Euler Integrals

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ABSTRACT

Euler's integral are two functions, called Beta and Gamma functions. They play important role in mathematics and its applications. These functions are defined through improper integrals and their properties depend on properties of improper integrals depending on parameter. In this thesis, proper and improper integrals are reviewed, Beta and Gamma functions are defined and their properties are presented.

Keywords: Euler integrals, Riemann integral, improper integral, Gamma function, Beta function.

ÖZ

Euler integralları Beta ve Gamma fonksyonlarıdır. Bunlar matematik ve onun uygulamalarında önemli rol alırlar. Bu fonksiyonlar belirsiz integrallar olarak tanımlanırlar ve özelliklerini parametreye bağlı belirsız integralların özelliklernden alırlar. Bu tezde belirli ve belirsiz integrallar incelenmidir, Beta ve Gamma fonksiyonları tanımlanmış ve özellikleri verilmelidir.

Anahtar Kelimeler: Euler integrallar, Riemann integralı, belirsiz integrallar, Gamma fonksiyonu, Beta fonksiyonu.

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INTRODUCTION

In year 1729 the Swiss mathematician Leon hard Euler (1707-1783) defined the Gamma function. This definition appears in his correspondence. He was 22 years old when he defined this function. Discovery of gamma function was in the intersection of two great problems of the 17th century. The first one is interpolation and the second one establishing integral calculus, mainly setting up formula of indefinite integration. To monitor interpolation problem consider the sums

$$T_1 = 1$$

$$T_2 = 1+2$$

$$T_3 = 1+2+3$$

$$T_4 = 1+2+3+4$$

$$T_5 = 1+2+3+4+5$$

It is know that the nth sum is calculated by formula

$$T_n = \frac{n(n+1)}{2}$$

This formula is spectacular, because it interpolates non-integer number (say, $n = \frac{5}{2}$) of numbers from an integer number ($n = 1, 2, 3, \cdots$) of them. For example,

$$T_{\frac{5}{2}} = \frac{\frac{5}{2}(\frac{5}{2}+1)}{2} = \frac{35}{8}\cdot$$

This type of questions are frequent in the studies of the 17th and 18th centuries. A familiar power function is defined for integer values of argument by

$$f(n) = a^n = a \cdot a \cdots a.$$

Newton extended $f(x)$ to any real x by using

$$a^0 = 1,\ a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad and \quad a^{-n} = \frac{1}{a^n}\cdot$$

This explains a basic idea of interpolation problem: definition of quantities which may have no real meaning by reasonably interpolating them by those which have a real meaning.

In this regard the Gamma function is spectaculars. Its properties $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$ allows to get $\Gamma(n) = (n-1)!$ if n is integer. Thus, the gamma function interpolates non integer factorials from integer factorials. Later, gamma function laid down on interpolation of non integer order differentiation by integer order differentiation.

Euler derived gamma function in the form

$$\Gamma(\alpha) = \int_0^1 (-\log x)^{(\alpha-1)} dx\ ,\ \alpha > 0.$$

Later this definition was modified by Adrien Marie Legendre (1752-1833) to the familiar form

$$\Gamma(\alpha) = \int_0^\infty (e)^{-t} t^{\alpha-1} dt.$$

Moreover, Legendre called gamma function as the second Euler's integral, regarding the first Euler's integral to be the beta function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

which is related to Gamma function as

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

In this thesis we review the Euler's integrals.

The Euler's integrals are improper Riemann integrals. Therefore, before starting the Euler's integrals, we review proper and improper Riemann integrals in this thesis.

Chapter 2

RIEMANN INTEGRAL

2.1 Proper Riemann Integral

2.1.1 Definition

Definition 2.1. Let f be a function defined and bounded on the closed interval $[a,b]$, let $P : x_0, x_1, ..., x_n$ be a partition of the interval $[a,b]$, such that $a = x_0 < x_1 < \cdots < x_n = b$ and take a point x_i^* in each sub interval $[x_i, x_{i-1}]$. Form the following sum

$$S(p,f) = f(x_0^*)(x_1 - x_0) + f(x_1^*)(x_2 - x_1) + ... + f(x_{n-1}^*)(x_n - x_{i-1}),$$

or

$$S(p,f) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

Such a sum is called a Riemann sum for the function f over the interval $[a,b]$.

Geometrically, it gives an approximation of the area under the curve $y = f(x)$ between $x = a$ and $x = b$. The Riemann integral of f over the interval $[a,b]$ is the limit:

$$\lim_{\triangle_P \to 0} S(p,f) = \int_a^b f(x)dt,$$

where $\triangle_p = max\{x_1 - x_0, ..., x_n - x_{n-1}\}$. More precisely, Riemann integral is defined as follows [3].

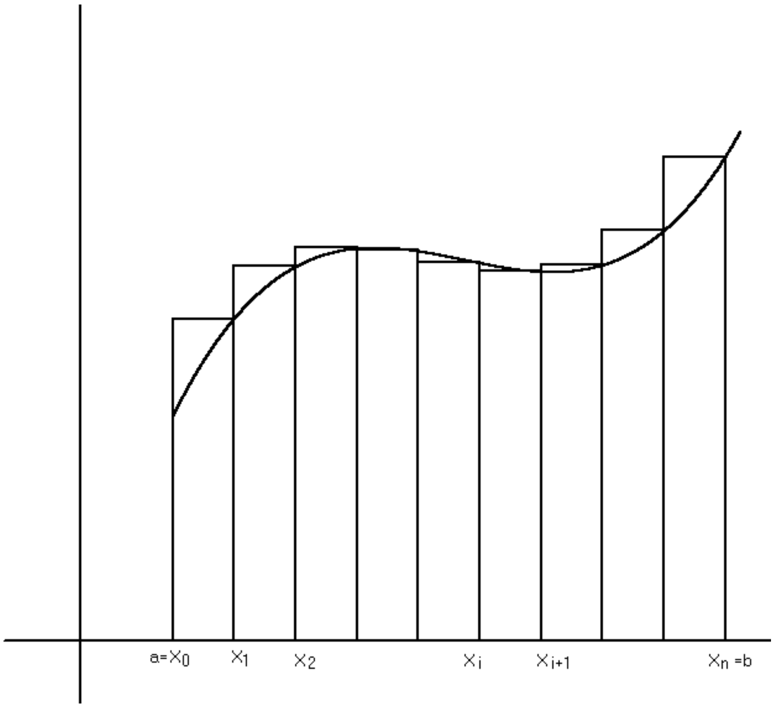

Figure 2.1: Upper integral sum.

Definition 2.2. The function $f(x)$ is said to be Riemann integrable over $[a,b]$ if a number m exists such that for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|m - S(p,f)| < \varepsilon$ for any partition p of the interval $[a,b]$, with a norm $\Delta_p < \delta$ where Δ_p = max $\{\Delta x_i, i = 1,2,...,n\}$ [3]. The number m is called the Riemann integral of $f(x)$ over integral $[a,b]$ and denoted by $m = \int_a^b f(x)dx$.

Note 2.3. The integration symbol $\int$ was first used by Gottfried Wilhelm Leibniz (1646-1716) to represent a sum.

Example 2.4. Find the Riemann integral of the function $f(x) = x^3 + 2x$ over the interval $[1,4]$.

Solution: First of all we divide the interval $[1,4]$ into n sub interval of length $\Delta = \frac{4-1}{n} = \frac{3}{n}$ and construct the partition p as follows:

$$p : x_0 = 1 < x_1 = 1 + \frac{3}{n} < x_2 = 1 + \frac{3}{n} + \frac{3}{n} < \cdots < x_n = 1 + \frac{3}{n} + \cdots + \frac{3}{n} = 4.$$

Then

$$\Delta x_i = x_i - x_{i-1} = 1 + \frac{3i}{n} - (1 + \frac{3(i-1)}{n}) = \frac{3}{n}.$$

Take

$$x_i^* = 1 + \underbrace{\frac{3}{n} + \cdots + \frac{3}{n}}_{i-times} = 1 + \frac{3i}{n}\cdot$$

Now

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n}\left[(1+\frac{3i}{n})^3 + 2(1+\frac{3i}{n})\right]\frac{3}{n}$$

$$= \frac{3}{n}\left[\sum_{i=1}^{n}(1 + \frac{9i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3} + 2 + \frac{6i}{n})\right]$$

$$= \frac{3}{n}\left[\sum_{i=1}^{n}1 + \frac{9}{n}\sum_{i=1}^{n}i + \frac{27}{n^2}\sum_{i=1}^{n}i^2 + \frac{27}{n^3}\sum_{i=1}^{n}i^3 + \sum_{i=1}^{n}2 + \frac{6}{n}\sum_{i=1}^{n}i\right]$$

$$= \frac{3}{n}\left[n + \frac{9}{n}\frac{n(n+1)}{2} + \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{27}{n^3}\frac{n^2(n+1)^2}{4} + 2n + \frac{6}{n}\frac{n(n+1)}{2}\right].$$

Simplifying, we get

$$= 3\underbrace{\left(3 + \frac{9}{2} + \frac{9}{2n} + \frac{27}{3} + \frac{27}{2n} + \frac{27}{6n} + \frac{27}{4} + \frac{27}{2n} + \frac{27}{4n} + 3 + \frac{3}{n}\right)}_{A(n)}$$

Hence, by taking limit as $x \to \infty$, we get

$$\int_1^4 (x^3 + 2x)dx = \lim_{x\to\infty} 3\Big(A(n)\Big) = \frac{315}{4}.$$

2.1.2 Existence of Riemann Integral

The following theorem establishes a necessary and sufficient condition for existence of Riemann integral.

Theorem 2.5. Let $f(x)$ be a bounded function on a finite interval $[a,b]$ and let $p = \{x_0, x_1, ..., x_n\}$ be any partition of $[a,b]$, let m_i and M_i be the infimum and supremum of $f(x)$ on the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, ..., n$, respectively. The function f is Riemann integrable on $[a,b]$ if and only if for a given $\varepsilon > 0$ there exist $\delta > 0$ such that

$$U(p, f) - L(p, f) < \varepsilon$$

whenever $\Delta_p < \delta$, where $\Delta_p = max\{\Delta x_i, i = 1, 2, ..., n\}$ is the norm of p,

$$U(p, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

and

$$L(p, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

To prove this theorem we will give several definitions and lemmas [4].

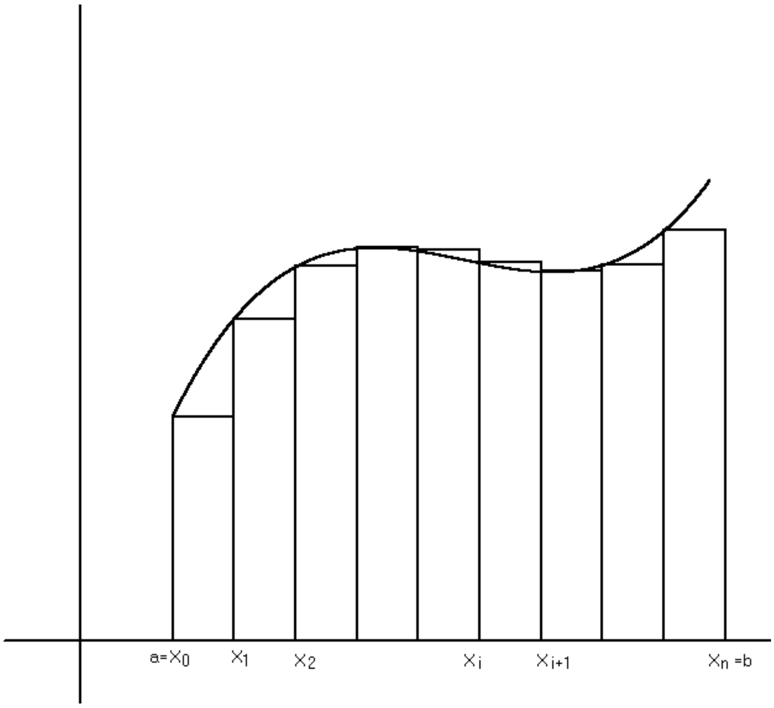

Figure 2.2: Lower integral sum.

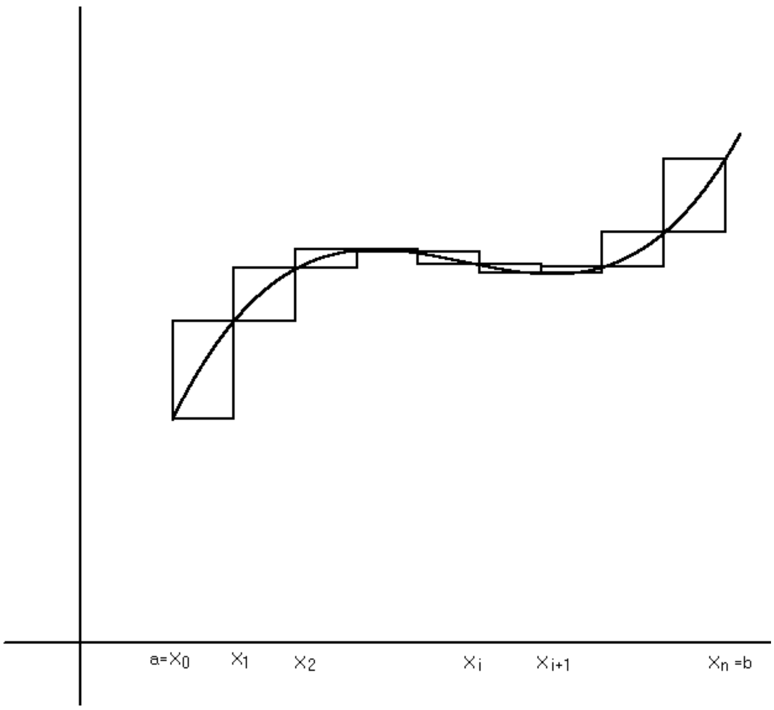

Figure 2.3: Difference of Upper and lower integral sums.

Definition 2.6. We say that a partition p_2 is a refinement of a partition p_1 or p_2 is finer than p_1 if $p_1 \subset p_2$ that is if every point of p_1 is used in p_2 [4].

Lemma 2.7 Let p and $p^{'}$ be two partitions of interval $[a,b]$ such that $p \subset p^{'}$ [4]. Then

$$U(p^{'},f) \leq U(p,f)$$

and

$$L(p,f) \leq L(p^{'},f).$$

Proof. Let $p = \{x_0, x_1, ..., x_n\}$. Without loss of generality, assume that $p^{'}$ differs from p by division of the i^{th} subinterval $[x_{i-1} - x_i]$ into T_i parts with the respective lengths $\Delta^1_{x_i}, \Delta^2_{x_i}, \cdots, \Delta^{T_i}_{x_i}$, where $T_i \geq 1$, $i = 1,2,...,n$. Now if $m_i^{(j)}$ and $M_i^{(j)}$ are the infimum and supremum of $f(x)$ over $\Delta_i^{(j)}$ respectively, then it's clear that $m_i \leq m_i^{(j)} \leq M_i^{(j)} \leq M_i$ for $j = 1,2,...,T_i$ and $i = 1,2,...,n$, where m_i and M_i are the infimum and supremum of $f(x)$ over $[x_{i-1}, x_i]$, respectively. This implies that

$$L(p,f) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} \sum_{j=1}^{T_i} m_i^{(j)} \Delta x_i^{(j)} = L(p^{'},f)$$

and

$$U(p^{'},f) = \sum_{i=1}^{n} \sum_{j=1}^{T_i} M_i^{(j)} \Delta x_i^{(j)} \leq \sum_{i=1}^{n} M_i \Delta x_i = U(p,f).$$

This proves the lemma.

Lemma 2.8. If p and $p^{'}$ be any two partitions of the interval $[a,b]$, then $L(p,f) \leq U(p^{'},f)$. Proof. Let $p^* = p \cup p^{'}$. Then the partition p^* is a refinement of both p and

$p^{'}$. By Lemma (2.7.)

$$L(p,f) \leq L(p^*,f) \leq U(p^*,f) \leq U(p,f).$$

This proves the lemma [4].

Now we are ready to prove Theorem 2.5.

Proof of Theorem 2.5. Let $\varepsilon > 0$ be given and suppose that for every $\varepsilon > 0$, there is $\varepsilon > 0$ such that

$$U(p,f) - L(p,f) < \varepsilon$$

holds for each partition p of the interval $[a,b]$ with $\Delta_p < \varepsilon$ and let

$$S(p,f) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

where x_i^* is any point in the interval $[x_{i-1}, x_i]$, $i = 1, ..., n$. By the definition of $L(p,f)$ and $U(p,f)$ we can write

$$L(p,f) \leq S(p,f) \leq U(p,f).$$

Let m and M be the infimum and supremum of $f(x)$ over interval $[a,b]$, respectively. Then

$$m(b-a) \leq L(p,f) \leq S(p,f) \leq U(p,f) \leq M(b-a).$$

By the least upper bound property of the system of real numbers

$$\sup_p L(p,f) \ \ and \ \ \inf_p U(p,f)$$

exist and satisfy

$$\sup_p L(p,f) \leq \inf_p U(p,f).$$

Now suppose that for the given $\varepsilon > 0$ there exist $\delta > 0$ such that $\Delta_p < \delta$ implies

$$U(p,f) - L(p,f) < \varepsilon.$$

For any partition of $[a,b]$ whose norm $\Delta_p < \delta$, we have

$$L(p,f) \leq \sup_p (p,f) \leq \inf_p U(p,f) \leq U(p,f).$$

Hence

$$\inf_p U(p,f) - \sup_p L(p,f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we conclude that

$$\inf_p U(p,f) = \sup_p L(p,f).$$

Denote is the common value of

$$\inf_p U(p,f) \qquad and \qquad \sup_p L(p,f)$$

by A. Then

$$|A - S(p, f)| < \varepsilon.$$

Thus, A is Riemann integral of $f(x)$ on the interval $[a,b]$.

Now we want to prove the converse of the theorem, that is, if $f(x)$ is Riemann integrable on $[a,b]$ then $U(p,f) - L(p,f) < \varepsilon$ is satisfied [4].

Let $f(x)$ be Riemann integrable. Then for each $\varepsilon > 0$, there exist $\delta > 0$ such that

$$\left| A - \sum_{i=1}^{n} f(x_i^{'})\Delta x_i \right| < \frac{\varepsilon}{3} \tag{2.1}$$

and

$$\left| A - \sum_{i=1}^{n} f(x_i^{''})\Delta x_i \right| < \frac{\varepsilon}{3} \tag{2.2}$$

for any partition p of $[a,b]$ with $\Delta_p < \delta$ and any choice of $x_i^{'}, x_i^{''} \in [x_{i-1}, x_i], i = 1, ..., n$ where $A = \int_a^b f(x)dx$. From Eq (2.1) and Eq (2.2) we obtain that

$$\left| \sum_{i=1}^{n} [f(x_i^{'}) - f(x_i^{''})]\Delta x_i \right| < \frac{2\varepsilon}{3}.$$

Now $M_i - m_i$ is the supremum of $f(x) - f(x^*)$ for x and x^* in $[x_{i-1}, x_i], i = 1, \ldots, n$. This means that for a given $\lambda > 0$ we can choose $x_i^{'}$ and $x_i^{''}$ in $[x_{i-1}, x_i]$, so that $f(x_i^{'}) - f(x_i^{''}) > M_i - m_i - \lambda$, $i = 1, \ldots, n$, otherwise $M_i - m_i - \lambda$ would be an upper bound for $f(x) - f(x^*)$ for all x and x^* in $[x_{i-1}, x_i]$ which is a contraduction. In particular, if

$\lambda = \frac{\varepsilon}{3(b-a)}$ then we can find $x_i^{'}$ and $x_i^{''} \in [x_{i-1}, x_i]$ such that

$$U(p,f) - L(p,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \sum_{i=1}^{n} ([f(x_i^{'}) - f(x_i^{''})](\Delta x_i + \lambda)$$

$$< \sum_{i=1}^{n} |f(x_i^{'}) - f(x_i^{''})\Delta x_i| + \lambda(b-a) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Theorem 2.9. If f is continous on $[a,b]$ then f is Riemann integrable [5]. Proof Since f is continous on $[a,b]$ this implies that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x^*) - f(x^{**})| < \frac{\varepsilon}{b-a} \tag{2.3}$$

whenever $|x^* - x^{**}| < \delta$.

Now, let $p = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of $[a,b]$ with $||p|| < \delta$. Since f is continous on $[x_{i-1}, x_i]$. Let m_i and M_i be the infimum and supremum of f respectively for each sub interval $[x_{i-1}, x_i]$ where $m_i = f(c_i)$ and $M_i = f(d_i)$ for some $d_i, c_i \in [x_{i-1}, x_i]$. Since $|c_r - d_r| < \delta$ it follows from Eq (2.3) that

$$M_i - m_i = |f(c_i) - f(d_i)| < \frac{\varepsilon}{b-a} \quad i = 1, 2, ..., n.$$

Hence

$$U(p,f) - L(p,f) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \Delta x_i = \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i$$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Now by Theorem 2.5 we get that f is Riemann integrable.

Theorem 2.10. If f is monotonic on $[a,b]$ then f is Riemann integrable [5].

Proof. Let f be monotonically non-decreasing on $[a,b]$ let $\varepsilon > 0$ be given and let $p =$ $a = x_0, x_1, ..., x_n = b$ be any partition of $[a,b]$ with

$$||p|| \leq \frac{\varepsilon}{f(b)-f(a)}.$$

Since f is nondecreasing, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$U(p,f) - L(p,f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i$$

$$= \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]\Delta x_i$$

$$\leq \frac{\varepsilon}{f(b)-f(a)} = \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$$

$$= \frac{\varepsilon}{f(b)-f(a)} = [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_0)]$$

$$= \frac{\varepsilon}{f(b)-f(a)} = [f(x_n) - f(x_0)]$$

$$= \frac{\varepsilon}{f(b)-f(a)} = [f(b) - f(a)] = \varepsilon.$$

Now by Theorem 2.5 f is Riemann integrable on $[a,b]$. Before progressing to further proporties of Riemann integrals, we have seen that boundedness is necessary but not sufficient for Reimann integrablity and that continuity is sufficient but not necessary.

With a view if denomestration a condition which is both necessary and sufficient we introduce at this point the concept of the zero set or a set of measure zero [4].

Definition 2.11. A subset A or $\mathbb{R}$ is said to be of mesure zero or (zero set) if for every $\varepsilon > 0$ there exist a finite or countable number of open intervals $I_1, I_2, ...$ such that

$$A \subset \bigcup_{i=1}^{\infty} I_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} |I_n| < \varepsilon.$$

Thus A is a zero set iff for every $\varepsilon > 0$, there exist a squence $\{I_n\}$ of open intervals which covers A and satisfy $\sum |I_n| < \varepsilon$ [4].

Theorem 2.12 (Necessary Condition). If f is Riemann integrable on $[a,b]$, then the set of its discontinuity points is a zero set [4].

Proof. Let f be Riemann integrable on $[a,b]$ and let D be the set of discontinuity points of f and, corresponding to each positive integer i, let D_i be the set of points of $[a,b]$ at each of which the fluctuation of f exceeds $\frac{1}{i}$. Then

$$D = \bigcup_{i=1}^{\infty} D_i.$$

Let us assume that D is not a zero set, then for some integer k the set $D_{\frac{1}{k}}$ is not a zero set, so there exist $\varepsilon > 0$ such every countable open covering $\{I_n\}$ of D_k satisfies $\sum_{n=1}^{\infty} |I_n| \geq \varepsilon$. Let $\{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ and suppose $[x_{t_1-1}, x_{t_1}], [x_{t_2-1}, x_{t_2}], ..., [x_{t_q-1}, x_{t_q}]$

$(q < n)$ are those segments of p that contain points of D_k with

$$\sum_{j=1}^{q} \Delta x_{tj} \geq \varepsilon.$$

In each of these segments the oscillation of f exceeds $\frac{1}{k}\cdot$ Therefore,

$$M_{tj} - m_{tj} > \frac{1}{k}, \qquad j = 1, 2, ..., q,$$

where

$$M_{tj} = \sup\{f(x) : x_{t_j - 1} \leq x \leq x_{t_j}\}$$

and

$$m_{tj} = \inf\{f(x) : x_{t_j - 1} \leq x \leq x_{t_j}\}.$$

This impies that

$$U(p, f) - L(p, f) > \frac{\varepsilon}{k}\cdot$$

Since $\frac{\varepsilon}{k}$ is independent of p , we get from Theorem 2.9 f is Riemann not integrable. This contraduction implies that D is a zero set.

Definition 2.13 [Oscillatory Sum]. Recall that

$$L(p, f) = \sum_{i=1}^{n} m_i \Delta x_i \quad , \quad U(p, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Let $\omega_i = M_i - m_i$. ω_i is called the oscillation of f on $[x_{i-1}, x_i]$ and denoted by $\omega(p, [x_{i-1}, x_i])$:

$$U(p,f) - L(p,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \omega_i \Delta x_i.$$

The sum $\sum_{i=1}^{n} \omega_i \Delta x_i$ is called oscillatary sum for the function f corresponding to the partition p and denoted by $\omega(p, f)$ [5].

Theorem 2.14(Sufficient Condition). If f is a bouded function having a zero set of discontinuoty points on $[a,b]$ then f is Riemann integrable on $[a,b]$ [4]. Proof. Let $\{c_1, c_2, ..., c_n\}$ be the finite (ordered) set of points of discontinouity of f in $[a,b]$. Let $\varepsilon > 0$ be given we enclose the points $c_1, c_2, ..., c_p$ respectively in p non-overlapping intervals

$$[a_1, b_1], [a_2, b_2], ..., [a_p, b_p], \tag{2.4}$$

such that

$$\omega(p,q) = \sum_{j=1}^{p} |[a_j, b_j]| < \frac{\varepsilon}{2(M-m)},$$

where as usual

$$M = \sup\{f(x) : a \le x < b\} \quad , \quad m = \inf\{f(x) : a \le x < b\}.$$

Now if f is continous on each of the subintervals

$$[a, a_1], [b_1, a_2], \cdots, [b_p, b], \tag{2.5}$$

there are partitions $p_r : r = 1,2,...,p+1$ respectively of the sub intervals in Eq (2.5) such that

$$\omega(p_r, f) < \frac{\varepsilon}{2(p+1)}\,, \qquad r = 1,2,...,p+1.$$

Now consider the partition $p = \cup\{p_r : r = 1,2,...,p+1\}$. The subintervals of p can be divided in to two groups:

(a) All the subintervals given in Eq (2.4)

$p_r : r = 1,2,\cdots,p+1;$

(b) All the subintervals of Eq (2.5).

The total construction to the oscillatory sum $\omega(p, f)$ of the subintervals in (a) is

$$\omega(p, f) < \frac{\varepsilon}{2(p+1)}(p+1) = \frac{\varepsilon}{2},$$

and the total contribution to $\omega(p, f)$ of the subintervals in (b) is

$$\omega(p, f) < \frac{\varepsilon}{2(M-m)}(M-m) = \frac{\varepsilon}{2}.$$

Thus there exist a partition p such that

$$\omega(p, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Now by Theorem 2.5 we get that f is Riemann integrable.

2.1.3 Properties of Riemann Integral

Theorem 2.15. If f_1 and f_2 are two Riemann integrable functions on $[a,b]$ and k_1 and k_2 are two real numbers [4], then $k_1 f_1 + k_2 f_2$ is also Riemann integrable on $[a,b]$ and

$$\int_a^b (k_1 f_1(x) + k_2 f_2(x))dx = k_1 \int_a^b f_1(x)dx + k_2 \int_a^b f_2(x)dx$$

Proof. This follows from the equality

$$S(P, k_1 f_1 + k_2 f_2) = k_1 S(P, f_1) + k_2 S(P, f_2).$$

Theorem 2.16. Let f be Riemann integrable on $[a,b]$ and let $c \in (a,b)$ then f is Riemann integrable on $[a,c]$ and on $[c,b]$ [4].

Theorem 2.17. Let f be Riemann integrable on $[a,c]$ and on $[c,d]$ then f is Riemann integrable on $[a,b]$ [4].

Theorem 2.18. If f_1 and f_2 are Riemann integrable on $[a,b]$ and $f_1 \leq f_2$ then [4]

$$\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx.$$

Proof. This follows from the ineqality

$$S(p, f_1) = \sum_{i=1}^{n} f_1(x_i^*)\Delta x_i \leq \sum_{i=1}^{n} f_2(x_i^*) = S(p, f_2)$$

Theorem 2.19. Let f be Riemann integrable on $[a,b]$ then $|f|$ is also Riemann integrable and on $[a,b]$ [4]

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx.$$

Proof. Let $\varepsilon > 0$ be given and p be a partition of $[a,b]$ then $U(p,f) - L(p,f) < \varepsilon$ (by Theorem 2.5 let M_i and m_i be supremum and infimum of $f(x)$ on $[x_{i-1}, x_i]$ with respect to partition p . And let $M_i^{'}$ and $m_i^{'}$ be supremum and infimum of $|f|$ with respect to partition p, respectively. It is clear that $M_i - m_i \geq M_i^{'} - m_i^{'}$ for $i = 1, ..., n$. So, as a consequence

$$U(p,|f|) - L(p,|f|) \leq U(p,f) - L(p,f) < \varepsilon.$$

Hence we get that $|f|$ is Riemann integrable.

Now it remains to show that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$. Since $-|f| \leq f \leq |f|$ then

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

This implies

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Theorem 2.20 (Mean Value Theorem for Integrals). Let f be continuous on $[a,b]$ and let M and m be supremum and infimum f on $[a,b]$ respectively then there is $c \in (a,b)$ such that [2]

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Proof. Since M and m are supremum and infimum of f on $[a,b]$ and f is continuous, there are $x_1, x_2 \in [a,b]$ such that

$$m = f(x_1) \qquad and \qquad M = f(x_2). \tag{2.6}$$

We know $m \leq f(x) \leq M$ Since

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a),$$

by dividing both sides by $(b-a)$, we get

$$m \leq \frac{\int_a^b f(x)dx}{b-a} \leq M$$

and from Eq (2.6)

$$f(x_1) \leq \frac{\int_a^b f(x)dx}{b-a} \leq f(x_2)$$

By intermediate value theorem there is $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

This proves the theorem.

Theorem 2.21. Let $f \in R(a,b)$, then the function F defined on (a,b) by $F(x) = \int_a^x f(t)dt$ is continous on $[a,b]$ Proof. Since $f \in R(a,b)$, f is bounded on (a,b), assume that $\exists\, M \in \mathbb{R}$ s.t $|f(t)| \leq M$, $\forall\, t \in [a,b]$. Let $a \leq x \leq y \leq b$. Then

$$|F(y) - F(x)| = \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right|$$

$$= \left| \int_a^y f(t)dt + \int_x^a f(t)dt \right| = \left| \int_x^y f(t)dt \right| \leq M|y-x| = M(y-x). \quad (2.7)$$

Let $\varepsilon > 0$ be given. Then if $|y-x| < \frac{\varepsilon}{M}$ we see that from Eq (2.7) that $|F(y) - f(x)| < \varepsilon$ which proves the continuty of F on (a,b).

Theorem 2.22 (Fundemental Theorem of Calculus). The followin statements hold [2]:

(a) If $f : [a,b] \to \mathbb{R}$ is differentiable and $f^{'} \in R(a,b)$, then $\int_a^b f^{'}(x)dx = f(b) - f(a)$.

(b) If $f \in R(a,b)$ and let $F(x)$ defined as $F(x) = \int_a^x f(t)dt$, $a \leq x \leq b$. If f is continous at $c \in [a,b]$ then F is differentiable at c and $F^{'}(c) = f(c)$.

Proof. For part (a), let $p = \{x_0, ..., x_n\}$ be any partition of $[a,b]$. Since f is differentiable, it is continous by mean value theorem of differentiability $\exists\ c_i \in (x_{i-1}, x_i)$ such that

$$f^{'}(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, \ i = 1, 2, ..., n.$$

Summation yields

$$\sum_{i=1}^{n} f^{'}(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Hence

$$L(f^{'}, p) \leq \sum_{i=1}^{n} f^{'}(c_i)(x_i - x_{i-1}) \leq U(f^{'}, p).$$

This implies that

$$L(f^{'}, p) \leq f(b) - f(a) \leq U(f^{'}, p).$$

Since $f^{'}$ is Riemman integrable , $L(f^{'}) = U(f^{'})$. Therefore,

$$\int_a^b f^{'}(x)dx = f(b) - f(a).$$

To prove part (b) take $\varepsilon > 0$. Since f is continous at c, we can find $\delta > 0$ such that $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ whenever $|x - c| < \delta$ and $x \in [a,b]$. Take t satisfying $|t| < \delta$ and $c + t \in [a,b]$. Then

$$\int_c^{c+t} (f(c) - \varepsilon)dx \leq \int_c^{c+t} f(x)dx \leq \int_c^{c+t} (f(c) + \varepsilon)dx,$$

or

$$(f(c) - \varepsilon)t \leq F(c+t) - F(c) \leq (f(c) + \varepsilon)t.$$

This implies that

$$\left| \frac{F(c+t) - F(c)}{t} - F(c) \right| < \varepsilon.$$

this proves the theorem.

Theorem 2.23 (Change of Variable). Let g be a real valued function on the closed bounded interval $[a,b]$ such that $g^{'} \in R(a,b)$. If $f(x)$ is a continous function on $[a,b]$, then [5]

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(u))g^{'}(u)\,du.$$

Proof. Since f is continous function on the closed bounded interval $[g(a), g(b)]$, the by Theorem 2.21 there exists a function $F(x)$ s.t $F^{'}(x) = f(x)$, $x \in [a,b]$. Now, let

$Q(u) = F(g(u))$ for $u \in [a,b]$ then

$$Q^{'}(u) = F^{'}(g(u)) \cdot g^{'}(u) = f(g(u)) \cdot g^{'}(u) = (f \circ g)g^{'}(u), \quad a \leq u \leq b.$$

Now, continuity of $g^{'}$ implies the continuity of g and continuity of f and g implies the continuity of $(f \circ g)g^{'}$ on $[a,b]$. By Theorem 2.21 F is continous and by fundamental theorem of calculus we have the conclusion of theorem.

Theorem 2.24 (Integration by Parts). If u and v be differentiable on $[a,b]$ with $u^{'}, v^{'} \in R(a,b)$ [5]. Then

$$\int_a^b uv^{'} dx = u(b)v(b) - u(a)v(a) - \int_a^b u^{'} v dx$$

Proof. We know that $(uv)^{'} = u^{'}v + v^{'}u$. So $u^{'}v + v^{'}u \in R(a,b)$ now by taking integral from a to b on both sides we get

$$\int_a^b (uv^{'}) dx = \int_a^b vu^{'} dx + \int_a^b uv^{'} dx$$

$$u(b)v(b) - u(a)v(a) = \int_a^b vu^{'} dx + \int_a^b uv^{'} dx$$

$$\int_a^b uv^{'} dx = u(b)v(b) - u(a)v(a) - \int_a^b u^{'} v dx.$$

2.1.4 Riemann Integral Depending on Parameter

The idea of an integral depending on a prameter is a considration of

$$F(y) = \int_{E_y} f(x,y) dx$$

where y is a parameter over a set T and for each $y \in T$ there corresponds a set E_y and a function $g_y(x) = f(x,y)$ which is Riemann integrable over E_y in the proper or improper senses, where T is a subset of $\mathbb{R}$.

Now, we consider a function $f(x,y)$ on $[a,b] \times [c,d]$ which is Riemann integrable on $[a,b]$. We study continuity, differentiability and integrability of $F(y)$ on $[a,b]$.

Theorem 2.25 (Interchange of limit and integral). Let f_n be a sequence of continous functions in $[a,b] \subset \mathbb{R}$, such that f_n converges to f uniformly as $n \to \infty$. Then f is Riemann integrable on $[a,b]$ and [3]

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof. Since uniform limit of continous functions is continous, f is continous. Therefore, f is Riemann integrable on $[a,b]$. Furthermore, from uniform continuity, for every $\varepsilon > 0$ there is N such that $\forall n > N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$, for all $x \in [a,b]$. This implies

$$\frac{-\varepsilon}{b-a} < f_n(x) - f(x) < \frac{\varepsilon}{b-a}$$

or

$$\int_a^b \frac{-\varepsilon}{b-a}dx < \int_a^b f_n(x)dx - \int_a^b f(x)dx < \int_a^b \frac{\varepsilon}{b-a}dx.$$

Then

$$\frac{-\varepsilon(b-a)}{b-a}dx < \int_a^b f_n(x)dx - \int_a^b f(x)dx < \frac{\varepsilon(b-a)}{b-a}dx.$$

Then, for all $n > N$.

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| < \varepsilon.$$

This means

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem 2.26 (Continuity of an integral depending on a parameter). Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous function [10]. Then

$$F(y) = \int_a^b f(x,y)dx \quad , \quad y \in [c,d],$$

is continous at every point $y \in [c,d]$.

Proof. Take $y_0 \in [a,b]$. By continuity of f, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x,y) - f(x,y_0)| < \frac{\varepsilon}{b-a}.$$

Then

$$\int_a^b \frac{-\varepsilon}{b-a}dx < \int_a^b (f(x,y)dx - f(x,y_0))dx < \int_a^b \frac{\varepsilon}{b-a}dx$$

or

$$\left| \int_a^b (f(x,y)dx - f(x,y_0))dx \right| < \varepsilon.$$

Thus

$$\lim_{y \to y_0} \int_a^b f(x,y)dx = \int_a^b f(x,y_0)dx.$$

This mean the continuity of F at y_0.

Theorem 2.27 (Differentiation of an integral depending on a paramter). Let $f(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ be a continous function on $[a,b] \times [c,d]$ and has a continous partial derivatives with respect to the parameter $y \in [c,d]$ then [10]

$$F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y} dx$$

Proof. Let $y_0 \in [c,d]$. Applying Theorem 2.26 we can calculate $F'(y)$ as follows

$$F'(y) = \lim_{y \to y_0} \frac{F(y) - F(y_0)}{y - y_0}$$

$$= \lim_{y \to y_0} \int_a^b \frac{f(x,y) - f(x,y_0)}{y - y_0} dx = \int_a^b \frac{\partial f(x,y)}{\partial y} dx.$$

Theorem 2.28 (Interchange the order of integration). If the function $f : [a,b] \times [c,d] \to \mathbb{R}$ is continous then the functions [12]

$$F(y) = \int_a^b f(x,y)dx, \quad c \le y \le d,$$

and

$$G(x) = \int_c^d F(x,y)dx, \quad a \le x \le b,$$

are integrable and

$$\int_c^d \left(\int_a^b f(x,y)dx \right) dy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

Proof. By Theorem 2.26 $F(y)$ and $G(x)$ are integrable. Let

$$F_0(t) = \int_a^t \left(\int_c^d f(x,y)dy \right) dx \; , \; G_0(t) = \int_c^d \left(\int_a^t f(x,y)dx \right) dy, \quad a \leq t \leq b$$

By Theorem 2.26 and (fundemental theorem of calculus) $F_0(t)$ and $G_0(t)$ are differentiable on $[a,b]$ and

$$F_0^{'}(t) = G_0^{'}(t) = \int_c^d f(t,y)dy.$$

We conclude that $F_0(t) = G_0(t)$, since $F_0(a) = G_0(a)$. In particular $F_0(b) = G_0(b)$. Hence the proof is complete.

2.2 Improper Riemann Integral

Previously, we have only considered integrals of bounded functions on a finite interval $[a,b]$. We now extend the Riemann integral to unbounded functions and functions on infinite intervals. In these cases the Riemann integral is called an improper integral.

There are two kinds of improper integrals. The first kind improper integrals considers bounded functions on unbounded intervals. If $f(x)$ is Riemann integrable on $[a,b]$ for any $b > a$, then

$$\int_a^\infty f(x)dx$$

is called improper integral of the first kind.

In other words, let I be an interval of the form $[a,\infty)$ or $(-\infty,b]$ and let f be a function defined on I and Riemann integrable on every bounded and closed subinterval of I. We informally let

1. $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$ if $I=[a,\infty)$,

2. $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$ if $I=(-\infty,b]$.

The second kind improper integrals considers functions $f(x)$ on a bounded interval $[a,b]$ where $f(x)$ is unbounded about finite number of points inside $[a,b]$. Then $\int_a^b f(x)dx$ is said to be improper integral of the second kind [5].

In other words, let I be an interval of the form $[a,b)$ or $(a,b]$ and let f be a function defined and unbounded on I and Riemann integrable on every closed subinterval of I, we informally let

1. $\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$ if $I=[a,b)$

2. $\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$ if $I=(a,b]$.

2.2.1 Definition

Definition 2.29. Let $F(y)=\int_a^y f(x)dx$ suppose that $F(y)$ exist for each $y>a$, if $F(y)$ has a finite limit L as $y\to\infty$, then the improper integral $\int_a^\infty f(x)dx$ is said to be converge to L, where L represents the Riemann integral value of $f(x)$ on $[a,\infty)$ and we write as $L=\int_a^\infty f(x)dx$. On the other hand, if $L=\pm\infty$ then $\int_a^\infty f(x)dx$ is said to be diverge. Similarly we define the integral

$$\int_{-\infty}^{a} f(x)dx \quad as \quad \int_{-y}^{b} f(x)dx \quad as \quad y\to\infty,$$

and also

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{u\to\infty}\int_{-u}^{b} f(x)dx + \lim_{v\to\infty}\int_{b}^{v} f(x)dx.$$

Definition 2.30 [Cauchy Sequence]. A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for any $\varepsilon > 0$ there exists a natural number $\mathbb{N}$ such that for both natural numbers p and q, where $p > \mathbb{N}$ and $q > \mathbb{N}$, $|a_p - a_q| < \varepsilon$ [4].

Theorem 2.31. The improper integral $\int_a^{\infty} f(x)dx$ converges if and only if for a given $\varepsilon > 0$, there exist t_0 such that $|\int_{t_1}^{t_2} f(x)dx| < \varepsilon$ whenever t_1 and t_2 exceed t_0 [3].

Proof. If $F(t) = \int_a^t f(x)dx$ has a limit L as $t \to \infty$ then for a given $\varepsilon > 0$ there exist t_0 such that $t > t_0$ implies

$$|F(t) - L| < \frac{\varepsilon}{2}.$$

Now since t_1 and t_2 exceed to,

$$\left|\int_{t_1}^{t_2} f(x)dx\right| = |F(t_2) - F(t_1)|$$

$$\leq |F(t_2) - L| + |F(t_1) - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely: suppose that for every $\varepsilon > 0$ there is t_0 such that for all $t_1 > t_0$ and $t_2 > t_0$, $\left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon$. We want to show that $F(t)$ has a limit as $t \to \infty$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence defined as follows

$$g_n = \int_a^{a+n} f(x)dx \qquad n = 1, 2, ...$$

This means that for each $\varepsilon > 0$,

$$g_n - g_m = \int_a^{a+n} f(x)dx - \int_a^{a+m} f(x)dx$$

$$= \int_a^{a+n} f(x)dx + \int_{a+m}^{a} f(x)dx$$

$$= \int_{a+m}^{a+n} f(x)dx.$$

This implies that

$$|g_n - g_m| = \left| \int_{a+m}^{a+n} f(x)dx \right| < \varepsilon.$$

If m and n are large enough, this implies that $\{g_n\}_{n=1}^{\infty}$ ia a Cauchy sequence.

Hence it converges by theorem which state ”the sequence $\{g_n\}_{n=1}^{\infty}$ converges iff for each $\varepsilon > 0$ there is and integer $\mathbb{N}$ such that $|a_m - a_n| < \varepsilon$ for all $m > \mathbb{N}$, $n > \mathbb{N}$ ”. Let

$$g = \lim_{n\to\infty} g_n.$$

To show that

$$\lim_{t\to\infty} F(t) = g,$$

we write

$$|F(t)-g| = |F(t)-g_n+g_n-g| \le |F(t)-g_n|+|g_n-g|. \tag{2.8}$$

Suppose that $\varepsilon > 0$ is given so there exist an integers N_1 and N_2 such that $|g_n - g| < \frac{\varepsilon}{2}$ for all $n > N_1$, and

$$|F(t)-g_n| = \left|\int_{a+n}^{t} f(x)dx\right| < \frac{\varepsilon}{2} \tag{2.9}$$

for all $n > N_2$. If $n > N_1$ and $t = a+n > N_2$ thus by choosing $t > a+n$, where $t > max\{N_1, N_2 - a\}$ from Eq (2.8) and Eq (2.9) we get that $|F(t)-g| < \varepsilon$.

Definition 2.32 [Absolute Convergence]. If $\int_a^\infty |f(x)|dx$ is convergent, then $\int_a^\infty f(x)dx$ is said to be absolutely convergent [3].

2.2.2 Properties of Improper Riemann Integral

Theorem 2.33 (Comparison Test). Let $f(x)$ and $g(x)$ be two functions which are bounded and integrable on $[a,\infty)$ and let $f(x)$ be positive and $|g(x)| \le f(x)$, $x \ge a$. Then if $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent and that $\int_a^\infty g(x)dx \le \int_a^\infty f(x)dx$ [5].

Proof. Since $|g(x)| \le f(x)$ it's clear that

$$\int_a^\infty (f-g)(x)dx \ge 0$$

because

$$U(f-g,p) \ge L(f-g,p) \ge 0.$$

Hence by Theorem 2.15

$$\int_a^\infty f(x)dx - \int_a^\infty g(x)dx \geq 0.$$

This implies that

$$\int_a^\infty g(x)dx \leq \int_a^\infty f(x)dx.$$

Now if

$$\int_a^\infty f(x)dx < \infty \qquad then \qquad \int_a^\infty g(x)dx < 0.$$

Example 2.34. Test the convergence of [5]

$$\int_0^\infty \frac{\cos x}{1+x^2}dx$$

Solution: Let

$$f(x) = \frac{\cos x}{1+x^2}$$

Since $|\cos x| \leq 1$, we have

$$\left|\frac{\cos x}{1+x^2}\right| \leq \frac{1}{1+x^2}.$$

Let $g(x) = \frac{1}{1+x^2}\cdot$ Then

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \left[\tan^{-1} x \right]_0^b$$

$$\lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 0 \right] = \frac{\pi}{2}.$$

So by comparison test we get that $\int_0^\infty \frac{\cos x}{1+x^2}\, dx$ is convergent.

Theorem 2.35 The improper integral $\int_a^\infty \frac{dx}{x^p}$, where $a \geq 0$, converges if and only if $p > 1$ [5].

Proof. We have, for $p \neq 1$,

$$\int_a^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_a^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p} \right]_a^b$$

$$= \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p} \right] - \frac{a^{1-p}}{1-p}.$$

Now if $p < 1$ then $\frac{b^{1-p}}{1-p} \to \infty$ as $b \to \infty$ in this case the integral is not convergent. If $p > 1$, then

$$\int_a^\infty \frac{dx}{x^p} = \frac{a^{1-p}}{p-1},$$

and the improper integral converges. Additionally, for $p = 1$, we have

$$\int_a^\infty \frac{dx}{x} = \lim_{b\to\infty} [\ln b - \ln a] = \infty$$

so, for $p = 1$, the improper integral diverges.

Note 2.36. This integral is one of the most important integrals for the application of comparison test.

Theorem 2.37 (Absolute convergence). If f is bounded and integrable on $[a,x]$ for each $x \geq a$ and if $\int_a^\infty |f(x)|dx$ is convergent then $\int_a^\infty f(x)dx$ is also convergent [5].

Proof. Let $f(x) = [f(x) + |f(x)|] - |f(x)|$ then

$$\int_a^\infty f(x)dx = \int_a^b [f(x) + |f(x)|]dx - \int_a^b |f(x)|dx, \qquad b \geq a \tag{2.10}$$

By assumption $\int_a^\infty |f(x)|dx$ converges as $b \to \infty$. Also

$$0 \leq f(x) + |f(x)| \leq 2|f(x)|.$$

Then

$$0 \leq \int_a^b [f(x) + |f(x)|]dx \leq \int_a^b 2|f(x)|dx.$$

Since

$$\int_a^b 2|f(x)|dx$$

is converges as $b \to \infty$, by comparison test, we get that

$$\int_a^b [f(x) + |f(x)|]dx$$

is also convergent as $b \to \infty$ and by Eq (2.10) we obtain that

$$\int_a^b f(x)dx$$

is convergent as $b \to \infty$

Example 2.38. Prove that $\int_1^\infty \frac{\sin x}{x^4}\, dx$ is absolutely convergent [5].

Solution: We have

$$\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx = \lim_{b\to\infty} \int_1^b \frac{|\sin x|}{|x^4|} dx$$

$$\leq \lim_{b\to\infty} \int_1^b \frac{1}{x^4} dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-3}}{-3}\right]_1^b$$

$$= \left[\frac{1}{3} - \lim_{b\to\infty} \frac{1}{3b^3}\right] = \frac{1}{3}.$$

Hence,

$$\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx$$

is convergent. This implies that

$$\int_1^\infty \frac{\sin x}{x^4} dx$$

is absolutely convergent.

Now we will consider a few examples of the second kind improper integrals

Example 2.39. Test the convergence of $\int_a^b \frac{dx}{(x-a)^n}$ · We have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{t\to 0} \int_{a+t}^b \frac{dx}{(x-a)^n}$$

$$= \lim_{t\to 0} \frac{1}{1-n}(x-a)^{1-n}$$

$$= \lim_{t\to 0} \frac{1}{1-n}\left[(b-a)^{1-n} - t^{1-n}\right]$$

$$\frac{1}{1-n}(b-a)^{1-n} \qquad if \qquad n < 1$$

and if $n = 1$ we have

$$\int_a^b \frac{dx}{(x-a)} = \lim_{t\to 0} \ln(b-a) - \ln t = \infty$$

So the integral is convergent if $n < 1$ and it's divergent (non-convergent) if $n \geq 1$.

Example 2.40. Test the convergence of $\int_0^1 \frac{\sec x}{x} dx$. Since $|\sec x| \geq 1$ for each values of x [5].

$$\left|\frac{\sec x}{x}\right| \geq \frac{1}{x}$$

and

$$\int_0^1 \frac{1}{x}\, dx = \lim_{t\to 0}\left[\ln 1 - \ln t\right] = \infty \cdot \tag{2.11}$$

This implies that $\int_0^1 \frac{dx}{x}$ is divergent and this means that the given integral is also divergent.

2.2.3 Improper Riemann Integral Depending on Parameter

Suppose that the improper integral

$$F(y) = \int_I f(x,y)dx$$

over $I \subset \mathbb{R}$ converges for all values of the parameter $y \in E \neq \emptyset$ and assume that the integral (2.11) has only one singularity that is either $I = [a, \infty)$ or $I = (-\infty, b]$ or f unbounded as a function of x in the interval $I = [a, b)$ or $I = (a, b]$

Definition 2.41. We say that the improper integral Eq (2.11) depending on the parameter $y \in E$ converges uniformly for $y \in E$ if there exist a function $g : E \to \mathbb{R}$ such that [12]

$$1)\ \lim_{y\to\infty} \sup_{y\in E} \left| g(y) - \int_a^b f(x,y)dx \right| = 0 \ if\ \ I = [a,\infty),$$

$$2)\ \lim_{a\to-\infty} \sup_{y\in E} \left| g(y) - \int_a^b f(x,y)dx \right| = 0 \ if\ \ I = (-\infty,b],$$

$$3)\ \lim_{c\to b^-} \sup_{y\in E} \left| g(y) - \int_a^c f(x,y)dx \right| = 0 \ if\ \ I = [a,b),$$

$$4)\ \lim_{c\to a^+} \sup_{y\in E} \left| g(y) - \int_c^b f(x,y)dx \right| = 0 \ if\ \ I = (a,b].$$

Theorem 2.42 (Weierstrass test). Let I be one of the four intervals in the definition (2.41) and let $E \neq \emptyset$. Assume that a function $I : I \times E \to \mathbb{R}$ [12]. Proof. Let $I = [a, \infty)$. From

$$\int_a^b |g(x,y)|dx \leq \int_a^b f(x)dx \leq \int_a^\infty f(x)dx.$$

We obtain that, under fixed $y \in E$, the function

$$\psi(b,y) = \int_a^b |g(x,y)|dx, \quad b \geq a$$

is bounded and increasing. Therefore, the improper integral $\int_a^\infty g(x,y)dx$ converges absolutly at every $y \in E$. Furthermore,

$$0 \leq \sup_{y\in E}\left|\int_a^\infty g(x,y)dx - \int_a^b g(x,y)dx\right|$$

$$0 = \sup_{y\in E}\left|\int_b^\infty g(x,y)dx\right| \leq \sup_{y\in E}\int_b^\infty |g(x,y)|dx \leq \int_b^{infty} f(x)dx.$$

Since $\int_b^\infty f(x)dx$ converges, $\lim_{b\to\infty}\int_b^\infty f(x)dx = 0$. This implies that the improper integral $\int_a^\infty g(x,y)dx$ converges uniformly for $y \in E$. The other cases of the interval I can be handled similarly.

Example 2.43. The integral $\int_1^\infty \frac{dx}{x^2+y^2}$ converges uniformly for each value of $y \in \mathbb{R}$ since $\forall y \in \mathbb{R}$

$$\int_a^\infty \frac{dx}{x^2+y^2} \leq \int_a^\infty \frac{dx}{x^2} = \frac{1}{a},$$

where the Weierstrass test is used [10].

Example 2.44. The integral $\int_1^\infty e^{-xy}dy$ converges only for $y > 0$ and moreover it converges uniformly. This follows from [10]

$$0 \leq \int_1^\infty e^{-xy}dx = \lim_{b\to\infty}\int_1^b e^{-xy}dx = \lim_{b\to\infty}\frac{1-e^{-by}}{y} = \frac{1}{y}\cdot$$

Chapter 3

EULER'S INTEGRALS

The Beta and Gamma functions are two functions which are famous as Euler's integrals. The Beta and Gamma functions are defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \qquad m,n \in \mathbb{R},$$

and

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \qquad \alpha \in \mathbb{R}.$$

These integrals are very important and have wide applications in mathematics, physics, statistics and other applied siences. Furthermore, Beta and Gamma functions are defined as two improper integrals , depending on a parameter [10].

3.1 Gamma Function

3.1.1 Definition

In 1729 the Gamma function was defined by Leonhard Euler for the first time as follows:

$$\Gamma(\alpha) = \int_0^1 (-\log x)^{\alpha-1}dx, \qquad \alpha > 0.$$

Then Euler defned the Gamma funtion as follows:

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n^{\alpha} n!}{\alpha(\alpha+1)\ldots(\alpha+n)} \cdot$$

Later the following definition of Gamma function was adopted:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

By integration by part we have

$$\int_0^\infty x^{\alpha-1} e^{-x} dx = \Big[-x^{\alpha-1} e^{-x} \Big]_0^\infty + (n+1) \int_0^\infty x^{\alpha-2} e^{-x} dx.$$

Here

$$\lim_{x\to 0} \frac{x^{\alpha-1}}{e^x} = \lim_{x\to 0} \frac{x^{\alpha-1}}{1+x+\frac{x^2}{2!}+\cdot+\frac{x^n}{n!}} = 0\,,$$

and

$$\lim_{x\to\infty} \frac{x^{\alpha-1}}{e^{x-1}} = 0.$$

So,

$$\Gamma(\alpha) = (\alpha-1) \int_0^\infty x^{\alpha-2} e^{-x} dx = (\alpha-1)\Gamma(\alpha-1)\,. \tag{3.1}$$

Assuming that α is a positive integer, we can repeat the formula in (3.1) α-times and obtain

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)\cdots 3\cdot 2\cdot \int_0^\infty e^{-x} dx,$$

implying

$$\Gamma(\alpha) = (\alpha - 1)!.$$

Therefore, sometimes, the Gamma function is called a generalized factorial function.

3.1.2 Properties of Gamma Function

The Gamma function has the following basic properties [10] [5]:

1. $\Gamma(1) = 1$
2. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \qquad \alpha > 0$
3. $\Gamma(n) = (n-1)! \qquad n \in \mathbb{N}$
4. $\lim_{\alpha \to 0^+} \Gamma(\alpha) = \infty, \qquad \alpha \in \mathbb{N}.$

   Proof. To prove this limit, we consider

   $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \geq \int_0^1 x^{\alpha-1} e^{-x} dx \geq \frac{1}{e} \int_0^1 x^{\alpha-1} dx. \tag{3.2}$$

   Hence,

   $$\lim_{\alpha \to 0^+} \Gamma(\alpha) \geq \frac{1}{e} \lim_{a \to 0^+} \int_0^1 x^{\alpha-1} dx$$

   $$= \frac{1}{e} \lim_{a \to 0^+} \left[\frac{x^\alpha}{\alpha} \right]_0^1$$

   $$= \frac{1}{e} \left[\frac{1}{\alpha} - 0 \right]$$

   Thus

   $$\lim_{\alpha \to 0^+} \Gamma(\alpha) = \lim_{\alpha \to 0^+} \frac{1}{\alpha} = \infty \cdot$$

5. $\Gamma(\alpha) = 2\int_0^\infty x^{2\alpha-1}e^{-x^2}dx$

Proof. Let $t = x^2$. Then $dt = 2x\,dx$ and $x = t^{\frac{1}{2}}\cdot$ We obtain that

$$2\int_0^\infty x^{2\alpha-1}e^{-x^2} = \cancel{2}\int_0^\infty t^{\frac{1}{2}(2\alpha-1)}e^{-t}\frac{dt}{\cancel{2}\,t^{\frac{1}{2}}} = \int_0^\infty t^{\alpha-1}e^{-t} = \Gamma(\alpha).$$

6. $\Gamma(\alpha) = \int_0^1(-\ln y)^{\alpha-1}dy$.

Proof. Let $x = -\ln y$. Then $dx = -\frac{dy}{y}$ and $y = e^{-x}$. We obtain that

$$\int_0^1(-\ln y)^{\alpha-1}dy = -\int_\infty^0 x^{\alpha-1}e^{-x}dx = \int_0^\infty x^{\alpha-1}e^{-x}dx = \Gamma(\alpha)$$

3.2 Beta Function

There is an important integral which can be expressed in terms of the Gamma function.

3.2.1 Definition

The function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}, \qquad m,n > 0.$$

is called the Beta function [10].

Theorem 3.1. For $m > 0$ and $n > 0$,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)} \tag{3.3}$$

Proof. We know that

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$$

and by Proposition 2.1 we have

$$\int_0^\infty t^{n-1}e^{-t}dt = 2\int_0^\infty x^{2n-1}e^{-x^2}dx, \tag{3.4}$$

and similarly

$$\Gamma(m) = 2\int_0^\infty y^{2m-1}e^{-y^2}dy. \tag{3.5}$$

Hence by multiplying Eq (3.4) and Eq (3.5) we get that

$$\Gamma(m)\Gamma(n) = \left(2\int_0^\infty x^{2n-1}e^{-x^2}dx\right)\left(2\int_0^\infty y^{2m-1}e^{-y^2}dy\right).$$

Since two integrals are independent, we can write

$$\Gamma(n)\Gamma(m) = 4\int_0^\infty\int_0^\infty x^{2n-1}y^{2m-1}e^{-(x^2+y^2)}dxdy.$$

Using polar cordinates letting $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$ and $r > 0$, we have

$$\Gamma(n)\Gamma(m) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(n+m)-1}e^{-r^2}\sin^{2m-1}\theta\cos^{2n-1}\theta drd\theta$$

$$= \left(2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta\right)\left(2\int_0^\infty r^{2(n+m)-1}e^{-r^2}dr\right)$$

$$= \left(2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta\right)\Gamma(m+n). \tag{3.6}$$

Hence we must to show that $2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = \beta(m,n)$. Let $t = \sin^2\theta$. If

$\theta = 0$, then $t = 0$ if $\theta = \frac{\pi}{2}$, then $t = 1$, and $dt = 2\cos\theta\sin\theta d\theta$. Therefore,

$$2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\ \cos^{2n-1}\theta d\theta$$

$$= 2\int_0^1 t^{m-\frac{1}{2}}\ (1-t)^{n-\frac{1}{2}}\ \frac{1}{2}(1-t)^{-\frac{1}{2}}\ t^{-\frac{1}{2}} dt$$

$$= \int_0^1 t^{m-1}\ (1-t)^{n-1}\ dt. \tag{3.7}$$

So from Eq (3.3), Eq (3.6) and Eq (3.7), we obtain that $\Gamma(n)\Gamma(m) = \beta(m,n)\Gamma(m,n)$, and this implies that $\beta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

3.2.2 Properties of Beta Function

The beta function has properties [5] [10]:

(a). $\beta(m,n) = \beta(n,m)$,

(b). $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$,

(c). $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$,

(d). $m\beta(m,n+1) = n\beta(m+1,n)$,

(e). $\beta(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} dt$,

(f). $\beta(m+1,n) = \frac{m}{m+n}\beta(m,n)$,

Proof (a). In

$$\beta(m,n) = \int_0^1 x^{m-1}\ \ (1-x)^{n-1} dx$$

substitute $1-x = y$. Then $x = y-1$, $dx = -dy$. Respectively,

$$\beta(m,n) = \int_0^1 x^{m-1}\ \ (1-x)^{n-1} dx$$

$$= -\int_0^1 (1-y)^{m-1}\, y^{n-1} dy$$

$$= \int_0^1 y^{n-1}\, (1-y)^{m-1} dy = \beta(n,m).$$

(b). This is Theorem 3.1

(c). We have

$$\beta(m+1,n)+\beta(m,n+1) = \int_0^1 x^m\, (1-x)^{n-1} dx + \int_0^1 x^{m-1}\, (1-x)^n dx$$

$$= \int_0^1 x^{m-1}\, (1-x)^{n-1}\, [x+(1-x)] dx$$

$$= \int_0^1 x^{m-1}\, (1-x)^{n-1}\, dx = \beta(m,n).$$

(d). We have

$$\beta(m+1,n) = \int_0^1 x^m\, (1-x)^{n-1} dx.$$

Let $u = x^m$ and $dv = (1-x)^{n-1} dx$. Then

$du = mx^{m-1} dx$ and $v = -\frac{1}{n}(1-x)^n$. By integration by parts formula,

$$\beta(m+1,n) = \int_0^1 x^m\, (1-x)^{n-1} dx = -\frac{x^m}{n}(1-x)^n\Big]_0^1 + \frac{m}{n} \int_0^1 x^{m-1}\, (1-x)^n dx$$

$$= \frac{m}{n} \int_0^1 x^{m-1}\, (1-x)^n dx = \frac{m}{n}\, \beta(m,n+1).$$

(e). Substitute $x = \frac{t}{1+t}$ then $dx = \frac{dt}{(1+t)^2}$ and $t = \frac{x}{1-x}\cdot$ This implies

$$\int_0^\infty \frac{t^{m-1}}{(1+t)m+n}dt$$

$$= \int_0^1 x^m \left(1+\frac{x}{1-x}\right)^{-n} \left(\frac{x}{1-x}\right)^{-1} (1-x)^{-2}dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m,n).$$

(f). See proof d.

3.3 Some Important Examples

Example 3.2. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. To prove we consider

$$\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}}\, e^{-x}dx = 2\int_0^\infty e^{-t^2}dt.$$

To go on, we need polar coordinates. Consider the double integral

$$I = \int_0^\infty \int_0^\infty e^{-x^2-y^2}dxdy.$$

By chainging to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, $0 < r < \infty$, $\theta \in [0, \frac{\pi}{2}]$,

$$I = \int_0^\infty \int_0^{\frac{\pi}{2}} e^{r^2} r\,dr\,d\theta = \frac{\pi}{2}\int_0^\infty re^{-r^2}dr = \frac{\pi}{4}\cdot$$

Also,

$$I = \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}dxdy = \left[\int_0^\infty e^{-x^2}dx\right]^2.$$

Therefore,

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Thus,

$$\Gamma(\frac{1}{2}) = 2\int_0^{\infty} e^{-t^2} dt = 2\,\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Example 3.3. Evaluate $\Gamma(\frac{3}{2})$. We have

$$\Gamma(\frac{3}{2}) = \Gamma(1+\frac{1}{2}) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

Example 3.4. Evaluate $\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta d\theta$. We have

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta d\theta = \frac{1}{2}\,\beta(\frac{5}{2},3)$$

$$= \frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\,\Gamma(3)}{\Gamma(\frac{11}{2})} = \frac{1}{2}\left[\frac{\frac{3}{2}\,\frac{\sqrt{\pi}}{2}\,2!}{\frac{9}{2}\,\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{\sqrt{\pi}}{2}}\right].$$

Example 3.5. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\tan x} dx$. We have

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x} dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\,\cos^{-\frac{1}{2}} dx = \frac{1}{2}\,\beta(\frac{3}{4},\frac{1}{4})$$

$$\frac{1}{2}\,\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{\Gamma(1)} = \frac{1}{2}\,\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4}) = \frac{1}{2}\,\frac{\pi}{\sin(\frac{\pi}{4})}.$$

Example 3.6. Prove $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(m\pi)}$. To prove [11], consider

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(m\pi)}$$

$$\int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t}\,dt = \int_0^{\infty} \frac{x^a}{1+x}\,\frac{dx}{x} = \int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin(m\pi)}.$$

Hence,

$$\Gamma(1-m) = \int_0^{\infty} x^{-m} e^{-x}\,dx$$

$$= t\int_0^{\infty} (xt)^{-m} e^{-xt}\,dx \qquad (by\ letting\ x = xt)$$

Then

$$\Gamma(m)\Gamma(1-m) = \left(\int_0^{\infty} t^{m-1} e^{-t}\,dt\right)\left(t\int_0^{\infty} (xt)^{-m} e^{-xt}\,dx\right)$$

$$= \int_0^{\infty} \left(t\int_0^{\infty} (xt)^{-m} e^{-xt}\,dx\right) e^{-t} t^{m-1}\,dt$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-t(x+1)} x^{-m}\,dx\,dt$$

$$= \int_0^{\infty} \frac{-1}{1+x}\left[e^{-t(1+x)}\right]_0^{\infty} x^{-m} dx$$

$$= \int_0^{\infty} \frac{-1}{1+x}\,x^{-m} dx$$

$$= \int_0^{\infty} \frac{x^{1-m}}{1+x}\,\frac{dx}{x}$$

$$= \frac{\pi}{\sin(\pi(1-m))} = \frac{\pi}{\sin(m\pi)}.$$

3.4 Some Properties of Gamma function

1. Show that $2^n\,\Gamma(n+\frac{1}{2}) = 1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}$ where $n \in \mathbb{N}$ [5].

2. Show that $\Gamma(\frac{3}{2}-x)\Gamma(\frac{3}{2}+x) = (\frac{1}{4}-x^2)\pi\sec(\pi x)$[5].

Proof (1). We have

$$\Gamma(n+\frac{1}{2}) = \Gamma(n-\frac{1}{2}+1) = (n-\frac{1}{2})\Gamma(n-\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\Gamma(n-\frac{3}{2})\cdot$$

Proessing this n-times, we obtain that

$$\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})\cdots\frac{3}{2}\,\frac{1}{2}\,\Gamma(\frac{1}{2})\cdot$$

So,

$$2^n\,\Gamma(n+\frac{1}{2}) = 1.2.3\cdots(2n-1)(\sqrt{\pi}).$$

(2) We have

$$\Gamma(\frac{3}{2}-x) = \Gamma(\frac{1}{2}-x+1) - (\frac{1}{2}-x)\Gamma(\frac{1}{2}-x) \tag{3.8}$$

and

$$\Gamma(\frac{3}{2}+x) = (\frac{1}{2}+x)\Gamma(\frac{1}{2}+x). \tag{3.9}$$

Multiplying (3.8) and (3.9) we obtain

$$\Gamma(\frac{1}{2}-x)\Gamma(\frac{3}{2}+x) = (\frac{1}{4}-x^2)\Gamma(\frac{1}{2}-x)\Gamma(\frac{1}{2}+x). \tag{3.10}$$

Since $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(m\pi)}$, we obtain

$$\Gamma(\frac{1}{2}-x)\Gamma(\frac{3}{2}+x) = (\frac{1}{4}-x^2)\ \frac{\pi}{\sin(\pi x)} = (\frac{1}{4}-x^2)\pi\sec\pi x. \tag{3.11}$$

Theorem 3.7. The Gamma function is convergent for $n > 0$ [5].

Proof. We have

$$\int_0^\infty t^{n-1}e^{-t}dt = \int_0^1 t^{n-1}e^{-t}dt + \int_1^\infty t^{n-1}e^{-t}dt.$$

Now we choose the first integral for convergence at 0 and the second integral for convergent at ∞ .

Test the convergence at 0 : Let $f(t) = t^{n-1}e^{-t}$ and let $g(t) = \frac{1}{t^{1-n}}$. We have $\frac{f(t)}{g(t)} = e^{-t}$ and we see that $e^{-t} \to 1$ as $t \to 0$. Thus

$$\int_0^1 g(t)dt = \int_0^1 \frac{1}{t^{1-n}}dt$$

is convergent iff $1-n < 0$. So $\int_0^1 t^{n-1}e^{-t}dt$ convergent at 0 for $n > 0$.

Test the convergence at ∞: Since $e^t > t^{n-1}$ for any value of n when t is heightly large. So $t^{n+1} < e^t$ implies that $t^{n+1}e^{-t} < 1$ and this means that $t^{n-1}e^{-t} < t^{-2} - \frac{1}{t^2}$. But $\int_1^\infty \frac{1}{t^2}\ dt$ is convergent by integral test. So $\int_1^\infty t^{n-1}e^{-t}$ is also convergent iff $n > 0$. So

$$\int_0^\infty t^{n-1}e^{-t}dt$$

converges for $n > 0$.

Theorem 3.8. The Beta function $\beta(m,n)$ is convergent for $m,n > 0$.

Proof. To test the convergence of the Beta function we have

$$\int_0^1 t^{m-1}(1-t)^{n-1}dt = \int_0^{\frac{1}{2}} t^{m-1}(1-t)^{n-1}dt + \int_{\frac{1}{2}}^1 t^{m-1}(1-t)^{n-1}dt$$

now to test

Convergence at 0: Let $f(t) = t^{m-1}(1-t)^{n-1}$ and $g(t) = \frac{1}{t^{1-m}}$. Then $\frac{f(t)}{g(t)} = (1-t)^{n-1}$. It is clear that $(1-t)^{n-1} \to 1$ as $t \to 0$ and $\int_0^{\frac{1}{2}} \frac{dt}{t^{n-1}}$ is convergent iff $1-m < 1$ $m > 0$. So

$$\int_0^{\frac{1}{2}} t^{m-1}(1-t)^{n-1}dt$$

is also converges at 0 for $m > 0$.

Convergence at (1): $f(t)$ can be written in the form $f(t) = \frac{t^{m-1}}{(1-t)^{1-n}}$. Let $g(t) = \frac{1}{(1-t)^{1-n}}$. Now $\frac{f(t)}{g(t)} = t^{m-1}$ and it's clear $t^{m-1} \to 1$ as $t \to 1$. So

$$\int_{\frac{1}{2}}^1 g(t)dt = \int_{\frac{1}{2}}^1 \frac{dt}{(1-t)^{1-n}}dt$$

converges iff $1-n < 1$ i.e $n > 0$. Thus we get that

$$\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

exists for all $m,n>0$.

Theorem 3.9. (Dirichlet's integral). The equality

$$\int\int\cdots\int x_1^{a_1-1}\,x_2^{a_2-1}\cdots\,x_n^{a_n-1}dx_1dx_2\cdots dx_n=\frac{\Gamma(a_1)\,\Gamma(a_2)\,\cdots\Gamma(a_n)}{\Gamma(a_1+a_2+\cdots+a_n+1)}$$

holds, where the integral is extended to all positive values of the variables subject to condition $x_1+x_2+\cdots+x_n\leq 1$ [5].

Proof. First consider the double integral $\int\int x_1^{a_1-1}\,x_2^{a_2-1}dx_1dx_2$ where $x_1+x_2\leq 1$ Let us denote the integral by I_2. We have

$$I_2=\int_0^1\int_0^{1-x_1}x_1^{a_1-1}\,x_2^{a_2-1}dx_1dx_2$$

$$=\int_0^1 x_1^{a_1-1}\frac{(1-x_1)^{a_2}}{a_2}\,dx_1=\frac{1}{a_2}\,\beta(a_1,a_2+1)=\frac{\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(a_1+a_2+1)}$$

If the condition be $x_1+x_2\leq h$, then putting $\frac{x_1}{h}=u$ and $\frac{x_2}{h}=v$ we see that

$$I_2=\int\int x_1^{a_1-1}x_2^{a_2-1}dx_2dx_1=h^{a_1+a_2}\int\int u^{a_1-1}v^{a_2-1}dudv$$

$$=\frac{\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(a_1+a_2+1)}h^{(a_1+a_2)}.$$

Now consider the triple integral

$$I_3=\int\int\int x_1^{a_1-1}\,x_2^{a_2-1}\,x_3^{a_3-1}dx_1\,dx_2\,dx_3,$$

where $x_1 + x_2 + x_3 \leq 1$. Then

$$I_3 = \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} x_1^{a_1-1}\, x_2^{a_2-1}\, x_3^{a_3-1} dx_1\, dx_2\, dx_3$$

$$= \int_0^1 x_1^{a_1-1} \frac{\Gamma(a_2)\,\Gamma(a_3)}{\Gamma(a_2+a_3+1)} (1-x_1)^{a_2+a_3} dx_1$$

$$= \frac{\Gamma(a_2)\,\Gamma(a_3)}{\Gamma(a_2+a_3+1)} \cdot \frac{\Gamma(a_1)\,\Gamma(a_2+a_3+1)}{\Gamma(a_1+a_2+a_3+1)}$$

$$= \frac{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)}{\Gamma(a_1+a_2+a_3+1)}.$$

Thus the theorem is true for double and triple integrals. Now assume that the theorem is true for n^{th} integrals, this means that

$$\int\int \cdots \int x_1^{a_1-1}\, x_2^{a_2-1} \cdots\, x_n^{a_n-1} dx_n dx_{n-1} \cdots dx_2 dx_1 = \frac{\Gamma(a_1)\,\Gamma(a_2)\,\cdots \Gamma(a_n)}{\Gamma(a_1+a_2+\cdots+a_n+1)}$$

where $x_1 + x_2 + \cdots + x_n \leq 1$.

Now we must prove that it's true for all $(n+1)$ integralse have

$$I_{n+1} = \int\int \cdots \int x_1^{a_1-1}\, x_2^{a_2-1} \cdots\, x_n^{a_n-1}\, x_{n+1}^{a_{n+1}-1}\, dx_{n+1}\, dx_n \cdots dx_2\, dx_1$$

$$= \int_0^1 x_1^{a_1-1} \frac{\Gamma(a_2)\,\Gamma(a_3) \cdots \Gamma(a_n)\,\Gamma(a_{n+1})}{\Gamma(a_2+a_3+\cdots+a_n+a_{n+1}+1)} (1-x_1)^{a_2+a_3+\cdots+a_{n+1}} dx_1$$

$$= \frac{\Gamma(a_2)\,\Gamma(a_3) \cdots \Gamma(a_{n+1})}{\Gamma(a_2+a_3+\cdots+a_{n+1}+1)} \cdot \frac{\Gamma(a_1)\,\Gamma(a_2+a_3+\cdots+a_{n+1}+1)}{\Gamma(a_1+a_2+\cdots+a_n+a_{n-1}+1)}$$

$$= \frac{\Gamma(a_1)\,\Gamma(a_2)\cdots\Gamma(a_{n+1})}{\Gamma(a_1+a_2+a_3+\cdots+a_{n+1}+1)}.$$

Hence, the theorem holds for all values of n.

3.5 Applications of Beta and Gamma Functions

As we mentioned previously, Gamma and Beta functions have a great usage and applicability in statistics. Before dealing with this issue, we introduce some definitions [7].

Definition 3.10 [Random Variable]. A random variable (statistical variable) is a function from the sample space into the system of real numbers. Random variables are divided into two groups:

1. Discrete random variable is a random variable which takes discrete values finite or (countable) such as $x = 0, 1, ...$
2. Continous random variable is a random variable which takes continous values or (uncountable) such as $(a < x < b)$ [7].

Probability distribution:

a. Discreate random variable: If the random variable x is defined on the discrete experiment, then its probability distribution is called discrete probability distribution.

b. Continous random variable: If the random variable x is defined on the continous expriment then its distribution is called a continous probability distribution [7].

Definition 3.11 [Probability density function].

A function $f(x)$ is said to be a probability distribution function, or the density function if it satisfies the following conditions:

1. $\sum f(x) = 1$ for discrete type;

2. $\int_{-\infty}^{\infty} f(x)dx = 1$ for continous type ($f(x)$ be Riemman integrable) [7].

Gamma distribution function:

In the Gamma function, substitute $y = \frac{x}{\beta}$ and obtain

$$\Gamma(\alpha) = \int_0^\infty (\frac{x}{\beta})^{\alpha-1} e^{-\frac{x}{\beta}} \frac{1}{\beta} dx,$$

where $dy = \frac{dx}{\beta}$. Then

$$\Gamma(\alpha) = \int_0^\infty \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^\alpha} dx = 1.$$

Since $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha) > 0$ we see that

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \beta^\alpha}, & 0 < x < \infty, \\ 0, & \text{else where,} \end{cases}$$

is a probability density function of a random variable of the continuous type.

A distribution of a random variable x that has a probability density function of this form is said to be Gamma distribution with parameters α and β [7].

Remark 3.12. The Gamma distribution is the probability model for waiting times. For instance, in life testing.

Chi-Square (χ_r^2):

A random variable χ of the continous type that has the p.d.f

$$f(x) = \begin{cases} \frac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}}, & 0 < x < \infty, \\ 0, & \text{else where,} \end{cases}$$

is said to have chi-square distribution. A function $f(x)$ of this form called a chi-square probability distribution function. This is a particular case of Gamma distribution when $\alpha = \frac{r}{2}$ and $\beta = 2$ [7].

Definition 3.13 [Degree of freedom]. A positive integer normally equal to the number of independent observations in a sample minus the number of population parameters to be estimated from the sample [7].

Example 3.14. Let x be $\chi^2_{(10)}$ with $r = 10$ degrees of freedom. Find $P(3.25 \leq x \leq 20.5)$ [7].

Solution: We have

$$P(3.25 \leq x \leq 20.5)$$

$$= P(x \leq 20.5) - P(x \leq 3.25)$$

$$= .975 - 0.025 = 0.95$$

Exponential distribution:

A random variable x is said to have on exponential distribution with parameter $\beta > 0$ if the density function of x is [1]

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & 0 \leq x < \infty, \\ 0, & \text{else where}, \end{cases}$$

The situations when we use exponential distribution:

The exponential density function is often useful for modeling the length of life of electronics. Suppose that the lenght of time a componenet already has operated does not affect its chance of operating for at least b additional time units, that is, the probability that the component will operate for more than $a+b$ time units, given that it has already operated for at least a time units, is the same as the probability that a new component will operate for at least b time units if the new component is put into service at time 0.

A fuse is an example of a component for which this assumption often is reasonable. We will see in the following example that the exponentioal distribution provides a modle for the distriution of the life time of such a component [1].

Example 3.15. The lifetime (in hours) x of an electronic component is a random variable with density function given by

$$f(x) = \begin{cases} \frac{1}{100}e^{-\frac{x}{100}}, & x > 0, \\ 0, & \text{else where,} \end{cases}$$

Three of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail. Find the probability that the equipment will operate for at least 200 hourse without failure [1].

Solution: Let

$$A = P\{y > 200\} = \int_{200}^{\infty} \frac{1}{100} e^{\frac{-y}{100}}\, dy = e^{-2}$$

$$P\{\text{work match}\} = P\{AAA + AA\}$$

$$= P\{AAA\} + P\{AA\}$$

$$= (P\{A\})^3 + (P\{A\})^2$$

$$= (e^{-2})^3 + (e^{-2})^2 = e^{-6} + e^{-4}$$

Example 3.16. Let x be an exponential probability density function, show that if $a > 0$ and $b > 0$ then $p(x > a+b|x > a) = p(x > b)$ [1]

Solution:

$$p(x > a+b|x > a) = \frac{p(x > a+b)}{p(x > a)}$$

because the intersection of the events $(x > a+b)$ and $(x > b)$ is the event $(x > a+b)$. Now

$$p(x > a+b) = \int_{a+b}^{\infty} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = e^{-\frac{x}{\beta}} \Big|_{a+b}^{\infty} = e^{-\frac{(a+b)}{\beta}}.$$

Similarly,

$$p(x > a) = \int_{a}^{\infty} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = e^{-\frac{x}{\beta}} \Big|_{a}^{\infty} = e^{-\frac{(a)}{\beta}}.$$

$$P(x > a+b|x > a) = \frac{e^{-(a+b)}/\beta}{e^{-a}/\beta} = e^{-b/\beta}$$

Note 3.17. This property of the exponential is often called memoryless property of the distribution.

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