Partial Complete Controllability of Semilinear Control Systems

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ABSTRACT

This work is devoted to examining the partially complete controllability for deterministic semilinear systems in Hilbert spaces. Besides reviewing briefly some existing results of controllability concepts, two main sets of sufficient conditions for partial controllability concepts are proved. The strategy in both results is based on the contraction mapping principle which has played an effective role as the cornerstone of studying controllability concepts for semilinear system, provided that the corresponding linear system is partially complete controllable. The first one is simply obtained by contraction mapping theorem. However, the second result uses the generalized contraction mapping theorem. In the first part, we study the partially complete controllability of deterministic semilinear systems for any positive time. The benefit of this result is demonstrated on some appropriate examples. In the second part, we deal with the same kind of deterministic semilinear systems but with additional condition on the nonlinear part. By this technique, we can defeat the improper integral which arises when we select a suitable control operator by which a generalized contraction mapping theorem can be applied.

Keywords: Contraction mapping principle, complete controllability, partial controllability, semilinear system.

ÖZ

Bu çalışma, ayrılabilir Hilbert uzaylarında, deterministik yarı-lineer sistemler için, kısmen tam kontrol edilebilirliği inceler. Bu tür kontrol edilebilirlik için, iki temel set yeterlilik koşulu ispatlanmıştır. Her iki sonuçtaki strateji, yarı-lineer sistemlerde kontrol edilebilirlik durumlarının incelenmesinde önemli rol oynayan büzülme dönüşüm esasına dayanmaktadır. İlk sonuç sadece büzülme dönüşüm teoremi ile elde edilmiştir. Ancak, ikinci sonuç genelleştirilmiş büzülme dönüşüm teoremini kullanır. İlk kısımda, herhangi bir pozitif zaman dilimi için, deterministik yarı-lineer sistemlerin kısmen tam kontrol edilebilirliği incelenmiştir. Bu sonucun yararı, bazı uygun örnekler üzerinde gösterilmiştir. İkinci bölümde ise, deterministik yarı-lineer sistemlerin farklı bir türü, lineer olmayan terimleri, zamana bağlı bir yardımcı terimle çarpılarak incelenmiştir. Bu teknik ile, 1'den küçük Lipschitz katsayısını elde edebilmek için, ardarda integral alımında ortaya çıkan, improper integral ortadan kaldırılmış olur.

Anahtar kelimeler: Daralma eşleme özelliği, tam kontrol edilebilirlik, kısmi kontrol edilebilirlik , yarı- lineer sistem.

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Chapter 1

INTRODUCTION

Controllability has played a tremendously significant role in designing the modern control systems whereby most of the problems have constantly arose in many widespread academical fields, for instance mechanics, electrics, fluids, chemistry, finance and even biology, can be abstractly represented in the state space as mathematical problems (see for example, Aris and Keller (1982), Erdi and Toth (1989), Lauffenburger, Coron (1996), Ammar-Khodja et al. (2011) also the bibliography therein, etc.). Indeed, regarding to the widely practical implementations of such systems in everyday life, from simple household televisions to very new modern technology such as fighter without pilot (it is an unmanned flight of a retired F-16 fighter plane), the controllability concept has been increasingly progressed and taken a broad range of interests and attentions among several outstanding researchers from over half a century till now. During this period on, many investigators have been duly engaged in examining the conditions of controllability for deterministic as well as stochastic control systems. In short, controllability concept for finite dimensional systems, as Kalman in 1960 defined, is a possession of attaining every state value from a given initial state value at a terminal positive time with the aid of an auxiliary process (Kalman, 1960). In other words, a given control system with a unique mild solution x_t is called controllable for the finite positive time T if for every arbitrary x_1 taken from the state space, this system can be affected by the so-called control process so that its solution x_t should have the ability to be reached by x_1 at the terminal positive time T from any initial state value. This concept of controllability was perfectly carried out for finite dimensional deterministic systems and presented in various books, see for example Brockett (1970), Balakrishnan (1976), Curtain and Pritchard (1978), Bensoussan et al. (1992), Zabczyk (1992). Afterwards, many attentive scientists have enormously extended this concept to infinite dimensional systems. However, not even all these systems are controllable according to Kalman's definition, and they still somewhat have the meaning of controllability if we weaken the Kalman's definition to approximate controllability which is more flexible and covering more useful control systems, especially in infinite dimensional spaces (see Fattorini (1967) and Russel (1967)). At a glance, approximate controllability concept, is the property of reaching every state value from any given initial state value at a terminal positive time with an error less than an arbitrarily small positive number ε. These two well-known concepts of controllability are equivalent for finite dimensional linear control systems and Kalman's rank condition accomplishes necessary and sufficient conditions for such control systems. At present, the necessary and sufficient conditions for complete and approximate controllability of deterministic linear systems are almost perfectly examined and investigated by many heedful authors, for example, Zabczyk (1981), Klamka (1991), Bensoussan (1992), Curtain and Zwart (1995), Bensoussan et al. (1993), Bashirov (2003). It is remarkable that most of studies dealing with the deterministic linear systems, the controllability operator plays a leading role in both complete and approximate controllability for them. In addition, Bashirov and Mahmudov (1999a, 1999b) discovered a new collection of necessary and sufficient conditions for complete and approximate controllability of deterministic linear systems based on resolvent operator. Similarly, but only the sufficient conditions of controllability (approximate and complete) for deterministic semilinear systems have been examined extensively by means of fixed point theorems, providing that the corresponding linear part is (approximately and completely) controllable and these results are contained in various papers including Chukwu and Lenhart (1991), Balachandran and Dauer (1987, 2002), Dauer and Mahmudov (2002), Do (1989), Klamka (2000), Mahmudov (2003a, 2003b), Naito (1987), Zhou (1983) etc. Moreover, fixed point theorems have been widely used to draw up sufficient conditions of controllability for several kinds of nonlinear systems, for instance, Naito (1992) carried out the complete and approximate controllability for nonlinear Volterra integrodifferential control systems. Klamka (1996, 2000) used Schauder fixed point theorem to derive the sufficient conditions of controllability for nonlinear systems. Balachandran and Sakthivel (2001). Sakthivel and Choi (2004) have examined the sufficient conditions of complete controllability for semilinear integrodifferential control systems utilizing Schaefer's fixed point theorem with additional condition that the linear operator A generates a compact semigroup. Recently, however, these concepts of controllability have been extended to different kind systems which are called fractional differential systems and many results for them have been detected and presented in several papers, for example, Sakthivel et al. (2011, 2012), Sakthivel and Mahmudov, and Nieto (2012), Yan (2012), Mahmudov (2013a, 2013b) and Ganesh et al.(2013) etc.

In the real life, the majority of natural events are stochastically and accidentally occurred. Besides, economics and businesses are actually randomly manipulated by external factors so that most of the economical and business problems can be modeled as stochastic systems. Therefore, drawing attention to explore controllability for stochastic control systems is extremely important and hence controllability concept was extended for them and still has been rarely investigated by varied authors such as Mahmudov (2001a, 2001b), Mahmudov and Denker (2000), Enrhardt and Kliemann (1982), Zabczyk (1981) and Dubov and Mordukhovich (1978) etc. for stochastic linear control systems and Mahmudov and Zorlu (2003, 2005), Socha (Sep, 1994) and Sunahara et al. (1974) for stochastic nonlinear control systems. Moreover, Mahmudov (2001a, 2001b) has established that in finite dimensional spaces the concepts of complete and approximate controllability are equivalent for both stochastic and deterministic linear control systems. However, Bashirov et al. (2010) has showed that the complete controllability of linear stochastic systems can never hold and this extraordinary result has led to insert a convenient notion of controllability for stochastic systems which is more flexible and suitable for such kind of systems. Therefore, the notions of S- and C-controllability for stochastic control systems were initiated by Bashirov and Kerimov (1997) with an intrinsic motivation from Sunahara et al. (1974). Roughly speaking, S-controllability of stochastic systems is a possession of reaching any small neighborhood of a given state value (random or not) from an initial state value at a terminal positive time T for probability quite near the one. Likewise, C-controllability of stochastic systems is simply S-controllable reinforced with some uniformity. The necessary and sufficient conditions of S- and C-controllability for partially observed linear stochastic systems are found and discussed completely in works of Bashirov (1996), Bashirov and Kerimov (1997) and Bashirov and Mahmudov (1999a, 1999b). Later in 2007, these concepts were extended to partial versions by Bashirov et al. (2007) and they are completely studied for partially observable stochastic linear systems in this work.

Recently, Bashirov et al. (2007) observed that there are several control systems such as higher-order differential equation, wave equation and delay equation, can be expressed in terms of standard systems (first order differential equation) which can be achieved simply by expanding the dimension of the state space. For these special systems, the so-called partial controllability concepts are strongly recommended, and which consequently conditions for these concepts become weaker and smoother since the conditions of controllability for the enlarged systems are too strong. Necessary and sufficient conditions for the concept of partial controllability for deterministic and stochastic linear control systems are almost perfectly found in a very analogous way of ordinary controllability and presented in the studies of Bashirov et al. (2007, 2010) with a very suitable examples. Thereafter, it is nearly fresh results, this concept is extended and well-motivated to semilinear deterministic systems and hence the only sufficient conditions of partially complete controllability for such systems are established and existed in very fresh work Bashirov and Jneid (2013) by means of contraction mapping theorem and Bashirov and Jneid (2014) using generalized contraction mapping theorem. Moreover, Bashirov and Noushin carried out the sufficient conditions for partially approximate controllability of semilinear control systems by using a different technique.

This dissertation is essentially intended to motivate partial controllability concepts and deeply emphasized on deriving a new series of sufficient conditions for deterministic semilinear systems. I do consider, for simplicity, only one kind of semilinear systems in this work. It is simply a basic semilinear deterministic control system given as

$$\begin{cases} x_t' = Ax_t + Bu_t + f(t, x_t, u_t), \ 0 < t \le T, \\ x_0 = \zeta \in X. \end{cases} \tag{1.0.1}$$

Hence, sufficient conditions of partially complete controllability for this system are given by means of contraction mapping theorem with an appropriate condition imposed on the Lipschitz coefficient of f. Whilst generalized contraction mapping theorem is not applicable to this system with this naturally reasonable conditions. Therefore, we play with the boundedness of f by adding an additional condition on it and then generalized contraction mapping theorem can be used to establish sufficient conditions of partially complete controllability for the control system (1.0.1). Both results are demonstrated on several useful examples.

The rest of this dissertation is organized as follows: In Chapter 2, we shortly display essential and useful facts from functional analysis which are very prerequisite for the following chapters. In Chapter 3, we present a large review of the basically existing results relating to the most important concepts of ordinary controllability for deterministic control systems in finite and infinite dimensional spaces. In Chapter 4, emphasis is placed on partial controllability concept for deterministic control systems, especially semilinear systems and it encompasses two sets of sufficient conditions for such control systems by using Contraction Mapping Theorem and its generalization. Finally, Chapter 5 is aimed to recap in a very few phrases the achievements of this dissertation.

Chapter 2

PRELIMINARIES

This chapter is devoted to providing a concise presentation of some essential facts from functional analysis that will be used later in the upcoming chapters. In fact, the main idea of putting it here is to provide with all basic information for clear reading the theorems in the following chapters. In other words, it contains an adequate packet of definitions, theorems, lemmas, corollaries, and remarks whereby the processes of explanation in next chapters will be easily realized. This chapter is composed mostly of six main sections. These sections are assigned to functional analysis and at a glance, useful facts are borrowed without proofs since they are included with thorough explanations in various books, one may prefer to the books Banach (1922), Yosida (1980), Siddigi (1986) and Kreyszig (1978).

2.1 Banach and Hilbert Spaces

In this thesis we frequently deal with Banach and Hilbert spaces. So, we shortly review some important facts about it.

Definition 2.1.1 (**Linear Space**) A linear space V on the field $\mathbb{R}$ is a set with two binary operations, called vector addition (+) defined on V and scalar multiplication ($\cdot$) defined from $\mathbb{R} \times V$ to V so that the following statements hold;

(1) (Commutativity) for all $u, v \in V$, $u + v = v + u$;

(2) (Associativity) for all $u,v,w \in V$, $(u + v) + w = u + (v + w)$;

(3) (Additive Inverse) for all $u \in V$, there is $-u \in V$, such that $(-u) + u = u + (-u) = 0$;

(4) (Additive Identity) for all $u \in V$, there exists $0 \in V$, such that $0 + u = u + 0 = u$;

(5) (Multiplicative Identity) For each $u \in V$ and $1 \in \mathbb{R}$, $1 \cdot u = u$;

(6) (Scalar Multiplicative Associativity) For all $k, l \in \mathbb{R}$ and $u \in V$, $l \cdot (k \cdot u) = (l \cdot k) \cdot u$;

(7) (Vector Distributivity) For all $k \in \mathbb{R}$ and $u, v \in V$, we have $k \cdot (u + v) = k \cdot u + k \cdot v$;

(8) (Scalar Distributivity) For all $k, l \in \mathbb{R}$ and $u \in V$, we have $(k + l) \cdot u = k \cdot u + l \cdot u$.

Definition 2.1.2 A mapping $\|\cdot\|$ from a linear space V to $\mathbb{R}^+ \cup \{0\}$

$$\|\cdot\| : V \to \mathbb{R}^+ \cup \{0\};\ x \longmapsto \|x\|,$$

possessing the three properties

(1) (Nonnegativity) $\|x\| > 0\ \forall\ x \in V$ and $\|x\| = 0 \Leftrightarrow x = 0$;

(2) (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|\ \forall\ x, y \in V$;

(3) (Positive Homogeneity) For every $k \in \mathbb{R}$, and $x \in V$, $\|k \cdot x\| = |k| \cdot \|x\|$.

is called a norm on V.

For any given norm $\|\cdot\|$, the distance from a vector x to a vector y can be simply defined by

$$d(x, y) = \|x - y\|$$

and $(V, \|\cdot\|)$ together is called a normed space. In what follows, if X is a linear space, a normed space $(X, \|\cdot\|)$ will be denoted by X.

Definition 2.1.3 Given a normed space X. If every Cauchy sequence in X is convergent in X, then X is said to be a Banach space.

Definition 2.1.4 Let V be a linear space on $\mathbb{R}$. The mapping which appoints to each couple $(x, y) \in V \times V$, a scalar in $\mathbb{R}$

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R};\ (x, y) \longmapsto \langle x, y \rangle$$

is called an inner product (scalar product) if it possesses the following three properties:

(1) (Nonnegativity) $\langle x, x \rangle \geq 0\ \forall\ x \in V$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;

(2) (Symmetry) For every $x, y \in V$, $\langle x, y \rangle = \langle y, x \rangle$;

(3) (Additivity and Homogeneity) $\forall\ x, y, z \in V$, and $l, k \in \mathbb{R}$, $\langle lx + ky, z \rangle = l\langle x, z \rangle + k\langle y, z \rangle$.

Following the definition of inner product, it is trivial to introduce a norm over V as $||x|| = \sqrt{\langle x, x \rangle}$. This norm will be called a norm generated by inner product $\langle \cdot, \cdot \rangle$. Moreover, a linear space V with a norm which comes from an inner product is called an inner product space.

Definition 2.1.5 A Hilbert space is a Banach space endowed with a norm generated by inner product. In other words, it is a complete inner product space.

Definition 2.1.6 If a Hilbert space X has a dense countable subset, then it is said to be separable.

2.2 Linear Operators

A linear operator is one of the essential parts in theory of controllability since the controllability operator plays a vital role in forming conditions for controllability. This section deals with some properties of linear operators. For more details I refer to the books Dunford and Schwartz (1959), , Li and Yong (1995) and Bashirov (2003).

Let K be an operator from a subspace $D(K)$ of a linear space X to a linear space Y, where $D(K)$ is the domain of K. Then the sets

$$\begin{cases} \mathrm{Ker}(K) & = \left\{x \in D(K) : Kx = 0\right\} \\ R(K) & = \{Kx : x \in D(K)\} \end{cases} \tag{2.2.1}$$

are called a kernel, and range of K, respectively.

Definition 2.2.1 Let X and Y be linear spaces. A linear operator K from X to Y is defined as a function from $D(K) \subseteq X$ into Y having the following properties

(1) (Denseness of Domain) $\overline{D(K)} = X$;

(2) (Linearity) For every $\alpha, \beta \in \mathbb{R}$, and $x, y \in X$, we have $K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$.

Definition 2.2.2 Let X and Y be Banach spaces. A linear operator $K : D(K) \subseteq X \longrightarrow Y$ is said to be bounded if

(1) (Denseness of Domain) $\overline{D(K)} = X$;

(2) (Boundedness) $\exists a > 0$ s.t. $\forall x \in X$, $||Kx|| \leq a||x||$.

Remark 2.2.3 Any linear bounded operator $K : D(K) \subseteq X \longrightarrow Y$ has a unique linear bounded extension $\tilde{K} : X \longrightarrow Y$ which preserves the norm

$$||\tilde{K}|| = ||K|| = \sup_{||x|| \neq 0} \frac{||Kx||}{||x||} \tag{2.2.2}$$

so that we can always define $K : X \longrightarrow Y$ without any modifications.

Definition 2.2.4 Let X and Y be Banach spaces. A linear operator $K : D(K) \subseteq X \longrightarrow Y$ is said to be closed if

(1) (Denseness of Domain) $\overline{D(K)} = X$;

(2) (Closedness) For any sequence $\{x_n\}$ in $D(K)$, $x_n \to x$ and $Kx_n \to z$ imply $x \in D(K)$ and $Kx = z$.

Proposition 2.2.5 Suppose that X and Y are two Banach spaces. If K is linear operator from X to Y, then the following statements are equivalent:

(i) K is continuous on X, i.e. $\lim_{x \to x_0} ||Kx - Kx_0|| = 0$;

(ii) K is bounded.

Let $\mathcal{L}(X, Y)$ be the class of all linear bounded operators from X into Y. Define the sum and product by real number in $\mathcal{L}(X, Y)$ as follows

(i) $(K + L)x = Kx + Lx$

(ii) $(tK)x = t(Kx)$

Under these operations, $\mathcal{L}(X, Y)$ is obviously a linear space. Furthermore, if we introduce a norm as follows

$$||K|| = \sup_{||x|| \leq 1} ||Kx|| = \sup_{x \neq 0} \frac{||Kx||}{||x||}, \quad \forall K \in \mathcal{L}(X, Y), \tag{2.2.3}$$

then $\mathcal{L}(X, Y)$ becomes a Banach space.

2.3 Adjoint Operator

The space $\mathcal{L}(X, \mathbb{R})$ is well-known as a dual space of X and denoted by $X^* = \mathcal{L}(X, \mathbb{R})$. An element $f \in \mathcal{L}(X, \mathbb{R})$ is called a linear bounded functional. Now, for any given two Banach spaces X and Y, let K be in $\mathcal{L}(X, Y)$. Then, the function K^* operating from the Banach space Y^* into the Banach space X^*, defined by

$$(K^* y^*)x = y^*(Kx) \quad \forall y^* \in Y^*, \ x \in X, \tag{2.3.1}$$

is called the adjoint of K. Obviously, K^* is linear and bounded. In the case, when X and Y are Hilbert spaces, the adjoint operator K^* is defined by

$$\langle Kx, y\rangle_Y = \langle x, K^*y\rangle_X \quad \forall x \in X,\ y \in Y. \tag{2.3.2}$$

Definition 2.3.1 Assume that X is a Hilbert space and $K \in \mathcal{L}(X,X) = \mathcal{L}(X)$. If $K^* = K$, then K is called self-adjoint. If, K is self-adjoint, then K is said to be

(1) Nonnegative, if $\forall\ z \in X$, $\langle Kz, z\rangle \geq 0$;

(2) Positive, if $\forall\ z \in X$ with $z \neq 0$, $\langle Kz, z\rangle > 0$;

(3) Coercive, if there is $\delta > 0$ such that $\langle Kz, z\rangle \geq \delta ||z||^2\ \forall\ z \in X$.

For simplification, we write $K \geq 0$ (respectively, $K > 0$) if K is nonnegative (respectively positive). We can present the norm of nonnegative operator K by one of the following formulas:

$$||K|| = \sup_{||z||=1} ||Kz|| = \sup_{||z||=1} \langle Kz, z\rangle$$

Theorem 2.3.2 (Riesz) Let X be a Hilbert space. Then, $X^* = X$. More precisely, for every $f \in X^*$, there is a $y \in X$, such that $f(z) = \langle z, y\rangle$ for all $z \in X$.

For any $B \subset X$, where X is Hilbert space, we define $B^\perp = \left\{z \in X : \langle z, y\rangle = 0,\ \forall y \in B\right\}$. $B^\perp$ is well-known as orthogonal complement of B.

Proposition 2.3.3 Let X, Y and Z be Hilbert spaces. Then, for all $K \in \mathcal{L}(X,Y)$ the following properties hold:

(1) K is invertible and $K^{-1} \in \mathcal{L}(Y,X)$ if and only if K^* is invertible and $(K^*)^{-1} \in \mathcal{L}(Y^*,X^*)$. Moreover, $(K^*)^{-1} = (K^{-1})^*$;

(2) $||K^*|| = ||K||$;

(3) If K is closed, then $(K^*)^* = K$;

(4) $N(K) = \mathrm{R}(K^*)^\perp$ and $N(K^*) = \mathrm{R}(K)^\perp$;

(5) $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y^*,X^*)$ are isometric.

Definition 2.3.4 Suppose that X and Y are two Banach spaces. Let $\{K_n\}$ be a sequence from $\mathcal{L}(X,Y)$. Then,

(1) K_n is uniformly convergent to $K \in \mathcal{L}(X,Y)$ if $||K_n - K||_{\mathcal{L}} \to 0$ as $n \to \infty$;

(2) K_n is strongly convergent to $K \in \mathcal{L}(X,Y)$ if $||K_n z - Kz||_Y \to 0$ as $n \to \infty$ for every $z \in X$;

(3) K_n is weakly convergent to $K \in \mathcal{L}(X,Y)$ if $y^*((K_n - K)z) \to 0$ as $n \to \infty$ for every $z \in X$ and $y^* \in Y^*$.

2.4 Basic Results from Functional Analysis

In this section we review most useful definitions, theorems and lemmas from functional analysis. They are concerning the concepts of controllability.

Definition 2.4.1 (Contraction Mapping) Let X be a Banach space and K be an operator mapping X into itself. K is called a contraction mapping if there is $0 \leq b < 1$ such that for every $y, z \in X$ we have

$$||K(y) - K(z)|| \leq b||y - z||.$$

Theorem 2.4.2 (Contraction Mapping Theorem) (Banach, 1922) Assume that X is a Banach space and $K : X \longrightarrow X$ is a contraction mapping. Then K has exactly one fixed point. More precisely, there exists a unique $x_0 \in X$ such that $K(x_0) = x_0$.

Theorem 2.4.3 (Generalized Contraction Mapping Theorem) Let X be a Banach space and K be a nonlinear operator mapping X into itself. Let $K^1 = K$, $K^2 = K \circ K$, $\cdots$, $K^n = K^{n-1} \circ K$ for any given $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, K^n is a contraction mapping, then K has exactly one fixed point in X.

Theorem 2.4.4 (Fubini's Theorem) Assume $k : D = [a,b] \times [c,d] \to \mathbb{R}$ is integrable with respect to total variable $(x,y) \in [a,b] \times [c,d]$. If for all $y \in [c,d]$, $k(x,y)$ is integrable in respect of $x \in [a,b]$, and $\int_a^b k(x,y)\,dx$ as a function of y is integrable on $[c,d]$, then

$$\int_D k(x,y)\,dD = \int_c^d \Big(\int_a^b k(x,y)\,dx \Big) dy = \int_a^b \Big(\int_c^d k(x,y)\,dy \Big) dx.$$

Furthermore, if $k(x,y)$ is given as the product of two independent functions $k(x,y) = h(x)g(y)$, then

$$\int_D k(x,y)\,dD = \Big(\int_a^b h(x)\,dx \Big) \Big(\int_c^d g(y)\,dy \Big).$$

The proof of the following three theorems can be found with required details in the book of Curtain and Pritchard (1978).

Theorem 2.4.5 Let X, Y and Z be Hilbert spaces and let $K \in \mathcal{L}(X,Z)$ and $L \in \mathcal{L}(Y,Z)$. Then, the following statements are equivalent:

(1) $\mathrm{R}(K) \subset \mathrm{R}(L)$;

(2) there is $\delta > 0$, such that

$$\|K^* x\| \leq \delta \|L^* x\| \quad \forall x \in Z.$$

Theorem 2.4.6 Let X, Y and Z be Hilbert spaces and let $K \in \mathcal{L}(X,Z)$ and $L \in \mathcal{L}(Y,Z)$. Then, the following statements are equivalent:

(1) $\overline{R(K)} \subset \overline{R(L)}$;

(2) $\mathrm{Ker}L^* \subset \mathrm{Ker}K^*$.

Theorem 2.4.7 (Orthogonal Decomposition) Let X be a Hilbert space. Then for every subspace $M \subset X$, the following identity holds

$$X = M^{\perp} \oplus \overline{M} = M^{\perp} \oplus M^{\perp\perp}.$$

Moreover,

$$\overline{M} = X \Leftrightarrow M^{\perp} = \{0\}.$$

For instance, if for $K \in \mathcal{L}(X,X)$ we assume $M = R(K)$, then

$$X = \overline{R(K)} \oplus \mathrm{Ker}K^*.$$

Lemma 2.4.8 (Holder's Inequality) Suppose that $f \in L_p(c,d)$ and $g \in L_q(c,d)$. Then, the following inequality is true

$$\int_c^d |f(r)g(r)|\,dr \leq \Big(\int_c^d |f(r)|^p\,dr\Big)^{\frac{1}{p}}\Big(\int_c^d |g(r)|^q\,dr\Big)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The case when $p = q = 2$ this inequality is well-known as the Cauchy-Schwartz inequality.

Lemma 2.4.9 (Gronwall's Inequality) Assume that f is a nonnegative function on $[c,d]$, satisfying

$$f(t) \leq g(t) + \delta \int_a^t f(r)\,dr, \quad c \leq t \leq d,$$

where $\delta \geq 0$ and g is integrable on $[c,d]$. Then,

$$f(t) \leq g(t) + \delta \int_a^t e^{\delta(t-s)} g(s)\,ds.$$

2.5 C_0-semigroups and Resolvent Operators

Semigroups play a significant role in the controllability concept so that we assigned this section to review some basic facts about this topic. For further information, one can refer for this book Pazzy (1983).

Definition 2.5.1 Let X be a Banach space. The collection $\{\mathcal{T}_t : \mathcal{T}_t \in \mathcal{L}(X),\ 0 \leq t < \infty\}$ is called a strongly continuous semigroup (or simply C_0-semigroup) if for every $t, s \geq 0$ and $x \in X$ the following hold

(i) $\mathcal{T}_0 = I$;

(ii) $\mathcal{T}_{t+s} = \mathcal{T}_t \mathcal{T}_s$;

(iii) $\lim_{t\to 0^+} \|\mathcal{T}_t x - x\| = 0$.

Where I is the identity operator on X. The second property (ii) is known as a semigroup property and the last property (iii) refers to the strong continuity. If $\lim_{t\to 0^+} \|\mathcal{T}_t - I\| = 0$, then the semigroup $\mathcal{T}_t$ is said to be uniformly continuous.

Definition 2.5.2 Let $\mathcal{T}_t$ be a semigroup on Banach space X. Then a linear operator A

is said to be an infinitesimal generator of $\mathcal{T}_t$ if

$$Az = \lim_{t\to 0^+} \frac{\mathcal{T}_t z - z}{t} = \frac{d^+\mathcal{T}_t z}{dt}, \quad \forall z \in D(A),$$

where

$$D(A) = \left\{ z \in X : \lim_{t\to 0^+} \frac{\mathcal{T}_t z - z}{t} \text{ exists} \right\}.$$

Theorem 2.5.3 Let $\mathcal{T}_t$ be a C_0-semigroup on a Banach space X, with infinitesimal generator A. Then,

(i) $z_0 \in D(A)$ yields $\mathcal{T}_t z_0 \in D(A)$ $\forall$ $t \geq 0$;

(ii) $\frac{d\mathcal{T}_t z}{dt} = A\mathcal{T}_t z = \mathcal{T}_t Az$, $\forall$ $z \in D(A)$, $t > 0$;

(iii) A is closed and $\overline{D(A)} = X$;

(iv) $\mathcal{T}_t z - z = \int_0^t \mathcal{T}_s Az\, ds$ $\forall$ $z \in D(A)$.

Remark 2.5.4 If $A \in \mathcal{L}(X)$, then the C_0-semigroup $\mathcal{T}_t$ generated by A can be explicitly expressed as

$$\mathcal{T}_t = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = e^{At}.$$

Therefore, the semigroup generated by a closed operator A is also denoted by e^{At}.

Proposition 2.5.5 Let A be linear closed operator on a Banach space X. If $\mathcal{T}_t$ is the C_0-semigroup generated by A, then $\mathcal{T}_t^*$ is a semigroup on X^*. If additionally, X is a Hilbert space, $\mathcal{T}_t^*$ becomes a C_0-semigroup on X^* with the generator A^*, that is, $\mathcal{T}_t^* = e^{A^* t}$.

Definition 2.5.6 (Resolvent of A**)** Let A and X be given as in previous preposition. The set of all complex numbers λ whereby $\lambda I - A$ is nonsingular (invertible), is called the resolvent set of A and denoted by $\rho(A)$. The set of the operators $(\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ is called the resolvent of A and denoted by $R(\lambda, A)$.

2.6 Review of Evolution Equations

Let X be a Banach space, and $f \in L_1(0,T;X)$. Consider a linear system

$$\begin{cases} x_t' = Ax_t + f(t), \ 0 < t \leq T, \\ x(0) = x_0 \in X, \end{cases} \tag{2.6.1}$$

where A is a bounded operator which generates a C_0-semigroup $\mathcal{T}_t = e^{At}$ on X.

Definition 2.6.1 A function, $x \in C(0,T;X)$, is called

(1) A strong solution of (2.6.1) if it has the following properties

- x is strongly differentiable almost everywhere on $[0,T]$;
- $x(t) \in D(A)$ for almost every $t \in [0,T]$;
- x satisfies the equation (2.6.1) almost everywhere with $x(0) = x_0$.

(2) A weak solution of (2.6.1) if for every $y^* \in D(A^*)$, $\langle x(\cdot), y^* \rangle$ is absolutely continuous on $[0,T]$ and

$$\langle x_t, y^* \rangle = \langle x_0, y^* \rangle + \int_0^t \Big(\langle x(r), A^* y^* \rangle + \langle f(r), y^* \rangle \Big) dr, \ \forall t \in [0,T].$$

(3) A mild solution of (2.6.1) if for every $t \in [0,T]$

$$x_t = e^{At} x_0 + \int_0^t e^{A(t-r)} f(r) dr.$$

Proposition 2.6.2 Let A be a generator of a C_0-semigroup on X. Then the system (2.6.1) has a mild solution if and only if it has a weak solution.

Now, consider a basic semilinear system

$$\begin{cases} x_t' = Ax_t + f(t,x_t), \ 0 < t \le T, \\ x(0) = x_0 \in X. \end{cases} \tag{2.6.2}$$

Theorem 2.6.3 Assume that $f : [0,T]\times X \longrightarrow X$ satisfies the following assumptions:

(1) $f(\cdot,x)$ is strongly measurable for every fixed $x \in X$;

(2) there exists $K \in L_1(0,T;\mathbb{R})$ so that

$$\begin{cases} \|f(t,x) - f(t,y)\| \le K(t)\|x-y\|, \\ \|f(t,0)\| \le K(t), \end{cases} \tag{2.6.3}$$

for all $x,y \in X$ and $t \in [0,T]$.

Then the semilinear system (2.6.2) has a unique mild solution $x \in C(0,T;X)$.

Theorem 2.6.4 [41] Let X be a Banach space. If the function f is continuous in t and Lipschitz in respect of the second variable, i.e. there is a positive constant C such that

$$\|f(t,y) - f(t,z)\| \le C\|y-z\|,$$

then the semilinear system (2.6.2) admits exactly one mild solution $x \in C(0,T;X)$.

Chapter 3

LITERATURE SURVEY

This chapter is dedicated to display a brief discussion about the most important concepts of controllability for deterministic control systems in finite and infinite dimensions. Actually, these results are heavily studied and adequately discussed in plentiful works (see, for example Kalman (1960), Triggiani (1975), Klamka and Socha (1977, 1980), Curtain and Pritchard (1978), Alekseev and Tikhomirov, and Fomin (1979), Zabczyk (1995) and Bashirov and Mahmudov (1999a) etc.). Two main sections are included in this chapter. In the first section, the conditions for complete and approximate controllability of deterministic linear systems in infinite and finite dimensional spaces are reviewed with some common examples. However, the second section is concentrated on sufficient conditions of complete and approximate controllability for semilinear deterministic systems.

3.1 Controllability Concepts for Linear Deterministic Systems

The necessary and sufficient conditions of complete and approximate controllability for linear deterministic systems are reviewed in this section. The proofs therein may be found in various papers since these notions of controllability for linear systems are vastly investigated by so many authors as it is mentioned above.

3.1.1 Complete Controllability of Linear Systems

Consider the following initial value linear system

$$\begin{cases} x_t' = Ax_t + Bu_t, \ 0 < t \leq T, \\ x_0 = \zeta \in X, \end{cases} \tag{3.1.1}$$

where x and u are state and control processes, respectively. Throughout this section we impose the following statements

(A) X and U are separable Hilbert spaces;

(B) A is a densely defined closed linear operator on X, generating a C_0-semigroup e^{At}, $t \geq 0$;

(C) B is a bounded linear operator from U to X;

(D) $U_{\text{ad}} = L_2(0,T;U)$ is the space of equivalence classes of all Lebesgue measurable and square integrable functions from $[0,T]$ to U (in theory of controllability, it is well-known as a set of admissible controls or sometimes called a transitive set).

Then, under the conditions (A)–(D) the system (3.1.1) admits a unique mild solution given by

$$x_t = e^{At}\zeta + \int_0^t e^{A(t-s)}Bu_s\,ds. \tag{3.1.2}$$

Now, for each $0 \leq t \leq T$, let us introduce the reachable set as follows

$$D_{\zeta,t} = \{h \in X : \exists u \in U_{\text{ad}} \text{ such that } h = x_t\}. \tag{3.1.3}$$

Definition 3.1.1 The system (3.1.1) is said to be completely controllable for the positive time T if for a given initial state value $\zeta \in X$ and arbitrary state value $x_1 \in X$, there exists a control $u \in L_2(0,T;U)$ whereby the solution x of control system (3.1.1) satisfies $x_T = x_1$. In a brief form, that merely means $X = D_{\zeta,T}$.

From now on, D^c-controllability would be stood for the complete controllability for the positive time T. Moreover, D^c_s-controllability would be stood for the complete controllability on $[0,s]$ for $0 < s \leq T$.

Let the linear operator Q on X be defined as

$$Q_t = \int_0^t e^{Ar} BB^* e^{A^* r}\, dr, \tag{3.1.4}$$

and Λ_t by

$$\Lambda_t : L_2(0,t;U) \longrightarrow X, \;\; \Lambda_t u = \int_0^t e^{Ar} Bu(r)\, dr. \tag{3.1.5}$$

for all $0 \leq t \leq T$. Clearly, Λ_t has an adjoint operator $\Lambda_t^* : X \longrightarrow L_2(0,t;U)$ given by

$$[\Lambda_t^*(x)](r) = B^* e^{A^*(r)} x, \; 0 \leq r \leq t. \tag{3.1.6}$$

Obviously, $Q_t = \Lambda_t \Lambda_t^*$. Furthermore, $(\Lambda_t \Lambda_t^*)^* = (\Lambda_t^*)^* \Lambda_t^* = \Lambda_t \Lambda_t^*$ and hence $Q_t^* = Q_t$. This means that the controllability operator is self-adjoint. Clearly, from the equality $Q_t = \Lambda_t \Lambda_t^*$, it can be easily shown that Q_t is nonnegative and hence the resolvent operator, $R(\lambda, -Q_t)$ is well-defined for all $\lambda > 0$.

Lemma 3.1.2 Given $T > 0$. If Λ_t and $D_{\zeta,t}$ are defined as above, then

$$R(\Lambda_t) + e^{At}\zeta = D_{\zeta,t} \;\text{ for all } 0 < t \leq T. \tag{3.1.7}$$

Proof. It clear that $\forall x \in D_{\zeta,t}, \zeta \in X$, $\exists u \in U_{\text{ad}}$ such that $x_t = e^{At}\zeta + \int_0^t e^{Ar}Bu_r\,dr = e^{At}\zeta + \Lambda_t(u)$. This gives $D_{\zeta,t} \subseteq R(\Lambda_t) + e^{At}\zeta$. Moreover, $\forall x \in R(\Lambda_t)$, $\exists u \in U_{\text{ad}}$ such that $\Lambda_t(u) = \int_0^t e^{Ar}Bu_r\,dr$ and $e^{At}\zeta + \Lambda_t(u) = x_t \in D_{\zeta,t}$. Therefore, $R(\Lambda_t) + e^{At}\zeta \subseteq D_{\zeta,t}$, proving the lemma. ■

Theorem 3.1.3 [10, 11, 22, 23, 41] Under conditions (A)–(D) and above notation, for all $T > 0$ and $x \in X$, the following assertions are equivalent:

(a) The system (3.1.1) is D^c-controllable;

(b) $R(\Lambda_T) = X$;

(c) Q_T is coercive;

(d) $\int_0^T \|[\Lambda^* x](s)\|_U^2\,ds = \|\Lambda_T^* x\|_{L_2}^2 \geq \gamma\|x\|^2$;

(e) $\text{Ker}(\Lambda_T^*) = 0$ and $R(\Lambda_T^*)$ is closed;

(f) $R(\lambda, -Q_T)$ converges uniformly as $\lambda \longrightarrow 0^+$;

(g) $R(\lambda, -Q_T)$ converges strongly as $\lambda \longrightarrow 0^+$;

(h) $R(\lambda, -Q_T)$ converges weakly as $\lambda \longrightarrow 0^+$;

(i) $\lambda R(\lambda, -Q_T) \longrightarrow 0$ uniformly as $\lambda \longrightarrow 0^+$.

Proof. The proof is long so that we prefer to separate it into two parts: (a)⇒(b) ⇒(c) ⇒(d) ⇒ (e) ⇒ (a) and (a) ⇒(f)⇒(g) ⇒(h)⇒ (i)⇒ (a). Let us get off on the first part.

Begin with (a)⇒ (b). If (a) is true, then by the definition of controllability $D_{\zeta,T} = X$. In virtue of Lemma 3.1.2 this gives that $R(\Lambda_T) = X$ since $e^{AT}\zeta$ is fixed in X for a given $\zeta \in X$. Hence (b) follows.

(b)⇒(c). By Lemma 3.1.2 and the equality $R(\Lambda_T) = X$, one can easily obtain $R(\Lambda_T) = D_{\zeta,T}$. According to Theorem 2.4.5 if we let $X = Z$, $Y = U_{\text{ad}}$, $K = I$, and $L = \Lambda_T$, then

$$
\begin{aligned}
X \subseteq D_{\zeta,T} &\implies R(I) \subseteq R(\Lambda_T) \\
&\implies \forall x \in X,\ \exists \delta > 0 \text{ such that } \|\Lambda_T^*(x)\|^2 \geq \frac{\|x\|^2}{\delta} \\
&\implies \forall x \in X,\ \langle \Lambda_T^*(x), \Lambda_T^*(x) \rangle \geq \frac{\|x\|^2}{\delta} \\
&\implies \forall x \in X,\ \langle Q_T(x), x \rangle \geq \frac{\|x\|^2}{\delta} \\
&\implies Q_T \text{ is coercive.}
\end{aligned}
$$

Hence (c) holds.

For the implication (c)⇒(d), we have

$$
\begin{aligned}
\langle Q_T x, x \rangle &= \langle \Lambda_T \Lambda_T^* x, x \rangle \\
&= \left\langle \int_0^T e^{Ar} BB^* e^{A^* r} x\, dr, x \right\rangle \\
&= \|\Lambda_T^* x\|_{L_2}^2.
\end{aligned}
$$

Then, using this identity and assertion (c) we obtain $\|\Lambda_T^* x\|_{L_2}^2 \geq \gamma \|x\|^2$. Therefore, (d) holds.

Next, show that (d)⇒ (e). In accordance with the Theorem 2.4.5 if we assume $X = Z$, $Y = U_{\text{ad}}$, $K = I$, and $L = \Lambda_T$, the condition

$$
\|\Lambda_T^* x\|_{L_2}^2 = \int_0^T \|[\Lambda_T x](s)\|_U^2\, ds \geq \gamma \|x\|_X^2.
$$

is equivalent to

$$X \subset R(\Lambda_T).$$

Therefore, $X = R(\Lambda)$. Now, using Theorem 2.4.7 (orthogonal decomposition), $R(\Lambda_T)^{\perp} = \{0\}$ and since $\mathrm{Ker}(\Lambda_T^*) = R(\Lambda_T)^{\perp}$ then $\mathrm{Ker}(\Lambda_T^*) = 0$ and $R(\Lambda_T^*)$ is closed. Therefore (e) holds.

To complete the first part, it remains to show that (e)⇒(a). Let (e) be true. By Theorem 2.4.7, $X = R(\Lambda_T)$. Using Lemma 3.1.2 and the equality $X = R(\Lambda_T)$ one can easily obtain $D_{\zeta,T} = X$ for all $\zeta \in X$. So (e)⇒(a).

Now, moving on to the second part of equivalence. To start with (a)⇒(f). Let (a) be true. Then Q_T is coercive. Therefore, for every $\lambda \geq 0$ and $x \in X$, there exists $\gamma > 0$ so that

$$\langle x, (\lambda I + Q_T)x\rangle = \lambda\|x\|^2 + \langle x, Q_T x\rangle \geq (\lambda + \gamma)\|x\|^2. \tag{3.1.8}$$

Clearly, $(\lambda I + Q_T)$ is a nonnegative bounded operator on X, it follows from Chapter 2 (see (2.3)) that

$$\|(\lambda I + Q_T)\| = \sup_{\|x\|=1} \langle x, Q_T x\rangle \geq (\lambda + \gamma).$$

Using properties of operator norm we obtain

$$\|(\lambda I + Q_T)^{-1}\| \leq \frac{1}{(\lambda + \gamma)} \leq \frac{1}{\gamma}.$$

Thus, for some $\gamma > 0$ the following inequality holds

$$\|R(\lambda, -Q_T)\| \leq \frac{1}{\gamma}. \tag{3.1.9}$$

Moreover, using (3.1.9) and the equality $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$, we obtain

$$\begin{aligned}
\|R(\lambda, -Q_T) - Q_T^{-1}\| &= \|(\lambda I + Q_T)^{-1} - Q_T^{-1}\| \\
&= \|(\lambda I + Q_T)^{-1}(Q_T - \lambda I - Q_T)Q_T^{-1})\| \\
&\leq \lambda \|(\lambda I + Q_T)^{-1}\| \cdot \|Q_T^{-1}\|. \\
&\leq \frac{\lambda}{\gamma^2} \text{ for all} \lambda \geq 0 \text{ and some } \gamma > 0.
\end{aligned}$$

Finally, by taking $\lambda \to 0^+$ we get $R(\lambda, -Q_T) \to Q_T^{-1}$ uniformly and (f) follows.

Borrowing the properties of convergent sequence of operators from Chapter 2 (see Definition 2.3.4), the implications (f)⇒(g) and (g)⇒(h) are trivial.

For (h)⇒(i), it comes straightforward from the boundedness of a weakly convergent sequence of operators.

(i)⇒(a). Assume (i) holds. This means that

$$\lambda \|(\lambda I + Q_T)^{-1}\| \to 0 \text{ as } \lambda \to 0^+. \tag{3.1.10}$$

By applying square root on both side of (3.1.10), we obtain

$$(\lambda)^{\frac{1}{2}} \|(\lambda I + Q_T)^{-\frac{1}{2}}\| \to 0 \text{ as } \lambda \to 0^+.$$

For a given $\epsilon = \frac{1}{\sqrt{2}}$, we can find a sufficiently small λ_1 so that

$$(\lambda_1)^{\frac{1}{2}} \|(\lambda_1 I + Q_T)^{-\frac{1}{2}}\| \leq \frac{1}{\sqrt{2}}. \tag{3.1.11}$$

Now, using (3.1.11), for every $x \in X$ we have

$$\begin{aligned}||x||^2 &= ||((\lambda_1)^{-\frac{1}{2}}(\lambda_1 I + Q_T)^{\frac{1}{2}})((\lambda_1)^{\frac{1}{2}}(\lambda_1 I + Q_T)^{-\frac{1}{2}})x||^2 \\ &\leq \frac{1}{2}||((\lambda_1)^{-\frac{1}{2}}(\lambda_1 I + Q_T)^{\frac{1}{2}})x||^2 \\ &= \frac{1}{2}\langle(\lambda_1)^{-1}(\lambda_1 I + Q_T)x, x\rangle.\end{aligned}$$

Which implies that

$$\langle Q_T x, x\rangle \geq \lambda_1 ||x||^2 \text{ for all x} \in \text{X}.$$

This yields that Q_T is coercive and by the first part of equivalences, (a) holds. This accomplishes the proof. ■

Theorem 3.1.4 [22] The linear control system (3.1.1) is D^c-controllable if and only if Q_T has a bounded inverse.

Proof. To start with the necessary condition, let (3.1.1) be D^c-controllable. By Theorem 3.1.3 Q_T is coercive. This means that for every $x \in X$

$$\langle Q_T x, x\rangle \geq \gamma ||x||^2 \text{ for some } \gamma > 0.$$

In particular, $Q_T \geq 0$ and $Q_T^{\frac{1}{2}}$ exists as an operator in $\mathcal{L}(X)$. Then

$$\begin{aligned}\langle Q_T^{\frac{1}{2}} Q_T^{\frac{1}{2}} x, x\rangle \geq \gamma||x||^2 &\implies \langle Q_T^{\frac{1}{2}} x, Q_T^{\frac{1}{2}} x\rangle \geq \gamma||x||^2 \\ &\implies ||Q_T^{\frac{1}{2}} x||^2 \geq \gamma||x||^2 \\ &\implies ||Q_T^{\frac{1}{2}} x|| \geq \sqrt{\gamma}||x||.\end{aligned}$$

Hence $Q_T^{-\frac{1}{2}} = (Q_T^{\frac{1}{2}})^{-1}$ exists. If we let $Q_T^{-1} = Q_T^{-\frac{1}{2}} Q_T^{-\frac{1}{2}}$ then

$$Q_T^{-1} Q_T = Q_T^{-\frac{1}{2}} (Q_T^{-\frac{1}{2}} Q_T^{\frac{1}{2}} Q_T^{\frac{1}{2}}) = Q_T^{-\frac{1}{2}} Q_T^{\frac{1}{2}} = I \tag{3.1.12}$$

Also

$$Q_T Q_T^{-1} = Q_T^{\frac{1}{2}} (Q_T^{\frac{1}{2}} Q_T^{-\frac{1}{2}} Q_T^{-\frac{1}{2}}) = Q_T^{\frac{1}{2}} Q_T^{-\frac{1}{2}} = I \tag{3.1.13}$$

By (3.1.12) and (3.1.13), Q_T^{-1} defined as $Q_T^{-\frac{1}{2}} Q_T^{-\frac{1}{2}}$ is a bounded inverse of Q_T .

For sufficient condition, let the control u be taken as

$$u(t) = B^* e^{A^*(T-t)} Q_T^{-1} (h - e^{AT} \zeta) \text{ for all } 0 \leq t \leq T, \tag{3.1.14}$$

where, h is any given state value in X and ζ is the initial state value in X. It is clear that $u \in L_2(0, T; U)$. Substituting u defined in (3.1.14) into (3.1.1) we get

$$x_t = e^{At} \zeta + Q_T^{-1} (h - e^{AT} \zeta) \int_0^t e^{A(t-s)} B B^* e^{A^*(T-s)} \, ds.$$

At $t = T$, $x_T = h$ which proves the system (3.1.1) is D^c-controllable since h was selected as an arbitrary state from X. ■

Example 3.1.5 Consider the system of linear differential equations

$$\begin{cases} x_t' = y_t, \; x_0 = 0, \\ y_t' = -x_t + u_t, \; y_0 = 1, \; 0 < t \leq 2\pi. \end{cases} \tag{3.1.15}$$

This system can easily be re-expressed as the standard form of the linear system given in this thesis

$$z_t' = A z_t + B u_t, \; 0 < t \leq 2\pi, \tag{3.1.16}$$

where,

$$z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ Z_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.1.17}$$

Now, to apply Theorem 3.1.3 of complete controllability we need to find the controllability operator Q_T. Since the operator A is a matrix, one can use algebraical method to find the fundamental matrix e^{At} (C_0-semigroup in finite dimensional space which is generated by a matrix A)

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

and obviously

$$e^{A^* t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Then, Q_T can be calculated as

$$\begin{aligned}
Q_T &= \int_0^{2\pi} e^{Ar} B B^* e^{A^* r}\, dr \\
&= \int_0^{2\pi} \begin{bmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix} dr \\
&= \int_0^{2\pi} \begin{bmatrix} \sin^2 r & \sin r \cos r \\ \sin r \cos r & \cos^2 r \end{bmatrix} dr \\
&= \pi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

which gives that the operator Q_T is coercive and the system (3.1.16) is D^c-controllable.

3.1.2 Approximate Controllability of Linear Systems

As Triggiani in 1975 established, the control systems can never be completely controllable in infinite dimensional space whenever the linear operator B is compact or the semigroup generated by A is compact (see Triggiani (1975) and Bensoussan (1993)). For this reason, we can relax the complete controllability and obtain a new concept of controllability suitable for wider types of systems, especially in infinite dimensional cases. Indeed, this notion is more flexible, owns a wide range of applications and it is called approximate controllability. In this section, a necessary and sufficient condition of approximate controllability for the deterministic linear systems are derived with a short investigation.

Definition 3.1.6 The control system (3.1.1) is said to be approximately controllable for the positive time T if $\overline{D_{\zeta,T}} = X$.

In what follows, the control system (3.1.1) will be called D^a-controllable if $\overline{D_{\zeta,T}} = X$ for the time $T > 0$. In addition, it will be called D^a_t-controllable for every $0 < t \leq T$ if $\overline{D_{\zeta,t}} = X$ for every $0 < t \leq T$.

Theorem 3.1.7 [22] The linear control system (3.1.1) is D^a-controllable if, and only if,

$$B^* e^{A^* t} z = 0 \;\text{ for all } 0 \leq t \leq T \;\text{ implies }\; z = 0. \tag{3.1.18}$$

Proof. Let $X = Z$, $Y = U_{\text{ad}}$, $K = I$, and $L = \Lambda_T$. By Theorem 2.4.6, the condition of approximate controllability $X = \overline{R(I)}$ is equivalent to

$$\text{Ker}(\Lambda_T^*) \subset \text{Ker}(I^*) = \{0\}. \tag{3.1.19}$$

Since $R(\Lambda_T^{\perp}) = \mathrm{Ker}(\Lambda_T^*)$, using Theorem 2.4.7 and (3.1.19) we obtain

$$\overline{R(\Lambda_T)} = X. \tag{3.1.20}$$

Using Lemma 3.1.2 this implies that $\overline{D_\zeta, T} = X$. Then it obvious that $\mathrm{Ker}(\Lambda_T^*) = \{0\}$ is equivalent to

$$B^* e^{A^* t} z = 0 \text{ for every } 0 \le t \le T \text{ yields } z = 0.$$

Therefore, theorem is proved. ■

Theorem 3.1.8 [22, 41] For any given $T > 0$, the following assertions are equivalent:

(i) The linear system (3.1.1) is D^a-controllable.

(ii) $\overline{\Lambda_T(U_{\mathrm{ad}})} = X$. i.e., the range of Λ_T is dense in X.

(iii) $\mathrm{Ker}(\Lambda_T^*) = \{0\}$. i.e., the linear operator Λ_T^* is one to one.

(iv) $Q_T > 0$. i.e., the controllability operator Q_T is positive.

Proof. For (i)⇒(ii). Given arbitrary $\zeta, y \in X$ and positive time T. Then, it is clear that $y + e^{AT}\zeta$ is an element in X. Now, using the definition of approximate controllability, there exists a control $u \in U_{\mathrm{ad}}$, so that for all $\epsilon > 0$

$$\|y - \Lambda_T(u)\| = \|y + e^{AT}\zeta - x_T\| < \epsilon,$$

which implies that $y \in \overline{\Lambda_T(U_{\mathrm{ad}})}$, and since y was chosen arbitrary it follows that $X \subseteq \overline{\Lambda_T(U_{\mathrm{ad}})}$. Then $\overline{\Lambda_T(U_{\mathrm{ad}})} = X$. This proves (i)⇒(ii).

Next, the equivalence (ii)⇔(iii) comes straightforward from Theorem 2.4.7 since $\mathrm{Ker}(\Lambda_T^*) = R(\Lambda_T^{\perp})$.

(iii)⇔(iv). Using the identities $\langle Q_T x, x\rangle = \|\Lambda_T^* x\|^2$ and $\mathrm{Ker}(\Lambda_T^*) = R(\Lambda_T^{\perp})$, we obtain

$$Q_T > 0 \Leftrightarrow \mathrm{Ker}(\Lambda_T^*) = 0. \tag{3.1.21}$$

Therefore, (iii)⇔ (iv).

To complete the proof, let (iv) be true and show that (i) holds. According to Theorem 2.4.7 and by assumption (iv) we obtain

$$\mathrm{Ker}(\Lambda_T^*) = 0 \Leftrightarrow R(\Lambda_T^{\perp}) = 0 \Leftrightarrow \overline{\Lambda_T(U_{\mathrm{ad}})} = X. \tag{3.1.22}$$

It remains to show that $\overline{\Lambda_T(U_{\mathrm{ad}})} = \overline{D_{\zeta,T}}$. From Lemma 3.1.2 it is shown that

$$D_{\zeta,T} = \Lambda_T(U_{\mathrm{ad}}) + e^{AT}\zeta. \tag{3.1.23}$$

Therefore, by (3.1.22) and (3.1.23) together it follows that $\overline{D_{\zeta,T}} = X$ and (i) follows. ■

Lemma 3.1.9 [10, 22] Let $\lambda > 0$ and $h \in X$. Then there is exactly one optimal control $u^\lambda \in U_{\mathrm{ad}}$ on which the functional

$$J(u) = \|x_T^u - h\|^2 + \lambda \int_0^T \|u_t\|^2\, dt,$$

subject to

$$\begin{cases} x_t' = Ax_t + Bu_t,\ 0 < t \le T, \\ x_0 = \zeta \in X. \end{cases} \tag{3.1.24}$$

takes its minimum value on U_{ad}. Moreover, for every $0 \le t \le T$,

$$u_t^\lambda = -B^* e^{A^*(T-t)} R(\lambda, -Q_T)(e^{AT}\zeta - h),\ \ a.e. \tag{3.1.25}$$

and

$$x_T^{u^\lambda} - h = \lambda R(\lambda, -Q_T)(e^{AT}\zeta - h). \tag{3.1.26}$$

Here, as usual, $R(\lambda, -Q_T)$ is the resolvent operator of $-Q_T$.

Proof. It is well-known that the functional J has a unique optimal control $u^\lambda \in U_{\mathrm{ad}}$. Computing the variation of J (see Mahmudov and Bashirov (1997), one can obtain an optimal solution u^λ satisfying

$$u_t^\lambda = -\frac{1}{\lambda} B^* e^{A^*(T-t)}(x_T^{u^\lambda} - h), \ \ a.e. \tag{3.1.27}$$

Substituting (3.1.27) in equation (3.1.24), we obtain

$$\begin{aligned} x_T^{u^\lambda} &= e^{AT}\zeta + \frac{1}{\lambda}\int_0^T e^{A(T-r)} BB^* e^{A^*(T-r)}(x_T^{u^\lambda} - h)\,dr \\ &= e^{AT}\zeta - \frac{1}{\lambda} Q_T(x_T^{u^\lambda} - h). \end{aligned}$$

Then,

$$\lambda x_T^{u^\lambda} = \lambda e^{AT}\zeta - Q_T(x_T^{u^\lambda} - h). \tag{3.1.28}$$

Rearranging, we have

$$(\lambda I + Q_T) x_T^{u^\lambda} = \lambda e^{AT}\zeta + Q_T h. \tag{3.1.29}$$

Since $(\lambda I + Q_T)^{-1}$ exists, this yields

$$\begin{aligned} x_T^{u^\lambda} &= (\lambda I + Q_T)^{-1}\lambda e^{AT}\zeta + (\lambda I + Q_T)^{-1}(\lambda I + Q_T - \lambda I)h \\ &= \lambda(\lambda I + Q_T)^{-1}(e^{AT}\zeta - h) + h. \end{aligned}$$

Therefore,

$$x_T^{u^\lambda} - h = \lambda R(\lambda, -Q_T)(e^{AT}\zeta - h). \tag{3.1.30}$$

This proves (3.1.26).

Next, substituting (3.1.26) into (3.1.27), we can easily obtain (3.1.25). ■

Theorem 3.1.10 [10, 11] For any given $T > 0$, the following assumptions are equivalent:

(i) The linear system (3.1.1) is D^a-controllable;

(ii) $\lambda R(\lambda, -Q_T) \to 0$ strongly as $\lambda \longrightarrow 0^+$;

(iii) $\lambda R(\lambda, -Q_T) \to 0$ weakly as $\lambda \longrightarrow 0^+$.

Proof. To begin with (i)⇔ (ii). Let (i) be true. Then following Lemma 3.1.9 for any $h \in X$, one can find a sequence $w^m \in U_{\text{ad}}$ so that

$$\|x_T^{w^m} - h\| \to 0 \ \ as \ \ m \to \infty. \tag{3.1.31}$$

Moreover, for a given $\lambda > 0$, and a control u^λ such that the functional given in Lemma 3.1.9 takes on its minimum value, we have

$$\begin{aligned} \|x_T^{u^\lambda} - h\|^2 &\leq \|x_T^{u^\lambda} - h\|^2 + \lambda \int_0^T \|u_t^\lambda\|^2 \, dt, \\ &\leq \|x_T^{w^m} - h\|^2 + \lambda \int_0^T \|w_t^m\|^2 \, dt. \end{aligned} \tag{3.1.32}$$

Now, let $\epsilon > 0$ be arbitrary, then by (3.1.31), one can select sufficiently large m such that

$$\|x_T^{w^m} - h\|^2 \leq \frac{\epsilon}{2}. \tag{3.1.33}$$

In addition, we can also pick sufficiently small $\theta > 0$ in that $0 < \lambda < \theta$ and

$$\lambda \int_0^T \|w_t^m\|^2 \, dt \leq \frac{\epsilon}{2}. \tag{3.1.34}$$

Hence, substituting (3.1.33) and (3.1.34) in (3.1.32) we obtain $\|x_T^{u^\lambda} - h\|^2 \leq \epsilon$ for every $0 < \lambda < \theta$. Then, using this estimation in Lemma 3.1.9 we get

$$\|x_T^{u^\lambda} - h\| = \|\lambda R(\lambda, -Q_T)(e^{AT}\zeta - h)\| \leq \epsilon,$$

for arbitrary $h \in X$ and $\epsilon > 0$. Therefore, $\lambda R(\lambda, -Q_T) \to 0$ strongly as $\lambda \to 0^+$, and (ii) holds. Conversely, for (ii) $\Rightarrow$ (i), letting (ii) be hold, for any given $h \in X$, using Lemma 3.1.9 one can select λ sufficiently small so that

$$\|x_T^{u^\lambda} - h\| = \|\lambda R(\lambda, -Q_T)(e^{AT}\zeta - h)\|. \tag{3.1.35}$$

By assumption (ii) the left norm goes to zero in the strong topology and consequently $x_T^{u^\lambda} \to h$ as $\lambda \to 0$. This implies that system (3.1.1) is D^a-controllable since h was selected arbitrarily from X. Then, (i) is proved.

For (ii)$\Leftrightarrow$(iii), the direct implication comes evidently from the fact in functional analysis. To show the converse implication, let $\lambda R(\lambda, -Q_T) \to 0$ weakly as $\lambda \longrightarrow 0^+$. This means that for every $x, y \in X$, $\langle \lambda R(\lambda, -Q_T)x, y\rangle \to 0$ as $\lambda \to 0^+$. As it is well-know from previous sections, $R(\lambda, -Q_T) \geq 0$ and self adjoint linear operator over X, hence

$$\begin{aligned}
\|\lambda R(\lambda, -Q_T)x\|^2 &= \langle R(\lambda, -Q_T)x, R(\lambda, -Q_T)x\rangle \\
&\leq (\|\lambda R(\lambda, -Q_T)\|^2)^{\frac{1}{2}} \lambda \langle R(\lambda, -Q_T)x, x\rangle \\
&\leq \langle \lambda R(\lambda, -Q_T)x, x\rangle \to 0 \text{ whenever } \lambda \to 0^+.
\end{aligned}$$

Since $x \in X$ is arbitrary, $\lambda R(\lambda, -Q_T) \to 0$ whenever $\lambda \to 0^+$ in the strong topology. Therefore, (ii) is verified and the proof is accomplished. ■

Example 3.1.11 Consider the linear deterministic control system

$$y_t' = Ay_t + Bu_t,\ 0 < t \le T,\ y_0 \in X. \tag{3.1.36}$$

In this example, let $X = \ell_2$ that is a Hilbert space consisting of numerical sequences $\{x_n\}$ which satisfy $\sum_{n=1}^{\infty} x_n^2 < \infty$. The inner product in ℓ_2 is given by

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n y_n. \tag{3.1.37}$$

Moreover, this space has a well-known basis set as follows:

$$S = \Big\{ e_1 = (1,0,0,\cdots),\ e_2 = (0,1,0,\cdots),\ e_3 = (0,0,1,0\cdots), \ldots \Big\}.$$

If we select $A = 0$, this gives $e^{At} = e^{A^*t} = I$. Let also B be a matrix as follows,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Clearly, $B = B^*$, and therefore,

$$Q_T = \int_0^T e^{Ar} BB^* e^{A^*r} dr = TB^2.$$

It can be easily calculated that

$$\sum_{n=1}^{\infty}\langle Be_n, Be_n\rangle = B^2\sum_{n=1}^{\infty}\langle e_n, e_n\rangle = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

This implies that the operator B is Hilbert-Schmidt and hence $B \in \mathcal{L}(\ell_2)$. Moreover, since $A = 0$ one can simply obtain $e^{A^*t} = I$ and then

$$B^*e^{A^*t}x = 0 \quad \text{yields} \quad Bx = 0$$

which by definition of B can easily imply $x = 0$. According to Theorem 3.1.7, control system (3.1.36) is D^a-controllable. However, it can never be D^c-controllable since

$$\langle Q_T e_n, e_n\rangle = T\langle B^2 e_n, e_n\rangle = \frac{T}{n^2} \to 0 \text{ as } n \to \infty.$$

For basis $\{e_n\}$ there is no positive quantity γ wherein Q_T satisfies $\langle Q_T e_n, e_n\rangle \geq \gamma\|e_n\|^2$. In other words, Q_T is not coercive and hence system (3.1.36) is not D^c-controllable.

3.1.3 Controllability Concepts for Linear Systems in Finite Dimensions

In finite dimensional spaces, the operators A and B can be represented by matrices say, $A \in M_{n,n}$ and $B \in M_{n,m}$, and in theory of linear algebra they are also called linear transformations. Moreover, it is known widely that every linear transformation on finite dimensional space is bounded and closed. In this section, let the state and admissible control processes in the given control system be taken in the Euclidean n-space $\mathbb{R}^n$ and m-space $\mathbb{R}^m$, respectively. For this reason, such systems are called finite dimensional systems. This section is focused on the condition of Kalman which is quite useful and more applicable in the finite dimensional spaces. Unfortunately, this condition is not valid for infinite dimensional systems. Therefore, Kalman's rank condition is given as a necessary and sufficient condition of complete and approximate controllability only for finite dimensional linear systems. Throughout the whole of present section, let $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ for some $n,m \in \mathbb{N}$. For any given matrices $A \in M_{n,n}$ and $B \in M_{n,m}$, $[A : B]$ stands for the matrix $[B, AB, \cdots, A^{n-1}B] \in M_{n,nm}$, that is, $[A : B]$ is the matrix columns of matrices $B, AB, \ldots, A^{n-1}B$.

Theorem 3.1.12 (Kalman's Rank Conditions) [67, 68] Given $T > 0$. The following assumptions are equivalent:

(i) $\operatorname{rank}[A : B] = n$.

(ii) $Q_T > 0$.

(iii) Q_T is coercive.

(iv) The system (3.1.1) is D^a-controllable.

(v) The system (3.1.1) is D^c-controllable.

Proof. Let start with (ii) ⇔(iii). Since the definite positiveness and coerciveness of any operator in finite dimensional space are equivalent; it follows that Q_T is coercive if and only if it is positive definite. Therefore, (ii)⇔ (iii). Consequently, according to the results given in foregoing sections the relation (iv) ⇔(v) is clearly deduced from (ii)⇔ (iii). The essential part in this proof is to show this equivalence (i) ⇔(ii). To begin with the necessary part, suppose that (i) is true, then rank $[A, B] = n$. Take $x \in \mathbb{R}^n$ so that

$$B^* e^{A^* t} x = 0 \ \forall t \in [0, T]. \tag{3.1.38}$$

Differentiate both sides of equation (3.1.38) k-times in respect of t, thus for all $k = 0, 1, \ldots,$ we obtain

$$B^*(A^*)^k e^{A^*t} x = 0 \ \forall t \in [0, T]. \tag{3.1.39}$$

Then, by (i), this equality is true only if $\forall t \in [0, T]$, $e^{A^*t}x = 0$. This yields that $x = 0$, and hence, $\Lambda_T^* x = 0$ implies $x = 0$. Furthermore, since $\langle Q_T x, x\rangle = \|\Lambda_T^* x\|^2$, it follows that for all $x \in \mathbb{R}^n$, $\langle Q_T x, x\rangle$ is nonnegative and equal to zero only when $x = 0$. Therefore, Q_T is definite positive.

Now, for the sufficient part let us assume the contrary, that is, Q_T is definite positive and rank$[A : B] \neq n$. From the definition of $[A : B]$, its rank can not exceed n and hence rank$[A : B] < n$. Next, based on some facts from linear algebra there exists $x \in \mathbb{R}^n$ with $x \neq 0$ and $x[A : B] = 0$. More precisely,

$$xB = xAB = xA^2B = \cdots = xA^{n-1}B = 0. \tag{3.1.40}$$

By Cayley-Hamilton theorem if the characteristic polynomial of A is given by

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n, \ a_0 \neq 0, \ \lambda \in \mathbb{C},$$

then

$$a_0A^n + a_1A^{n-1} + \cdots + a_n = 0. \tag{3.1.41}$$

Multiplying both sides of (3.1.41) by xB we obtain

$$xA^nB = \frac{-1}{a_0}(a_1A^{n-1}xB + \cdots + a_nxB) = 0. \tag{3.1.42}$$

Similarly, $xA^{n+1}B = 0$ and mathematical induction on k gives the following

$$xA^k B = 0 \text{ for all } k = 0, 1, 2, \dots, \tag{3.1.43}$$

which obviously yields

$$xe^{At}B = x\Big(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\Big)B = \sum_{k=0}^{\infty} \frac{xA^k B t^k}{k!} = 0. \tag{3.1.44}$$

Transposing (3.1.44), it follow that $B^* e^{A^* t} x = 0$, which implies that $\Lambda_T^* x = 0$ and consequently,

$$\langle Q_T x, x\rangle = \|\Lambda_T^* x\|^2 = 0. \tag{3.1.45}$$

Then, by (3.1.45), there exists $x \neq 0$ on which $\langle Q_T x, x\rangle = 0$. This means that Q_T is not positive definite which contradicts the assumption at the beginning. Therefore, $\text{rank}[A : B] = n$ and (ii) follows. Finally, (i)$\Leftrightarrow$ (ii). ■

Theorem 3.1.13 (Resolvent Conditions) [10, 11] Given $T > 0$. The following assertions are equivalent:

(i) $R(\lambda, -Q_T)$ converges uniformly as $\lambda \longrightarrow 0^+$.

(ii) $R(\lambda, -Q_T)$ converges strongly as $\lambda \longrightarrow 0^+$.

(iii) $R(\lambda, -Q_T)$ converges weakly as $\lambda \longrightarrow 0^+$.

(iv) $\lambda R(\lambda, -Q_T) \longrightarrow 0$ uniformly as $\lambda \longrightarrow 0^+$.

(v) $\lambda R(\lambda, -Q_T) \rightarrow 0$ strongly as $\lambda \longrightarrow 0^+$.

(vi) $\lambda R(\lambda, -Q_T) \rightarrow 0$ weakly as $\lambda \longrightarrow 0^+$.

Proof. This theorem is proved in the previous sections in the case of infinite dimensional spaces. ■

Remark 3.1.14 To sum up, in finite dimensional space splitting the controllability concept into two kind (approximation and completeness) is meaningless since they are equivalent. Furthermore, Kalman's rank condition is valuable and more applicable in the case of finite dimension.

Example 3.1.15 Let the control system (3.1.1) be given in $\mathbb{R}^2$ with matrix A and vector B introduced as follows:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Clearly, by simple computations

$$\text{rank}[A : B] = \text{rank} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 2 = \dim \mathbb{R}^2.$$

Therefore, the system (3.1.1) is D^c-controllable as well as D^a-controllable since it is satisfied the conditions of Theorem 3.1.12 .

Example 3.1.16 Let the control system (3.1.1) be given in $\mathbb{R}^2$ with matrix A and vector B determined as

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

It is obvious that,

$$\text{rank}[A : B] = \text{rank} \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} = 1 \neq \dim \mathbb{R}^2 = 2,$$

which, in accordance with Theorem 3.1.12, implies that system (3.1.1) is neither D^c-controllable nor D^a-controllable.

Example 3.1.17 Let us consider the same linear control system (3.1.15) given in the previous subsection in Example 3.1.5. Then,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.1.46}$$

and this system is evidently defined in $\mathbb{R}^2$. Therefore, Kalman's Rank Conditions are very suitable for it. Now, by easy calculating one can obtain

$$\text{rank}[A : B] = \text{rank}[B, AB] = \text{rank} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 = \dim\mathbb{R}^2 = 2,$$

which, by Theorem 3.1.12 implies that the system (3.1.15) is D^c-controllable as well as D^a-controllable.

3.2 Controllability Concepts for Semilinear Deterministic Systems

In spite of the plenty of results which have been studying sufficient conditions of controllability for semilinear systems, necessary conditions have remained obscure for such type of systems and no results have come out related to the necessity. This may push us to avoid putting ourselves in enormous risk once thinking about the necessary conditions and because of the short time we have in PhD thesis we prefer not enter in this topic so only sufficient conditions will be considered in our study. This section is aimed to revise some studies about sufficient conditions. In addition, it is remarkable that fixed point theorems are widely-used in exploring the sufficient conditions of controllability for semilinear systems and yet it depends on several relative theorems such as: contraction mapping, generalized contraction mapping, Schauder, Schaefer, Leray-Schauder, Darboux and Nussbaum theorems. Because of the huge studies, this section would be briefly concerned with a few results especially that was done by means of contraction mapping principles. In fact, we shall split this section into two main subsections one for examining the complete controllability and the other for the approximate controllability and in both we focus on the results given via contraction mapping principles.

Consider the basic semilinear control system

$$\begin{cases} x'_t = Ax_t + Bu_t + f(t, x_t, u_t), \ 0 < t \le T, \\ x(0) = \zeta \in X. \end{cases} \tag{3.2.1}$$

Here, as usual, $x \in X$ and $u \in U_{\text{ad}}$ are state and control processes. Assume the following conditions

(A0) X and U are separable Hilbert spaces;

(A1) A and B are the same as defined in the corresponding linear system in previous Subsection;

(A2) f is Lipschitz continuous with respect to x and u, that is, for all $t \in [0, T]$, $u, v \in U$ and $x, y \in X$,

$$||f(t, x, u) - f(t, y, v)|| \le K(||x - y|| + ||u - v||)$$

for some $K \ge 0$;

(A3) f is continuous on $[0,T]\times X\times U$ and bounded, that is,

$$\|f(t,g,u)\| \leq L \text{ for all } (t,g,u) \in [0,T]\times X\times U$$

for some $L > 0$;

(A4) $U_{\text{ad}} = C(0,T;U)$;

(E0) $\lambda R(\lambda, -Q_T) \to 0$ strongly as $\lambda \longrightarrow 0^+$;

(E1) $\lambda R(\lambda, -Q_t) \to 0$ uniformly as $\lambda \longrightarrow 0^+$ for all $0 < t \leq T$;

(F0) Q_T is coercive. That is, there exists $\gamma > 0$ such that $\langle Q_T x, x\rangle \geq \gamma\|x\|^2$ for all $x \in X$.

Note that the condition (E0) means that the linear system (3.1.1) associated with (3.2.1) (the case when $f = 0$) is D^a-controllable. Similarly, the condition (F0) implies the existence of bounded operator Q_T^{-1} which satisfies this relation $\|Q_T^{-1}\| \leq \frac{1}{\gamma}$. Respectively, the linear system (3.1.1) associated with (3.2.1) (the case when $f = 0$) is D^c-controllable.

The above conditions imply the existence of a unique continuous function that satisfies the equation (3.2.1) in the mild sense for every $u \in U_{\text{ad}}$ and $\zeta \in X$ (see, Byszewski (1991) and Li and Yong (1995)), that is, there is a function $x \in C(0,T;X)$ such that

$$x_t = e^{At}\zeta + \int_0^t e^{A(t-r)}(Bu_r + f(r,x_r,u_r))\,dr. \tag{3.2.2}$$

for all $u \in U_{\text{ad}}$ and $\zeta \in X$

3.2.1 Complete Controllability of Semilinear Systems

In this subsection, sufficient conditions of complete controllability for semilinear deterministic systems are given using contraction mapping theorem.

Denote $\tilde{X} = C(0,T;X)$. Then $(\tilde{X} \times U_{\text{ad}}, \|(\cdot,\cdot)\|)$ is a Banach space where

$$\|(\cdot,\cdot)\| = \|(\cdot,\cdot)\|_{\tilde{X}\times U_{\text{ad}}} = \|\cdot\|_{\tilde{X}} + \|\cdot\|_{U_{\text{ad}}}.$$

Lemma 3.2.1 [12] Under the conditions (A0) and (A1), the following inequality holds

$$\|Q_t\| \le \|Q_T\|, \quad 0 \le t \le T.$$

Proof. It is simple to show that $Q_t = Q_t^*$ and $\langle Q_t x, x\rangle \ge 0 \ \forall x \in X$. Hence,

$$\|Q_t\| = \sup_{\|x\|=1} \langle Q_t x, x\rangle.$$

Then

$$\begin{aligned}
\langle Q_T x, x\rangle &= \int_0^T \langle e^{As} B B^* e^{A^* s} x, x\rangle\, ds \\
&= \langle Q_t x, x\rangle + \int_t^T \langle e^{As} B B^* e^{A^* s} x, x\rangle\, ds \\
&= \langle Q_t x, x\rangle + \int_t^T \langle B^* e^{A^* s} x, B^* e^{A^* s} x\rangle\, ds \\
&= \langle Q_t x, x\rangle + \int_t^T \|B^* e^{A^* s} x\|^2\, ds \\
&\ge \langle Q_t x, x\rangle.
\end{aligned}$$

This implies $\|Q_t\| \le \|Q_T\|$. ■

The proof of the following lemmas and theorem given in this subsection can be found in various form in different papers with a minor change, see for example, Mahmudov (2003) and Dauer and Mahmudov (2002).

Lemma 3.2.2 Assume that the assumptions (A0)-(A2) and (F0) hold. Then for any arbitrary $h \in X$, the nonlinear operator $G : \tilde{X} \times U_{\text{ad}} \to \tilde{X} \times U_{\text{ad}}$, which is defined by

$$G(y,v)(t) = (Y(t), V(t)), \ \forall t \in [0,T], \tag{3.2.3}$$

where

$$\begin{aligned} Y(t) &= Q_t e^{A^*(T-t)} Q_T^{-1}\Big(h - e^{AT}\zeta - \int_0^T e^{A(T-s)} f(s, y_s, v_s)\,ds\Big) \\ &\quad + e^{At}\zeta + \int_0^t e^{A(t-s)} f(s, y_s, v_s)\,ds \qquad (3.2.4) \\ V(t) &= B^* e^{A^*(T-t)} Q_T^{-1}(h - e^{AT}\zeta) \\ &\quad - B^* e^{A^*(T-t)} Q_T^{-1} \int_0^T e^{A(T-s)} f(s, y_s, v_s)\,ds, \qquad (3.2.5) \end{aligned}$$

satisfies the following inequality

$$\|G(y,v)(t) - G(z,w)(t)\| \le \Big(\frac{1 + \|Q_T\|N + \|B\|N}{\gamma}\Big) NKT(\|y-z\| + \|v-w\|), \tag{3.2.6}$$

where

$$N = \sup_{0 \le t \le T} \|e^{At}\|.$$

Proof. Let (y,v) and (z,w) be two functions in $\tilde{X} \times U_{\text{ad}}$ such that $G(y,v) = (Y,V)$ and $G(z,w) = (Z,W)$. Then,

$$\|G(y,v) - G(z,w)\|_{\tilde{X} \times U_{\text{ad}}} = \|Y - Z\|_{\tilde{X}} + \|V_\lambda - W_\lambda\|_{U_{\text{ad}}}.$$

Let us start with estimating $\|Y-Z\|_{\tilde{X}}$ as follows:

$$\begin{aligned}
\|Y-Z\| &= \max_{t\in[0,T]}\bigg\|\int_0^t e^{A(t-s)}(f(s,y_s,v_s)-f(s,z_s,w_s))\,ds \\
&\quad -\int_0^t e^{A(t-r)}BB^*e^{A^*(t-r)}e^{A^*(T-t)}Q_T^{-1} \\
&\quad \times\int_0^T e^{A(T-s)}(f(s,y_s,v_s)-f(s,z_s,w_s))\,ds\,dr\bigg\| \\
&= \max_{t\in[0,T]}\bigg\|\int_0^t e^{A(t-s)}(f(s,y_s,v_s)-f(s,z_s,w_s))\,ds \\
&\quad -\int_0^T\int_0^t e^{A(t-r)}BB^*e^{A^*(t-r)}e^{A^*(T-t)}Q_T^{-1}e^{A(T-s)} \\
&\quad \times(f(s,y_s,v_s)-f(s,z_s,w_s))\,dr\,ds\bigg\| \\
&= \max_{t\in[0,T]}\bigg\|\int_0^t e^{A(t-s)}(f(s,y_s,v_s)-f(s,z_s,w_s))\,ds \\
&\quad -\int_0^T Q_t e^{A^*(T-t)}Q_T^{-1}e^{A(T-s)}(f(s,y_s,v_s)-f(s,z_s,w_s))\,dr\,ds\bigg\| \\
&\le \max_{t\in[0,T]}(N+\|Q_t\|N^2)\int_0^T \|Q_T^{-1}f(s,y_s,v_s)-f(s,z_s,w_s)\|\,ds \\
&\le \frac{1+\|Q_T\|N}{\gamma}N\int_0^T \|f(s,y_s,v_s)-f(s,z_s,w_s)\|\,ds \\
&\le \frac{1+\|Q_T\|N}{\gamma}NK\int_0^T (\|y_s-z_s\|+\|v_s-w_s\|)\,ds \\
&\le \frac{1+\|Q_T\|N}{\gamma}NKT(\|y-z\|+\|v-w\|). \qquad (3.2.7)
\end{aligned}$$

Similarly, for $\|V-W\|_{U_{\text{ad}}}$ we have

$$\begin{aligned}
\|V-W\| &= \max_{t\in[0,T]}\bigg\|-B^*e^{A^*(T-t)}\int_0^t Q_T^{-1}e^{A(T-s)}(f(s,y_s,v_s)-f(s,z_s,w_s))\,ds\bigg\| \\
&\le \frac{|B\|N^2}{\gamma}\int_0^T \|f(s,y_s,v_s)-f(s,z_s,w_s)\|\,ds \\
&\le \frac{\|B\|N}{\gamma}NK\int_0^T (\|y_s-z_s\|+\|v_s-w_s\|)\,ds \\
&\le \frac{\|B\|N}{\gamma}NKT(\|y-z\|+\|v-w\|). \qquad (3.2.8)
\end{aligned}$$

Gathering (3.2.7) and (3.2.8), we get

$$\begin{aligned}||G(y,v)(t)-G(z,w)(t)|| &\le \Big(\frac{1+||Q_T||N}{\lambda}NKT+\frac{||B||N}{\gamma}NKt\Big)(||y-z||+||v-w||)\\ &=\Big(\frac{1+||Q_T||N+||B||N}{\gamma}\Big)NKT(||y-z||+||v-w||). \qquad (3.2.9)\end{aligned}$$

This proves the result. ■

For simplification, let denote the large coefficient in (3.2.9) by P as shown below

$$P=\Big(\frac{1+||Q_T||N+||B||N}{\gamma}\Big)NKT. \qquad (3.2.10)$$

Lemma 3.2.3 Assume that the conditions (A0)-(A3) hold. If, additionally,

$$P<1, \qquad (3.2.11)$$

then the operator G, which transforms $\tilde{X}\times U_{\mathrm{ad}}$ into $\tilde{X}\times U_{\mathrm{ad}}$, has a unique fixed point $(x,u)\in\tilde{X}\times U_{\mathrm{ad}}$.

Proof. First it is clear that the operator G transforms $\tilde{X}\times U_{\mathrm{ad}}$ into $\tilde{X}\times U_{\mathrm{ad}}$. Then, by virtue of Lemma 3.2.2, G is a contraction mapping on the Banach space $\tilde{X}\times U_{\mathrm{ad}}$. Therefore, G has a unique fixed point $(x,u)\in\tilde{X}\times U_{\mathrm{ad}}$. ■

Theorem 3.2.4 Assume the conditions (A0)-(A3) and (F0) hold. If the inequality

$$P<1 \qquad (3.2.12)$$

holds, then the semilinear system (3.2.1) is D^c-controllable.

Proof. Take any $\zeta\in X$ and $h\in X$. Show that there is $u\in U_{\mathrm{ad}}$ such that $h=x_T$. To this end, consider u, defined as follows:

$$u_t = B^* e^{A^*(T-t)} Q_T^{-1}(h - e^{AT}\zeta) \\ - \int_0^T B^* e^{A^*(T-t)} Q_T^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds. \tag{3.2.13}$$

Substituting (3.2.13) into (3.2.2) and applying Fubini's theorem (see Bashirov (2003), p. 45), we get

$$\begin{aligned} x_t &= e^{At}\zeta + \int_0^t e^{A(t-s)} BB^* e^{A^*(t-s)} e^{A^*(T-t)} Q_T^{-1}(h - e^{AT}\zeta)\,ds \\ &- \int_0^t e^{A(t-r)} BB^* e^{A^*(t-r)} e^{A^*(T-t)} \int_0^T Q_T^{-1} e^{A(T-s)}\,ds\,dr \\ &+ \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\ &= e^{At} x_0 + Q_t e^{A^*(T-t)} Q_T^{-1}(h - e^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\ &- \int_0^T Q_t e^{A^*(T-t)} Q_T^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds. \end{aligned} \tag{3.2.14}$$

According to Lemma 3.2.3, there exists a unique couple $(x, u) \in \tilde{X} \times U_{\text{ad}}$, satisfying (3.2.13) and (3.2.14). So, $u \in U_{\text{ad}}$. Moreover, at $t = T$, we have

$$\begin{aligned} x_T &= Q_T Q_T^{-1}\Big(h - e^{AT}\zeta - \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds\Big) \\ &\quad e^{AT}\zeta + \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\ &= h. \end{aligned}$$

Therefore, the semilinear control system (3.2.1) is D^c-controllable as desired. ■

As it is shown in Theorem 3.2.4, to apply contraction mapping theorem we need to strengthen the conditions imposing on Lipschitz coefficient. However, instead, in the following subsection we will prove the sufficient conditions of approximate controllability only by additional condition on controllability operator using generalized contraction mapping theorem. This is a normal inspiration since the complete controllability concept is stronger than approximate controllability concept.

3.2.2 Approximate Controllability of Semilinear Systems

Unlike the complete controllability, the generalized contraction mapping theorem is very quite suitable for investigation of the approximate controllability for semilinear deterministic systems. In the current subsection, we shall follow the same notation and assumptions imposed in the whole of this chapter.

Lemma 3.2.5 Assume the conditions (A0)-(A3) hold. Then for any arbitrary $h \in X$ and $\lambda > 0$, the operator $G_\lambda : \tilde{X} \times U_{\text{ad}} \to \tilde{X} \times U_{\text{ad}}$, which is defined by

$$G_\lambda(y,v)(t) = (Y_\lambda(t), V_\lambda(t)), \ \forall t \in [0,T], \tag{3.2.15}$$

where

$$\begin{aligned} Y_\lambda(t) &= e^{At}\zeta + Q_t e^{A^*(T-t)}(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) \\ &\quad - \int_0^t Q_{t-s} e^{A^*(T-t)}(\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, y_s, v_s)\,ds \\ &\quad + \int_0^t e^{A(t-s)} f(s, y_s, v_s)\,ds, \end{aligned} \tag{3.2.16}$$

$$\begin{aligned} V_\lambda(t) &= B^* e^{A^*(T-t)}(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) \\ &\quad - \int_0^t B^* e^{A^*(T-t)}(\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, y_s, v_s)\,ds, \end{aligned} \tag{3.2.17}$$

has exactly one fixed point in $\tilde{X} \times U_{\text{ad}}$.

Proof. Let (y,v) and (z,w) be two functions in $\tilde{X} \times U_{\text{ad}}$ such that $G_\lambda(y,v) = (Y_\lambda, V_\lambda)$ and $G_\lambda(z,w) = (Z_\lambda, W_\lambda)$. Following the same process of demonstrating Lemma 3.2.2, one can readily obtain

$$
\begin{aligned}
\|G_\lambda(y,v)(t)-G_\lambda(z,w)(t)\| &\le \Big(\frac{1+\|Q_T\|N+\|B\|N}{\lambda}\Big)NK\int_0^t(\|y_s-z_s\|+\|v_s-w_s\|)\,ds \\
&=\Big(\frac{1+\|Q_T\|N+\|B\|N}{\lambda}\Big)NKt(\|y-z\|+\|v-w\|) \\
&=P_\lambda t(\|y-z\|+\|v-w\|). \qquad (3.2.18)
\end{aligned}
$$

Now, by repeating the same argument on G_λ^2 we get

$$
\begin{aligned}
\Big\|G_\lambda^2(y,v)(t)-G_\lambda^2(z,w)(t)\Big\| &\le P_\lambda\int_0^t\|G_\lambda(y,v)(s)-G_\lambda(z,w)(s)\|\,ds \\
&\le P_\lambda^2(\|y-z\|+\|v-w\|)\int_0^t s\,ds \\
&= P_\lambda^2\frac{t^2}{2!}(\|y-z\|+\|v-w\|). \qquad (3.2.19)
\end{aligned}
$$

Then,

$$
\Big\|G_\lambda^2(y,v)-G_\lambda^2(z,w)\Big\| \le P_\lambda^2\frac{T^2}{2!}(\|y-z\|+\|v-w\|). \qquad (3.2.20)
$$

Consequently, applying induction principle on $n \ge 1$, we obtain

$$
\Big\|G_\lambda^n(y,v)-G_\lambda^n(z,w)\Big\| \le P_\lambda^n\frac{T^n}{n!}(\|y-z\|+\|v-w\|). \qquad (3.2.21)
$$

Since

$$
\lim_{n\to\infty}(P_\lambda)^n\frac{T^n}{n!}=0, \qquad (3.2.22)
$$

the following relation holds for sufficiently large n,

$$
0\le (P_\lambda)^n\frac{T^n}{n!}<1. \qquad (3.2.23)
$$

Then for sufficiently great n, G_λ^n is a contraction mapping on $\tilde{X} \times U_{\text{ad}}$, and does so G_λ. Therefore, G_λ has exactly one fixed point $(x,u) \in \tilde{X} \times U_{\text{ad}}$ and x associated with this u here is a solution of the control system (3.2.1) . ■

Theorem 3.2.6 Under the conditions (A0)-(A3) and (E1), the semilinear system (3.2.1) is D^a-controllable.

Proof. Let $\zeta \in X$ and $h \in X$. we need to demonstrate the existence of control $u \in U_{\text{ad}}$ so that $\|h - x_T\| \to 0$ as $\lambda \to 0^+$ where x_T is a solution of system (3.2.1) at the terminal time T . To this end, consider u, defined as follows:

$$\begin{aligned} u_t &= B^* e^{A^*(T-t)}(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) \\ &\quad - \int_0^t B^* e^{A^*(T-t)}(\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, y_s, u_s)\, ds. \end{aligned} \tag{3.2.24}$$

Substituting (3.2.24) into (3.2.2) and applying Fubini's Theorem (see Bashirov (2003), p. 45), we obtain

$$\begin{aligned} x_t &= e^{At}\zeta + \int_0^t e^{A(t-s)} BB^* e^{A^*(t-s)} e^{A^*(T-t)} (\lambda I + Q_T)^{-1}(h - e^{AT}\zeta)\, ds \\ &\quad - \int_0^t e^{A(t-r)} BB^* e^{A^*(t-r)} e^{A^*(T-t)} \int_0^r (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\, ds dr \\ &\quad + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\, ds \\ &= e^{At}\zeta + Q_t e^{A^*(T-t)}(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\, ds \\ &\quad - \int_0^t \int_s^t e^{A(t-r)} BB^* e^{A^*(t-r)} e^{A^*(T-t)} (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\, dr ds \\ &= e^{At}\zeta + Q_t e^{A^*(T-t)}(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\, ds \\ &\quad - \int_0^t Q_{t-s} e^{A^*(T-t)} (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\, ds. \end{aligned} \tag{3.2.25}$$

By virtue of Lemma 3.2.5, there exists unique couple $(x,u) \in \tilde{X} \times U_{\text{ad}}$, fulfilling (3.2.24) and (3.2.25). Hence, $u \in U_{\text{ad}}$. Furthermore, we have

$$\begin{aligned}
x_T &= e^{AT}\zeta + Q_T(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) + \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad - \int_0^T Q_{T-s}(\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&= e^{AT}\zeta + Q_T(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) + \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad - \int_0^T Q_{T-s}(\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&= e^{AT}\zeta + Q_T(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) + \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad + \lambda(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) - \lambda(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) \\
&\quad + \lambda \int_0^T (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad - \lambda \int_0^T (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&= h - \lambda(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) - \lambda \int_0^T (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_T - h\| &= \left\| \lambda(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta) - \lambda \int_0^T (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds \right\| \\
&\leq \lambda \|(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta)\| + \|\lambda \int_0^T (\lambda I + Q_{T-s})^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds\| \\
&\leq \|\lambda(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta)\| + M \int_0^T \|\lambda(\lambda I + Q_{T-s})^{-1} f(s, x_s, u_s)\,ds\|.
\end{aligned}$$

Applying Lebesgue dominated convergence theorem on the integral term, $M\int_0^T \|\lambda(\lambda I + Q_{T-s})^{-1}\| \cdot \|f(r, x_r, u_r)\|\,dr \to 0$ as $\lambda \to 0^+$, for all $0 \leq s < T$, since $\|\lambda(\lambda I + Q_{T-s})^{-1}\| \to 0$ as $\lambda \to 0^+$ (condition (E1)) and $\|\lambda(\lambda I + Q_T)^{-1}(h - e^{AT}\zeta)\| \to 0$ as $\lambda \to 0^+$ as$\lambda \to 0^+$ (condition (E0)). Thus, $\|x_T - h\| \to 0$ as $\lambda \to 0^+$. Therefore, the semilinear system (3.2.1) is D^a-controllable. ■

Chapter 4

PARTIAL CONTROLLABILITY CONCEPTS FOR DETERMINISTIC SYSTEMS

Notion of ordinary controllability has been pointedly received a great deal of attentions for more than a half of century and today is almost adequately examined by so many authors for both deterministic and stochastic control systems in finite and infinite dimensional spaces. Therefore, it has not sounded easy to push forward a new result on this concept, whereas Bashirov (2003) observed that there are several control systems can be expressed in terms of standard systems (first order differential equations) which can be achieved simply by extending the dimension of the state space. For such special systems, the so-called partial controllability concept is strongly recommended and hence conditions for this concept can be weaker. Hence, in this chapter, we roughly review some results of this notion of controllability for deterministic linear systems. Moreover, we discusses the sufficient conditions of partial controllability for semilinear deterministic systems by means of contraction mapping theorem as well as generalized contraction mapping theorem. In fact, these two consequences are the main results of my thesis.

4.1 Motivation

This section is appointed to present the needing definitions and sets related to the new concept of controllability which is called partial controllability. Moreover, one most important question will be answered in the end of this section. "Why partial controllability is valuable?" or simply "Why partial concept of controllability is required?" As it is well-recognized, controllability operator which is presented in the previous chapter has played a significant role in theory of controllability, partial version of this operator also has almost an identical role in theory of partial controllability and so does resolvent operator. Therefore, these two operators are defined in partial sense in this section with some beneficial notifications about them.

As usual, assume that X and U are real separable Hilbert spaces. Take L to be a linear projection operator from X into $\mathbb{H}$ provided that the range $\mathbb{H}$ of L is a closed subspace in X.

Now, let us recall that from precedent chapter the controllability operator Q_t given by

$$Q_t = \int_0^t e^{Ar} BB^* e^{A^*r} \, dr, \tag{4.1.1}$$

and Λ_t given by

$$\Lambda_t : L_2(0,t;U) \longrightarrow X, \ \ \Lambda_t u = \int_0^t e^{Ar} Bu(r) \, dr.$$

for all $0 \le t \le T$. Hence, Λ_t possesses an adjoint operator $\Lambda_t^* : X \longrightarrow L_2(0,t;U)$ as

$$[\Lambda_t^*(x)](r) = B^* e^{A^*(r)} x, \ 0 \le r \le t.$$

Obviously, $Q_t = \Lambda_t \Lambda_t^*$ and hence $Q_t^* = Q_t$. Furthermore, this shows that Q_t is non-negative and respectively the resolvent operator, $R(\lambda, -Q_t)$ is well-defined for all $\lambda > 0$. Expanding the controllability operator Q_t into partial version is happening just by multiplying it by L from left and L^* from right as Bashirov et al. (2007) defined

$$\tilde{Q}_t = LQ_tL^*,\ 0 \le t \le T. \tag{4.1.2}$$

Similar to the properties of controllability operator above, for every $0 \le t \le T$, $\tilde{Q}_{T-t} \ge 0$ and self-adjoint, hence the resolvent operator $R(\lambda, -\tilde{Q}_{T-t}) = (\lambda + \tilde{Q}_{T-t})^{-1}$, is well-defined for every $\lambda > 0$.

The purpose of drawing attention to and examining the notions of partial controllability is that there are several control systems that can be expressed as a standard form ,i.e, as a first order differential equation, simply by extending the original state space. Hence, the notions of partial controllability have become very useful and more adapted for such systems using the projection operator L which mapping the expanding space into the main one. Furthermore, the great advantages of this concept of controllability are powerfully manifested in the following examples:

Example 4.1.1 Consider the n^{th}-order differential equation

$$z_t^{(n)} = f(t, z_t, z_t', \ldots, z_t^{(n-1)}, u_t),\ \ z \in \mathbb{R}. \tag{4.1.3}$$

As usual, $\mathbb{R}$ is the real number space which is taken as a state space of the system (4.1.3). By the definition, the concepts of controllability for this system are the equality to or denseness in $\mathbb{R}$ of the appropriate attainable set. This system can be easily written in terms of the standard form as the first order differential equation

$$x_t' = Ax_t + F(t, x_t, u_t) \tag{4.1.4}$$

if

$$x_t = \begin{pmatrix} z_t \\ z_t' \\ \vdots \\ z_t^{(n-2)} \\ z_t^{(n-1)} \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$F(t,x,u) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t,z,z',\ldots,z^{(n-1)},u) \end{pmatrix}.$$

The state space of the system (4.1.4) is the n-dimensional Euclidean space $\mathbb{R}^n$ and correspondingly, its attainable set becomes a subset of $\mathbb{R}^n$. Therefore, the concepts of controllability for the system (4.1.4) are stronger than the same for the system (4.1.3). However, using the projection operator L defined as

$$L = [\,1 \quad 0 \quad \cdots \quad 0 \quad 0\,] : \mathbb{R}^n \to \mathbb{R},$$

would make the concepts of L-partial controllability for the system (4.1.4) the same as the concepts of ordinary controllability for the system (4.1.3).

Example 4.1.2 Consider the nonlinear wave equation

$$\frac{\partial^2 x_{t,\theta}}{\partial t^2} = \frac{\partial^2 x_{t,\theta}}{\partial \theta^2} + f(t, x_{t,\theta}, \partial x_{t,\theta}/\partial t, , u_t), \tag{4.1.5}$$

where x is a real-valued function of two variables $t \geq 0$ and $0 \leq \theta \leq 1$. The state space of this system is $L_2(0,1)$. This system can be re-expressed as the first order abstract differential equation

$$y'_t = Ay_t + F(t, y_t, u_t) \tag{4.1.6}$$

if

$$y_t = \begin{bmatrix} x_{t,\theta} \\ \partial x_{t,\theta}/\partial t \end{bmatrix}, \; A = \begin{bmatrix} 0 & I \\ d^2/d\theta^2 & 0 \end{bmatrix}, \; F(t,y,u) = \begin{bmatrix} 0 \\ f(t, y_1, y_2, u) \end{bmatrix},$$

where $y \in L_2(0,1) \times L_2(0,1)$. The state space $L_2(0,1) \times L_2(0,1)$ of the system (4.1.6) is the expending of the state space $L_2(0,1)$ for the system (4.1.5). This is actually a price what is paid to get the wave equation (4.1.5) as the form of first order differential equation (4.1.6). The notions of controllability for the system (4.1.6) are strong comparable with the same notions for the original system (4.1.5). If

$$L = [\, I \quad 0 \,] : L_2(0,1) \times L_2(0,1) \to L_2(0,1),$$

then the concepts of L-partial controllability for the system (4.1.6) become the concepts of ordinary controllability for the system (4.1.5).

Example 4.1.3 Consider a nonlinear delay system

$$x_t' = f(t, x_t, \int_{-\varepsilon}^{0} x_{t+\theta}\, d\theta, u_t), \tag{4.1.7}$$

where, the distributed delay is included in the nonlinear term, and x is a real-valued function. Obviously, the state space of this system is $\mathbb{R}$. To remove the delay from this system, one can enlarge $\mathbb{R}$ to $\mathbb{R} \times L_2(-\varepsilon, 0)$ and introduce $L_2(-\varepsilon, 0)$-valued function as

$$[\bar{x}_t]_\theta = x_{t+\theta},\ t \geq 0,\ -\varepsilon \leq \theta \leq 0, \tag{4.1.8}$$

and operator Γ as

$$\Gamma h = \int_{-\varepsilon}^{0} h_\theta, h \in L_2(-\varepsilon, 0). \tag{4.1.9}$$

Therefore, for

$$y_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix},\ A = \begin{bmatrix} 0 & 0 \\ 0 & d/d\theta \end{bmatrix},\ F(t, y, u) = \begin{bmatrix} f(t, x, \Gamma \bar{x}, u) \\ 0 \end{bmatrix},$$

the system (4.1.7) can be represented in terms of the abstract system

$$y_t' = Ay_t + f(t, y_t, u_t). \tag{4.1.10}$$

Then, it can be observed that the concepts of controllability for the system (4.1.10) are strong for the same of the system (4.1.7), however, the concepts of L-partial controllability for the system (4.1.10) with

$$L = [\ I \quad 0\ ] : \mathbb{R} \times L_2(0, 1) \to \mathbb{R}, \tag{4.1.11}$$

are exactly the ordinary concepts of controllability for the original system (4.1.7).

Example 4.1.4 Consider the basic semilinear integro-differential system

$$x_t' = Ax_t + Bu_t + f(t, x_t, \int_0^t g(s, x_s)\,ds, u_t), \tag{4.1.12}$$

where, the integral form is included in the nonlinear term, and x is a real-valued function. obviously, the state space of this system is $\mathbb{R}$. To remove the integral from this system , one can enlarge $\mathbb{R}$ to $\mathbb{R} \times L_2(0,t)$ and introduce $L_2(0,t)$-valued function as

$$\bar{x}_t = \int_0^t g(t, x_s)\,ds \; t \geq 0.$$

Therefore, the system (4.1.12) can be written in terms of the abstract system

$$y_t' = Ay_t + Bu_t + F(t, y_t, u_t). \tag{4.1.13}$$

Where,

$$y_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix},\; A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{d}{dt} \end{bmatrix},\; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; F(t, y, u) = \begin{bmatrix} f(t, x, \bar{x}, u) \\ 0 \end{bmatrix},$$

Then, it can be observed that the concepts of controllability for the system (4.1.13) are too strong for the same of the original system (4.1.12), however, the concepts of L-partial controllability for the system (4.1.13) with

$$L = [\, I \quad 0 \,] : \mathbb{R} \times L_2(0,1) \to \mathbb{R},$$

are exactly the ordinary concepts of controllability for the original system (4.1.12).

Consequently, the foregoing examples have been strongly motivating and encouraging the study of the of partially controllability concepts.

4.2 Partially Complete Controllability of Deterministic Linear Systems

As we have seen in ordinary sense, conditions of partial controllability for linear control systems are very similar to what they have been done in ordinary controllability in the precedent chapter. More precisely, if the controllability operator is replaceable by its partial version as defined above, the most results given in ordinary controllability would be extended into the partial controllability as well. This can be happened only by imposing the appropriate conditions on the partial controllability operator $\tilde{Q}$. In this section, necessary and sufficient conditions for partial controllability of deterministic linear systems are given only with mentioning the location of the full explanations which are done in ordinary controllability in the foregoing Chapter.

Now, consider the basic linear deterministic system

$$\begin{cases} x_t' = Ax_t + Bu_t, \ 0 < t \le T, \\ x(0) = \zeta \in X. \end{cases} \tag{4.2.1}$$

Here, as usual, x and u are state and control processes, respectively. Assume X, U, A and B are as usual in the previous sections for all the given systems therein. Let $U_{\text{ad}} = L_2(0,T;U)$. For every $u \in U_{\text{ad}}$ and $\zeta \in X$, there exists a unique mild solution $x(t;\zeta,u)$ of the control system (4.2.1) described by the integral form as

$$x(t;\zeta,u) = x_t = e^{At}\zeta + \int_0^t e^{A(t-s)}Bu_s\,ds, 0 < t \le T. \tag{4.2.2}$$

Let

$$D_{\zeta,T} = \{h \in \mathbb{H} : x_0 = \zeta \text{ and } \exists u \in U_{\text{ad}} \text{ such that } h = Lx_T\}.$$

Following Bashirov et al. (2007), the control system (4.2.1) is said to be L-partially D^c-controllable if $D_{\zeta,T} = \mathbb{H}$ for all $\zeta \in X$. In a similar way, the control system (4.2.1) is said to be L-partially D^a-controllable if $\overline{D_{\zeta,T}} = \mathbb{H}$ for all $\zeta \in X$. In the particular case, when $L = I$, we return back to the ordinary controllability which was studied in the previous chapter.

Theorem 4.2.1 [7] Under the conditions and notation in this section, the given assertions below are equivalent:

(i) The deterministic system (4.2.1) with the admissible space U_{ad} is L-partially D^c-controllable;

(ii) The partial controllability operator $\tilde{Q}_T$ is coercive ;

(iii) $R(\lambda, -\tilde{Q}_T)$ is uniformly convergent as $\lambda \longrightarrow 0^+$;

(iv) $R(\lambda, -\tilde{Q}_T)$ is strongly convergent as $\lambda \longrightarrow 0^+$;

(v) $R(\lambda, -\tilde{Q}_T)$ is weakly convergent as $\lambda \longrightarrow 0^+$;

(vi) $\lambda R(\lambda, -\tilde{Q}_T) \longrightarrow 0$ uniformly as $\lambda \longrightarrow 0^+$.

Proof. This proof is very analogous to the proof of Theorem 3.1.3 for ordinary sense in the case when $X = \mathbb{H}$. This can be done simply by replacing Q_T with its partial version $\tilde{Q}_T$. Therefore, there is no need to repeat the proof here. ■

4.3 Partially Approximate Controllability of Linear Deterministic Systems

As shown early in foregoing Section 4.2, it is not complicated to extend the outcomes of approximate controllability to the partially approximate controllability for deterministic linear system which can be merely happen by replacing the controllability operator with its partial version. In this section, we shall employ the same notations as well as the same system given in the previous section. Moreover, results in this section would be arranged in similar way to those in precedent section.

Theorem 4.3.1 [7] Under the conditions and notation in this section, the assertions given below are equivalent:

(i) The deterministic system (4.2.1) with the admissible space U_{ad} is L-partially D^a-controllable

(ii) The partial controllability operator $\tilde{Q}_T$ is positive;

(iii) $\tilde{B}^* e^{\tilde{A}^* t} z = 0$ for all $0 \le t \le T$, yields $z = 0$;

(iv) $\lambda R(\lambda, -\tilde{Q}_T)$ is strongly convergent to zero as $\lambda \longrightarrow 0^+$;

(v) $\lambda R(\lambda, -\tilde{Q}_T)$ is weakly convergent to zero as $\lambda \longrightarrow 0^+$.

4.4 Partially Complete Controllability via Contraction Mapping Theorem

This section comprises one of my main results concerning with partially complete controllability of semilinear deterministic systems by using contraction mapping principle. In general, as a fixed point theorem is a very dynamic tool used in ordinary controllability for semilinear systems, it also has the similar role in concepts of partial controllability. Therefore, we apply fixed point theorem obtaining a set of new sufficient conditions for partially complete controllability of semilinear deterministic control systems. In fact the proofs given in this section are minor changes from which were given in ordinary sense.

Consider the semilinear deterministic control system

$$\begin{cases} x_t' = Ax_t + Bu_t + f(t, x_t, u_t),\ 0 < t \le T, \\ x_0 = \zeta \in X, \end{cases} \tag{4.4.1}$$

where x and u are the state and control processes, respectively. Let the following conditions be assumed in the current section.

(A1) X and U are real separable Hilbert spaces, $\mathbb{H}$ is a closed subspace of X and L is a projection operator from X into $\mathbb{H}$;

(A2) A and B are as usual;

(A3) f is a continuous nonlinear function from $[0,T] \times X \times U$ to X, satisfying

- f is Lipschitz continuous with respect to x and u, that is, there exists some $K \ge 0$ such that

$$\|f(t,x,u) - f(t,y,v)\| \le K(\|x-y\| + \|u-v\|) \tag{4.4.2}$$

for all $t \in [0,T]$, $u, v \in U$ and $x, y \in X$,;

(A4) $U_{\text{ad}} = C(0,T;U)$;

(A5) $\tilde{Q}_T$ is coercive on $\mathbb{H}$.

Note that from the previous sections the condition (F) implies the existence of bounded linear operator $\tilde{Q}_T^{-1}$ such that $\|\tilde{Q}_T^{-1}\| \le \frac{1}{\gamma}$. Respectively, the linear system associated with (4.4.1) (the case when $f = 0$) is L-partially complete controllable on $[0,T]$ (see, Bashirov et al.(2007).

Under the above conditions the semilinear system (4.4.1) has a unique mild continuous solution for every $u \in U_{\text{ad}}$ and $\zeta \in X$ (see, Byszewski (1991) and Li and Yong (1995)), that is, there exists a unique function $x \in C(0,T;U)$ satisfies

$$x_t = e^{At}\zeta + \int_0^t e^{A(t-r)}(Bu_r + f(r, x_r, u_r))\,dr. \tag{4.4.3}$$

In what follows, let $\mathbb{X} = C(0,T;X)$. Then $\mathbb{X} \times U_{\text{ad}}$ is a Banach space endowed with the following norm

$$\|(\cdot,\cdot)\|_{\mathbb{X}\times U_{\text{ad}}} = \|\cdot\|_{\mathbb{X}} + \|\cdot\|_{U_{\text{ad}}}. \tag{4.4.4}$$

Lemma 4.4.1 Under the conditions (A1) and (A2), for all $0 \le t \le T$ the following inequalities hold

$$\|Q_t\| \le \|Q_T\| \;\text{ and }\; \|\tilde{Q}_t\| \le \|\tilde{Q}_T\|. \tag{4.4.5}$$

Proof. This inequality $\|Q_t\| \le \|Q_T\|$ was established in previous Chapter. For the second part of this lemma follows from $\langle \tilde{Q}_t x, x\rangle = \langle Q_t L^* x, L^* x\rangle$. ■

Lemma 4.4.2 Let the conditions (A1)-(A5) be fulfilled and $h \in \mathbb{H}$. Then for the non-linear operator $F : \mathbb{X} \times U_{\text{ad}} \to \mathbb{X} \times U_{\text{ad}}$, which is defined by

$$F(y,v)(t) = (Y(t), V(t)),\; \forall t \in [0,T], \tag{4.4.6}$$

where

$$Y(t) = e^{At}\zeta + Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, y_s, v_s)\, ds$$
$$- Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L \int_0^T e^{A(T-s)} f(s, y_s, v_s)\, ds \tag{4.4.7}$$

$$V(t) = B^* e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}\Big(h - Le^{AT}\zeta - L\int_0^T e^{A(T-s)} f(s, y_s, v_s)\, ds\Big), \tag{4.4.8}$$

the following inequality holds:

$$\|F(y,v) - F(z,w)\| \le \Big(1 + \frac{\|Q_T\|N}{\gamma} + \frac{\|B\|N}{\gamma}\Big) NKT(\|y-z\| + \|v-w\|).$$

Proof. Let (y,v) and (z,w) be two functions in $\mathbb{X} \times U_{\text{ad}}$ such that $F(y,v) = (Y,V)$ and $F(z,w) = (Z,W)$. Then,

$$\|F(y,v) - F(z,w)\|_{\mathbb{X}\times U_{\text{ad}}} = \|Y - Z\|_{\mathbb{X}} + \|V - W\|_{U_{\text{ad}}}. \tag{4.4.9}$$

Here, $\|Y - Z\|_{\mathbb{X}}$ can be estimated as follows:

$$\begin{aligned}
\|Y - Z\| = \max_{t\in[0,T]} \Big\| &\int_0^t e^{A(t-s)}(f(s, y_s, v_s) - f(s, z_s, w_s))\, ds \\
&- Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L \int_0^T e^{A(T-s)}(f(s, y_s, v_s) - f(s, z_s, w_s))\, ds \Big\| \\
&\le \max_{t\in[0,T]} (N + \|Q_t\|\, \|\tilde{Q}_T^{-1}\| N^2) \int_0^T \|f(s, y_s, v_s) - f(s, z_s, w_s)\|\, ds \\
&\le (1 + \|Q_T\|\, \|\tilde{Q}_T^{-1}\| N) N \int_0^T \|f(s, y_s, v_s) - f(s, z_s, w_s)\|\, ds \\
&\le \Big(1 + \frac{\|Q_T\|N}{\gamma}\Big) NK \int_0^T (\|y_s - z_s\| + \|v_s - w_s\|)\, ds \\
&\le \Big(1 + \frac{\|Q_T\|N}{\gamma}\Big) NKT(\|y - z\| + \|v - w\|).
\end{aligned} \tag{4.4.10}$$

In the same way, for $||V-W||_{U_{\rm ad}}$ we have

$$
\begin{aligned}
||V-W|| &= \max_{t\in[0,T]} \left\| B^* e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L \int_0^T e^{A(T-s)} (f(s,y_s,v_s) - f(s,z_s,w_s))\, ds \right\| \\
&\le ||B|| ||\tilde{Q}_T^{-1}|| N^2 \int_0^T ||f(s,y_s,v_s) - f(s,z_s,w_s)||\, ds \\
&\le ||B|| \frac{1}{\gamma} N^2 K \int_0^T (||y_s - z_s|| + ||v_s - w_s||)\, ds \\
&\le \frac{||B||N}{\gamma} NKT(||y-z|| + ||v-w||).
\end{aligned}
\tag{4.4.11}
$$

Gathering (4.4.10) and (4.4.11), we obtain the required inequality. ■

Lemma 4.4.3 Assume that the conditions (A1)-(A5) hold. If, additionally,

$$
\left(1 + \frac{||Q_T||N}{\gamma} + \frac{||B||N}{\gamma}\right) NKT < 1, \tag{4.4.12}
$$

then the operator F, which transforms $\mathbb{X} \times U_{\rm ad}$ into $\mathbb{X} \times U_{\rm ad}$, has a unique fixed point $(x,u) \in \mathbb{X} \times U_{\rm ad}$.

Proof. First it is clear that the operator F transforms $\mathbb{X} \times U_{\rm ad}$ into $\mathbb{X} \times U_{\rm ad}$. Then, by virtue of Lemma 4.4.2, F is a contraction mapping on the Banach space $\mathbb{X} \times U_{\rm ad}$. Therefore, F has a unique fixed point $(x,u) \in \mathbb{X} \times U_{\rm ad}$. ■

Theorem 4.4.4 Under the conditions (A1)-(A5) and (4.4.12), the semilinear control system (4.4.1) is L-partially D^c-controllable.

Proof. Take arbitrary $\zeta \in X$ and $h \in \mathbb{H}$. Let us prove that there is $u \in U_{\rm ad}$ such that $h = Lx_T$. For this purpose, consider u, defined as the following form

$$
u_t = B^* e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} \left(h - Le^{AT}\zeta - L \int_0^T e^{A(T-s)} f(s,x_s,u_s)\, ds \right). \tag{4.4.13}
$$

Substituting (4.4.13) into (4.4.3) and using Fubini's theorem (see Bashirov (2003), p. 45), we obtain

$$
\begin{aligned}
x_t &= e^{At}\zeta + \int_0^t e^{A(t-s)}BB^* e^{A^*(t-s)} e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta)\,ds \\
&\quad - \int_0^t \int_0^T e^{A(t-s)}BB^* e^{A^*(t-s)} e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L e^{A(T-r)} f(r, x_r, u_r)\,dr ds \\
&\quad + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\
&= e^{At}\zeta + Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\
&\quad - \int_0^T \int_0^t e^{A(t-s)}BB^* e^{A^*(t-s)} e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L e^{A(T-r)} f(r, x_r, u_r)\,ds dr \\
&= e^{At}\zeta + Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\
&\quad - \int_0^T Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L e^{A(T-r)} f(r, x_r, u_r)\,dr \\
&= e^{At}\zeta + Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\
&\quad - Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1} L \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds. \qquad (4.4.14)
\end{aligned}
$$

In accordance with Lemma 4.5.3, there exists a unique pair $(x, u) \in \mathbb{X} \times U_{\text{ad}}$, satisfying (4.4.13) and (4.4.14). Therefore, $u \in U_{\text{ad}}$. Moreover, we have

$$
\begin{aligned}
Lx_T &= L\Big(e^{AT}\zeta + Q_T L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad - Q_T L^* \tilde{Q}_T^{-1} L \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds\Big) \\
&= Le^{AT}\zeta + LQ_T L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + L\int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad - LQ_T L^* \tilde{Q}_T^{-1} L \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds = h.
\end{aligned}
$$

Thus, there is $u \in U_{\text{ad}}$ steering ζ to x_T such that $Lx_T = h$. This proves the theorem. ■

Remark 4.4.5 Decomposing Q_T to the following form

$$
Q_T = \begin{bmatrix} \tilde{Q}_T & R_T \\ R_R^* & P_T \end{bmatrix}, \qquad (4.4.15)
$$

where $R_T : \mathbb{H}^\perp \to \mathbb{H}$ and $P_T : \mathbb{H}^\perp \to \mathbb{H}^\perp$ are other components of Q_T aside from $\tilde{Q}_T$, hence one can obtain

$$\langle Q_T h, h\rangle = \langle \tilde{Q}_T h_1, h_1\rangle + 2\langle R_T h_2, h_1\rangle + \langle P_T h_2, h_2\rangle, \tag{4.4.16}$$

where $h_1 = Lh \in \mathbb{H}$ and $h_2 = h - Lh \in \mathbb{H}^\perp$. Then, the coerciveness of Q_T yields the same of $\tilde{Q}_T$. However, the converse is not always valid. Theorem 4.4.4 is too strong of the cases in which $\tilde{Q}_T$ is coercive but P_T is not.

Example 4.4.6 In Theorem 4.4.4, we demonstrate only a sufficient condition of L-partial complete controllability. However, this example will prove that this can not be a necessary condition. To this end, let consider the state of $L = I$ when L-partial complete controllability reduces to ordinary complete controllability. Consider a basic one-dimensional control system

$$x_t' = 2x_t + 2u_t,\ \zeta \in \mathbb{R}. \tag{4.4.17}$$

Clearly, it is a linear system and the controllability operator Q_T of this system is equal to

$$\int_0^T 4e^{4t}\,dt = e^{4T} - 1 > 0 \text{ for all } T > 0.$$

In view of the theory of controllability for linear systems given in precedent sections, the control system (4.4.17) is completely controllable for the time $T > 0$. The system (4.4.17) can be written as

$$x_t' = x_t + u_t + f(x_t, u_t),\ \zeta \in \mathbb{R},$$

where

$$f(x,u) = x + u.$$

Here, f is a continuous Lipschitz function with $K = 1$. Moreover, $A = B = 1$, which yields $\|B\| = 1$ and

$$N = \sup_{[0,T]} \|e^{At}\| = e^{T}. \tag{4.4.18}$$

Furthermore,

$$Q_T = \int_0^T e^{2t}\, dt = \frac{e^{2T} - 1}{2}.$$

Then, $\|Q_T\| = \gamma = (e^{2T} - 1)/2$. Therefore, the inequality (4.4.12) becomes

$$\left(1 + e^{T} + \frac{2e^{T}}{e^{T} - 1}\right) e^{T} T < 1.$$

Taking $T \to \infty$ on both sides we obtain that the left hand side in this inequality goes to ∞. This implies that for a sufficiently large T, the conditions of Theorem 4.4.4 can never hold for this T, whilst the system (4.4.17) as we have seen early in this example is completely controllable. Thus, Theorem 4.4.4 establishes just a sufficient condition which is not a necessary condition.

The following examples reveal the significant advantages of using L-partial complete controllability for some appropriate control systems.

Example 4.4.7 Consider the system of deterministic equations

$$\begin{cases} x_t' = y_t + bu_t, \ x_0 = \zeta \in \mathbb{R}, \\ y_t' = f(t, x_t, y_t, u_t), \ y_0 \in \mathbb{R} \end{cases} \tag{4.4.19}$$

on $[0,T]$, where $u \in U_{\text{ad}} = C(0,T;\mathbb{R})$ as a control process and $(x,y) \in \mathbb{R}\times\mathbb{R}$ as a state process. The complete controllability property is interpreted in $\mathbb{R}\times\mathbb{R}$ as

$$\{(x,y) \in \mathbb{R}^2 : \exists u \in U_{\text{ad}} \text{ such that } (x_T, y_T) = (x,y)\} = \mathbb{R}^2, \tag{4.4.20}$$

However, the partially complete controllability property is investigated in $\mathbb{R}$ as

$$\{x \in \mathbb{R} : \exists u \in U_{\text{ad}} \text{ such that } x_T = x\} = \mathbb{R}. \tag{4.4.21}$$

Rewriting the system (4.4.19) in $\mathbb{R}^2$ as the following semilinear system

$$z_t' = Az_t + F(t, z_t, u_t) + Bu_t, \tag{4.4.22}$$

where

$$z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \ F(t,z,u) = \begin{bmatrix} 0 \\ f(t,x,y,u) \end{bmatrix}, \tag{4.4.23}$$

assuming that

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}\times\mathbb{R}. \tag{4.4.24}$$

It can be computed the fundamental matrix e^{At} as

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}. \tag{4.4.25}$$

Hence,

$$\|e^{At}\| \leq 1 + t \leq 1 + T,\ 0 \leq t \leq T. \tag{4.4.26}$$

The controllability operator is

$$Q_T = \int_0^T e^{At} BB^* e^{A^* t}\, dt = b^2 T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.4.27}$$

Clearly, Q_T is not coercive. Therefore, the conditions for complete controllability , based on coerciveness of Q_T, fail for the control system (4.4.19). However, the system (4.4.19) can be completely controllable for appropriate nonlinear functions f, if we investigate the partial complete controllability for this system being interested only in the first component x_t of z_t.

Let $L = [1 \ \ 0]$. Then

$$\tilde{Q}_T = LQ_T L^* = b^2 T > 0. \tag{4.4.28}$$

This implies that the linear control system associated to the semilinear system (4.4.22) is L-partially complete controllable. Moreover, the inequality (4.4.12) becomes

$$\left(1 + \frac{b^2 T(1+T)}{b^2 T} + \frac{b(1+T)}{b^2 T}\right)(1+T)TK < 1, \tag{4.4.29}$$

or, equivalently,

$$K < \frac{b}{(1+T)(1+T+2bT+bT^2)}. \tag{4.4.30}$$

This estimation sets up a strong relation between Lipschitz coefficient K and terminal time moment T. Regarding (4.4.30), T must be taken sufficiently large in order to get a proper Lipschitz coefficient $K < 1$. Then the control system (4.4.19) is L-partially D^c-controllable if the Lipschitz coefficient K, related to f, satisfies (4.4.30).

Example 4.4.8 Delay equations are by all means typical of the most systems for employing the concepts of partial controllability. Consider the semilinear control delay equation

$$\begin{cases} x_t' = ax_t + bu_t + f(t, x_t, \int_{-\varepsilon}^0 x_{t+\theta}\, d\theta, u_t), \\ x_0 = \zeta,\ x_\theta = \eta_\theta,\ -\varepsilon \le \theta \le 0, \end{cases} \tag{4.4.31}$$

on $[0,T]$, where $0 < \varepsilon < T$, $\eta \in L_2(-\varepsilon, 0; \mathbb{R})$ and $u \in U_{\mathrm{ad}} = C(0,T;\mathbb{R})$.

Let introduce the function $\bar{x} : [0,T] \to L_2(-\varepsilon, 0; \mathbb{R})$ by

$$[\bar{x}_t]_\theta = x_{t+\theta},\ 0 \le t \le T,\ -\varepsilon \le \theta \le 0. \tag{4.4.32}$$

This function satisfies

$$\bar{x}_t' = (d/d\theta)\bar{x}_t,\ \bar{x}_0 = \eta,\ 0 < t \le T. \tag{4.4.33}$$

Denote by $\mathcal{T}_t$, $t \ge 0$, the semigroup generated by the linear differential operator $d/d\theta$

and let Γ be the linear integral operator from $L_2(-\varepsilon, 0; \mathbb{R})$ to $\mathbb{R}$, defined by

$$\Gamma h = \int_{-\varepsilon}^{0} h_\theta \, d\theta, \; h \in L_2(-\varepsilon, 0; \mathbb{R}). \tag{4.4.34}$$

Note that $\|\Gamma\| \leq \sqrt{\varepsilon}$, and for

$$y_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix}, \; \zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}), \tag{4.4.35}$$

the system (4.4.31) can be written as

$$y_t' = Ay_t + F(t, y_t, u_t) + Bu_t, \; y_0 = \zeta, \tag{4.4.36}$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & d/d\theta \end{bmatrix}, \; B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \; F(t, y, u) = \begin{bmatrix} f(t, x, \Gamma \bar{x}, u) \\ 0 \end{bmatrix}, \tag{4.4.37}$$

assuming that

$$y = \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \in \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}). \tag{4.4.38}$$

The semigroup, generated by A, is expressed as

$$e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & \mathcal{T}_t \end{bmatrix}, \quad t \geq 0. \tag{4.4.39}$$

Therefore, one can calculated the controllability operator for the system (4.4.31) as

$$Q_T = \int_0^T e^{At}B^*Be^{A^*t}\,dt = \int_0^T \begin{bmatrix} b^2e^{2at} & 0 \\ 0 & 0 \end{bmatrix} dt = \begin{bmatrix} b^2(e^{2aT}-1)/2a & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.4.40}$$

and this is absolutely not a coercive operator. Taking into consideration that the main system is given by (4.4.31), and (4.4.36) is just the illustration of (4.4.31) in the standard form whereby the original state space $\mathbb{R}$ is enlarged to $\mathbb{R}\times L_2(-\varepsilon,0;\mathbb{R})$, we notice that sufficient conditions for complete controllability of the system (4.4.31) are in fact the same as for L-partial complete controllability of the system (4.4.36) if

$$L = [\,1 \quad 0\,] : \mathbb{R}\times L_2(-\varepsilon,0;\mathbb{R}) \to \mathbb{R}. \tag{4.4.41}$$

Calculating partial controllability operator of the enlarging system (4.4.36), we obtain

$$\tilde{Q}_T = LQ_TL^* = \frac{b^2(e^{2aT}-1)}{2a} > 0, \tag{4.4.42}$$

which is obviously coercive. Furthermore, using this estimation and simplifying

$$\|e^{At}\| \leq 1 + e^{aT}, \;\; \|Q_T\| = \gamma = \frac{b^2(e^{2aT}-1)}{2a}, \tag{4.4.43}$$

the inequality (4.4.12) can be estimated as

$$\left(1 + 1 + e^{aT} + \frac{2a}{b(e^{aT}-1)}\right)(1+e^{aT})TK < 1. \tag{4.4.44}$$

Therefore, if the Lipschitz coefficient K of the function F and terminal time T satisfying the inequality (4.4.44), the system (4.4.31)is completely controllable, which in other words means that the system (4.4.36) is L-partially complete controllable.

4.5 Partially Complete Controllability via Generalized Contraction Mapping Theorem

Consider the semilinear control system as

$$\begin{cases} x_t' = Ax_t + Bu_t + f(t, x_t, u_t), \ 0 < t \le T, \\ x_0 = \zeta \in X, \end{cases} \tag{4.5.1}$$

where x is a state process and u is a control. We assume that the following conditions hold.

(A) X, U, $\mathbb{H}$, A and B are defined as in the previous section;

(B) f is a nonlinear function from $[0,T]\times X\times U$ to X, satisfying

- f is continuous on $[0,T]\times X\times U$;
- f is bounded on $[0,T]\times X\times U$, and satisfies

$$\|f(t,x,u)\| \le \phi_t \text{ for all } (t,x,u) \in [0,T]\times X\times U;$$

- f is Lipschitz in respect of x and u, that is, there is $K \ge 0$ such that

$$\|f(t,x,u) - f(t,y,v)\| \le K(\|x-y\| + \|u-v\|) \forall t \in [0,T], x,y \in X \text{ and } u,v \in U;$$

(C) ϕ is a continuous nonnegative real-valued function on $[0,T]$.

Here, ϕ is a some sort adjusting function. Below we will put additional condition on ϕ so that to get a controllability property of the system (4.5.1).

Under these conditions the semilinear system (4.5.1) has a unique continuous solution in the mild sense for every $u \in C(0,T;U)$ and $\zeta \in X$ (see, Li and Yong (1995)), that is, there is a unique function $x \in C(0,T;X)$ such that

$$x_t = e^{At}\zeta + \int_0^t e^{A(t-r)}(Bu_r + f(r, x_r, u_r))\,dr. \tag{4.5.2}$$

Denote $U_{\text{ad}} = C(0,T;U)$, regarding this space as a set of admissible controls.

Let us recall that the controllability operator by

$$Q_t = \int_0^t e^{Ar}BB^* e^{A^* r}\,dr,\ 0 \le t \le T, \tag{4.5.3}$$

and its partial version $\tilde{Q}$ by

$$\tilde{Q}_t = LQ_tL^*,\ 0 \le t \le T.$$

The L-partial controllability operator $\tilde{Q}$ posses the following natural properties, which will be used without reference:

(i) $\tilde{Q}_0 = 0$;

(ii) $\tilde{Q}_t \ge 0\ \forall\ 0 \le t \le T$, that is $\langle \tilde{Q}_t h, h\rangle \ge 0\ \forall h \in \mathbb{H}$;

(iii) $\tilde{Q}_t - \tilde{Q}_s = \tilde{Q}_{t-s}\ \forall\ 0 \le s \le t \le T$;

(iv) $\tilde{Q}$ is increasing on $[0,T]$, that is $\langle (\tilde{Q}_t - \tilde{Q}_s)h, h\rangle \ge 0$ for all $h \in \mathbb{H}$ and $0 \le s \le t \le T$;

(v) $\|\tilde{Q}_s\| \le \|\tilde{Q}_t\|$ for every $0 \le s \le t \le T$;

(vi) $\tilde{Q}$ is a uniformly continuous function on $[0,T]$.

Here, (i) and (ii) are evident, (iii) comes straightforward from

$$\tilde{Q}_t - \tilde{Q}_s = \int_s^t Le^{Ar}BB^* e^{A^* r}L^*\,dr = \tilde{Q}_{t-s},$$

and (iv)–(vi) follow (iii). In a particular case, when $L = I$, then $Q = \tilde{Q}$ and (i)–(vi) hold for Q as well.

We will additionally assume that

(D) $\tilde{Q}_t$ is coercive for every $0 < t \leq T$.

The condition (D) means that there exists a $\gamma_t > 0$ such that

$$\langle \tilde{Q}_t h, h \rangle \geq \gamma_t \|h\|^2, \ \forall h \in \mathbb{H},$$

which, clearly from Chapter 2 implies the existence of the bounded inverse $\tilde{Q}_t^{-1}$ with

$$\|\tilde{Q}_t^{-1}\| \leq \frac{1}{\gamma_t} \text{ for every } 0 < t \leq T.$$

Obviously γ_t is not a unique number. Let $\gamma_t = \max S$, where S is defined by

$$S = \left\{ \beta > 0 : \langle \tilde{Q}_t h, h \rangle \geq \beta \|h\|^2 \text{ for all } h \in \mathbb{H} \right\}. \tag{4.5.4}$$

So, it is clear by condition (D) that the set S is nonempty, closed and bounded above. Therefore, $\gamma_t = \max S$ exists. Furthermore, the function $\gamma : (0, T] \to (0, \infty)$ have the following properties:

(a) γ is increasing, i.e., if $s < t$, then $\gamma_s \leq \gamma_t$;

(b) γ is a continuous function;

(c) $\lim_{t \to 0^+} \gamma_t = 0$.

Obviously, the integral $\int_0^T \frac{ds}{\gamma_s}$ is an improper integral since $\lim_{t \to 0^+} \gamma_t = 0$. Therefore, we additionally put an auxiliary function in this improper integral as a rate of convergence as follows.

(E) $\int_0^T (\phi_s/\gamma_{T-s})\,ds < \infty$.

As a special case, if there exists $\alpha > 0$ such that $\gamma_{T-s} \geq (T-s)^{1-\alpha}\phi_s \;\forall\; 0 \leq s \leq T$, then

$$\int_0^T \frac{\phi_s\,ds}{\gamma_{T-s}} = \int_0^T (T-s)^{\alpha-1}\,ds = \frac{T^\alpha}{\alpha} < \infty.$$

Therefore, the condition (E) holds.

Now, select $0 < \sigma < T$ and let $\tilde{U}_\sigma = C(0,\sigma;U)$ and $\tilde{X}_\sigma = C(0,\sigma;X)$. Then $\tilde{X}_\sigma \times \tilde{U}_\sigma$ is a Banach space endowed the norm

$$\|(\cdot,\cdot)\|_{\tilde{X}_\sigma\times\tilde{U}_\sigma} = \|\cdot\|_{\tilde{X}_\sigma} + \|\cdot\|_{\tilde{U}_\sigma}. \tag{4.5.5}$$

Let the operator $G_\sigma : \tilde{X}_\sigma \times \tilde{U}_\sigma \to \tilde{X}_\sigma \times \tilde{U}_\sigma$ be defined as

$$G_\sigma(y,v)_t = (Y_t, V_t),\; 0 \leq t \leq \sigma, \tag{4.5.6}$$

where

$$\begin{aligned} Y_t = e^{At}\zeta + Q_t e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^t e^{A(t-s)} f(s,y_s,v_s)\,ds \\ - \int_0^t Q_{t-s} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} f(s,y_s,v_s)\,ds \end{aligned} \tag{4.5.7}$$

and

$$V_t = B^* e^{A^*(T-t)} L^* \Big(\tilde{Q}_T^{-1}(h - Le^{AT}\zeta) - \int_0^t \tilde{Q}_{T-s}^{-1} e^{A(T-s)} f(s,y_s,v_s)\,ds \Big), \tag{4.5.8}$$

where $\zeta \in X$ and $h \in \mathbb{H}$.

Lemma 4.5.1 Assume that the conditions (A)–(D) hold. Then, for every $0 < \sigma < T$, $\zeta \in X$, and $h \in \mathbb{H}$, the nonlinear operator G_σ, which is defined by (4.5.6)–(4.5.8), has exactly one fixed point in $\tilde{X}_\sigma \times \tilde{U}_\sigma$.

Proof. It is evident that G_σ transforms $\tilde{X}_\sigma \times \tilde{U}_\sigma$ into $\tilde{X}_\sigma \times \tilde{U}_\sigma$. To complete the proof, it remains to demonstrate that the operator G_σ is a contraction mapping. To this end, Take two pair functions $(y, v), (z, w) \in \tilde{X}_\sigma \times \tilde{U}_\sigma$. Let

$$N = \sup_{0 \le t \le T} \|e^{At}\|. \tag{4.5.9}$$

Denote $G_\sigma(y, v) = (Y, V)$ and $G_\sigma(z, w) = (Z, W)$. Then

$$G_\sigma(y, v)_t - G_\sigma(z, w)_t = (Y_t - Z_t, V_t - W_t). \tag{4.5.10}$$

Here, $\|Y_t - Z_t\|_X$ can be evaluated as follows:

$$\begin{aligned}
\|Y_t - Z_t\| &\le \int_0^t \|Q_{t-s} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)}\| \cdot \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\
&\quad + \int_0^t \|e^{A(t-s)}\| \cdot \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\
&\le N^2 \int_0^t \frac{\|Q_{t-s}\|}{\gamma_{T-s}} \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\
&\quad + N \int_0^t \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\
&\le N^2 \int_0^t \frac{\|Q_T\|}{\gamma_{T-\sigma}} K(\|y_s - z_s\| + \|v_s - w_s\| ds \\
&\quad + N \int_0^t K(\|y_s - z_s\| + \|v_s - w_s\| ds \\
&\le \Big(1 + \frac{\|Q_T\| N}{\gamma_{T-\sigma}}\Big) NK \int_0^t (\|y_s - z_s\| + \|v_s - w_s\|) ds.
\end{aligned} \tag{4.5.11}$$

In a similar way, for $\|V_t - W_t\|$ we have

$$\begin{aligned}\|V_t - W_t\| &\le \|B^* e^{A^*(T-t)} L^*\| \int_0^t \|\tilde{Q}_{T-s}^{-1} e^{A(T-s)}\| \cdot \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\ &\le N^2 \|B\| \int_0^t \frac{1}{\gamma_{T-s}} \|f(s, y_s, v_s) - f(s, z_s, w_s)\| ds \\ &\le \frac{\|B\| N}{\gamma_{T-\sigma}} NK \int_0^t (\|y_s - z_s\| + \|v_s - w_s\|) ds. \qquad (4.5.12)\end{aligned}$$

From (4.5.11) and (4.5.12) together, we obtain the following inequality

$$\|G_\sigma(y, v)_t - G_\sigma(z, w)_t\| \le \mathbb{k} \int_0^t \|(y_s, v_s) - (z_s, w_s)\| ds, \qquad (4.5.13)$$

where

$$\mathbb{k} = \Big(1 + \frac{\|Q_T\| N}{\gamma_{T-\sigma}} + \frac{\|B\| N}{\gamma_{T-\sigma}}\Big) NK. \qquad (4.5.14)$$

Let $G_\sigma^0(y, v) = (y, v)$ and define $G_\sigma^n(y, v) = G_\sigma(G_\sigma^{n-1}(y, v))$. Then (4.5.14) implies

$$\begin{aligned}\|G_\sigma^2(y, v)_t - G_\sigma^2(z, w)_t\| &\le \mathbb{k} \int_0^t \|G_\sigma^1(y, v)_s - G_\sigma^1(z, w)_s\| ds \\ &\le \mathbb{k}^2 \int_0^t \int_0^s \|(y_r, v_r) - (z_r, w_r)\| dr ds.\end{aligned}$$

Repeating this process on G_σ, n times, we obtain

$$\begin{aligned}\|G_\sigma^n(y, v)_t - G_\sigma^n(z, w)_t\| &\le \mathbb{k}^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|(y_{t_n}, v_{t_n}) - (z_{t_n}, w_{t_n})\| dt_n \cdots dt_2 dt_1 \\ &\le \mathbb{k}^n \|(y, v) - (z, w)\|_{\tilde{X}_\sigma \times \tilde{U}_\sigma} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_2 dt_1 \\ &\le \frac{\mathbb{k}^n t^n}{n!} \|(y, v) - (z, w)\|_{\tilde{X}_\sigma \times \tilde{U}_\sigma}.\end{aligned}$$

Hence,

$$\|G_\sigma^n(y, v) - G_\sigma^n(z, w)\|_{\tilde{X}_\sigma \times \tilde{U}_\sigma} \le \frac{(\mathbb{k} T)^n}{n!} \|(y, v) - (z, w)\|_{\tilde{X}_\sigma \times \tilde{U}_\sigma}.$$

Since

$$\lim_{n\to\infty}\frac{(\Bbbk T)^n}{n!}=0,$$

one can easily conclude that for sufficiently large n

$$0<\frac{(\Bbbk T)^n}{n!}<1,$$

and hence for sufficiently large n, G_σ^n is a contraction mapping. Then by generalized contraction mapping theorem, the operator G_σ admits only one fixed point. ∎

Lemma 4.5.2 Under the conditions (A)–(E), let $0<\tau<\sigma<T$, $\zeta\in X$, and $h\in\mathbb{H}$. Define the operator G_σ by (4.5.6)–(4.5.8). Let (x,u) be a fixed point of G_σ in $\tilde{X}_\sigma\times\tilde{U}_\sigma$. Then the restriction $(x|_{[0,\tau]},u|_{[0,\tau]})$ of (x,u) from the interval $[0,\sigma]$ upon the interval $[0,\tau]$, is a fixed point of G_τ in $\tilde{X}_\tau\times\tilde{U}_\tau$.

Proof. From (4.5.6)–(4.5.8), one can easily notice that if $(Y,V)=G_\sigma(y,v)$, then

$$(Y|_{[0,\tau]},V|_{[0,\tau]})=G_\sigma(y,v)|_{[0,\tau]}=G_\tau(y|_{[0,\tau]},v|_{[0,\tau]}).$$

Therefore, if (x,u) is a fixed point of G_σ, then $(x|_{[0,\tau]},u|_{[0,\tau]})$ is a fixed point of G_τ. ∎

Lemma 4.5.3 Under the conditions (A)–(F), let $\zeta\in X$, and $h\in\mathbb{H}$. Then the operator G_T, defined by (4.5.6)–(4.5.8) for $\sigma=T$, has a unique fixed point in $\tilde{X}_T\times\tilde{U}_T$.

Proof. By virtue of Lemma 4.5.2, one can find a unique pair (x,u) of X- and U-valued continuous functions on $[0,T)$ satisfying

$$\begin{aligned}x_t&=e^{At}\zeta+Q_te^{A^*(T-t)}L^*\tilde{Q}_T^{-1}(h-Le^{AT}\zeta)+\int_0^t e^{A(t-s)}f(s,x_s,u_s)\,ds\\&\quad-\int_0^t Q_{t-s}e^{A^*(T-t)}L^*\tilde{Q}_{T-s}^{-1}e^{A(T-s)}f(s,x_s,u_s)\,ds\end{aligned}\tag{4.5.15}$$

and

$$u_t = B^* e^{A^*(T-t)} L^* \Big(\tilde{Q}_T^{-1}(h - Le^{AT}\zeta) - \int_0^t \tilde{Q}_{T-s}^{-1} e^{A(T-s)} f(s, x_s, u_s)\, ds \Big). \qquad (4.5.16)$$

Since

$$\|Q_{t-s} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} f(s, x_s, u_s)\| \leq \frac{\|Q_T\| N^2 \phi_s}{\gamma_{T-s}}, \; 0 \leq s \leq T,$$

and by condition (F) the improper integral

$$\int_0^T \frac{\phi_s}{\gamma_{T-s}} ds$$

is convergent. Hence, by equation (4.5.15) x_t is well-defined with $\lim_{t \to T} x_t = x_T$. The same argument as above applies, is used for (4.5.16) as well, and consequently the pair point (x, u) is well-defined in $\tilde{X}_T \times \tilde{U}_T = C(0, T; X) \times C(0, T; U)$. Furthermore, $(x|_{[0,\sigma]}, u|_{[0,\sigma]}) = G_\sigma(x|_{[0,\sigma]}, u|_{[0,\sigma]})$ for all $0 < \sigma < T$ and the continuity of (x, u) gives $G_T(x, u) = (x, u)$. Therefore, the couple (x, u), defined by (4.5.15)–(4.5.16) on $[0, T]$, is a fixed point of G_T. On the other hand, if G_T has two fixed points, then G_σ must have two distinct fixed points for some $0 < \sigma < T$ which contradicts Lemma 4.5.1. Thus the couple (x, u), given by (4.5.15)–(4.5.16), is a unique fixed point of G_T. ■

Theorem 4.5.4 Suppose that the conditions (A)–(E) hold. Then the semilinear control system (4.5.1) is L-partially D^c- controllable.

Proof. Let $\zeta \in X$ and $h \in \mathbb{H}$ be arbitrary. We have to demonstrate that there exists $u \in U_{\text{ad}}$ such that $Lx_T = h$, where x is a mild solution of (4.5.2) corresponding to this u. Let (x, u) be a fixed point of G_T, defined by (4.5.6)–(4.5.8) for $\sigma = T$. Then (x, u)

satisfies (4.5.15)–(4.5.16). So, x can be represented as follows:

$$\begin{aligned}
x_t &= e^{At}\zeta + \int_0^t e^{A(t-s)} f(s, x_s, u_s)\,ds \\
&\quad + \int_0^t e^{A(t-s)} BB^* e^{A^*(t-s)} e^{A^*(T-t)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta)\,ds \\
&\quad - \int_0^t \int_s^t e^{A(t-r)} BB^* e^{A^*(t-r)} e^{A^*(T-t)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} f(s, x_s, u_s)\,dr ds \\
&= e^{At}\zeta + \int_0^t e^{A(t-r)} f(r, x_r, u_r)\,dr \\
&\quad + \int_0^t e^{A(t-r)} BB^* e^{A^*(T-r)} L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta)\,dr \\
&\quad - \int_0^t \int_0^r e^{A(t-r)} BB^* e^{A^*(T-r)} L^* \tilde{Q}_{T-s}^{-1} e^{A(T-s)} f(s, x_s, u_s)\,ds dr \\
&= e^{At}\zeta + \int_0^t e^{A(t-r)} f(r, x_r, u_r)\,dr + \int_0^t e^{A(t-r)} B u_r\,dr.
\end{aligned}$$

Hence, the fixed point (x, u) of G_T can be interpreted as that $u \in U_{\text{ad}}$ and $x \in \tilde{X}_T$ is a mild solution of the equation (4.5.1), corresponding to u. Moreover, we have

$$\begin{aligned}
Lx_T &= Le^{AT}\zeta + LQ_T L^* \tilde{Q}_T^{-1}(h - Le^{AT}\zeta) + \int_0^T e^{A(T-s)} f(s, x_s, u_s)\,ds \\
&\quad - \int_0^T LQ_{T-s} L^* \tilde{Q}_{T-s}^{-1} f(s, x_s, u_s)\,ds = h.
\end{aligned}$$

Therefore, for every $\zeta \in X$ and $h \in \mathbb{H}$, there exists $u \in U_{\text{ad}}$ such that $Lx_T = h$. This means that the system (4.5.1) is L-partially complete controllable. ■

We investigate the result of Theorem 4.5.4 in the following examples of some appropriate control systems.

Example 4.5.5 The condition (F) in Theorem 4.5.4 is very hard condition. For instance, if we select $\phi_t = 1$, then condition (F) becomes

$$\int_0^T \frac{\phi_t\,dt}{\gamma_t} = \int_0^T \frac{dt}{\gamma_{T-t}} < \infty.$$

This improper integral diverges even for very simple examples. Consider a one-dimensional linear equation

$$x_t' = ax_t + bu_t,\ 0 \le t \le T, \tag{4.5.17}$$

where $a \neq 0$ and $b \neq 0$. Clearly, the control system (4.5.17) is completely controllable for the time $T > 0$ since

$$Q_T = \int_0^T e^{at} b b e^{at}\, dt = \frac{b^2(e^{2aT}-1)}{2a} > 0. \tag{4.5.18}$$

Here $\gamma_t = \frac{b^2(e^{2at}-1)}{2a}$. Hence

$$\begin{aligned}\int_0^T \frac{dt}{\gamma_t} &= \frac{1}{b^2}\int_0^T \frac{2a}{e^{2at}-1}\, dt = -\frac{1}{b^2}\int_0^T \Big(2a + \frac{2ae^{2at}}{1-e^{2at}}\Big) dt \\ &= -\frac{2aT + \ln|1-e^{2aT}|}{b^2} + \frac{\lim_{t\to 0^+}\ln|1-e^{2at}|}{b^2} = \infty.\end{aligned}$$

Therefore, the function ϕ should be selected so as to be able to guarantee the convergence of the improper integral in (F).

Remark 4.5.6 It is clear that Example 4.5.5 proves that Theorem 4.5.4 establishes just sufficient condition of L- partial complete controllability. In that example L was taken as an identity operator I and $f = 0$ satisfies condition (B).

Example 4.5.7 Consider the system of differential equations

$$\begin{cases} x_t' = y_t + bu_t,\ x_0 \in \mathbb{R}, \\ y_t' = f(t, x_t, y_t, u_t),\ y_0 \in \mathbb{R}, \end{cases} \tag{4.5.19}$$

on $[0, T]$, where $u \in U_{\text{ad}} = C(0, T; \mathbb{R})$. This control system can be written in $\mathbb{R}^2$ as the following standard semilinear system

$$z_t' = Az_t + F(t, z_t, u_t) + Bu_t, \tag{4.5.20}$$

where

$$z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \; F(t, z, u) = \begin{bmatrix} 0 \\ f(t, x, y, u) \end{bmatrix}, \tag{4.5.21}$$

assuming that

$$z = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4.5.22}$$

One can calculate that

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \tag{4.5.23}$$

The controllability operator is

$$Q_t = \int_0^t e^{As} BB^* e^{A^* s} ds = b^2 t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; 0 < t \leq T. \tag{4.5.24}$$

Hence, Q_t is not coercive and the conditions for complete controllability, based on coercivity of Q_t, fail for this example. We can examine the partial complete controllability for this system being interested in just the first component x_t of z_t.

Let $L = [\ 1 \quad 0\ ]$. Then

$$\tilde{Q}_t = LQ_T L^* = b^2 t > 0. \tag{4.5.25}$$

Therefore, $\tilde{Q}_t$ is coercive for all $0 < t \leq T$. Here $\gamma_t = b^2 t$. So, if $\phi_t \leq (T - t)^{\alpha}$ for some

$\alpha > 0$, then

$$\int_0^T \frac{\phi_t}{\gamma_{T-t}} dt \le \frac{1}{b^2} \int_0^T \frac{(T-t)^\alpha}{(T-t)} dt = \frac{1}{b^2} \int_0^T t^{\alpha-1} dt = \frac{T^\alpha}{b^2 \alpha} < \infty.$$

Thus if, additionally, f satisfies condition (B), then the system (4.5.20) is L-partially complete controllable on $[0, T]$.

Example 4.5.8 Delay equations are typical for application of partial controllability concepts. Consider the semilinear delay equation

$$\begin{cases} x_t' = a x_t + b u_t + f(t, x_t, \int_{-\varepsilon}^0 x_{t+\theta} d\theta, u_t), \\ x_0 = \zeta, \ x_\theta = \eta_\theta, \ -\varepsilon \le \theta \le 0, \end{cases} \tag{4.5.26}$$

on $[0, T]$, where $a \neq 0$, $b \neq 0$, $0 < \varepsilon < T$, $\zeta \in \mathbb{R}$, $\eta \in L_2(-\varepsilon, 0; \mathbb{R})$ (the space of square integrable functions) and $u \in U_{\text{ad}} = C(0, T; \mathbb{R})$.

Introduce the function $\bar{x} : [0, T] \to L_2(-\varepsilon, 0; \mathbb{R})$ by

$$[\bar{x}_t]_\theta = x_{t+\theta}, \ 0 \le t \le T, \ -\varepsilon \le \theta \le 0. \tag{4.5.27}$$

This function is a solution of

$$\bar{x}_t' = (d/d\theta)\bar{x}_t, \ \bar{x}_0 = \eta, \ 0 < t \le T. \tag{4.5.28}$$

Denote by $\mathcal{T}_t$, $t \ge 0$, the semigroup generated by the differential operator $d/d\theta$ and let Γ be the integral operator from $L_2(-\varepsilon, 0; \mathbb{R})$ to $\mathbb{R}$, defined by

$$\Gamma h = \int_{-\varepsilon}^0 h_\theta d\theta, \ h \in L_2(-\varepsilon, 0; \mathbb{R}).$$

Then for

$$y_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix}, \ \xi = \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \in \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}),$$

the system (4.5.26) can be written as

$$y_t' = Ay_t + Bu_t + F(t, y_t, u_t), \ y_0 = \xi, \tag{4.5.29}$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & d/d\theta \end{bmatrix}, \ B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \ F(t, y, u) = \begin{bmatrix} f(t, x, \Gamma\bar{x}, u) \\ 0 \end{bmatrix},$$

where the variable y consists of two components:

$$y = \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \in \mathbb{R} \times L_2(-\varepsilon, 0; \mathbb{R}).$$

The semigroup, e^{At} has the form

$$e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & \mathcal{T}_t \end{bmatrix}, \ t \geq 0.$$

Then

$$Q_t = \int_0^t e^{As} B^* B e^{A^* s} \, ds = \int_0^t \begin{bmatrix} b^2 e^{2as} & 0 \\ 0 & 0 \end{bmatrix} dt = \frac{b^2(e^{2at} - 1)}{2a} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is obviously not a coercive operator.

Taking into consideration that the original system is given by (4.5.26), and (4.5.29) is just an illustration of (4.5.26) in the standard form, enlarging the original state space $\mathbb{R}$ to $\mathbb{R}\times L_2(-\varepsilon,0;\mathbb{R})$, one can see that the complete controllability of the system (4.5.26) is in fact L-partial complete controllability of the system (4.5.29) if

$$L = [\,1 \quad 0\,] : \mathbb{R}\times L_2(-\varepsilon,0;\mathbb{R}) \to \mathbb{R}.$$

Calculating partial controllability operator $\tilde{Q}_t$, we have

$$\tilde{Q}_t = LQ_tL^* = \frac{b^2(e^{2at}-1)}{2a} > 0,$$

So, $\tilde{Q}_t$ is coercive for $0 < t \le T$, and evidently

$$\gamma_t = \frac{b^2(e^{2at}-1)}{2a}.$$

Hence, if $\phi_t \le a(e^{2a(T-t)}-1)(T-t)^{1-\alpha}$ for some $\alpha > 0$, then

$$\int_0^T \frac{\phi_t}{\gamma_{T-t}}\,dt \le \frac{2a^2}{b^2}\int_0^T (T-t)^{1-\alpha}\,dt = 2\frac{a^2T^\alpha}{b^2\alpha} < \infty.$$

On the other hand, if additionally f satisfies condition (B), then the system (4.5.26) is L-partially complete controllable on $[0,T]$ and consequently the system (4.5.20) is completely controllable for the time $T > 0$.

Chapter 5

CONCLUSION

In this thesis two sufficient conditions for the partially complete controllability of semilinear deterministic control systems are derived, provided that the associated linear control system is partially complete controllable. The first of them uses the contraction mapping theorem and very similar to non-partial case. The second one is based on generalized contraction mapping theorem. In this part, in order to apply the generalized contraction mapping theorem, the convergence of the improper integral in (F) is required and for this purpose an additional condition on the nonlinear part of the given control in (D) is needed to select a suitable control.

These sufficient conditions are often applied to higher order differential equations, wave equations and delay equations. Moreover, there are also different kinds of control systems which besides semilinearity include impulsiveness, fractional derivatives, randomness etc. The results of this thesis can be extended in some cases to cover these systems as well.

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