$

$$\Big(\frac{\partial^6 u(x_0,y_0)}{\partial x^6}+6\frac{\partial^6 u(x_0,y_0)}{\partial x^5\partial y}+15\frac{\partial^6 u(x_0,y_0)}{\partial x^4\partial y^2}+20\frac{\partial^6 u(x_0,y_0)}{\partial x^3\partial y^3}+$$

$$15\frac{\partial^6 u(x_0,y_0)}{\partial x^2\partial y^4}+6\frac{\partial^6 u(x_0,y_0)}{\partial x\partial y^5}+\frac{\partial^6 u(x_0,y_0)}{\partial y^6}\Big)\frac{h^6}{6!}+\Big(-\frac{\partial^7 u(x_0,y_0)}{\partial x^7}-$$

$$7\frac{\partial^7 u(x_0,y_0)}{\partial x^6\partial y}-21\frac{\partial^7 u(x_0,y_0)}{\partial x^5\partial y^2}-35\frac{\partial^7 u(x_0,y_0)}{\partial x^4\partial y^3}-35\frac{\partial^7 u(x_0,y_0)}{\partial x^3\partial y^4}-$$

$$21\frac{\partial^7 u(x_0,y_0)}{\partial x^2\partial y^5}-7\frac{\partial^7 u(x_0,y_0)}{\partial x\partial y^6}-\frac{\partial^7 u(x_0,y_0)}{\partial y^7}\Big)\frac{h^7}{7!}\Big)/20=$$

$$u(x_0,y_0)+\Big(\frac{2}{5}\frac{\partial^2 u(x_0,y_0)}{\partial x^2}\frac{h^2}{2!}+\frac{2}{5}\frac{\partial^4 u(x_0,y_0)}{\partial x^4}\frac{h^4}{4!}+\frac{2}{5}\frac{\partial^6 u(x_0,y_0)}{\partial x^6}\frac{h^6}{6!}+$$

$$\frac{2}{5}\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\frac{h^2}{2!}+\frac{2}{5}\frac{\partial^4 u(x_0,y_0)}{\partial y^4}\frac{h^4}{4!}+\frac{2}{5}\frac{\partial^6 u(x_0,y_0)}{\partial y^6}\frac{h^6}{6!}\Big)+\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}\frac{h^2}{10}+$$

$$\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\frac{h^2}{10}+\frac{\partial^4 u(x_0,y_0)}{\partial x^4}\frac{h^4}{120}+\frac{\partial^4 u(x_0,y_0)}{\partial x^2\partial y^2}\frac{h^4}{20}+\frac{\partial^4 u(x_0,y_0)}{\partial y^4}\frac{h^4}{120}+$$

$$\frac{\partial^6 u(x_0,y_0)}{\partial x^6}\frac{h^6}{5.6!}+\frac{\partial^6 u(x_0,y_0)}{\partial x^4\partial y^2}\frac{3h^6}{6!}+\frac{\partial^6 u(x_0,y_0)}{\partial x^2\partial y^4}\frac{3h^6}{6!}+\frac{\partial^6 u(x_0,y_0)}{\partial y^6}\frac{h^6}{5.6!}\Big)=$$

$$u(x_0,y_0)+\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{3h^2}{10}+\Big(\frac{\partial^4 u(x_0,y_0)}{\partial x^4}+$$

$$\frac{\partial^4 u(x_0,y_0)}{\partial y^4}\Big)\frac{3h^4}{5!}+\Big(\frac{\partial^6 u(x_0,y_0)}{\partial x^6}+\frac{\partial^6 u(x_0,y_0)}{\partial y^6}\Big)\frac{3h^6}{5.6!}+\frac{\partial^4 u(x_0,y_0)}{\partial x^2\partial y^2}\frac{h^4}{20}+$$

$$\Big(\frac{\partial^6 u(x_0,y_0)}{\partial x^4\partial y^2}+\frac{\partial^6 u(x_0,y_0)}{\partial x^2\partial y^4}\Big)\frac{3h^6}{6!}=u(x_0,y_0)+\frac{\partial^2}{\partial x^2}\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+$$

$$\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{h^4}{40}+\frac{\partial^2}{\partial y^2}\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{h^4}{40}+\frac{\partial^4}{\partial x^4}\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+$$

$$\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{3h^6}{5.6!}+\frac{\partial^4}{\partial y^4}\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{3h^6}{5.6!}+$$

$$\frac{\partial^4}{\partial x^2\partial y^2}\Big(\frac{\partial^2 u(x_0,y_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0)}{\partial y^2}\Big)\frac{2h^6}{5.5!}=u(x_0,y_0)\Rightarrow Bp_7(x_0,y_0)=u(x_0,y_0)$$

■

Lemma 2.1.5 *The inequality holds*

$$\max_{(x,y)\in\left(\Pi^{1h}\cup\Pi^{2h}\right)} |Bu-u| \leq ch^6, \tag{2.1.15}$$

where u is a solution of problem (2.1.1).

Proof. Let (x_0,y_0) be a point of Π^{1h}, and let

$$R_0 = \{(x,y): \ |x-x_0|<h, |y-y_0|<h\}, \tag{2.1.16}$$

be an elementary square, some sides of which lie on the boundary of the rectangle Π. On the vertices of R_0, and on the mid-points of its sides lie the nodes of which the function values are used to evaluate $Bu(x_0,y_0)$. We represent a solution of problem (2.1.1) in some neighborhood of $(x_0,y_0)\in\Pi^{1h}$ by Taylor's formula

$$u(x,y) = p_7(x,y) + r_8(x,y), \tag{2.1.17}$$

where $p_7(x,y)$ is the seventh order Taylor's polynomial, $r_8(x,y)$ is the remainder term. By using Proposition (2.1.4)

$$Bp_7(x_0,y_0) = u(x_0,y_0) \tag{2.1.18}$$

Now, we estimate r_8 at the nodes of the operator B. We take a node (x_0+h, y_0+h) which is one of the eight nodes of B, and consider the function

$$\widetilde{u}(s) = u\left(x_0+\frac{s}{\sqrt{2}}, y_0+\frac{s}{\sqrt{2}}\right), \quad -\sqrt{2}h \leq s \leq \sqrt{2}h \tag{2.1.19}$$

of one variable s. By virtue of Lemma 2.1.3, we have

$$\left|\frac{d^8\widetilde{u}(s)}{ds^8}\right| \leq c(\sqrt{2}h-s)^{-2}, \ \ 0\leq s<\sqrt{2}h. \tag{2.1.20}$$

We represent function (2.1.19) around the point $s=0$ by Taylor's formula

$$\widetilde{u}(s) = \widetilde{p}_7(s) + \widetilde{r}_8(s),$$

where $\widetilde{p}_7(s) \equiv p_7\left(x_0+\frac{s}{\sqrt{2}}, y_0+\frac{s}{\sqrt{2}}\right)$ is the seventh order Taylor's polynomial of the variable s, and

$$\widetilde{r}_8(s) \equiv r_8\left(x_0+\frac{s}{\sqrt{2}}, y_0+\frac{s}{\sqrt{2}}\right), \quad 0 \le |s| < \sqrt{2}h, \tag{2.1.21}$$

is the remainder term. On the basis of (2.1.20) and the integral form of the remainder term of Taylor's formula, we have

$$\left|\widetilde{r}_8(\sqrt{2}h-\varepsilon)\right| \le c\frac{1}{7!}\int\limits_0^{\sqrt{2}h-\varepsilon}\left(\sqrt{2}h-\varepsilon-t\right)^7(\sqrt{2}h-t)^{-2}dt \le c_1h^6,\ 0<\varepsilon\le\frac{h}{\sqrt{2}}. \tag{2.1.22}$$

Taking into account the continuity of the function $\widetilde{r}_8(s)$ on $\left[-\sqrt{2}h, \sqrt{2}h\right]$, from (2.1.21) and (2.1.22), we obtain

$$|r_8(x_0+h, y_0+h)| \le c_1h^6, \tag{2.1.23}$$

where c_1 is a constant independent of the taken point (x_0, y_0) on Π^{1h}. Estimation (2.1.23) is obtained analogously for the remaining seven nodes of operator B. Since the norm of the operator is equal to one in uniform metric, by using (2.1.23), we have

$$|Br_8(x_0, y_0)| \le c_2h^6. \tag{2.1.24}$$

Hence, on the basis of (2.1.17), (2.1.18), (2.1.20) and linearity of the operator B, we obtain

$$|Bu(x_0, y_0) - u(x_0, y_0)| \le ch^6,$$

for any $(x_0, y_0) \in \Pi^{1h}$. Now, let (x_0, y_0) be a point of Π^{2h}, and let in the Taylor formula (2.1.17) corresponding to this point, the remainder term $r_8(x, y)$ be represented in the Lagrange form.

Moreover,

$$M^{(8)} = \frac{\partial^8 u(x^*, y^*)}{\partial l^8}, \partial/\partial l \equiv \alpha\partial/\partial x + \beta\partial/\partial y, \alpha^2+\beta^2 = 1$$

hence,

$$|r_8(x,y)| = c_3 \left|M^{(8)}\right| h^8. \tag{2.1.25}$$

where c_3 is a constant independent of the point $(x_0,y_0) \in \Pi^{2h}$.Then

$$\begin{aligned} Br_8(x_0,y_0) &= (r_8(x_0+h,y_0) + r_8(x_0-h,y_0) + r_8(x_0,y_0+h) + \\ &\quad r_8(x_0,y_0-h))/5 + (r_8(x_0+h,y_0+h) + r_8(x_0+h,y_0-h) \\ &\quad + r_8(x_0-h,y_0+h) + r_8(x_0-h,y_0-h))/20, \end{aligned} \tag{2.1.26}$$

contains eighth order derivatives of the solution of problem (2.1.1) at some points of the open square R_0 defined by (2.1.16), when $(x_0,y_0) \in \Pi^{2h}$. The square R_0 lies at a distance from the boundary γ of the rectangle Π not less than h. Therefore, by using (2.1.25) and (2.1.26), we obtain

$$|Br_8(x_0,y_0)| = c_4 \left|M^{(8)}\right| h^8,$$

on the basis of Lemma 2.1.3, we obtain

$$|Br_8(x_0,y_0)| \le c_4 \rho^{-2} h^8 \le c_4 \frac{h^8}{(2h)^2} = c_4 \frac{h^6}{4}, \tag{2.1.27}$$

where c_4 is a constant independent of the point $(x_0,y_0) \in \Pi^{2h}$. Again, on the basis of (2.1.17), (2.1.18) and (2.1.27) follows estimation (2.1.15) at any point $(x_0,y_0) \in \Pi^{2h}$. Lemma 2.1.5 is proved. ■

We represent two more Lemmas. Consider the following systems

$$q_h = Bq_h + g_h \text{ on } \Pi^h, \ q_h = 0 \text{ on } \gamma^h, \tag{2.1.28}$$

$$\overline{q}_h = B\overline{q}_h + \overline{g}_h \text{ on } \Pi^h, \ \overline{q}_h \ge 0 \text{ on } \gamma^h, \tag{2.1.29}$$

where g_h and $\overline{g}_h$ are given functions, and $|g_h| \le \overline{g}_h$ on Π^h.

Lemma 2.1.6 *The solutions q_h and $\overline{q}_h$of systems* (2.1.28) *and* (2.1.29) *satisfy the inequality*

$$|q_h| \le \overline{q}_h \ \ on\ \overline{\Pi}^h.$$

The proof of Lemma 2.1.6 follows from the comparison theorem (see Chapter 4 in [59]).

Lemma 2.1.7 *For the solution of the problem*

$$q_h = Bq_h + h^6 \ \ on\ \Pi^h, \ \ q_h = 0 \ \ on\ \gamma^h, \tag{2.1.30}$$

the inequality holds

$$q_h \le \frac{5}{3}\rho d h^4 \ \ on\ \overline{\Pi}^h,$$

where $d = \max\{a,b\}$, $\rho = \rho(x,y)$ *is the distance from the current point* $(x,y) \in \overline{\Pi}^h$ *to the boundary of the rectangle* Π.

Proof. We consider the functions

$$\overline{q}_h^{(1)}(x,y) = \frac{5}{3}h^4(ax - x^2) \ge 0,\ \overline{q}_h^{(2)}(x,y) = \frac{5}{3}h^4(by - y^2) \ge 0 \text{ on } \overline{\Pi},$$

Let $\overline{q}_h(x,y) = \overline{q}_h^{(1)}(x,y)$, then

$$B\overline{q}_h(x_0,y_0) = (\overline{q}_h(x_0+h,y_0) + \overline{q}_h(x_0-h,y_0) + \overline{q}_h(x_0,y_0+h) + \overline{q}_h(x_0,y_0-h))/5+$$
$$(\overline{q}_h(x_0+h,y_0+h) + \overline{q}_h(x_0+h,y_0-h) + \overline{q}_h(x_0-h,y_0+h) + \overline{q}_h(x_0-h,y_0-h))/20$$
$$= \left(\frac{5}{3}h^4(a(x_0+h) - (x_0+h)^2) + \frac{5}{3}h^4(a(x_0-h) - (x_0-h)^2 + \frac{5}{3}h^4(ax_0 - x_0^2) +\right.$$
$$\left.\frac{5}{3}h^4(ax_0 - x_0^2)\right)/5 + \left(\frac{5}{3}h^4(a(x_0+h) - (x_0+h)^2) + \frac{5}{3}h^4(a(x_0+h) - (x_0+h)^2 +\right.$$
$$\left.\frac{5}{3}h^4(a(x_0-h) - (x_0-h)^2 + \frac{5}{3}h^4(a(x_0-h) - (x_0-h)^2\right)/20 =$$

$$\frac{h^4}{3}\left(ax_0+ah-x_0^2-2x_0h-h^2+ax_0-ah-x_0^2+2x_0h-h^2+ax_0-x_0^2+ax_0-x_0^2\right)+$$
$$\frac{h^4}{12}\left(ax_0+ah-x_0^2-2x_0h-h^2+ax_0+ah-x_0^2-2x_0h-h^2+ax_0-ah-x_0^2+2x_0h-\right.$$
$$\left.h^2+ax_0-ah-x_0^2+2x_0h-h^2\right)=\frac{h^4}{3}\left(4ax_0-4x_0^2-2h^2\right)+\frac{h^4}{12}\left(4ax_0-4x_0^2-4h^2\right)$$
$$=h^4\frac{\left(20ax_0-20x_0^2-12h^2\right)}{12}=\frac{5}{3}h^4(ax_0-x_0^2)-h^6=\overline{q}_h\,(x_0,y_0)-h^6,$$

Similarly let $\overline{q}_h\,(x,y)=\overline{q}_h^{(2)}(x,y)$ then

$$B\overline{q}_h\,(x_0,y_0)=(\overline{q}_h(x_0+h,y_0)+\overline{q}_h(x_0-h,y_0)+\overline{q}_h(x_0,y_0+h)+\overline{q}_h(x_0,y_0-h))\,/5+$$

$$(\overline{q}_h(x_0+h,y_0+h)+\overline{q}_h(x_0+h,y_0-h)+\overline{q}_h(x_0-h,y_0+h)+\overline{q}_h(x_0-h,y_0-h))\,/20$$
$$=\Bigg(\frac{5}{3}h^4(b\,(x_0+h)-(x_0+h)^2)+\frac{5}{3}h^4(b\,(x_0-h)-(x_0-h)^2+\frac{5}{3}h^4(bx_0-x_0^2)+$$
$$\frac{5}{3}h^4(bx_0-x_0^2)\Bigg)\,/5+\Bigg(\frac{5}{3}h^4(b\,(x_0+h)-(x_0+h)^2)+\frac{5}{3}h^4(b\,(x_0+h)-(x_0+h)^2+$$
$$\frac{5}{3}h^4(b\,(x_0-h)-(x_0-h)^2+\frac{5}{3}h^4(b\,(x_0-h)-(x_0-h)^2\Bigg)\,/20=$$
$$\frac{h^4}{3}\left(bx_0+bh-x_0^2-2x_0h-h^2+bx_0-bh-x_0^2+2x_0h-h^2+bx_0-x_0^2+bx_0-x_0^2\right)+$$
$$\frac{h^4}{12}\left(bx_0+bh-x_0^2-2x_0h-h^2+bx_0+bh-x_0^2-2x_0h-h^2+bx_0-bh-x_0^2+2x_0h-\right.$$
$$\left.h^2+bx_0-bh-x_0^2+2x_0h-h^2\right)=\frac{h^4}{3}\left(4bx_0-4x_0^2-2h^2\right)+\frac{h^4}{12}\left(4bx_0-4x_0^2-4h^2\right)=$$
$$h^4\frac{\left(20bx_0-20x_0^2-12h^2\right)}{12}=\frac{5}{3}h^4(bx_0-x_0^2)-h^6=\overline{q}_h\,(x_0,y_0)-h^6,$$

which are solutions of the equation $\overline{q}_h=B\overline{q}_h+h^6$ on Π^h. By virtue of Lemma 2.1.6, we obtain

$$q_h\le\min_{i=1,2}\overline{q}_h^{(i)}(x,y)\le\frac{5}{3}\rho dh^4\ \text{ on }\overline{\Pi}^h.$$

■

Theorem 2.1.8 *Assume that the boundary functions* $\varphi_j,\ j=1,2,3,4$ *satisfy conditions (2.1.2) and (2.1.3). Then*

$$\max_{\overline{\Pi}^h}|u_h-u|\le c\rho h^4, \tag{2.1.31}$$

where u is the exact solution of problem (2.1.1)), and u_h is the solution of the finite difference problem (2.1.13.

Proof. Let

$$\varepsilon_h = u_h - u \ \text{ on } \overline{\Pi}^h. \tag{2.1.32}$$

Then

$$B\varepsilon_h = Bu_h - Bu \Rightarrow Bu_h = B\varepsilon_h + Bu$$

Moreover,

$$u_h = \varepsilon_h + u$$

By considering problem (2.1.13) it is obvious that

$$\varepsilon_h = B\varepsilon_h + (Bu - u) \ \text{ on } \Pi^h, \ \ \varepsilon_h = 0 \ \text{ on } \gamma^h. \tag{2.1.33}$$

By virtue of estimation (2.1.15) for $(Bu - u)$, and by applying Lemma 2.1.6 to the problems (2.1.30) and (2.1.33), on the basis of Lemma 2.1.7 we obtain

$$\max_{\overline{\Pi}^h} |\varepsilon_h| \leq c\rho h^4. \tag{2.1.34}$$

From (2.1.32) and (2.1.34) follows the proof of Theorem 2.1.8. ■

2.2 Approximation of the First Derivative

We denote by $\Psi_j = \frac{\partial u}{\partial x}$ on γ_j, $j = 1, 2, 3, 4$, and consider the boundary value problem:

$$\Delta v = 0 \ \text{ on } \Pi, \ \ v = \Psi_j \ \text{ on } \gamma_j, \ j = 1, 2, 3, 4, \tag{2.2.1}$$

where u is a solution of the boundary value problem (2.1.1).

We put

$$\Psi_{1h}(u_h) = \frac{1}{12h}(-25\varphi_1(y) + 48u_h(h,y) - 36u_h(2h,y) + 16u_h(3h,y) - 3u_h(4h,y)) \quad \text{on } \gamma_1^h, \tag{2.2.2}$$

$$\Psi_{3h}(u_h) = \frac{1}{12h}(25\varphi_3(y) - 48u_h(a-h,y) + 36u_h(a-2h,y) - 16u_h(a-3h,y) + 3u_h(a-4h,y)) \quad \text{on } \gamma_3^h, \tag{2.2.3}$$

$$\Psi_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x} \quad \text{on } \gamma_p^h,\ p = 2,4, \tag{2.2.4}$$

where u_h is the solution of the finite difference boundary value problem (2.1.13).

Lemma 2.2.1 *The inequality is true*

$$|\Psi_{kh}(u_h) - \Psi_{kh}(u)| \leq c_5 h^4,\ \ k = 1,3, \tag{2.2.5}$$

where u_h is the solution of problem (2.1.13)*, u is the solution of problem* (2.1.1).

Proof. On the basis of (2.2.2), (2.2.3) and Theorem 2.1.8, Then if $k = 1$,

$$|\Psi_{1h}(u_h) - \Psi_{1h}(u)| = \left| \frac{1}{12h}((-25\varphi_1(y) + 48u_h(h,y) - 36u_h(2h,y) + 16u_h(3h,y) - 3u_h(4h,y)) - (-25\varphi_1(y) + 48u(h,y) - 36u(2h,y) + 16u(3h,y) - 3u(4h,y)) \right| \leq$$

$$\frac{1}{12h}(48\,|u_h(h,y) - u(h,y)| - 36\,|u_h(2h,y) - u(2h,y)| + 16\,|u_h(3h,y) - u(3h,y)| - 3\,|u_h(4h,y) - u(4h,y)|) \leq \frac{1}{12h}\left(48\,(ch)\,h^4 + 36\,(c2h)\,h^4 + 16\,(c3h)\,h^4 + 3\,(c4h)\,h^4\right) \leq c_5 h^4$$

Similarly if $k = 3$,

$$|\Psi_{3h}(u_h) - \Psi_{3h}(u)| = \left| \frac{1}{12h}((25\varphi_1(y) \ - 48u_h(a-h,y) + 36u_h(a-2h,y) \right.$$

$$- 16u_h(a-3h,y) + 3u_h(a-4h,y)) + (25\varphi_1(y) - 48u(a-h,y) + 36u(a-2h,y)$$

$$- 16u(a-3h,y) + \ 3u(4h,y))\,| \le \frac{1}{12h}(48\,|u_h(a-h,y) - u(a-h,y)|$$

$$- 36\,|u_h(a-2h,y) - u(a-2h,y)| + 16\,|u_h(a-3h,y) - u(a-3h,y)|$$

$$- 3\,|u_h(a-4h,y) - u(a-4h,y)|) \le \frac{1}{12h}\left(48\,(ch)\,h^4 + 36\,(c2h)\,h^4 + 16\,(c3h)\,h^4 + \right.$$

$$\left. 3\,(c4h)\,h^4\right) \le c_5 h^4$$

hence

$$|\Psi_{kh}(u_h) - \Psi_{kh}(u)| \ \le \ \frac{1}{12h}\left(48\,(ch)\,h^4 + 36\,(c2h)\,h^4 + 16\,(c3h)\,h^4 + 3\,(c4h)\,h^4\right) \le c_5 h^4, \ \ k = 1,3.$$

■

Lemma 2.2.2 *The inequality holds*

$$\max_{(x,y)\in\gamma_k^h} |\Psi_{kh}(u_h) - \Psi_k| \le c_6 h^4, \ \ k = 1,3. \tag{2.2.6}$$

Proof. From Lemma 2.1.1 follows that $u \in C^{5,0}(\overline{\Pi})$. Then, at the end points $(0,\nu h) \in \gamma_1^h$ and $(a,\nu h) \in \gamma_3^h$ of each line segment $\{(x,y) : 0 \le x \le a, 0 < y = \nu h < b\}$ expressions (2.2.2) and (2.2.3) give the fourth order approximation of $\frac{\partial u}{\partial x}$, respectively. From the truncation error formulae (see [61]) follows that

$$\frac{\max_{(x,y)\in\gamma_k^h}\left|\frac{\partial^5 u}{\partial x^5}\right|}{(n+1)!}\prod_{\substack{k=0\\k\neq j}}^{n}(x_j - x_k)$$

hence,

$$\max_{(x,y)\in\gamma_k^h} |\Psi_{kh}(u) - \Psi_k| \le \frac{\max_{(x,y)\in\overline{\Pi}}\left|\frac{\partial^5 u}{\partial x^5}\right|}{5!} h(2h)(3h)(4h) = \frac{h^4}{5}\max_{(x,y)\in\overline{\Pi}}\left|\frac{\partial^5 u}{\partial x^5}\right| \le c_7 h^4, \ \ k = 1,3. \tag{2.2.7}$$

On the basis of Lemma 2.2.1 and estimation (2.2.7) follows (2.2.6),

$$\max_{(x,y)\in\gamma_k^h} |\Psi_{kh}(u_h) - \Psi_k| = \max_{(x,y)\in\gamma_k^h} |\Psi_{kh}(u_h) - \Psi_{kh}(u) + \Psi_{kh}(u) - \Psi_k| \leq$$

$$\max_{(x,y)\in\gamma_k^h} |\Psi_{kh}(u_h) - \Psi_{kh}(u)| + \max_{(x,y)\in\gamma_k^h} |\Psi_{kh}(u) - \Psi_k| \leq c_6 h^4, \ \ k = 1,3.$$

■

We consider the finite difference boundary value problem

$$v_h = Bv_h \ \text{ on } \Pi^h, \ \ v_h = \Psi_{jh} \ \text{ on } \gamma_j^h, \ j = 1,2,3,4, \tag{2.2.8}$$

where Ψ_{jh}, $j = 1,2,3,4$, are defined by (2.2.2) -(2.2.4)

Theorem 2.2.3 *The estimation is true*

$$\max_{(x,y)\in\overline{\Pi}^h} \left| v_h - \frac{\partial u}{\partial x} \right| \leq ch^4, \tag{2.2.9}$$

where u is the solution of problem (2.1.1)*, v_h is the solution of the finite difference problem* (2.2.8).

Proof. Let

$$\varepsilon_h = v_h - v \ \text{ on } \overline{\Pi}^h, \tag{2.2.10}$$

where $v = \frac{\partial u}{\partial x}$. From (2.2.8) and (2.2.10), we have

$$\varepsilon_h = B\varepsilon_h + (Bv - v) \text{ on } \Pi^h, \ \varepsilon_h = \Psi_{kh}(u_h) - v \text{ on } \gamma_k^h, \ k = 1,3, \ \ \varepsilon_h = 0 \text{ on } \gamma_p^h, \ p = 2,4. \tag{2.2.11}$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{2.2.12}$$

where

$$\varepsilon_h^1 = B\varepsilon_h^1 \text{ on } \Pi^h, \tag{2.2.13}$$

$$\varepsilon_h^1 = \Psi_{kh}(u_h) - v \text{ on } \gamma_k^h,\ k = 1,3,\ \varepsilon_h^1 = 0 \text{ on } \gamma_p^h,\ p = 2,4; \tag{2.2.14}$$

$$\varepsilon_h^2 = B\varepsilon_h^2 + (Bv - v) \text{ on } \Pi^h,\ \varepsilon_h^2 = 0 \text{ on } \gamma_j^h,\ j = 1,2,3,4. \tag{2.2.15}$$

By Lemma 2.2.2 and by maximum principle, for the solution of system (2.2.13), (2.2.14), we have

$$\max_{(x,y)\in\overline{\Pi}^h} \left|\varepsilon_h^1\right| \le \max_{q=1,3} \max_{(x,y)\in\gamma_q^h} \left|\Psi_{qh}(u_h) - v\right| \le c_6 h^4. \tag{2.2.16}$$

The solution ε_h^2 of system (2.2.15) is the error of the approximate solution obtained by the finite difference method for problem (2.2.1), when the boundary values satisfy the conditions

$$\Psi_j \in C^{4,\lambda}(\gamma_j),\ 0 < \lambda < 1,\ j = 1,2,3,4, \tag{2.2.17}$$

$$\Psi_j^{(2q)}(s_j) = (-1)^q \Psi_{j-1}^{(2q)}(s_j),\ q = 0,1. \tag{2.2.18}$$

Since the function $v = \frac{\partial u}{\partial x}$ is harmonic on Π with the boundary functions $\Psi_j,\ j = 1,2,3,4$, on the basis of (2.2.17), (2.2.18), and Remark 15 in [62], we have

$$\max_{(x,y)\in\overline{\Pi}^h} \left|\varepsilon_h^2\right| \le c_8 h^4. \tag{2.2.19}$$

If the solution of problem (2.1.1) $u \in \widetilde{C}^{4,\lambda}(\overline{\Pi})$, $0 < \lambda < 1$, then

$$\max_{(x,y)\in\overline{\Pi}^h} |u - u_h| \le ch^4$$

where u_h is the solution of the finite difference problem (2.1.13).) By (2.2.12), (2.2.16) and (2.2.19) inequality (2.2.9) follows. ■

2.3 Approximation of the Pure Second Derivatives

We denote by $\omega = \frac{\partial^2 u}{\partial x^2}$. The function ω is harmonic on Π, on the basis of Lemma 2.1.1 is continuous on $\overline{\Pi}$, and is a solution of the following Dirichlet problem

$$\Delta\omega = 0 \ \text{ on } \Pi, \ \ \omega = \digamma_j \ \text{ on } \gamma_j, \ j = 1,2,3,4, \tag{2.3.1}$$

where

$$\digamma_\tau = \frac{\partial^2 \varphi_\tau}{\partial x^2}, \ \ \tau = 2,4, \tag{2.3.2}$$

$$\digamma_\nu = -\frac{\partial^2 \varphi_\nu}{\partial y^2}, \ \ \nu = 1,3. \tag{2.3.3}$$

From the continuity of the function ω on $\overline{\Pi}$, and from (2.1.2), (2.1.3) and (2.3.2), (2.3.3) it follows that

$$\digamma_j \in C^{4,\lambda}(\gamma_j), \ 0 < \lambda < 1, \ j = 1,2,3,4, \tag{2.3.4}$$

$$\digamma_j^{(2q)}(s_j) = (-1)^q \digamma_{j-1}^{(2q)}(s_j), \ q = 0,1, \ j = 1,2,3,4. \tag{2.3.5}$$

Let ω_h be a solution of the finite difference problem

$$\omega_h = B\omega_h \ \text{ on } \Pi^h, \ \ \omega_h = \digamma_j \ \text{ on } \gamma_j^h \cup \dot{\gamma}_j, \ j = 1,2,3,4, \tag{2.3.6}$$

where $\digamma_j$, $j = 1,2,3,4$, are the functions determined by (2.3.2) and (2.3.3).

Theorem 2.3.1 *The estimation holds*

$$\max_{\overline{\Pi}^h} |\omega_h - \omega| \le ch^4, \tag{2.3.7}$$

where $\omega = \frac{\partial^2 u}{\partial x^2}$, *u is the solution of problem* (2.1.1) *and* ω_h *is the solution of the finite difference problem* (2.3.6).

Proof. On the basis of conditions (2.3.4) and (2.3.5), the exact solution of problem (2.3.1) belongs to the class of functions $\widetilde{C}^{4,\lambda}(\overline{\Pi})$ (see [62]). Hnece, inequality (2.3.7) follows from the results in ([62]) (Remark 15), as the case of the Dirichlet problem. ■

Chapter 3

ON A HIGHLY ACCURATE APPROXIMATION OF THE FIRST AND PURE SECOND DERIVATIVES OF THE LAPLACE EQUATION IN A RECTANGULAR PARALLELEPIPED

3.1 The Dirichlet Problem on a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3) : 0 < x_i < a_i, i = 1,2,3\}$ be an open rectangular parallelepiped; $\Gamma_j (j = 1,2,\ldots,6)$ be its faces including the edges; Γ_j for $j = 1,2,3$ (for $j = 4,5,6$) belongs to the plane $x_j = 0$ (to the plane $x_{j-3} = a_{j-3}$), and let $\Gamma = \cup_{j=1}^{6}\Gamma_j$ be the boundary of R; $\gamma_{\mu\nu} = \Gamma_\mu \cap \Gamma_\nu$ be the edges of the parallelepiped R. If f has k-th derivatives on D satisfying a Hölder condition, we say that $f \in C^{k,\lambda}(D)$, where exponent $\lambda \in (0,1)$.

We consider the following boundary value problem

$$\Delta u = 0 \text{ on } R, \; u = \varphi_j \text{ on } \Gamma_j, \; j = 1,2,\ldots,6, \tag{3.1.1}$$

where $\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, φ_j are given functions. Assume that

$$\varphi_j \in C^{6,\lambda}(\Gamma_j), \; 0 < \lambda < 1, \; j = 1,2,\ldots,6, \tag{3.1.2}$$

$$\varphi_\mu = \varphi_\nu \text{ on } \gamma_{\mu\nu}, \tag{3.1.3}$$

$$\frac{\partial^2\varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2\varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2\varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{3.1.4}$$

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4\varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4\varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4\varphi_\nu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}. \tag{3.1.5}$$

where $1 \le \mu < \nu \le 6$, $\nu - \mu \neq 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, and t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

Lemma 3.1.1 *The solution u of the problem* (3.1.1) *is from* $C^{5,\lambda}(\overline{R})$,

The proof of Lemma 3.1.1 follows from Theorem 2.1 in [55].

Lemma 3.1.2 *The inequality is true*

$$\max_{0\le p\le 3}\max_{0\le q\le 3-p}\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_1^{2p}\partial x_2^{2q}\partial x_3^{6-2p-2q}}\right|\le c<\infty, \qquad (3.1.6)$$

where u is the solution of the problem (3.1.1).

Proof. From Lemma 3.1.1 it follows that the functions $\frac{\partial^4 u}{\partial x_1^4}$, $\frac{\partial^4 u}{\partial x_2^4}$ and $\frac{\partial^4 u}{\partial x_3^4}$ are continuous on $\overline{R}$. We put $w=\frac{\partial^4 u}{\partial x_1^4}$. The function w is harmonic in R, and is the solution of the problem

$$\Delta w=0 \text{ on } R,\ \ w=\Psi_j \text{ on } \Gamma_j,\ j=1,2,\dots,6,$$

where

$$\Psi_\tau=\frac{\partial^4\varphi_\tau}{\partial x_2^4}+\frac{\partial^4\varphi_\tau}{\partial x_3^4}+2\frac{\partial^4\varphi_\tau}{\partial x_2^2\partial x_3^2},\ \ \tau=1,4$$
$$\Psi_\nu=\frac{\partial^4\varphi_\nu}{\partial x_1^4},\ \ \nu=2,3,5,6.$$

where Ψ_τ when $\tau=1,4$ is calculated by following,

$$\Psi_\tau=\frac{\partial^4\varphi_\nu}{\partial x_1^4}=\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2\varphi_\tau}{\partial x_1^2}\right)=\frac{\partial^2}{\partial x_1^2}\left(-\frac{\partial^2\varphi_\tau}{\partial x_2^2}-\frac{\partial^2\varphi_\tau}{\partial x_3^2}\right)=-\frac{\partial^4\varphi_\tau}{\partial x_1^2\partial x_2^2}-\frac{\partial^4\varphi_\tau}{\partial x_1^2\partial x_3^2}=$$
$$-\frac{\partial^2}{\partial x_2^2}\left(-\frac{\partial^2\varphi_\tau}{\partial x_2^2}-\frac{\partial^2\varphi_\tau}{\partial x_3^2}\right)-\frac{\partial^2}{\partial x_3^2}\left(-\frac{\partial^2\varphi_\tau}{\partial x_2^2}-\frac{\partial^2\varphi_\tau}{\partial x_3^2}\right)=\frac{\partial^4\varphi_\tau}{\partial x_2^4}+\frac{\partial^4\varphi_\tau}{\partial x_3^4}+2\frac{\partial^4\varphi_\tau}{\partial x_2^2\partial x_3^2}$$

From conditions (3.1.2)-(3.1.5) it follows that

$$\Psi_j \in C^{2,\lambda}(\Gamma_j),\ 0<\lambda<1,\ j=1,2,...,6$$

$$\Psi_\mu = \Psi_\nu,\ \text{on } \gamma_{\mu\nu},\ 1\le\mu<\nu\le 6,\ \nu-\mu\ne 3.$$

Hence, on the basis of Theorem 4.1 in [55], we have

$$\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_1^6}\right| = \sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^2 w}{\partial x_1^2}\right| < \infty, \tag{3.1.7}$$

$$\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_1^4 \partial x_2^2}\right| = \sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^2 w}{\partial x_2^2}\right| < \infty, \tag{3.1.8}$$

$$\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_1^4 \partial x_3^2}\right| = \sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^2 w}{\partial x_3^2}\right| < \infty, \tag{3.1.9}$$

Similarly, when $w = \frac{\partial^4 u}{\partial x_2^4}$. The function w is harmonic in R, and is the solution of the problem

$$\Delta w = 0 \text{ on } R, \; w = \Psi_j \text{ on } \Gamma_j, \; j = 1,2,\ldots,6,$$

where

$$\Psi_\tau = \frac{\partial^4 \varphi_\tau}{\partial x_1^4} + \frac{\partial^4 \varphi_\tau}{\partial x_3^4} + 2\frac{\partial^4 \varphi_\tau}{\partial x_1^2 \partial x_3^2}, \;\; \tau = 2,5$$

$$\Psi_\nu = \frac{\partial^4 \varphi_\nu}{\partial x_2^4}, \;\; \nu = 2,3,5,6.$$

From conditions (3.1.2)-(3.1.5) it follows that

$$\Psi_j \in C^{2,\lambda}(\Gamma_j), \; 0 < \lambda < 1, \; j = 1,2,...,6$$

$$\Psi_\mu = \Psi_\nu, \text{ on } \gamma_{\mu\nu}, \; 1 \le \mu < \nu \le 6, \; \nu - \mu \ne 3.$$

Hence, on the basis of Theorem 4.1 in [55], we have

$$\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_2^4 \partial x_1^2}\right| = \sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^2 w}{\partial x_1^2}\right| < \infty, \tag{3.1.10}$$

$$\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_2^6}\right| = \sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^2 w}{\partial x_2^2}\right| < \infty, \tag{3.1.11}$$

$$\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^6 u}{\partial x_2^4 \partial x_3^2}\right| = \sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^2 w}{\partial x_3^2}\right| < \infty, \tag{3.1.12}$$

and when $w = \frac{\partial^4 u}{\partial x_2^4}$. The function w is harmonic in R, and is the solution of the problem

$$\Delta w = 0 \text{ on } R, \; w = \Psi_j \text{ on } \Gamma_j, \; j = 1,2,\ldots,6,$$

where

$$\Psi_\tau = \frac{\partial^4 \varphi_\tau}{\partial x_1^4} + \frac{\partial^4 \varphi_\tau}{\partial x_2^4} + 2\frac{\partial^4 \varphi_\tau}{\partial x_1^2 \partial x_2^2}, \quad \tau = 3,6$$

$$\Psi_\nu = \frac{\partial^4 \varphi_\nu}{\partial x_3^4}, \quad \nu = 2,3,5,6.$$

From conditions (3.1.2)-(3.1.5) it follows that

$$\Psi_j \in C^{2,\lambda}(\Gamma_j),\ 0<\lambda<1,\ j=1,2,...,6$$

$$\Psi_\mu = \Psi_\nu, \text{ on } \gamma_{\mu\nu},\ 1 \le \mu < \nu \le 6,\ \nu - \mu \ne 3.$$

Hence, on the basis of Theorem 4.1 in [55], we have

$$\sup_{(x_1,x_2,x_3)\in R} \left| \frac{\partial^6 u}{\partial x_3^4 \partial x_1^2} \right| = \sup_{(x_1,x_2,x_3)\in R} \left| \frac{\partial^2 w}{\partial x_3^2} \right| < \infty, \tag{3.1.13}$$

$$\sup_{(x_1,x_2,x_3)\in R} \left| \frac{\partial^6 u}{\partial x_3^4 \partial x_2^2} \right| = \sup_{(x_1,x_2,x_3)\in R} \left| \frac{\partial^2 w}{\partial x_2^2} \right| < \infty, \tag{3.1.14}$$

$$\sup_{(x_1,x_2,x_3)\in R} \left| \frac{\partial^6 u}{\partial x_3^6} \right| = \sup_{(x_1,x_2,x_3)\in R} \left| \frac{\partial^2 w}{\partial x_3^2} \right| < \infty, \tag{3.1.15}$$

From (3.1.7) – (3.1.15), estimation (3.1.6) follows. ■

Lemma 3.1.3 *Let $\rho(x_1,x_2,x_3)$ be the distance from the current point of the open parallelepiped R to its boundary and let $\partial/\partial l \equiv \alpha_1 \partial/\partial x_1 + \alpha_2 \partial/\partial x_2 + \alpha_3 \partial/\partial x_3$, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Then the next inequality holds*

$$\left| \frac{\partial^8 u(x_1,x_2,x_3)}{\partial l^8} \right| \le c\rho^{-2}(x_1,x_2,x_3), \quad (x_1,x_2,x_3) \in R \tag{3.1.16}$$

where c is a constant independent of the direction of differentiation $\partial/\partial l$, u is a solution of the problem (3.1.1).

Proof. Since the sixth order derivatives of the solution u of the form $\partial^6/\partial x_1^{2p} \partial x_2^{2q} \partial x_3^{6-2p-2q}$, $p+q+s=3$ are harmonic and by Lemma 3.1.2 are bounded in R, any eighth order derivative can be obtained by twice differentiating some of these derivatives. The Lemma 3 from [57] (Chap. 4, Sec. 3) is illustrated in following,

Let u is a bounded and harmonic function in R, ($|u| \leq M$). Then any derivative $D^{\beta}u$ of the oder $|\beta| = k$, $k = 1, 2, \ldots$, at the point $x \in R$ satisfies the following inequality,

$$|D^{\alpha}u| \leq M\left(\frac{n}{\rho}\right)^{k} k^{k}$$

where ρ is the distance from the current point to the boundary of R,hence, we have

$$\max_{0\leq\mu\leq 8}\max_{0\leq\nu\leq 8-\mu}\left|\frac{\partial^{8}u(x_1,x_2,x_3)}{\partial x_1^{\mu}\partial x_2^{\upsilon}\partial x_3^{8-\mu-\upsilon}}\right| \leq c_1\rho^{-2}(x_1,x_2,x_3), \quad (x_1,x_2,x_3) \in R. \quad (3.1.17)$$

From inequality (3.1.17), inequality (3.1.16) follows. ■

Let $h > 0$, and $a_i/h \geq 6, i = 1, 2, 3$, integers. We assign R^h, a cubic grid on R, with step h, obtained by the planes $x_i = 0, h, 2h, \ldots$, $i = 1, 2, 3$. Let D_h be a set of nodes of this grid, $R_h = R \cap D_h$ (see Fig. (3.1)); $\Gamma_{jh} = \Gamma_j \cap D_h$, and $\Gamma_h = \Gamma_{1h} \cup \Gamma_{2h} \cup \ldots \cup \Gamma_{6h}$.

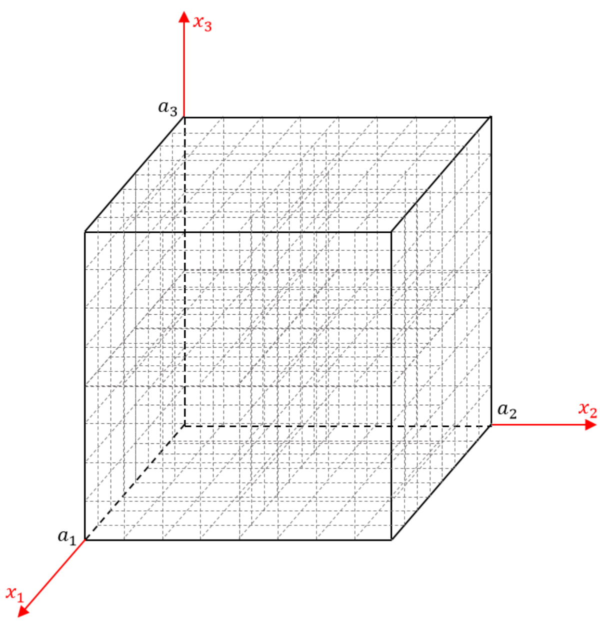

Figure 3.1. $R_h = R \cap D_h$

Let the Averaging operator $\Re$ with twenty seven points be defined as follows (see [58])

$$\Re u(x_1,x_2,x_3) = \frac{1}{128}\left(14\sum_{p=1}^{6}{}_{(1)} u_p + 3\sum_{q=7}^{18}{}_{(2)} u_p + \sum_{r=19}^{26}{}_{(3)} u_r\right), \quad (x_1,x_2,x_3) \in R,$$

where the sum $\sum_{(k)}$ is taken over the grid nodes that are at a distance of $\sqrt{k}h$ from the point (x_1,x_2,x_3),(see Fig. (3.2)), and u_p, u_q, and u_r are the values of u at the corresponding grid points.

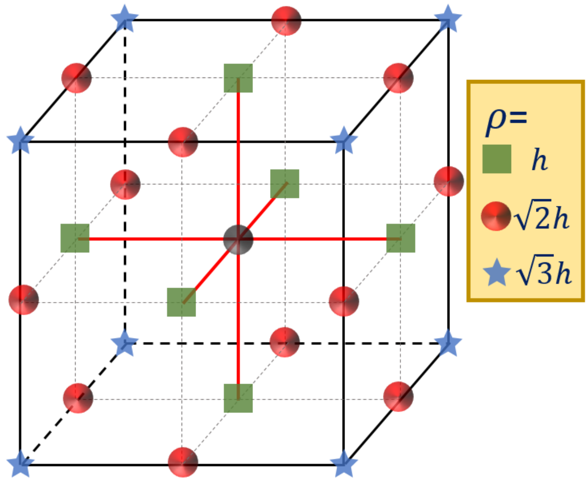

Figure 3.2. Twenty six points arount center using in operator $\Re$.Each point has a distance of $\sqrt{k}h$ from the point (x_1,x_2,x_3).

We consider the finite difference approximations of problem (3.1.1):

$$u_h = \Re u_h \ \text{ on } R^h, \ \ u_h = \varphi_j \ \text{ on } \Gamma_{jh}, \ j = 1,2,\ldots,6. \tag{3.1.18}$$

By the maximum principle (see, [59], Chap.4), problem (3.1.18) has a unique solution.

In what follows and for simplicity, we will denote by $c, c_1, c_2, \ldots$ constants which are independent of h and the nearest factor, the identical notation will be used for various constants.

Let R^{kh} be the set of nodes of the grid R^h whose distance from Γ is kh. It is obvious that $1 \leq k \leq N(h)$, where

$$N(h) = [\min\{a_1, a_2, a_3\}/(2h)]. \tag{3.1.19}$$

We define for $1 \leq k \leq N(h)$

$$f_h^k = \begin{cases} 1, & (x_1, x_2, x_3) \in R^{kh}, \\ 0, & (x_1, x_2, x_3) \in R^h \backslash R^{kh} \end{cases}$$

Lemma 3.1.4 *The solution of the system*

$$v_h^k = \Re v_h^k + f_h^k \quad on\ R^h, \quad v_h^k = 0 \ \ on\ \Gamma_h,$$

satisfies the inequality

$$\max_{(x_1,x_2,x_3)\in R^h} v_h^k \leq 6k, \ \ 1 \leq k \leq N(h). \tag{3.1.20}$$

Proof. Let w_h^k is the function defined on $R_h \cup \Gamma_h$ and defined as a conditional funcion

$$w_h^k = \begin{cases} 0, & (x_1, x_2, x_3) \in \Gamma_h, \\ 6m, & (x_1, x_2, x_3) \in R_h^m, \ \ 1 \leq m < k, \\ 6k, & (x_1, x_2, x_3) \in R_h^l, \ \ k \leq l < N(h). \end{cases} \tag{3.1.21}$$

It is clear that

$$\max_{(x_1,x_2,x_3)\in R_h} w_h^k \leq 6k.$$

we have

$$w_h^k - \Re w_h^k \geq f_h^k \quad \text{on} \ \ R_h, \ k = 1, 2, \ldots, N(h). \tag{3.1.22}$$

The correctness of inequality in (3.1.22) is shown for some examples in following:

Example 1: If $m = k$ then $f_h^k = 1$. In consider of Fig. (3.3), Fig. (3.4) and Fig. (3.2) $\Re w_h^k$ is:

$$\Re w_h^k = \frac{1}{128}\left[14\left(6\left(k-1\right)+5\left(6k\right)\right)+3\left(4\left(6\left(k-1\right)\right)+8\left(6k\right)\right)+\left(4\left(6\left(k-1\right)\right)+4\left(6k\right)\right)\right]$$

$$= 6k - \frac{45}{32} = w_h^k - \frac{45}{32} \Rightarrow w_h^k - \Re w_h^k = \frac{45}{32} > 1 = f_h^k$$

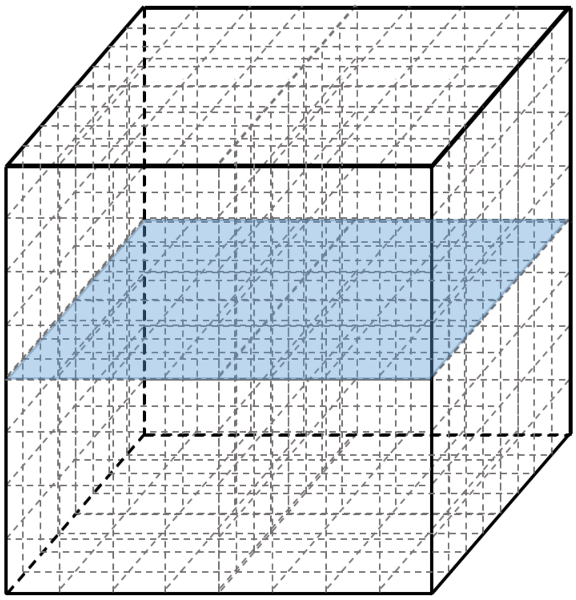

Figure 3.3. The selected plane from R_h used in Fig. (3.2).

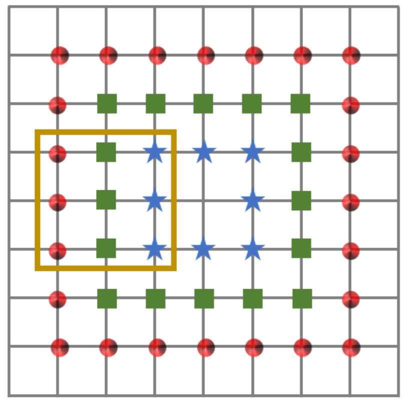

Figure 3.4. The selected plane with 9-point scheme in a square.

$$\Re w_h^k = \frac{1}{128}\left[14\left(6\left(m-1\right)+4\left(6m\right)+6\left(m+1\right)\right)+\right.$$

$$\left.3\left(4\left(6\left(m-1\right)\right)+4\left(6m\right)+4\left(6\left(m+1\right)\right)\right)+\left(4\left(6\left(m-1\right)\right)+4\left(6\left(m+1\right)\right)\right)\right] =$$

$$6m = w_h^k \Rightarrow w_h^k - \Re w_h^k = 0 = f_h^k$$

If $m \neq k$ and $m > k$ then $f_h^k = 0$ and $\Re w_h^k$ is:

$$\Re w_h^k = \frac{1}{128}\left[14\left(6k\right)+3\left(12\left(6k\right)\right)+8\left(6k\right)\right] = 6k = w_h^k \Rightarrow w_h^k - \Re w_h^k = 0 = f_h^k$$

Example 2: If $m = k$ then $f_h^k = 1$. In consider of Fig. (3.5) $\Re w_h^k$ is:

$$\Re w_h^k = \frac{1}{128}\left[14\left(2\left(k-1\right)+4\left(6k\right)\right)+3\left(7\left(6\left(k-1\right)\right)+5\left(6k\right)\right)+\left(6\left(6\left(k-1\right)\right)+\right.\right.$$

$$\left.\left.2\left(6k\right)\right) = 6k - \frac{165}{64} = w_h^k - \frac{165}{64} \Rightarrow w_h^k - \Re w_h^k = \frac{165}{64} > 1 = f_h^k\right.$$

If $m \neq k$ and $m < k$ then $f_h^k = 0$ and $\Re w_h^k$ is:

$$\Re w_h^k = \frac{1}{128}[14(2(6(m-1))+4(6m))+$$

$$3(7(6(m-1))+4(6m)+6(m+1))+(6(6(m-1))+2(6(m+1)))] =$$

$$6k - \frac{75}{64} = w_h^k - \frac{75}{64} \Rightarrow w_h^k - \Re w_h^k = \frac{75}{64} > 0 = f_h^k$$

If $m \neq k$ and $m > k$ then $f_h^k = 0$ and $\Re w_h^k$ is:

$$\Re w_h^k = \frac{1}{128}[14(6k)+3(12(6k))+8(6k)] = 6k = w_h^k \Rightarrow w_h^k - \Re w_h^k = 0 = f_h^k$$

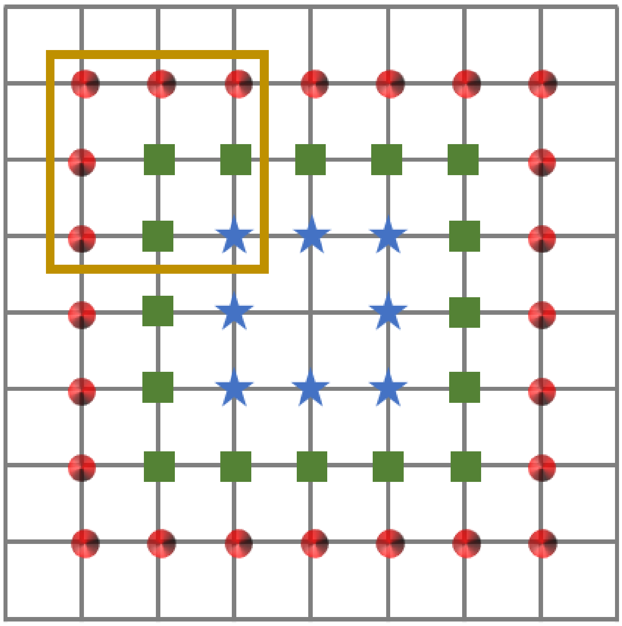

Figure 3.5. The selected plane with 9-point scheme in a square.

Then by the comparison theorem (see Chapter 4 in [59]), and by (3.1.21), we obtain

$$v_h^k \leq w_h^k \leq 6k \quad \text{on} \quad R_h,$$

this follows the inequality (3.1.20). ■

Proposition 3.1.5 *The equation holds*

$$\Re p_7(x_0, y_0, z_0) = u(x_0, y_0, z_0) \tag{3.1.23}$$

where $p_7(x_{10}, x_{20}, x_{30})$ *is the seventh order Taylor's polynomial at* (x_{10}, x_{20}, x_{30}) *and* u

is a harmonic function.

Proof. Here it has a similar proof of proposition (2.1.4) in Chapter (2) and taking into account that the function u is harmonic, by exhaustive calculations, we have

$$\Re p_7(x_{10},x_{20},x_{30}) = \frac{1}{128}(14(p_7(x_{10}+h,x_{20},x_{30}) + p_7(x_{10}-h,x_{20},x_{30}) +$$

$$p_7(x_{10},x_{20}+h,x_{30}) + p_7(x_{10},x_{20}-h,x_{30}) + p_7(x_{10},x_{20},x_{30}+h) +$$

$$p_7(x_{10},x_{20},x_{30}-h)) + 3(p_7(x_{10}+h,x_{20}+h,x_{30}) + p_7(x_{10}+h,x_{20}-h,x_{30}) +$$

$$p_7(x_{10}+h,x_{20},x_{30}+h) + p_7(x_{10}+h,x_{20},x_{30}-h) + p_7(x_{10}-h,x_{20}+h,x_{30}) +$$

$$p_7(x_{10}-h,x_{20}-h,x_{30}) + p_7(x_{10}-h,x_{20},x_{30}+h) + p_7(x_{10}-h,x_{20},x_{30}-h) +$$

$$p_7(x_{10},x_{20}+h,x_{30}+h) + p_7(x_{10},x_{20}+h,x_{30}-h) + p_7(x_{10},x_{20}-h,x_{30}+h) +$$

$$p_7(x_{10},x_{20}-h,x_{30}-h)) + (p_7(x_{10}+h,x_{20}+h,x_{30}+h) +$$

$$p_7(x_{10}+h,x_{20}+h,x_{30}-h) + p_7(x_{10}+h,x_{20}-h,x_{30}+h) +$$

$$p_7(x_{10}+h,x_{20}-h,x_{30}-h) + p_7(x_{10}-h,x_{20}+h,x_{30}+h) +$$

$$p_7(x_{10}-h,x_{20}+h,x_{30}-h) + p_7(x_{10}-h,x_{20}-h,x_{30}+h) +$$

$$p_7(x_{10}-h,x_{20}-h,x_{30}-h)) = u(x_0,y_0,z_0) +$$

$$\frac{15}{64}h^2\sum_{i=1}^{3}\frac{\partial^2 u(x_{10},x_{20},x_{30})}{\partial x_i^2} + \frac{1}{128}h^4\sum_{i=1}^{3}\frac{\partial^4 u(x_{10},x_{20},x_{30})}{\partial x_i^4} +$$

$$\frac{1}{64}h^4\sum_{i=1}^{2}\sum_{j=i+1}^{3}\frac{\partial^4 u(x_{10},x_{20},x_{30})}{\partial x_i^2\partial x_j^2} + \frac{1}{1536}h^6\sum_{i=1}^{3}\frac{\partial^6 u(x_{10},x_{20},x_{30})}{\partial x_i^6} +$$

$$\frac{5}{1536}h^6\sum_{i=1}^{2}\sum_{j=i+1}^{3}\left(\frac{\partial^6 u(x_{10},x_{20},x_{30})}{\partial x_i^2\partial x_j^4} + \frac{\partial^6 u(x_{10},x_{20},x_{30})}{\partial x_i^4\partial x_j^2}\right) +$$

$$\frac{12}{1536}h^6\frac{\partial^6 u(x_{10},x_{20},x_{30})}{\partial x_1^2\partial x_2^2\partial x_3^2} = u(x_0,y_0,z_0) +$$

$$\frac{2}{256}\left(\frac{\partial}{\partial x^2}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2} + \frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2} + \frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right) +\right.$$

$$\frac{\partial}{\partial y^2}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2} + \frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2} + \frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right) +$$

$$\frac{\partial}{\partial z^2}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)\Bigg)+$$

$$\frac{1}{1536}\left(\frac{\partial}{\partial x^4}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)+\right.$$

$$\frac{\partial}{\partial y^4}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)+$$

$$\frac{\partial}{\partial z^4}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)+$$

$$4\frac{\partial}{\partial x^2\partial y^2}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)+$$

$$4\frac{\partial}{\partial x^2\partial z^2}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)+$$

$$\left.4\frac{\partial}{\partial y^2\partial z^2}\left(\frac{\partial^2 u(x_0,y_0,z_0)}{\partial x^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial y^2}+\frac{\partial^2 u(x_0,y_0,z_0)}{\partial z^2}\right)\right)=u(x_0,y_0,z_0)$$

■

Lemma 3.1.6 *Let u be a solution of problem* (3.1.1). *The inequality holds*

$$\max_{(x_1,x_2,x_3)\in R^{kh}}|\Re u-u|\le c\frac{h^6}{k^2},\ k=1,2,...,N(h) \tag{3.1.24}$$

Proof. Let (x_{10},x_{20},x_{30}) be a point of R^{1h}, and let

$$R_0=\{(x_1,x_2,x_3):\ |x_i-x_{i0}|<h,i=1,2,3\}, \tag{3.1.25}$$

be an elementary cube, some faces of which lie on the boundary of the rectangular parallelepiped R. On the vertices of R_0, and on the center of its faces and edges lie the nodes of which the function values are used to evaluate $\Re u(x_{10},x_{20},x_{30})$. We represent a solution of problem (3.1.1) in some neighborhood of $x_0=(x_{10},x_{20},x_{30})\in R^{1h}$ by Taylor's formula

$$u(x_1,x_2,x_3)=p_7(x_1,x_2,x_3;x_0)+r_8(x_1,x_2,x_3;x_0), \tag{3.1.26}$$

where $p_7(x_1,x_2,x_3)$ is the seventh order Taylor's polynomial, $r_8(x_1,x_2,x_3)$ is the remainder term. Taking into account that the function u is harmonic, we have

$$\Re p_7(x_{10},x_{20},x_{30};x_0)=u(x_{10},x_{20},x_{30}) \tag{3.1.27}$$

Now, we estimate r_8 at the nodes of the operator $\Re$. We take a node $(x_{10}+h, x_{20}, x_{30}+h)$ which is one of the twenty six nodes of $\Re$, and consider the function

$$\widetilde{u}(s) = u\left(x_{10}+\frac{s}{\sqrt{2}}, x_{20}, x_{30}+\frac{s}{\sqrt{2}}\right), \quad -\sqrt{2}h \leq s \leq \sqrt{2}h \tag{3.1.28}$$

of single variable s, which is the arc length along the straight line through the points $(x_{10}-h, x_{20}, x_{30}-h)$ and $(x_{10}+h, x_{20}, x_{30}+h)$. By virtue of Lemma 3.1.3, we have

$$\left|\frac{d^8\widetilde{u}(s)}{ds^8}\right| \leq c(\sqrt{2}h-s)^{-2}, \;\; 0 \leq s < \sqrt{2}h. \tag{3.1.29}$$

We represent the function (3.1.28) around the point $s=0$ by Taylor's formula

$$\widetilde{u}(s) = \widetilde{p}_7(s) + \widetilde{r}_8(s),$$

where $\widetilde{p}_7(s) \equiv p_7\left(x_{10}+\frac{s}{\sqrt{2}}, x_{20}, x_{30}+\frac{s}{\sqrt{2}}\right)$ is the seventh order Taylor's polynomial of the variable s, and

$$\widetilde{r}_8(s) \equiv r_8\left(x_{10}+\frac{s}{\sqrt{2}}, x_{20}, x_{30}+\frac{s}{\sqrt{2}}; x_0\right), \;\; |s| < \sqrt{2}h, \tag{3.1.30}$$

is the remainder term. On the basis of the continuity of $\widetilde{r}_8(s)$ on the interval $\left[-\sqrt{2}h, \sqrt{2}h\right]$ and estimation (3.1.29),we obtain

$$\begin{aligned} r_8\left(x_{10}+h, x_{20}, x_{30}h; x_0\right) &= \lim_{\varepsilon\to+0} \widetilde{r}_8(\sqrt{2}h-\varepsilon) \\ &\leq \lim_{\varepsilon\to+0}\left[c\frac{1}{7!}\int\limits_0^{\sqrt{2}h-\varepsilon}\left(\sqrt{2}h-\varepsilon-t\right)^7(\sqrt{3}h-t)^{-2}dt\right] \\ &\leq c_1h^6, \; 0<\varepsilon\leq\frac{\sqrt{2}h}{2} \end{aligned} \tag{3.1.31}$$

where c_1 is a constant independent of the choice of $(x_{10}, x_{20}, x_{30}) \in R^{kh}$. Estimation (3.1.31) is obtained analogously for the remaining twenty five nodes on the closed cube (3.1.25). Since the norm of the operator $\Re$ in the uniform metric is equal to one, by virtue of (3.1.31), we have

$$|\Re r_8\left(x_{10}, x_{20}, x_{30}\right)| \leq c_2h^6. \tag{3.1.32}$$

From (3.1.26), (3.1.27) and (3.1.32), we obtain

$$|\Re u(x_{10},x_{20},x_{30})-u\,(x_{10},x_{20},x_{30})|\leq ch^{6},$$

for any $(x_{10},x_{20},x_{30})\in R^{1h}$. Now, let (x_{10},x_{20},x_{30}) be a point of R^{kh}, for $2\leq k\leq N(h)$. By Lemma 3.1.3 for any k, $2\leq k\leq N(h)$, we obtain

$$|\Re r_{8}\,(x_{10},x_{20},x_{30})|\leq c_{3}\frac{h^{6}}{k^{2}}, \tag{3.1.33}$$

where c_3 is a constant independent of k, $2\leq k\leq N(h)$, and the choice of $(x_{10},x_{20},x_{30})\in R^{kh}$. On the basis of (3.1.26), (3.1.27), and (3.1.33) estimation (3.1.24) follows. ■

Lemma 3.1.7 *Assume that the boundary functions φ_j, $j=1,2,\ldots,6$, satisfy conditions (3.1.2)-(3.1.5). Then*

$$\max_{\overline{R}^h}|u_h-u|\leq ch^{6}(1+|\ln h|), \tag{3.1.34}$$

where u_h is the solution of the finite difference problem (3.1.18), and u is the exact solution of problem (3.1.1).

Proof. Let

$$\varepsilon_{h}=u_{h}-u\ \text{ on }\overline{R}^{h}. \tag{3.1.35}$$

By (3.1.18) and (3.1.35) the error function satisfies the system of equations

$$\varepsilon_{h}=\Re\varepsilon_{h}+(\Re u-u)\ \text{ on }R^{h},\ \ \varepsilon_{h}=0\ \text{ on }\Gamma^{h}. \tag{3.1.36}$$

We represent a solution of the system (3.1.36) as follows

$$\varepsilon_{h}=\sum_{k=1}^{N(h)}\varepsilon_{h}^{k}, \tag{3.1.37}$$

where ε_h^k, $1\leq k\leq N(h)$, $N(h)$ defined by (3.1.19), is a solution of the system

$$\varepsilon_{h}^{k}=\Re\varepsilon_{h}^{k}+\nu^{k}\ \text{ on }R^{h},\ \ \varepsilon_{h}^{k}=0\ \text{ on }\Gamma^{h}, \tag{3.1.38}$$

when

$$v^k = \begin{cases} \Re u - u & \text{on } R^{kh} \\ 0 & \text{on } R^h \backslash R^{kh}. \end{cases}.$$

Then for the solution of (3.1.38) by applying Lemmas 3.1.4 and 3.1.6, we have

$$\max_{(x_1,x_2,x_3)\in R^h} \left|\varepsilon_h^k\right| \le c\frac{h^6}{k}, \ \ 1 \le k \le N(h). \tag{3.1.39}$$

On the basis of (3.1.35), (3.1.37), and (3.1.39), we obtain

$$\max_{(x_1,x_2,x_3)\in R^h} |u_h - u| \le ch^6 (1 + |\ln h|).$$

■

Let ω be a solution of the problem

$$\Delta\omega = 0 \ \text{ on } \ R, \ \ \omega = \psi_j \ \text{ on } \Gamma_j, \ j = 1,2,\ldots,6, \tag{3.1.40}$$

where ψ_j, $j = 1,2,...,6$ are given functions and

$$\psi_j \in C^{4,\lambda}(\Gamma_j), \ 0 < \lambda < 1, \ j = 1,2,\ldots,6, \tag{3.1.41}$$

$$\psi_\mu = \psi_\nu \text{ on } \gamma_{\mu\nu}, \tag{3.1.42}$$

$$\frac{\partial^2\psi_\mu}{\partial t_\mu^2} + \frac{\partial^2\psi_\nu}{\partial t_\nu^2} + \frac{\partial^2\psi_\mu}{\partial t_{\mu\nu}^2} = 0 \ \text{ on } \gamma_{\mu\nu}. \tag{3.1.43}$$

Lemma 3.1.8 *The estimation holds*

$$\max_{\overline{R}^h} |\omega_h - \omega| \le ch^4, \tag{3.1.44}$$

where ω is the exact solution of problem (3.1.40), ω_h is the exact solution of the finite difference problem

$$\omega_h = \Re\omega_h \ \ \textit{on } R^h, \ \ \omega_h = \psi_j \ \ \textit{on } \Gamma_{jh}, \ j = 1,2,\ldots,6. \tag{3.1.45}$$

Proof. It follows from Lemma 1.2 in [54] that

$$\max_{0\le p\le q}\max_{0\le q\le 2-p}\sup_{(x_1,x_2,x_3)\in R}\left|\frac{\partial^4\omega(x_1,x_2,x_3)}{\partial x_1^{2p}\partial x_2^{2q}\partial x_3^{4-2p-2q}}\right|<\infty,$$

where u is the solution of problem (3.1.40). Then, instead of inequality (3.1.17), we have

$$\max_{0\le\mu\le 8}\max_{0\le\nu\le 8-\mu}\left|\frac{\partial^8\omega(x_1,x_2,x_3)}{\partial x_1^{\mu}\partial x_2^{\nu}\partial x_3^{8-\mu-\nu}}\right|\le c\rho^{-4}(x_1,x_2,x_3),\ \ (x_1,x_2,x_3)\in R, \qquad (3.1.46)$$

where $\rho(x_1,x_2,x_3)$ is the distance from $(x_1,x_2,x_3)\in R$ to the boundary Γ. On the basis of estimation (3.1.46) and Taylor's formula, by analogy with the proof of Lemma 3.1.6 we have

$$\max_{(x_1,x_2,x_3)\in R^{kh}}|\Re\omega-\omega|\le c\frac{h^4}{k^4},\ k=1,2,...,N(h).$$

We put

$$\varepsilon_h=\omega_h-\omega \ \text{ on } R^h\cup\Gamma_h.$$

Then, as the proof of Lemma 3.1.7, we obtain

$$\max_{\overline{R}^h}|\omega_h-\omega|\le c_4h^4\sum_{k=1}^{N(h)}\frac{1}{k^3}\le ch^4.$$

■

3.2 Approximation of the First Derivative

Let $v=\frac{\partial u}{\partial x_1}$ and let $\Phi_j=\frac{\partial u}{\partial x_1}$ on $\Gamma_j,\ j=1,2,\ldots,6$, and consider the boundary value problem:

$$\Delta v=0 \ \text{ on } R,\ \ v=\Phi_j \ \text{ on } \Gamma_j,\ j=1,2,\ldots,6, \qquad (3.2.1)$$

where u is a solution of the boundary value problem (3.1.1).

We define the following operators $\Phi_{\nu h}$, $\nu=1,2,\ldots,6$,

$$\Phi_{1h}(u_h) = \frac{1}{12h}(-25\varphi_1(x_2,x_3)+48u_h(h,x_2,x_3)-36u_h(2h,x_2,x_3)$$
$$+16u_h(3h,x_2,x_3)-3u_h(4h,x_2,x_3)) \quad \text{on } \Gamma_1^h, \tag{3.2.2}$$
$$\Phi_{4h}(u_h) = \frac{1}{12h}(25\varphi_4(x_2,x_3)-48u_h(a_1-h,x_2,x_3)+36u_h(a_1-2h,x_2,x_3)$$
$$-16u_h(a_1-3h,x_2,x_3)+3u_h(a_1-4h,x_2,x_3)) \quad \text{on } \Gamma_4^h, \tag{3.2.3}$$
$$\Phi_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x_1} \quad \text{on } \Gamma_p^h,\ p=2,3,5,6, \tag{3.2.4}$$

where u_h is the solution of finite difference problem (3.1.18).

Lemma 3.2.1 *The inequality is true*

$$|\Phi_{kh}(u_h)-\Phi_{kh}(u)| \le c_3 h^5(1+|\ln h|), \quad k=1,4, \tag{3.2.5}$$

where u_h is the solution of problem (3.1.18)*, u is the solution of problem* (3.1.1).

Proof. It is obvious that $\Phi_{ph}(u_h)-\Phi_{ph}(u)=0$ for $p=2,3,5,6$. For $k=1$, by (3.2.2) and Lemma 3.1.7, we have

$$|\Phi_{1h}(u_h)-\Phi_{1h}(u)| = \Big|\frac{1}{12h}((-25\varphi_1(x_2,x_3)+48u_h(h,x_2,x_3)-36u_h(2h,x_2,x_3)$$
$$+16u_h(3h,x_2,x_3)-3u_h(4h,x_2,x_3))-(-25\varphi_1(x_2,x_3)+48u(h,x_2,x_3)$$
$$-36u(2h,x_2,x_3)+16u(3h,x_2,x_3)-3u(4h,x_2,x_3)))\Big|$$
$$\le \frac{1}{12h}(48\,|u_h(h,x_2,x_3)-u(h,x_2,x_3)|+36\,|u_h(2h,x_2,x_3)-u(2h,x_2,x_3)|$$
$$+16\,|u_h(3h,x_2,x_3)-u(3h,x_2,x_3)|+3\,|u_h(4h,x_2,x_3)-u(4h,x_2,x_3)|)$$
$$\le c_5 h^5(1+|\ln h|).$$

In following shown the same inequality is true when $k=4$ also,

$$|\Phi_{1h}(u_h) - \Phi_{1h}(u)| = \left| \frac{1}{12h}((-25\varphi_4(x_2,x_3) \ + 48u_h(a_1-h,x_2,x_3) \right.$$

$$- 36u_h(a_1-2h,x_2,x_3) + 16u_h(a_1-3h,x_2,x_3) - 3u_h(a_1-4h,x_2,x_3))$$

$$-(-25\varphi_4(x_2,x_3) + 48u(a_1-h,x_2,x_3) - 36u(a_1-2h,x_2,x_3) + 16u(a_1-3h,x_2,x_3)$$

$$- \ 3u(a_1-4h,x_2,x_3)))| \leq \frac{1}{12h}(48\,|u_h(a_1-h,x_2,x_3) - u(a_1-h,x_2,x_3)|$$

$$+ 36\,|u_h(a_1-2h,x_2,x_3)| - u(a_1-2h,x_2,x_3) + 16\,|u_h(a_1-3h,x_2,x_3)$$

$$-u(a_1-3h,x_2,x_3) + 3\,|u_h(4h,x_2,x_3) - u(4h,x_2,x_3)|) \leq c_5 h^5(1+|\ln h|).$$

■

Lemma 3.2.2 *The inequality holds*

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Phi_{kh}(u_h) - \Phi_k| \leq c_4 h^4, \ \ k = 1,4. \tag{3.2.6}$$

where Φ_{kh}, $k = 1,4$ *are defined by (3.2.2), (3.2.3), and* $\Phi_k = \frac{\partial u}{\partial x_1}$ *on* Γ_k, $k = 1,4$.

Proof. From Lemma 3.1.1 it follows that $u \in C^{5,0}(\overline{R})$. Then, at the end points $(0,\nu h,\omega h) \in \Gamma_1^h$ and $(a_1,\nu h,\omega h) \in \Gamma_4^h$ of each line segment $\{(x_1,x_2,x_3) : 0 \leq x_1 \leq a_1, 0 < x_2 = \nu h < a_2, 0 < x_3 = \omega h < a_3\}$, expressions (3.2.2) and (3.2.3) give the fourth order approximation of $\frac{\partial u}{\partial x_1}$, respectively. From the truncation error formulas it (see [61]) follows that

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Phi(u) - \Phi_k| \leq c_5 h^4, \ \ k = 1,4. \tag{3.2.7}$$

On the basis of Lemma 3.2.1 and estimation (3.2.7), (3.2.6) follows,

$$\begin{aligned}\max_{(x,y)\in\gamma_k^h} |\Phi_{kh}(u_h) - \Phi_k| &= \max_{(x,y)\in\gamma_k^h} |\Phi_{kh}(u_h) - \Phi_{kh}(u) + \Phi_{kh}(u) - \Phi_k| \\ &\leq \max_{(x,y)\in\gamma_k^h} |\Phi_{kh}(u_h) - \Phi_{kh}(u)| + \max_{(x,y)\in\gamma_k^h} |\Phi_{kh}(u) - \Phi_k| \\ &\leq c_4 h^4, \ \ k = 1,4.\end{aligned}$$

We consider the finite difference boundary value problem

$$v_h = \Re v_h \text{ on } R^h, \ \ v_h = \Phi_{jh} \text{ on } \Gamma_j^h, \ j = 1,2,\ldots,6, \tag{3.2.8}$$

where $\Psi_{jh}, \ j = 1,2,\ldots,6$, are defined by (3.2.2)-(3.2.4).

Theorem 3.2.3 *The estimation is true*

$$\max_{(x_1,x_2,x_3)\in \overline{R}^h} \left| v_h - \frac{\partial u}{\partial x_1} \right| \leq ch^4, \tag{3.2.9}$$

where u is the solution of problem (3.1.1)*, v_h is the solution of the finite difference problem* (3.2.8).

Proof. Let

$$\varepsilon_h = v_h - v \text{ on } \overline{R}^h, \tag{3.2.10}$$

where $v = \frac{\partial u}{\partial x_1}$. From (3.2.8) and (3.2.10), we have

$$\begin{aligned} \varepsilon_h &= \Re \varepsilon_h + (\Re v - v) \text{ on } R^h, \\ \varepsilon_h &= \Phi_{kh}(u_h) - v \text{ on } \Gamma_k^h, \ k = 1,4, \ \ \varepsilon_h = 0 \text{ on } \Gamma_p^h, \ p = 2,3,5,6. \end{aligned}$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{3.2.11}$$

where

$$\varepsilon_h^1 = \Re \varepsilon_h^1 \text{ on } R^h, \tag{3.2.12}$$

$$\varepsilon_h^1 = \Phi_{kh}(u_h) - v \text{ on } \Gamma_k^h, \ k = 1,4, \ \ \varepsilon_h^1 = 0 \text{ on } \Gamma_p^h, \ p = 2,3,5,6; \tag{3.2.13}$$

$$\varepsilon_h^2 = \Re \varepsilon_h^2 + (\Re v - v) \text{ on } R^h, \ \ \varepsilon_h^2 = 0 \text{ on } \Gamma_j^h, \ j = 1,2,\ldots,6. \tag{3.2.14}$$

By Lemma 3.2.2 and by the maximum principle, for the solution of system (3.2.12), (3.2.13), we have

$$\max_{(x_1,x_2,x_3)\in \overline{R}^h} \left| \varepsilon_h^1 \right| \leq \max_{q=1,4} \max_{(x_1,x_2,x_3)\in \Gamma_q^h} \left| \Phi_{qh}(u_h) - v \right| \leq c_4 h^4. \tag{3.2.15}$$

The solution ε_h^2 of system (3.2.14) is the error of the approximate solution obtained by the finite difference method for problem (3.2.1), when on the boundary nodes Γ_{jh}, the approximate values are defined as the exact values of the functions Φ_j in (3.2.1). It is obvious that Φ_j, $j = 1,2,\dots,6$, satisfy the conditions

$$\Phi_j \in C^{5,\lambda}(\Gamma_j),\ 0 < \lambda < 1,\ j = 1,2,\dots,6, \tag{3.2.16}$$

$$\Phi_\mu = \Phi_\nu \text{ on } \gamma_{\mu\nu}, \tag{3.2.17}$$

$$\frac{\partial_\mu^2 \Phi}{\partial t_\mu^2} + \frac{\partial_\nu^2 \Phi}{\partial t_\nu^2} + \frac{\partial_\mu^2 \Phi}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}. \tag{3.2.18}$$

Since the function $v = \frac{\partial u}{\partial x_1}$ is harmonic on R with the boundary functions Ψ_j, $j = 1,2,\dots,6$, on the basis of (3.2.16)- (3.2.18), and Lemma 3.1.8 we obtain

$$\max_{(x_1,x_2,x_3)\in \overline{R}^h} \left|\varepsilon_h^2\right| \le c_6 h^4. \tag{3.2.19}$$

By (3.2.11), (3.2.15) and (3.2.19) inequality (3.2.9) follows. ■

Remark 3.2.4 *On the basis of Lemma 3.1.2 the sixth order pure derivatives are bounded in R. Therefore, if we replace the formulae* (3.2.2) *and* (3.2.3) *by the fifth order forward and backward numerical differentiation formulae (see Chap.2 in [41]), then by analogy to the proof of estimation* (3.2.9), *we obtain*

$$\max_{(x_1,x_2,x_3)\in \overline{R}^h} \left| v_h - \frac{\partial u}{\partial x_1} \right| \le ch^5(1 + |\ln h|).$$

3.3 Approximation of the Pure Second Derivatives

We denote by $\omega = \frac{\partial^2 u}{\partial x_1^2}$. The function ω is harmonic on R, on the basis of Lemma 3.1.1 is continuous on $\overline{R}$, and is a solution of the following Dirichlet problem

$$\Delta \omega = 0 \text{ on } R,\ \ \omega = \chi_j \text{ on } \Gamma_j,\ j = 1,2,\dots,6, \tag{3.3.1}$$

where

$$\chi_\tau = \frac{\partial^2 \varphi_\tau}{\partial x_1^2}, \quad \tau = 2,3,5,6, \tag{3.3.2}$$

$$\chi_\nu = -\left(\frac{\partial^2 \varphi_\nu}{\partial x_2^2} + \frac{\partial^2 \varphi_\nu}{\partial x_3^2}\right), \quad \nu = 1,4. \tag{3.3.3}$$

Let ω_h be the solution of the finite difference problem

$$\omega_h = \Re\omega_h \text{ on } R^h, \quad \omega_h = \chi_j \text{ on } \Gamma_j^h, \ j = 1,2,\ldots,6, \tag{3.3.4}$$

where χ_j, $j = 1,2,\ldots,6$ are the functions determined by (3.3.2) and (3.3.3).

Theorem 3.3.1 *The estimation holds*

$$\max_{\overline{R}^h} |\omega_h - \omega| \le ch^4, \tag{3.3.5}$$

where $\omega = \frac{\partial^2 u}{\partial x_1^2}$, *u is the solution of problem* (3.1.1) *and* ω_h *is the solution of the finite difference problem* (3.3.4).

Proof. From the continuity of the function ω on $\overline{R}$, and from (3.1.2)-(3.1.5) and (3.3.2), (3.3.3) it follows that

$$\chi_j \in C^{4,\lambda}(\Gamma_j), \ 0 < \lambda < 1, \ j = 1,2,\ldots,6, \tag{3.3.6}$$

$$\chi_\mu = \chi_\nu \text{ on } \gamma_{\mu\nu}, \tag{3.3.7}$$

$$\frac{\partial^2 \chi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \chi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \chi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}. \tag{3.3.8}$$

The boundary functions χ_j, $j = 1,2,\ldots,6$, in (2.3.1) on the basis of (3.3.6)-(3.3.8) satisfy all conditions of Lemma 3.1.8 in which follows the proof of the error estimation (3.3.5). ■

Chapter 4

NUMERICAL EXPERIMENTS

In this chapter we present the numerical results obtained in support of the theorical part. Our aim is to show the high order accurate approximation of the first and pure second derivatives of the Laplace equation on a rectangle and a rectangular parallelepiped. Further, we show how these results are obtained and their application for different boundary functions and different dimensional domains.

All results are obtained by using the Strongly Implicit Procedure.

4.1 The Strongly Implicit Procedure (SIP)

The Strongly Implicit Procedure is a method for finding the approximate solution of sparse linear system of equations. The linear system of equations can be shown in matrix form ($Au = q$), for which SIP is used effectively when matix A has many zero entries and the non-zero entries lie on a finite number of diagonals. In SIP Incomplete LU decomposition is used, which is the approximation of the exact LU decomposition solution.

In our studies A is related to eight-point averaging operator when applied on a rectangle and to twenty-six point difference operator on a rectangular parallelepiped. Vector u is a vector of unknown variables in the finite difference approximation of the boundary value problem. On the right hand side of the equation, vector q corresponds to the value of the boundary function, when the averaging operator is applied for the approximate solution of Laplace's equation on each point of domain grid. In the two dimensional case if the rectangular mesh is dimensions of $M \times N$, then the matrix A has order $MN \times MN$. Each row of the matrix A has the coefficients of the unknown variables of an equation corresponding to each point of grid. In the three dimensional case if the mesh has $M \times N \times Q$ points then matrix A has dimensions of $MNQ \times MNQ$. If the mesh size is h in any direction of domain then by choosing a small value of h the grid will have many points which results in a large number equations related to each point. Thus, it is required to solve a large system of linear equations $Au = q$, and using the method of LU decomposition takes impractical processing time and amount of memory. The SIP helps to improve the CPU time as regards to the non-zero entries it lies only on finite number of digonals. The following figure (Figure (4.1)) shows the matrix A for the nine point scheme with 9 diagonals. The main diagonal is called by $A_{[0]}$ and the adjacent diagonals are $\{A_{[-1]}, A_{[1]}\}$. There are two diagonal with a distance of M from main diagonal shown by $\{A_{[-M]}, A_{[M]}\}$ and the adjacent diagonals of these diagonals $\{A_{[-M-1]}, A_{[-M+1]}\}$, $\{A_{[M-1]}, A_{[M+1]}\}$ are nine different diagonals (see Figure (4.2)). The label of each diagonal is chosen taking into account the distance from main diagonal (the number of entries needed to move to the right to reach the upper side diagonals as a positive value and number of entries needed move to down to reach the loweside diagonals as a negative value used for subscrpt of A). In the nine point scheme, each point has eight neighbor points which are used by averaging operator for the approximate solution of Laplace's equation and illustrates the coefficients of the unknown variables in the correct row of matrix A correspondting to the index grid points (see Figure (4.3)).

In the method of LU decomposition matrix A can be factorized as a $A = LU$ where L and U are lower and upper tiangular matrices respectively. These L and U can be computed with partial pivoiting. The solution of $LUu = q$ can be found by assuming $Uu = y$, then by forward substitution $Ly = q$ calculates y, and u is computed by back-substitution using the system $Uu = y$. The new matrix $\bar{A}$ in SIP is defined instead of matrix A, with a negligible matrix N ($\|N\| << \|A\|$). Hence the new $\bar{A}$ can be factorized by $\bar{L}$ and $\bar{U}$ ($\bar{A} = \bar{L}\bar{U}$) and $\bar{A}$ has exactly same diagonal of A with some additional new diagonals, Figure (4.4). Further we show how we can find the entries of $\bar{L}$ and $\bar{U}$ by using different averaging operators. Now the iterative procedure can be done by using the following equation:

$$(A+N)u = (A+N)u + (q - Au)$$

The iterative procedure is:

$$(A+N)u^{(n+1)} = (A+N)u^{(n)} + (q - Au^{(n)})$$

The $(n+1)$ the iterative step $u^{(n+1)}$of u can be computed by previous iterative information $u^{(n)}$ (Chapter 5 [71]). The matrix $\bar{L}\bar{U}$ is used instead of the matrix $A+N$ (see Figure (4.4)) and calculated before starting iterative procedure only once. The difference of $u^{(n+1)}$ and $u^{(n)}$ is defined by $d^{(n+1)}$:

$$d^{(n)} = u^{(n+1)} - u^{(n)}$$

and the residual $R^{(n)}$ is defined by:

$$R^{(n)} = q - Au^{(n)}$$

Then in the iterative procedure the following solutions are applied:

$$\bar{L}\bar{U}d^{(n)} = R^{(n)}, \tag{4.1.1}$$

followed by:

$$u^{(n+1)} = d^{(n)} + u^{(n)}$$

Equation (4.1.1) can be solved by using forward and backward substitutions

$$\bar{L}y^{(n)} = R^{(n)} \tag{4.1.2}$$

and

$$\bar{U}d^{(n)} = y^{(n)}. \tag{4.1.3}$$

we demonstrate how the entries of $\bar{L}$, $\bar{U}$ and the solution of linear systems (4.1.2) and (4.1.3) for different averaging operators on different domain can be obtained.

$$A = \left[\begin{array}{ccccccccccccccccccccc}
20&-4&&&&&&-4&-1&&&&&&&&&&&& \\
-4&20&-4&&&&&-1&-4&-1&&&&&&&&&&& \\
&-4&20&-4&&&&&-1&-4&-1&&&&&&&&&& \\
&&-4&20&-4&&&&&-1&-4&-1&&&&&&&&& \\
&&&-4&20&-4&&&&&-1&-4&-1&&&&&&&& \\
&&&&-4&20&-4&&&&&-1&-4&-1&&&&&&& \\
&&&&&-4&20&&&&&&-1&-4&&&&&&& \\
-4&-1&&&&&&20&-4&&&&&&-4&-1&&&&& \\
-1&-4&-1&&&&&-4&20&-4&&&&&-1&-4&-1&&&& \\
&-1&-4&-1&&&&&-4&20&-4&&&&&-1&-4&-1&&& \\
&&-1&-4&-1&&&&&-4&20&-4&&&&&-1&-4&-1&& \\
&&&-1&-4&-1&&&&&-4&20&-4&&&&&-1&-4&-1& \\
&&&&-1&-4&-1&&&&&-4&20&-4&&&&&-1&-4&-1 \\
&&&&&-1&-4&&&&&&-4&20&&&&&&-1&-4 \\
&&&&&&&-4&-1&&&&&&20&-4&&&&& \\
&&&&&&&-1&-4&-1&&&&&-4&20&-4&&&& \\
&&&&&&&&-1&-4&-1&&&&&-4&20&-4&&& \\
&&&&&&&&&-1&-4&-1&&&&&-4&20&-4&& \\
&&&&&&&&&&-1&-4&-1&&&&&-4&20&-4& \\
&&&&&&&&&&&-1&-4&-1&&&&&-4&20&-4 \\
&&&&&&&&&&&&-1&-4&&&&&&-4&20
\end{array}\right]$$

Figure 4.1. The coefficients of unknown variables of the equations corresponding to each point of the grid when nine-point scheme is applied.

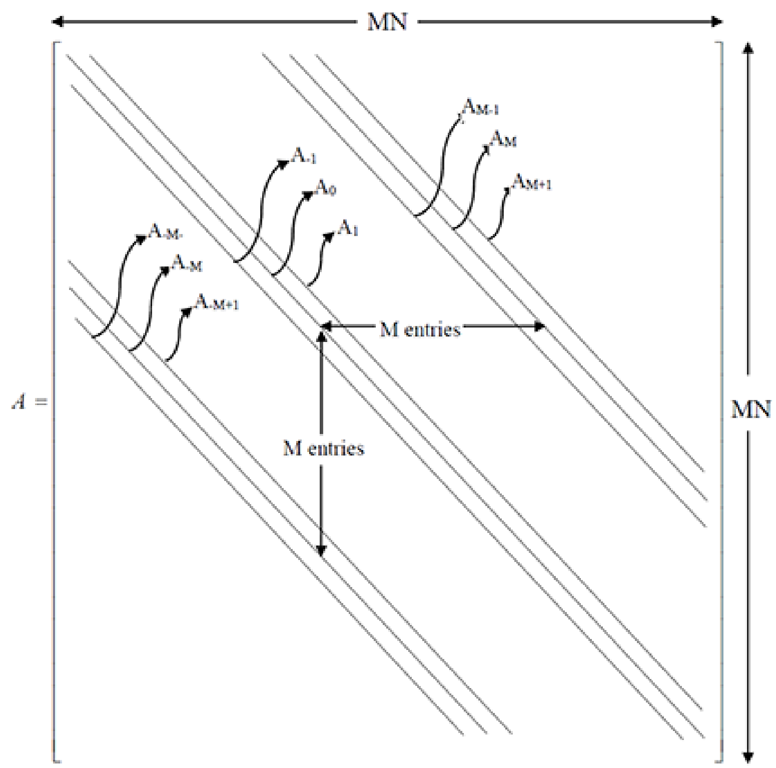

Figure 4.2. Matrix A for nine point scheme with 9 diagonals.

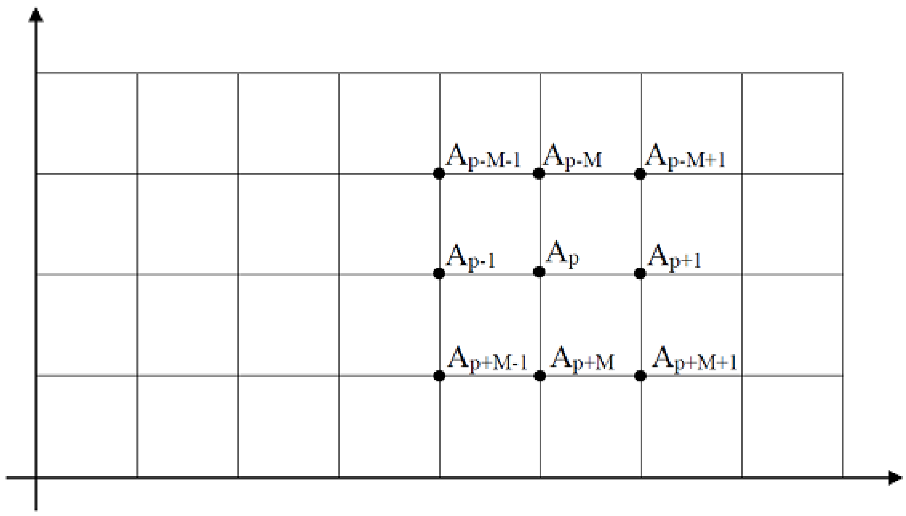

Figure 4.3. 8 neighboring points around point A_p in nine point scheme.

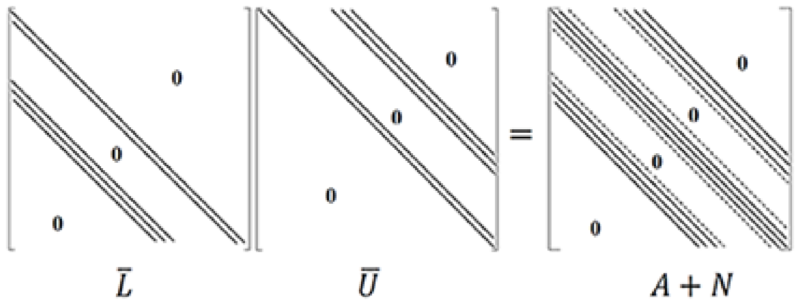

Figure 4.4. $\bar{L}\bar{U} = A + N$

4.2 Rectangular

The entries of $\bar{L}$ and $\bar{U}$ for nine point difference scheme is calculated by the following recurrence relations [72]:

$$\bar{L}_{[-M-1]}(i) = A_{[-M-1]}(i)$$

$$\bar{L}_{[-M-1]}(i)\bar{U}_{[1]}(i-M-1) + \bar{L}_{[-M]}(i) = A_{[-M]}(i)$$

$$\bar{L}_{[-M]}(i)\bar{U}_{[1]}(i-M) + \bar{L}_{[-M+1]}(i) = A_{[-M+1]}(i)$$

$$\bar{L}_{[-M-1]}(i)\bar{U}_{[M]}(i-M-1) + \bar{L}_{[-M]}(i)\bar{U}_{[M-1]}(i-M) + \bar{L}_{[-1]}(i) = A_{[-1]}(i)$$

$$\bar{L}_{[-M-1]}(i)\bar{U}_{[M+1]}(i-M-1) + \bar{L}_{[-M]}(i)\bar{U}_{[M]}(i-M) +$$

$$\bar{L}_{[-M+1]}(i)\bar{U}_{[M-1]}(i-M+1) + \bar{L}_{[-1]}(i)\bar{U}_{[1]}(i-1) + \bar{L}_{[0]}(i) = A_{[0]}(i)$$

$$\bar{L}_{[-M]}(i)\bar{U}_{[M+1]}(i-M) + \bar{L}_{[-M+1]}(i)\bar{U}_{[M]}(i-M+1) + \bar{L}_{[0]}(i)\bar{U}_{[1]}(i) = A_{[1]}(i)$$

$$\bar{L}_{[-1]}(i)\bar{U}_{[M]}(i-1) + \bar{L}_{[0]}(i)\bar{U}_{[M-1]}(i) = A_{[M-1]}(i)$$

$$\bar{L}_{[-1]}(i)\bar{U}_{[M+1]}(i-1) + \bar{L}_{[0]}(i)\bar{U}_{[M]}(i) = A_{[M]}(i)$$

where $\bar{L}_{[M]}(i)$ and $\bar{U}_{[M]}(i)$ are related to the value of the i-th row, on diagonals which have a distance M from the main diagonal in matrix $\bar{L}$ and $\bar{U}$, respectively.

Hence, the entries of $\bar{L}$ and $\bar{U}$ can be obtained by:

m

$$\bar{L}_{[-M-1]}(i) = A_{[-M-1]}(i)$$

$$\bar{L}_{[-M]}(i) = A_{[-M]}(i) - \bar{L}_{[-M-1]}(i)\bar{U}_{[1]}(i-M-1)$$

$$\bar{L}_{[-M+1]}(i) = A_{[-M+1]}(i) - \bar{L}_{[-M]}(i)\bar{U}_{[1]}(i-M)$$

$$\bar{L}_{[-1]}(i) = A_{[-1]}(i) - \bar{L}_{[-M-1]}(i)\bar{U}_{[M]}(i-M-1) - \bar{L}_{[-M]}(i)\bar{U}_{[M-1]}(i-M)$$

$$\bar{L}_{[0]}(i) = A_{[0]}(i) - \bar{L}_{[-M-1]}(i)\bar{U}_{[M+1]}(i-M-1) - \bar{L}_{[-M]}(i)\bar{U}_{[M]}(i-M) -$$

$$\bar{L}_{[-M+1]}(i)\bar{U}_{[M-1]}(i-M+1) - \bar{L}_{[-1]}(i)\bar{U}_{[1]}(i-1)$$

$$\bar{U}_{[1]}(i) = \left(A_{[1]}(i) - \bar{L}_{[-M]}(i)\bar{U}_{[M+1]}(i-M) - \bar{L}_{[-M+1]}(i)\bar{U}_{[M]}(i-M+1)\right)/\bar{L}_{[0]}(i)$$

$$\bar{U}_{[M-1]}(i) = \left(A_{[M-1]}(i) - \bar{L}_{[-1]}(i)\bar{U}_{[M]}(i-1)\right)/\bar{L}_{[0]}(i)$$

$$\bar{U}_{[M]}(i) = \left(A_{[M]}(i) - \bar{L}_{[-1]}(i)\bar{U}_{[M+1]}(i-1)\right)/\bar{L}_{[0]}(i)$$

$$\bar{U}_{[M+1]}(i) = A_{[M+1]}(i)\,/\,\bar{L}_{[0]}(i)$$

Next step is the iterative procedure. It means the forward and backward substiution to find the solution of $y^{(n)}$ in (4.1.2) and $d^{(n)}$ in (4.1.3) where directly $y^{(n)}(i)$ and $d^{(n)}(i)$ ($y^{(n)}(i)$ and $d^{(n)}(i)$ related to point i) is computed by the following:

$$\begin{aligned} y^{(n)}(i) &= \Big(R^{(n)}(i) - L_{[-M-1]}(i)\,.y^{(n)}(i-M-1) - L_{[-M]}(i)\,.y^{(n)}(i-M) - \\ &\quad L_{[-M+1]}(i)\,.y^{(n)}(i-M+1) - L_{[-1]}(i)\,.y^{(n)}(i-1)\Big)\,/\,L_{[0]}(i) \end{aligned}$$

and

$$\begin{aligned} d^{(n)}(i) &= y^{(n)}(i) - \bar{U}_{[1]}(i)\,.d^{(n)}(i+1) - \bar{U}_{[M-1]}(i)\,.d^{(n)}(i+M-1) - \\ &\quad \bar{U}_{[M]}(i)\,.d^{(n)}(i+M) - \bar{U}_{[M+1]}(i)\,.d^{(n)}(i+M+1) \end{aligned}$$

4.3 Rectangular Parallelepiped

The entries of $\bar{L}$ and $\bar{U}$ for the nine-point difference scheme is calculated by the following recurrence relations:

$\bar{L}_{[-MN-M-1]}(i) = A_{[-MN-M-1]}(i)$

$\bar{L}_{[-MN-M-1]}(i)\bar{U}_{[1]}(i-MN-M-1) + \bar{L}_{[-MN-M]}(i) = A_{[-MN-M]}(i)$

$\bar{L}_{[-MN-M]}(i)\bar{U}_{[1]}(i-MN-M) + \bar{L}_{[-MN-M+1]}(i) = A_{[-MN-M+1]}(i)$

$\bar{L}_{[-MN-M-1]}(i)\bar{U}_{[M]}(i-MN-M-1) + \bar{L}_{[-MN-M]}(i)\bar{U}_{[M-1]}(i-MN-M) +$

$\bar{L}_{[-MN-1]}(i) = A_{[-MN-1]}(i)$

$\bar{L}_{[-MN-M-1]}(i)\bar{U}_{[M+1]}(i-MN-M-1) + \bar{L}_{[-MN-M]}(i)\bar{U}_{[M]}(i-MN-M) +$

$\bar{L}_{[-MN-M+1]}(i)\bar{U}_{[M-1]}(i-MN-M+1) + \bar{L}_{[-MN-1]}(i)\bar{U}_{[1]}(i-MN-1) +$

$\bar{L}_{[-MN]}(i) = A_{[-MN]}(i)$

$\bar{L}_{[-MN-M]}(i)\bar{U}_{[M+1]}(i-MN-M) + \bar{L}_{[-MN-M+1]}(i)\bar{U}_{[M]}(i-MN-M+1) +$

$\bar{L}_{[-MN]}(i)\bar{U}_{[1]}(i-MN) + \bar{L}_{[-MN+1]}(i) = A_{[-MN+1]}(i)$

$\bar{L}_{[-MN-1]}(i)\bar{U}_{[M]}(i-MN-1) + \bar{L}_{[-MN]}(i)\bar{U}_{[M-1]}(i-MN) + \bar{L}_{[-MN+M-1]}(i) =$

$A_{[-MN+M-1]}(i)$

$\bar{L}_{[-MN-1]}(i)\bar{U}_{[M+1]}(i-MN-1) + \bar{L}_{[-MN]}(i)\bar{U}_{[M]}(i-MN) +$

$\bar{L}_{[-MN+1]}(i)\bar{U}_{[M-1]}(i-MN+1) + \bar{L}_{[-MN+M-1]}(i)\bar{U}_{[1]}(i-MN+M-1) +$

$\bar{L}_{[-MN+M]}(i) = A_{[-MN+M]}(i)$

$\bar{L}_{[-MN]}(i)\bar{U}_{[M+1]}(i-MN) + \bar{L}_{[-MN+1]}(i)\bar{U}_{[M]}(i-MN+1) +$

$\bar{L}_{[-MN+M]}(i)\bar{U}_{[1]}(i-MN+M) + \bar{L}_{[-MN+M+1]}(i) = A_{[-MN+M+1]}(i)$

$\bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN]}(i-MN-M-1) + \bar{L}_{[-MN-N]}(i)\bar{U}_{[MN-1]}(i-MN-M) +$

$\bar{L}_{[-MN-1]}(i)\bar{U}_{[MN-M]}(i-MN-1) + \bar{L}_{[-MN]}(i)\bar{U}_{[MN-M-1]}(i-MN) +$

$\bar{L}_{[-M-1]}(i) = A_{[-M-1]}(i)$

$\bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN+1]}(i-MN-M-1) + \bar{L}_{[-MN-M]}(i)\bar{U}_{[MN]}(i-MN-M) +$

$\bar{L}_{[-MN-M+1]}(i)\bar{U}_{[MN-1]}(i-MN-M+1) + \bar{L}_{[-MN-1]}(i)\bar{U}_{[MN-M+1]}(i-MN-1) +$

$\bar{L}_{[-MN]}(i)\bar{U}_{[MN-M]}(i-MN)+\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN-M-1]}(i-MN+1)+$

$\bar{L}_{[-M-1]}(i)\bar{U}_{[1]}(i-M-1)+\bar{L}_{[-M]}(i)=A_{[-M]}(i)$

$\bar{L}_{[-MN-M]}(i)\bar{U}_{[MN+1]}(i-MN-M)+\bar{L}_{[-MN-M+1]}(i)\bar{U}_{[MN]}(i-MN-M+1)+$

$\bar{L}_{[-MN]}(i)\bar{U}_{[MN-M+1]}(i-MN)+\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN-M]}(i-MN+1)+$

$\bar{L}_{[-M]}(i)\bar{U}_{[1]}(i-M)+\bar{L}_{[-M+1]}(i)=A_{[-M+1]}(i)$

$\bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN+M]}(i-MN-M-1)+$

$\bar{L}_{[-MN-M]}(i)\bar{U}_{[MN+M-1]}(i-MN-M)+\bar{L}_{[-MN-1]}(i)\bar{U}_{[MN]}(i-MN-1)+$

$\bar{L}_{[-MN]}(i)\bar{U}_{[MN-1]}(i-MN)+\bar{L}_{[-MN+M-1]}(i)\bar{U}_{[MN-M]}(i-MN+M-1)+$

$\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN-M-1]}(i-MN+M)+\bar{L}_{[-M-1]}(i)\bar{U}_{[M]}(i-M-1)+$

$\bar{L}_{[-M]}(i)\bar{U}_{[M-1]}(i-M)+\bar{L}_{[-1]}(i)=$

$A_{[-1]}(i)\ \bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN+M+1]}(i-MN-M-1)+$

$\bar{L}_{[-MN-M]}(i)\bar{U}_{[MN+M]}(i-MN-M)+$

$\bar{L}_{[-MN-M+1]}(i)\bar{U}_{[MN+M-1]}(i-MN-M+1)+$

$\bar{L}_{[-MN-1]}(i)\bar{U}_{[MN+1]}(i-MN-1)+\bar{L}_{[-MN]}(i)\bar{U}_{[MN]}(i-MN)+$

$\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN-1]}(i-MN+1)+\bar{L}_{[-MN+M-1]}(i)\bar{U}_{[MN-M+1]}(i-MN+M-1)+$

$\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN-M]}(i-MN+M)+$

$\bar{L}_{[-MN+M+1]}(i)\bar{U}_{[MN-M-1]}(i-MN+M+1)+$

$\bar{L}_{[-M-1]}(i)\bar{U}_{[M+1]}(i-M-1)+\bar{L}_{[-M]}(i)\bar{U}_{[M]}(i-M)+$

$\bar{L}_{[-M+1]}(i)\bar{U}_{[M-1]}(i-M+1)+\bar{L}_{[-1]}(i)\bar{U}_{[1]}(i-1)+\bar{L}_{[0]}(i)=A_{[0]}(i)$

$\bar{L}_{[-MN-M]}(i)\bar{U}_{[MN+M+1]}(i-MN-M)+$

$\bar{L}_{[-MN-M+1]}(i)\bar{U}_{[MN+M]}(i-MN-M+1)+\bar{L}_{[-MN]}(i)\bar{U}_{[MN+1]}(i-MN)+$

$\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN]}(i-MN+1)+\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN-M+1]}(i-MN+M)+$

$\bar{L}_{[-MN+M+1]}(i)\bar{U}_{[MN-M]}(i-MN+M+1)+$

$\bar{L}_{[-M]}(i)\bar{U}_{[M+1]}(i-M)+\bar{L}_{[-M+1]}(i)\bar{U}_{[M]}(i-M+1)+\bar{L}_{[0]}(i)\bar{U}_{[1]}(i)=A_{[1]}(i)$

$\bar{L}_{[-MN-1]}(i)\bar{U}_{[MN+M]}(i-MN-1)+\bar{L}_{[-MN]}(i)\bar{U}_{[MN+M-1]}(i-MN)+$

$\bar{L}_{[-MN+M-1]}(i)\bar{U}_{[MN]}(i-MN+M-1)+\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN-1]}(i-MN+M)+$

$\bar{L}_{[-1]}(i)\bar{U}_{[M]}(i-1)+\bar{L}_{[0]}(i)\bar{U}_{[M-1]}(i)=A_{[M-1]}(i)$

$\bar{L}_{[-MN-1]}(i)\bar{U}_{[MN+M+1]}(i-MN-1)+\bar{L}_{[-MN]}(i)\bar{U}_{[MN+M]}(i-MN)+$

$\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN+M-1]}(i-MN+1)+\bar{L}_{[-MN+M-1]}(i)\bar{U}_{[MN+1]}(i-MN+M-1)+$

$\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN]}(i-MN+M)+\bar{L}_{[-MN+M+1]}(i)\bar{U}_{[MN-1]}(i-MN+M+1)+$

$\bar{L}_{[-1]}(i)\bar{U}_{[M+1]}(i-1)+\bar{L}_{[0]}(i)\bar{U}_{[M]}(i)=A_{[M]}(i)$

$\bar{L}_{[-MN]}(i)\bar{U}_{[MN+M+1]}(i-MN)+\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN+M]}(i-MN+1)+$

$\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN+1]}(i-MN+M)+\bar{L}_{[-MN+M+1]}(i)\bar{U}_{[MN]}(i-MN+M+1)+$

$\bar{L}_{[0]}(i)\bar{U}_{[M+1]}(i)=A_{[M+1]}(i)$

$\bar{L}_{[-M-1]}(i)\bar{U}_{[MN]}(i-M-1)+\bar{L}_{[-M]}(i)\bar{U}_{[MN-1]}(i-M)+$

$\bar{L}_{[-1]}(i)\bar{U}_{[MN-M]}(i-1)+\bar{L}_{[0]}(i)\bar{U}_{[MN-M-1]}(i)=A_{[MN-M-1]}(i)$

$\bar{L}_{[-M-1]}(i)\bar{U}_{[MN+1]}(i-M-1)+\bar{L}_{[-M]}(i)\bar{U}_{[MN]}(i-M)+$

$\bar{L}_{[-M+1]}(i)\bar{U}_{[MN-1]}(i-M+1)+\bar{L}_{[-1]}(i)\bar{U}_{[MN-M+1]}(i-1)+$

$\bar{L}_{[0]}(i)\bar{U}_{[MN-M]}(i)=A_{[MN-M]}(i)$

$\bar{L}_{[-M]}(i)\bar{U}_{[MN+1]}(i-M)+\bar{L}_{[-M+1]}(i)\bar{U}_{[MN]}(i-M+1)+$

$\bar{L}_{[0]}(i)\bar{U}_{[MN-M+1]}(i)=A_{[MN-M+1]}(i)$

$\bar{L}_{[-M-1]}(i)\bar{U}_{[MN+M]}(i-M-1)+\bar{L}_{[-M]}(i)\bar{U}_{[MN+M-1]}(i-M)+$

$\bar{L}_{[-1]}(i)\bar{U}_{[MN]}(i-1)+\bar{L}_{[0]}(i)\bar{U}_{[MN-1]}(i)=A_{[MN-1]}(i)$

$\bar{L}_{[-M-1]}(i)\bar{U}_{[MN+M+1]}(i-M-1)+\bar{L}_{[-M]}(i)\bar{U}_{[MN+M]}(i-M)+$

$$\bar{L}_{[-M+1]}(i)\,\bar{U}_{[MN+M-1]}(i-M+1)+\bar{L}_{[-1]}(i)\,\bar{U}_{[MN+1]}(i-1)+$$

$$\bar{L}_{[0]}(i)\,\bar{U}_{[MN]}(i)=A_{[MN]}(i)$$

$$\bar{L}_{[-M]}(i)\,\bar{U}_{[MN+M+1]}(i-M)+\bar{L}_{[-M+1]}(i)\,\bar{U}_{[MN+M]}(i-M+1)+$$

$$\bar{L}_{[0]}(i)\,\bar{U}_{[MN+1]}(i)=A_{[MN+1]}(i)$$

$$\bar{L}_{[-1]}(i)\,\bar{U}_{[MN+M]}(i-1)+\bar{L}_{[0]}(i)\,\bar{U}_{[MN+M-1]}(i)=A_{[MN+M-1]}(i)$$

$$\bar{L}_{[-1]}(i)\,\bar{U}_{[MN+M+1]}(i-1)+\bar{L}_{[0]}(i)\,\bar{U}_{[MN+M]}(i)=A_{[MN+M]}(i)$$

$$\bar{L}_{[0]}(i)\,\bar{U}_{[MN+M+1]}(i)=A_{[MN+M+1]}(i)$$

where $\bar{L}_{[M]}(i)$ and $\bar{U}_{[M]}(i)$ are related to the value of i-th row, on diagonals which have a distance M from the main diagonal in matrices $\bar{L}$ and $\bar{U}$, respectively.

Hence, the entries of $\bar{L}$ and $\bar{U}$ can obtain by:

$$\bar{L}_{[-MN-M-1]}(i)=A_{[-MN-M-1]}(i)$$

$$\bar{L}_{[-MN-M]}](i)=A_{[-MN-M]}(i)-\bar{L}_{[-MN-M-1]}(i)\,\bar{U}_{[1]}(i-MN-M-1)$$

$$\bar{L}_{[-MN-M+1]}(i)=A_{[-MN-M+1]}(i)-\bar{L}_{[-MN-M]}(i)\,\bar{U}_{[1]}(i-MN-M)$$

$$\bar{L}_{[-MN-1]}(i)=A_{[-MN-1]}(i)-\bar{L}_{[-MN-M-1]}(i)\,\bar{U}_{[M]}(i-MN-M-1)+$$

$$\bar{L}_{[-MN-M]}(i)\,\bar{U}_{[M-1]}(i-MN-M)\ \bar{L}_{[-MN]}(i)=A_{[-MN]}(i)-$$

$$\bar{L}_{[-MN-M-1]}(i)\,\bar{U}_{[M+1]}(i-MN-M-1)+\bar{L}_{[-MN-M]}(i)\,\bar{U}_{[M]}(i-MN-M)+$$

$$\bar{L}_{[-MN-M+1]}(i)\,\bar{U}_{[M-1]}(i-MN-M+1)+\bar{L}_{[-MN-1]}(i)\,\bar{U}_{[1]}(i-MN-1)$$

$$\bar{L}_{[-MN+1]}(i)=A_{[-MN+1]}(i)-\bar{L}_{[-MN-M]}(i)\,\bar{U}_{[M+1]}(i-MN-M)+$$

$$\bar{L}_{[-MN-M+1]}(i)\,\bar{U}_{[M]}(i-MN-M+1)+\bar{L}_{[-MN]}(i)\,\bar{U}_{[1]}(i-MN)$$

$$\bar{L}_{[-MN+M-1]}(i)=A_{[-MN+M-1]}(i)-\bar{L}_{[-MN-1]}(i)\,\bar{U}_{[M]}(i-MN-1)+$$

$$\bar{L}_{[-MN]}(i)\,\bar{U}_{[M-1]}(i-MN)$$

$$\bar{L}_{[-MN+M]}(i) = A_{[-MN+M]}(i) - \bar{L}_{[-MN-1]}(i)\bar{U}_{[M+1]}(i-MN-1) +$$

$$\bar{L}_{[-MN]}(i)\bar{U}_{[M]}(i-MN) + \bar{L}_{[-MN+1]}(i)\bar{U}_{[M-1]}(i-MN+1) +$$

$$\bar{L}_{[-MN+M-1]}(i)\bar{U}_{[1]}(i-MN+M-1)$$

$$\bar{L}_{[-MN+M+1]}(i) = A_{[-MN+M+1]}(i) - \bar{L}_{[-MN]}(i)\bar{U}_{[M+1]}(i-MN) +$$

$$\bar{L}_{[-MN+1]}(i)\bar{U}_{[M]}(i-MN+1) + \bar{L}_{[-MN+M]}(i)\bar{U}_{[1]}(i-MN+M)$$

$$\bar{L}_{[-M-1]}(i) = A_{[-M-1]}(i) - \bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN]}(i-MN-M-1) +$$

$$\bar{L}_{[-MN-N]}(i)\bar{U}_{[MN-1]}(i-MN-M) + \bar{L}_{[-MN-1]}(i)\bar{U}_{[MN-M]}(i-MN-1) +$$

$$\bar{L}_{[-MN]}(i)\bar{U}_{[MN-M-1]}(i-MN)$$

$$\bar{L}_{[-M]}(i) = A_{[-M]}(i) - \bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN+1]}(i-MN-M-1) +$$

$$\bar{L}_{[-MN-M]}(i)\bar{U}_{[MN]}(i-MN-M) + \bar{L}_{[-MN-M+1]}(i)\bar{U}_{[MN-1]}(i-MN-M+1) +$$

$$\bar{L}_{[-MN-1]}(i)\bar{U}_{[MN-M+1]}(i-MN-1) + \bar{L}_{[-MN]}(i)\bar{U}_{[MN-M]}(i-MN) +$$

$$\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN-M-1]}(i-MN+1) + \bar{L}_{[-M-1]}(i)\bar{U}_{[1]}(i-M-1)$$

$$\bar{L}_{[-M+1]}(i) = A_{[-M+1]}(i) - \bar{L}_{[-MN-M]}(i)\bar{U}_{[MN+1]}(i-MN-M) +$$

$$\bar{L}_{[-MN-M+1]}(i)\bar{U}_{[MN]}(i-MN-M+1) + \bar{L}_{[-MN]}(i)\bar{U}_{[MN-M+1]}(i-MN) +$$

$$\bar{L}_{[-MN+1]}(i)\bar{U}_{[MN-M]}(i-MN+1) + \bar{L}_{[-M]}(i)\bar{U}_{[1]}(i-M)$$

$$\bar{L}_{[-1]}(i) = A_{[-1]}(i) - \bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN+M]}(i-MN-M-1) +$$

$$\bar{L}_{[-MN-M]}(i)\bar{U}_{[MN+M-1]}(i-MN-M) + \bar{L}_{[-MN-1]}(i)\bar{U}_{[MN]}(i-MN-1) +$$

$$\bar{L}_{[-MN]}(i)\bar{U}_{[MN-1]}(i-MN) + \bar{L}_{[-MN+M-1]}(i)\bar{U}_{[MN-M]}(i-MN+M-1) +$$

$$\bar{L}_{[-MN+M]}(i)\bar{U}_{[MN-M-1]}(i-MN+M) + \bar{L}_{[-M-1]}(i)\bar{U}_{[M]}(i-M-1) +$$

$$\bar{L}_{[-M]}(i)\bar{U}_{[M-1]}(i-M)$$

$$\bar{L}_{[0]}(i) = A_{[0]}(i) - \bar{L}_{[-MN-M-1]}(i)\bar{U}_{[MN+M+1]}(i-MN-M-1) +$$

$$\bar{L}_{[-MN-M]}(i)\,\bar{U}_{[MN+M]}(i-MN-M)+$$

$$\bar{L}_{[-MN-M+1]}(i)\,\bar{U}_{[MN+M-1]}(i-MN-M+1)+$$

$$\bar{L}_{[-MN-1]}(i)\,\bar{U}_{[MN+1]}(i-MN-1)+\bar{L}_{[-MN]}(i)\,\bar{U}_{[MN]}(i-MN)+$$

$$\bar{L}_{[-MN+1]}(i)\,\bar{U}_{[MN-1]}(i-MN+1)+\bar{L}_{[-MN+M-1]}(i)\,\bar{U}_{[MN-M+1]}(i-MN+M-1)+$$

$$\bar{L}_{[-MN+M]}(i)\,\bar{U}_{[MN-M]}(i-MN+M)+$$

$$\bar{L}_{[-MN+M+1]}(i)\,\bar{U}_{[MN-M-1]}(i-MN+M+1)+$$

$$\bar{L}_{[-M-1]}(i)\,\bar{U}_{[M+1]}(i-M-1)+\bar{L}_{[-M]}(i)\,\bar{U}_{[M]}(i-M)+$$

$$\bar{L}_{[-M+1]}(i)\,\bar{U}_{[M-1]}(i-M+1)+\bar{L}_{[-1]}(i)\,\bar{U}_{[1]}(i-1)$$

$$\bar{U}_{[1]}(i)=\big(A_{[1]}(i)-\bar{L}_{[-MN-M]}(i)\,\bar{U}_{[MN+M+1]}(i-MN-M)+$$

$$\bar{L}_{[-MN-M+1]}(i)\,\bar{U}_{[MN+M]}(i-MN-M+1)+\bar{L}_{[-MN]}(i)\,\bar{U}_{[MN+1]}(i-MN)+$$

$$\bar{L}_{[-MN+1]}(i)\,\bar{U}_{[MN]}(i-MN+1)+\bar{L}_{[-MN+M]}(i)\,\bar{U}_{[MN-M+1]}(i-MN+M)+$$

$$\bar{L}_{[-MN+M+1]}(i)\,\bar{U}_{[MN-M]}(i-MN+M+1)+\bar{L}_{[-M]}(i)\,\bar{U}_{[M+1]}(i-M)+$$

$$\bar{L}_{[-M+1]}(i)\,\bar{U}_{[M]}(i-M+1)\big)\,/\,\bar{L}_{[0]}(i)$$

$$\bar{U}_{[M-1]}(i)=\big(A_{[M-1]}(i)-\bar{L}_{[-MN-1]}(i)\,\bar{U}_{[MN+M]}(i-MN-1)+$$

$$\bar{L}_{[-MN]}(i)\,\bar{U}_{[MN+M-1]}(i-MN)+\bar{L}_{[-MN+M-1]}(i)\,\bar{U}_{[MN]}(i-MN+M-1)+$$

$$\bar{L}_{[-MN+M]}(i)\,\bar{U}_{[MN-1]}(i-MN+M)+\bar{L}_{[-1]}(i)\,\bar{U}_{[M]}(i-1)\big)\,/\bar{L}_{[0]}(i)$$

$$\bar{U}_{[M]}(i)=\big(A_{[M]}(i)-\bar{L}_{[-MN-1]}(i)\,\bar{U}_{[MN+M+1]}(i-MN-1)+$$

$$\bar{L}_{[-MN]}(i)\,\bar{U}_{[MN+M]}(i-MN)+\bar{L}_{[-MN+1]}(i)\,\bar{U}_{[MN+M-1]}(i-MN+1)+$$

$$\bar{L}_{[-MN+M-1]}(i)\,\bar{U}_{[MN+1]}(i-MN+M-1)+\bar{L}_{[-MN+M]}(i)\,\bar{U}_{[MN]}(i-MN+M)+$$

$$\bar{L}_{[-MN+M+1]}(i)\,\bar{U}_{[MN-1]}(i-MN+M+1)+\bar{L}_{[-1]}(i)\,\bar{U}_{[M+1]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$$

$$\bar{U}_{[M+1]}(i)=\big(A_{[M+1]}(i)-\bar{L}_{[-MN]}(i)\,\bar{U}_{[MN+M+1]}(i-MN)+$$

$$\bar{L}_{[-MN+1]}(i)\,\bar{U}_{[MN+M]}(i-MN+1)+\bar{L}_{[-MN+M]}(i)\,\bar{U}_{[MN+1]}(i-MN+M)+$$

$\bar{L}_{[-MN+M+1]}(i)\,\bar{U}_{[MN]}(i-MN+M+1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN-M-1]}(i) = \big(A_{[MN-M-1]}(i) - \bar{L}_{[-M-1]}(i)\,\bar{U}_{[MN]}(i-M-1) +$

$\bar{L}_{[-M]}(i)\,\bar{U}_{[MN-1]}(i-M) + \bar{L}_{[-1]}(i)\,\bar{U}_{[MN-M]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN-M]}(i) = \big(A_{[MN-M]}(i) - \bar{L}_{[-M-1]}(i)\,\bar{U}_{[MN+1]}(i-M-1) +$

$\bar{L}_{[-M]}(i)\,\bar{U}_{[MN]}(i-M) + \bar{L}_{[-M+1]}(i)\,\bar{U}_{[MN-1]}(i-M+1) +$

$\bar{L}_{[-1]}(i)\,\bar{U}_{[MN-M+1]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN-M+1]}(i) = \big(A_{[MN-M+1]}(i) - \bar{L}_{[-M]}(i)\,\bar{U}_{[MN+1]}(i-M) +$

$\bar{L}_{[-M+1]}(i)\,\bar{U}_{[MN]}(i-M+1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN-1]}(i) = \big(A_{[MN-1]}(i) - \bar{L}_{[-M-1]}(i)\,\bar{U}_{[MN+M]}(i-M-1) +$

$\bar{L}_{[-M]}(i)\,\bar{U}_{[MN+M-1]}(i-M) + \bar{L}_{[-1]}(i)\,\bar{U}_{[MN]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN]}(i) = \big(A_{[MN]}(i) - \bar{L}_{[-M-1]}(i)\,\bar{U}_{[MN+M+1]}(i-M-1) +$

$\bar{L}_{[-M]}(i)\,\bar{U}_{[MN+M]}(i-M) + \bar{L}_{[-M+1]}(i)\,\bar{U}_{[MN+M-1]}(i-M+1) +$

$\bar{L}_{[-1]}(i)\,\bar{U}_{[MN+1]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN+1]}(i) = \big(A_{[MN+1]}(i) - \bar{U}_{[MN+M+1]}(i-M) +$

$\bar{L}_{[-M+1]}(i)\,\bar{U}_{[MN+M]}(i-M+1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN+M-1]}(i) = \big(A_{[MN+M-1]}(i) - \bar{L}_{[-1]}(i)\,\bar{U}_{[MN+M]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN+M]}(i) = \big(A_{[MN+M]}(i) - \bar{L}_{[-1]}(i)\,\bar{U}_{[MN+M+1]}(i-1)\big)\,/\,\bar{L}_{[0]}(i)$

$\bar{U}_{[MN+M+1]}(i) = A_{[MN+M+1]}(i)\,/\,\bar{L}_{[0]}(i)$

Next step is the iterative procedure. It means the forward and backward substiution to find the solution of $y^{(n)}$ in (4.1.2) and $d^{(n)}$ in (4.1.3) which directly $y^{(n)}(i)$ and $d^{(n)}(i)$ ($y^{(n)}(i)$ and $d^{(n)}(i)$ related to point i) is computed by following:

$$y^{(n)}(i) = \Big(R^{(n)}(i) - L_{[-MN-M-1]}(i).y^{(n)}(i-MN-M-1) -$$
$$L_{[-MN-M]}(i).y^{(n)}(i-MN-M) - L_{[-MN-M+1]}(i).y^{(n)}(i-MN-M+1) -$$
$$L_{[-MN-1]}(i).y^{(n)}(i-MN-1) - L_{[-MN]}(i).y^{(n)}(i-MN) -$$
$$L_{[-MN+1]}(i).y^{(n)}(i-MN+1) - L_{[-1]}(i).y^{(n)}(i-1) -$$
$$L_{[-MN+M-1]}(i).y^{(n)}(i-MN+M-1) - L_{[-MN+M]}(i).y^{(n)}(i-MN+M) -$$
$$L_{[-MN+M+1]}(i).y^{(n)}(i-MN+M+1) - L_{[-M-1]}(i).y^{(n)}(i-M-1) -$$
$$L_{[-M]}(i).y^{(n)}(i-M) - L_{[-M+1]}(i).y^{(n)}(i-M+1) - L_{[-1]}(i).y^{(n)}(i-1)\Big) \,/\, L_{[0]}(i)$$

and

$$d^{(n)}(i) = y^{(n)}(i) - \bar{U}_{[1]}(i).d^{(n)}(i+1) - \bar{U}_{[M-1]}(i).d^{(n)}(i+M-1) -$$
$$\bar{U}_{[M]}(i).d^{(n)}(i+M) - \bar{U}_{[M+1]}(i).d^{(n)}(i+M+1) -$$
$$\bar{U}_{[MN-M-1]}(i).d^{(n)}(i+MN-M-1) - \bar{U}_{[MN-M]}(i).d^{(n)}(i+MN-M) -$$
$$\bar{U}_{[MN-M+1]}(i).d^{(n)}(i+MN-M+1) - \bar{U}_{[MM-1]}(i).d^{(n)}(i+M-1) -$$
$$\bar{U}_{[MN]}(i).d^{(n)}(i+MN) - \bar{U}_{[MN+1]}(i).d^{(n)}(i+MN+1) -$$
$$\bar{U}_{[MN+M-1]}(i).d^{(n)}(i+MN+M-1) - \bar{U}_{[MN+M]}(i).d^{(n)}(i+MN+M) -$$
$$\bar{U}_{[MN+M+1]}(i).d^{(n)}(i+MN+M+1)$$

4.4 Numerical Examples

In the following it support the theorical part by numerical results are obtained in a rectangle and rectangular parallepiped by using incomplete LU decomposition. The results in each domain has three part:

- The approximate results for the solution of the Dirichlet problem of Laplace's equation

- The approximate results for the first derivative of the solution

- The approximate results for the pure second derivative of the solution

The grid spacing (difference step size) h is defined by $h = \frac{1}{2^n}$, $n = 3, 4, \ldots, 7$.

4.4.1 Domain in the Shape of a Rectangle

Let $\Pi = \{(x,y) : -1 < x < 1, 0 < y < 1\}$, and let γ be the boundary of Π. We consider the following problem:

$$\Delta u = 0 \text{ on } \Pi, \ u = \phi(x,y) \text{ on } \gamma_j, \ j = 1,2,3,4, \qquad (4.4.1)$$

where ϕ is the exact solution of this problem.

Let U be the exact solution and U_h be its approximate values on $\overline{\Pi}^h$ (which contains the nodes using on the square grids formed on Π) of the Dirichlet problem on the rectangular domain Π. We denote by $\|U - U_h\|_{\overline{\Pi}^h} = \max_{\overline{\Pi}^h} |U - U_h|$, $\Re_U^m = \frac{\|U - U_{2^{-m}}\|_{\overline{\Pi}^h}}{\left\|U - U_{2^{-(m+1)}}\right\|_{\overline{\Pi}^h}}$.

4.4.1.1 Fourth Order Accurate Forward and Backward Formulae

In the following examples the results are demonstrated in three tables. The first table is related to the approximate of problem (4.4.1) , the second and third tables is corresponds to the approximate values of $\upsilon = \frac{\partial u}{\partial x}$, $\omega = \frac{\partial^2 u}{\partial x^2}$, respectively. For instance in the first example,Table (4.1) it shows the approximateion of problem (4.4.1) and in Figure (4.5) the graphs show the approximate and exact solution. Table (4.2) and Table (4.3) show the approximate solutions which, converges as $O(h^4)$. Also in support of the numerical part the shapes of $\upsilon = \frac{\partial u}{\partial x}$, $\omega = \frac{\partial^2 u}{\partial x^2}$ and their approximations are shown in Figure (4.6) and Figure (4.7), respectively.

These results are obtained for different boundary functions which are given below.

Example: Let $\phi \in C^{6,\frac{1}{30}}$ on γ_j, $j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{181}{60}} \cos\left(\frac{181}{30} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.2}$$

Table 4.1. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{30}}$.

h	$\|u-u_h\|$	$\mathfrak{R}_U^m$
$\frac{1}{8}$	$1.46195706270007758884 6E-8$	62.01
$\frac{1}{16}$	$2.3576031502310490115 33E-10$	63.77
$\frac{1}{32}$	$3.6969565337572363887 34E-12$	64.63
$\frac{1}{64}$	$5.7204188708771637862 17E-14$	65.06
$\frac{1}{128}$	$8.7926871761968870580 66E-16$	

Table 4.2. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{30}}$.

h	$\|\upsilon-\upsilon_h\|$	$\mathfrak{R}_U^m$
$\frac{1}{8}$	$2.2999960647643250096 57E-2$	12.14
$\frac{1}{16}$	$1.8940591045681605251 04E-3$	14.08
$\frac{1}{32}$	$1.3448807937014745537 83E-4$	15.01
$\frac{1}{64}$	$8.9606632499776449869 27E-6$	15.46
$\frac{1}{128}$	$5.7963938638735426927 74E-7$	

Table 4.3. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{30}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$3.149059928597543772878E-6$	16.31
$\frac{1}{16}$	$1.931058119052719414451E-7$	16.36
$\frac{1}{32}$	$1.180485369727342048019E-8$	16.37
$\frac{1}{64}$	$7.211217140499053022025E-10$	16.37
$\frac{1}{128}$	$4.404326492162507264392E-11$	

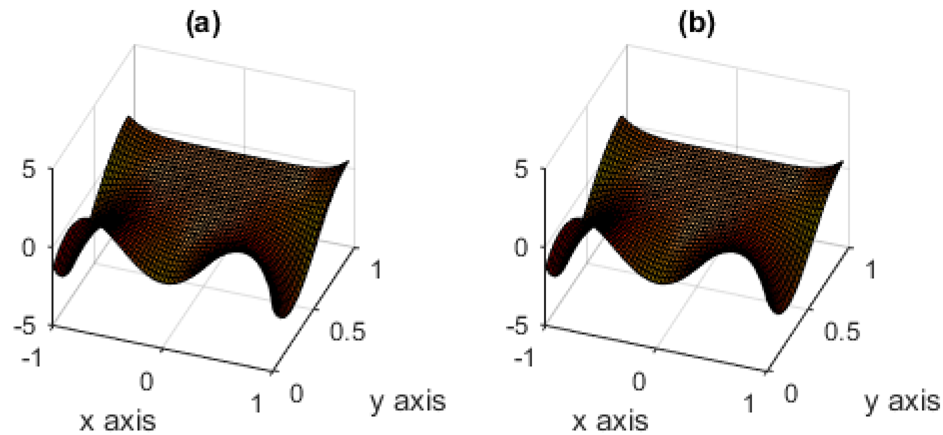

Figure 4.5. The graph of the approximate (a) and exact (b) solutions of u

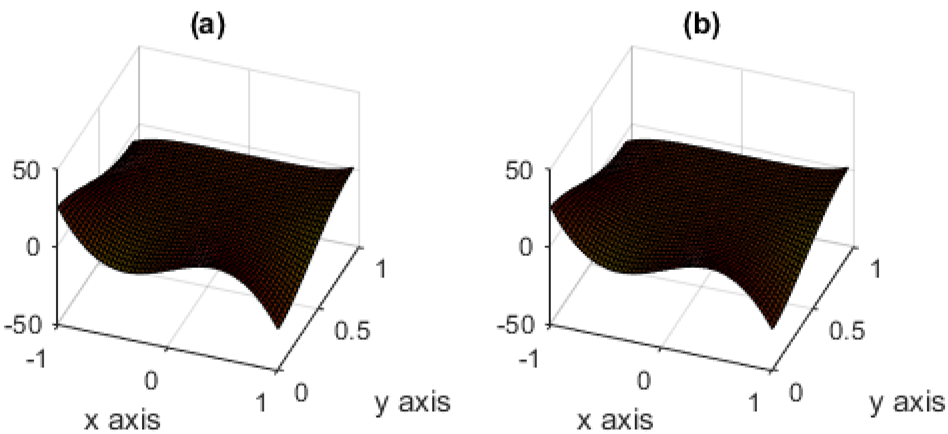

Figure 4.6. The graph of the approximate (a) and exact (b) solutions of $\frac{\partial u}{\partial x}$

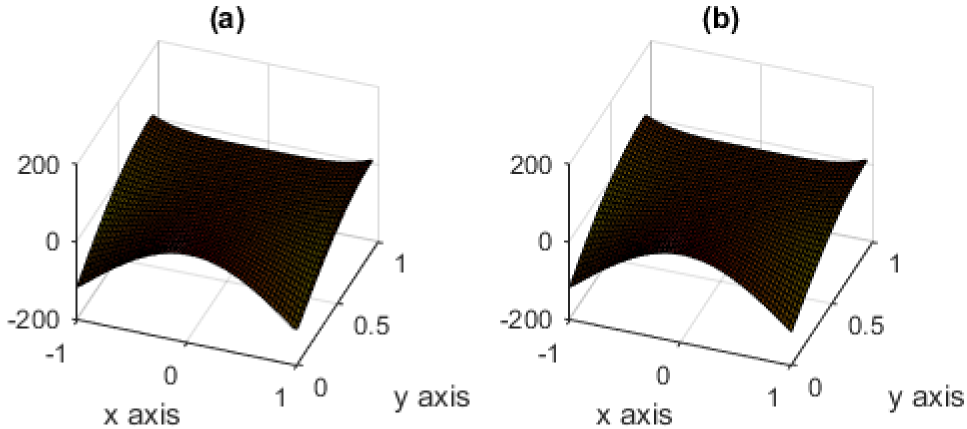

Figure 4.7. The graph of the approximate (a) and exact (b) solutions of $\frac{\partial^2 u}{\partial x^2}$

Example: Let $\phi \in C^{6,\frac{1}{20}}$, on γ_j, $j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{121}{40}} \cos\left(\frac{121}{20}\arctan\left(\frac{y}{x}\right)\right) \tag{4.4.3}$$

Table 4.4. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{20}}$.

h	$\|u-u_h\|$	$\Re_U^m$
$\frac{1}{8}$	$2.18531147375844958471 7E-8$	62.61
$\frac{1}{16}$	$3.49013493791441333999 4E-10$	64.44
$\frac{1}{32}$	$5.41648602370275030068 4E-12$	65.33
$\frac{1}{64}$	$8.29104132736262396623 1E-14$	65.78
$\frac{1}{128}$	$1.26035253998100098673 4E-15$	

Table 4.5. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{20}}$.

h	$\|\upsilon - \upsilon_h\|$	$\Re_U^m$
$\frac{1}{8}$	$2.4028248698353258208 62E-2$	12.12
$\frac{1}{16}$	$1.9820994952891614154 12E-3$	14.04
$\frac{1}{32}$	$1.4113353729303720446 43E-4$	14.97
$\frac{1}{64}$	$9.4305803248559583622 78E-6$	15.42
$\frac{1}{128}$	$6.1171589757028339447 66E-7$	

Table 4.6. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{20}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$4.5961165004043514098 68E-6$	16.50
$\frac{1}{16}$	$2.7859788135611530891 14E-7$	16.55
$\frac{1}{32}$	$1.6835370108548073316 51E-8$	16.56
$\frac{1}{64}$	$1.0166057199067431046 51E-9$	16.56
$\frac{1}{128}$	$6.1377058145400742347 58E-11$	

Example: Let $\phi \in C^{6,\frac{1}{10}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{61}{20}} \cos\left(\frac{61}{10} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.4}$$

Table 4.7. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{10}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$4.29246425222476886387 2E-8$	64.440
$\frac{1}{16}$	$6.66123514571846886493 8E-10$	66.282
$\frac{1}{32}$	$1.00499223436588179197 6E-11$	64.763
$\frac{1}{64}$	$1.55176623826828667166 7E-13$	63.789
$\frac{1}{128}$	$2.43267403214279385770 4E-15$	

Table 4.8. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{10}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$2.71791841997341171555 6E-2$	12.048
$\frac{1}{16}$	$2.25585824529134288750 5E-3$	13.931
$\frac{1}{32}$	$1.61930147136838917460 0E-4$	14.842
$\frac{1}{64}$	$1.09099338488883106474 6E-5$	15.313
$\frac{1}{128}$	$7.12476392627632804299 7E-7$	

Table 4.9. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{10}}$.

h	$\|\|\omega-\omega_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$8.39420634662260834209 6E-6$	17.080
$\frac{1}{16}$	$4.91457207963591269575 1E-7$	17.132
$\frac{1}{32}$	$2.86859874883613209844 6E-8$	17.144
$\frac{1}{64}$	$1.67319101012914715620 0E-9$	17.147
$\frac{1}{128}$	$9.75768555544362974164 2E-11$	

Example: Let $\phi \in C^{6,\frac{1}{4}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{25}{8}} \cos\left(\frac{25}{4}\arctan\left(\frac{y}{x}\right)\right) \tag{4.4.5}$$

Table 4.10. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{4}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$9.46866147925862623807 0E-8$	63.652
$\frac{1}{16}$	$1.48756598661322027926 4E-9$	63.638
$\frac{1}{32}$	$2.33751090836644478681 8E-11$	63.989
$\frac{1}{64}$	$3.65295136905845384663 3E-13$	63.971
$\frac{1}{128}$	$5.71026503194339390433 3E-15$	

Table 4.11. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{4}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$3.62151093511044239967 3E-2$	11.825
$\frac{1}{16}$	$3.06257782748510623581 5E-3$	13.611
$\frac{1}{32}$	$2.25011567831133410524 5E-4$	14.561
$\frac{1}{64}$	$1.54531622102049992068 2E-5$	15.096
$\frac{1}{128}$	$1.02363564279643188540 6E-6$	

Table 4.12. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{4}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.4728546193358705552 55E-5$	18.954
$\frac{1}{16}$	$7.7704855623604968158 62E-7$	19.010
$\frac{1}{32}$	$4.0874760595758206431 23E-8$	19.023
$\frac{1}{64}$	$2.1486690066687345195 96E-9$	19.026
$\frac{1}{128}$	$1.1293130666102268230 51E-10$	

Example: Let $\phi \in C^{6,\frac{1}{2}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{13}{4}} \cos\left(\frac{13}{2}\arctan\left(\frac{y}{x}\right)\right) \tag{4.4.6}$$

Table 4.13. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{2}}$.

h	$\|u - u_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.5228381902172219331 12E-7$	64.007
$\frac{1}{16}$	$2.3790929888702748102 70E-9$	63.769
$\frac{1}{32}$	$3.7308188774579812888 76E-11$	63.919
$\frac{1}{64}$	$5.8368348196221806775 42E-13$	63.975
$\frac{1}{128}$	$9.1234908247776188156 64E-15$	

Table 4.14. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{2}}$.

h	$\|\upsilon - \upsilon_h\|$	$\Re_U^m$
$\frac{1}{8}$	$4.37570952962073832499 1E-2$	11.42
$\frac{1}{16}$	$3.83240792253039900838 2E-3$	13.18
$\frac{1}{32}$	$2.90755703923332125442 7E-4$	14.163
$\frac{1}{64}$	$2.05290592943303156432 9E-5$	14.810
$\frac{1}{128}$	$1.38620430428717591093 1E-6$	

Table 4.15. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{2}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.18012437576371009772 3E-5$	22.539
$\frac{1}{16}$	$5.23571985861761315409 7E-7$	22.608
$\frac{1}{32}$	$2.31589087494236386484 3E-8$	22.623
$\frac{1}{64}$	$1.02369375059681868933 3E-9$	22.627
$\frac{1}{128}$	$4.52434336875006269111 7E-11$	

Example: Let $\phi \in C^{6,\frac{3}{4}}$, on γ_j, $j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{27}{8}} \cos\left(\frac{27}{4} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.7}$$

Table 4.16. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{3}{4}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$1.19200692438296186277 2E-7$	63.338
$\frac{1}{16}$	$1.8805886587748656041 65E-9$	63.764
$\frac{1}{32}$	$2.9492608307537960707 19E-11$	64.001
$\frac{1}{64}$	$4.6081525073184818858 52E-13$	63.977
$\frac{1}{128}$	$7.2028791313974493623 47E-15$	

Table 4.17. The approximate results for the first derivative when $\phi \in C^{6,\frac{3}{4}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$3.7023859762194513214 86E-2$	10.073
$\frac{1}{16}$	$3.6757226373512477010 79E-3$	12.899
$\frac{1}{32}$	$2.8496079389513554195 49E-4$	14.354
$\frac{1}{64}$	$1.9852045543650598335 81E-5$	15.106
$\frac{1}{128}$	$1.3142029913518935052 03E-6$	

Table 4.18. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{3}{4}}$.

h	$\|\|\omega-\omega_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$3.4852360702568963129 09E-6$	26.787
$\frac{1}{16}$	$1.3011058287666364467 75E-7$	26.882
$\frac{1}{32}$	$4.8402203344832000940 82E-9$	26.902
$\frac{1}{64}$	$1.7991805379598754908 03E-10$	26.908
$\frac{1}{128}$	$6.6866087501084466799 55E-12$	

Example: Let $\phi \in C^{6,\frac{9}{10}}$, on γ_j, $j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{69}{20}} \cos\left(\frac{69}{10} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.8}$$

Table 4.19. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{9}{10}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$5.305772553320321766766E-8$	63.986
$\frac{1}{16}$	$8.292059050053952313561E-10$	63.879
$\frac{1}{32}$	$1.298108357990300947143E-11$	63.946
$\frac{1}{64}$	$2.029993411923961018394E-13$	63.967
$\frac{1}{128}$	$3.173490528879122759159E-15$	

Table 4.20. The approximate results for the first derivative when $\phi \in C^{6,\frac{9}{10}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$4.003831300894382179437E-2$	9.547
$\frac{1}{16}$	$4.193649547657526600890E-3$	12.673
$\frac{1}{32}$	$3.308965269815573493929E-4$	14.218
$\frac{1}{64}$	$2.327302763652494792622E-5$	15.026
$\frac{1}{128}$	$1.548884402678707364254E-6$	

Table 4.21. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{9}{10}}$.

h	$\|\omega-\omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$9.668117711892401268900E-7$	29.846
$\frac{1}{16}$	$3.239307807806365124242E-8$	29.825
$\frac{1}{32}$	$1.086123915091408323509E-9$	29.850
$\frac{1}{64}$	$3.638534025142625161660E-11$	29.856
$\frac{1}{128}$	$1.218707932851867080331E-12$	

4.4.1.2 Sixth Order Accurate Forward and Backward Formulae

In the following examples we used forward and backward formulae for sixth order accuracy to find a new boundry values on the sides when $x=-1$ and $x=1$ for the first derivative problem.

Example: Let $\phi \in C^{6,\frac{1}{10}}$, on γ_j, $j=1,2,3,4$, where

$$\phi(x,y)=(x^2+y^2)^{\frac{61}{20}}\cos\left(\frac{61}{10}\arctan\left(\frac{y}{x}\right)\right) \tag{4.4.9}$$

Table 4.22. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{10}}$.

h	$\|u-u_h\|$	$\frac{\varepsilon^i}{\varepsilon^{i+1}}$
$\frac{1}{8}$	$4.292464252224768863872E-8$	64.440
$\frac{1}{16}$	$6.661235145718468864938E-10$	66.282
$\frac{1}{32}$	$1.004992234365881791976E-11$	64.763
$\frac{1}{64}$	$1.551766238268286671667E-13$	63.789
$\frac{1}{128}$	$2.432674032142793857704E-15$	

Table 4.23. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{10}}$.

h	$\|\upsilon - \upsilon_h\|$	$\Re_U^m$
$\frac{1}{8}$	$4.04555152255190072900 8E-5$	58.806
$\frac{1}{16}$	$6.87938359170739905402 6E-7$	61.467
$\frac{1}{32}$	$1.11919733792779277613 7E-8$	62.356
$\frac{1}{64}$	$1.79484405952878452935 0E-10$	62.852
$\frac{1}{128}$	$2.85567885567237596872 5E-12$	

Table 4.24. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{10}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$8.39420634662260834209 6E-6$	17.080
$\frac{1}{16}$	$4.91457207963591269575 1E-7$	17.132
$\frac{1}{32}$	$2.86859874883613209844 6E-8$	17.144
$\frac{1}{64}$	$1.67319101012914715620 0E-9$	17.147
$\frac{1}{128}$	$9.75768555544362974164 2E-11$	

Example: Let $\phi \in C^{6,\frac{1}{4}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{25}{8}} \cos\left(\frac{25}{4}\arctan\left(\frac{y}{x}\right)\right) \tag{4.4.10}$$

Table 4.25. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{4}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$9.46866147925862623807 0E-8$	63.652
$\frac{1}{16}$	$1.48756598661322027926 4E-9$	63.638
$\frac{1}{32}$	$2.33751090836644478681 8E-11$	63.989
$\frac{1}{64}$	$3.65295136905845384663 3E-13$	63.971
$\frac{1}{128}$	$5.71026503194339390433 3E-15$	

Table 4.26. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{4}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$1.41156751962072999236 0E-4$	58.960
$\frac{1}{16}$	$2.39409774858750317810 3E-6$	61.272
$\frac{1}{32}$	$3.90735531587028191221 3E-8$	62.283
$\frac{1}{64}$	$6.27353353007139854128 0E-10$	62.856
$\frac{1}{128}$	$9.98079017884346982753 5E-12$	

Table 4.27. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{4}}$.

h	$\|\|\omega-\omega_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$1.47285461933587055525 5E-5$	18.954
$\frac{1}{16}$	$7.77048556236049681586 2E-7$	19.010
$\frac{1}{32}$	$4.08747605957582064312 3E-8$	19.023
$\frac{1}{64}$	$2.14866900666873451959 6E-9$	19.026
$\frac{1}{128}$	$1.12931306661022682305 1E-10$	

Example: Let $\phi \in C^{6,\frac{1}{2}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{13}{4}} \cos\left(\frac{13}{2}\arctan\left(\frac{y}{x}\right)\right) \tag{4.4.11}$$

Table 4.28. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{1}{2}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$1.5228381902172219331 12E-7$	64.007
$\frac{1}{16}$	$2.3790929888702748102 70E-9$	63.769
$\frac{1}{32}$	$3.7308188774579812888 76E-11$	63.919
$\frac{1}{64}$	$5.8368348196221806775 42E-13$	63.975
$\frac{1}{128}$	$9.1234908247776188156 64E-15$	

Table 4.29. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{2}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$4.6078032182943783652 26E-4$	58.261
$\frac{1}{16}$	$7.9088918765852166760 46E-6$	60.908
$\frac{1}{32}$	$1.2985066441377258880 31E-7$	62.223
$\frac{1}{64}$	$2.0868480232968987041 92E-9$	62.881
$\frac{1}{128}$	$3.3187484630181058653 45E-11$	

Table 4.30. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{2}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.18012437576371009772 3E-5$	22.539
$\frac{1}{16}$	$5.23571985861761315409 7E-7$	22.608
$\frac{1}{32}$	$2.31589087494236386484 3E-8$	22.623
$\frac{1}{64}$	$1.02369375059681868933 3E-9$	22.627
$\frac{1}{128}$	$4.52434336875006269111 7E-11$	

Example: Let $\phi \in C^{6,\frac{3}{4}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{27}{8}} \cos\left(\frac{27}{4} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.12}$$

Table 4.31. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{3}{4}}$.

h	$\|u - u_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.19200692438296186277 2E-7$	63.338
$\frac{1}{16}$	$1.88058865877486560416 5E-9$	63.764
$\frac{1}{32}$	$2.94926083075379607071 9E-11$	64.001
$\frac{1}{64}$	$4.60815250731848188585 2E-13$	63.977
$\frac{1}{128}$	$7.20287913139744936234 7E-15$	

Table 4.32. The approximate results for the first derivative when $\phi \in C^{6,\frac{3}{4}}$.

h	$\|\upsilon - \upsilon_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.05867934577977077814E-3$	56.715
$\frac{1}{16}$	$1.86666248292406604317E-5$	60.516
$\frac{1}{32}$	$3.08458347379907154469E-7$	62.149
$\frac{1}{64}$	$4.96323126186465492331E-9$	62.939
$\frac{1}{128}$	$7.88579223896987444657E-11$	

Table 4.33. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{3}{4}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$3.48523607025689631290E-6$	26.787
$\frac{1}{16}$	$1.30110582876663644677E-7$	26.882
$\frac{1}{32}$	$4.84022033448320009408E-9$	26.902
$\frac{1}{64}$	$1.79918053795987549080E-10$	26.908
$\frac{1}{128}$	$6.68660875010844667995E-12$	

Example: Let $\phi \in C^{6,\frac{9}{10}}$, on $\gamma_j, j = 1,2,3,4$, where

$$\phi(x,y) = (x^2+y^2)^{\frac{69}{20}} \cos\left(\frac{69}{10} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.13}$$

Table 4.34. The approximate of solution in problem (4.4.1) when the boundary function is in $C^{6,\frac{9}{10}}$.

h	$\|u - u_h\|$	$\Re_U^m$
$\frac{1}{8}$	$5.30577255332032176 6766E-8$	63.986
$\frac{1}{16}$	$8.29205905005395231 3561E-10$	63.879
$\frac{1}{32}$	$1.29810835799030094 7143E-11$	63.946
$\frac{1}{64}$	$2.02999341192396101 8394E-13$	63.967
$\frac{1}{128}$	$3.17349052887912275 9159E-15$	

Table 4.35. The approximate results for the first derivative when $\phi \in C^{6,\frac{9}{10}}$.

h	$\|\upsilon - \upsilon_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.61229382162723752 5847E-3$	55.941
$\frac{1}{16}$	$2.88214609690200192 8266E-5$	60.124
$\frac{1}{32}$	$4.79369452959413367 4179E-7$	62.088
$\frac{1}{64}$	$7.72079638080588915 2056E-9$	62.998
$\frac{1}{128}$	$1.22557078160685936 3419E-10$	

Table 4.36. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{9}{10}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$9.66811771189240126 8900E-7$	29.846
$\frac{1}{16}$	$3.23930780780636512 4242E-8$	29.825
$\frac{1}{32}$	$1.08612391509140832 3509E-9$	29.850
$\frac{1}{64}$	$3.63853402514262516 1660E-11$	29.856
$\frac{1}{128}$	$1.21870793285186708 0331E-12$	

4.4.2 Domain in the Shape of a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3) : 0 < x_1 < 1, 0 < x_2 < 1, 0 < x_3 < 0.5, i = 1,2,3\}$, and let Γ be the boundary of R. We consider the following boundary-value problem:

$$\Delta u = 0 \text{ on } R, \ u = \varphi(x,y,z) \text{ on } \Gamma_j, \ j = 1,2,\ldots,6, \qquad (4.4.14)$$

where φ is the exact solution of this problem.

Let U denote the exact solution and U_h be its approximate values on $\overline{R}^h$ (contains the nodes of the cubic grid formed in R) of the Dirichlet problem for laplace's equation on the rectangular parallelepiped domain R. We denote by $||U - U_h||_{\overline{R}^h} = \max_{\overline{R}^h} |U - U_h|$, $\Re_U^m = \frac{||U - U_{2^{-m}}||_{\overline{R}^h}}{\left\|U - U_{2^{-(m+1)}}\right\|_{\overline{R}^h}}$.

In Table (4.37), the approximate results for the solution of the Dirichlet problem for the Laplace's equation are presented. Table (4.38) shows the maximum errors and convergence order of the first derivative when $4-th$ order accuracy forward backward formula is used, and in Table (4.39), the maximum errors and the convergence order of the approximations of the pure second derivatives of problem (4.4.14) for different step size h are presented.

4.4.2.1 Fourth Order Accurate Forward and Backward Formulae

In the following examples forward and backward formulae is used for fourth order accuracy to find a new boundary values on faces when $x = -1$ or $x = 1$ for the first derivative problem.

The results show that the approximate solutions converge as $O(h^4)$.

Example: Let $\phi \in C^{6,\frac{1}{30}}$, on Γ_j, $j = 1,2,\ldots,6$, where

$$\phi(x,y) = ((z-\frac{1}{2})^2 - \frac{(x^2+y^2)}{2}) + (x^2+y^2)^{\frac{181}{60}} \cos\left(\frac{181}{30} \arctan\left(\frac{y}{x}\right)\right) \tag{4.4.15}$$

Table 4.37. The approximate of solution in problem (4.4.14) when the boundary function is in $C^{6,\frac{1}{30}}$.

h	$\|u-u_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.364190306380006E-9$	54.95
$\frac{1}{16}$	$2.482778220153542E-11$	62.64
$\frac{1}{32}$	$3.963714104714007E-13$	63.14
$\frac{1}{64}$	$6.277272419478622E-15$	63.77
$\frac{1}{128}$	$9.843687691215732E-17$	

Table 4.38. The approximate results for the first derivative when $\phi \in C^{6,\frac{1}{30}}$.

h	$\|\upsilon-\upsilon_h\|$	$\Re_U^m$
$\frac{1}{8}$	$1.499307742596606E-2$	9.78
$\frac{1}{16}$	$1.532700715023690E-3$	12.93
$\frac{1}{32}$	$1.185409560760095E-4$	14.50
$\frac{1}{64}$	$8.177080034789001E-6$	15.25
$\frac{1}{128}$	$5.360495371756569E-7$	

Table 4.39. The approximate results for the pure second derivative when $\phi \in C^{6,\frac{1}{30}}$.

h	$\|\omega - \omega_h\|$	$\Re_U^m$
$\frac{1}{8}$	$9.824337972038735E-7$	15.21
$\frac{1}{16}$	$6.458728915909570E-8$	16.21
$\frac{1}{32}$	$3.985011206044048E-9$	16.36
$\frac{1}{64}$	$2.436150564128134E-10$	16.37
$\frac{1}{128}$	$1.487939094224197E-11$	

4.4.2.2 Fifth Order Accurate Forward and Backward Formulae

In the following examples it used forward and backward formulae are used for sixth order accuracy to find new boundary values on the faces when $x = 0$ and $x = 1$ for the first derivative problem.

The results shows that the approximate solutions converge as $O(h^5)$.

Example: Let $\phi \in C^{5,\frac{1}{30}}$, on Γ_j, $j = 1, 2, \ldots, 6$, where

$$\phi(x,y) = ((z - \frac{1}{2})^2 - \frac{(x^2 + y^2)}{2}) + (x^2 + y^2)^{\frac{151}{60}} \cos\left(\frac{151}{30} \arctan\left(\frac{y}{x}\right)\right) \qquad (4.4.16)$$

Table 4.40. The approximate of solution in problem (4.4.14) when the boundary function is in $C^{5,\frac{1}{30}}$.

h	$\|\|u-u_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$1.364190306380006E-9$	54.95
$\frac{1}{16}$	$2.482778220153542E-11$	62.64
$\frac{1}{32}$	$3.963714104714007E-13$	63.14
$\frac{1}{64}$	$6.277272419478622E-15$	63.77
$\frac{1}{128}$	$9.843687691215732E-17$	

Table 4.41. The approximate results for the first derivative when $\phi \in C^{5,\frac{1}{30}}$.

h	$\|\|\upsilon-\upsilon_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$2.046960510985336E-3$	22.08
$\frac{1}{16}$	$9.272500731110548E-5$	27.35
$\frac{1}{32}$	$3.390258952082138E-6$	29.78
$\frac{1}{64}$	$1.138244315600217E-7$	30.91
$\frac{1}{128}$	$3.682339328876473E-9$	

Table 4.42. The approximate results for the pure second derivative when $\phi \in C^{5,\frac{1}{30}}$.

h	$\|\|\omega-\omega_h\|\|$	$\Re_U^m$
$\frac{1}{8}$	$9.824337972038735E-7$	15.21
$\frac{1}{16}$	$6.458728915909570E-8$	16.21
$\frac{1}{32}$	$3.985011206044048E-9$	16.36
$\frac{1}{64}$	$2.436150564128134E-10$	16.37
$\frac{1}{128}$	$1.487939094224197E-11$	

CONCLUSION

The obtained results can be used to highly approximate the derivatives for the solution of Laplace's equation by the finite difference method, in various combined and composite grid methods, as well as some versions of the domain decomposition methods for obtaining an approximation of the derivative of the solution of the Dirichlet problem for Laplace's equation on polygons covered by overlapping rectangles (see [46], [63], [64], [65]).

Also for rectangular parallelepiped domain a highly accurate difference schemes are proposed and investigated under the conditions imposed on the given boundary functions to approximate the solution of the 3D Laplace equation, its first and pure second derivatives on a cubic grid. The uniform convergence for the approximate solution at the rate of $O(h^6 |\ln h|)$, for the first and pure second derivatives at the rate of $O(h^4)$ is proved. It is shown that the accuracy for the approximate value of the first derivatives can be improved up to $O(h^5 |\ln h|)$ for the same boundary functions by using the fifth order formulae on some faces of the parallelepiped. The obtained results can be used to justify finding the above mentioned derivatives of the solution of the 3D Laplace's boundary value problems on domains described as a union or as an intersection of a finite number of rectangular parallelepipeds by the difference method, with the use of Schwarz's or Schwarz-Neumann iterations (see [67], [68], [46], [63], [64]).

REFERENCES

[1] Basset, A. B. (1888). *A Treatise on Hydrodynamics*. New York, Dover Publications, 1-2.

[2] Batchelor, G. K. (1967). *Introduction to Fluid Dynamies*. Cambridge, Cambridge University Press.

[3] Bear, J. (1972). *Dynamics of Fluids in Porous Media*. New York.

[4] Bear, J. (1979). *Hydraulies of Groundwater*. New York, McGraw-Hill.

[5] Bergman, S. & Schiffer, M. (1953). *Kernel Functions and Elliptic Differential Equations in Mathematical Physicsr*. New York, Academic Press.

[6] Chorlton, F. (1967). *Textbook of Fluid Dynamies*. London, Van Nostrand.

[7] Collins, RE. (1961). *Flow of Fluids Through Porous Media*. New York, Reinhold.

[8] Dullien, F. A. L. (1992). *Porous Media: Fluid Transport and Pore Structure*. New York, Academic Press.

[9] Fichera, G. (1965). *Linear Elliptic Differential Systems and Eigen-value Problems*. Berlin, Lecture Notes in Mathematics, Springer-Verlag, 8.

[10] Gilbarg, D. & Trudinger, N. (1977). *Elliptic Partial Differential Equations of the Second Order*. Berlin, Springer-Verlag.

[11] Greenkorn, R. A. (1983). *Flow Phenomena in Porous Media*. New York, Marcel-Dekker.

[12] Harr, M. E. (1962). *Groundwater and Seepage*. New York, McGraw-Hill.

[13] Hermance, J. F. (1991). *A Mathematical Primer on Groundwater Flow*. Englewood, Prentice-Hall, Cliffs, N. J.

[14] Kellogg, O. D. (1953). *Foundations of Potential Theory*. Englewood, Dover Publications.

[15] Kirkham, D. & Powers, W. L. (1972). *Advanced Soil Physics*. New York, John Wiley.

[16] Kleinstreuer, C. (1997). *Engineering Fluid Dynamics*. Cambridge, Cambridge University Press.

[17] Lamb, Sir Horace (1993). *Hydrodynamics*. Cambridge, Cambridge University Press.

[18] Lin, C. C. & Segel, L.A. (1974). *Mathematics Applied to Deterministic Problems in the Applied Sciences*. New York, Macmillan.

[19] Lions, J. L. (1957). *Lectures on Elliptic Partial Differential Equations*. Bombay, Tata.

[20] Miranda, C. (1970). *Partial Differential Equations of Elliptic Type*. New York,

Springer-Verlag, 2nd edition.

[21] Muskat, M. (1937). *The Flow of Homogeneous Fluids Through Porous Media*. New York, McGrawHill.

[22] Nikolaevskii, V. N. (1990). *Mechanics of Porous and Fractured Media*. Singapore, World Scientific.

[23] Ozisik, M. N. (1993). *Heat Conduction*. New York, John Wiley.

[24] Philips, O. M. (1991). *Flow and Reactions in Permeable Rocks*. Cambridge, Cambridge University Press.

[25] Polubarinova-Kochina, P. Ya. (1962). *Theory of Ground Water Movement*. Princeton, Princeton University Press, N. J.

[26] Sahimi, M. (1995). *Flow and Transport in Porous Media and Fractured Rock*. Weinheim, VCH Verlagsgesselschaft.

[27] Scheidegger, A. E. (1974). *The Physics of Flow Through Porous Media*. Toronto, University of Toronto Press.

[28] Strack, O. D. L. (1989). *Groundwater Mechanics*. Englewood Cliffs, Prentice-Hall, N. J.

[29] Verruijt, A. (1970). *Theory of Groundwater Flow*. New York, Gordon and Breach.

[30] Vukovich, M. (1997). *Groundwater Dynamics*.Littleton, Water Resources Publications, Steady Flow.

[31] White, F. M. (1984). *Heat Transfer*. Mass, Reading, Water Resources Publications, Addison-Wesley.

[32] Xiao, S. (1992). *Flow and Transport in Porous Media*. Singapore, World Scientific Publishing.

[33] Yih, C. -S. (1969). *Fluid Mechanics*. New York.

[34] Kantorvich, L. V. & Krylov V. I. (1958). *Approximate Method of Higher Analysis*. New York, Interscience Publisher, Inc.

[35] Volkov, E. A. (1994). *Block method for solving the Laplace equation and constructing conformal mappings*. USA, CRC Press.

[36] Li, Z. C. & Lu, T. T. (2000). *Singularities and treatments of elliptic boundary value problems* (pp. 97-145). Mathematical and Computer Modelling 31(8).

[37] Volkov, E. A. (1976). *On the method of composite meshes for Laplace's equation on polygons,* (pp. 68-102). Trudy Mat. Inst. Steklov. 140.

[38] Dosiyev, A. A. (1994). *A block-grid method of increased accuracy for solving Dirichlet's problem for Laplace's equation on polygons* (pp. 591-604), Comp. Maths Math. Phys., 34(5).

[39] Motz, H. (1946). *The treatment of singularities of partial differential equations by relaxation methods* (pp. 371-377). Quart. Appl. Math 4.

[40] Volkov, E. A. (1965). *On differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle*(pp. 101-126). Proc. Steklov Inst. Math. Issue 77.

[41] Bakhvalov, N. S. (1959). *Numerical solution of the Dirichlet problem for Laplace's equation* (pp. 171-195). Vestnik Moskow. Uni. Ser. Mat. Meh. Astr. Fiz. Him.

[42] Bramble, J. H. & Hubbard B. E. & Zlamal, M. (1968). *Discrete analogs of Dirichlet problem with isolated singularities* (pp. 1-25) Siam S. Numer. Anal. V. 5..

[43] Andreev V. B. (1977). *The mixed problem for Laplace's mesh equation in a half-plane* (pp. 997-1000). Dokl. Akad. Nauk SSSR 234.

[44] Dosiyev, A. A.& Mamedov, Ja. D. (1978). *On the solution by the grid method of a mixed boundary-value problem for nonlinear elliptic equations* (pp. 1186-1190). Dokl. Akad. Nauk SSSR 242.

[45] Wigley, N. M. (1988). *An efficient method for subtracting off singularities at corners for Laplace's equation* (pp. 368-376). J. Comput. Phys., 78.

[46] Dosiyev, A. A. (2004). *The high accurate block-grid method for solving Laplace's boundary value problem with singularities* (pp. 153-178). SIAM Jour-

nal on Numerical Analysis, 42(1).

[47] Wasow, W. (1952). *On the trunction error in the solution of Laplace's equation by finite differences* (pp. 345-348). J. Res. Nat. Bur. Standards 48.

[48] Laasonen, P. (1957). *On the degree of convergence of discrete approximations for the solution of the Dirichlet Problem* (pp. 1-19). Ann. Acad. Sci. fen. A. I., 246.

[49] Hackbusch, W. *Elliptic Differential Equations*. Theory and Numerical Treatment, Springer-Verlag, Berlin (1992)

[50] Volkov E. A. (1966). *Effective error estimates for grid method solutions of boundary-value problems for Laplace's and Poisson's equations on a rectangle and certain triangles* (pp. 55-85). Tr. Mat. Inst. Akad. Nauk SSSR, 74.

[51] Lebedev, V. I. (1960). *Evaluation of the error involved in the grid method for Newmann's two dimensional problem*(pp. 703-705). Sov. Math. Dokl., 1.

[52] Volkov, E. A. (1999). *On convergence in C_2 of a difference solution of the Laplace equation on a rectangle* (pp. 291-298). Russ. J. Numer. Anal. Math Modelling, 14, No. 3.

[53] Volkov, E. A. (2005). *On the convergence in C_h^1 of the difference solution to the Laplace equation in a rectangular parallelepiped* (pp. 1531-1537). Comput. Math. Math. Phys., Vol.45, No. 9.

[54] Volkov, E. A. (2004). *the grid method for approximating the derivatives of the solution of the Dirichlet problem for the Laplace equation on the rectangular parallelepiped* (pp. 269-278). Russ. J. Numer. Anal. Math Modelling, 19, No. 3.

[55] Volkov, E. A. (1969). *On differential properties of solutions of the Laplace and Poisson equations on a parallelepiped and efficient error estimates of the method of nets* (pp. 105, 54-78). Proc. Steklov Inst. Math.

[56] Volkov, E. A. (1957). *On the solution by the grid method of the inner Dirichlet problem for the Laplace equation* (pp. 34-61). Vycisl. Mat.,1, English trans. in: Amer. Math. Soc. Transl., 24, No.2. (1963) (pp. 279-307).

[57] Mikhailov, V. P. (1978). *Partial Differential Equations*Mir, Moscow,

[58] Mikeladze, Sh. E. (1938). *On the numerical solution of Laplace's and Poisson's differential equations* (pp. 271-292). Izv. Acad. Nauk SSSR, Ser. Mat., No. 2.

[59] Samarskii, A. A. (2001). *The theory of difference schemes*, Marcel Dekker, Inc.

[60] Volkov, E. A. & Dosiyev, A. A. (2012). *A highly accurate homogeneous scheme for solving the Laplace equation on a rectangular parallelepiped with boundary values in* $C^{k,1}$ (pp. 879-886). Comput. Math. Math. Phys., Vol.52, No.6.

[61] Burder, Richard, L. & Dougla Faires, J. (2011). *Numerical Analysis*. Brooks/Cole, Cengage Learning.

[62] Dosiyev, A. A. (2003). *On the maximum error in the solution of Laplace equation by finite difference method* (pp. 229-241). Intern. Journal of Pure and Appl. Math. Vol. 7, No. 2.

[63] Volkov, E. A. & Dosiyev, A. A. (2007). *A high accurate composite grid method for solving Laplace's boundary value problems with singularities* (pp. 291-307). Russ. J. Numer. Anal. Math. Modelling, 22, No. 3.

[64] Dosiyev, A. A. (2014). *The block-grid method for the approximation of the pure second order derivatives for the solution of Laplace's equation on a staircase polygon* (pp. 14-23). J. Comput. Appl. Math. 259.

[65] Volkov, E. A. (2007). *Grid apprximation of the first derivatives of the solution to the Dirichlet problem for the Laplace equation on a polygon* (pp. 92-107). Proc. Steklov Inst. Math., Vol. 255.

[66] Volkov, E. A. (2007). *On a combined grid method for solving the Dirichlet problem for the Laplace equation in a rectangular parallelepiped* (pp. 638-643). Comput. Math. Math. Phys., Vol. 47, No. 4.

[67] Kantorovich, L. V. & Krylov, V. I. (1958). *Approximate Methods of Higher Analysis*. Noordhoff Ltd., Netherlands.

[68] Badea, Lori (2004). *On the Schwarz-Neumann method with an arbitrary number of domains* (pp. 215-238). IMA Journal of Numer. Anal., 24.

[69] Dosiyev, A. A. & M. Sadeghi, H. (2015). *A fourth order accurate approximation of the first and pure second derivatives of the Laplace equation on a rectangle, Advances in Difference Equations*. 2015:67 DOI 10.1186/s13662-015-0408-8

[70] Dosiyev, A. A. & Mir-Mohammad Sadeghi, Hamid (2016). *On a highly accurate approximation of the first and pure second derivatives of the Laplace equation in a rectangular parallelepiped*. 2016:145, DOI: 10.1186/s13662-016-0868-5

[71] Smith, G. D. (1985). *Numerical solution of partial differential equations: finite difference methods*. Oxford University Press.

[72] Peysson, Y. & Shoucri, M. (1998). *An approximate factorization procedure for solving nine-point elliptic difference equations Application for a fast 2-D relativistic Fokker-Planck solver*. Comput. Phys. Commun., 109(55).