On A Comparative Study of Direct Solution Methods of the Discrete Poisson's Equation on A Rectangle

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ABSTRACT

The solution of systems of algebraic equations arising from the 5-point discretization of Poisson's equation on a rectangle with Dirichlet boundary conditions is analyzed by direct solution methods. Special emphasis is given for block direct methods, such as block elimination, block decomposition and block cyclic reduction methods. For this purpose block elimination algorithms, orthogonal block decomposition algorithms, cyclic odd even reduction method, (CORF) algorithm and Buneman version of the CORF algorithm is also studied. A test problem is constructed for the Laplace equation and solved by these block methods for the mesh size $h = \frac{1}{4}$.
Comparisons are given based on the computational complexity of the methods.

Keywords: Block elimination methods, block cyclic reduction method, block decomposition methods, Thomas algorithm, discrete Poisson's equation, 5-point scheme.

ÖZ

Poisson denkleminin dikdörtgen üzerindeki Dirichlet sınır değer probleminin 5-nokta çözümlemesi ile elde edilen cebirsel denklem takımlarının çözümü doğrudan yöntemler ile incelendi. Blok yoketme yöntemleri, blok ayrıştırma yöntemleri, ve blok döngüsel indirgeme yöntemleri gibi blok doğrudan yöntemlere özel önem verildi. Bu amaç doğrultusunda blok yoketme algorithmaları, dik blok ayrıştırma algorithmaları, tek çift döngüsel indirgeme metodu, (CORF) algorithması ve Buneman versiyonu çalışıldı. Laplace denklemi için bir test proplemi oluşturuldu ve adım büyüklüğü $h = \frac{1}{4}$ için verilen yöntemler ile çözüldü. Karşılaştırmalar yöntemlerin hesaplama karmaşasına göre verildi.

Anahtar kelimeler: Blok yoketme yöntemi, blok döngüsel indirgeme yöntemi, blok ayrıştırma yöntemi, Thomas algorıthması, Poisson denklemi, 5-nokta şeması.

To God and my family.

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INTRODUCTION

Many problems in Science and Engineering need the solution of the Poisson's equation,

$$\Delta u = y \ in \ R,$$
$$u = w \ on \ \partial R, \tag{1.1}$$

where R is a rectangle, ∂R is the boundary of R, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, and y, w are known functions.

Finite differences with 5-point, 9-point or 7-point schemes may be used for approximating the partial differential equation to find the numerical solution of (1.1). These schemes for (1.1) results in algebraic linear systems of equations which are usually in large dimensions and are sparse systems. Classical direct methods such as Gaussian elimination, LU decomposition methods are inefficient both in storage and computational complexity. Therefore, iterative methods such as point successive over relaxation (SOR) and Peaceman-Rachford alternating direction implicit iteration (ADI) method were used for the solution of such discrete problems. In general, iterative methods have some pitfalls which includes;

1. Initial guess to generate successive approximations to a solution,

2. Total computational complexity increases as iteration number increases,

3. Convergence rate, which sometimes depends on the spectral properties of the coefficient matrix,

4. In general, accuracy which is usually determined by the convergence test is inferior to the accuracy of the direct method and is limited as the exact solution of the equation cannot be obtained in finite number of steps.

However, in the second half of the 20th century, direct methods which utilize the special block structure of these linear system of equations have been proposed [1]. Some of these methods are; block elimination methods, block decomposition methods, cyclic reduction methods, tensor product methods and the Fourier series methods.

In this thesis, we consider systems of algebraic simultaneous equations arising from the 5-point discretization of Poisson's equation on a rectangle with Dirichlet boundary conditions, which results to symmetric block tridiagonal matrices. Block direct methods such as block elimination methods, block decomposition methods, and cyclic reduction methods will be analyzed and a comparative study will be provided based on their computational complexity.

In Chapter 2, the 5-point finite difference analogue of the Poisson's equation on a rectangle with Dirichlet boundary conditions is reviewed and remarked that the resulting system of equations possess block tridiagonal coefficient matrix. A test problem is considered and a system of equations is obtained when the mesh size h is $h = 1/4$.

In Chapter 3, block elimination methods are analyzed for the solution to the non-singular block tridiagonal systems and particularly for the solution of the obtained block tridiagonal system from the 5-point difference analogue of the Poisson's equation. The test problem is solved by block Gaussian elimination method, block polynomial form, Schechter form and the simplified Schechter form algorithm.

In Chapter 4, the matrix decomposition methods are analyzed for the solution of the general block tridiagonal systems. Orthogonal block decomposition algorithm is studied, which requires the eigenvalues and eigenvectors of the blocks in the main block diagonal for the obtained block tridiagonal matrices. Therefore, we also reviewed the power method for finding eigenvalues and eigenvectors.

In Chapter 5, we considered the cyclic odd-even reduction and factorization (CORF) algorithm to solve the systems. Due to the difficulties encountered in using the CORF algorithm, the Buneman version of CORF algorithm is also studied.

In Chapter 6, comparisons are given according to the computational complexity.

Chapter 2

DISCRETE POISSON'S EQUATION ON A RECTANGLE

2.1 Introduction

One of the forms in which a second order partial differential equation in two variables can be classified is the elliptic form. In general, it is of the form;

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0, \tag{2.1}$$

which satisfies the condition $B^2 - AC < 0$, for $u_{xy} = u_{yx}$. Its basic example is the Laplace Equation;

$$\nabla^2 u = 0. \tag{2.2}$$

But if the equation is non-homogeneous, then it is called Poisson's Equation. The Poisson's Equation, named after a French mathematician, physicist and geometer Simeon Denis Poisson, is required to solve many physical problems, e.g. the steady-state temperature distribution on a heated plate. It is usually written in the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{2.3}$$

where u is some scalar potential which is to be determined and $f(x, y)$ is a known function.

2.2 Finite Difference Analogue of the Poisson's Equation

A finite difference analogue of the Poisson's equation is the Discrete Poisson's equation. For computational purposes, finite difference analogue based on treating the plate as a grid of discrete points are substituted for the partial derivatives in (2.3). Let's consider a rectangle $R = (0, a) \times (0, b)$ and define mesh spacing $\Delta x = \frac{a}{N+1}$ and $\Delta y = \frac{b}{M+1}$ (M and N are integers). The mesh points $x_i = i\Delta x \; and \; y_j = j\Delta y$ are used to define the discrete interior R_h and discrete boundary ∂R_h such that

$$R_h = \{(x_i, y_j) \big| 1 \leq i \leq N, 1 \leq j \leq M\},$$
$$\partial R_h = \partial R \cap \{(x_i, y_j) \big| 0 \leq i \leq N+1, 0 \leq j \leq M+1\}. \tag{2.4}$$

With the notation $U_{ij} = U(x_i, y_j)$, we define the usual 5-point approximation and obtain the discrete operator Δ_h and $\Delta_h U_{ij}$ as:

$$\Delta_h U_{ij} = \frac{1}{(\Delta x)^2}\left(U_{i-1,j} - 2U_{ij} + U_{i+1,j}\right) + \frac{1}{(\Delta y)^2}\left(U_{i,j-i} - 2U_{ij} + U_{i,j+1}\right). \tag{2.5}$$

If $\Delta x = \Delta y = h$, then we have

$$U_{i+1,j} + U_{i-1,j} + U_{i.j+1} U u_{i,j-1} - 4U_{i,j} = h^2 f_{i,j}. \tag{2.6}$$

This is also known as the five-point difference formula. For Laplace's equation, the right hand side is zero, i.e.

$$U_{i+1,j} + U_{i-1,j} + U_{i.j+1} + U_{i,j-1} - 4U_{i,j} = 0. \tag{2.7}$$

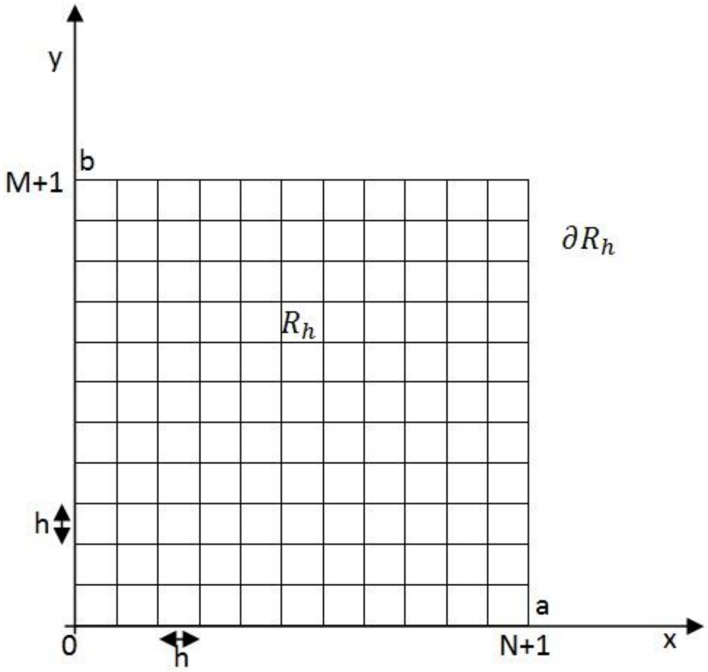

Figure 1. A rectangular grid used for finite difference equation.

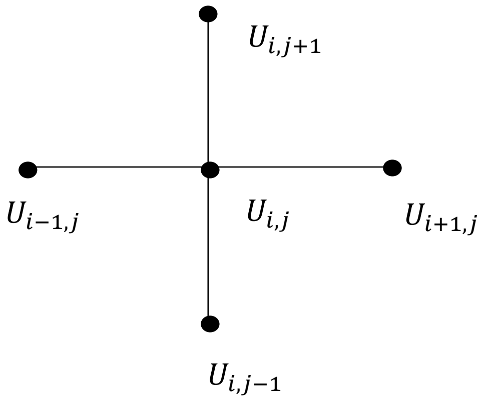

Figure 2. A 5-point Stencil.

2.3 The Dirichlet Boundary Condition

A Dirichlet Boundary Condition is a continuous function given on the boundary ∂R of the domain that the solution satisfies. Consider the boundary value problem (1.1);

Let $R = \{(x, y): 0 < x < a, 0 < y < b\}$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \; on \; R,$$

$$u(x, y) = \varphi_j \; on \; \gamma_j, \qquad j = 1,2,3,4 \,, \tag{2.8}$$

where R is a rectangle, φ_j are known boundary conditions on the boundaries γ_j, counted in anticlockwise direction, where γ_1 is the boundary on the side $x = 0$.

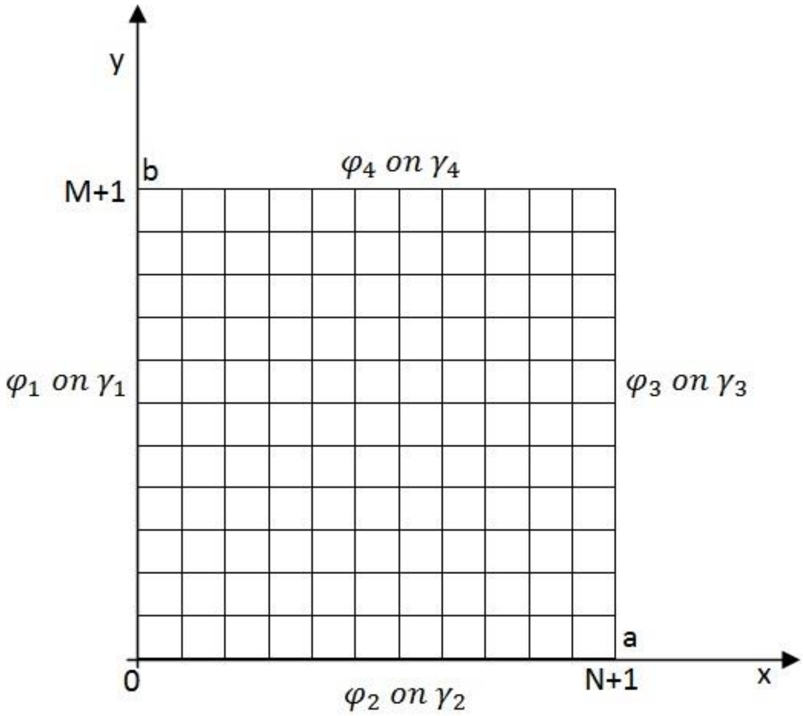

Figure 3. A rectangular grid indicating its boundary conditions.

Using the 5-point difference analogue of the Poison's equation given in Section 2.2 and employing the boundary conditions, we get the discrete Poisson's problem

$$\Delta_h U = h^2 f_{ij} \; on \; R_h,$$

$$U = \varphi_j \; on \; \gamma_j. \tag{2.9}$$

For the approximate solution at the interior grids, we need to solve the algebraic system of equations,

$$AU = Y, \tag{2.10}$$

obtained from (2.9) which can be written in block tridiagonal matrices form as:

$$\begin{bmatrix} D_1 & C_1 & 0 & \cdots & 0 \\ A_2 & D_2 & C_2 & 0 & 0 \\ 0 & A_3 & D_3 & C_3 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & & D_{N-1} & C_{N-1} \\ 0 & \cdots & 0 & A_N & D_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}, \tag{2.11}$$

where $A = \begin{bmatrix} D_1 & C_1 & 0 & \cdots & 0 \\ A_2 & D_2 & C_2 & 0 & 0 \\ 0 & A_3 & D_3 & C_3 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & & D_{N-1} & C_{N-1} \\ 0 & \cdots & 0 & A_N & D_N \end{bmatrix}, U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}$ $and\ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}$.

In this case, A_j and C_j are identity matrices and $D_j = D$ for $j = 1,2, \ldots, N$, so we have

$$A = [I \;\; D \;\; I]_{N\times N},$$

$$I = [0 \;\; 1 \;\; 0]_{M\times M},$$

$$D = [\lambda \;\; -2(1+\lambda) \;\; \lambda]_{M\times M},$$

$$\lambda = \left(\frac{\Delta x}{\Delta y}\right)^2.$$

When $\Delta x = \Delta y$, $\lambda = 1$ and $D = [1 \;\; -4 \;\; 1]_{M\times M}$. We define u_i as the vector with components comprising of the ith vertical line of the array U,

$$u_i = \begin{bmatrix} U_{i1} \\ U_{i2} \\ U_{i3} \\ U_{i4} \\ \vdots \\ U_{iM} \end{bmatrix}, \quad 1 \le i \le N.$$

2.3.1 Test Problem

Let $R = \{(x,y): 0 < x < 1, 0 < y < 1\}$, considering a case of (2.8) for $f(x,y) = 0$, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; on \; R,$$

with boundary conditions:

$$\varphi_1(y) = \sin y \;\; on \; \gamma_1,$$

$$\varphi_2(x) = 0 \; on \; \gamma_2,$$

$$\varphi_3(y) = e^1 \sin y \;\; on \; \gamma_{3,}$$

$$\varphi_4(x) = e^x \sin 1 \;\; on \; \gamma_4,$$

where the exact solution is $u(x, y) = e^x \sin y$.

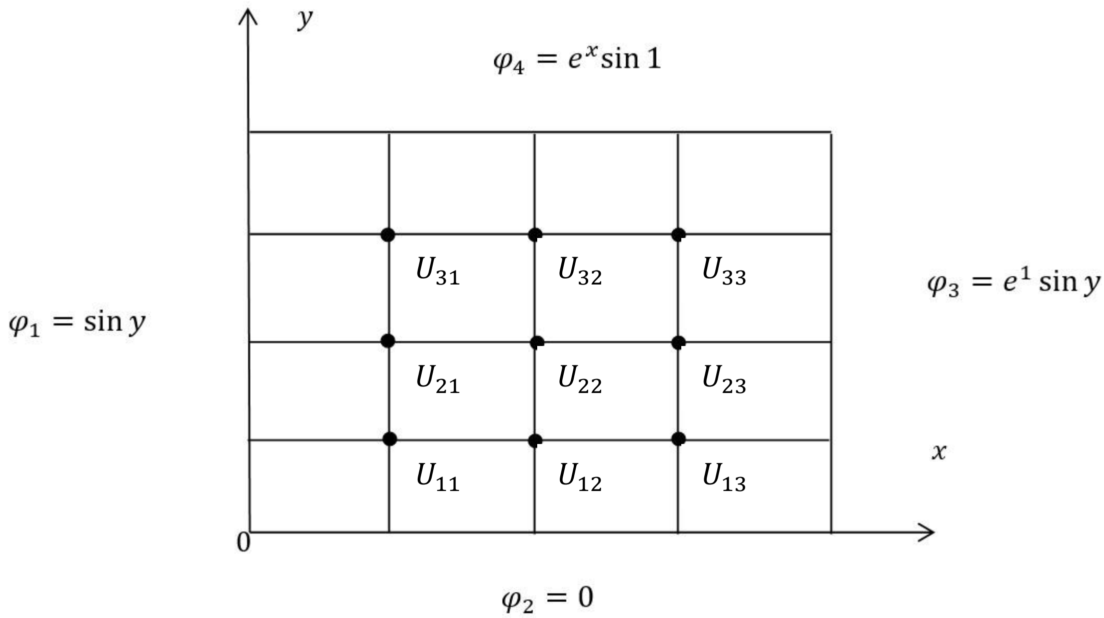

Figure 4. A square grid for the case where $N = M = 3$, and $h = 1/4$.

Taking $h = 1/4$, a balance node for U_{11} according to (2.7) is $-4U_{11} + U_{12} + U_{21} = -\varphi_2(h) - \varphi_1(h)$. For U_{12}, the equation is $U_{11} - 4U_{12} + U_{13} + U_{22} = -\varphi_2(2h)$. The balance nodes for other interior points can be generated accordingly and (2.7) results to the matrix equation below:

$$\begin{bmatrix} D & I & 0 \\ I & D & I \\ 0 & I & D \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

where $y_1 = \begin{bmatrix} -\varphi_2(h) - \varphi_1(h) \\ -\varphi_2(2h) \\ -\varphi_2(3h) - \varphi_3(h) \end{bmatrix}, y_2 = \begin{bmatrix} -\varphi_1(2h) \\ 0 \\ -\varphi_3(2h) \end{bmatrix}, y_3 = \begin{bmatrix} -\varphi_1(3h) - \varphi_4(h) \\ -\varphi_4(2h) \\ -\varphi_3(3h) - \varphi_4(3h) \end{bmatrix}$.

Let V be the trace of the exact solution u on the grid,

$$V_1(h,h) = e^h \sin h = e^{0.25} \sin 0.25 = 0.317673$$

$$V_2(2h,h) = e^{2h} \sin h = e^{0.5} \sin 0.25 = 0.407900$$

$$V_3(3h,h) = e^{3h} \sin h = e^{0.75} \sin 0.25 = 0.523754$$

$$V_4(h,2h) = e^h \sin 2h = e^{0.25} \sin 0.5 = 0.615595$$

$$V_5(2h,2h) = e^{2h} \sin 2h = e^{0.5} \sin 0.5 = 0.790439$$

$$V_6(3h,2h) = e^{3h} \sin 2h = e^{0.75} \sin 0.5 = 1.014944$$

$$V_7(h,3h) = e^h \sin 3h = e^{0.25} \sin 0.75 = 0.875241$$

$$V_8(2h,3h) = e^{2h} \sin 3h = e^{0.5} \sin 0.75 = 1.123832$$

$$V_9(3h,3h) = e^{3h} \sin 3h = e^{0.75} \sin 0.75 = 1.443029.$$

Representing the solution in block form:

$$v_1 = \begin{bmatrix} 0.317673 \\ 0.407900 \\ 0.523754 \end{bmatrix}, v_2 = \begin{bmatrix} 0.615595 \\ 0.790439 \\ 1.014944 \end{bmatrix}, v_3 = \begin{bmatrix} 0.875241 \\ 1.123832 \\ 1.443029 \end{bmatrix},$$

the right-hand side results to:

$$Y_1 = -\varphi_1(h) - \varphi_2(h) = -\sin 0.25 - 0 = -0.247404$$

$$Y_2 = -\varphi_2(2h) = 0$$

$$Y_3 = -\varphi_2(3h) - \varphi_3(h) = 0 - e^1 \sin 0.25 = -0.672514$$

$$Y_4 = -\varphi_1(2h) = -\sin 0.5 = -0.479426$$

$$Y_5 = 0$$

$$Y_6 = -\varphi_3(2h) = -e^1 \sin 0.5 = -1.303214$$

$$Y_7 = -\varphi_1(3h) - \varphi_4(h) = -\sin 0.75 - e^{o.25} \sin 1 = -1.762109$$

$$Y_8 = -\varphi_4(2h) = -\,e^{0.5}\sin 1 = -1.387351$$

$$Y_9 = -\varphi_3(3h) - \varphi_4(3h) = -e^1 \sin 0.75 - e^{0.75}\sin 1 = -3.634280.$$

The right-hand side in block form is:

$$y_1 = \begin{bmatrix} -0.247404 \\ 0 \\ -0.672514 \end{bmatrix}, y_2 = \begin{bmatrix} -0.479426 \\ 0 \\ -1.303214 \end{bmatrix}, y_3 = \begin{bmatrix} -1.762109 \\ -1.387351 \\ -3.634280 \end{bmatrix}.$$

Chapter 3

BLOCK- ELIMINATION METHODS

3.1 Introduction

The difference analogue of the Poisson's equation produces a set of algebraic equations (2.10). In this Chapter, the block-Gaussian elimination, the block-polynomial form and the Schechter form will be reviewed to find solution to the system (2.10).

3.2 Block-Gaussian Elimination Method

Considering the block tridiagonal matrix equation in its general form, where A_j and C_j may not be identity matrix,

$$AU \equiv [A_j \;\; D_j \;\; C_j]U = Y, \tag{3.1}$$

where A is a block matrix with dimension N and A_j, D_j and C_j are $M \times M$ matrices. This method depends upon the calculation of matrix inverse, which is also stored as it is used recursively. The procedure for block-Gaussian elimination for the solution of (2.10) can be written in the form [2], [1]:

Algorithm: Block Gaussian Elimination [1]

$$
\begin{aligned}
f_1 &= D_1^{-1} y_1, \\
R_1 &= -D_1^{-1} \mathrm{C}_{1,} \\
f_j &= \left(A_j R_{j-1} + D_j\right)^{-1} \left(y_j - A_j f_{j-1}\right), \qquad 2 \le j \le N, \\
R_j &= -\left(A_j R_{j-1} + D_j\right)^{-1} C_j, \qquad 2 \le j \le N-1,
\end{aligned}
\tag{3.2}
$$

$$u_N = f_N,$$

$$u_j = f_j + R_j u_{j+1}\,, \qquad\qquad 1 \le j \le N-1.$$

This procedure is stable and will produce an exact solution of the equation (relative to the increase of round off error) provided the matrices

$$A_j R_{j-1} + D_j \qquad\qquad (1 \le j \le N, R_0 = 0)$$

are non-singular (i.e. they have determinant to be non-zero). But for large values of M and N, this procedures may not be too satisfactory in terms of time execution and memory requirements for the storage.

3.2.1 Solution of the Test Problem by Block Gaussian Elimination Method

For the test problem given in Section 2.3.1, $M = N = 3$ and $A_2 = A_3 = I_{3\times3}$,

$$C_1 = C_2 = I_{3\times3},\ D_j = D = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix},\ j = 1,2,3$$

$$f_1 = D_1^{-1} y_1,$$

$$D_1^{-1} = \begin{bmatrix} -0.267857 & -0.071429 & -0.017857 \\ -0.071429 & -0.285714 & -0.071429 \\ -0.017857 & -0.071429 & -0.267857 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} -0.247404 \\ 0 \\ -0.672514 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0.078278 \\ 0.065709 \\ 0.184555 \end{bmatrix}$$

$$R_1 = -D_1^{-1} C_1 = -D_1^{-1}$$

$$R_1 = -D_1^{-1} = \begin{bmatrix} 0.267857 & 0.071429 & 0.017857 \\ 0.071429 & 0.285714 & 0.071429 \\ 0.017857 & 0.071429 & 0.267857 \end{bmatrix}$$

$$f_j = \left(R_{j-1} + D_j\right)^{-1}\left(y_j - f_{j-1}\right), \qquad\qquad j = 2,3$$

$$f_2 = (R_{2-1} + D_2)^{-1}(y_2 - f_{2-1}) = (R_1 + D_2)^{-1}(y_2 - f_1)$$

$$y_2 = \begin{bmatrix} -0.479426 \\ 0 \\ -1.303214 \end{bmatrix}$$

$$f_2 = \begin{bmatrix} 0.212438 \\ 0.211795 \\ 0.460455 \end{bmatrix}$$

$$R_j = -\left(R_{j-1} + D_j\right)^{-1}, \qquad j = 2, 3$$

$$R_2 = -(R_1 + D_2)^{-1} = \begin{bmatrix} 0.294824 & 0.093168 & 0.028157 \\ 0.093168 & 0.322981 & 0.093168 \\ 0.028157 & 0.093168 & 0.294824 \end{bmatrix}$$

$$f_3 = (R_{3-1} + D_3)^{-1}(y_3 - f_{3-1}) = (R_2 + D_3)^{-1}(y_3 - f_2)$$

$$y_3 = \begin{bmatrix} -1.762109 \\ -1.387351 \\ -3.634280 \end{bmatrix}$$

$$f_3 = \begin{bmatrix} 0.875621 \\ 1.124380 \\ 1.443528 \end{bmatrix}$$

$$u_N = f_N,$$

$$u_3 = f_3 = \begin{bmatrix} 0.875621 \\ 1.124380 \\ 1.443528 \end{bmatrix}$$

$$u_j = f_j + R_j u_{j+1} \qquad j = 1, 2$$

$$u_2 = f_2 + R_2 u_3$$

$$u_2 = \begin{bmatrix} 0.615994 \\ 0.791018 \\ 1.015453 \end{bmatrix}$$

$$u_1 = f_1 + R_1 u_2$$

$$u_1 = \begin{bmatrix} 0.317911 \\ 0.408246 \\ 0.524053 \end{bmatrix}.$$

The solution obtained from the block-Gaussian elimination procedures are compared with the exact solution in Table 1:

Table 1. Results of the test problem by Block Gaussian Elimination Method

Unknowns	Exact solution	Block-Gaussian Form	Absolute error
U_{11}	0.317673	0.317911	0.000238
U_{12}	0.407900	0.408246	0.000346
U_{13}	0.523754	0.524053	0.000299
U_{21}	0.615595	0.615994	0.000399
U_{22}	0.790439	0.791018	0.000579
U_{23}	1.014944	1.015453	0.000509
U_{31}	0.875241	0.875621	0.000380
U_{32}	1.123832	1.124380	0.000548
U_{33}	1.443029	1.443528	0.000499

3.3 Block Polynomial Form

The block polynomial form is a simplification of the block-Gaussian elimination method in the case of Poisson's equation, with Dirichlet boundary conditions. Here we have $A_j = C_j = I_{M\times M}$ and $D_j = D$.

Algorithm: Block Polynomial [1]

$$f_j = P_j^{-1}(D)\sum_{q=1}^{j}(-1)^{q+j}P_{q-1}(D)y_q\,, \qquad 1 \le j \le N$$
$$R_j = -P_j^{-1}(D)P_{j-1}(D), \;\; 1 \le j \le N \tag{3.3}$$

where $P_j(D)$ is the polynomial in D of degree j, given by:

$$P_0(D) = I$$
$$P_j(D) = \prod_{q=1}^{j}\left[D - x_q(j)I\right], \qquad j \ge 1 \tag{3.4}$$

where $x_q(j) = 2\cos\frac{q\pi}{j+1}$.

All of the matrices $\left[D - x_q(j)I\right]$ are diagonally dominant, which makes them non singular because the solution of $x_q(j)$ will remain between -2 and 2, i.e. $x_q(j) \in (-2,2)$. Therefore, there exist $P_j^{-1}(D)$ with the algorithm written in the form:

$$u_N = P_N^{-1}(D) \sum_{q=1}^{N} (-1)^{q+N} P_{q-1}(D) y_q,$$

$$u_j = P_j^{-1}(D) \left[\sum_{q=1}^{j} (-1)^{q+j} P_{q-1}(D) y_q - P_{j-1}(D) u_{j+1} \right], \qquad (3.5)$$

$$1 \leq j \leq N-1.$$

Since calculating matrix inverse is not preferable, (3.5) can be rewritten as:

$$P_N(D) u_N = \sum_{q=1}^{N} (-1)^{q+N} P_{q-1}(D) y_q,$$

$$P_j(D) u_j = \left[\sum_{q=1}^{j} (-1)^{q+j} P_{q-1}(D) y_q - P_{j-1}(D) u_{j+1} \right], \qquad (3.6)$$

$$1 \leq j \leq N-1.$$

Each of the polynomials $P_j(D), (1 \leq j \leq N)$ is expressed by solving a diagonally dominant tridiagonal matrix equation, therefore, cyclic reduction methods can be used. For the application, we employed Thomas algorithm to find the solution of the tridiagonal systems of size $M \times M$ each.

3.3.1 The Thomas Algorithm

The Thomas algorithm, also known as tridiagonal matrix algorithm, is an effective way of finding the solution of tridiagonal matrix system. It depends on LU decomposition in which a matrix system $Gz = w$ is rewritten as $LUz = w$, where G is decomposed by LU, L is the lower triangular matrix and U is the upper triangular matrix. Therefore, all the advantages of LU decomposition can be achieved if the algorithm is applied properly. The solution of this system $Gz = w$ is obtained by putting $L\beta = w$ for the solution of β and $Uz = \beta$ for the solution of z. The algorithm consists of three steps which are: decomposition, forward substitution and backward substitution.

Thomas Algorithm [3]**:**

Given $G = [a_q \ d_q \ c_q]$, $G = LU$

$L = [e_q \ \ 1 \ \ 0]$ and $U = [0 \ f_q \ c_q]$, $1 \leq q \leq M$.

To solve $z, L\beta = w$ and $Uz = \beta$

$\beta_q = w_q - e_q\beta_{q-1}, 1 < q \leq M.$

where $\beta_1 = w_1$, β_q is solved by forward substitution.

$z_q = f_q^{-1}(\beta_q - c_q z_{q+1}), \qquad 1 \leq q \leq M - 1.$

$z_M = f_M^{-1}\beta_M$, z_q is solved by backward substitution.

3.3.2 Solution of the Test Problem by Block Polynomial Form

Given that $P_0(D) = I$ and $P_j(D) = \prod_{q=1}^{j}[D - x_q(j)I]$, $\qquad j \geq 1$

$$P_1(D) = [D - x_1(1)I],$$

$$P_2(D) = [D - x_1(2)I][D - x_2(2)I],$$

$$P_3(D) = [D - x_1(3)I][D - x_2(3)I][D - x_3(3)I].$$

Since $x_q(j) = 2\cos\frac{q\pi}{j+1}$,

$x_1(1) = 0$

$x_1(2) = 1$

$x_2(2) = -1$

$x_1(3) = 1.414214$

$x_2(3) = 0$

$x_3(3) = -1.414214$

For $N = 3$,

$P_3(D)u_3 = \sum_{q=1}^{3}(-1)^{q+3}P_{q-1}(D)y_q$

$P_3(D)u_3 = P_0(D)y_1 - P_1(D)y_2 + P_2(D)y_3$

Let $P_0(D)y_1 - P_1(D)y_2 + P_2(D)y_3 = w_1$

$P_3(D)u_3 = w_1$

$P_3(D) = [D - 1.414214I][D][D + 1.414214I]$

$[D - 1.414214I][D][D + 1.414214I]u_3 = w_1$

Let $[D + 1.414214]u_3 = M_1$

$[D - 1.414214I][D]M_1 = w_1$

Let $[D]M_1 = M_2$

$[D - 1.414214I]M_2 = w_1$.

$[D - 1.414214I]$ results into a tridiagonal block matrix which can be solved by Thomas algorithm to find the solution of vector matrix M_2. M_2 is substituted to find $M_1$and $M_1$is substituted to find u_3, all solved using the Thomas algorithm. The result of the block vector $u_3 = \begin{bmatrix} 0.875621 \\ 1.124380 \\ 1.443528 \end{bmatrix}$.

$P_j(D)u_j = \left[\sum_{q=1}^{j}(-1)^{q+j}P_{q-1}(D)y_q - P_{j-1}(D)u_{j+1}\right]$,

$P_2(D)u_2 = \left[\sum_{q=1}^{2}(-1)^{q+2}P_{q-1}(D)y_q - P_{2-1}(D)u_{2+1}\right] = -P_0(D)y_1 + P_1(D)y_2 - P_1(D)u_3$,

Let $-P_0(D)y_1 + P_1(D)y_2 - P_1(D)u_3 = w_2$,

$P_2(D)u_2 = w_2$,

$P_2(D) = [D - I][D + I]$,

$[D - I][D + I]\, u_2 = \ w_2$.

Let $M_3 = [D + I]\, u_2$, $[D - I]M_3 = w_2$. Here also, $[D - I]$ is a tridiagonal block matrix, M_3 is solved for and substituted to find the solution of u_2 by Thomas algorithm.

$$u_2 = \begin{bmatrix} 0.615994 \\ 0.791018 \\ 1.015453 \end{bmatrix}.$$

$$P_1(D)u_1 = \left[\sum_{q=1}^{1}(-1)^{q+1}P_{q-1}(D)y_q - P_{1-1}(D)u_{1+1}\right] = P_0(D)y_1 - P_0(D)u_2,$$

Let $P_0(D)y_1 - P_0(D)u_2 = w_3$, so $P_1(D)\, u_1 = w_3$

Since $P_1(D) = D$, $Du_1 = w_3$ and D is a tridiagonal matrix, this system is solved by Thomas algorithm to find the solution of u_1.

$$u_1 = \begin{bmatrix} 0.317911 \\ 0.408246 \\ 0.524053 \end{bmatrix}.$$

We get exactly the same table as in Table 1 which represents the block Gaussian elimination method.

Table 2. Results of the test problem by Block Polynomial Method

Unknowns	Exact solution	Block-Polynomial Form	Absolute error
U_{11}	0.317673	0.317911	0.000238
U_{12}	0.407900	0.408246	0.000346
U_{13}	0.523754	0.524053	0.000299
U_{21}	0.615595	0.615994	0.000399
U_{22}	0.790439	0.791018	0.000579
U_{23}	1.014944	1.015453	0.000509
U_{31}	0.875241	0.875621	0.000380
U_{32}	1.123832	1.124380	0.000548
U_{33}	1.443029	1.443528	0.000499

3.4 Block Schechter Form

From the block polynomial form, we have that $y = P_j^{-1}(D)x$ exists. So Schechter [4] used an alternative method to solve this equation, proposing the algorithm;

Algorithm: Block Schechter Form [4]

$$D = BSB,$$

$$S = [0 \ \ \lambda_j \ \ 0]_{M\times M}$$

$$\lambda_j = 2\left[\left(\cos\frac{j\pi}{M+1} - 1\right) - 1\right] \tag{3.7}$$

$$(B)_{ij} = \sqrt{\frac{2}{M+1}}\sin\frac{ij\pi}{M+1}\ , \quad 1 \le i, j \le M$$

since $B^2 = I$, we have

$$P_k^{-1}(D)x = B\left[0 \ \ \frac{1}{P_k(\lambda_j)} \ \ 0\right]Bx$$

$$u_j = P_N^{-1}(D)\left[P_{N-j}(D)\sum_{q=1}^{j}(-1)^{q+j}P_{q-1}(D)y_q\right.$$

$$\left. + P_{j-1}(D)\sum_{q=j+1}^{N}(-1)^{q+j}P_{N-q}(D)y_q\right], \quad 1 \le j \le N.$$

But this procedure is not as efficient as the recursion method (3.5), (3.6) defined in the block polynomial form due to its computational complexity which involves a larger operation count, therefore, Schechter proposed a more simplified procedure for these problem:

Algorithm: Simplified Block Schechter Form [4]

$$\begin{aligned} f_1 &= y_1, \\ f_j &= P_{j-i}(D)y_j - f_{j-1}, \quad 2 \le j \le N, \\ u_N &= P_N^{-1}(D)f_N, \\ u_{N-1} &= y_N - Du_N, \\ u_j &= y_{j+1} - Du_{j+1} - u_{j+2}, \quad 1 \le j \le N-2. \end{aligned} \tag{3.8}$$

This procedure, derived by modifying the block Gaussian elimination formulas has a lower operation count when compared to the block polynomial form (3.5), (3.6) and the initial Schechter form (3.7).

3.4.1 Solution of the Test Problem by the Schechter's Algorithm

We use the algorithm (3.7).

Taking $\lambda_j = 2\left[\left(\cos\frac{j\pi}{M+1} - 1\right) - 1\right]$, $j = 1,2,3$ and $M = 3$

$\lambda_1 = 2\left[\left(\cos\frac{\pi}{4} - 1\right) - 1\right] = -2.585786,$

$\lambda_2 = 2\left[\left(\cos\frac{2\pi}{4} - 1\right) - 1\right] = -4,$

$\lambda_3 = 2\left[\left(\cos\frac{3\pi}{4} - 1\right) - 1\right] = -5.414214,$

$(B)_{ij} = \sqrt{\frac{2}{M+1}}\sin\frac{ij\pi}{M+1}$, $i, j = 1,2,3$

$(B)_{11} = \sqrt{\frac{2}{4}}\sin\frac{\pi}{4} = 0.5$,

$(B)_{12} = \sqrt{\frac{2}{4}}\sin\frac{2\pi}{4} = 0.707107$,

$(B)_{13} = \sqrt{\frac{2}{4}}\sin\frac{3\pi}{4} = 0.5$

$(B)_{21} = \sqrt{\frac{2}{4}}\sin\frac{2\pi}{4} = 0.707107,$

$(B)_{22} = \sqrt{\frac{2}{4}}\sin\frac{4\pi}{4} = 0,$

$(B)_{23} = \sqrt{\frac{2}{4}}\sin\frac{6\pi}{4} = -0.707107,$

$(B)_{31} = \sqrt{\frac{2}{4}}\sin\frac{3\pi}{4} = 0.5,$

$(B)_{32} = \sqrt{\frac{2}{4}}\sin\frac{6\pi}{4} = -0.707107,$

$(B)_{33} = \sqrt{\frac{2}{4}}\sin\frac{9\pi}{4} = 0.5.$

$$B = \begin{bmatrix} 0.5 & 0.707107 & 0.5 \\ 0.707107 & 0 & -0.707107 \\ 0.5 & -0.707107 & 0.5 \end{bmatrix} = B^T,$$

$$P_3^{-1}(D) = B \begin{bmatrix} 0 & \frac{1}{P_3(\lambda_j)} & 0 \end{bmatrix} B,$$

$$P_3^{-1}(D) = B \begin{bmatrix} {}^1/_{p_3(\lambda_1)} & 0 & 0 \\ 0 & {}^1/_{p_3(\lambda_2)} & 0 \\ 0 & 0 & {}^1/_{p_3(\lambda_3)} \end{bmatrix} B,$$

$$P_3^{-1}(D) = \begin{bmatrix} -0.031250 & -0.026786 & -0.013393 \\ -0.026786 & -0.044643 & -0.026786 \\ -0.013393 & -0.026786 & -0.031250 \end{bmatrix},$$

$$u_1 = P_3^{-1}(D)[P_2(D)P_0(D)y_1 + P_0(D)(-P_1(D)y_2 + P_0(D)y_3)],$$

$$u_1 = \begin{bmatrix} 0.317912 \\ 0.408250 \\ 0.524053 \end{bmatrix},$$

$$u_2 = P_3^{-1}(D)[P_1(D)(-P_0(D)y_1 + P_1(D)y_2) + P_1(D)(-P_0(D)y_3)],$$

$$u_2 = \begin{bmatrix} 0.615995 \\ 0.791032 \\ 1.015451 \end{bmatrix},$$

$$u_3 = P_3^{-1}(D)[P_0(D)(P_0(D)y_1 - P_1(D)y_2 + P_2(D)y_3)],$$

$$u_3 = \begin{bmatrix} 0.875622 \\ 1.124399 \\ 1.443525 \end{bmatrix}.$$

Table 3. Results of the test problem by Block Schechter Form

Unknowns	Exact solution	Block-Schechter Form	Absolute error
U_{11}	0.317673	0.317912	0.000239
U_{12}	0.407900	0.408250	0.000350
U_{13}	0.523754	0.524053	0.000299
U_{21}	0.615595	0.615995	0.000400
U_{22}	0.790439	0.791032	0.000593
U_{23}	1.014944	1.015451	0.000507
U_{31}	0.875241	0.875622	0.000381
U_{32}	1.123832	1.124399	0.000567
U_{33}	1.443029	1.443525	0.000496

3.4.2 Solution of the Test Problem by the Simplified Block Schechter Form

We use the algorithm (3.8)

$$f_1 = y_1 = \begin{bmatrix} -0.247404 \\ 0 \\ -0.672514 \end{bmatrix}$$

$$f_2 = P_1(D)y_2 - f_1 = \begin{bmatrix} 2.165108 \\ -1.782640 \\ 5.885370 \end{bmatrix}$$

$$f_3 = P_2(D)y_3 - f_2 = \begin{bmatrix} -22.894324 \\ 21.368785 \\ -54.697151 \end{bmatrix}$$

$$u_3 = P_3^{-1}(D)f_3 = \begin{bmatrix} 0.875621 \\ 1.124381 \\ 1.443529 \end{bmatrix}$$

$$u_2 = y_3 - Du_3 = \begin{bmatrix} 0.615994 \\ 0.791022 \\ 1.015456 \end{bmatrix}$$

$$u_1 = y_3 - Du_2 - u_3 = \begin{bmatrix} 0.317908 \\ 0.408258 \\ 0.524057 \end{bmatrix}$$

Table 4. Results of the test problem by Simplified Block-Schechter Form

Unknowns	Exact solution	Simplified Block-Schechter Form	Absolute error
U_{11}	0.317673	0.317908	0.000235
U_{12}	0.407900	0.408258	0.000358
U_{13}	0.523754	0.524057	0.000303
U_{21}	0.615595	0.615994	0.000399
U_{22}	0.790439	0.791022	0.000583
U_{23}	1.014944	1.015456	0.000512
U_{31}	0.875241	0.875621	0.000380
U_{32}	1.123832	1.124381	0.000549
U_{33}	1.443029	1.443529	0.000500

Chapter 4

MATRIX DECOMPOSITION METHODS

4.1 Introduction

The solution of general block tridiagonal systems (2.10) arising from a finite difference equation can be given by matrix decomposition method. The block triangular decomposition of A can be expressed as:

$$A = LU, \tag{4.1}$$

where L is the lower triangular matrix and U is the upper triangular matrix which is expressed respectively as

$$L = [L_N \quad I \quad 0], \qquad\qquad U = [0 \quad U_N \quad C_N].$$

The recurrence for L_i and U_i , $i = 2,3, \dots, N$ is given in [5]

$$\begin{aligned} U_1 &= D_1, \\ L_i &= A_i U_{i-1}^{-1}, \\ U_i &= D_i - L_i C_{i-1}. \end{aligned} \tag{4.2}$$

In this Chapter, we will consider majorly the orthogonal block decomposition method, given in [6].

4.2 Orthogonal Block Decomposition Method

This method involves finding the eigenvalues and eigenvectors of the matrices D and T, which is used to find the orthogonal matrix Q required for the solution of the tridiagonal systems. From the system A of block dimension N,

$$A = [T \quad D \quad T]. \tag{4.3}$$

4.2.1 Case when D commutes with T

In this case, the matrix D commutes with matrix T, i.e. $DT = TD$ and D and T are $M \times M$ symmetric matrices. Since these matrices are symmetric and they commute, then there exist an orthogonal matrix Q such that,

$$Q^T DQ = \Lambda\,, \qquad\qquad Q^T TQ = \Omega, \tag{4.4}$$

where Q is the matrix containing the set of eigenvectors of D and T, Λ is a real diagonal matrix of eigenvalues of D and Ω is also a real diagonal matrix of eigenvalues of T.

Similar to the matrix A, we have the vectors U and Y, written as:

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \;\; Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

We will represent the entries of the block u_j and y_j as

$$u_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{Mj} \end{bmatrix}, \;\; y_j = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{Mj} \end{bmatrix}, \quad j = 1,2,3, \dots, N. \tag{4.5}$$

From the system (4.3), we have

$$\begin{gathered} Du_1 + Tu_2 = y_1, \\ Tu_{j-1} + Du_j + Tu_{j+1} = y_j, \quad j = 2,3, \dots, N-1, \\ Tu_{N-1} + Du_N = y_N. \end{gathered} \tag{4.6}$$

Using (4.4), (4.6) becomes

$$\begin{gathered} \Lambda\bar{u}_1 + \Omega\bar{u}_2 = \bar{y}_1, \\ \Omega\bar{u}_{j-1} + \Lambda\bar{u}_j + \Omega\bar{u}_{j+1} = \bar{y}_j, \quad j = 2,3, \dots, N-1, \\ \Omega\bar{u}_{N-1} + \Lambda\bar{u}_N = \bar{y}_{N,} \end{gathered} \tag{4.7}$$

where $\bar{u}_j = Q^T u_j$ and $\bar{y}_j = Q^T y_j$, $j = 1,2,3, \dots, N$.

The components of $\bar{u}_j$ and $\bar{y}_j$ are labeled as:

$$\bar{u}_j = \begin{bmatrix} \bar{u}_{1j} \\ \bar{u}_{2j} \\ \vdots \\ \bar{u}_{Mj} \end{bmatrix}, \quad \bar{y}_j = \begin{bmatrix} \bar{y}_{1j} \\ \bar{y}_{2j} \\ \vdots \\ \bar{y}_{Mj} \end{bmatrix}$$

(4.7) can be written for $p = 1,2,\dots,M$ as

$$\lambda_p \bar{u}_{p1} + \omega_p \bar{u}_{p2} = \bar{y}_{p1},$$

$$\omega_p \bar{u}_{pj-1} + \lambda_p \bar{u}_{pj} + \omega_p \bar{u}_{pj+1} = \bar{y}_{pj}, \quad j = 2,3,\dots,N-1, \tag{4.8}$$

$$\omega_p \bar{u}_{pN-1} + \lambda_p \bar{u}_{pN} = \bar{y}_{pN}.$$

From this, we have the system

$$\Gamma_p \hat{u}_p = \hat{y}_p, \tag{4.9}$$

where $\Gamma_p = [\omega_p \;\; \lambda_p \;\; \omega_p]_{N\times N}$, $\hat{u}_p = \begin{bmatrix} \bar{u}_{p1} \\ \bar{u}_{p2} \\ \vdots \\ \bar{u}_{pN} \end{bmatrix}, \quad \hat{y}_p = \begin{bmatrix} \bar{y}_{p1} \\ \bar{y}_{p2} \\ \vdots \\ \bar{y}_{pN} \end{bmatrix}.$

From finding $\hat{u}_p$ which can be computed by Thomas algorithm, it is then possible to solve for $u_j = Q\hat{u}_j$.

Therefore, we have the algorithm as [6] [7]:

1. Find the eigenvalues and eigenvectors of D and T
2. Compute $\bar{y}_j = Q^T y_j \;\; , j = 1,2,\dots,N$
3. Solve $\Gamma_p \hat{u}_p = \hat{y}_p \;\; , p = 1, 2, \dots, M$
4. Compute $u_j = Q\bar{u}_j \;\; , \; j = 1, 2, \dots N$

4.2.2 Case when *D* and *T* do not commute

D and T may not need to commute. If we assume that T is symmetric and positive definite, then there exist a matrix P, such that [8]

$$T = PP^T, \qquad D = P\Delta P^T, \tag{4.10}$$

where Δ is the diagonal matrix of eigenvalues $T^{-1}D$ and the matrix of the eigenvectors of $T^{-1}D$ is P^{-T}. Using (4.10), we have the following algorithm [5]:

1. Find the eigenvalues and eigenvectors of $T^{-1}D$
2. Compute $\hat{y}_j = P^{-1}y_j$
3. Solve $\Gamma_p \hat{u}_p = \hat{y}_p$, where $\Gamma_p = [1 \quad \delta_p \quad 1]$
4. Compute $u_j = P^{-T}\bar{u}_j$

4.2.3 The Power method

The power method is an iterative method for approximating eigenvalues and eigenvectors. Normally, the power method only determines the largest eigenvalue, also known as the dominant eigenvalue. But with slight modification, it can be used to determine the non-dominant eigenvalues, that is the intermediate and the smallest eigenvalues.

Definition 1 [9]: If λ_1 is an eigenvalue of A that is larger in absolute value than any other eigenvalue, it is called the dominant eigenvalue. An eigenvector V_1 corresponding to λ_1 is called a dominant eigenvector.

The power method can be used when the eigenvalues of an $n \times n$ matrix A is ordered in magnitude as

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n|.$$

It is also used when $A_{n\times n}$ has n linearly independent eigenvectors. To apply this method, the analyzed matrix system should be in the form:

$$Ax = \lambda x.$$

A non-zero vector x_0 is chosen as an initial approximation and the sequence is given by

$$x_1 = Ax_0,$$
$$x_2 = Ax_1 = A(Ax_0) = A^2x_0,$$
$$x_3 = Ax_2 = A(A^2x_0) = A^3x_0,$$
$$\vdots$$
$$x_m = Ax_{m-1} = A(A^{m-1}x_0) = A^mx_0.$$

If the sequence is correctly scaled, a good approximation of the dominant eigenvector of A is obtained and the Rayleigh quotient is used to determine the corresponding eigenvalue.

Theorem 1 [10]: If x is an eigenvector of a matrix A, then its corresponding eigenvalue is given by:

$$\lambda = \frac{Ax \cdot x}{x \cdot x}.$$

This quotient is called the Rayleigh quotient.

We observe that this method produces approximate eigenvectors with large components, therefore each approximation can be scaled down and the scaled vector is used in the next iteration. The advantage of this method is that the eigenvalue is obtained alongside the eigenvector.

4.2.4 Solution of the Test Problem by Orthogonal Block Decomposition Method

The matrix $D = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}$ and $T = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We calculate the matrix $V = \begin{bmatrix} -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$, which the columns are the eigen vectors of D.

Normalizing V, we have

$$\hat{V}_1 = \begin{bmatrix} -{}^{1}/_{\sqrt{2}} \\ 0 \\ {}^{1}/_{\sqrt{2}} \end{bmatrix}, \quad \hat{V}_2 = \begin{bmatrix} {}^{1}/_{2} \\ {}^{\sqrt{2}}/_{2} \\ {}^{1}/_{2} \end{bmatrix}, \ \hat{V}_3 = \begin{bmatrix} {}^{1}/_{2} \\ -{}^{\sqrt{2}}/_{2} \\ {}^{1}/_{2} \end{bmatrix}$$

The orthogonal Matrix Q is

$$\begin{bmatrix} -{}^{1}/_{\sqrt{2}} & {}^{1}/_{2} & {}^{1}/_{2} \\ 0 & {}^{\sqrt{2}}/_{2} & -{}^{\sqrt{2}}/_{2} \\ {}^{1}/_{\sqrt{2}} & {}^{1}/_{2} & {}^{1}/_{2} \end{bmatrix}$$

$$Q^T A Q = \begin{bmatrix} -4 & 0 & 0 \\ 0 & \sqrt{2}-4 & 0 \\ 0 & 0 & -\sqrt{2}-4 \end{bmatrix} = \Lambda$$

$$\bar{y}_j = Q^T y_j, \quad j = 1, 2, 3$$

$$\bar{y}_1 = \begin{bmatrix} -0.300598 \\ -0.459959 \\ -0.459959 \end{bmatrix}$$

$$\bar{y}_2 = \begin{bmatrix} -0.582506 \\ -0.891320 \\ -0.891320 \end{bmatrix}$$

$$\bar{y}_3 = \begin{bmatrix} -1.323825 \\ -3.679200 \\ -1.717189 \end{bmatrix}$$

To solve $\Gamma_p \hat{u}_p = \hat{y}_p$, $p = 1,2,3$

$$\Gamma_1 \hat{u}_1 = \hat{y}_1 = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} \bar{u}_{11} \\ \bar{u}_{12} \\ \bar{u}_{13} \end{bmatrix} = \begin{bmatrix} -0.300598 \\ -0.582506 \\ -1.323825 \end{bmatrix}$$

Using the Thomas algorithm to solve this system, $\hat{u}_1$ results to

$$\hat{u}_1 = \begin{bmatrix} \bar{u}_{11} \\ \bar{u}_{12} \\ \bar{u}_{13} \end{bmatrix} = \begin{bmatrix} 0.145765 \\ 0.282461 \\ 0.401571 \end{bmatrix}$$

$$\Gamma_2 \hat{u}_2 = \hat{y}_2 = \begin{bmatrix} \sqrt{2}-4 & 1 & 0 \\ 1 & \sqrt{2}-4 & 1 \\ 0 & 1 & \sqrt{2}-4 \end{bmatrix} \begin{bmatrix} \bar{u}_{21} \\ \bar{u}_{22} \\ \bar{u}_{23} \end{bmatrix} = \begin{bmatrix} -0.459959 \\ -0.891320 \\ -3.679200 \end{bmatrix}$$

Solving $\hat{u}_2$, using the Thomas algorithm results to

$$\hat{u}_2 = \begin{bmatrix} \bar{u}_{21} \\ \bar{u}_{22} \\ \bar{u}_{23} \end{bmatrix} = \begin{bmatrix} 0.709656 \\ 1.375059 \\ 1.954631 \end{bmatrix}$$

$$\Gamma_3 \hat{u}_3 = \hat{y}_3 = \begin{bmatrix} -\sqrt{2}-4 & 1 & 0 \\ 1 & -\sqrt{2}-4 & 1 \\ 0 & 1 & -\sqrt{2}-4 \end{bmatrix} \begin{bmatrix} \bar{u}_{31} \\ \bar{u}_{32} \\ \bar{u}_{33} \end{bmatrix} = \begin{bmatrix} -0.459959 \\ -0.891320 \\ -1.717189 \end{bmatrix}$$

Solving $\hat{u}_3$, using the Thomas algorithm results to

$$\hat{u}_3 = \begin{bmatrix} \bar{u}_{31} \\ \bar{u}_{32} \\ \bar{u}_{33} \end{bmatrix} = \begin{bmatrix} 0.132309 \\ 0.256389 \\ 0.364518 \end{bmatrix}$$

To compute $u_j = Q\bar{u}_j$, $j = 1, 2, \dots N$,

$$u_1 = Q\bar{u}_1 = \begin{bmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0.145765 \\ 0.709656 \\ 0.132309 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0.317911 \\ 0.408246 \\ 0.524054 \end{bmatrix}$$

$$u_2 = Q\bar{u}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0.282461 \\ 1.375059 \\ 0.256389 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0.615994 \\ 0.791019 \\ 1.015454 \end{bmatrix}$$

$$u_3 = Q\bar{u}_3 = \begin{bmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0.401571 \\ 1.954631 \\ 0.364518 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0.875621 \\ 1.124380 \\ 1.443528 \end{bmatrix}$$

Table 5. Results of the test problem by Orthogonal Block Decomposition Form

Unknowns	Notations	Exact solution	Orthogonal block decomposition form	Absolute error
U_{11}	u_{11}	0.317673	0.317911	0.000238
U_{12}	u_{21}	0.407900	0.408246	0.000346
U_{13}	u_{31}	0.523754	0.524054	0.000300
U_{21}	u_{12}	0.615595	0.615994	0.000399
U_{22}	u_{22}	0.790439	0.791019	0.000580
U_{23}	u_{32}	1.014944	1.015454	0.000510
U_{31}	u_{13}	0.875241	0.875621	0.000380
U_{32}	u_{23}	1.123832	1.124380	0.000548
U_{33}	u_{33}	1.443029	1.443528	0.000499

Chapter 5

BLOCK CYCLIC REDUCTION METHOD

5.1 Introduction

Consider the matrix equation

$$AU \equiv \begin{bmatrix} I & D \\ C & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{5.1}$$

The solution to (5.1) can be written in the form

$$\begin{aligned} u_1 &= (I - DC)^{-1}(y_1 - Dy_2), \\ u_2 &= y_2 - Cy_1. \end{aligned} \tag{5.2}$$

Thus, we reduce the problem to solving for u_1 only. Assuming I, D, C are square matrices, this reduces the number of unknown by half. A similar method to this is the cyclic odd-even reduction. We give here, the presentation due to Buzbee, Golub and Nielson [6].

5.2 Cyclic Reduction Methods

From the matrix system (4.3), where $[T \quad D \quad T]$ is of block dimension N, we assume still that D and T are symmetric and they commute. We assume also that $N = S - 1$, where $S = 2^{k+1}$ and k is some positive integer. We rewrite the second equation in (4.6) as follows:

$$Tu_{j-2} + Du_{j-1} + Tu_j = y_{j-1},$$

$$Tu_{j-1} + Du_j + Tu_{j+1} = y_j,$$

$$Tu_j + Du_{j+1} + Tu_{j+2} = y_{j+1}.$$

The first and third equations are multiplied by T and the second equation is multiplied by $-D$. Adding them all, the result is the equation:

$$T^2 u_{j-2} + (2T^2 - D^2)u_j + T^2 u_{j+2} = Ty_{j-1} - Dy_j + Ty_{j+1}.$$

If j is even, the new system of matrix equation involves u_j's that have even indices,

$$[T^2 \quad (2T^2 - D^2) \quad T^2][u_{2j}] = [Ty_{2j-1} - Dy_{2j} + Ty_{2j+1}], \tag{5.3}$$

and the eliminated equation will be written as the system:

$$[0 \quad D \quad 0][u_{2j+1}] = [-Ty_{2j} + y_{2j+1} - Ty_{2j+2}]. \tag{5.4}$$

The block dimension of (5.3) is now $2^k - 1$ while that of (5.4) is 2^k, [1].

The matrix decomposition method can be used to solve (5.4), or the reduction technique is applied repeatedly to the system until we have one block. However, we can stop the process at any step and use the method in section 4 to solve the resulting matrix, as this will reduce its subjection to round-off errors [7].

Applying the same technique to reduce (5.3), we define the sequence:

$$D^{(0)} = D, \qquad T^{(0)} = T, \qquad y_j^{(0)} = y_j, \qquad u_j^{(0)} = u_j, \qquad j = 1,2, \dots, N$$

For $i = 0,1,2, \dots k$

$$\begin{aligned} T^{(i+1)} &= \left(T^{(i)}\right)^2 \\ D^{(i+1)} &= 2\left(T^{(i)}\right)^2 - \left(D^{(i)}\right)^2 \\ u_j^{(i+1)} &= u_{2j}^{(i)}, \\ y_j^{(i+1)} &= T^{(i)}\left(y_{j-2^i}^{(i)} + y_{j+2^i}^{(i)}\right) - D^{(i)} y_j^{(i)} \end{aligned} \tag{5.5}$$

At each stage, we observe that we have a new system of equations to solve, in the form

$$M^{(i)} Z^{(i)} = F^{(i)} \tag{5.6}$$

where $M^{(i)} = \begin{bmatrix} T^{(i)} & D^{(i)} & T^{(i)} \end{bmatrix}$, $\quad Z^{(i)} = \begin{bmatrix} u_{2^i} \\ u_{2^{i+1}} \\ \vdots \\ u_{j2^i} \\ \vdots \end{bmatrix}$, $\quad F^{(i)} = \begin{bmatrix} y_{2^i}^{(i)} \\ y_{2^{i+1}}^{(i)} \\ \vdots \\ y_{j2^i}^{(i)} \\ \vdots \end{bmatrix}$

and the eliminated equation written in the form

$$P^{(i)} W^{(i)} = G^{(i)}, \tag{5.7}$$

where $P^{(i)} = \begin{bmatrix} 0 & D^{(i-1)} & 0 \end{bmatrix}$, $\quad W^{(i)} = \begin{bmatrix} u_{2^i-2^{i-1}} \\ u_{2^{i+1}-2^{i-1}} \\ \vdots \\ u_{j2^i-2^{i-1}} \\ \vdots \end{bmatrix}$,

$$G^{(i)} = \begin{bmatrix} y_{2^i-2^{i-1}}^{(i-1)} - T u_{2^i}^{(i-1)} \\ y_{2^{i+1}-2^{i-1}}^{(i-1)} - T\left(u_{2^{i+1}}^{(i-1)} + u_{2^i}^{(i-1)}\right) \\ \vdots \\ y_{j2^i-2^{i-1}}^{(i-1)} - T\left(u_{j2^i}^{(i-1)} + u_{(j-1)2^i}^{(i-1)}\right) \\ \vdots \end{bmatrix}.$$

To solve (5.6), we can use the methods reviewed in previous sections, or we continue to compute $M^{(i+1)}$ and eliminate half of its unknowns. This matrix is of block dimension $2^{k+1-i} - 1$, [1]. After k steps, the matrices reduce to one block with dimension M,

$$D^{(k)} u_{2^k} = y_{2^k}^{(k)}. \tag{5.8}$$

This algorithm is known as cyclic reduction method [6]. Next, we consider the factorization of $D^{(i)}$. From (5.3)-(5.5), we observe that $D^{(i)}$ is a polynomial of degrees 2^i in D and T, such that [6]

$$D^{(i)} = \sum_{j=0}^{2^{i-1}} c_{2j}^{(i)} D^{2j} T^{2^i-2j} \equiv P_{2^i}(D, T).$$

To find the linear factors of $P_{2^i}(D, T)$, let

$p_{2^i}(d,t) = \sum_{j=0}^{2^{i-1}} c_{2j}^{(i)} d^{2j} t^{2^i-2j}, \qquad c_{2^i}^{(i)} = -1$

For $t \neq 0$, we use the substitution

$$\frac{d}{t} = -2\cos\theta. \tag{5.9}$$

Note that from (5.5),

$$p_{2^{i+1}}(d,t) = 2t^{2^{i+1}} - \left(p_{2^i}(d,t)\right)^2. \tag{5.10}$$

Then, using (5.9) and (5.10), we have that

$$p_{2^i}(d,t) = -2t^{2^i}\cos 2^i\theta,$$

and consequently,

$p_{2^i}(d,t) = 0$, when $\frac{d}{2t} = -\cos(\frac{2j-1}{2^{i+1}})\,\pi$ for $j = 1,2,3,\dots,2^i$.

Therefore, we can write

$$p_{2^i}(d,t) = -\prod_{j=1}^{2^i}\left[d + 2t\cos\left(\frac{2j-1}{2^{i+1}}\right)\pi\right].$$

Hence,

$$D^{(i)} = -\prod_{j=1}^{2^i}\left(D + 2\cos\theta_j^{(i)}\,T\right),$$

where $\theta_j^{(i)} = \frac{2j-1}{2^{i+1}}\pi$.

Denote

$$D_j^{(k)} = D + 2\cos\theta_j^{(k)}\,T, \tag{5.11}$$

so, to solve $D^{(k)}u_{2^k} = y_{2^k}^{(k)}$, we set $Z_1 = -y_{2^k}^{(k)}$ and solve

$$\begin{aligned} D_j^{(k)} Z_{j+1} &= Z_j, \quad \text{for } j = 1,2,\dots,2^k, \\ Z_{2^k+1} &= u_{2^k}. \end{aligned} \tag{5.12}$$

To solve (5.7), a similar algorithm can be used with

$$D^{(i)} = -\prod_{j=1}^{2^i} D_j^{(i)}. \tag{5.13}$$

This algorithm above is called the cyclic odd-even reduction factorization (CORF) algorithm. The numerical calculation of y_j^k in (5.5) depends on the rounding errors in many applications. Buneman algorithm [11] stabilize the cyclic odd-even reduction factorization.

5.2.1 Solution of the Test Problem by Cyclic Reduction Method

From the test problem in section (2.3.1), we obtained the matrix equation

$$\begin{bmatrix} D & I & 0 \\ I & D & I \\ 0 & I & D \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Where $D = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, y_1 = \begin{bmatrix} -0.247404 \\ 0 \\ -0.672514 \end{bmatrix},$

$$y_2 = \begin{bmatrix} -0.479426 \\ 0 \\ -1.303214 \end{bmatrix}, y_3 = \begin{bmatrix} -1.762109 \\ -1.387351 \\ -3.634280 \end{bmatrix}$$

The first and third equation is multiplied by I and the second equation is multiplied by $-D$. This results to

$$Du_1 + u_2 = y_1,$$

$$-Du_1 - D^2u_2 - Du_3 = -Dy_2,$$

$$u_2 + Du_3 = y_3.$$

Adding the three equations together, we have

$$(2I - D^2)u_2 = y_1 - Dy_2 + y_3 = \left(2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}\right)^2\right)u_2 = y_1 - \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}y_2 + y_3$$

$$\begin{bmatrix} -15 & 8 & -1 \\ 8 & -16 & 8 \\ -1 & 8 & -15 \end{bmatrix} u_2 = \begin{bmatrix} -3.927217 \\ 0.395289 \\ -9.519650 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0.615994 \\ 0.791018 \\ 1.015453 \end{bmatrix}.$$

To solve the eliminated equations,

$$Du_1 = y_1 - u_2,$$

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} u_1 = y_1 - \begin{bmatrix} 0.615994 \\ 0.791018 \\ 1.015453 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0.317911 \\ 0.408246 \\ 0.524053 \end{bmatrix},$$

$$Du_3 = y_3 - u_2,$$

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} u_3 = y_3 - \begin{bmatrix} 0.615994 \\ 0.791018 \\ 1.015453 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0.875621 \\ 1.124379 \\ 1.443528 \end{bmatrix}$$

Table 6. Results of the Test problem by Block Cyclic Reduction Method

Unknowns	Exact solution	Block cyclic reduction method	Absolute error
U_{11}	0.317673	0.317911	0.000238
U_{12}	0.407900	0.408246	0.000346
U_{13}	0.523754	0.524053	0.000299
U_{21}	0.615595	0.615994	0.000399
U_{22}	0.790439	0.791018	0.000579
U_{23}	1.014944	1.015453	0.000509
U_{31}	0.875241	0.875621	0.000380
U_{32}	1.123832	1.124379	0.000547
U_{33}	1.443029	1.443528	0.000499

5.3 The Buneman Algorithm

The Buneman algorithm is a more stable algorithm compared to CORF algorithm. It is possible to use this algorithm to solve (2.10) arising from a 5 point difference approximation of Poisson's equation on a rectangular region using Dirichlet boundary condition [6]. The Buneman algorithm has a distinct approach in the calculation of the right hand side of the system at each phase of the reduction process, which differentiates it from the CORF algorithm. In the case of Buneman algorithm, we assume that in the system (4.6), the matrix T is an identity matrix of order g, i.e. $T = I_g$.

Let us consider the system (4.6) with dimension $N = 2^{k+1} - 1$, one stage of cyclic reduction process results to

$$u_{j-2} + \left(2I_g - D^2\right)u_j + u_{j+2} = y_{j-1} - Dy_j + y_{j+1}, \tag{5.14}$$

for $j = 2,4, \dots, N-1$, where $u_0 = u_{N+1} = \theta$, a null vector. The right hand side of (5.9) can be written as follows

$$y_{j-1} - Dy_j + y_{j+1} = D^{(1)}D^{-1}y_j + y_{j-1} - 2D^{-1}y_j + y_{j+1} = y_j^{(1)}, \tag{5.15}$$

where $D^{(1)} = (2I_g - D^2)$. Let $P_j^{(1)} = D^{-1}y_j$, $Q_j^{(1)} = y_{j-1} + y_{j+1} - 2P_j^{(1)}$, so that

$$y_j^{(1)} = D^{(1)}P_j^{(1)} + Q_j^{(1)}. \tag{5.16}$$

After i reductions, by (5.5), we have

$$y_j^{(i+1)} = y_{j-2^i}^{(i)} - D^{(i)}y_j^{(i)} + y_{j+2^i}^{(i)}. \tag{5.17}$$

Similar to (5.16), we write

$$y_j^{(i)} = D^{(i)}P_j^{(i)} + Q_j^{(i)}. \tag{5.18}$$

From (5.5), we can say that $\left(D^{(i)}\right)^2 = 2I_g - D^{(i+1)}$. Making use of this identity and substituting (5.18) into (5.17), we have

$$P_j^{(i+1)} = P_j^{(i)} - \left(D^{(i)}\right)^{-1}\left(P_{j-2^i}^{(i)} - Q_j^{(i)} + P_{j+2^i}^{(i)}\right),$$
$$Q_j^{(i+1)} = Q_{j-2^i}^{(i)} - 2P_j^{(i+1)} + Q_{j+2^i}^{(i)}. \tag{5.19}$$

To compute $\left(D^{(i)}\right)^{-1}\left(P_{j-2^i}^{(i)} - Q_j^{(i)} + P_{j+2^i}^{(i)}\right)$ in (5.19) above, we have that

$P_j^{(i)} - P_j^{(i+1)} = \left(D^{(i)}\right)^{-1}\left(P_{j-2^i}^{(i)} - Q_j^{(i)} + P_{j+2^i}^{(i)}\right)$. Multiply both sides of the equation by $\left(D^{(i)}\right)$, we get $\left(D^{(i)}\right)\left(P_j^{(i)} - P_j^{(i+1)}\right) = \left(P_{j-2^i}^{(i)} - Q_j^{(i)} + P_{j+2^i}^{(i)}\right)$. We solve this system of equation, where $\left(D^{(i)}\right)$ is calculated by factorization in (5.13), that is

$$D^{(i)} = -\prod_{j=1}^{2^i}\left(D + 2\cos\theta_j^{(i)}\, I_g\right),$$

$$\theta_j^{(i)} = \frac{2j-1}{2^{i+1}}\pi.$$

After k reductions, we have $D^{(k)}u_{2^k} = D^{(k)}P_{2^k}^{(k)} + Q_{2^k}^{(k)}$, therefore,

$$u_{2^k} = P_{2^k}^{(k)} + (D^{(k)})^{-1}Q_{2^k}^{(k)}.$$

Again, we factorize $D^{(k)}$, using it to compute $(D^{(k)})^{-1}Q_{2^k}^{(k)}$. To do back substitution, we use the relationship

$$u_{j-2^i} + D^{(i)}u_j + u_{j+2^i} = D^{(i)}P_j^{(i)} + Q_j^{(i)}.$$

For $j = r2^i, r = 1,2,\dots,2^{k+1-i} - 1$, with $u_0 = u_{2^{k+1}} = \theta$(null vector).

$$D^{(i)}\left(u_j - P_j^{(i)}\right) = Q_j^{(i)} - \left(u_{j-2^i} + u_{j+2^i}\right). \tag{5.20}$$

Let $u_j - P_j^{(i)} = v,\ D^{(i)}v = Q_j^{(i)} - \left(u_{j-2^i} + u_{j+2^i}\right),$

to solve for u_j, first solve for v using the factorization of $D^{(i)}$ in (5.13), then

$$u_j = P_j^{(i)} + v. \tag{5.21}$$

Chapter 6

COMPARISON

In the second half of the 20th century, direct methods which utilizes the special block structure of the algebraic linear systems were developed. In this thesis, we analysed block elimination methods, block decomposition methods and block cyclic reduction methods. We present operation counts for some of these methods with $M = N$. Terms of lower order in N will not be included, therefore, the operation count is only valid for large N.

The operation counts given by Dorr [1] indicates that the methods which have been discussed in this thesis offer economic significance over the older techniques. Table 7 presents the operational counts of some methods discussed as given in [1].

Table 7. Operation counts for some methods discussed

Method	Order of Operations
Block Polynomial Form	$6N^3$
Block Schechter Form	$12N^3$
Simplified Block Schechter Form	$\frac{3}{2}N^3$
Odd-Even Reduction (CORF)	$\frac{9}{2}N^2 \log_2 N$

We note that the orthogonal matrix decomposition method discussed in Section 4 is carefully analyzed by Hockney [7] for solving Poisson's equation on a rectangle. In this case, Q is known. He took advantage of this and the fact that fast Fourier transform [12] can be used to solve steps (ii) and (iv), taking into account also that fast Fourier transform requires $2N \log_2 N$ operations [12]. Therefore, Orthogonal matrix decomposition method is a valuable method. Comparing block elimination methods, simplified block Schechter form requires less operation count which is $\frac{3}{2}N^3$, than block polynomial form and block Schechter form.

CORF algorithm when used as studied in section (5.2) and given in [6] for the numerical calculation of (5.5) represents some degree of instability. This also occurs when the method presented in this way is applied to solve the algebraic systems of equations arising from 5-point discretization of the Laplacian equation on a rectangle as stated in Section 10 of [6]. On the other hand, Hockney noted that there could be a better advantage to applying just the cyclic reduction method until the size of the matrix is reduced such that other methods which are more stable and direct can be applied to solve the already reduced matrix [1].

Therefore, Buneman algorithm, Hockney algorithm (when used carefully) and orthogonal matrix decomposition methods are powerful direct methods for solving discrete Poisson's equation on a rectangle with Dirichlet boundary conditions.

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