Fractional Mixed Volttera-Fredholm Integrodifferential Equation

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ABSTRACT

In mathematics, an integrodifferential equation is an equation that involves both integral and derivative of a function. These equations model many situation ranging from science and engineering. A particular rich source is electrical circuit analysis.
Different techniques have been evolved for finding the solution of these differential equations under certain conditions. One of them is to prove the existence and uniqueness of mixed Volteraa-Fredholm type integral equation with the integral boundary conditions in Banach Space. This has been worked on by some authors such as S A Murad from Iraq, H J Zekri from Iraq, S Hadid from UAE.

Keywords: fixed point theorems; sequential fractional derivative; integral boundary conditions; fractional differential equation

ÖZ

Matematikte integrodifferansiyel denklem, bir fonksiyonun hem integralini hem de türevini içeren bir denklemdir. Bu denklemler birçok durumu bilim ve mühendislikten modüle eder. Bu diferansiyel denklemlerin belirli koşullar altında çözümünün bulunması için farklı teknikler geliştirilmiştir. Bunlardan biri, karışık Volteraa-Fredholm tipi integral denkleminin Banach Uzayındaki integral sınır koşullarıyla varlığını ve tekliğini kanıtlamaktır. Bu, Irak'tan S A Murad, yine Irak'tan H J Zekri, Birleşik Arap Emirliği'nden S. Hadid gibi bazı yazarlarca çalışılmıştır.

Anahtar Kelimeler: sabit nokta teoremleri, kesirli türev, integral sınır koşulları, integrodiferansiyel kesirli denklemler

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Chapter 1

INTRODUCTION

Theoretical and practical approach of the fractional differential equations to solving various mathematical problems has been extensively worked on. The theoretical part has been investigated by Momani and Hadid in regards with local and global existence theorom of both fractional differential equations and fractional integral differential equations.

Integral differential equations with integral boundary conditins have been successfully applied to various application such as population dynamics and cellular systems. It involves the research work of various researchers who have been able to investigate the behaviour of these equations in the different ways and have successfully proved different phenomenon in relation with these equations under various initial and boundary conditions, such as:

- Tidke has studied the existence of global solutions for non-linear mixed Volteraa-Fredholm involving non-local conditions.

- Ahmed and Nieto have studied the results for boundary value problem for non-linear integral differential equations of fractional order with integral equation.

- N Guerekata has studied the existence of solution of fractional abstract differential equations using non-local initial conditions.

So we will extract the relevant information from the work of these researchers and apply them to study the existence and uniqueness theorem for non-linear fractional mixed Volteraa-Fredholm integral differential equations with integral boundary conditions in banach space using the banach fixed point and Krasnosel'skii fixed point theorems for the case $1 < \gamma \leq 2$.

Chapter 2

REVIEW OF FRACTIONAL CALCULUS

Fractional Calculus is a modern and expanding domain of mathematical analysis. This applies to any real number with operations similar to usual differentiation. The order of the derivative may also be variable, distributed or complex.

Basically, Fractional Calculus includes more information in the model then offered by the classical integer order calculus. Besides an essential mathematical interest, its overall goal is general improvement of the physical world models for the purpose of computer simulation, analysis, design and control in practical applications. In the last four decades fractional calculus became an acceptable tool for a large number of diverse scientific communities due to more adequate modelling in various fields of mechanics, electricity, chemistry, biology, medicine, economics, control theory, as well as signal and image processing. This list can be extended since Fractional Community is still rapidly growing as can be seen at the websites of previous FDA events.

2.1 Some Basic Concepts

The integer order differential equation is,

$$D^p f(z) = \lim_{q \to 0} q^{-p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} f(z + (p - j)q).$$

Where $p \in \mathbb{N}.$ and,

$$\binom{p}{j} = \frac{p!}{j!\left(p - j\right)!}.$$

For non-integer order differential equation, it becomes,

$$D^{\gamma} f(l) = \lim_{q \to 0} q^{-\gamma} \sum_{j=0}^{\infty} (-1)^j \binom{\gamma}{j} f(l + (\gamma - j)q).$$

where $\gamma \in \mathbb{R}^+$ and in case of complex numbers $\mathrm{Re}(\gamma) > 0,$ and,

$$\binom{\gamma}{j} = \frac{\Gamma(\gamma + 1)}{\Gamma((j+1)(\gamma - j + 1))}.$$

There are some properties that can be required to the fractional derivative such as,

$$D^{\gamma+\beta} f(l) = D^{\gamma} D^{\beta} f(l).$$

where $\gamma, \beta \in \mathbb{R}^+$ and in case of complex numbers, $\mathrm{Re}(\gamma),\ \mathrm{Re}(\beta) > 0$. Also,

$$D^{\gamma}(af(l) + bg(l)) = aD^{\gamma} f(l) + bD^{\gamma} g(l).$$

where a, b are arbitrary constants.

2.2 Exponentials

Let us consider the $\gamma - order$ derivative for e^{az} where a is arbitrary constant.

$$\begin{aligned} D^{\gamma} e^{az} &= \lim_{q \to 0} q^{-\gamma} \sum_{j=0}^{\infty} (-1)^j \binom{\gamma}{j} e^{a(z+(\gamma-j)q)} \\ &= a^{\gamma} e^{az}. \end{aligned}$$

Similarly, we consider,

$$\begin{aligned} D^{\gamma} \cos(z) + iD^{\gamma} \sin(z) = D^{\gamma} e^{iz} &= i^{\gamma} e^{iz} \\ &= e^{\frac{\gamma\pi i}{2}} e^{iz} = e^{i(z+\frac{\gamma\pi}{2})} \\ &= \cos(z + \tfrac{\gamma\pi}{2}) + i\sin(z + \tfrac{\gamma\pi}{2}). \end{aligned} \tag{2.1}$$

and

$$
\begin{aligned}
D^{\gamma}\cos(z) - iD^{\gamma}\sin(z) = D^{\gamma}e^{-iz} &= (-i)^{\gamma}e^{-iz} \\
&= e^{\frac{-\gamma\pi i}{2}}e^{-iz} = e^{-i(z+\frac{\gamma\pi}{2})} \\
&= \cos(z+\tfrac{\gamma\pi}{2}) - i\sin(z+\tfrac{\gamma\pi}{2}).
\end{aligned} \tag{2.2}
$$

Solving (2.1) and (2.2) we get,

$$D^{\gamma}\sin(z) = \sin(z+\tfrac{\gamma\pi}{2}).$$

$$D^{\gamma}\cos(z) = coz(z+\tfrac{\gamma\pi}{2}).$$

The following rules can be proved in the same way,

$$D^{\gamma}\sin(az) = a^{\gamma}\sin(az+\tfrac{\gamma\pi}{2}).$$

$$D^{\gamma}coz(az) = a^{\gamma}\sin(az+\tfrac{\gamma\pi}{2}).$$

where a is any arbitrary constant.

Now consider the following function

$$f(z) = \sum_{p=-\infty}^{\infty} a_p e^{ipz}.$$

it can be solved as follows,

$$D^{\gamma}f(z) = \sum_{p=-\infty}^{\infty} a_p D^{\gamma}e^{ipz} = \sum_{p=-\infty}^{\infty} a_p (pi)^{\gamma}e^{ipz} = \sum_{p=-\infty}^{\infty} a_p p^{\gamma}e^{\frac{\gamma\pi i}{2}}e^{ipz} = \sum_{p=-\infty}^{\infty} a_p p^{\gamma}e^{i(pz+\frac{\gamma\pi}{2})}.$$

2.3 Powers

Now consider

$$D^{p}z^{a} = \frac{a!}{(a-p)!}z^{a-p}.$$

Where $p \in \mathbb{N}$ and in case of fractional order γ, it can be generalized to,

$$D^{\gamma}z^{a} = \frac{\Gamma(a+1)}{\Gamma(a-\gamma+1)}z^{a-\gamma}. \;\; \gamma > 0.$$

So it can be used to solve any function that can be expanded in power of *z* as follows, Let $f(z)=\sum_{v=-\infty}^{\infty} a_v z^v$, then

$$D^{\gamma} f(z)=\sum_{v=-\infty}^{\infty} a_v D^{\gamma} z^{v} = \sum_{v=-\infty}^{\infty} a_v \frac{\Gamma(v+1)}{\Gamma(v-\gamma+1)} z^{v-\gamma} \quad .$$

Expanding in Taylor series results in,

$$D^{\gamma} f(z+a)=\sum_{v=0}^{\infty} \frac{D^{v} f(a)}{\Gamma(v-\gamma+1)} z^{v-\gamma} \quad \gamma>0.$$

2.4 Binomial Formula

The exponential functions allow the substitution of binomial formula which requires the following operator,

$$d_q f(z)=f(z+q)$$

And for $\gamma \in R^{+}$,

$$d_q^{\gamma} f(z)=f(z+\gamma q) \,.$$

Now for $p \in \mathbb{N}$,

$$\begin{aligned} D^{p} f(z) &= \lim_{q\to 0} q^{-p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} d_q^{\,p-j} f(z) \\ &= \lim_{q\to 0} \left(\frac{d_q - 1}{q} \right)^{p} f(z). \end{aligned} \tag{2.3}$$

For fractional or complex number $\gamma>0$, or $\mathrm{Re}(\gamma)>0$ it becomes,

$$\begin{aligned} D^{\gamma} f(z) &= \lim_{q\to 0} \left(\frac{d_q - 1}{q} \right)^{\gamma} f(z) \\ &= \lim_{q\to 0} q^{-\gamma} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\gamma+1)}{j!\,\Gamma(\gamma-j+1)} f(z+(\gamma-j)q) \end{aligned} \tag{2.4}$$

2.5 Functions of the Derivative

Equation (2.3) reveals that the derivative itself is

$$D = \lim_{q \to 0} \frac{d_q - 1}{q} \ .$$

Let a function of the derivative can be expanded in power of z.

$$\text{g(z)} = \sum_{p=-\infty}^{\infty} a_p z^p \Rightarrow g(D) f(z) = \sum_{p=-\infty}^{\infty} a_p D^p f(z).$$

then the result is a weighted sum of derivatives of various orders. These functions are called formal differential operations.

Now let,

$$f(z) = e^{az} \text{such that } e^{az} = \sum_{j=0}^{\infty} \frac{a^j z^j}{j!}, \text{ then } e^{aD} e^{bz} = e^{ab} e^{bz} = e^{b(z+a)}.$$

where a, b are arbitrary constants..

Now consider the following functions.

$$g(z) = \sum_{n=0}^{\infty} a_n z^n.$$

$$f(z) = \sum_{n=0}^{\infty} b_n z^n$$

$$\Rightarrow g(D) f(z) = \sum_{n=0}^{\infty} a_n D^n \sum_{r=0}^{\infty} b_r z^r$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n \sum_{r=0}^{\infty} b_r D^n z^r = \sum_{n=0}^{\infty} p_n z^n.$$

where

$$p_n = \frac{1}{n!} \sum_{r=0}^{\infty} (n+r)! a_r b_{n+r}$$

2.6 Grunwald-Letnikov Derivative

It is well known that,

$$D^p f(z) = \lim_{q \to 0} q^{-p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} f(z + (p-j)q) \ .$$

According to Grunwald-Letnikov derivative, if $q \to 0\ as\ p \to \infty$, then

$$D^p f(z) = \lim_{q \to 0} q^{-p} \sum_{j=0}^{\frac{z-a}{q}} (-1)^j \binom{p}{j} f(z - jq). \ \ p \in \mathbb{N}.$$

where any value of a less than z can be chosen.

For non-integer γ, it becomes,

$$D^{\gamma} f(z) = \lim_{q \to 0} q^{-p} \sum_{j=0}^{\frac{z-a}{q}} (-1)^j \frac{\Gamma(\gamma+1)}{j!\Gamma(\gamma - j + 1)} f(z - jq). \ \ \gamma > 0.$$

Or equivalently,

$$D^{\gamma} f(z) = \lim_{j \to \infty} \left(\frac{j}{z-a} \right)^{\gamma} \sum_{k=0}^{j} (-1)^k \frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma - k + 1)} f\left(z - k\left(\frac{z-a}{j} \right) \right).$$

This is true for positive values of γ and for negative values we have,

$$D^{-\gamma} f(z) = \lim_{q \to 0} q^{-\gamma} \sum_{j=0}^{\frac{z-a}{q}} \frac{\Gamma(\gamma + j)}{j!\Gamma(\gamma)} f(z - jq).$$

2.7 Riemann-Liouville Derivative

This is the one that is used widely. Let us consider,

$$D^{-1} f(l) = \int_0^l k(\omega) d\omega.$$

Then iteration yields,

$$D^{-p} f(l) = \frac{1}{(p-1)!} \int_0^l f(\omega)(l-\omega)^{p-1} d\omega. \ \ p \in \mathbb{N}.,$$

In case of non-integer γ it becomes,

$$D^{\gamma} f(l) = \frac{1}{\Gamma(-\gamma)} \int_0^l \frac{f(\omega)}{(l-\omega)^{\gamma+1}} d\omega.$$

with the lower integration limit it becomes,

$$ {}_aD_l^{\gamma} f(l) = \frac{1}{\Gamma(-\gamma)} \int_a^l \frac{f(\omega)}{(l-\omega)^{\gamma+1}} d\omega. \ . $$

So the derivative using binomial formula in (2.3) and (2.4) is equivalent to the Reimann-Liouville derivative with infinity as starting limit provided the real part of γ is negative.

$$ {}_{-\infty}D_l^{\gamma} f(l) = \lim_{q\to 0}\left(\frac{d_q - 1}{q}\right)^{\gamma} f(l) = \frac{1}{\Gamma(-\gamma)} \int_{-\infty}^l \frac{f(\omega)}{(l-\omega)^{\gamma+1}} d\omega. \ . $$

This kind of Riemann-Liouville derivative with a lower limit of negative infinity is known as Weyl derivative.

2.8 Domain Transform

The transform of the vth derivative of the function $f(l)$ is,

$$ \mathcal{L}\left\{D^{v} f(l)\right\} = p^{v}\mathcal{L}\left\{f(l)\right\} - \sum_{r=0}^{v-1} p^{r}(D^{v-r-1}f)(0).. $$

If the all the terms in the sum above are 0, then it can written for non-integer values as well,

$$ D^{\gamma} f(l) = \mathcal{L}^{-1}\left\{s^{\gamma}\mathcal{L}\left\{f(l)\right\}\right\}. $$

Thus we have,

$$ \begin{aligned} D^{\gamma} f(l) &= D^{\gamma}\left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{sl}\int_{0}^{\infty} e^{-sl} ds dl\right) \\ &= \mathcal{L}^{-1}\left\{s^{\gamma}\mathcal{L}\left\{f(l)\right\}\right\}. \end{aligned} $$

Where a is choosen such that it is greater than the real part of any of the singularities of $f(l)$.

Now the Fourier transforms and its inverse is,

$$ \mathcal{F}\left\{f(l)\right\} = \int_{-\infty}^{\infty} e^{-itl} f(l) dl. $$

$$\mathcal{F}^{-1}\{f(l)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itl} f(l)dl.$$

which can be generalized in the same way to be used for the solution of fractional differential equation of non-integer order γ.

$$D^{\gamma} f(l) = \mathcal{F}^{-1}\left\{(it)^{\gamma}\mathcal{F}\{f(l)\}\right\}.$$

So we can observe that in case of laplace transform the generalized derivative is a Riemann-Liouville derivative with the lower starting limit of zero and in case of Fourier transform it is Weyl derivative.

2.9 Convolution

Convolution is a very important and simple process used to deal with the problems involving frequency spaces with the help of Laplace and Fourier transform. The derivative of a function is its convolution with certain function. For example let us consider the following function

$$\begin{aligned}\Phi_{\gamma}(x) * f(x) &= \int_{0}^{x} f(l)\Phi_{\gamma}(x-l)dl = \int_{0}^{x}\frac{(x-l)^{\gamma-1}}{\Gamma(-\gamma)} f(l)dl \\ &= D^{-\gamma} f(l).\end{aligned}$$

where

$$\Phi_{\gamma}(x) = \frac{x^{\gamma-1}}{\Gamma(\gamma)}.$$

Now consider,

$$\mathcal{L}\left\{x^{\gamma}\right\} = \frac{\Gamma(\gamma+1)}{l^{\gamma+1}} \Rightarrow \mathcal{L}\left\{\Phi_{\gamma}\right\} = \mathcal{L}\left\{\frac{x^{\gamma-1}}{\Gamma(\gamma)}\right\} = l^{-\gamma}.$$

So we will have,

$$\mathcal{L}\left\{D^{\gamma} f(x)\right\} = \mathcal{L}\left\{\Phi_{\gamma}(x) * f(x)\right\} = \mathcal{L}\left\{\Phi_{\gamma}(x)\right\}\mathcal{L}\{f(x)\} = l^{-\gamma}\mathcal{L}\{f(x)\}. \tag{2.5}$$

So it proves the fact. If any function other than (2.5) is chosen then it would be simply the function of derivative. For example consider the following function that can be expanded in the power of *l*,

$$g(l)=\sum_{v=-\infty}^{\infty} a_v l^v..$$

$$\Rightarrow g(D)f(x)=\sum_{v=-\infty}^{\infty} a_v D^v f(x)=\sum_{v=-\infty}^{\infty} a_v \Phi_{-v}(x)*f(x)=h(x)*f(x).$$

where

$$h(x)=\sum_{v=-\infty}^{\infty} b_v x^v.$$

$$b_v=\frac{a_{-v-1}}{\Gamma(v+1)}.$$

Now consider the laplace transform.

$$\mathcal{L}\{g(D)f(x)\}=\mathcal{L}\{h(x)*f(x)\}=\mathcal{L}\{h(x)\}\mathcal{L}\{f(x)\}=g(l)\mathcal{L}\{f(x)\}.$$

So clearly,

$$g(D)f(x)=\mathcal{L}^{-1}\{g(t)\mathcal{L}\{f(x)\}\}. \tag{2.6}$$

In case of Fourier transform

$$g(D)f(x)=\mathcal{F}^{-1}\{g(it)\mathcal{F}\{f(x)\}\}. \tag{2.7}$$

(2.6) and (2.7) are very useful tools for calculating the functions of derivatives. For example consider,

$$\begin{aligned}\cos(D)\sin(z)&=\mathcal{F}^{-1}\{\cos(it)\mathcal{F}\{\sin(z)\}\}=\mathcal{F}^{-1}\{\cos(it)\pi i(\delta(t+1)-\delta(t-1))\}\\&=\cosh(1)\sin(z).\end{aligned}$$

2.10 Fractional integral equations

Definition 2.10.1. Cauchy integral formula is also used to generalize derivative of non-integral order that becomes very important rule in case of complex analysis. For integer order $n \in \mathbb{N}$, it becomes,

$$D^n f(l) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-l)^{n+1}} dz.$$

which in case of non-integral order becomes,

$${}_{z_0}D_z^{\gamma} f(z) = \frac{\Gamma(\gamma+1)}{2\pi i} \oint_{C(z_0,z^+)} \frac{f(l)}{(l-z)^{\gamma+1}} dl.$$

but we have to be careful in choosing the integration contour because there is an isolated singularity for $l = z$.

Definition 2.10.2. The Cauchy formula for the fractional integral is

$$J^q f(l) = \frac{1}{(q-1)!} \int_0^l (l-\omega)^{q-1} f(\omega) d\omega.$$

where $l > 0$ and $q \in \mathbb{N}$.

So using the gamma function we can define the fractional integral of order γ in the following way.

$$J^{\gamma} f(l) = \frac{1}{\Gamma(\gamma)} \int_0^l (l-\omega)^{\gamma-1} f(\omega) d\omega. \qquad (2.8)$$

where $l > 0$ and $\gamma \in R^+$.

This fractional integral has the following important properties.

$$J^{\gamma} J^{\beta} = J^{\gamma+\beta} \text{ where } \gamma, \beta > 0.$$

which implies the following commutative property,

$$J^{\gamma} J^{\beta} = J^{\beta} J^{\gamma} \text{ where } \gamma, \beta > 0.$$

The fractional integral operator on power function is.

$$J^{\gamma} v^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma+1)} v^{\alpha+\gamma}.$$

where

$$\gamma > 0,\ \alpha > -1,\ v > 0.$$

The proof is based on the Gamma and Beta functions

$$\Gamma(t) = \int_0^\infty e^{-u} u^{t-1} du, \quad \Gamma(t+1) = t\Gamma(t). \quad \text{where } \operatorname{Re}(t) > 0.$$

and,

$$B(h, j) = \int_0^1 u^{h-1}(1-u)^{j-1} du = \frac{\Gamma(h)\Gamma(j)}{\Gamma(h+j)} = B(j,h).$$

where $\operatorname{Re}(h)$ and $\operatorname{Re}(j) > 0$.

Now consider the following notation.

$$\Phi_{\gamma}(l) = \frac{l_{+}^{\gamma-1}}{\Gamma(\gamma)}, \quad \text{where } \gamma > 0$$

where + sign is used to show that function vanishes at $\gamma > 0$.

Therefore fractional integral of order $\gamma > 0$ can be represented as the Laplace convolution between $\Phi_{\gamma}(l)$ and $f(l)$ as,

$$J^{\gamma} f(l) = \Phi_{\gamma}(l) * f(l) = \frac{1}{\Gamma(\gamma)} \int_0^l (l-\omega)^{\gamma-1} f(\omega) d\omega.$$

and also,

$$\Phi_{\gamma}(l) * \Phi_{\beta}(l) = \Phi_{\gamma+\beta}(l) \quad \gamma, \beta > 0.$$

Definition 2.10.3. Let D^q denotes the derivative operator of order q then we have,

$$D^q J^q = I, \quad \text{but } J^q D^q \neq I \quad \text{where } q \in \mathbb{N}.$$

which means that D^q is left operator on corresponding integral operator so let us introduce integer p such that $p-1<\gamma<p$, then we can write fractional derivative of order γ in the following way,

$$D^{\gamma} f(l)=\begin{cases} \dfrac{d^p}{dl^p}\left[\dfrac{1}{\Gamma(p-\gamma)}\displaystyle\int_0^l \dfrac{f(\omega)}{(l-\omega)^{\gamma+1-p}}d\omega\right], & p-1<\gamma<p \\ \dfrac{d^p}{dl^p} f(l) & \gamma=p. \end{cases} \tag{2.9}$$

so we can easily recognize that,

$$D^{\gamma} f(l)=D^p J^{p-\gamma} f(l).\ \ p-1<\gamma<p.\ .$$

Hence the fractional derivative of order γ on the power function can be defined as,

$$D^{\gamma} l^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)} l^{\beta-\gamma},\quad \gamma>0,\ \ \beta>-1,\ \ l>0. \tag{2.10}$$

One important fact of the fractional derivative of order γ is that it is not zero in case of constant function $f(l)=1$ if $\gamma \notin N$. Let $\beta=0$ in (2.10) , then we have,

$$D^{\gamma} 1=\frac{l^{-\gamma}}{\Gamma(1-\gamma)},\quad \gamma\geq 0,\ \ l>0.$$

which is of course zero if $\gamma\in\mathbb{N}$ because of the poles of gamma function in the points 0, -1, -2,…….

Definition 2.10.4. Another definition of fractional derivative of order γ has been defined by Caputo called Caputo fractional derivative of order γ . It is defined as

$$D_*^{\gamma} f(l)=J^{p-\gamma} D^p k(l),\ \ p-1<\gamma\leq p. \tag{2.11}$$

$$D_*^{\gamma} f(l)=\begin{cases} \dfrac{1}{\Gamma(p-\gamma)}\displaystyle\int_0^l \dfrac{f^p(\omega)}{(l-\omega)^{\gamma+1-p}}d\omega, & p-1<\gamma<p \\ \dfrac{d^p}{dl^p} f(l) & \gamma=p. \end{cases}$$

provided that absolute integrability of derivative of order p is met. Now in general,

$$D^{\gamma} f(l) = D^{p} J^{p-\gamma} f(l) \neq J^{p-\gamma} D^{\gamma} f(l) = D_{*}^{\gamma} f(l)$$

unless function vanishes at $t = 0^{+}$ for the first $(p-1)$ derivative because for $p-1 < \gamma < p$ and $l > 0$, we have,

$$D^{\gamma} f(l) = D_{*}^{\gamma} f(l) + \sum_{q=0}^{p-1} \frac{l^{q-\gamma}}{\Gamma(q-\gamma+1)} f^{(q)}(0^{+}). \qquad (2.12)$$

Thus using (2.10), we can write

$$D_{*}^{\gamma} f(l) = D^{\gamma} \left(f(l) - \sum_{q=0}^{p-1} \frac{l^{q}}{q!} f^{(q)}(0^{+}) \right).$$

Hence according to this definition, the fractional derivative of a constant is still 0,

$$D_{*}^{\gamma} 1 = 0, \quad \gamma > 0.$$

Now as

$$D^{\gamma} z^{\gamma-1} = 0, \quad \gamma > 0, \ z > 0.$$

We thus recognize the following statements about functions which for $l > 0$ admit the same fractional derivative of order γ with $p-1 < \gamma < p$. $p \in \mathbb{N}$.

$$D^{\gamma} f(l) = D^{\gamma} g(l) \Leftrightarrow f(l) = g(l) + \sum_{i=1}^{p} c_i l^{\gamma-i}.$$

$$D_{*}^{\gamma} f(l) = D_{*}^{\gamma} g(l) \Leftrightarrow f(l) = g(l) + \sum_{i=1}^{p} c_i l^{p-i}.$$

where c_j are arbitrary constants. So we observe that,

$$J^{\gamma} D^{\gamma} l^{\gamma-1} = 0, \text{ but } D^{\gamma} J^{\gamma} l^{\gamma-1} = l^{\gamma-1}$$

where $\gamma > 0, \quad l > 0$.

Now let $\gamma \to (p-1)^{+}$, then from (2.9) ,

$$\gamma \to (p-1)^{+} \Rightarrow D^{\gamma} f(l) \to D^{p} J f(l) = D^{p-1} f(l).$$

and from (2.11),

$$\gamma \to (p-1)^{+} \Rightarrow D_{*}^{\gamma} f(l) \to JD^{p} f(l) = D^{p-1} f(l) - f^{p-1}(0^{+}).$$

We will now show that both (2.9) and (2.11) can be derived using convolution of $\Phi_{-\gamma}(l)$ with $f(l)$. For this purpose we will take Gel's and Shilov functions into count,

$$\Phi_{-v}(l) = \frac{l_{+}^{-v-1}}{\Gamma(-v)} = \delta^{(v)}(l), \quad v = 0,1,......$$

where $\delta^{(v)}(l)$ denotes the generalized derivative of order v of the Dirac delta distribution. So,

$$\frac{d^{p}}{dl^{p}} f(l) = f^{(p)}(l) = \Phi_{-p}(l) * f(l) = \int_{0^{-}}^{l^{+}} f(\omega)\delta^{(p)}(l-\omega)d\omega, \; l > 0.$$

which is based on the following properties,

$$\int_{0^{-}}^{l^{+}} f(\omega)\delta^{(p)}(l-\omega)d\omega = (-1)^{p} f^{(p)}(l).$$

and

$$\delta^{(p)}(l-\omega) = (-1)^{p} \delta^{(p)}(\omega - l).$$

In the above equations the limits of integration are extended for the possibility of impulse functions centred at extremes.

Then the formal definition of the fractional derivative of order γ is given by,

$$\Phi_{-\gamma}(l) * f(l) = \frac{1}{\Gamma(-\gamma)} \int_{0^{-}}^{l^{+}} \frac{f(\omega)}{(l-\omega)^{1+\gamma}} d\omega, \quad \gamma \in \mathbb{R}^{+}..$$

But the kernel $\Phi_{-\gamma}(l)$ is not locally absolutely integrable, and then the above integral is divergent in general so we need to regularize the divergent integral in some way.

Let $p-1<\gamma<p$, then we may write $-\gamma=-p+(p-\gamma)$ or $-\gamma=(p-\gamma)-p$, so we obtain,

$$\left[\Phi_{-p}(l)*\Phi_{p-\gamma}(l)\right]*f(l)=\Phi_{-p}(l)*\left[\Phi_{p-\gamma}(l)*f(l)\right]=D^{p}J^{p-\gamma}f(l).$$

Or,

$$\left[\Phi_{p-\gamma}(l)*\Phi_{-p}(l)\right]*f(l)=\Phi_{p-\gamma}(l)*\left[\Phi_{-p}(l)*f(l)\right]=J^{p-\gamma}D^{p}f(l).$$

which gives the two alternative definitions of (2.9) and (2.11).

Definition 2.10.5. Riemann has generalized the integral Cauchy formula starting at $l=0$ and Liouville choose to start it at $-\infty$ defined by,

$$J_{-\infty}^{\gamma}f(l)=\frac{1}{\Gamma(\gamma)}\int_{-\infty}^{l}(l-\omega)^{\gamma-1}k(\omega)d\omega,\quad \gamma\in\mathbb{R}^{+}. \tag{2.13}$$

so for $p-1<\gamma\leq p,\ p\in N,\ D_{-\infty}^{\gamma}f(l)=D^{p}J_{-\infty}^{p-\gamma}f(l).$, we have

$$D_{-\infty}^{\gamma}f(l)=\begin{cases}\dfrac{d^{p}}{dl^{p}}\left[\dfrac{1}{\Gamma(p-\gamma)}\displaystyle\int_{-\infty}^{l}\frac{f(\omega)}{(l-\omega)^{\gamma+1-p}}d\omega\right], & p-1<\gamma<p\\ \dfrac{d^{p}}{dl^{p}}f(l) & \gamma=p.\end{cases}$$

In case $f(l)$ vanishes at $l\to-\infty$ along with its first $p-1$ derivatives, then

$$D^{p}J_{-\infty}^{p-\gamma}f(l)=J_{-\infty}^{p-\gamma}D^{\gamma}f(l).$$

The sufficient condition that (2.8) converges is that,

$$f(l)=O(l^{\varepsilon-1}),\quad \varepsilon>0,\ \text{and } l\to o^{+} \tag{2.14}$$

and sufficient condition that (2.13) converges is,

$$f(l)=O(|l|^{-\gamma-\varepsilon}),\quad \varepsilon>0,\ \text{and } l\to-\infty.. \tag{2.15}$$

Integrable functions which satisfy (2.14) and (2.15) are sometimes referred as Riemann class and Liouville class respectively. For example power function l^{β} with $\beta > -1$ and $l > 0$ are of Riemann class and on the other hand the functions with, $|l|^{-\delta}$ with $\delta > \gamma > 0$ and $l < 0$ and $\exp(cl)$ with $\mathrm{c} > 0$ is of Liouville class. So we have,

$$J_{-\infty}^{\gamma} |l|^{-\delta} = \frac{\Gamma(\delta-\gamma)}{\Gamma(\delta)} |l|^{-\delta+\gamma}, \;\; \mathrm{D}_{-\infty}^{\gamma} |l|^{-\delta} = \frac{\Gamma(\delta+\gamma)}{\Gamma(\delta)} |l|^{-\delta-\gamma}.$$

and

$$J_{-\infty}^{\gamma} e^{cl} = c^{-\gamma} e^{cl}, \;\; \mathrm{D}_{-\infty}^{\gamma} e^{cl} = c^{\gamma} e^{cl}.$$

Now in case of jump discontinuities at t=0, it is worthwhile to write,

$\int_{-\infty}^{l} (....) d\omega = \int_{0^-}^{l} (....) d\omega,$ for example consider for $0 < \gamma < 1$.

$$\frac{1}{\Gamma(1-\gamma)} \int_{0^-}^{l^+} \frac{f^{/}(\omega)}{(l-\omega)^{\gamma}} d\omega = \frac{l^{-\gamma}}{\Gamma(1-\gamma)} f(0^+) + \frac{1}{\Gamma(1-\gamma)} \int_{0}^{l} \frac{f^{/}(\omega)}{(l-\omega)^{\gamma}} d\omega.$$

$$= \frac{l^{-\gamma}}{\Gamma(1-\gamma)} f(0^+) + D_*^{\gamma} f(l) = D^{\gamma} f(l).$$

where we have used (2.12) with $p = 1$.

Definition 2.10.6. Weyl has defined another complementary integral with respect to usual Reimann-Liouville integral as follows,

$$W_{\infty}^{\gamma} f(l) = \frac{1}{\Gamma(\gamma)} \int_{l}^{\infty} (\omega - l)^{\gamma - 1} f(\omega) d\omega, \;\; \gamma \in \mathbb{R}^+. . \qquad (2.16)$$

The relation between (2.13) and (2.16) is

$$J_{-\infty}^{\gamma} f(l) = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{l} (l-\omega)^{\gamma-1} k(\omega) d\omega = -\frac{1}{\Gamma(\gamma)} \int_{\infty}^{-l} (l+\omega^{/})^{\gamma-1} f(-\omega^{/}) d\omega^{/}.$$

$$= \frac{1}{\Gamma(\gamma)} \int_{l'}^{\infty} (\omega' - l')^{\gamma - 1} f(-\omega') d\omega' = W_{\infty}^{\gamma} g(l')$$

where $g(l') = f(-l'), l' = -l$ and we have also carried out change of variable as

$$\omega \to \omega' = -\omega \text{ and } l \to l' = -l.$$

Definition 2.10.7. The operation of integration and differentiation with respect to laws of commutation and additivity are,

$$J^p J^q = J^q J^p = J^{p+q}, \quad D^p D^q = D^q D^p = D^{p+q}.$$

where $p, q = 0, 1, 2, ...$

In general both D^{γ} and D_*^{γ} do not satisfy either semi-group property or commutative property.

$$D^{\gamma} D^{\beta} f(l) = D^{\beta} D^{\gamma} f(l) \neq D^{\gamma+\beta} f(l).$$
$$D^{\gamma} D^{\beta} g(l) \neq D^{\beta} D^{\gamma} g(l) = D^{\gamma+\beta} g(l).$$

For example let $f(l) = l^{-1/2}$ and $\beta = \alpha = 1/2$. Then using (2.10), we get,

$$D^{1/2} f(l) = 0, \quad D^{1/2} D^{1/2} f(l) = 0,$$

but

$$D^{1/2+1/2} f(l) = Df(l) = -z^{-3/2}/2.$$

Similary $g(l) = l^{1/2}$ and $\gamma = 1/2, \beta = 3/2$, then again using (2.10) we get,

$$D^{1/2} g(l) = \sqrt{\pi}/2, \; D^{3/2} g(l) = 0.$$

but

$$D^{1/2} D^{3/2} g(l) = 0, \quad D^{3/2} D^{1/2} g(l) = -l^{3/2} / 4.$$

$$D^{1/2+3/2} = D^2 g(l) = -l^{3/2} / 4.$$

Chapter 3

FRACTIONAL INTEGRAL EQUATIONS

3.1 Abel integral equation of first kind

Let us consider Abel integral equation of first kind as follows.

$$\frac{1}{\Gamma(\gamma)}\int_0^l \frac{u(\omega)}{(l-\omega)^{1-\gamma}}d\omega = f(l), \quad 0<\gamma<1. \tag{3.1}$$

where $f(l)$ is the given function. Its fractional integral form is,

$$J^{\gamma}u(l) = f(l).$$

which can be solved in terms of fractional derivative,

$$u(l) = D^{\gamma} f(l).$$

Now using the fact that $D^{\gamma}J^{\gamma} = I$, we will solve (3.1) using Laplace transform. As we know that

$$J^{\gamma}u(l) = \Phi_{\gamma}(l) * u(l) \div \tilde{u}(s)/s^{\gamma}.$$

where $\div$ denotes the Laplace transform pair and,

$$\Phi_{\gamma}(l) = \frac{l_{+}^{\gamma-1}}{\Gamma(\gamma)}.$$

So we have,

$$\frac{\tilde{u}(s)}{s^{\gamma}} = \tilde{f}(s) \Rightarrow \tilde{u}(s) = s^{\gamma}\,\tilde{f}(s). \tag{3.2}$$

Now we will follow two ways to get the inverse Laplace transform from (3.2) in the following manner.

Our first approach is to use the standard rules, so we can write (3.2) as

$$\tilde{u}(s) = s\left[\frac{\tilde{f}(s)}{s^{1-\gamma}}\right].$$

Thus we obtain,

$$u(l) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dl}\int_0^l \frac{f(\omega)}{(l-\omega)^{\gamma}}d\omega. \tag{3.3}$$

writing (3.2) in the following way, we get,

$$\tilde{u}(s) = \frac{1}{s^{1-\gamma}}\left[s\,\tilde{f}(\iota s) - f(0^+)\right] + \frac{f(0^+)}{s^{1-\gamma}}. \tag{3.4}$$

So we will have,

$$u(l) = \frac{1}{\Gamma(1-\gamma)}\int_0^l \frac{f'(\omega)}{(l-\omega)^{\gamma}}d\omega + f(0^+)\frac{l^{-\gamma}}{\Gamma(1-\gamma)}.\,. \tag{3.5}$$

Hence (3.3) and (3.5) are the solution of (3.1) in view of fractional derivative D^{γ} *and* D_*^{γ}.

Second approach needs that $f(l)$ should be differentiable with Laplace transformable derivative with $0 \le \left|f(0^+)\right| < \infty$, then from (3.4) it is noticeable that $u(0^+)$ is infinite if $f(0^+) \ne 0$, being $\mathrm{u}(l) = O(l^{-\gamma})$, as $l \to 0^+$. Weaker condition is needed for the 1st approach that the integral at R.H.S of (3.3) must vanish as $l \to 0^+$ which shows that with $f(l) = O(l^{-\nu}), 0 < \nu < 1-\gamma$ as $l \to 0^+$, can be infinite as $l \to 0^+$. So from (3.3) and the fact that $\Phi_{1-\gamma} * \Phi_{1-\nu} = \Phi_{2-\gamma-\nu}$, we observe that $u(0^+)$ can be infinite if $f(0^+)$ is infinite, as $u(l) = O(l^{-(\gamma+\nu)})$, as $l \to 0^+$.

3.2 Abel integral equation of second kind

Let us consider the Abel equation of second kind,

$$u(l) + \frac{\vartheta}{\Gamma(\gamma)}\int_0^l \frac{u(\omega)}{(l-\omega)^{1-\gamma}}d\omega = f(l), \quad \gamma > 0, \quad \vartheta \in C. \tag{3.6}$$

In terms of fractional integral operator, it becomes,

$$(1+\vartheta J^{\gamma})u(l)=f(l). \tag{3.7}$$

which can be solved as follows,

$$u(l)=(1+\vartheta J^{\gamma})^{-1}f(l)=\left(1+\sum_{p=1}^{\infty}(-\vartheta)^{p}J^{\gamma p}\right)f(l). \tag{3.8}$$

using Laplace convolution, $J^{\gamma p}$ can be written as,

$$J^{\gamma p}f(l)=\Phi_{\gamma p}(l)*f(l)=\frac{l_{+}^{\gamma p-1}}{\Gamma(\gamma p)}*f(l).$$

So the formal solution is,

$$u(l)=f(l)+\left(\sum_{p=1}^{\infty}(-\vartheta)^{p}\frac{l_{+}^{\gamma p-1}}{\Gamma(\gamma p+1)}\right),\quad l>0,\ \gamma>0.$$

Now as we know

$$e_{\gamma}(l;\partial):=E_{\gamma}(-\partial l^{\gamma})=\sum_{p=0}^{\infty}\frac{(-\partial l^{\gamma})^{p}}{\Gamma(\gamma p+1)},\quad l>0,\ \gamma>0,\ \partial\in\mathbb{C}.$$

where E_{γ} is Mittage-Leffler function of order γ, so we have

$$\sum_{p=0}^{\infty}(-\partial)^{p}\frac{l_{+}^{\gamma p-1}}{\Gamma(\gamma p)}=\frac{d}{dl}E_{\gamma}(-\partial l^{\gamma})=e'_{\gamma}(l;\partial),\quad l>0.$$

So the final solution is

$$u(l)=f(l)+e'_{\gamma}(l;\partial)*f(l). \tag{3.9}$$

The result can also be obtained in the following way.

Applying Laplace transform to (3.6), we get,

$$\left[1+\frac{\partial}{s^{\gamma}}\right]\tilde{u}(s)=\tilde{f}(s)\Rightarrow\tilde{u}(s)=\frac{s^{\gamma}}{s^{\gamma}+\partial}\tilde{f}(s). \tag{3.10}$$

Now using the following Laplace transform pair we will get the inverse Laplace transform,

$$e_\gamma(l;\partial) := E_\gamma(-\partial l^\gamma) \div \frac{s^{\gamma-1}}{s^\gamma + \partial}.$$

Now we can choose two different ways to get the inverse Laplace transform. According to the standard rule, (3.10) becomes,

$$\tilde{u}(s) = s\left[\frac{s^{\gamma-1}}{s^\gamma + \partial}\tilde{f}(s)\right].$$

So we have,

$$u(l) = \frac{d}{dl}\int_0^l f(l-\omega)e_\gamma(\omega;\partial)d\omega.$$

If we write (3.10) as,

$$\tilde{u}(s) = \frac{s^{\gamma-1}}{s^\gamma + \partial}\left[s\tilde{f}(s) - f(0^+)\right] + f(0^+)\frac{s^{\gamma-1}}{s^\gamma + \partial}.$$

We obtain,

$$u(l) = \int_0^l f^{/}(l-\omega)e_\gamma(\omega;\partial)d\omega + f(0^+)e_\gamma(l;\partial).$$

Now we can observe that $e_\gamma(l;\partial)$ is a differentiable function w.r.t. l with

$$e_\gamma(0^+;\partial) = E_\gamma(0^+) = 1.$$

Then there exists another possibility to write (3.10) as,

$$\tilde{u}(s) = \left[s\frac{s^{\gamma-1}}{s^\gamma + \partial} - 1\right]\tilde{f}(s) + \tilde{f}(s).$$

Then we write,

$$u(l) = \int_0^l f(l-\omega)e_\gamma^{/}(\omega;\partial)d\omega + f(l).$$

which is in agreement with (3.9).

3.3 Some Applications of Abel integral equation

Abel integral equations are very useful for the calculation of various physical measurements, most of them deal with the radius of a circle or a sphere as independent variable which gives J^{γ} after the change of variable usually $\gamma = 1/2$ and the equation becomes the first kind equation.

Another important field is inverse boundary value problems in partial differential equations which involves Abel integral equations, in particular parabolic ones in which the independent variable has the meaning of time. We will discuss the problem of heating of a semi-infinite rod by influx or efflux across the boundary into the internal side.

Let us consider the following heat equation

$$r_l - r_{xx} = 0, \quad r = r(x,l).$$

The interval is semi-infinite $0 < x < \infty$ and $0 < l < \infty$ of space and time respectively and $r(x,l)$ means temperature.

Let the initial temperature of rod is 0,

$$r(x,0) = 0 \text{ for } 0 < x < \infty.$$

Assume the influx across the boundary $x = 0$ from $x < 0$ to $x > 0$ is given by

$$-r(0,l) = p(l).$$

Then under the suitable regularity condition, we can find $r(x,l)$ using the following formula,

$$r(x,l) = \frac{1}{\sqrt{\pi}} \int_0^l \frac{p(\omega)}{\sqrt{(l-\omega)}} e^{-x^2/[4(l-\omega)]} d\omega \quad x > 0 \text{ and } l > 0. \qquad (3.11)$$

We will deal with interior boundary temperature $\Psi(l) := r(0^+, l) \;\; l > 0$ which can be evaluated using (3.11)

$$\Psi(l)=\frac{1}{\sqrt{\pi}}\int_0^l \frac{p(\omega)}{\sqrt{(l-\omega)}}d\omega = J^{1/2}p(l),\ l>0. \tag{3.12}$$

This is Abel integral equation of first kind and can be used to find the unknown influx $p(l)$ if the internal boundary temperature $\Psi(l)$ is given by measurements or can be determined by controlling the influx.

Using (2.9) with $p=1$ and $\gamma=1/2$, we can have

$$p(l)=D^{1/2}\Psi(l)=\frac{1}{\sqrt{\pi}}\frac{d}{dl}\int_0^l \frac{\Psi(\omega)}{\sqrt{(l-\omega)}}d\omega.$$

Now consider the following special cases

$$(i)\quad \Psi(l)=l \Rightarrow p(l)=\frac{1}{2}\sqrt{\pi l}.$$

$$(ii)\quad \Psi(l)=1 \Rightarrow p(l)=\frac{1}{\sqrt{\pi l}}.$$

Both results have been achieved using (2.10). So the required influx is continuous and increases from $0\rightarrow\infty$ for the linear increase in internal boundary temperature with unbounded derivative at $l=0^{+}$ but it decreases from $\infty\rightarrow 0$ as $l\rightarrow\infty$.

Now we will modify the problem using Abel integral of second kind for which we will assume that the given rod is just touching the source of a liquid in $x<0$ and the external boundary temperature is strictly being controlled such that

$$r(0^{-},l):=\Omega(l)$$

Now according to Newton's law of radiation, the influx of heat from 0^{-} to 0^{+} that is proportional to the difference of internal and external temperature is given by

$$p(t)=\partial\left[\Omega(l)-\Psi(l)\right],\quad \partial>0..$$

Putting this into (3.12), we get

$$\Psi(l)=\frac{\partial}{\sqrt{\pi}}\int_0^l \frac{\Omega(\omega)-\Psi(\omega)}{\sqrt{l-\omega}}d\omega.$$

which can be represented in terms of operator notation as follows,

$$(1+\partial J^{1/2})\Psi(l)=\partial J^{1/2}\Omega(l). \tag{3.13}$$

Now let the external boundary temperature is given but the internal boundary temperature is unknown, then (3.13) is the Abel integral equation of second kind that can be used to evaluate $\Psi(l)$.

Let us take $\gamma=1/2$, then (3.13) becomes,

$$\Psi(l)=\partial(1+\partial J^{1/2})^{-1}J^{1/2}\Omega(l).$$

that is of the form (3.7) and by (3.8) its solution is as follows,

$$\Psi(l)=-\sum_{p=0}^{\infty}(-\partial)^{p+1}J^{p+1/2}\Omega(l).$$

Now we will consider the following case, with $\Omega(l)=1$.

then with $\alpha=0$ and using the following relation, we obtain,

$$J^{\gamma}l^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma+1)}l^{\alpha+\gamma} \quad \alpha>-1,\ \gamma>0,\ l>0.$$

$$J^{p+1/2}\Omega(l)=\frac{l^{(p+1)/2}}{\Gamma\left[(p+1)/2+1\right]}.$$

Hence,

$$\Omega(l)=-\sum_{p=0}^{\infty}(-\partial)^{p+1}\frac{l^{(p+1)/2}}{\Gamma\left[(p+1)/2+1\right]}=1-\sum_{q=0}^{\infty}(-\partial)^{q}\frac{l^{q/2}}{\Gamma\left[q/2+1\right]}.$$

So that,

$$\Omega(l)=1-E_{1/2}\left(-\partial l^{1/2}\right)=1-e_{1/2}(l;\partial).$$

3.4 Fractional Differential Equations

Now we will discuss simple fractional relaxation and Oscillation equations.

The relaxation differential equation is

$$v^{/}(l) = -v^{/}(l) + g(l).$$

$g(l)$ is a continuous function and $l > 0$.

Using initial condition $v(0^{+}) = b_0,$ the solution is,

$$v(l) = b_0 e^{-l} + \int_0^l g(l-\omega) e^{-\omega} d(\omega). \tag{3.14}$$

Similarly the oscillation differential equation reads

$$v^{//}(l) = -v^{/}(l) + g(l).$$

The solution of the above equation under the initial condition $v(0^{+}) = b_0,$ and $v^{/}(0^{+}) = b_1,$ is

$$v(l) = b_0 \cos l + b_1 \sin l + \int_0^l g(l-\omega) \sin \omega d(\omega). \tag{3.15}$$

Now consider,

$$D_*^{\gamma} v(l) = D^{\gamma}\left(v(l) - \sum_{k=0}^{p-1} \frac{l^k}{k!} v^{(k)}(0^{+}) \right) = -v(l) + g(l) \quad l > 0. \tag{3.16}$$

p is a positive integer such that $p-1 < \gamma \le p$ that provides the number of prescribed initial values $v^{(k)}(0^{+}) = b_k$, where $k = 0,1,2,........, p-1$. Now consider,

$$v(l) = \sum_{k=0}^{p-1} b_k v_k(l) + \int_0^l g(l-\omega) v_{\delta}(\omega) d(\omega).$$

$$v_k(l) = J^k v_0(l), \quad v_k^{(h)}(0^{+}) = \delta_{kh}, \quad h,k = 0,1,2,...p-1 \tag{3.17}$$

$$v_{\delta}(l) = -v_0^{/}(l), \quad l > 0.$$

where $v_k(l)$ represents those linearly independent solutions of the differential equation of order p satisfying the initial conditions (3.17). The function $v_{\delta}(l)$ is called impulse-response solution representing the particular solution of inhomogeneous equation with all $b_k = 0$ and with $g(l) = \delta(l)$. So in case of the ordinary relaxation and oscillation, we observe that

$$v_0(l) = e^{-l} = v_\delta(l).$$

and,

$$v_0(l) = \cos l, \quad v_1(l) = Jv_0(l) = \sin l = \cos(l - \pi/2) = v_\delta(l).$$

Now we will solve (3.16) by Laplace transform. We will first reduce the equation with the given initial condition into an equivalent fractional integral equation and then apply the Laplace transform. .

Apply J^γ operator on both sides of (3.16), we have

$$v(l) = \left(\sum_{k=0}^{p-1} b_k \frac{l^k}{k!} - J^\gamma v(l) + J^\gamma g(l) \right).$$

Then we apply the Laplace transform that will yield,

$$\tilde{v}(l) = \left(\sum_{k=0}^{p-1} \frac{b_k}{s^{k+1}} - \frac{1}{s^\gamma} \tilde{v}(l) + \frac{1}{s^\gamma} \tilde{g}(l) \right).$$

hence we have,

$$\tilde{v}(l) = \left(\sum_{k=0}^{p-1} b_k \frac{s^{\gamma-k-1}}{s^\gamma + 1} - \frac{1}{s^\gamma + 1} \tilde{g}(l) \right). \qquad (3.18)$$

Now we will introduce Mittag-Leffler functions

$$e_\gamma(l) = e_\gamma(l;1) := E_\gamma(-l^\gamma) \div \frac{s^{\gamma-1}}{s^\gamma + 1}.$$

$$v_k(l) := J^k e_\gamma(l) \div \frac{s^{\gamma-k-1}}{s^\gamma + 1}, \quad k = 0,1,2....p-1.$$

From inverse Laplace transform of equation (3.18), we find,

$$v(l) = \left(\sum_{k=0}^{p-1} b_k v_k(l) - \int_0^l g(l - \omega) v_0^{/}(\omega) d(\omega) \right).$$

where we have used the following fact with $v_0(0^+) = e_\gamma(0^+) = 1$.

$$\frac{1}{s^\lambda + 1} = -\left(s\frac{s^{\gamma-1}}{s^\gamma + 1} - 1 \right) \div -v_0^{/}(l) = -e_\gamma^{/}(l).$$

That encompasses the solution (3.14) and (3.15) found for $\gamma = 1, 2$ respectively.

3.5 The Mittag-Leffler Type Functions

The function $E_\gamma(g)$ with $\gamma > 0$ is defined as,

$$E_\gamma(g) := \sum_{p=0}^{\infty} \frac{g^p}{\Gamma(\gamma p + 1)}, \quad \gamma > 0, \ g \in \mathbb{C}. \tag{3.19}$$

For $\gamma \to 0^+$, the entire complex plane will lose analyticity since,

$$E_0(g) := \sum_{p=0}^{\infty} g^p = \frac{1}{1-g}, \quad |g| < 1.$$

The Mittag-Leffler function can be used to give the generalization of exponential function,

$$v! = \Gamma(v+1), \quad \text{so } (\gamma v)! = \Gamma(\gamma v + 1) \ v \in \mathbb{N}.$$

The particular cases are,

$$E_2(+g^2) = \cosh g, \ \ E_2(-g^2) = \cos g, \ \ \mathrm{g} \in C.$$

and

$$E_{1/2}(\pm g^{1/2}) = e^g \left[1 + erf(\pm g^{1/2}) \right] = e^g erfc(\mp g^{1/2}), \ \mathrm{g} \in C. \tag{3.20}$$

where erf (erfc) denotes the error function as follows,

$$erf(g) = \frac{2}{\sqrt{\pi}} \int_0^g e^{-v^2} dv, \quad erfc(g) = 1 - erf(g).$$

Now we will prove (3.20). First we will use g instead of $\pm g^{1/2}$ to avoid polidromy, so we will write,

$$E_{1/2}(g) = \sum_{b=0}^{\infty} \frac{g^{2b}}{\Gamma(b+1)} + \sum_{b=0}^{\infty} \frac{g^{2b+1}}{\Gamma(b+3/2)} = v(g) + w(g).$$

where it can been recognized as,

$$v(g)=\sum_{b=0}^{\infty}\frac{g^{2b}}{\Gamma(b+1)}=\exp(g^2)$$

and

$$w(g)=\sum_{b=0}^{\infty}\frac{g^{2b}+1}{\Gamma(b+3/2)}=\exp(g^2)erf(g).$$

It is given in the Bateman Project [70], the error function can be represented as the following series,

$$erf(g)=\frac{2}{\sqrt{\pi}}e^{-g^2}\sum_{b=0}^{\infty}\frac{2^b}{\Gamma(2b+1)!!}g^{2b+1}\quad g\in\mathbb{C}.$$

where

$$(2b+1)!!=1\cdot 3\cdot 5....(2b+1)=2^{b+1}\Gamma\left(b+3/2\right)\sqrt{\pi}.$$

Another proof can be given if $w(g)$ satisfies the following differential equation.

$$w^{/}(g)=2\left[\frac{1}{\sqrt{\pi}}+gw(g)\right],\quad w(0)=0.$$

whose solution can be immediately obtained as follows

$$w(g)=\frac{2}{\sqrt{\pi}}e^{g^2}\int_0^g e^{-v^2}dv=e^{g^2}erf(g).$$

Now consider the following function

$$E_{\gamma,\beta}(g)=\sum_{b=0}^{\infty}\frac{g^b}{\Gamma\left(\gamma b+\beta\right)},\quad \gamma>0,\quad \beta>0,\ g\in\mathbb{C}. \tag{3.21}$$

Particular cases are

$$E_{1,2}(g)=\frac{e^g-1}{g},\quad E_{2,2}(g)=\frac{\sinh(g^{1/2})}{g^{1/2}}.$$

Now we will discuss the general functional relations for generalized Mittage-Leffler function (3.21) which involves both parameters γ and β.

$$E_{\gamma,\beta}(g)=\frac{1}{\Gamma(\beta)}+gE_{\gamma,\beta+\gamma}(g).$$

$$E_{\gamma,\beta}(g)=\beta E_{\gamma,\beta+1}(g)+\gamma g\frac{d}{dg}E_{\gamma,\beta+1}(g).$$

$$\left(\frac{d}{dg}\right)^m\left[g^{\beta-1}E_{\gamma,\beta}(g^\gamma)\right]=g^{\beta-m-1}E_{\gamma,\beta-m}(g^\gamma),\ m\in N.$$

Now we will prove the following different relations,

$$\left(\frac{d}{dt}\right)^m E_m(t^m)=E_m(t^m),\quad m\in\mathbb{N}. \tag{3.22}$$

$$\frac{d^m}{dt^m}E_{m/n}(t^{m/n})=E_{m/n}(t^{m/n})+\sum_{i=0}^{n-1}\frac{t^{-im/n}}{\Gamma(1-im/n)},\quad n=2,3,...., \tag{3.23}$$

$$E_{m/n}(t)=\frac{1}{m}\sum_{j=0}^{m-1}E_{1/n}\left(t^{1/m}e^{i2\pi j/m}\right). \tag{3.24}$$

and

$$E_{1/n}(t^{1/n})=e^t\left[1+\sum_{i=0}^{n-1}\frac{\alpha(1-i/n,t)}{\Gamma(1-i/n)}\right]\quad n=2,3,......, \tag{3.25}$$

where $\alpha(a;t)=\int_0^t e^{-x}x^{a-1}dx$ denotes the incomplete gamma function.

Now it is clear that the (3.22) and (3.23) can be easily derived from (3.19). So we will prove (3.24).

We know that,

$$\sum_{j=0}^{m-1}e^{i2\pi jk/m}=\begin{cases}m & \text{if } k=0\\ 0 & \text{if } k\neq 0\end{cases}$$

So from (3.19) and using the above identity we have

$$\sum_{j=0}^{m-1}E_\gamma(te^{i2\pi j/m})=mE_{\gamma m}(t^m)\quad m\in\mathbb{N}.$$

Replacing γ by γ/m and t by $t^{1/m}$ in the above equation we get

$$E_\gamma(t\ )=\frac{1}{m}\sum_{j=0}^{m-1}E_{\gamma/m}(t^{1/m}e^{i2\pi j/m})\ \ m\in\mathbb{N}.\ .$$

Replacing γ by m/n, we get the required equation (3.24).

Now we will prove (3.25)

In order to prove the required relation we will first consider (3.23) for $m=1$. Then we will multiply both sides by e^{-t} to get the following relation.

$$\frac{d}{dt}\left[e^{-t}E_{1/n}(t^{1/n})\right]=e^{-t}\sum_{j=1}^{n-1}\frac{t^{-j/n}}{\Gamma(1-j/n)}.$$

Integrating both sides and using the concept of incomplete gamma function, proves the relation (3.25).

Chapter 4

EXISTENCE AND UNIQUENESS THEOREM OF FRACTIONAL MIXED VOLTERAA-FREDHOLM INTEGRODIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

Definition 4.1. Caputo fractional order derivative of the function *f* defined on the interval (a, b) is given by

$$ {}_a^tD^{\gamma} f(z) = J^{p-\gamma} \frac{d^p}{dt^p} f(z) = \frac{1}{\Gamma(p-\gamma)} \int_a^z (z-u)^{p-\gamma-1} f^{(p)}(u) du. $$

Where γ is complex number with $\gamma > 0$ or $\operatorname{Re}(\gamma) > 0$ and $p = [\gamma] + 1$, $[\gamma]$ is the integer part of γ.

Lemma 4.2. Let γ is greater than 0 then

$$ {}_a^tD^{-\gamma}\, {}_a^tD^{\gamma} f(t) = f(t) + h_0 + h_1 t + h_2 t^2 + ... + h_{p-1} t^{p-1} $$

For some $h_i \in \mathbb{R}, i = 0,1,......, p-1, p = [\gamma]+1.$

Definition 4.3. If *f* is defined everywhere on the given interval (a, b) then,

$$ {}_a^bD^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^b (b-t)^{\gamma-1} f(t) dt, $$

provided that $\gamma > 0$ and the integral exists.

Theorem 4.4 (Krasnpsel'skii fixed point theorem). Let *M* be a non-empty, bounded and closed-convex subset of a Banach space *X* and let *A* and *B* be the two operators such that,

1. $Ax + By \in M$, where $x, y \in M$.

2. A is continuous and compact.

3. B is a contraction mapping.

According to this theorem, we have $z \in M$ such that $\mathrm{z} = Az + Bz$

Let X be a Banach space with the norm $\|.\|$. Let $c = ([0,T], X)$ be the Banach space of all the continuous functions such that,

$$\varphi : [0,T] \to X,$$

with norm $\|\varphi\| = \sup \|\varphi(u)\|$, $\mathrm{u} \in [0,T]$.

Now the fractional mixed Volttera-Fredholm integrodifferential equation is,

$$D^{\gamma} y(t) = f\left(t, y(t), \int_0^t k(t,u,y(u))du, \int_0^T h_1(t,u,y(u))du\right). \quad (4.1)$$

with the following boundary conditions,

$$y(0) - y^{/}(0) = \int_0^T g(y(u))du, \quad \mathrm{y}(T) - y^{/}(T) = \int_0^T h(y(u))du. \quad (4.2)$$

where $1 < \gamma \le 2$, D^{γ} is the Caputo fractional derivative and the non-linear function

$$f : [0,T] \times X \times X \times X \to X, k, h_1 : [0,T] \times [0,T] \times X \to X \text{ and } g, h : X \to X.$$

with the following conditions satisfied,

$(C1)$ there exists constants G_1 and G_2 such that $\|h(y)\| \le G_1$, and $\|g(\mathrm{y})\| \le G_2$ $y \in X$.

(C2) there exists constants a_1 and a_2 such that $\|h(x_1) - \Psi(x_2)\| \le a_1 \|x_1 - x_2\|$ and $\|g(x_1) - g(x_2)\| \le a_2 \|x_1 - x_2\|$, $\forall x_1, x_2 \in X$.

$(C3)$ there exists $D_r = \{y \in C : \|y\| \le r\}$ with the following condition statisfied,

$$\le G_1(1+T) + G_2(T-1) + \frac{F_0(N_1 + M_1 K(r))}{\Gamma(\gamma+1)T^{2-\gamma}} \le r$$

where $F_0 = 2T^2 + T + \gamma(T+1)$

$(C4)$ there exists $D_r = \{y \in C : \|y\| \le r\}$ with the following condition statisfied,

$a_1(1+T)+a_2(T-1)+(dV^*C_1(1+T))/(\Gamma(\gamma+1)T^{2-\gamma})<1,$
where $C_1 = 2T^2+T+\gamma(T+1)$ and $V^* = \sup\{L(u)(1+p(u)+q(u);\ \ u\in[0,T]\}$

$(C5)$ there exists a continuous function $p:[0,T]\to\mathbb{R}^+ and\ \ p_1:[0,T]\to\mathbb{R}^+ such$ that
$\left\|\int_0^t (k(t,u,y_1)-k(t,u,y_2))du\right\| \le p(t)\,\|\,y_1-y_2\,\|$ and $\left\|\int_0^t k(t,u,y)du\right\| \le p_1(t)\|y\|,$
$\forall t,u\in[0,T]$ and $y_1,y_2,y\in X.$

$(C6)$ there exists a continuous function $q:[0,T]\to\mathbb{R}^+ and\ \ q_1:[0,T]\to\mathbb{R}^+ such$ that
$\left\|\int_0^t (h_1(t,u,y_1)-k(t,u,y_2))du\right\| \le q(t)\,\|\,y_1-y_2\,\|$ and $\left\|\int_0^t h_1(t,u,y)du\right\| \le q_1(t)\|y\|,$
$\forall t,u\in[0,T]$ and $y_1,y_2,y\in X.$

$(C7)$ there exists continuous functions $L:[0,T]\to\mathbb{R}^+ and\ \ N_1$ is positive constant such that $\|\,f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\|\le L(t)K\left(\|x_1-x_2\|+\|y_1-y_2\|+\right.$ $\left.\|z_1-z_2\|\right)$ and $N_1=\sup_{t\in[0,T]}\|\,f(t,0,0,0)\|$ $\forall t\in[0,T]$ and $x_1,y_1,z_1,x_2,y_2,$ $z_2\in X$ and $K:R^+\to(0,\infty)$ is continuous non decreasing function satisfying $K(\alpha(\mathrm{t})y)\le\alpha(t)K(y),$ where $\alpha:[0,T]\to R^+$ is a continuous function, $y\in X$.

Lemma 4.5. Let,

$$1<\gamma\le 2 \text{ and } f: J\ \mathrm{x}\ X\ \mathrm{x}\ X\to X \text{ where } J=[0,T].$$

be a continuous function, then the solution of fractional differential equation (4.1) with the boundary conditions (4.2) is,

$$\begin{aligned}
y(t) &= \frac{(1+t)}{T}\int_0^T h(y(u))du + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(u))du \\
&-\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du \\
&+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du \\
&+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du.
\end{aligned}$$

Proof. Using lemma (4.2), we can solve (4.1) to get the following equation,

$$y(t)=\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du + B_1 + B_2 t. \quad (4.3)$$

Integrating both sides with respect to t , we get

$$y'(t)=\int_0^t \frac{(t-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du + B_2.$$

Now we will apply the boundary conditions given in (4.2) as follows

$$y(0)=B_1 \text{ and } y'(0)=B_2.$$

and,

$$y(T)=\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du + B_1 + B_2 T.$$

$$y'(T)=\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du + B_2.$$

Now consider the following boundary conditions

$$y(0)-y'(0)=\int_0^T g(y(u))du, \qquad y(T)-y'(T)=\int_0^T h(y(u))du.$$

Using these boundary conditions we will get two equations solving which yields the value for B_1 and B_2 given below

$$\begin{aligned}B_2 &= \frac{1}{T}\int_0^T h(y(u))du - \frac{1}{T}\int_0^T g(y(u))du \\ &\quad - \frac{1}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du \\ &\quad + \frac{1}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du.\end{aligned}$$

$$B_2 = \frac{1}{T}\int_0^T h(y(u))du + \left(1 - \frac{1}{T}\right)\int_0^T g(y(u))du$$
$$-\frac{1}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du$$
$$+\frac{1}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du.$$

Putting these values in (4.3), we will get the following result

$$y(t) = \frac{(1+t)}{T}\int_0^T h(y(u))du + \left(1 - \frac{(1+t)}{T}\right)\int_0^T g(y(u))du$$
$$-\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du$$
$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du$$
$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du.$$

which completes the proof.

Theorem 4.6 If the conditions (C_1 to C_5) are satisfied, then the equations (4.1) has a unique solution on $J = [0,T]$.

Proof. Let us consider an operator $F : C \to C$, then for any $y \in C$ we have,

$$F(y(t)) = \frac{(1+t)}{T}\int_0^T h(y(u))du + \left(1 - \frac{(1+t)}{T}\right)\int_0^T g(y(u))du$$
$$-\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du$$
$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du$$
$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du.$$

Now we need to show that F has a fixed point on D_r where $D_r = \{y \in C : \| y \| \le r\}$. This fixed point will be the solution of fractional mixed Volttera-Fredholm integrodifferential equation.

Let us first prove that $F(D_r) \subset D_r$. For this consider any ordinary $y \in D_r$, and then we have,

$$\|F(y(t))\| \le \frac{(1+t)}{T}\int_0^T \|h(y(u))\|du + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(u))\|du$$
$$+\frac{(1+t)}{T}\int_0^\sigma \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$
$$+\frac{(1+t)}{T}\int_0^\sigma \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$
$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du.$$

$$\|F(y(t))\| \le \frac{(1+t)}{T}\int_0^T \|h(y(u))\|du + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(u))\|du$$
$$+\frac{(1+t)}{T}\int_0^\sigma \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$
$$+\frac{(1+t)}{T}\int_0^\sigma \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$
$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du.$$

$$\|F(y(t))\| \le \frac{(1+t)}{T}\int_0^T \|h(y(u))\|du + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(u))\|du$$
$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right.$$
$$\left. - f(u,0,0,0) + f(u,0,0,0)\right\| du$$
$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}\left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right.$$

$$-f(u,0,0,0)+f(u,0,0,0)\Big\| du$$

$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} \left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right.$$

$$-f(u,0,0,0)+f(u,0,0,0)\Big\| du.$$

Let $G_1 = \|h(y(u))\|$ and $G_2 = \|g(y(u))\|$, so the above takes the following form,

$$\|F(y(t))\| \le \frac{(1+t)}{T}\int_0^T \|h(y(u))\| du + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(u))\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} \left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right.$$

$$-f(u,0,0,0)\Big\| du + \frac{(1+t)}{T}\int_0^\sigma \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} \| f(t,0,0,0)\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} \left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right.$$

$$-f(u,0,0,0)\Big\| du + \frac{(1+t)}{T}\int_0^\sigma \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} \| f(t,0,0,0)\| du$$

$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} \left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right.$$

$$-f(u,0,0,0)\Big\| du + \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} \| f(t,0,0,0)\| du.$$

using C_5 , we get,

$$\le \frac{(1+t)}{T} G_1 T + \left(1-\frac{(1+t)}{T}\right) G_2 T + \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} L(u)$$

$$\text{x } K\left(\|y(u)\| + \left\|\int_0^u k(u,\lambda,y(\lambda))d\lambda\right\| + \left\|\int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right\|\right) du$$

$$+\frac{(1+t)N_1}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} + \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)$$

$$\text{x } K\left(\|y(u)\| + \left\|\int_0^u k(u,\lambda,y(\lambda))d\lambda\right\| + \left\|\int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right\|\right) du$$

$$+\frac{(1+t)N_1}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} + \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)$$

$$\leq \frac{(1+t)}{T}G_1T + \left(1-\frac{(1+t)}{T}\right)G_2T + \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}L(u)$$

$$\text{x } K\left(\|y(u)\| + \left\|\int_0^u k(u,\lambda,y(\lambda))d\lambda\right\| + \left\|\int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right\|\right)du$$

$$+\frac{(1+t)N_1}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} + \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}L(u)$$

$$\text{x } K\left(\|y(u)\| + \left\|\int_0^u k(u,\lambda,y(\lambda))d\lambda\right\| + \left\|\int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right\|\right)du$$

$$+\frac{(1+t)N_1}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} + \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}L(u)$$

$$\text{x } K\left(\|y(u)\| + \left\|\int_0^u k(u,\lambda,y(\lambda))d\lambda\right\| + \left\|\int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right\|\right)du$$

$$+N_1\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}du.$$

Now re-arranging the terms and applying C_4, we get,

$$\leq (1+t)G_1 + \left(1-\frac{(1+t)}{T}\right)G_2T + \frac{(1+t)N_1}{T}$$

$$\text{x } \left(\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}du + \int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}du\right) + N_1\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}L(u)K\left(\|y\| + p_1(u)\|y\| + q_1(u)\|y\|\right)du$$

$$\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}L(u)K\left(\|y\| + p_1(u)\|y\| + q_1(u)\|y\|\right)du$$

$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}L(u)K\left(\|y\| + p_1(u)\|y\| + q_1(u)\|y\|\right)du.$$

$$\|F(y(t))\| \leq (1+t)G_1 + \left(1-\frac{(1+t)}{T}\right)G_2T$$

$$+\frac{(1+t)N_1}{T}\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)} + \frac{T^{\gamma}}{\Gamma(\gamma+1)}\right) + N_1\frac{T^{\gamma}}{\Gamma(\gamma+1)}$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}L(u)(1+p_1(u)+q_1(u))K\left(\|y\|\right)du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}L(u)(1+p_1(u)+q_1(u))K\left(\|y\|\right)du$$

$$\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}L(u)(1+p_1(u)+q_1(u))K\left(\|y\|\right)du.$$

Let $V_1 = \sup\{L(u)(1+p_1(u)+q_1(u);\ \ u\in[0,T]\}$ and $\left(1-\left(\frac{1+t}{T}\right)\right)<\left(1-\frac{1}{T}\right)$,

we get,

$$\|F(y(t))\| \le (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2T + \frac{(1+t)N_1}{T}\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)}+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)+N_1\frac{T^{\gamma}}{\Gamma(\gamma+1)}$$
$$+\frac{(1+t)V_1}{T}\int_0^T\frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}K\|y\|du+\frac{(1+t)V_1}{T}\int_0^T\frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}K\|y\|du$$
$$+V_1\int_0^t\frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}K(\|y\|)du.$$

$$\le (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2T + \frac{(1+t)N_1}{T}\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)}+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)+N_1\frac{T^{\gamma}}{\Gamma(\gamma+1)}$$
$$+\frac{(1+t)V_1K(r)}{T}\int_0^T\frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}du+\frac{(1+t)V_1K(r)}{T}\frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}du$$
$$+\mathrm{V}_1K(r)\int_0^t\frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}du.$$

$$\le (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2T + \frac{(1+t)N_1}{T}\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)}+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)+N_1\frac{T^{\gamma}}{\Gamma(\gamma+1)}$$
$$+\frac{(1+t)V_1K(r)}{T}\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)}+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)+\frac{V_1K(r)T^{\gamma}}{\Gamma(\gamma+1)}.$$

$$\le (1+t)G_1+(T-1)G_2T+\frac{(1+t)}{T}\left(N_1+M_1K(r)\right)\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)}+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)$$
$$+\left(N_1+M_1K(r)\right)\frac{T^{\gamma}}{\Gamma(\gamma+1)}.$$

$$\|F(y(t))\| \le (1+t)G_1+(T-1)G_2T+\frac{(1+T)}{T}\left(N_1+M_1K(r)\right)\left(\frac{T^{\gamma-1}}{\Gamma(\gamma)}+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)$$
$$+\left(N_1+M_1K(r)\right)\frac{T^{\gamma}}{\Gamma(\gamma+1)}.$$

$$\le G_1(1+T)+G_2(T-1)+\frac{F_0(N_1+M_1K(r))}{\Gamma(\gamma+1)T^{2-\gamma}}$$

where $F_0 = 2T^2+T+\gamma(T+1)$.

Now let x and $y\in C$ and let $t\in[0,T]$, then

$$\|F(x(t))-F(y(t))\| \le \frac{(1+t)}{T}\int_0^T \|h(x)-h(y)\| du + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(x)-g(y)\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}$$

$$\text{x} \left\| f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)\right.$$

$$\left. -f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$\text{x} \left\| f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)\right.$$

$$\left. -f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$

$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}$$

$$\text{x} \left\| f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)\right.$$

$$\left. -f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du.$$

Now using C_1 to C_5, we will get the following,

$$\|F(x(t))-F(y(t))\| \le +\frac{a_1(1+t)}{T}\int_0^T \|(x-y)\| du + a_2\left(1-\frac{(1+t)}{T}\right)\int_0^T \|(x-y)\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} L(u)$$

$$\text{x } K\left(\|x(u)-y(u)\| + \left\|\int_0^u (k(u,\lambda,x(\lambda))-k(u,\lambda,y(\lambda)))d\lambda\right\|\right.$$

$$\left. +\left\|\int_0^T (h_1(u,\lambda,x(\lambda))-h_1(u,\lambda,y(\lambda)))d\lambda\right\|\right) du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)$$

$$\text{x } K\left(\|x(u)-y(u)\| + \left\| \int_0^u (k(u,\lambda,x(\lambda)) - k(u,\lambda,y(\lambda)))d\lambda \right\| \right.$$

$$\left. + \left\| \int_0^T (h_1(u,\lambda,x(\lambda)) - h_1(u,\lambda,y(\lambda)))d\lambda \right\| \right) du + \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)$$

$$\text{x } K\left(\|x(u)-y(u)\| + \left\| \int_0^u (k(u,\lambda,x(\lambda)) - k(u,\lambda,y(\lambda)))d\lambda \right\| \right.$$

$$\left. + \left\| \int_0^T (h_1(u,\lambda,x(\lambda)) - h_1(u,\lambda,y(\lambda)))d\lambda \right\| \right) du.$$

$$\leq \frac{(1+t)a_1}{T}\int_0^T \|x-y\| du + a_2\left(1-\frac{(1+t)}{T}\right)\int_0^T \| x-y \| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} L(u)K\left(\|(x-y)\| + p(u)\|(x-y)\| + q(u)(x-y)\right) du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)K\left(\|(x-y)\| + p(u)\|(x-y)\| + q(u)(x-y)\right) du$$

$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)K\left(\|(x-y)\| + p(u)\|(x-y)\| + q(u)(x-y)\right) du.$$

$$\leq \frac{(1+t)a_1}{T}\int_0^T \|(x-y)\| du + a_2\left(1-\frac{1}{T}\right)\int_0^T \|(x-y)\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} L(u)(1+p(u)+q(u))K\left(\|x-y\|\right) du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)(1+p(u)+q(u))K\left(\|x-y\|\right) du$$

$$+\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)(1+p(u)+q(u))K\left(\|x-y\|\right) du.$$

Now let Let $V^* = \sup\{L(u)(1+p(u)+q(u); \ u \in [0,T]\}$ and let $K\left(\|x-y\|\right) \leq d\|x-y\|$ where $d > 0,$ then we have,

$$\|F(x(t))-F(y(t))\| \le \frac{(1+t)a_1}{T}\int_0^T \|x-y\|\,du + a_2\left(1-\frac{1}{T}\right)\int_0^T \|x-y\|\,du$$
$$+\frac{dV^*(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}\|x-y\|\,du$$
$$+\frac{dV^*(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}\left(\|x-y\|\right)du$$
$$+dV^*\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}\left(\|x-y\|\right)du.$$

Solving the integrals we get,

$$\|F(x(t))-F(y(t))\| \le \left[a_1(1+T)+a_2(T-1)+\frac{dV^*C_1(1+T)}{\Gamma(\gamma+1)T^{2-\gamma}}\right]\|x-y\|,$$

where $M_1 = 2T^2 + T + \gamma(1+T)$.

Now as $a_1(1+T)+a_2(T-1)+(dV^*C_1(1+T))/(\Gamma(\gamma+1)T^{2-\gamma})<1$, so F is contraction mapping which proves that the given integrodifferential equation has a unique solution on $[0,T]$.

Theorom 4.7 Let the conditions $C_1 - C_5$ hold for the following,

$$\left\| f\left(t, y(t), \int_0^t k(t,\lambda,y(\lambda))d\lambda, \int_0^T h_1(t,\lambda,y(\lambda))d\lambda\right)\right\| \le \varphi(t), \quad \text{where } \varphi(t)\in L_1(J).$$

Then the boundary value problem (4.1) – (4.2) has at least one element in $[0,T]$

Proof Let us consider $D_r = \{y \in C : \|y\| \le r\}$, so we introduce operators E_1 and E_2 as,

$$A(x(t)) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1} f\left(t, x(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du.$$

$$B(x(t)) = \frac{(1+t)}{T}\int_0^T x(y(u))du + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(x(u))du + \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$\text{x}\, f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, x(u), \int_0^u k(\theta,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)du.$$

Now let us prove that if $x, y \in D_r$, then $Ax + By \in D_r$,

$$\|Ax+By\| = \left\|\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, x(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du\right.$$

$$+\frac{(1+t)}{T}\int_0^T h(y(u))du + \left(1-\frac{(1+t)}{T}\right)\int_0^T g(y(u))du + \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$\text{x}\, f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du + \frac{(1+t)}{T}$$

$$\left.\text{x} \int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, y(u), \int_0^u k(\theta,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)du\right\|.$$

$$\leq \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} \left\| f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)\right\| du$$

$$+\frac{(1+t)}{T}\int_0^T \|h(y(u))\|du + \left(1-\frac{(1+t)}{T}\right)\int_0^T \|g(y(u))\|du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$\text{x} \left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}$$

$$\text{x} \left\| f\left(u, y(u), \int_0^u k(u,\lambda,y(\lambda))d\lambda, \int_0^T h_1(u,\lambda,y(\lambda))d\lambda\right)\right\| du$$

$$\leq \|\varphi\|_{L_1} \int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} du + (1+t)G_1$$

$$+\left(1-\frac{(1+t)}{T}\right)G_2 T + \frac{(1+t)\|\varphi\|_{L_1}}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} du$$

$$+\frac{(1+t)\|\varphi\|_{L_1}}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} du.$$

$$\leq \frac{\|\varphi\|_{L_1} T^{\gamma}}{\Gamma(\gamma+1)} + (1+t)G_1 + \left(1-\frac{1}{T}\right)G_2 T + \frac{(1+t)T^{\gamma-1}\|\varphi\|_{L_1}}{T\Gamma(\gamma)} + \frac{(1+t)T^{\gamma}\|\varphi\|_{L_1}}{T\Gamma(\gamma+1)}.$$

$$\leq \frac{\|\varphi\|_{L_1} T^{\gamma}}{\Gamma(\gamma+1)} + \frac{(1+t)\|\varphi\|_{L_1} T^{\gamma-1}}{T\Gamma(\gamma)} + \frac{(1+t)\|\varphi\|_{L_1} T^{\gamma}}{T\Gamma(\gamma+1)} + (1+T)G_1 + (T-1)G_2.$$

$$\leq (1+T)G_1 + (T-1)G_2 + \frac{F_2 T^{\gamma-2}}{\Gamma(\gamma+1)}\|\varphi\|_{L_1}.$$

where $F_2 = 2T^2 + T(\gamma+1) + T.$

Now we need to prove that $B(x)$ is a contraction mapping.

$$\|Bx_1 - Bx_2\| \leq \frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} \left\| f\left(u, x_1(u), \int_0^u k(u,\lambda,x_1(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x_1(\lambda))d\lambda\right)\right.$$

$$\left. - f\left(u, x_2(u), \int_0^u k(u,\lambda,x_2(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x_2(\lambda))d\lambda\right)\right\| du$$

$$+\frac{(1+t)}{T}\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} \left\| f\left(u, x_1(u), \int_0^u k(u,\lambda,x_1(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x_1(\lambda))d\lambda\right)\right.$$

$$\left. - f\left(u, x_2(u), \int_0^u k(u,\lambda,x_2(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x_2(\lambda))d\lambda\right)\right\| du$$

$$\leq \frac{(1+t)}{T}\left[\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)} L(u)(1+p(\theta)+q(\theta))K\left(\|x_1-x_2\|\right)du\right.$$

$$\left. +\int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)} L(u)(1+p(\theta)+q(\theta))K\left(\|x_1-x_2\|\right)du\right].$$

Let $K\left(\|x_1-x_2\|\right) \leq v\|x_1-x_2\|,$ we have,

$$\|Bx_1 - Bx_2\| \le \frac{(1+t)vV^*}{T}\|x_1 - x_2\|\left[\int_0^T \frac{(T-u)^{\gamma-2}}{\Gamma(\gamma-1)}du + \int_0^T \frac{(T-u)^{\gamma-1}}{\Gamma(\gamma)}du\right].$$

$$\le \frac{(1+t)vV^*}{T}\|x_1 - x_2\|\left[\frac{T^{\gamma-1}}{\Gamma(\gamma)} + \frac{T^{\gamma}}{\Gamma(\gamma+1)}\right].$$

$$\le \frac{vV^*(1+T)(\gamma+T)}{\Gamma(\gamma+1)T^{2-\gamma}}\|x_1 - x_2\|.$$

So B is a contraction mapping since $x(t)$ is continuous, hence $A(x)$ is continuous.

Now consider the following,

$$\|A(x(t))\| = \left\|\frac{1}{\Gamma(\gamma)}\int_0^t (t-u)^{\gamma-1} f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)du\right\|.$$

$$\le \|\varphi\|_{L_1}\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}du.$$

$$\le \frac{T^{\gamma}\|\varphi\|_{L_1}}{\Gamma(\gamma+1)}$$

which clearly shows that A is uniformly bounded on D_r Now we will prove that $Ax(t)$ is equicontinuous. For this purpose we take any two elements from $[0,T]$,say t_1, t_2 and $x \in D_r$. Then as f is bounded on compact set $J \times D_r$ thus,

$\sup_{(t,u)\in W\times D_r}\left\|f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)\right\| = \alpha_0 < \infty$, so we have

$$\|Ax(t_1) - Ax(t_2)\| = \left\|\int_0^{t_1}\frac{(t_1-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)du\right.$$
$$\left. - \int_0^{t_2}\frac{(t_2-u)^{\gamma-1}}{\Gamma(\gamma)} f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda\right)du\right\|.$$

$$\leq \frac{1}{\Gamma(\gamma)} \left\| \int_0^{t_1} \left[(t_1 - u)^{\gamma-1} - (t_2 - u)^{\gamma-1} \right] \right.$$

$$\left. \text{x } f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda \right) du \right\|$$

$$+ \left\| \int_{t_1}^{t_2} (t_2 - u)^{\gamma-1} f\left(u, x(u), \int_0^u k(u,\lambda,x(\lambda))d\lambda, \int_0^T h_1(u,\lambda,x(\lambda))d\lambda \right) du \right\|.$$

$$\leq \frac{\alpha_0}{\Gamma(\gamma+1)} \left[2(t_2 - t_1)^{\gamma} + (t_1^{\gamma} - t_2^{\gamma}) \right].$$

So A is relatively compact. By Arzela-Ascoli theorem, it is compact that concludes the result of Krasnosel'skii theorem.

Example: 4.8

$$y^{(1.5)}(t) = \frac{1}{10} + \frac{1}{10 + |y(s)|} + \int_0^t \frac{|y(s)|}{10e^{|y(s)|} + t} ds + \int_0^1 \frac{|y(s)| e^{-t}}{10 + |y(s)|^2} dt.$$

And the integral boundary consitons are

$$y(0) - y^{/}(0) = \int_0^1 \frac{1}{10 + |y(s)|} ds, \quad y(1) - y^{/}(1) = \int_0^1 \frac{1}{10 + e^{-|y(s)|}} ds.$$

So we have

$$\|g(y(t)\| = \left\| \frac{1}{10 + |y(t)|} \right\| \leq \frac{1}{10}$$

$$\|g(x) - g(y)\| =\leq \frac{1}{100} \|x - y\|.$$

$$\|h(y(t)\| = \left\| \frac{1}{10 + e^{-|y(t)|}} \right\| \leq \frac{1}{10}.$$

$$\|h(x) - h(y)\| =\leq \frac{1}{100} \|x - y\|.$$

$$\left\| \int_0^t (k(t,s,x) - k(t,s,y))ds \right\| \leq \frac{1}{10e^t} \|x - y\|, \quad \left\| \int_0^t (k(t,s,y)ds \right\| \leq \frac{1}{10 + t} \|y(t)\|.$$

$$\left\|\int_0^t (h_1(t,s,x)-h_1(t,s,y))ds\right\| \le \frac{1}{10e^t}\|x-y\|, \quad \left\|\int_0^t (h_1(t,s,y)ds\right\| \le \frac{1}{10+t}\|y(t)\|.$$

$$\| f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le \frac{1}{10+t}\left(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|\right).$$

and $f(t,0,0,0)=1/10$. So all the conditions are stratified with $G_1 = G_2 = 0.1$, $a_1 = a_2 = 0.01$, $M_1^* = 0.12$, $\omega = 0.1$, $C_0 = 6$, $N_1 = 0.1$, $M^* = 0.12$ and $C_1 = 6$,therefore we have,

$$a_1(1+T) + a_2(T-1) + \frac{\omega M^* C_1(1+T)}{\Gamma(\gamma+1)T^{2-\gamma}} < 1 \Leftrightarrow 0.01(2) + \frac{(0.1))0.12)6(2)}{\Gamma(2.5)} < 1$$

which proves that integrodifferential equation has a unique solution.

CONCLUSION

It concludes that under the certain condition, there exists a unique solution for integrodifferential equation with integral boundary values with $1 < \gamma \leq 2$ in Banach space. Integrodifferential equations play an important role in developing various applications such as cellular systems.

To reach this conclusion we have used the concept of Banach fixed point theorem and Krasnosel'skii fixed point theorem. First we proved the existence and then uniqueness of solution of given integrodifferential equation under the given integral boundary conditions.

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