Determination of an Unknown Diffusion Coefficient in a Parabolic Inverse Problem

Kabiru Garba Ibrahim

Submitted to the
Institute of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Masters of Science
in
Applied Mathematics And Computer Science

Eastern Mediterranean University
July, 2016
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Cem Tanova
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Applied Mathematics and Computer Science.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Applied Mathematics and Computer Science.

Assoc. Prof. Dr. Derviş Subaşı
Supervisor

Examining Committee

1. Assoc. Prof. Dr. Derviş Subaşı

2. Asst. Prof. Dr Suzan Cival Buranay

3. Asst. Prof. Dr. Pembe Sabancıgil Özder

ABSTRACT

In this thesis we studied the finite difference approximation for the solution of one dimensional parabolic inverse problem of finding the function $\phi(x,t)$ and the unknown positive coeffient $b(t)$. The Backward time centered space (BTCS) which is unconditionally stable is studied and it's convergent is proved using application of discrete maximum principle. Error estimates for $\phi(x,t)$ and $b(t)$ is studied and to give clear overview of the methodology several model problems are solved numerically. According to the experimental numerical results the concluding remark are presented.

Keywords: finite difference methods, parabolic inverse problem, convergence, Error estimates, maximum principle.

ÖZ

Bu tez tek boyutlu parabolik ters problemlerinin sayısal analiz tekniği kullanılarak çözülmesi ile ilgilidir. Çözüm esnasında klasik geri zaman merkezli sonlu farklar tekniği kullanılarak $\phi(x.t)$ fonksiyonu ve yayılma katsayısı $b(t)$ hesplanmıştır. Kullanılan sonlu farklar tekniğinin yakınsaması ayrık maksimum prensibi ile hesplanmış ayrıca $\phi(x.t)$ ve $b(t)$ bilinmeyenlerinin hata tahminleri çalışılmıştır. Sayısal analiz hesaplarında iki farklı denklem üzerinde çalışılmış ve sonuçlar ile düşünceler yazılmıştır.

Anahtar kelimeler: sonlu fark yöntemleri, parabolik ters problemi sorun, yakınsama, hata tahminleri, maksimum ilkesi.

To My Family

ACKNOWLEDGEMENT

I give thank and praise to Almighty Allah for his marcy, guidance and blessing upon me for making this thesis work to reality. My deepest gratitude is to my supervior Ass.prof. Dervis Subasi I have been amazingly privileged to have an advisor who give me the sovereignty to studay on my own and at the same time the assist to recuperate when my step weakened. His tolerance and support help me overwhelmed various crises condition and finish this thesis. I hope that one day I would become as good an advisor to my student as Subasi has been to me.

Most essentially, none of this would have been likely possible without the prayer and support of my family. My family to whome this thesis is dedicate to, has been a constant source of love, concern and support all these period. I would like to express my heart-felt gratitude to my family, I have to give a special mention, for the support given by: my parent and my brother Engr. Sani G Ibrahim I really I appreciate the generosity, love, guidance and counseling may almighty Allah reward them abundently.

Finally I wish to thanks all my friends who helped and encouraged me during the period of my studies.

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INTRODUCTION

In this thesis we analyse the problem of solving two unknown functions $\phi(x,t)$ and the diffusion coefficient $b(t)$ in the parabolic inverse problem

$$\phi_t = b(t)\phi_{xx} \qquad in \;\; '\Omega^T \tag{1.1}$$

$$\phi(x,0) = \psi(x), \qquad 0 \le x \le 1 \tag{1.2}$$

$$\phi(0,t) = \mathrm{H}_1(t), \qquad 0 \le t \le T \tag{1.3}$$

$$\phi(1,t) = \mathrm{H}_2(t), \qquad 0 \le t \le T \tag{1.4}$$

Where $'\Omega^T = \{(x,t): x \in (0,1),\ t \in (0,T)\}, T > 0, and\ \psi, \mathrm{H}_1, \mathrm{H}_2$ are well-known function, while $\phi(x,t)$ and $b(t)$ are unknown it is clear that with the data mentioned above this problem is under-determined, so to solve the inverse problem we most introduce a supplementary boundary condition such that the one and only solution of $\phi(x,t)$ and $b(t)$ are obtained. In particular, this may take form of the heat flux $q(t)$ at a given point $x^* = 0\ or\ 1$, that is,

$$-b(t)\phi_x(x^*,t) = q(t), \qquad 0 \le t \le T. \tag{1.5}$$

As a matter of choice, one may recommend other function, say

$$\phi(x^*,t) = q(t), \qquad 0 \le t \le T, \tag{1.6}$$

where $x^* \in (0,1)$ thus a resumption of the function $b(t)$ together with the solution $\phi(x,t)$ can be formed.

The problem of restoring a time dependent coefficient in a parabolic inverse problem has drawn so many interest and considered by many scientist, and mathematians.In the past decennary a countless covenant of attentiveness has been focused towards the resolve of unknown diffusion coefficients in partial differential equation. One of the motivation behind this thesis is to regulate the unknown variables in a section by quantifying only the data on the boundary and specific consideration has been concentrated on coefficients that denote the physical quantities, such as, the conductivity of a medium. The approaches used depend toughly on the nature of the equations and variables on which the unknown quantity is projected a priori to depend. A significant but challenging situation is when the new conductivity build upon the dependent variable of the solution $\phi(x,t)$.

For a heat energy challenging, this has a physical clarification in which the temperature reliant on conductivity. The spatial transformation of the function $\phi(x,t)$ is insignificant in association with the variation in time, then a rational estimate to this state of actions may be to consider the coefficient to be the function only of the time variable. The mathematical solution of the problem $(1.1)-(1.4)$ has been talk over by numerous authors. For parabolic inverse problem of discovering $b(t)$ Azari [1] studied $\phi_t = b(t)\phi_{xx}$ with the respect to initial-boundary and over quantified condition to regulate the time reliant on coefficient and then converted the inverse problem to nonclassical equation. The maximum principle was then applied to this problem and global existence clarification to these problems where achieved from the continuity techniques. In [2] the numerical solution of $(1.1)-(1.6)$ are also debated using Explicit, Implicit and Crank Nicolson numerical schemes and higher order was recommended to determine the function ϕ and the unknown time reliant on coefficient $b(t)$, in which so many numerical investigation were obtained to examine the effectiveness and accuracy of the numerical consequence, error approximation and numerical solution of $\phi(x,t)$ and $b(t)$ were developed. In [3] Pseudospectral Legendre scheme is engaged to solve problem $(1.1)-(1.4)$ where the Errors of $\phi(x,t)$ and $b(t)$ are acquired by using Explicit, implicit, Crank Nicolson, Saulyev's first and second kind. In [4] the author discussed over the problem of determining concurrent time reliant on thermal diffusivity and the temperature circulation in one dimensional parabolic equation in nonlocal boundary and integral over resolve conditions, the uniqueness and existence condition of classical clarification of the problem were also discussed. In [5] finite difference estimate to an inverse problems $(1.1)-(1.6)$ were also deliberated, the Implicit Euler scheme is considered and is shown that the scheme is stable using maximum norm and convergence are proved using discrete maximum principle. The error estimation and numerical investigation of $\phi(x,t)$ and $b(t)$, and some newly projected procedure are presented. Author in [6] also researchED on the problem $(1.1)-(1.4)$, but the numerical results of the investigation are far-off from tolerable. In [7] Cannon and Jones studied $\phi_t = b(t)\phi_{xx}$ subject to time reliant on boundary conditions. The foremost target of the research is to decrease the problematic case to nonlinear integral equation for the quantity $b(t)$. This suggestion, which depends on the explicit arrangement of elementary solution of the heat equation, does not simply lead to the separation of m space variable for $m \geq 2$. In [8] Cannon and William verified the fortitude of a time reliant on conductivity for potential arbitrary field in $\mathbb{R}^n$, their technique can be labelled as a "lenient" amendment of the methodology of Jones, and depends on the compactness and maximum principle of a convinced smoothing to produce a desire effect by sequential estimates.

This thesis is prepared as follows. In Chapter 2, the finite difference method is expressed from the renovation of parabolic inverse problem and several elementary basis are indicated in the form of lemmas. The backward time centered space (BTCS) is considered and it is shown to be stable in the maximum norm by means of discrete form of the maximum principle for parabolic finite difference scheme . In Chapter 3, the convergence and error estimate of the numerical method of the transformed parabolic inverse problem is discoursed. In chapter 4, two numerical investigations accessede to determine or to check the correctness and efficiency of the backward Euler estimates by presenting the errors for $\phi(x,t)\ and\ b(t)$ of each models. Finally conclusion and observation are given for each experiment.

Chapter 2

FINITE DIFFERENCE TECHNIQUE

In this chapter the numerical methods of one dimensional parabolic inverse problem is advanced to solve (1.1)-(1.4) The finite difference consequent from substituting the space and the time derivative. The parabolic space domain $[0,1] \times [0, T]$ is derived in to mesh of $M \times N$ with the spatial step size $h = \frac{1}{M}$ and the time step size $k = \frac{1}{N}$.

Now we can design the grid points (x_i, t_i) by

$$x_i = i \times h, \qquad i = 0,1,2, \dots \dots M$$

$$t_j = j \times k, \qquad j = 0,1,2, , \dots, N$$

where M and N are any integers, the notation ϕ_i^j , b_j are used to designate the finite difference estimates of

$$\phi(i \times h, j \times k) \quad and \;\; b(j \times k).$$

2.1 Transformation of the Inverse Problem

Taking the derivative of equation (1.6) with the respect to t, we obtained

$$q'(t) = \phi_t(x^*, t). \tag{2.2}$$

Substituting equation (2.2) in eqution (1.1) we have

$$q'(t) = b(t)\phi_{xx}(x^*, t), \tag{2.3}$$

this yields to

$$b(t) = \frac{q'(t)}{\phi_{xx}(x^*, t)}, \qquad t \in [0, T] \tag{2.4}$$

provided that $\phi_{xx}(x^*, t) \neq 0$;

Now equations $(1.1) - (1.6)$ change to the resulting problem

$$\phi_t = \frac{q'(t)}{\phi_{xx}(x^*, t)} \phi_{xx} \quad in \ \ '\Omega^T \tag{2.5}$$

$$\phi(x, 0) = \psi(x), \quad 0 \leq x \leq 1, \tag{2.6}$$

$$\phi(0, t) = \mathrm{H}_1(t), \quad 0 \leq t \leq T, \tag{2.7}$$

$$\phi(1, t) = \mathrm{H}_1(t), \quad 0 \leq t \leq T. \tag{2.8}$$

Our process is based on the following alteration, by setting

$$w(x, t) = \phi_{xx}(x, t), \tag{2.9}$$

taking derivative of (2.9) with respect to t we have

$$w_t(x, t) = \phi_{xxt}(x, t),$$

$$= \frac{\partial}{\partial x^2}\left(\frac{q'(t)}{\phi_{xx}(x^*, t)} \phi_{xx}(x, t)\right),$$

$$w_t(x, t) = \frac{\partial}{\partial x^2}\left(\frac{q'(t)}{w(x^*, t)} w(x, t)\right).$$

Therefore

$$w_t(x, t) = \frac{q'(t)}{w(x^*, t)} w_{xx}(x, t).$$

For initial condition at $t = 0$,

$$\phi(x, 0) = \psi(x).$$

Second derivative of $\phi(x, 0)$ with the respect to x, yield to

$$\phi_{xx}(x, 0) = \psi_{xx}(x), \qquad \text{then}$$

$$w(x,0) = \psi_{xx}(x). \tag{2.10}$$

Transformation of the left and right boundary condition at $x = 0$ and $x = 1$, respectively, we know

$$b(t) = \frac{q'(t)}{\phi_{xx}(x^*,t)}.$$

That is

$$b(t) = \frac{q'(t)}{w(x^*,t)},$$

for the left boundary condition

$$\phi(0,t) = \mathrm{H}_1(t), \qquad \text{then}$$

$$\phi_{xx}(0,t) = w(0,t) = \frac{1}{b(t)}\phi_t(0,t) = \frac{1}{b(t)}\mathrm{H}_1'(t).$$

It implies that

$$w(0,t) = \frac{\mathrm{H}_1'(t)}{q'(t)}w(x^*,t). \tag{2.11}$$

Similarly, for the right boundary condition $x = 1$,

$$\phi(1,t) = \mathrm{H}_2(t) \qquad \text{then}$$

$$\phi_{xx}(1,t) = w(1,t) = \frac{\phi_t(1,t)}{b(t)} = \frac{1}{b(t)}\mathrm{H}_2'(t).$$

Then it implies that

$$w(1,t) = \frac{\mathrm{H}_2'(t)}{q'(t)}w(x^*,t). \tag{2.12}$$

Where $w(x,t)$ is the solution of the following problem

$$w_t = \frac{q'(t)}{w(x^*,t)} w_{xx} \quad in \ \Omega^T \tag{2.13}$$

$$w(x,0) = \psi_{xx}(x), \qquad 0 \le x \le 1 \tag{2.14}$$

$$w(0,t) = \frac{\mathrm{H}_1{}'(t)}{q'(t)} w(x^*,t), \quad 0 \le t \le T\,, \tag{2.15}$$

$$w(1,t) = \frac{\mathrm{H}_2{}'(t)}{q'(t)} w(x^*,t), \quad 0 \le t \le T. \tag{2.16}$$

We make some assumptions that holds throughout this Thesis:

P(1) Let $\psi(x) \in C^{4+\partial}[0,1]$ and $\psi_{xx}(x) > 0$, $\psi_{xx}(x^*) = \frac{\varepsilon}{2} > 0$ and $\psi_{xxxx}(x) > 0$ on $[0,1]$.

P(2) Let $\mathrm{H}_1(t)$, $\mathrm{H}_2(t)$ and $q(t) \in C^{l+\frac{\partial}{2}}[0,T]$. Furthermore, $q'(t) > 0$ on $[0,T]$,

$$0 < \frac{\mathrm{H}_1(t)}{q'(t)} < 1, 0 < \frac{\mathrm{H}_2(t)}{q'(t)} < 1, and \ \left(\frac{\mathrm{H}_1(t)}{q'(t)}\right)' > 0, \left(\frac{\mathrm{H}_2(t)}{q'(t)}\right)' > 0, \qquad t \in [0,T].$$

Theorem 2.1: [7] Under the hypothesis P(1) and P(2), equation $(1.1) - (1.5)$ has a unique solution $(\phi(x,t), b(t))$ in $\Omega^T\ \forall\, T > 0)$ and for the problem $(2.13) - (2.16)$ we have

$$0 < \psi_{xx}(x) \le w(x,t) \le \max_{0 \le x \le 1} \psi_{xx}(x), \qquad (x,t) \in \Omega^T$$

Consequently there exist $\varepsilon > 0$ such that

$$w(x^*,t) \ge \psi_{xx}(x^*) = \frac{\varepsilon}{2} > 0, \qquad t \in [0,1] \tag{2.17}$$

Note that Our estimation now is to solved $(2.13) - (2.16)$ for $w(x,t)$, and follow by $b(t)$, then the unknown functions can easily be solve. If the solution $\phi(x,t)$ is also needed, then there is needs of solving an additional boundary value problem.

2.2 Backward Time Centered Space (BTCS)

Backward time centered space can be defined using the forward derivative approximation for the time derivative ϕ_t and second order approximation for the spatial derivative ϕ_{xx} defined at the point (x_i, t_{j+1}). Then the overall approximation is called Backward Time Centered space or Backward Euler scheme.

Lemma 2.1 [6] Suppose that $f(x) \in C^2[0,1]$ and there exist i_0 such that $x^* \in [x_{i_0}, x_{i_0+1})$.

$$f(x^*) = \frac{h-\varepsilon_x}{h} f\left(x_{i_0}\right) + \frac{\varepsilon_x}{h} f\left(x_{i_{0+1}}\right) + O(h^2), \tag{2.18}$$

where

$$\varepsilon_x = x^* - x_{i_0}.$$

Now the backward time centered space (see Figure 1) can now be defined by

$$\frac{w_i^{j+1} - w_i^j}{k} = \frac{B^{j+1}}{w_*^j} \frac{w_{i+1}^{j+1} + w_{i-1}^{j+1} - 2w_i^{j+1}}{h^2}, \qquad i = 0,1,..M-1, j \geq 0, \tag{2.19}$$

$$w_i^0 = \psi_{xxi} \qquad i = 0,1,\dots M, \tag{2.20}$$

$$w_i^{j+1} = Q_i^{j+1} w_*^j, \qquad i = 0 \text{ or } M, \qquad j \geq 1 \tag{2.21}$$

where

$$w_*^j = \phi_{xx}\left(x^*, t^j\right) = \frac{1}{h^2}\left[\phi\left(x^* + h, t^j\right) - 2\phi\left(x^*, t^j\right) + \phi(x^* - h, t^j)\right]$$

equivalently

$$w_*^{j+1} = \frac{h-\varepsilon_x}{h} w_{i_0}^{j+1} + \frac{\varepsilon_x}{h} w_{i_0+1}^{j+1} \tag{2.22}$$

and $B^{j+1} = q'\left(t_j\right)$, $Q_0^{j+1} = \left(\frac{\mathrm{H}_1\left(t_{j+1}\right)}{q(t)}\right)'$ and $Q_M^{j+1} = \left(\frac{\mathrm{H}_2\left(t_{j+1}\right)}{q(t)}\right)'$.

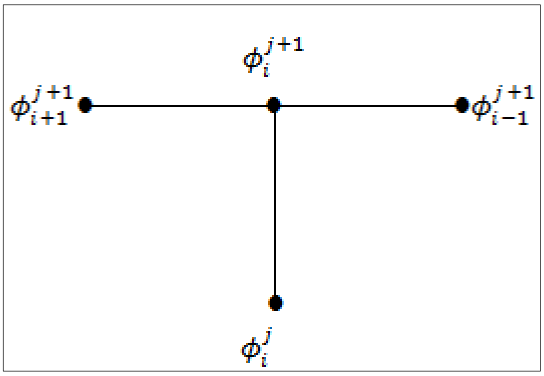

Figure 1. Computational molecule for BTCS

It's easy to see that $(2.19)-(2.22)$ is a semi$-$ implicit scheme because $w(x^*,t)$ is approximated using values at the previous time level. The scheme (2.19) result in a truncation error $O(h^2+k)$, which is the same as the standard backward finite difference scheme for parabolic equations. It can easily be seen that any standard numerical solver for parabolic equations can be used to solve $(2.19)-(2.21)$.

Let us define

$$\nabla^+ w_i = \frac{w_{i+1}-w_i}{h}, \qquad \nabla^- w_i = \frac{w_i - w_{i-1}}{h}. \tag{2.23}$$

Therefore we have

$$\nabla^2 w_i = \nabla^-\nabla^+ w_i = \frac{w_{i+1}-2w_i+w_{i-1}}{h^2} \tag{2.24}$$

and for some $\{d_i\}_{i=1}^{p}$ it follows

$$\nabla^2(d_i w_i) = d_i\nabla^2 w_i + w_{i+1}\nabla^2 d_{i+1} + \nabla^+ d_{i-1}\nabla^+ w_i + \nabla^+ d_{i-1}\nabla^+ w_{i-1} \tag{2.25}$$

Lemma 2.2 [7] The following inequality hold:

$$w_*^{j+1} \le \max_{0\le i\le n} w_i^{j+1}, \qquad j \ge 0 \tag{2.26}$$

$$0 < w_i^{j+1} \le \max_{0\le i\le M} d_i e^{\omega T} \max_{0\le i\le M} |\psi_{xxi}| \qquad i = 0,1,2\ldots M,\; j\ge 0 \tag{2.27}$$

Proof: The inequality (2.26) follows straight from (2.22) and the definition ε_x, let $d(x) = (1 + (x + x^*)^2)^n$, where $n > 0$ is any constant and

let

$$w_i^{j+1} = d_i e^{\omega t_j} R_i^{j+1},\ i = 0,1 \dots.,M, 1 \le j \le N, \tag{2.28}$$

where $d_i = d(x_i)$ and $\omega > 0$. Using the transformation (2.28) with some simple computation, we found that R_i^j satisfies

$$e^{-\omega k}\frac{R_i^{j+1}-R_i^j}{k} = \frac{B^{j+1}}{e^{\omega t_j}R_*^j}\left(\nabla^2 R_i^{j+1} + R_{i+1}^{j+1}\frac{\nabla^2 d_i}{d_i} + \nabla^+ R_i^{j+1}\frac{\nabla^+ d_{i-1}}{d_i} + \nabla^+ R_{i-1}^{j+1}\frac{\nabla^+ d_{i-1}}{d_i}\right)$$

$$-\frac{1-e^{-\omega k}}{k}R_i^{j+1}, \qquad i = 1,2,,\dots M-1,\ j \ge 1, \tag{2.29}$$

$$R_i^0 = \frac{\psi_{xxi}}{d_i} \qquad i = 0,1,\dots\dots\dots..M \tag{2.30}$$

$$R_i^{j+1} = \frac{Q_i^{j+1}}{d_i}R_*^j e^{-\omega k} \quad i = 0\ or\ M,\ j \ge 1 \tag{2.31}$$

as $d_i \ge 1$ for $i = 0,1,\dots\dots,M,$ we choose ω large enough and k sufficiently small such that

$$\frac{1-e^{-\omega k}}{k} \ge \frac{\omega}{2} \ge \frac{\nabla^2 d_i}{d_i}.$$

Using that discrete maximum principle [1] R_i^{j+1} can not have it's maximum in inside of $'\Omega^T$. If R_i^{j+1} reach a positive maximum at some point say (i^*, j^*) on the horizontal boundary condition, then by the boundary condition,

$$R_*^{j^*} = e^{\omega k}\frac{d_i}{Q_{i^*}^{j^*+1}}R_i^{j^*+1} \ge \frac{d_{i_*}}{Q_{i^*}^{j^*+1}}R_{i^*}^{j^*+1} > R_{i^*}^{j^*+1}.$$

For the fact that k is very large such that $\frac{d_{i^*}}{Q_{i^*}^{j^*+1}} > 1$ (this can be true since $d_{i^*} > 1$). It's contradiction since $R_{i^*}^{j^*}$ is the positive maximum over $\overline{'\Omega^T}$. It follows that

$$R_i^{j+1} \leq \max_{0\leq i\leq M}|\psi_{xxi}|, \qquad i = 0,1,2.M, \quad j = 0,1,2 \dots \dots, N. \qquad (2.32)$$

Hence, we obtain

$$w_i^{j+1} \leq \max_{0\leq i\leq M} d_i e^{\omega T}, \ \max_{0\leq i\leq M}|\psi_{xxi}|, \quad i = 0,1, \dots M, \ j = 0,1, . N. \qquad (2.33)$$

Now we need to drive the lower bound. We know that $\psi_{xxi} > 0$ for $i = 0,1, . . M$ and $Q_i^j > 0$ for $i = 0$ or M and $j \geq 0$, we assume that $w_i^j > 0$ for $i = 0,1, . M$ and $j \geq 0$. Let

$$j^* = \sup\{j_0: w_i^{j_0} > 0 \text{ for } i = 0,1, \dots . ., N, j_0 \geq 0\},$$

for $j^* = N$, we got the desired result. If $j^* < N$, then w_i^{j+1} attains it's minimum zeros at (i^*, j^*) . By the maximum principle, (i^*, j^*) can not be in the interior of $'\Omega^{T*}$. Hence (i^*, j^*) is in the horizontal boundary. However, at this point by the boundary condition we have $w_*^{j^*} = 0$, which implies that w_i^j reachs the minimum at the center point (x^*, j^*) of $'\Omega^{T*}$. This is contradiction. Therefore it follows that

$$w_i^{j+1} > 0, \qquad i = 0,1,2, \dots \dots \dots . M, \ j = 0,1 \dots . N.$$

In order to obtain further priori estimates, we consider to obtain a priori lower bound for w_*^j. Therefore, from lemma 2.2 we can express the following result.

Corollary 2. 1. [6] we have

$$w_*^{j+1} \geq \psi_{xx}(x^*) = \frac{\varepsilon_0}{2} > 0, \quad \text{for } j = 0,1, . . N \qquad (2.34)$$

Proof : Under the propositions $P(1)$ and $P(2)$, We assume $\left(w_i^{j+1} - w_i^j\right)\Big/_k \geq 0$.

It is clear that

$$v_i^{j+1} = \frac{w_i^{j+1} - w_i^j}{k}, \qquad i = 0,1,..M, \quad j \geq 2$$

satisfies

$$\frac{v_i^{j+1} - v_i^j}{k} = \frac{B^{j+1}}{w_*^j}\nabla^2 v_i^{j+1} + \frac{1}{k}\left(\frac{B^{j+1}}{w_*^j} - \frac{B^j}{w_*^{j-1}}\right)\frac{w_*^{j-1}}{B^j} v_i^j, \qquad i = 1,..M-1, j \geq 1,$$

$$v_i^0 = \frac{B^0}{w_*^0}\psi_{xxxxi}, \qquad i = 0,1,,\dots\dots.M.$$

$$v_i^{j+1} = Q_i^j v_*^j + \frac{Q_i^{j+1} - Q_i^j}{k} v_*^{j-1}, \qquad i = 0 \; or \; M, \qquad j \geq 1$$

By the estimate and condition $w_*^{j+1} > 0$ for $j \geq 0$, we know that

$$\frac{Q_i^{j+1} - Q_i^j}{k} w_*^{j-1} \geq 0.$$

Since $\frac{B^0}{w_*^0}\psi_{xxxxi} > 0$, $i = 0,1,..,M$ and $v_i^{j+1} \geq Q_i^{j+1} v_*^j$, $i = 0$ or 1, $j \geq 1$,

We finalize by applying the same argument as that proof of Lemma 2. That $v_i^{j+1} \geq 0$, for $i = 0,1,\dots\dots.M$ and $j \geq 0$. Consequently, it follows that

$$w_i^{j+1} \geq w_i^j \geq \cdots \geq w_i^0 = \psi_{xxi} > 0, \qquad i = 0,1,2 \dots..M \quad j \geq 0.$$

Precisely, we have

$$w_*^{j+1} \geq \psi_{xx}(x^*) = \frac{\varepsilon_0}{2} > 0, \qquad j \geq 0.$$

Chapter 3

CONVERGENCE

This chapter is apprehensive with the conditions that must be gratified if the solution of the finite difference equation is to be sensibly correct estimate to the solution of the correspondent parabolic partial differential equation.These conditions are attendant with the two different but interrelated problems. The first concerns the convergence of the exact solution of the approximating difference equations to the solution of the differential equation, the second concerns the unbounded growth or measured decay of any errors associated with the solution of the finite difference equations. Therefore convergence estimate theorem can be stated below.

3.1 Convergent Estimate Theorem

Theorem 3.2 [1, 7]: Suppose that $\phi \in C^{4,2}('\Omega)$. Then there exist $h_0 > 0$ and $k_0 > 0$, dependent upon the data $\mathrm{H}_1, \mathrm{H}_2, q, s = \max\{x^*, 1 - x^*\}$, and $T > 0$, such that $h \in (0, h_0)$ and $k \in (0, k_0)$, $\forall\ C > 0$, which is depending on s, T and $C^{4,2}$ norm of w, such that

$$\max_{i,j}\left|w\left(x_i, t_{j+1}\right) - w_i^{j+1}\right| \leq C(h^2 + k) \tag{3.2}$$

Proof: of the above theorem comprise of several steps.

Step 1: Let $e_i^{j+1} = w_i^{j+1} - w(x_i - t_{j+1})$ .then from $(2.13) - (2.16)$ and $(2.19) - (2.21)$ that e_i^{j+1} satisfies

$$\frac{e_i^{j+1} - e_i^j}{k} = {A_1}^{j+1}\nabla^2 e_i^{j+1} + G_i^{j+1} e_*^j + \epsilon_i^{j+1}, \quad i = 1,,.M - 1,\ j \geq 1\ , \tag{3.3}$$

$$e_i^0 = 0 \qquad\qquad i = 0,1,2..M, \tag{3.4}$$

$$e_i^{j+1} = e_i^j Q_i^{j+1} + k_i^{j+1} \qquad\qquad i = 0 \text{ or } M, \qquad\qquad j \geq 1 \qquad\qquad (3.5)$$

where

$$A_1{}^j = \frac{B^{j+1}}{w_*^j}\ , \qquad\qquad G_i^{j+1} = \frac{w_t(x_i, t_j)}{w_*^i},$$

and ϵ_i^{j+1} and k_i^{j+1} are the truncation errors induced by the discretizations of the differential equation and boundary conditions respectively.

Lemma 3.1: [4] Suppose that $w \in C^{4,2}\overline{\Omega}^T$ and the data are smooth. Then there exist a positive constant L , $C_0 = C_0(\|w\|_{C^{4,2}})$ $\quad 0 < h \leq h_0$, and $0 < k \leq k_0$ such that

$$\max_{0\leq i\leq M, 0\leq j\leq N} |\epsilon_i^j| \leq C_0(h^2 + k), \quad \max_{0\leq i\leq M, 0\leq j\leq N} |k_i^j| \leq C_0(h^2 + k), \qquad (3.6)$$

$$\max_{0\leq i\leq M, 0\leq j\leq N} |Q_i^j| \leq L, \qquad\qquad \max_{0\leq i\leq M, 0\leq j\leq N} |B_i^{j+1}| \leq L, \qquad (3.7)$$

$$\max_{0\leq i\leq n, 0\leq j\leq N} |A_1{}^j| \leq \frac{L}{\varepsilon_0} \qquad\qquad \max_{0\leq i\leq M, 0\leq j\leq N} |G_i^j| \leq \frac{G_0}{\varepsilon_0} \qquad (3.8)$$

Proof: The inequality (3.6) follows from Taylor's expansion, also the inequality (3.7) hold from smoothness of the data B, Q and $q' \neq 0$. Finally, the inequalities (3.8) follows from (3.7), corollary 2.1 and $w \in C^{4,2}\overline{\Omega}^T$.

Step 2: Let $g(x) = 1 + \rho(x - x^*)^2$, where $\rho > 0$, is a constant to be chosen.

Let

$$e_i^{j+1} = g_i u_i^{j+1}, \qquad\qquad i = 0,1, \ldots\ldots M \qquad 0 \leq j \leq N, \qquad (3.9)$$

where $g_i = g(x_i)$. Then from (2.22) we have

$$e_*^{j+1} = \frac{h - \varepsilon_x}{h} g_{i_0} u_{i_0}^{j+1} + \frac{\varepsilon_x}{h} g_{i_0+1} u_{i_0}^{j+1} \qquad i = 0,1, \ldots. M, \qquad (3.10)$$

where

$$u_{**}^{j+1} = \frac{h - \varepsilon_x}{h} \varepsilon_x^2 u_{i_0}^{j+1} + \frac{\varepsilon_x}{h} (h - \varepsilon_x)^2 u_{i_0+1}^{j+1} \qquad (3.11)$$

Lemma 3.2: [1] we have

$$\left|u_{**}^{j+1}\right| \leq h^2 \max_{0\leq i\leq M}\left|u_i^{j+1}\right|, \qquad j \geq 0 \tag{3.12}$$

Proof: the inequality follows from (3.11) and the definition of ε_x,

Upon using the transformation of (3.9) and (3.10) with some simple calculation, we find out that u_i^{j+1} satisfies

$$\frac{u_i^{j+1}-u_j^i}{k} = {A_1}^{j+1}\left(\nabla^2 u_i^{j+1} + u_{i+1}^{j+1}\frac{\nabla^2 g_i}{g_i} + \nabla^+ u_i^{j+1}\frac{\nabla^+ g_{i-1}}{g_i} + \nabla^+ u_{i-1}^{j+1}\frac{\nabla^+ g_{i-1}}{g_i}\right)$$

$$+G_i^{j+1}\left(u_*^j + \rho u_{**}^j\right) + \frac{\epsilon_i^{j+1}}{g_i}, \quad i = 1,2,\dots,M-1, \;\; j \geq 1 \tag{3.13}$$

$$u_i^0 = 0 \qquad i = 0,1,\dots\dots M \tag{3.14}$$

$$u_i^{j+1} = \frac{Q_i^{j+1}}{g_i}\left(u_*^j + \rho y_{**}^j\right) + \frac{k_i^{j+1}}{g_i}, \quad i = 0 \text{ or } M, \qquad j \geq 1 \tag{3.15}$$

Step 3: Let $\mu > 0$ and

$$u_i^{j+1} = e^{\mu t_j} z_i^{j+1}, \qquad i = 1,2,\dots\dots M, \;\; 1 \leq j \leq N, \tag{3.16}$$

then z_i^{j+1} satisfies

$$e^{-\mu k}\frac{z_i^{j+1}-z_i^j}{k} = {A_1}^{j+1}\left(\nabla^2 z_i^{j+1} + z_{i+1}^{j+1}\frac{\nabla^2 g_i}{g_i} + \nabla^+ z_i^{j+1}\frac{\nabla^+ g_{i-1}}{g_i}\right.$$

$$\left. + \nabla^+ z_{i-1}^{j+1}\frac{\nabla^+ g_{i-1}}{g_i}\right) + e^{-\mu k}\frac{G_i^{j+1}}{g_i}\left(z_*^j + \rho z_{**}^j\right)$$

$$+ e^{-\mu t_j}\frac{\epsilon_i^{j+1}}{g_i} - \frac{1-e^{-\mu k}}{k} z_i^{j+1}, \quad i = 1,\dots M-1, j \geq 1, \tag{3.17}$$

$$z_i^0 = 0, \qquad i = 0,1,\dots M, \tag{3.18}$$

$$z_i^{j+1} = e^{-\mu k}\frac{Q_i^{j+1}}{g_i}\left(z_*^j + \rho z_{**}^j\right) + e^{-\mu t_j}\frac{k_i^{j+1}}{g_i}, \qquad i = 0 \; or \; M, \tag{3.19}$$

Lemma 3.3: [1] $\forall\ i = 0,1,...M$, we have

$$\left|\frac{\nabla^2 g_i}{g_i}\right| \leq 2\rho, \qquad \left|\frac{\nabla^+ g_{i-1}}{g_i}\right| \leq 2\rho, \tag{3.20}$$

$$\left|\frac{\nabla^+ g_i}{g_i}\right| \leq 2\rho, \qquad \left|\frac{1}{g_i}\right| \leq 1, \qquad c = \max\{x^*, 1 - x^*\}. \tag{3.21}$$

Step 4: Now we can use maximum principle to show that there exist

$h_0 = h_0(c, L) > 0,\ \ k_0 = k_0(s, L) > 0$, such that $\forall\ 0 < h < h_0\,, 0 < k \leq k_0$ holds

$$\max_{0\leq i\leq M, 0\leq j\leq N}\left|z_i^{j+1}\right| \leq C_0(h^2 + k) \tag{3.22}$$

Suppose that the maximum of $\left|z_i^{j+1}\right|$ reached at (i^*, j^*) and that $z_{i^*}^{j^*+1} > 0$. Then there exist two cases to be consider as follows

Case 1: Let $N^* = z_{i^*}^{j^*}$ and let i^* be a boundary point, then it follows from $(3.18), (3.19)$, Lemma 3.1, Lemma3.2 and (3.21) that for any $\mu > 0$,

$$N^* \leq \frac{L(1 + \rho h^2)}{1 + \rho s} N^* + C_0(h^2 + k) \tag{3.23}$$

where C_0 depends only on w and T. If we chose ρ and h such that

$$\rho = \frac{2L}{s}, \quad h = \sqrt{\frac{s}{2L}}, \tag{3.24}$$

we obtained that $((1 - L + \rho(s - Lh^2)) \geq 1$ and

$$N^* \leq \frac{C_0(h^2 + k)}{\left(1 - L + \rho(c - Lh^2)\right)} \leq C_0(h^2 + k) \tag{3.25}$$

Case 2: Assume that i^* is an interior point, then

$$h \leq \min\left(\frac{1}{2\rho}, \sqrt{\frac{s}{2L}}\right), \tag{3.26}$$

from the the discrete maximum principle [4] and (3.17) it follows that

$$\frac{1-e^{-\mu k}}{k}N^* \leq \frac{2L}{\varepsilon_0}\rho N^* + \frac{C_0}{\varepsilon_0}(N^* + \rho h^2 N^*) + C_0(h^2+k), \tag{3.27}$$

or for $\mu > 0$ sufficiently large and for some $k_\delta \in (0,k)$ and $k > 0$ suffiently small such that

$$N^* \leq \frac{C_0(h^2+k)}{\mu e^{-\mu k_\delta} - \left(\frac{4L}{\varepsilon_0}\rho + \frac{2C_0}{\varepsilon_0}(1+\rho h^2)\right)}. \tag{3.28}$$

Choose

$$\mu = 2\left(\frac{4L}{\varepsilon_0}\rho + \frac{2C_0}{\varepsilon_0}(1+\rho h^2) + 1\right)$$

and $k_0 > 0$ such that for $k \leq k_{0,}$

$$e^{-\mu k_\delta} \geq e^{-\mu k} \geq e^{-\mu k_0} = \frac{1}{2} \quad or \quad k \leq k_0 = \frac{\ln 2}{\mu}. \tag{3.29}$$

Therefore we can clearly see that

$$\mu e^{-\mu k_\delta} - \left(\frac{4L}{\varepsilon_0}\rho + \frac{2C_0}{\varepsilon_0}(1+\rho h^2)\right) \geq 1. \tag{3.30}$$

It implies that $N^* \leq C_0(h^2+k)$, where C_0 depends on w, T, L and s. By the same argument we can deal with $z_{i^*}^{j^*+1} > 0$ and we can obtain the equivalent inequality.

Step 5: It's enough to see from (3.16) and (3.22) that

$$\max_{0\leq i\leq M, 0\leq j\leq N}\left|u_i^{j+1}\right| \leq C(h^2+k), \tag{3.31}$$

where $C > 0$ depends on w, L, s and $T > 0$. Finally from equation (3.3) and (3.31) gives

$$\max_{0\leq i\leq M, 0\leq j\leq N}\left|\epsilon_i^{j+1}\right| \leq C(h^2+k), \tag{3.32}$$

which completes the proof.

3.2: Error Estimate for $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{t})$ and $\boldsymbol{b}(\boldsymbol{t})$

It is easily to observe that the numerical solution of w_i^{j+1} is not the solution of initial inverse value problem. To solve the original problem we must recover $\phi(x,t)$ from $w(x,t)$. We need to resolve the resulting boundary value problem by dealing with time t as a parameter form

$$\phi_{xx}(x,t) = w(x,t) \qquad x \in (0,1), \tag{3.33}$$
$$\phi(0,t) = \mathrm{H}_1(t),$$
$$\phi(1,t) = \mathrm{H}_2(t).$$

Our goal is to obtain the function $\phi(x,t)$. The differntiability of $\phi(x,t)$ with respect to t is also obvious. By using maximum principle [7], we obtained

$$|\phi(x,t)| \leq 2 \max_{0\leq x\leq 1, 0\leq t\leq 1} \left(|\mathrm{H}_1|, |\mathrm{H}_2|, \frac{1}{m} |w(x,t)| \right), \tag{3.34}$$

where $$m = \min_{0\leq x 1}(\tau^2 e^{-\tau x}), \qquad \tau > 0. \tag{3.35}$$

Now the finite difference solution of ϕ_i^{j+1} from w_i^{j+1} is define by (3.33)

$$w_i^{j+1} = \frac{\phi_{i+1}^{j+1} - 2\phi_i^{j+1} + \phi_{i-1}^{j+1}}{h^2}, \qquad 1 \leq i \leq M-1, j \geq 0, \tag{3.36}$$
$$\phi_0^{j+1} = \mathrm{H}_1^{j+1} \qquad j \geq 0,$$
$$\phi_i^{j+1} = \mathrm{H}_2^{j+1} \qquad j \geq 0,$$

observe that w_i^{j+1} is the solution of $(2.5) - (2.9)$. by applying discrete maximum principle for two point boundary problem, we have the following approximation to $\phi(x_i, t_j)$:

$$\left|\phi_i^{j+1}\right| \leq 2 \max_{0\leq i\leq M, 0\leq j\leq N} \left(|\mathrm{H}_1|, |\mathrm{H}_2|, \frac{1}{\eta} \left|\phi_i^{j+1}\right| \right) \tag{3.37}$$

Where

$$\eta = \min_{0\le i\le M}\left(\theta e^{-\theta i h}\right), \qquad \theta > 0.$$

From (3.1) and equation, (3.36) and , (3.37) we obtained

$$\left|\phi(x_i, t_j) - \phi_i^{j+1}\right| = O(h^2 + k), \quad i = 0,1,,..M, \ j = 0,1,N, \tag{3.38}$$

For every h and k sufficiently small. We generalize the above statement in to the following theorem.

Theorem 3.3: [8] Assume that the unique solution of $\phi(x,t)$ and $b(t)$ of $(1.1)-(1.6)$ exist, and ϕ is in $C^{4,2}('\Omega)$. There exist $h_0 > 0$ and $k_0 > 0$, dpendent upon the data H_1, H_2, h, and $T > 0$, Such that $\forall h \in (0, h_0)$ and $\forall k \in (0, k_0)$, (3.38) holds and ϕ_i^{j+1}satisfies (3.37).

Now recovering $b(t)$, from equation (2.4) we have

$$b(t) = \frac{q'(t)}{\phi_{xx}(x^*, t)} = \frac{q'(t)}{w(x^*, t)}.$$

Furthermore, approximation of $b(t)$ consist of numerical calculation of $q'(t)$ and $w(x^*, t)$, as

$$\frac{q'(t_{j+1})}{w(x^*, t_{j+1})} = \frac{q'(t)}{w_*^{j+1} + w(x^*, t) - w_*^{j+1}}$$

$$= \frac{q'(t_{j+1})}{w_*^{j+1}}\left(1 + O(h^2 + k)\right). \tag{3.39}$$

Hence we have

$$\left|\frac{q'(t_{j+1})}{w(x^*, t_{j+1})} - \frac{q^{j+1} - q^j}{w_*^{j+1} k}\right| = O(h^2 + k), \tag{3.40}$$

Using

$$b^{j+1} = \frac{q^{j+1} - q^j}{k w_*^{j+1}}\,, \tag{3.41}$$

as a numerical estimate for $b(t_{j+1})$. Then from Theorem 3.3 we obtained the following result.

Corollary 3. 1: for every h and k small enough, we have

$$\left|b(t_{j+1}) - b^{j+1}\right| = O(h^2 + k), \qquad j = 0,1..N \tag{3.42}$$

$$\left|\phi(x_i, t_{j+1}) - \phi_{j+1}^i\right| = O(h^2 + k) \qquad i = 0..M \tag{3.43}$$

Equation (3.42) and (3.43) are known as error estimate of $\phi(x,t)$ and $b(t)$.

Chapter 4

NUMERICAL RESULTS AND DISCUSSION

In this chapter, we will present the numerical experiment from solving two model problems by using the numerical procedures discussed in the previous chapters in order to give clear overview of the approaches. Each model problem we used various values of h with the fixed k and the point $x^* = 0.25$ choose as an interior point of the domain $'\Omega^T$ for the two model problems. In order to verify the accuracy of $b(t)$ and $\phi(x,t)$ using proposed finite difference schemes the following error calculation are used

$$e_\phi = \left|\phi(x_i, t_j) - \phi_j^i\right|,$$

and

$$e_\phi = \left\|\phi(x_i, t_j) - \phi_j^i\right\|_\infty.$$

Similarly

$$e_b = \left\|b(t_j) - b_j^i\right\|_\infty,$$

and

$$e_b = \left|b(t_j) - b_j^i\right|.$$

4.1. Problem 1

Consider the problem $(1.1) - (1.5)$ with

subject to the given initial condition

$$\psi(x) = 2e^{x}, \qquad 0 \le x \le 1$$

and boundary conditions

$$\mathrm{H}_1(t) = 1 + \frac{1 + 2t^3}{1 + t^3}, \qquad 0 \le t \le 1$$

$$\mathrm{H}_2(t) = e^{1} + \frac{e^{1}(1 + 2t^3)}{1 + t^3}, \qquad 0 \le t \le 1$$

with fixed point $x^{*} = \frac{1}{4}$,

$$q(t) = 1.28403 + \frac{1.28403(1 + 2t^3)}{1 + t^3}, \qquad 0 \le t \le 1,$$

for which the exact solution is

$$\phi(x,t) = e^{x} + \frac{e^{x}(1 + 2t^3)}{1 + t^3}, \qquad 0 \le t \le 1,$$

$$b(t) = \frac{3t^2}{2 + 5t^3 + 3t^6}, \qquad 0 \le t \le 1.$$

Table 1: Exact and Approximate values of ϕ with $\Delta t = 0.00025$, and $T = 1$

$\boldsymbol{x}$	**Exact** $\boldsymbol{\phi(x,t)}$	$\boldsymbol{\Delta x = 0.1}$	$\boldsymbol{\Delta x = 0.01}$	$\boldsymbol{\Delta x = 0.001}$
0.1	2.628177740940	2.6304302555089	2.6304947084293	2.636700615141
0.2	2.762927295189	2.7652952986495	2.7653630561419	2.772134284105
0.3	2.904585606820	2.9070750204148	2.9071462519080	2.914277838170
0.4	3.053506895400	3.0561239439588	3.0561988275688	3.063696543677
0.5	3.210063541719	3.2128147692266	3.2128934922013	3.220775637080
0.6	3.374647018940	3.3775393048979	3.3776220640859	3.385908334890
0.7	3.547668871483	3.5507094481131	3.5507964504554	3.559507539422
0.8	3.729561744103	3.7327582144306	3.7328496774784	3.742006304184
0.9	3.920780463725	3.9241408205911	3.9242369730496	3.933833087411
1.0	4.121803176750	4.1253358227967	4.1254369050972	4.134955543870

Table2: Exact and Approximate values for ϕ with the $\Delta t = 0.000025$, and $T = 1$

$\boldsymbol{x}$	**Exact** $\boldsymbol{\phi(x,t)}$	$\boldsymbol{\Delta t = 0.0001}$	$\boldsymbol{\Delta t = 0.00005}$	$\boldsymbol{\Delta t = 0.000025}$
0.1	2.628177740940	2.630690030051	2.630559814008	2.630494708429
0.2	2.762927295189	2.765568392117	2.765431499755	2.765363056141
0.3	2.904585606820	2.907362115683	2.907218204701	2.907146251908
0.4	3.053506895400	3.056425758917	3.056274469460	3.056198827568
0.5	3.210063541719	3.213132058568	3.212973012335	3.212893492201
0.6	3.374647018940	3.377872862012	3.377705661304	3.377622064085
0.7	3.547668871483	3.551060107066	3.550884333795	3.550796450455
0.8	3.729561744103	3.733126852053	3.732942066693	3.732849677478
0.9	3.920780463725	3.924528358668	3.924334099161	3.924236973049
1.0	4.121803176750	4.125743230376	4.125539010970	4.125436905097

Table 3: Exact and Approximat values for $b(t)$ with the $\Delta x = 0.01$ and $T = 1$

t	Exact b(t)	$\boldsymbol{\Delta t = 0.000025}$	$\boldsymbol{\Delta t = 0.00005}$	$\boldsymbol{\Delta t = 0.0001}$
0.1	0.0037488284	0.0055149706	0.00639881100	0.0080428688
0.2	0.0149625711	0.0167950443	0.01771296228	0.2046174532
0.3	0.0334670499	0.0352670499	0.03616557041	0.0388769044
0.4	0.0588179936	0.0606179936	0.05858923999	0.0642164877
0.5	0.0901937757	0.0920202710	0.09294666634	0.0956826646
0.6	0.1263342890	0.1282676223	0.12923426891	0.1321231781
0.7	0.1655487586	0.1683987565	0.16839685353	0.1712476475
0.8	0.2058064870	0.2076398203	0.20855699662	0.2113061677
0.9	0.2449067167	0.2466733834	0.24755766761	0.2502165551
1.0	0.2807017544	0.2825018842	0.28344457107	0.2861011238

Table 4: Exact and Approximate values for $b(t)$ with $\Delta t = 0.000025$ at $T = 1$

t	Exact b(t)	$\boldsymbol{\Delta x = 0.001}$	$\boldsymbol{\Delta x = 0.01}$	$\boldsymbol{\Delta x = 0.1}$
0.1	0.0037488284	0.0019213333	0.0010988100	0.0015511716
0.2	0.0149625711	0.0131291666	0.0122128633	0.0094266548
0.3	0.0334670499	0.0316632234	0.0307696470	0.0280435443
0.4	0.0588179936	0.0570175453	0.0534136161	0.0534167675
0.5	0.0901937757	0.0883597778	0.0874453459	0.0846108763
0.6	0.1263342890	0.1243966550	0.1234332450	0.1205477702
0.7	0.1655487586	0.1638590106	0.1626984449	0.1598373434
0.8	0.2058064870	0.2039553148	0.2420569876	0.2394062424
0.9	0.2449067167	0.2431393330	0.2422563455	0.2396507841
1.0	0.2807017544	0.2789248712	0.2786433109	0.2753688249

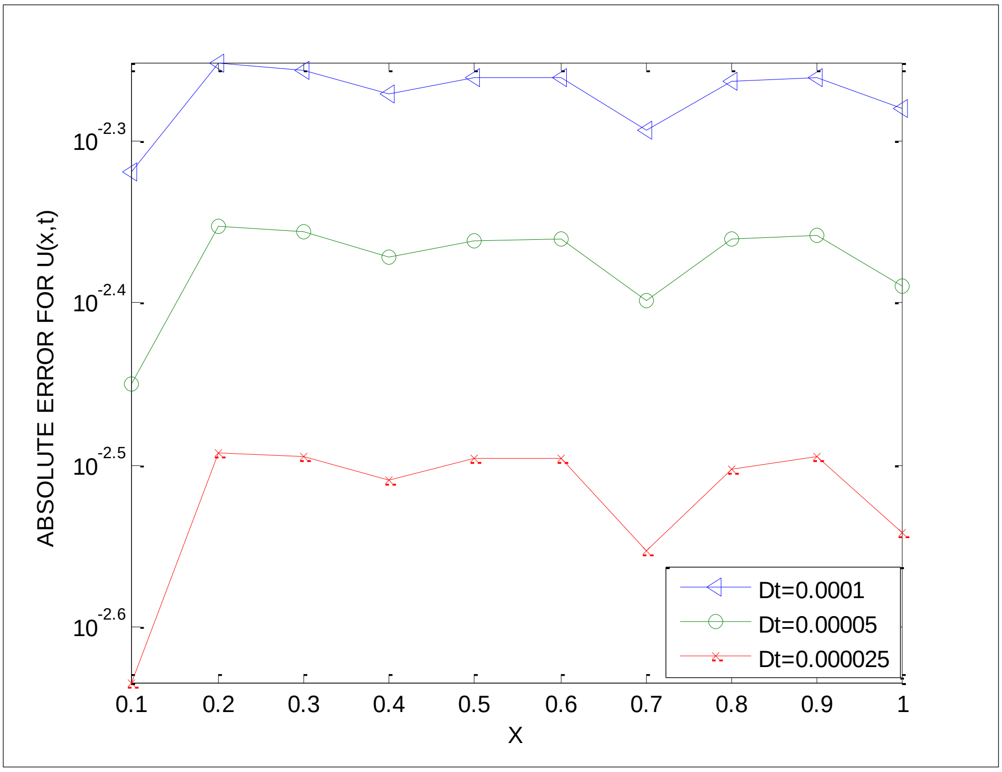

Figure 2: Absolute error for $\phi(x,t)$ and $\Delta x = 0.01, T = 1$

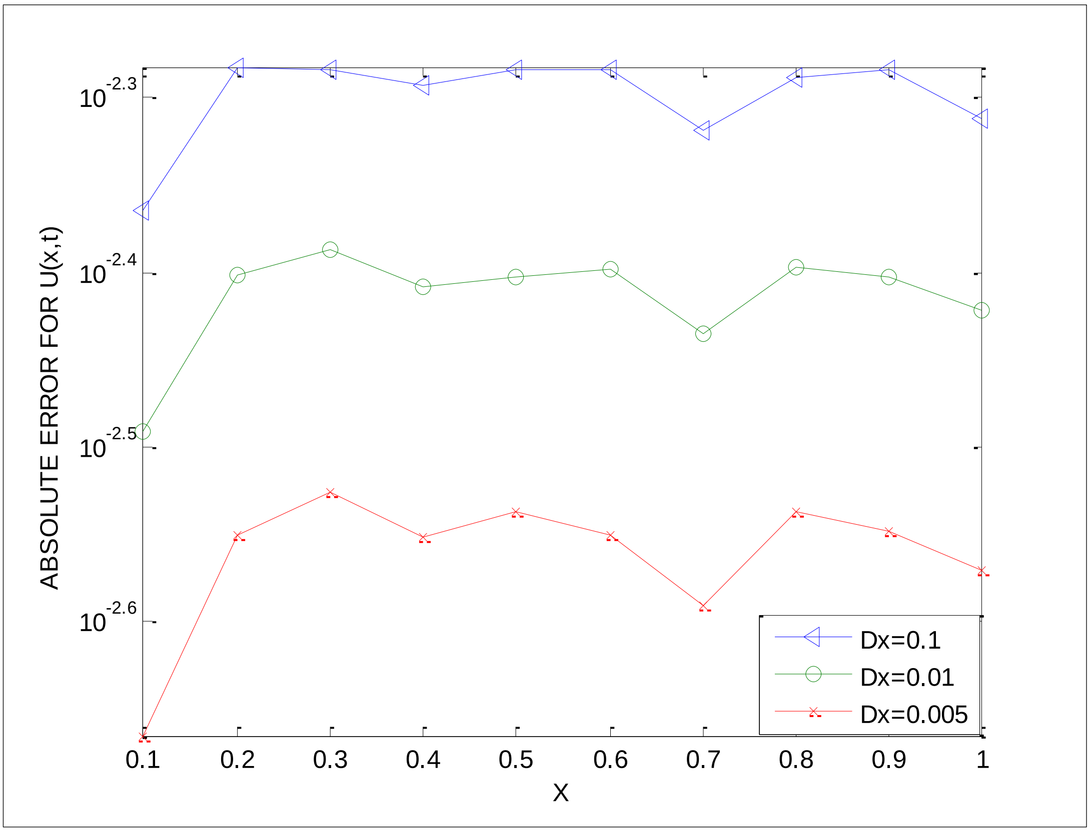

Figure 3: Absolute error for $\phi(x,t)$ and $\Delta t = 0.000025$, at $T = 1$

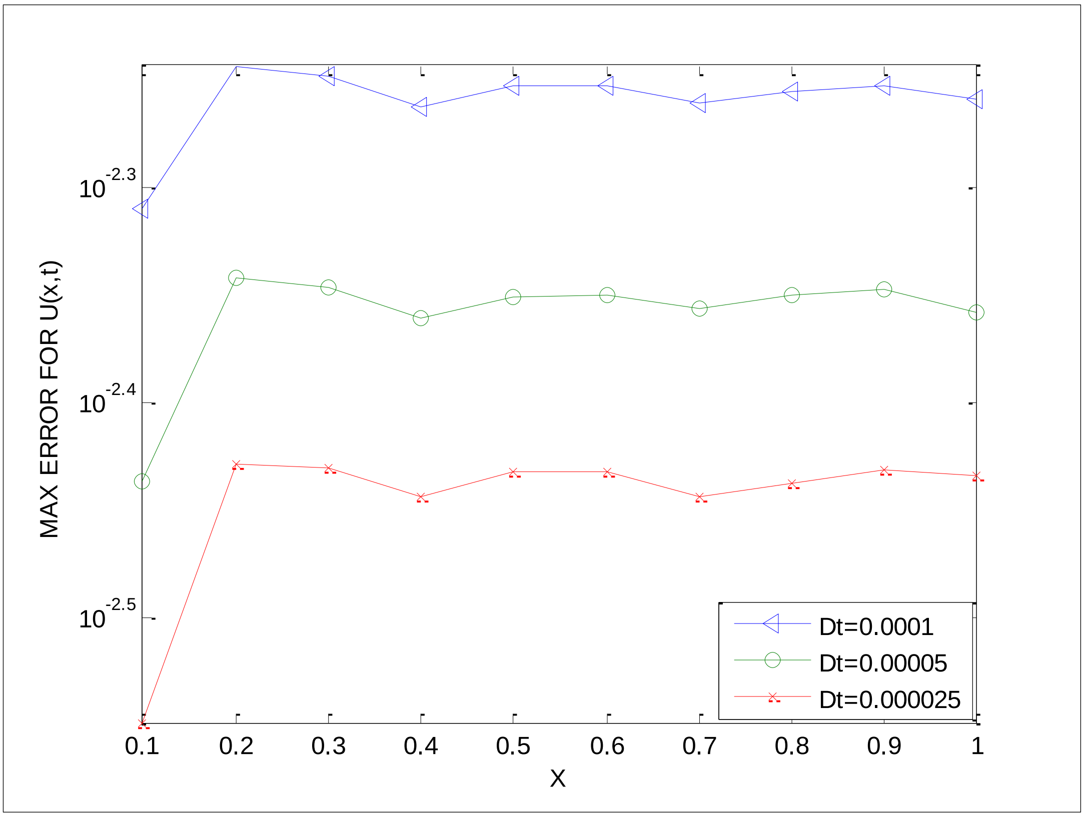

Figure 4: Maximum error for $\phi(x,t)$ and $\Delta x = 0.01$, at each time level

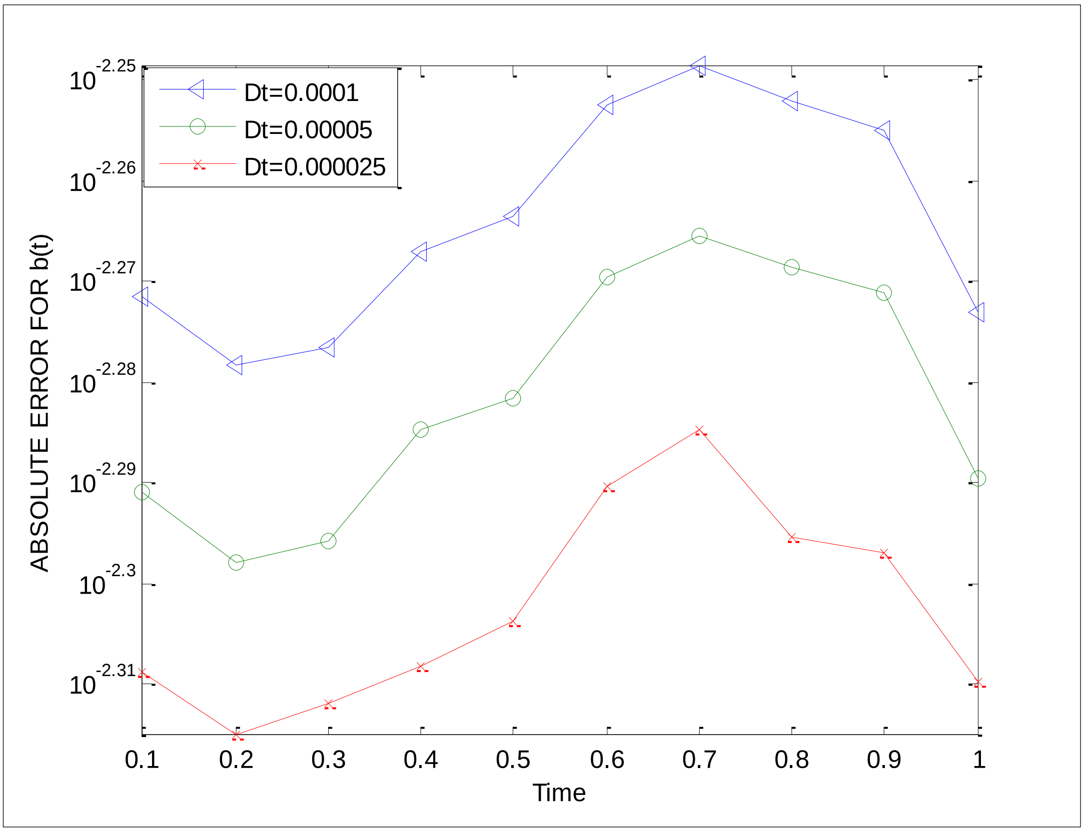

Figure 5: Absolute error for $b(t)$ and $\Delta x = 0.01$ at each time level

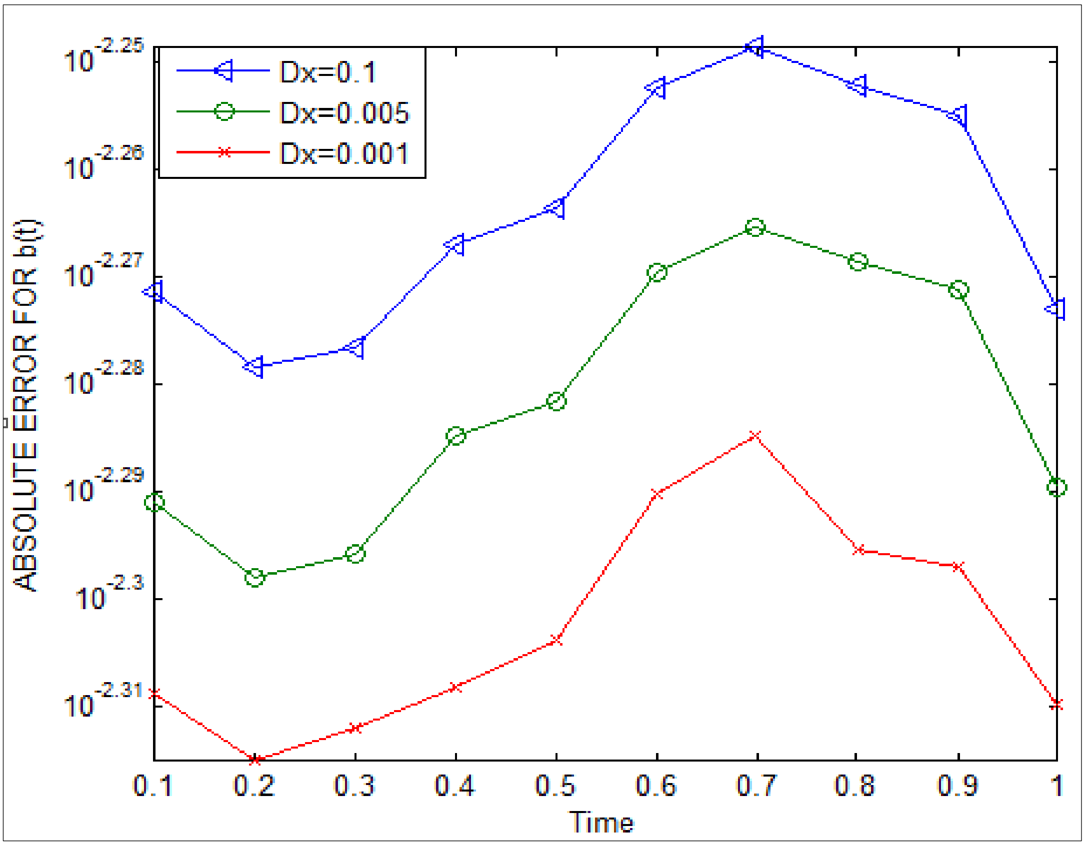

Figure 6: Absolute error for $b(t)$ and $\Delta t = 0.0001$, at each time step

4.2. Problem 2

Consider the problem $(1.1) - (1.5)$ with

$$\phi_t = b(t)\phi_{xx}, \qquad \text{in} \quad \Omega^T$$

subject to the initial condition

$$\psi(x) = e^{\frac{x}{2}}, \qquad x \in [0,1],$$

and the boundary conditions

$$\mathrm{H}_1(t) = \frac{1+2t^3}{1+t^3} + \sin\left(\frac{t}{2}\right), \qquad t \in [0,1],$$

$$\mathrm{H}_2(t) = \sqrt{e^1}\left[\frac{1+2t^3}{1+t^3} + \sin\left(\frac{t}{2}\right)\right]. \qquad t \in [0,1]$$

With fixed point $x^* = 0.25$,

$$q(t) = \frac{1.13315(1 + 2t^3)}{1 + t^3} + 1.13315 \sin\left(\frac{t}{2}\right),$$

for which the exact solution is

$$\phi(x, t) = e^{\frac{x}{2}}\left[\frac{1 + 2t^3}{1 + t^3} + \sin\left(\frac{t}{2}\right)\right],$$

$$b(t) = \frac{2\left[6t^2 + (1 + t^3)^2 \cos\left(\frac{t}{2}\right)\right]}{(1 + t)^3\left[1 + 2t^3 + (1 + t^3)\sin\left(\frac{t}{2}\right)\right]}.$$

Table 5: Exact and approximate values of ϕ with $\Delta x = 0.01$, $x^* = 0.25$ and $T = 1$

$\boldsymbol{x}$	**Exact $\boldsymbol{\phi}$**	$\boldsymbol{\Delta t = 0.0001}$	$\boldsymbol{\Delta t = 0.00005}$	$\boldsymbol{\Delta t = 0.000025}$
0.1	2.080912856	2.0799698300	2.0793100351	2.0798145576
0.2	2.187603540	2.1865573436	2.0793100351	2.1863920237
0.3	2.299764373	2.2981636342	2.2979172032	2.2983145576
0.4	2.417675813	2.4159052213	2.4148335668	2.4162854439
0.5	2.541632703	2.5395335842	2.5396375432	2.5403305592
0.6	2.671944999	2.6701207451	2.6700438147	2.6707356278
0.7	2.808938548	2.8067963407	2.8087237854	2.8078141151
0.8	2.952955907	2.9509682421	2.9516299007	2.2947399658
0.9	3.104357192	3.1026559118	3.1060514115	3.1026144625
1.0	3.263520991	3.2614316231	3.2654559224	3.2617257632

Table 6 : Exact and approximate values of ϕ with $\Delta t = 0.000025$ and $T = 1$

$\boldsymbol{x}$	**Exact** $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{t})$	$\boldsymbol{\Delta x = 0.1}$	$\boldsymbol{\Delta x = 0.01}$	$\boldsymbol{\Delta x = 0.001}$
0.1	2.080912856	2.0798145506	2.0793333151	2.0799698312
0.2	2.187603540	2.1863920226	2.0798122350	2.1865573433
0.3	2.299764373	2.2983145577	2.2969172032	2.2981636342
0.4	2.417675813	2.4162854743	2.4148335668	2.4159052213
0.5	2.541632703	2.5403305592	2.5396375432	2.5395335842
0.6	2.671944999	2.6707356278	2.6700438147	2.6701207451
0.7	2.808938548	2.8078141151	2.8087237854	2.8067963407
0.8	2.952955907	2.2947399658	2.9516299007	2.9509682423
0.9	3.104357192	3.1026144625	3.1060514115	3.1026559117
1.0	3.263520991	3.2617257666	3.2654559500	3.2614316241

Table 7: Exact and approximate values of $b(t)$ for $\Delta x = 0.01$ at $T = 1$

t	**Exact** $\boldsymbol{b}$	$\boldsymbol{\Delta t = 0.0001}$	$\boldsymbol{\Delta t = 0.00005}$	$\boldsymbol{\Delta t = 0.000025}$
0.1	1.979634405	1.976432650	1.977233816	1.973481614
0.2	2.014562188	2.011267253	2.012114123	2.011564574
0.3	2.098282728	2.094776455	2.094786440	2.095352100
0.4	2.222861861	2.219497253	2.221629098	2.220161956
0.5	2.378381379	2.374881765	2.376814069	2.371243057
0.6	2.552887320	2.569284432	2.559333432	2.549777645
0.7	2.732893051	2.729096564	2.729422650	2.729689346
0.8	2.904389231	2.901266201	2.905432431	2.901565492
0.9	3.054179413	3.050364431	3.046972533	3.051177134
1.0	3.171252165	3.167971771	3.167856549	3.168307690

Table 8: Exact and approximate values of $b(t)$ for $\Delta t = 0.000025$ at $T = 1$

t	Exact $b(t)$	$\Delta x = 0.1$	$\Delta x = 0.01$	$\Delta x = 0.001$
0.1	1.979634405	1.978233300	1.976432622	1.985240054
0.2	2.014562188	2.013114126	2.011267221	2.018562132
0.3	2.098282728	2.097786440	2.094776413	2.098776432
0.4	2.222861861	2.221629342	2.225497219	2.228618612
0.5	2.378381379	2.374814236	2.375881794	2.379881765
0.6	2.552887320	2.559334752	2.569284421	2.569284432
0.7	2.732893051	2.724422134	2.726096531	2.729396564
0.8	2.904389231	2.903432442	2.901266239	2.907266201
0.9	3.054179413	3.045974532	3.056369346	3.057944316
1.0	3.171252165	3.164856698	3.165972351	3.169971997

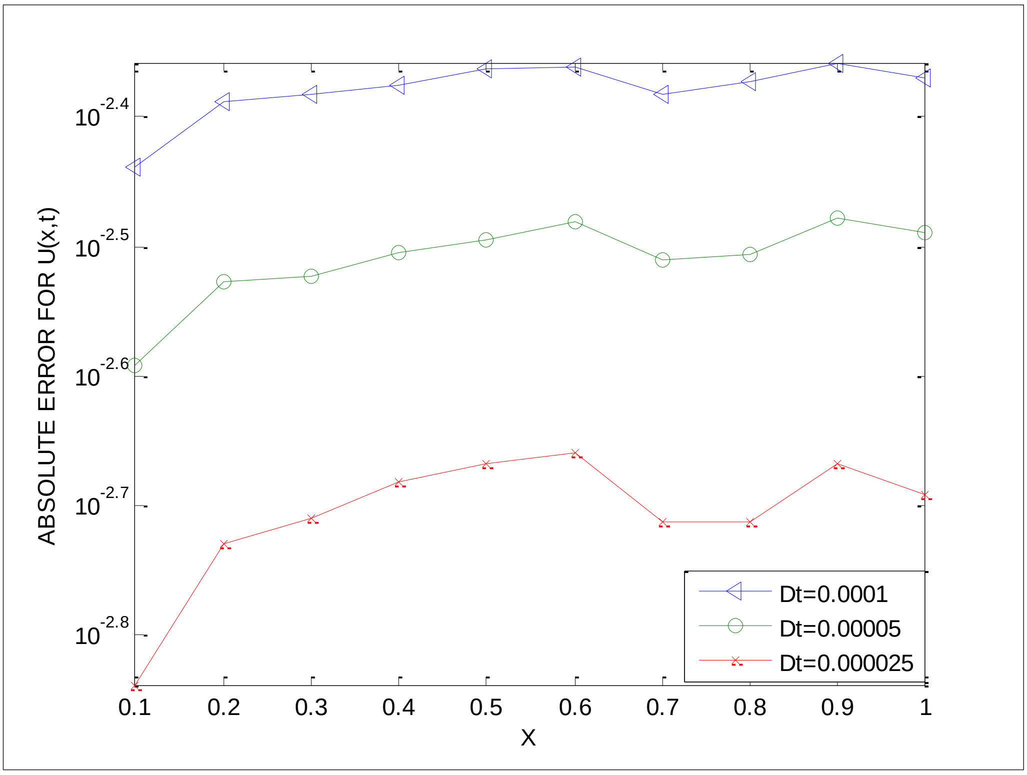

Figure 7: Absolute errors for $\phi(x,t)$ with $\Delta x = 0.01$ at $T = 1$

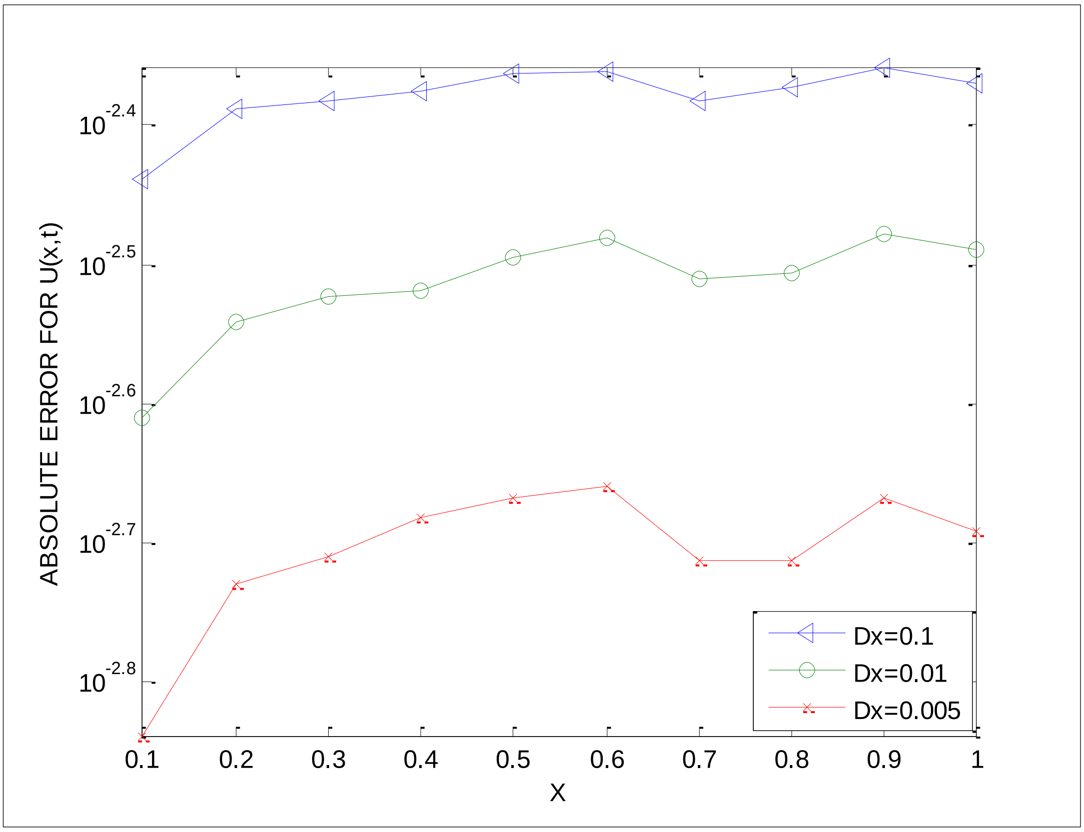

Figure 8: Absolute errors for $\phi(x,t)$ and $\Delta t = 0.000025$, at $T = 1$

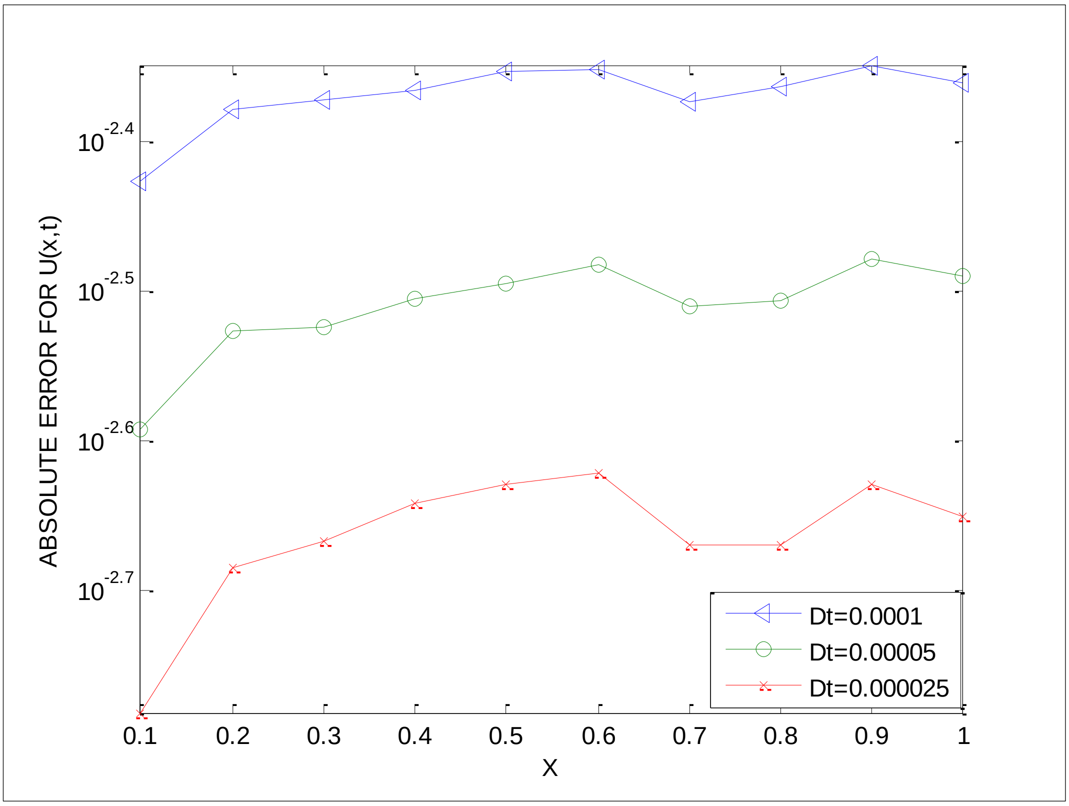

Figure 9: Maximum error of $\phi(x,t)$ for $\Delta x = 0.01$ at each time step

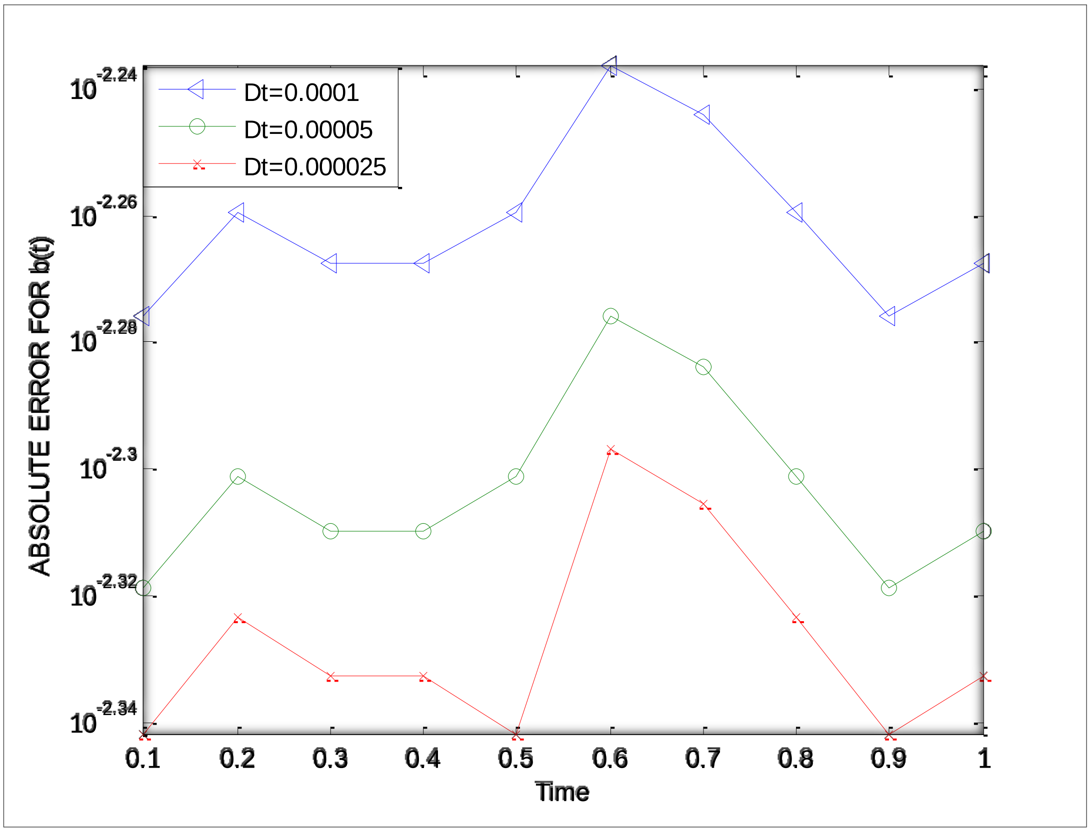

Figure 10: Absolute errors for $b(t)$ for $\Delta x = 0.01$ at each time level

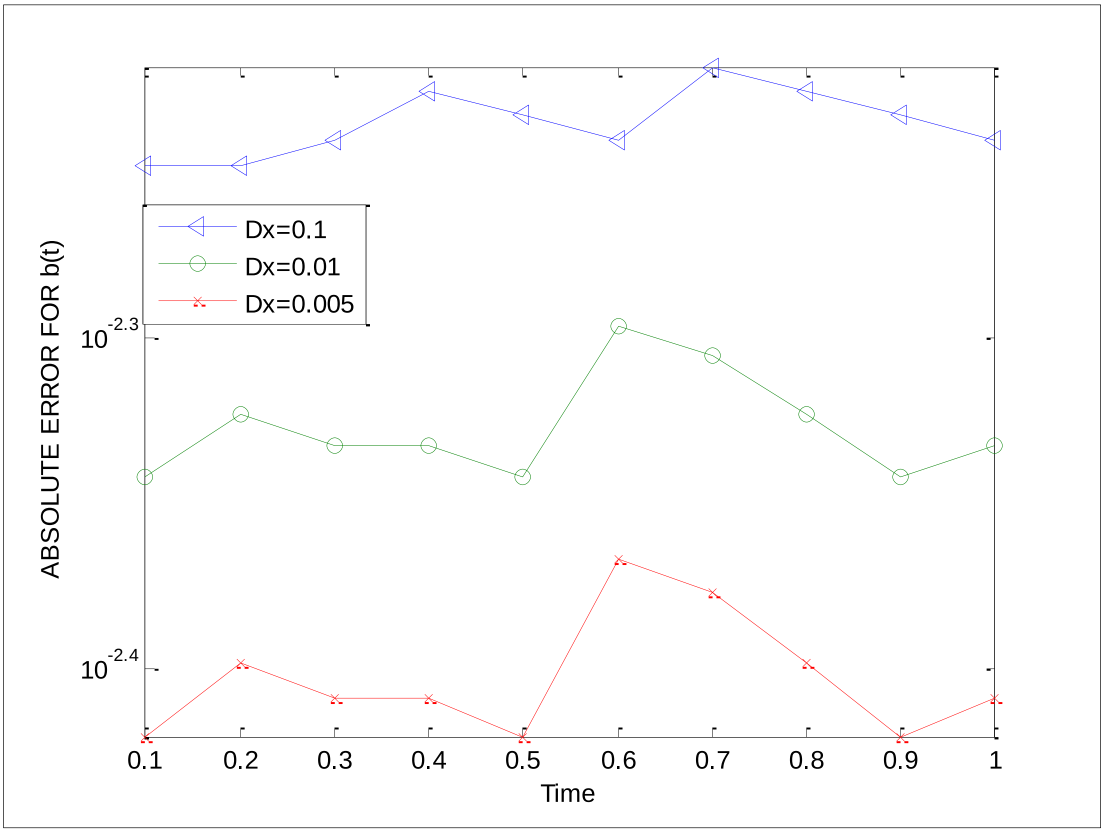

Figure 11: Absolute errors of $b(t)$ for $\Delta t = 0.00025$ at each time step

4.3: Overall Conclusion

The following conclusion can be drawn from the two presented numerical problems**:**

We used backward time centered space to compute the numerical solution to $\phi(x,t)$ at $T = 1$, with the various value of Δt and Δx. The errors plotted in Figure.2, Figure.4, Figure.8 and Figure.10 respectively, we can easily observed that the errors decreases rapidly when Δt is decreases with the fixed value of $\Delta x = 0.01$. it is obvious to see that on the both side of the boundary of Figure 2. and Figure 8, the errors is nearly zero because of the existing of boundary conditions on both side, Therefore the error at the boundary points are sufficiently small.

In Figure 3 and Figure 9, we can also observed that if Δt is fixed as $\Delta t = 0.000025$, the error of $\phi(x,t)$ decreases rapidly when Δx decreases.

The numerical errors for the diffusion coeffient $b(t)$ at different time level that are plotted In Figure 5, 7, 11 and Figure 13, respectively it was observed that if Δx is fixed as $\Delta x = 0.01$, the error of $b(t)$ decreases rapidly when Δt decreases. Likewise when Δt is fixed the as $\Delta t = 0.000025$, the error for $b(t)$ decrease rapidly when Δx is decreasing . It was also obvserve that the error is nearly zero when $(t \to 0)$. This is sensible because the initial condition is logically existing so therefor when $t = 0$ the errors disappears.

Chapter 5

CONCLUSION AND FUTURE WORK

In this thesis Backward time centered space of the finite difference scheme were applied for recovering time dependent diffusion coeffient in one-dimensional parabolic inverse problem. The suggested numerical approaches for solving these two model problems are very reasonable and these test experiment backed our theoretical expectation. Using the backward time centered space formula for the one –dimensional diffusion problem with an additional measurement define our model well. Various of issues can be tendent as subject for future examinations in this field. We can mentioned some of them in the following: We can extend this research to two or three dimensional problems, Employing Crank Nicolson finite difference techniques to solve the current problems, we can also extend to higher-order accurate finite difference methods, we can also apply on explicit formula which is conditionally stable, dealing with the more difficult extra measuments, using new numerical measures for solving Backward time centered space problems by using the descrived methods for simplifying the present problem with the Neumann's boundary condition.

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