A Comprehensive Study on the Class of *q*-Appell Polynomials

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ABSTRACT

This thesis is aimed to study the *q*-analogue of the class of so called Appell polynomials from different aspects and using various algebraic as well as analytic approaches. To achieve this aim, not only many new results are found based on a proposed general generating function for all members belonging to the aforementioned family of polynomials, but also various relations between famous members of this family are derived. *2D q*-Appell polynomials as the *q*-Appell polynomials in two variables can be considered as another new achievement of this thesis. In addition to the definition of the class of *q*-Appell polynomials by means of their generating function, a determinantal representation, for the first time, is proposed for indicating different members of the class of *q*-Appell polynomials. Moreover, it is shown that how easy some results can be proved by using the new proposed linear algebraic indication and applying basic properties of determinant. In the sequel, this family of *q*-polynomials are studied also from *q*-umbral point of view and many interesting results are found based on this algebraic approach.

Keywords: *q*-Appell, *q*-Calculus, Determinatal, *q*-Umbral, *q*-Polynomilas, *q*-Apostol, *q*-Bernoulli, *q*-Euler, *q*-Genocchi, *q*-Hermite.

ÖZ

Bu tez farklı açılardan ve çeşitli cebirsel yanı sıra analitik yaklaşımlar kullanarak *q*-Appell polinomların sınıfının incelenmesini amaçlanmaktadır. Bu amaca ulaşmak için, yukarıda belirtilen *q*-Appell polinomlar ailesine ait tüm üyeler üyeleri arasında çeşitli ilişkiler elde edilmektedir. İki değişkenli *q*-Appell polinomları olarak *2D q*-Appell polinomları bu tezin yeni bir başarı olarak kabul edilebilir. Ayrıca, bazı sonuçlar yeni önerilen lineer cebirsel gösterge kullanılarak ve determinantın temel özelliklerini uygulanarak ispat edilebilir. Ayrıca, bu tezde *q*-polinomların birçok ilginç özellikleri *q*-umbral açısından da incelenmiştir.

Anahtar Kelimeler: *q*-Appell, *q*-Matematik, *q*-Umbral, *q*-Polynomlar, *q*-Apostol, *q*-Bernoulli, *q*-Euler, *q*-Genocchi, *q*-Hermite.

TO My Compassionate Parents,

To My Lovely Brother and Sisters,

and

To My Everlasting Love

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Chapter 1

INTRODUCTION

Over the last centuries the study on various kinds of polynomials has been a significant part of mathematical research. Polynomials are important since not only they can be considered as some algebraic objects, but also they can be looked as functions in one or more variables. Among all various types of polynomials, the class of Appell polynomials has attracted the notice of many mathematicians because of their interesting characteristics. Since 1880, when Paul mile Appel for the first time defined a new class of polynomials which later became famous upon his name until the present day, a wide range of research has been conducted on the various members of the family of Appell polynomials as well as their q-analogues. The vast literature in this subject consists of various definitions, relations, properties, as well as extensions and generalizations. The study on these polynomials not only is vital in different mathematical branches such as theory of orthogonal polynomials and special functions, analytic number theory, combinatorics, probability and so on, but also they have many applications in some other research fields such as mathematical physics, signals and image processing, as well as electrical and computer engineering.

This research is basically purposed to study the class of q-Appell Polynomials. To do this, the first section in chapter two is provided in order to make the general reader familiar with the frequently used definitions and notations in this thesis. Moreover. the second section of chapter two gives a brief information about the classical Appell and Sheffer polynomials.

In chapter three 2D q-Appell polynomials are introduced as the class of q-Appell polynomials in two variable. As some famous examples of this family, $2D$ q-Bernoulli polynomials, $2D$ q-Euler polynomials, and $2D$ q-Genocchi polynomials are introduced and a various important properties and relations such as the explicit relation between q-Bernoulli and q-Euler polynomials as well as q-Genocchi and the q-Bernoulli Polynomials are obtained. Indeed, all the obtained facts in this section can be considered as the generalization of the formerly defined q-Appell polynomials.

In chapter four, the main attempt is to specify the characteristics and to show the properties of a family involving the q-analogue of Apostol type polynomials. The q-analogue of the Luo-Srivastava addition theorem is one of the most important results of this chapter.

In chapter five, a determinantal representation is proposed for indicating the family of q-Appell polynomials. Next, it is shown that this new representation how well coincides with the original definition of the aforementioned family of polynomials. Based on this new linear algebraic approach, also, it shown that many interesting results can be obtained easily, only by applying the elementary properties of determinant. At the end of this chapter, the coefficients used for writing the determinantal representation of some specific families of q-Appell polynomials, as some examples, are calculated.

Eventually, in chapter six, q-Appell polynomials are viewed from q-Umbral per-spective. Inspired by this algebraic approach, some obtained properties of q-Appell polynomials in the previous chapters are recast. Also, using q-Umbral techniques some new interesting results are obtained for the family of q-Genocchi polynomials. Indeed, similar results can be derived for the other members of the class of q-Appell polynomials. The essence of the results in this part of the study is concealed behind the fact that any arbitrary polynomial can be written based on a linear combination of q-Genocchi polynomials.

Chapter 2

PRELIMINARIES AND DEFINITIONS

2.1 Introduction

The main aim of this chapter is to make the general reader familiar with the expressions and notations which will appear quite frequently in the following chapters. One of the simple but important sections of this chapter is devoted to introduce q-Calculus related notations and miscellaneous q-formulas. Next, as the foundation of the q-Appell polynomials and their generalizations, the corresponding definitions of the classical polynomials to them will be introduced.

2.2 q-Calculus and its Commonly Used Notations

Since the first attempts in the appearance of q-calculus, the eighteenth century, while Leonard Euler defined the number q in his book, [1], up to the nowadays broad range of researches, q-Calculus has attracted a great interest of mathematicians as well as physicists because of its wide domain of application not only in mathematics, but also in some other fields such as theoretical physics, engineering, computer sciences, and so on, [2], [3], [4]. Nonetheless, the work on q-calculus day by day is progressing, there is still much to do in this arena and q-calculus has the capacity to be developed more.

The theory of q-calculus is embedded in the theory of q-analysis and q-special functions. As the result, before starting the main discussion, which is clearly related to various members of the family of q-Appell polynomials and lies in q-analysing and studying q-analogues of them, in the following a brief introductory about the q-numbers

q-notations is given. For all the definitions related to this section the interested readers are referred to [5], [6].

Definition 2.1. The q-number a is defined as

$$[a]_q = \frac{1-q^a}{1-q}, \qquad q \in \mathbb{C}\backslash\{1\}, a \in \mathbb{C}, q^a \neq 1. \tag{2.2.1}$$

Particularly, for $n \in \mathbb{N}$, the above definition changes to

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}. \tag{2.2.2}$$

In this case as $\lim_{q\to 1}[n]_q = \lim_{q\to 1}(1 + q + q^2 + ... + q^{n-1}) = n$, $[n]_q$ is called the q-analogue of n.

Definition 2.2. The q-factorial is defined as

$$[0]! = 1, \quad [n]_q! = [1]_q[2]_q \dots [n]_q, \qquad n \in \mathbb{N}, \tag{2.2.3}$$

also,

$$[2n]_q!! = [2n]_q[2n-2]_{q...}[2]_q \tag{2.2.4}$$

Remark 2.3. Clearly,

$$\lim_{q\to 1}[a]_q = a, \qquad \lim_{q\to 1}[n]_q! = n!. \tag{2.2.5}$$

Definition 2.4. The q-shifted factorial is defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1-q^j a), \quad n \in \mathbb{N}, \tag{2.2.6}$$

and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-q^j a), \quad |q| < 1, a \in \mathbb{C}. \tag{2.2.7}$$

Definition 2.5. The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad k,\ n \in \mathbb{N} \tag{2.2.8}$$

Proposition 2.6. *The following facts hold true for the q-binomial coefficient*

a) $\left[\begin{array}{c} n \\ k \end{array}\right]_q = \dfrac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$,

b) $\left[\begin{array}{c} n \\ k \end{array}\right]_q = \left[\begin{array}{c} n \\ n-k \end{array}\right]_q$,

c) For $\quad k < l < n$

$$\left[\begin{array}{c} n \\ l \end{array}\right]_q \left[\begin{array}{c} l \\ k \end{array}\right]_q = \left[\begin{array}{c} n \\ k \end{array}\right]_q \left[\begin{array}{c} n-k \\ n-l \end{array}\right]_q .$$

Definition 2.7. The q-analogue of the function $(x+y)^n$, is defined as

$$(x+y)_q^n := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k, \qquad n \in \mathbb{N}_0. \tag{2.2.9}$$

Definition 2.8. The q-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1} (1-q^j a) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} (-1)^k a^k. \tag{2.2.10}$$

Definition 2.9. In the standard approach to the q-calculus, the two following q-exponential functions are used:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \ |z| < \frac{1}{|1-q|}, \tag{2.2.11}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1+(1-q)q^k z), \quad 0 < |q| < 1, \ z \in \mathbb{C}. \tag{2.2.12}$$

Definition 2.10. The q-derivative of a function f at point $0 \neq z \in \mathbb{C}$ is defined as

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1. \tag{2.2.13}$$

Proposition 2.11. *Consider two arbitrary functions $f(z)$, and $g(z)$. The following relations hold true for their q-derivatives, [6]:*

a) if f is differentiable,

$$lim_{q\to 1} D_q f\left(z\right) = \frac{df(z)}{dz},$$

where $\frac{d}{dz}$ indicates the ordinary derivative defined in Calculus.

b) D_q is a linear operator; that is for arbitrary constants a and b

$$D_q(af(z) + bg(z)) = aD_q(f(z)) + bD_q(g(z)),$$

c) $D_q(f(z)g(z)) = f(qz)D_q g(z) + g(z)D_q f(z),$

d) $D_q(\dfrac{f(z)}{g(z)}) = \dfrac{g(qz)D_q f(z) - f(qz)D_q g(z)}{g(z)g(qz)}.$

Remark 2.12. As the direct result of definition (2.9) we have $e_q(z)E_q(-z) = 1$. Moreover, from the definition (2.10), it can be seen easily that

$$D_q e_q\left(z\right) = e_q\left(z\right), \quad D_q E_q\left(z\right) = E_q\left(qz\right). \tag{2.2.14}$$

Definition 2.13. The q-analogue of Taylor series expansion of an arbitrary function $f(z)$ for $0 < q < 1$, is defined as, [7]

$$f(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} D_q^n f(a)(z-a)_q^n, \tag{2.2.15}$$

$f(a)$ is the n^{th} q-derivative of the function f at point a.

Definition 2.14. Jakson integral of an arbitrary function $f(x)$ is defined as

where $D_q^n \int f(x) d_q x = (1-q) \sum_{j=0}^{\infty} x q^j f(xq^j), \ \ 0 < q < 1.$ (2.2.16)

2.3 The Main Classical Appell Polynomials

The study on various classes of polynomials has been a significant part of researches in algebra as well as other related mathematical branches such as real and complex analysis, orthogonal polynomials and special functions. Polynomials are important since not only they can be considered as some algebraic objects, but also they can be looked as functions in one or more variables. Generally, when we talk about polynomials we mean a linear combination $\sum\limits_{i=0}^{n} a_i x^i$, for real or complex coefficients a_i and arbitrary variable x. The purpose of this section is to introduce some of the classical polynomials such as Bernoulli, Euler, Genocchi, Apostol type, and Hermite polynomials as famous members of the class of Appell and Sheffer polynomials and some of their basic generalizations and properties, in order to give a bird's-eye view to the general readers for a better understanding of the concepts of the next chapters.

2.3.1 Classical Bernoulli, Euler, and Genocchi Polynomials

Since the seventeenth century until the present day a wide range of research has been conducted on the classical Bernoulli, Euler and Genocchi as well as Hermite numbers and polynomials. Among the vast publications in this subject, various definitions, relations, properties, as well as generalizations can be found. These polynomials not only are important in the theory of orthogonal polynomials and special functions, but also they have various applications in many other mathematical fields such as analytic number theory, combinatorics, probability and so on.

Definition 2.15. Classical Bernoulli polynomials $B_n(x)$, and numbers $B_n = B_n(0)$ are defined by means of the following generating functions, [8]-[12]

$$\frac{t}{e^t-1}e^{tx} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{2.3.1}$$

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{2.3.2}$$

respectively.

Definition 2.16. Classical Euler polynomials $E_n(x)$, and numbers $E_n = E_n(0)$ are defined by means of the following generating functions, [8]-[12]

$$\frac{2}{e^t+1}e^{tx} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \qquad |t| < \pi, \qquad \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}, \qquad |t| < \pi, \tag{2.3.3}$$

respectively.

Definition 2.17. Classical Genocchi polynomials $G_n(x)$, and numbers $G_n = G_n(0)$ are defined by means of the following generating functions, [10],[12], [13]

$$\frac{2t}{e^t+1}e^{tx} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \qquad |t| < \pi, \tag{2.3.4}$$

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n\frac{t^n}{n!}, \qquad |t| < \pi, \tag{2.3.5}$$

respectively.

Remark 2.18. As the direct results of the above definitions, for the classical Bernoulli, Euler, and Genocchi numbers we have

$$B_n(0) = B_n = (-1)^n B_n(1) = (2^{1-n}-1)^{-1}B_n(\frac{1}{2}),$$

$$E_n(0) = E_n = 2^n E_n(\frac{1}{2}),$$

$$G_n(0) = G_n, G_1 = 1, G_3 = G_5 = G_7 = \ldots = 0, \ and$$

$$G_{2n} = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}(0).$$

respectively, [8], [14], [15].

Remark 2.19. The classical Bernoulli, Euler, and Genocchi numbers also can be defined by the following recurrence relations, [16], [17]

$$B_0 = 1, \ (n+1)B_n = -\sum_{k=0}^{n+1} B_k, \tag{2.3.6}$$

$$E_n + 2^{n-1}\sum_{k=0}^{n-1}\binom{n}{k}\frac{E_k}{2^k} = 1, \quad n \geq 1, \tag{2.3.7}$$

$$2G_n + \sum_{k=0}^{n-1}\binom{n}{k}G_k = 0, \quad n \geq 2, \tag{2.3.8}$$

, respectively.

Proposition 2.20. *The following relations hold true for the classical Bernoulli, Euler, and Genocchi polynomials, respectively.*

$$\begin{aligned} B_n(x+1) - B_n(x) &= nx^{n-1}, \\ E_n(x+1) + E_n(x) &= 2x^n, \\ G_n(x+1) + G_n(x) &= 2nx^{n-1}. \end{aligned}$$

Proof. Since in the proofs the same technique is applied, only the proof of the third relation is given. The proof is based on the following identity

$$\begin{aligned} \sum_{n=0}^{\infty}(G_n(x+1) + G_n(x))\frac{t^n}{n!} &= \frac{2t}{e^t+1}e^{t(x+1)} + \frac{2t}{e^t+1}e^{tx} \\ &= \frac{2t}{e^t+1}e^{tx}(e^t+1) = 2te^{tx} \\ &= 2t\sum_{n=0}^{\infty}\frac{t^n x^n}{n!} = \sum_{n=1}^{\infty}2nx^{n-1}\frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, gives the desired result. □

2.3.2 Apostol Type Polynomials

In 1951, Apostol introduced an analogue for the classical Bernoulli polynomials and numbers and obtained some interesting relations for them including their elementary properties as well as the recursion formula for the Apostol-Bernoulli numbers, [18]. Later this analogue appeared in a wide range of mathematical publications as Apostol-Bernoulli polynomials, [19]-[22].

Definition 2.21. The Apostol-Bernoulli polynomials $B_n(x;\lambda)$, and numbers $B_n(0;\lambda) = B_n(\lambda)$ are defined by means of the following generating functions, [18]

$$\frac{t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x;\lambda) \frac{t^n}{n!}, \qquad |t| < 2\pi, \text{ when } \lambda = 1; |t| < |\log \lambda|, \text{ when } \lambda \neq 1, \tag{2.3.9}$$

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}, \qquad |t| < 2\pi, \text{ when } \lambda = 1; |t| < |\log \lambda|, \text{ when } \lambda \neq 1, \tag{2.3.10}$$

respectively.

In 2005, inspired by the Apostol's analogue for the Bernoulli polynomials, Luo introduced Apostol-Euler polynomials, [23]. Next, Luo and Srivastava generalized these definitions to the Apostol-Bernoulli and Apostol Euler polynomials of order α, [24], [25]. In 2009, Luo gradually, defined Apostol-Genocchi polynomilas and numbers and developed his definition to order α, [26]. Recently, many interesting results and generalizations have been obtained for the Apostol-Bernoulli and Apostol-Euler polynomials as well as Apostol-Genocchi polynomials, [27], [29], [30]-[33]. In the following the corresponding definitions to the above mentioned polynomials are given.

Definition 2.22. The Generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x;\lambda)$, and numbers $B_n^{\alpha}(0;\lambda) = B_n^{\alpha}(\lambda)$ of order α are defined by means of the following generating functions, [18]

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} B_n^{\alpha}(x;\lambda)\frac{t^n}{n!},$$
$$|t| < 2\pi, \text{ when } \lambda = 1;\ |t| < |\log\lambda|, \text{ when } \lambda \neq 1, \tag{2.3.11}$$

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{\alpha}(\lambda)\frac{t^n}{n!},$$
$$|t| < 2\pi, \text{ when } \lambda = 1;\ |t| < |\log\lambda|, \text{ when } \lambda \neq 1, \tag{2.3.12}$$

respectively.

Definition 2.23. The Generalized Apostol-Euler polynomials $E_n^{\alpha}(x;\lambda)$, and numbers $E_n^{\alpha}(0;\lambda) = E_n^{\alpha}(\lambda)$ of order α are defined by means of the following generating functions, [23]

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} E_n^{\alpha}(x;\lambda)\frac{t^n}{n!}, \qquad |t| < |\log(-\lambda)|;\ 1^{\alpha} := 1, \tag{2.3.13}$$
$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} = \sum_{n=0}^{\infty} E_n^{\alpha}(\lambda)\frac{t^n}{n!}, \qquad |t| < |\log(-\lambda)|;\ 1^{\alpha} := 1, \tag{2.3.14}$$

respectively.

Definition 2.24. The Generalized Apostol-Genocchi polynomials $G_n^{\alpha}(x;\lambda)$, and numbers $G_n^{\alpha}(0;\lambda) = G_n^{\alpha}(\lambda)$ of order α are defined by means of the following generating functions, [26]

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} G_n^{\alpha}(x;\lambda)\frac{t^n}{n!}, \qquad |t| < |\log(-\lambda)|;\ 1^{\alpha} := 1, \tag{2.3.15}$$
$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} = \sum_{n=0}^{\infty} G_n^{\alpha}(\lambda)\frac{t^n}{n!}, \qquad |t| < |\log(-\lambda)|;\ 1^{\alpha} := 1, \tag{2.3.16}$$

respectively.

Remark 2.25. Indeed, taking $\alpha = 1$, $\lambda = 1$, and $x = 0$, in the definitions of the Generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials of order α, we obtain

$$B_n^{\alpha}(0;1) = B_n, \quad E_n^{\alpha}(0;1) = E_n, \quad G_n^{\alpha}(0;1) = G_n,$$

respectively.

2.3.3 Appell Polynomials

In 1880, Appell defined a set of interesting polynomials which later was called the set of Appell polynomials upon his name.

Definition 2.26. The set of any n-degree polynomials $\{A_n(x)\}_{n=0}^{\infty}$ are called the set of Appell polynomials if they satisfy the following recurrence relation, [34].

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x), \quad n = 1, 2, ... \tag{2.3.17}$$

Remark 2.27. Appell polynomials also can be defined by means of the following generating function

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{2.3.18}$$

where $A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$, with real coefficients $A_n, n = 0, 1, 2, ...$ and $A_0 \neq 0$, [34].

Remark 2.28. Based on the different selections of $A(t)$ in the above definition various Appell type polynomials are obtained. In the following table some of them are mentioned.

Table 2.1: Various members of the family of Appell polynomials

Number	$A(t)$	$A_n(x)$	Polynomials
1	$\frac{t}{e^t-1}$	$B_n(x)$	Classical Bernoulli Polynomilas
2	$\frac{2}{e^t+1}$	$E_n(x)$	Classical Euler Polynomilas
3	$\frac{2t}{e^t+1}$	$G_n(x)$	Classical Genocchi Polynomilas
4	$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}$	$B_n^{\alpha}(x;\lambda)$	The generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x;\lambda)$ of order α
5	$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}$	$E_n^{\alpha}(x;\lambda)$	The generalized Apostol-Euler polynomials $B_n^{\alpha}(x;\lambda)$ of order α
6	$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}$	$G_n^{\alpha}(x;\lambda)$	The generalized Apostol-Genocchi polynomials $B_n^{\alpha}(x;\lambda)$ of order α
7	$\left(\frac{t}{e^t-\sum_{k=0}^{m-1}\frac{t^k}{k!}}\right)^{\alpha}$		The new Generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x;\lambda)$ of order α
8	$\left(\frac{2}{e^t+\sum_{k=0}^{m-1}\frac{t^k}{k!}}\right)^{\alpha}$		The new generalized Apostol-Euler polynomials $B_n^{\alpha}(x;\lambda)$ of order α
9	$\left(\frac{2t}{e^t+\sum_{k=0}^{m-1}\frac{t^k}{k!}}\right)^{\alpha}$		The new generalized Apostol-Genocchi polynomials $B_n^{\alpha}(x;\lambda)$ of order α

2.3.4 Sheffer Polynomials

In 1939, Sheffer generalized the definition of Appell polynomials and as the result introduced and studied a new family of polynomials, under the title the set of polynomials of type zero. Later, some other mathematicians introduced this class of polynomials in different ways and showed that their definitions coincide exactly with the original definition proposed by him. One of these novel definitions is proposed in a creative way by Roman and Rota, which will be explained in Chapter six.

Definition 2.29. Sheffer A-type zero polynomials, $S_n(x)$, by means of generating function are defined as, [35]

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}, \tag{2.3.19}$$

where $A(t)$ and $H(t)$ are in the form of the two following formal series

$$A(t) = \sum_{n=0}^{\infty} A_n\frac{t^n}{n!}, \qquad A_0 \neq 0, \tag{2.3.20}$$

and

$$H(t) = \sum_{n=1}^{\infty} H_n\frac{t^n}{n!}, \qquad H_1 \neq 0, \tag{2.3.21}$$

respectively.

Remark 2.30. Based on different choices of $A(t)$ and $H(t)$ in the above definition various Sheffer type polynomials are obtained. As one of the most important members of this family, Hermite polynomials can be considered by taking $A(t) = e^{-t^2}$ and $H(t) = 2t$, [36].

Definition 2.31. Hermite polynomials $H_n(x)$ can be defined by means of the following generating function

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}. \tag{2.3.22}$$

Chapter 3

THE CLASS OF GENERALIZED 2D q-APPELL POLYNOMIALS

3.1 Introduction

Carlitz, for the first time, extended the classical Bernoulli and Euler numbers and polynomials, introducing them as q-Bernoulli and q-Euler numbers and polynomials [37]-[39]. There are numerous recent investigations on this subject by, among many other authors, Cenki et al. ([40]-[42]), Choi et al. ([43] and [44]), Kim et al. ([45]-[48]), Ozden and Simsek [49], Ryoo et al. [50], Simsek ([51]-[53]), and Luo and Srivastava [54], Srivastava et al. [55], Mahmudov [56], [57]. Recently, Natalini and Bernardini [58], Bretti et al.[59],[60] Kurt [61], [62], Tremblay et al [63], [64] studied properties of the following generalized Bernoulli and Euler polynomials.

$$\left(\frac{t^m}{e^t - \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x) \frac{t^n}{n!}, \tag{3.1.1}$$

$$\left(\frac{2^m}{e^t + \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x) \frac{t^n}{n!}, \quad \alpha \in \mathbb{C},\ 1^{\alpha} := 1. \tag{3.1.2}$$

Applying the same approach which is used in the definitions (3.1.1) and (3.1.2), the classical Genocchi polynomials can be generalized as follows.

$$\left(\frac{2^m t^m}{e^t + \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x) \frac{t^n}{n!}, \quad \alpha \in \mathbb{C},\ 1^{\alpha} := 1. \tag{3.1.3}$$

Motivated by the generalizations in (3.1.1), (3.1.2), and (3.1.3) of the classical Bernoulli, Euler, and Genocchi polynomials, we introduce and investigate here the so-called generalized two dimensional q-Bernoulli, q-Euler, and q-Genocchi polynomials which are definedas follow.

Definition 3.1. Let $q, \alpha \in \mathbb{C},\ m \in \mathbb{N},\ 0 < |q| < 1$. The generalized two dimensional q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function:

$$\left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\, E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!}, \tag{3.1.4}$$

where $T_{m-1,q}(t) = \sum_{k=0}^{m-1} \dfrac{t^k}{[k]_q!}$.

Definition 3.2. Let $q, \alpha \in \mathbb{C},\ 0 < |q| < 1,\ m \in \mathbb{N}$. The generalized two dimensional q-Euler polynomials $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions:

$$\left(\frac{2^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\, E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!}. \tag{3.1.5}$$

Definition 3.3. Let $q, \alpha \in \mathbb{C},\ 0 < |q| < 1,\ m \in \mathbb{N}$. The generalized two dimensional q-Genocchi polynomials $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions:

$$\left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\, E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!}. \tag{3.1.6}$$

Remark 3.4. It is obvious that

$$\lim_{q\to 1^-} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = B_n^{[m-1,\alpha]}(x+y),\ \mathfrak{B}_{n,q}^{[m-1,\alpha]} = \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,0),$$

$$\lim_{q\to 1^-} \mathfrak{B}_{n,q}^{[m-1,\alpha]} = B_n^{[m-1,\alpha]},$$

$$\lim_{q\to 1^-} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = E_n^{[m-1,\alpha]}(x+y),\ \mathfrak{E}_{n,q}^{[m-1,\alpha]} = \mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,0),$$

$$\lim_{q\to 1^-} \mathfrak{E}_{n,q}^{[m-1,\alpha]} = E_n^{[m-1,\alpha]},$$

$$\lim_{q\to 1^-} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y) = G_n^{[m-1,\alpha]}(x+y),\ \mathfrak{G}_{n,q}^{[m-1,\alpha]} = \mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,0),$$

$$lim_{q\to 1^-} \mathfrak{G}_{n,q}^{[m-1,\alpha]} = G_n^{[m-1,\alpha]},$$

and also

$$\lim_{q\to 1^-}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)=B_n^{[m-1,\alpha]}(x)\,,\quad \lim_{q\to 1^-}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)=B_n^{[m-1,\alpha]}(y)\,,$$

$$\lim_{q\to 1^-}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)=E_n^{[m-1,\alpha]}(x)\,,\quad \lim_{q\to 1^-}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)=E_n^{[m-1,\alpha]}(y)\,,$$

$$\lim_{q\to 1^-}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)=G_n^{[m-1,\alpha]}(x)\,,\quad \lim_{q\to 1^-}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y)=G_n^{[m-1,\alpha]}(y)\,.$$

Here $B_n^{[m-1,\alpha]}(x)$, $E_n^{[m-1,\alpha]}(x)$, and $G_n^{[m-1,\alpha]}(y)$ denote the generalized Bernoulli, Euler and Genocchi polynomials defined in (3.1.1), (3.1.2), and (3.1.3). Notice that $B_n^{[m-1,\alpha]}(x)$ was introduced by Natalini [58], and $E_n^{[m-1,\alpha]}(x)$ and $G_n^{[m-1,\alpha]}(x)$were introduced by Kurt [61], and [62].

In fact, Definitions (3.1), (3.2), and (3.3) define the two different types $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Bernoulli polynomials, $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Euler polynomials, and $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Genocchi polynomials. Both polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)$, $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)$, $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y)$ coincide with the classical higher order generalized Bernoulli, Euler, and Genocchi polynomials in the limiting case $q\to 1^-$, respectively.

3.2 Generalized 2D q-Appell polynomials

Inspired by the above definitions, we define 2D q-Appell polynomials $\{A_{n,q}(x,y)\}_{n=0}^{\infty}$ by means of the following generating function

$$A_q(x,y;t):=A_q(t)e_q(tx)E_q(ty)=\sum_{n=0}^{\infty}A_{n,q}(x,y)\frac{t^n}{[n]_q!},\tag{3.2.1}$$

where

$$A_q(t):=\sum_{n=0}^{\infty}A_{n,q}\frac{t^n}{[n]_q!},\quad A_q(t)\neq 0,\tag{3.2.2}$$

is an analytic function at $t=0$, and $A_{n,q}(0,0):=A_{n,q}$.

3.3 Preliminaries and Lemmas

In this section some basic formulae are provided for the generalized q-Bernoulli and q-Euler polynomials to obtain the main results of this part of the study in the next section. The following result is q-analogue of the addition theorem for the classical Bernoulli and Euler polynomials.

Lemma 3.5. *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k}, \tag{3.3.1}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k}, \tag{3.3.2}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{G}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k}, \tag{3.3.3}$$

and also

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{(n-k)(n-k-1)}{2}} \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\, y^{n-k} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\, x^{n-k}, \tag{3.3.4}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{(n-k)(n-k-1)}{2}} \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x,0)\, y^{n-k} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(0,y)\, x^{n-k}, \tag{3.3.5}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{(n-k)(n-k-1)}{2}} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)\, y^{n-k} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y)\, x^{n-k}. \tag{3.3.6}$$

Proof. Because of applying the same technique in the proofs, only the relations (3.3.3) and (3.3.6) are proved. To show the identity (3.3.3), starting from the definition (3.3) we have

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} e_q(tx)\, E_q(ty). \tag{3.3.7}$$

Using the definitions of two exponential functions in (2.2.11) and (2.2.12), we can continue as

$$= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} \sum_{n=0}^{\infty} \frac{t^n x^n}{[n]_q!} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} t^k y^k}{[k]_q!}.$$

Using Cauchy product for series we obtain

$$= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k \frac{t^n}{[n]_q!}.$$

Clearly, the first part of the obtained coincides with the generalized two dimensional q-Genocchi numbers $\mathfrak{G}_{n,q}^{[m-1,\alpha]}$, and the second part is exactly the definition of $(x+y)_q^n$ given in (2.2.9). So we have

$$= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} (x+y)_q^n,$$

once more applying Cauchy product for series we get

$$= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{G}_{k,q}^{[m-1,\alpha]} (x+y)_q^{n-k}. \tag{3.3.8}$$

Consequently, comparing the coefficient of $\frac{t^n}{[n]_q!}$ in the left hand side of relation (3.3.7) with relation (3.3.8), leads to obtain the desired result.

To show the first part of identity (3.3.6), starting from the definition (3.3), we have

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx) E_q(ty) \tag{3.3.9}$$

Using the definition of exponential function $E_q(ty)$ given in (2.2.12), we can continue as

$$= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx) \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} t^k y^k}{[k]_q!},$$

where equivalently can be written as

$$= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} t^k y^k}{[k]_q!}.$$

Applying Cauchy product for series we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{(n-k)(n-k-1)}{2}}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)\,y^{n-k}\frac{t^n}{[n]_q!}. \tag{3.3.10}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in the left hand side of relation (3.3.9) with relation (3.3.10), leads to obtain the desired result.

To show the second part of identity (3.3.6), we follow a similar procedure to the above, starting from the definition below

$$\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\,e_q(tx), \tag{3.3.11}$$

and replacing the expressions $\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)$ and $e_q(tx)$ with $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y)$ and $\sum_{n=0}^{\infty}\frac{x^n t^n}{[n]_q!}$, respectively. □

Remark 3.6. In particular, setting $x=0$ and $y=0$ in (3.3.4), (3.3.5) and (3.3.6), we get the following formulae for the generalized q-Bernoulli and q-Euler and q-Genocchi polynomials,

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}x^{n-k}, \tag{3.3.12}$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{(n-k)(n-k-1)}{2}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}y^{n-k}, \tag{3.3.13}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\mathfrak{E}_{k,q}^{[m-1,\alpha]}x^{n-k}, \tag{3.3.14}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{(n-k)(n-k-1)}{2}}\mathfrak{E}_{k,q}^{[m-1,\alpha]}y^{n-k}, \tag{3.3.15}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\mathfrak{G}_{k,q}^{[m-1,\alpha]}x^{n-k}, \tag{3.3.16}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{(n-k)(n-k-1)}{2}}\mathfrak{G}_{k,q}^{[m-1,\alpha]}y^{n-k}, \tag{3.3.17}$$

respectively.

Remark 3.7. Setting $y = 1$ and $x = 1$ in (3.3.4), (3.3.5) and (3.3.6), we obtain

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,1) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{(n-k)(n-k-1)}{2}} \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0), \tag{3.3.18}$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y), \tag{3.3.19}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,1) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{(n-k)(n-k-1)}{2}} \mathfrak{E}_{k,q}^{(\alpha)}(x,0), \tag{3.3.20}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(1,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(0,y), \tag{3.3.21}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,1) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{(n-k)(n-k-1)}{2}} \mathfrak{G}_{k,q}^{[m-1,\alpha]}(x,0), \tag{3.3.22}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(1,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{G}_{k,q}^{[m-1,\alpha]}(0,y), \tag{3.3.23}$$

respectively.

Clearly relations (3.3.18), (3.3.20) and (3.3.22) are the generalization of q-analogues of the following identites

$$B_n(x+1) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) B_k(x), \tag{3.3.24}$$

$$E_n(x+1) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) E_k(x), \tag{3.3.25}$$

$$G_n(x+1) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) G_k(x), \tag{3.3.26}$$

respectively.

Lemma 3.8. *The generalized q-Bernoulli, q-Euler and q-Genocchi polynomials satisfy the following relations*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\, \mathfrak{B}_{n-k,q}^{[m-1,\beta]}(0,y), \tag{3.3.27}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x,0)\, \mathfrak{E}_{n-k,q}^{[m-1,\beta]}(0,y), \tag{3.3.28}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \mathfrak{G}_{k,q}^{[m-1,\alpha]}(x,0)\, \mathfrak{G}_{n-k,q}^{[m-1,\beta]}(0,y), \tag{3.3.29}$$

respectively.

Proof. Because of applying the same technique in the proofs, only the last relation is proved. To prove, we start from the following summation

$$\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha+\beta]}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha+\beta}e_q(tx)E_q(ty), \tag{3.3.30}$$

which, clearly, can be written as

$$=\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}e_q(tx)\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\beta}E_q(ty). \tag{3.3.31}$$

According to definition (3.3), we obtain

$$=\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\beta]}(0,y)\frac{t^n}{[n]_q!}. \tag{3.3.32}$$

Using Cauchy product for series we can write

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\mathfrak{G}_{k,q}^{[m-1,\alpha]}(x,0)\,\mathfrak{G}_{n-k,q}^{[m-1,\beta]}(0,y)\frac{t^n}{[n]_q!}. \tag{3.3.33}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in the left hand side of relation (3.3.30) with relation (3.3.33), leads to obtain the desired result. □

Lemma 3.9. *The following identities hold true for the q-derivatives of the q-Bernoulli, q-Euler and q-Genocchi polynomials with respect to the two variables x, and y*

$$D_{q,x}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)=[n]_q\,\mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(x,y), \tag{3.3.34}$$

$$D_{q,y}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)=[n]_q\,\mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(x,qy), \tag{3.3.35}$$

$$D_{q,x}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)=[n]_q\,\mathfrak{E}_{n-1,q}^{[m-1,\alpha]}(x,y), \tag{3.3.36}$$

$$D_{q,y}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)=[n]_q\,\mathfrak{E}_{n-1,q}^{[m-1,\alpha]}(x,qy), \tag{3.3.37}$$

$$D_{q,x}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)=[n]_q\,\mathfrak{G}_{n-1,q}^{[m-1,\alpha]}(x,y), \tag{3.3.38}$$

$$D_{q,y}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)=[n]_q\,\mathfrak{G}_{n-1,q}^{[m-1,\alpha]}(x,qy), \tag{3.3.39}$$

respectively.

Proof. Because of applying a similar technique in the proofs, only the last two relations are proved. To prove the first relation in (3.3.39), we consider the q-derivative of the following summation with respect to variable x

$$D_{q,x}\left(\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}\right) \tag{3.3.40}$$

as mentioned in part (c) of proposition (2.11), $D_{q,x}$ is a linear operator. So, we may write

$$\begin{aligned} &= \sum_{n=0}^{\infty}D_{q,x}\left(\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\right)\frac{t^n}{[n]_q!} \qquad (3.3.41)\\ &= D_{q,x}\left(\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}e_q(tx)\,E_q(ty)\right). \end{aligned}$$

Clearly, since $\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}$ and $E_q(ty)$ are independent from variable x, we only take q-derivative of $e_q(tx)$ with respect to x. So, we have

$$\begin{aligned} &= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\,D_{q,x}\left(e_q(tx)\right)\\ &= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\,D_{q,x}\left(\sum_{n=0}^{\infty}\frac{x^n t^n}{[n]_q!}\right). \end{aligned}$$

Again, because of linear property of $D_{q,x}$ mentioned in part (c) of proposition (2.11), we can continue as

$$\begin{aligned} &= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\left(\sum_{n=0}^{\infty}D_{q,x}(x^n)\frac{t^n}{[n]_q!}\right)\\ &= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\left(\sum_{n=1}^{\infty}[n]_q x^{n-1}\frac{t^n}{[n]_q!}\right)\\ &= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\left(\sum_{n=0}^{\infty}x^n\frac{t^{n+1}}{[n]_q!}\right)\\ &= \sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^{n+1}}{[n]_q!},\\ &= \sum_{n=1}^{\infty}[n]_q\,\mathfrak{G}_{n-1,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}. \qquad (3.3.42) \end{aligned}$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ in the left hand side of relation (3.3.41) with relation (3.3.42), leads to obtain the desired result.

Similar to the proof above, to show the second relation in (3.3.39), we consider the q-derivative of the following summation with respect to the variable y

$$D_{q,y}\left(\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}\right).$$

As it is mentioned in part (c) of proposition (2.11), $D_{q,y}$ is a linear operator. So, we can write

$$= \sum_{n=0}^{\infty} D_{q,y}\left(\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\right)\frac{t^n}{[n]_q!} \tag{3.3.43}$$

$$= D_{q,y}\left(\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,E_q(ty)\right) \tag{3.3.44}$$

clearly since $\left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}$ and $e_q(tx)$ are independent from variable y, we only take q-derivative of $E_q(ty)$ with respect to y. So, we have

$$\begin{aligned}
&= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,D_{q,y}\left(E_q(ty)\right)\\
&= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,D_{q,y}\left(\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}t^n y^n}{[n]_q!}\right).
\end{aligned}$$

Again because of the linear property of $D_{q,y}$, which is mentioned in part (c) of proposition (2.11), we may continue as

$$\begin{aligned}
&= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} D_{q,y}(y^n)\frac{t^n}{[n]_q!}\right)\\
&= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\left(\sum_{n=1}^{\infty} q^{\frac{n(n-1)}{2}} [n]_q y^{n-1})\frac{t^n}{[n]_q!}\right)\\
&= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} y^n\frac{t^{n+1}}{[n]_q!}\right)\\
&= \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} q^n y^n\frac{t^{n+1}}{[n]_q!}\right)\\
&= \sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,qy)\frac{t^{n+1}}{[n]_q!}\\
&= \sum_{n=1}^{\infty}[n]_q\,\mathfrak{G}_{n-1,q}^{[m-1,\alpha]}(x,qy)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{3.3.45}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in the left hand side of the relation (3.3.43) with relation (3.3.45), leads to obtain the desired result. □

Lemma 3.10. *The generalized q-Bernoulli, q-Euler and q-Genocchi polynomials satisfy the following relations:*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y)-\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)=\frac{[n]_q!}{[n-m]_q!}\mathfrak{B}_{n-m,q}^{[m-1,\alpha-1]}(0,y)\,,\quad n\geq m,\tag{3.3.46}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(1,y)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)=2^m\mathfrak{E}_{n,q}^{[m-1,\alpha-1]}(0,y)\,,\tag{3.3.47}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(1,y)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{G}_{n-k,q}^{[m-1,\alpha]}(0,y)=2^m\frac{[n]_q!}{[n-m]_q!}\mathfrak{G}_{n,q}^{[m-1,\alpha-1]}(0,y)\,,\quad n\geq m,\tag{3.3.48}$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)-\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,-1)=\frac{[n]_q!}{[n-m]_q!}\mathfrak{B}_{n-m,q}^{[m-1,\alpha-1]}(x,-1)\,,\quad n\geq m,\tag{3.3.49}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,-1)=2^m\mathfrak{E}_{n,q}^{[m-1,\alpha-1]}(x,-1)\,,\tag{3.3.50}$$

$$\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{G}_{n-k,q}^{[m-1,\alpha]}(x,-1)=2^m\frac{[n]_q!}{[n-m]_q!}\mathfrak{G}_{n-m,q}^{[m-1,\alpha-1]}(x,-1)\,\frac{t^n}{[n]_q!},\quad n\geq m.\tag{3.3.51}$$

Proof. We prove only the relations (3.3.48) and (3.3.51). The proof of relation (3.3.48) is based on the following equality

$$\sum_{n=0}^{\infty}\left(\mathfrak{G}_{n,q}^{[m-1,\alpha]}(1,y)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{G}_{n-k,q}^{[m-1,\alpha]}(0,y)\right)\frac{t^n}{[n]_q!},\tag{3.3.52}$$

which is equivalent to the following identities

$$
\begin{aligned}
&= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(t)\, E_q(ty) + T_{m-1,q}(t)\left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} E_q(ty) \\
&= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} E_q(ty)\,(e_q(t) + T_{m-1,q}(t)) \\
&= 2^m t^m \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha-1} E_q(ty) \\
&= \sum_{n=0}^{\infty} 2^m \frac{[n+m]_q!}{[n]_q!} \mathfrak{B}_{n,q}^{[m-1,\alpha-1]}(0,y) \frac{t^{n+m}}{[n+m]_q!} \\
&= \sum_{n=m}^{\infty} 2^m \frac{[n]_q!}{[n-m]_q!} \mathfrak{G}_{n,q}^{[m-1,\alpha-1]}(0,y) \frac{t^n}{[n]_q!}.
\end{aligned} \tag{3.3.53}
$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in relation (3.3.52) with relation (3.3.53), leads to obtain the desired result.

Here we used the following relation

$$
\begin{aligned}
&T_{m-1,q}(t)\left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} E_q(ty) = \sum_{n=0}^{m-1} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y) \left(\frac{t^n}{[n]_q!} + \frac{t^{n+1}}{[n]_q!} + \frac{t^{n+2}}{[n]_q!\,[2]_q!} + \dots + \frac{t^{n+m-1}}{[n]_q!\,[m-1]_q!}\right) \\
&= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} [n]_q\, \mathfrak{G}_{n-1,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&+ \sum_{n=0}^{\infty} \frac{[n]_q\,[n-1]_q}{[2]_q!} \mathfrak{G}_{n-2,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} + \dots \\
&+ \sum_{n=0}^{\infty} \frac{[n]_q \dots [n-m+2]_q}{[m-1]_q!} \mathfrak{G}_{n-m+1,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n,m-1)} \left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{G}_{n-k,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!}.
\end{aligned}
$$

In order to prove the relation (3.3.51), we start from the following equality

$$
\sum_{n=0}^{\infty} \left(\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0) + \sum_{k=0}^{\min(n,m-1)} \left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{G}_{n-k,q}^{[m-1,\alpha]}(x,-1)\right) \frac{t^n}{[n]_q!} \tag{3.3.54}
$$

which is equivalent to the following relation

$$
= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx) + T_{m-1,q}(t)\left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\, E_q(-t)\,.
$$

Noting to the fact that $E_q(-t) = \frac{1}{e_q(t)}$, which is mentioned in remark(2.12), we obtain

$$
\begin{aligned}
&= \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\left(1 + \frac{T_{m-1,q}(t)}{e_q(t)}\right) \\
&= 2^m t^m \left(\frac{2^m t^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha-1} e_q(tx)\, E_q(-t) \\
&= \sum_{n=0}^{\infty} 2^m \frac{[n+m]_q!}{[n]_q!} \mathfrak{G}_{n,q}^{[m-1,\alpha-1]}(x,-1) \frac{t^{n+m}}{[n+m]_q!},
\end{aligned}
$$

which is equivalent to write

$$
= \sum_{n=m}^{\infty} 2^m \frac{[n]_q!}{[n-m]_q!} \mathfrak{G}_{n-m,q}^{[m-1,\alpha-1]}(x,-1) \frac{t^n}{[n]_q!}. \tag{3.3.55}
$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ in (3.3.54) with (3.3.55) leads to obtain the desired result. □

Corollary 3.11. *Taking $q \to 1^-$ we have the following results*

$$
\mathfrak{B}_n^{[m-1,\alpha]}(x+1) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k}^{[m-1,\alpha]}(x) = \frac{[n]_q!}{[n-m]_q!} \mathfrak{B}_{n-m}^{[m-1,\alpha-1]}(x), \quad n \geq m,
$$

$$
\mathfrak{E}_n^{[m-1,\alpha]}(x+1) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_n^{[m-1,\alpha]}(x) = 2^m \mathfrak{E}_n^{[m-1,\alpha-1]}(x), \quad n \geq m,
$$

$$
\mathfrak{G}_n^{[m-1,\alpha]}(x+1) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{G}_{n-k}^{[m-1,\alpha]}(x) = 2^m \frac{[n]_q!}{[n-m]_q!} \mathfrak{G}_n^{[m-1,\alpha-1]}(x), \quad n \geq m.
$$

Lemma 3.12. *The generalized q-Bernoulli polynomials satisfy the following relations*

$$
\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) = [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\, \mathfrak{B}_{n-1-k,q}^{[0,-1]}. \tag{3.3.56}
$$

Proof. Indeed, we know that

$$\sum_{n=0}^{\infty}\left(\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y)-\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)\right)\frac{t^n}{[n]_q!}$$
$$=\left(\frac{t^m}{e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}e_q(t)\,E_q(ty)-T_{m-1,q}(t)\left(\frac{t^m}{e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)$$
$$=\left(\frac{t^m}{e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\,\frac{e_q(t)-T_{m-1,q}(t)}{t}t$$
$$=\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[0,-1]}\frac{t^{n+1}}{[n]_q!},$$

which is equivalent to write

$$=\sum_{n=1}^{\infty}[n]_q\sum_{k=0}^{n-1}\left[\begin{array}{c}n-1\\k\end{array}\right]_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\,\mathfrak{B}_{n-1-k,q}^{[0,-1]}\frac{t^n}{[n]_q!}.$$

□

Remark 3.13. Note to the fact that taking limit in relation (3.3.56) as $q\to 1^-$, leads to obtain

$$\mathfrak{B}_{n}^{[m-1,\alpha]}(y+1)-\sum_{k=0}^{\min(n,m-1)}\left(\begin{array}{c}n\\k\end{array}\right)\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y)$$
$$=n\sum_{k=0}^{n-1}\left(\begin{array}{c}n-1\\k\end{array}\right)\mathfrak{B}_{k}^{[m-1,\alpha]}(y)\,\mathfrak{B}_{n-1-k}^{[0,-1]}.$$

It is a correct form of formula (2.7) from [63] for $\lambda=1$.

Lemma 3.14. *We have*

$$x^n=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{[k]_q!}{[k+m]_q!}\mathfrak{B}_{n-k,q}^{[m-1,1]}(x,0), \tag{3.3.57}$$

$$y^n=\frac{1}{q^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{[k]_q!}{[k+m]_q!}\mathfrak{B}_{n-k,q}^{[m-1,1]}(0,y), \tag{3.3.58}$$

$$x^n=\frac{1}{2^m}\left(\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{k,q}^{[m-1,1]}(x,0)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{k,q}^{[m-1,1]}(x,0)\right), \tag{3.3.59}$$

$$y^n=\frac{1}{2^mq^{\frac{n(n-1)}{2}}}\left(\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{k,q}^{[m-1,1]}(0,y)+\sum_{k=0}^{\min(n,m-1)}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{n,q}^{[m-1,1]}(0,y)\right). \tag{3.3.60}$$

Proof. To prove relation (3.3.57) consider the following statement

$$\left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)}\right) e_q(tx)\,(e_q(t) - T_{m-1,q}(t)), \tag{3.3.61}$$

which is, clearly, equal to the summation below

$$= \sum_{n=0}^{\infty} x^n \frac{t^{n+m}}{[n]_q!}. \tag{3.3.62}$$

According to the Definition (3.1), statement (3.3.61) can be written as

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,1]}(x,0)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!} - \sum_{n=0}^{m-1}\frac{t^n}{[n]_q!}\right)\\
&= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,1]}(x,0)\frac{t^n}{[n]_q!}\sum_{n=m}^{\infty}\frac{t^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,1]}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{n+m}}{[n+m]_q!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathfrak{B}_{n-k,q}^{[m-1,1]}(x,0)\frac{[k]_q!\,[n]_q!}{[n-k]_q!\,[k+m]_q!\,[k]_q!} \times \frac{t^{n+m}}{[n]_q!},
\end{aligned}$$

which is equivalent to write

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!}\mathfrak{B}_{n-k,q}^{[m-1,1]}(x,0)\right)\frac{t^{n+m}}{[n]_q!}. \tag{3.3.63}$$

Comparing the coefficients of $\frac{t^{n+m}}{[n]_q!}$ in relation (3.3.62) with relation (3.3.63), leads to obtain the desired result.

To prove relation (3.3.58), we start with the following statement

$$\left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)}\right) E_q(ty)\,(e_q(t) - T_{m-1,q}(t)),$$

which is clearly equal to the summation below

$$= \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} y^n \frac{t^{n+m}}{[n]_q!}.$$

According to the Definition (3.1), relation above can be written as

$$= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,1]}(0,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{n+m}}{[n+m]_q!}.$$

Applying a similar process to the proof of relation (3.3.57), makes this part of the proof complete.

In order to prove relation (3.3.59), we consider the following identity

$$
\begin{aligned}
2^m \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} &= \left(\frac{2^m}{e_q(t) + T_{m-1,q}(t)} \right) e_q(tx) \left(e_q(t) + T_{m-1,q}(t) \right) \\
&= \sum_{n=0}^{\infty} \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) \frac{t^n}{[n]_q!} \left(\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} + \sum_{n=0}^{m-1} \frac{t^n}{[n]_q!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) + \sum_{k=0}^{\min(n,m-1)} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) \right) \frac{t^n}{[n]_q!}.
\end{aligned}
$$

The rest of proof will be similar to the proof of relation (3.3.57). Also, because of applying a similar technique in the proof of relation (3.3.60), we pass its proof. $\square$

Remark 3.15. From Lemma (3.14) we obtain the list of generalized q-Bernoulli and q-Euler polynomials as follows

$$
\begin{aligned}
\mathfrak{B}_{0,q}^{[m-1,1]}(x,0) &= [m]_q!, \\
\mathfrak{B}_{1,q}^{[m-1,1]}(x,0) &= [m]_q! \left(x - \tfrac{1}{[m+1]_q} \right), \\
\mathfrak{B}_{2,q}^{[m-1,1]}(x,0) &= x^2 - \tfrac{[2]_q [m]_q!}{[m+1]_q} x + \tfrac{[2]_q q^{m+1} [m]_q!}{[m+1]_q^2 [m+2]_q}.
\end{aligned}
,
$$

$$
\begin{aligned}
\mathfrak{B}_{0,q}^{[m-1,1]}(0,y) &= [m]_q!, \\
\mathfrak{B}_{1,q}^{[m-1,1]}(0,y) &= [m]_q! \left(y - \tfrac{1}{[m+1]_q} \right), \\
\mathfrak{B}_{2,q}^{[m-1,1]}(0,y) &= qy^2 - \tfrac{[2]_q [m]_q!}{[m+1]_q} y + \tfrac{[2]_q q^{m+1} [m]_q!}{[m+1]_q^2 [m+2]_q}.
\end{aligned}
,
$$

$$
\begin{aligned}
\mathfrak{E}_{0,q}^{[m-1,1]}(x,0) &= 2^{m-1}, \\
\mathfrak{E}_{1,q}^{[0,1]}(x,0) &= 2x - 2, \\
\mathfrak{E}_{1,q}^{[m-1,1]}(x,0) &= 2^{m-1}(x-1), \quad m \geq 2, \\
\mathfrak{E}_{2,q}^{[0,1]}(x,0) &= 2x^2 - 2[2]_q x - 2 + 2[2]_q, \\
\mathfrak{E}_{2,q}^{[1,1]}(x,0) &= 4x^2 - 4[2]_q x - 4 + 4[2]_q, \\
\mathfrak{E}_{2,q}^{[m-1,1]}(x,0) &= 2^{m-1}(x^2 - [2]_q x + [2]_q), \quad m \geq 3.
\end{aligned}
,
$$

$$
\begin{aligned}
\mathfrak{E}_{0,q}^{[m-1,1]}(0,y) &= 2^{m-1}, \\
\mathfrak{E}_{1,q}^{[0,1]}(0,y) &= 2y - 2, \\
\mathfrak{E}_{1,q}^{[m-1,1]}(0,y) &= 2^{m-1}(y-1), \quad m \geq 2, \\
\mathfrak{E}_{2,q}^{[0,1]}(0,y) &= 2qy^2 - 2[2]_q y + 2[2]_q - 2, \\
\mathfrak{E}_{2,q}^{[1,1]}(0,y) &= 4qy^2 - 4[2]_q y + -4[2]_q - 4, \\
\mathfrak{E}_{2,q}^{[m-1,1]}(0,y) &= 2^{m-1}(qy^2 - [2]_q y + [2]_q - 1), \quad m \geq 3,
\end{aligned}
$$

respectively.

3.4 Explicit relationship between the q-Bernoulli and q-Euler polynomials mials

In this section, some generalizations of the Srivastava-Pint´er addition theorem are derived. Also, some new formulae and some of their special cases are given. These results are the natural q-extensions of the main results of the researches which can be found in the references [65], [66].

Theorem 3.16. *The following relationships hold true between the generalized q-Bernoulli polynomials and q-Euler polynomials.*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)= \tag{3.4.1}$$
$$\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q \frac{1}{l^{n-k}}\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)+\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q \frac{1}{l^{k-j}}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\right]\mathfrak{E}_{n-k,q}(0,ly),$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)= \tag{3.4.2}$$
$$\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{1}{l^{n-k}}\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)+\mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\right]\mathfrak{E}_{n-k,q}(lx,0).$$

Proof. First, we prove (3.4.1). Using the following identity

$$\left(\frac{t^m}{e_q(t)-\sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx)\,E_q(ty)=$$
$$\frac{2}{e_q\left(\frac{t}{l}\right)+1}\times E_q\left(\frac{t}{l}ly\right)\times\frac{e_q\left(\frac{t}{l}\right)+1}{2}\times\left(\frac{t^m}{e_q(t)-\sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx).$$

We have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,ly)\frac{t^n}{l^n[n]_q!}\sum_{k=0}^{\infty}\frac{t^k}{l^k[k]_q!}\sum_{j=0}^{\infty}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\frac{t^j}{[j]_q!}\\&+\frac{1}{2}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\\&=:I_1+I_2.\end{aligned}$$

It is clear that

$$I_2=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\,\mathfrak{E}_{n-k,q}(0,ly)\frac{t^n}{[n]_q!}.$$

On the other hand

$$I_1=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}\sum_{j=0}^{\infty}\frac{t^j}{l^j[j]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\mathfrak{E}_{j,q}(0,ly)\frac{t^k}{l^k[k]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\sum_{j=0}^{n-k}\left[\begin{array}{c}n-k\\j\end{array}\right]_q\frac{1}{l^{n-k}}\mathfrak{E}_{j,q}(0,ly)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right]_q\mathfrak{E}_{j,q}(0,ly)\sum_{k=0}^{n-j}\left[\begin{array}{c}n-j\\k\end{array}\right]_q\frac{1}{l^{n-k}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}.$$

Therefore,

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right]_q\frac{1}{l^{n-k}}$$
$$\times\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)+\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\frac{1}{l^{k-j}}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\right]\mathfrak{E}_{n-k,q}(0,ly)\frac{t^n}{[n]_q!}.$$

Next, we prove relation (3.4.2) using the following identity

$$\left(\frac{t^m}{e_q(t)-\sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha}e_q(tx)\,E_q(ty)$$
$$=\frac{2}{e_q\left(\frac{t}{l}\right)+1}\times e_q\left(\frac{t}{l}lx\right)\times\frac{e_q\left(\frac{t}{l}\right)+1}{2}\times\left(\frac{t^m}{e_q(t)-\sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha}E_q(ty).$$

We have

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(lx,0)\frac{t^n}{l^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\frac{t^n}{[n]_q!}$$
$$+\frac{1}{2}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\frac{t^k}{l^k[k]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{t^n}{[n]_q!}$$
$$=:I_1+I_2.$$

It is clear that

$$
\begin{aligned}
I_2 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\frac{t^k}{l^k[k]_q!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\,\mathfrak{E}_{n-k,q}(lx,0)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
I_1 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\frac{t^k}{m^k[k]_q!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\mathfrak{E}_{n-k,q}(lx,0)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q l^{k-n}\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)+\mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\right]\mathfrak{E}_{n-k,q}(lx,0)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

Next, we discuss some special cases of Theorem (3.16).

Theorem 3.17. *The following relationship holds true between the generalized q-Bernoulli polynomials and the q-Euler polynomials.*

$$
\begin{aligned}
\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q &\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q\mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)\right.\\
&\left.+[k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q\mathfrak{B}_{j,q}^{[m-1,\alpha]}(0,y)\,\mathfrak{B}_{k-1-j,q}^{[0,-1]}\right]\mathfrak{E}_{n-k,q}(x,0).
\end{aligned}
$$

Remark 3.18. Taking $q \to 1^-$ in Theorem 3.17, we obtain Srivastava–Pintér addition theorem for the generalized Bernoulli and Euler polynomials.

$$
\begin{aligned}
\mathfrak{B}_{n}^{[m-1,\alpha]}(x+y) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}&\left[\mathfrak{B}_{k}^{[m-1,\alpha]}(y)+\sum_{k=0}^{\min(n,m-1)}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y)\right.\\
&\left.+k\sum_{j=0}^{k-1}\binom{k-1}{j}\mathfrak{B}_{j}^{[m-1,\alpha]}(y)\,\mathfrak{B}_{k-1-j}^{[0,-1]}\right]\mathfrak{E}_{n-k}(x). \qquad (3.4.3)
\end{aligned}
$$

Notice that Srivastava–Pintér addition theorem for the generalized Apostol–Bernoulli polynomials and the Apostol–Euler polynomials was given in [63]. The formula (3.4.3) is a correct version that of Theorem (3) in the reference [63] for $\lambda = 1$.

3.5 Explicit Relation between q-Genocchi and q-Bernoulli Polynomials

Theorem 3.19. *The following relation holds true between the generalized q-Genocchi and the generalized q-Bernoulli polynomials*

$$\mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \times \Bigg(\sum_{j=0}^{k-l+1} \begin{bmatrix} k+1 \\ j+l \end{bmatrix}_q m^{j+l} 2^l \frac{[j+l]_q!}{[j+2l]_q!} \mathfrak{G}_{j,q}^{[l-1,\alpha-1]}(x,-1)$$

$$- \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \sum_{i=0}^{l-1} \begin{bmatrix} j \\ i \end{bmatrix}_q \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}(x,-1)\Bigg)$$

$$- m^{k+1} \mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(x,0)\Bigg) \mathfrak{B}_{n-k,q}(0,my). \quad (3.5.1)$$

Proof. The proof is based on the following identity

$$\left(\frac{2^l t^l}{e_q(t) + \sum_{i=0}^{l-1} \frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx) E_q(ty)$$

$$= \left(\frac{2^l t^l}{e_q(t) + \sum_{i=0}^{l-1} \frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx) \times \frac{e_q\left(\frac{t}{m}\right) - 1}{\frac{t}{m}} \times \frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right) - 1} \times E_q\left(\frac{t}{m} my\right).$$

Consequently,

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,y) \frac{t^n}{[n]_q!}$$

$$= \frac{m}{t}\left(\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,0) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \frac{1}{m^k} \times \frac{t^k}{[k]_q!} - \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,0) \frac{t^n}{[n]_q!}\right) \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(0,my) \frac{t^n}{m^n [n]_q!}$$

$$= \frac{m}{t} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^{n-k}} \mathfrak{G}_{k,q}^{[l-1,\alpha]}(x,0) - \mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,0)\right) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(0,my) \frac{t^n}{m^n [n]_q!}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^{n-k-1}} \mathfrak{G}_{k,q}^{[l-1,\alpha]}(x,0) \right.$$

$$\left. - m\mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,0) \right) \frac{t^{n-1}}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(0,my) \frac{t^n}{m^n [n]_q!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q m^k \mathfrak{G}_{k,q}^{[l-1,\alpha]}(x,0) - m^{n+1}\mathfrak{G}_{n+1,q}^{[l-1,\alpha]}(x,0) \right)$$

$$\times \frac{t^n}{m^n [n+1]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(0,my) \frac{t^n}{m^n [n]_q!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \mathfrak{G}_{j,q}^{[l-1,\alpha]}(x,0) - m^{k+1}\mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(x,0) \right)$$

$$\times \frac{t^k}{[k+1]_q [k]_q!} \mathfrak{B}_{n-k,q}(0,my) \frac{t^{n-k}}{m^{n-k} [n-k]_q!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left(\sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \mathfrak{G}_{j,q}^{[l-1,\alpha]}(x,0) \right.$$

$$\left. \times -m^{k+1}\mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(x,0) \right) \mathfrak{B}_{n-k,q}(0,my) \frac{t^n}{[n]_q!}.$$

Now, we use relation (3.3.51) from Lemma (3.10); that is

$$\mathfrak{G}_{j,q}^{[l-1,\alpha]}(x,0) = 2^l \frac{[j]_q!}{[j+l]_q!} \mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}(x,-1) - \sum_{i=0}^{l-1} \begin{bmatrix} j \\ i \end{bmatrix}_q \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}(x,-1).$$

So, we have

$$\mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,y)$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left(\sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j (2^l \frac{[j]_q!}{[j+l]_q!} \mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}(x,-1) \right.$$
$$\left. - \sum_{i=0}^{l-1} \begin{bmatrix} j \\ i \end{bmatrix}_q \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}(x,-1) \right) - m^{k+1}\mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(x,0)) \mathfrak{B}_{n-k,q}(0,my)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left(\sum_{j=l}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j 2^l \frac{[j]_q!}{[j+l]_q!} \mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}(x,-1) \right.$$
$$\left. - \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \sum_{i=0}^{l-1} \begin{bmatrix} j \\ i \end{bmatrix}_q \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}(x,-1) \right)$$
$$- m^{k+1}\mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(x,0)) \mathfrak{B}_{n-k,q}(0,my),$$

which can be written as

$$
\begin{aligned}
&=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{1}{m^n\left[k+1\right]_q}\left(\sum_{j=0}^{k-l+1}\left[\begin{array}{c} k+1 \\ j+l \end{array}\right]_q m^{j+l}2^l\frac{\left[j+l\right]_q!}{\left[j+2l\right]_q!}\mathfrak{G}_{j,q}^{[l-1,\alpha-1]}\left(x,-1\right)\right.\\
&-\left.\sum_{j=0}^{k+1}\left[\begin{array}{c} k+1 \\ j \end{array}\right]_q m^j\sum_{i=0}^{l-1}\left[\begin{array}{c} j \\ i \end{array}\right]_q \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}\left(x,-1\right)\right)\\
&-m^{k+1}\mathfrak{G}_{k+1,q}^{[l-1,\alpha]}\left(x,0\right))\mathfrak{B}_{n-k,q}\left(0,my\right).
\end{aligned}
$$

□

Lemma 3.20. *The following relation holds true for the generalized q-Genocchi polynomials*

$$
\begin{aligned}
&\mathfrak{G}_{k,q}^{[l-1,\alpha]}\left(\frac{1}{m},y\right)+\sum_{j=0}^{k}\sum_{i=0}^{l-1}\left[\begin{array}{c} k \\ j \end{array}\right]_q\left[\begin{array}{c} j \\ i \end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{G}_{j-i,q}^{[l-1,\alpha]}\left(0,y\right)=\\
&\qquad\frac{2^m\left[k\right]_q!}{\left[k-l\right]_q!}\sum_{j=0}^{k-l}\left[\begin{array}{c} k-l \\ j \end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j-l}\mathfrak{G}_{j,q}^{[l-1,\alpha-1]}\left(0,y\right).
\end{aligned}
\tag{3.5.2}
$$

Proof. From relation (3.3.48) of Lemma (3.10), for $0\leq j\leq k$ we have

$$
\mathfrak{G}_{j,q}^{[l-1,\alpha]}\left(1,y\right)+\sum_{i=0}^{l-1}\left[\begin{array}{c} j \\ i \end{array}\right]_q\mathfrak{G}_{j-i,q}^{[l-1,\alpha]}\left(0,y\right)=2^l\frac{\left[l\right]_q!}{\left[j-l\right]_q!}\mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}\left(0,y\right). \tag{3.5.3}
$$

Multiplying both sides of the relation (3.5.3) by $\left[\begin{array}{c} k \\ j \end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}$, for 0≤$j$≤k and then adding the $k+1$ obtained equalities together, will lead to obtain

$$
\begin{aligned}
&\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{G}_{j,q}^{[l-1,\alpha]}\left(1,y\right)+\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j} \qquad (3.5.4)\\
&\times\sum_{i=0}^{l-1}\left[\begin{array}{c} j \\ i \end{array}\right]_q\mathfrak{G}_{j-i,q}^{[l-1,\alpha]}\left(0,y\right)\\
&=\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}2^l\frac{\left[l\right]_q!}{\left[j-l\right]_q!}\mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}\left(0,y\right). \qquad (3.5.5)
\end{aligned}
$$

From one hand we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[l-1,\alpha]}\left(\frac{1}{m},y\right)\frac{t^n}{\left[n\right]_q!}\\
&=\left(\frac{2^lt^l}{e_q\left(t\right)+T_{l-1,q}\left(t\right)}\right)^{\alpha}e_q\left(t\frac{1}{m}\right)E_q\left(ty\right),
\end{aligned}
$$

which can be written as

$$= \left(\frac{2^l t^l}{e_q(t) + T_{l-1,q}(t)} \right)^\alpha e_q(t) E_q(ty) e_q\left(t\frac{1}{m}\right) E_q(-t)$$
$$= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(1,y) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \frac{1}{m^k} . \frac{t^k}{[k]_q!} \sum_{l=0}^{\infty} \frac{q^{\frac{1}{2}l(l-1)}(-t)^l}{[l]_q!}.$$

It is equivalent to write that

$$= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(1,y) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{1}{m^l} . \frac{t^l}{[l]_q!} \frac{q^{\frac{(k-l)(k-l-1)}{2}}(-1)^{k-l} t^{k-l}}{[k-l]_q!}$$

$$= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(1,y) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left[\begin{array}{c} k \\ l \end{array} \right]_q \frac{1}{m^l}(-1)^{k-l} q^{\frac{(k-l)(k-l-1)}{2}} \frac{t^k}{[k]_q!}$$
$$= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]\alpha}(1,y) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \left(\frac{1}{m} - 1 \right)_q^k \frac{t^k}{[k]_q!},$$

which leads to obtain

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n-k} \mathfrak{G}_{k,q}^{[l-1,\alpha]}(1,y) \frac{t^k}{[k]_q!} \left(\frac{1}{m} - 1 \right)_q^{n-k} \frac{t^{n-k}}{[n-k]_q!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n-k} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{G}_{k,q}^{[l-1,\alpha]}(1,y) \left(\frac{1}{m} - 1 \right)_q^{n-k} \frac{t^n}{[n]_q!}.$$

This means that

$$\sum_{k=0}^{n-k} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{G}_{k,q}^{[l-1,\alpha]}(1,y) \left(\frac{1}{m} - 1 \right)_q^{n-k} = \mathfrak{G}_{n,q}^{[l-1,\alpha]}\left(\frac{1}{m}, y \right). \tag{3.5.6}$$

From another hand we have

$$2^m \sum_{j=0}^{k} \left[\begin{array}{c} k \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k-j} \frac{[n]_q!}{[n-m]_q!} \mathfrak{G}_{j-l,q}^{[m-1,\alpha-1]}(0,y)$$
$$= 2^m \sum_{j=l}^{k} \frac{[k]_q!}{[k-j]_q! [j]_q!} \frac{[j]_q!}{[j-l]_q!} \left(\frac{1}{m} - 1 \right)_q^{k-j} \mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}(0,y),$$

from which it can be written that

$$= 2^m \sum_{j=l}^{k} \frac{[k]_q!}{[k-j]_q! [j-l]_q!} \left(\frac{1}{m} - 1 \right)_q^{k-j} \mathfrak{G}_{j-l,q}^{[l-1,\alpha-1]}(0,y)$$
$$= 2^m \frac{[k]_q!}{[k-l]_q!} \sum_{j=0}^{k-l} \frac{[k-l]_q!}{[k-j-1]_q! [j]_q} \left(\frac{1}{m} - 1 \right)_q^{k-j-l} \mathfrak{G}_{j,q}^{[l-1,\alpha-1]}(0,y)$$
$$= \frac{2^m [k]_q!}{[k-l]_q!} \sum_{j=0}^{k-l} \left[\begin{array}{c} k-l \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k-j-l} \mathfrak{G}_{j,q}^{[l-1,\alpha-1]}(0,y).$$

This means that

$$2^m \sum_{j=0}^{k} \left[\begin{array}{c} k \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k-j} \frac{[n]_q!}{[n-m]_q!} \mathfrak{G}_{j-l,q}^{[m-1,\alpha-1]}(0,y)$$

$$= \frac{2^m [k]_q!}{[k-l]_q!} \sum_{j=0}^{k-l} \left[\begin{array}{c} k-l \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k-j-l} \mathfrak{G}_{j,q}^{[l-1,\alpha-1]}(0,y). \quad (3.5.7)$$

Substituting the results (3.5.6) and (3.5.7) in the identity (3.5.5) gives the desired result.

□

Theorem 3.21. *For $n \in \mathbb{N}_0$, the following relation holds true between the generalized q-Genocchi and the generalized q-Bernoulli polynomials:*

$$\mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,y) =$$

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{1}{m^{n-k-1}[k+1]_q} \Big(\frac{2^m [k]_q!}{[k-l]_q!} \sum_{j=0}^{k-l} \left[\begin{array}{c} k-l \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k-j-l} \mathfrak{G}_{j,q}^{[l-1,\alpha-1]}(0,y)$$

$$- \sum_{j=0}^{k} \sum_{i=0}^{l-1} \left[\begin{array}{c} k \\ j \end{array} \right]_q \left[\begin{array}{c} j \\ i \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k-j} \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}(0,y) - \mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(0,y) \Big) \mathfrak{B}_{n-k,q}(mx,0). \quad (3.5.8)$$

Proof. Using the following identity

$$\left(\frac{2^l t^l}{e_q(t) + \sum_{i=0}^{l-1} \frac{t^i}{[i]_q!}} \right)^{\alpha} e_q(tx) E_q(ty) =$$

$$\left(\frac{2^l t^l}{e_q(t) + \sum_{i=0}^{l-1} \frac{t^i}{[i]_q!}} \right)^{\alpha} E_q(ty) \times \frac{e_q\left(\frac{t}{m}\right) - 1}{\frac{t}{m}} \times \frac{\frac{t}{m}}{e_q\left(\frac{t}{m}\right) - 1} \times e_q\left(\frac{t}{m} mx \right),$$

we have

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(x,y) \frac{t^n}{[n]_q!} = \frac{m}{t} \Big(\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}\left(\frac{1}{m}, y \right) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{[l-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \Big)$$

$$\times \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx,0) \frac{t^n}{m^n [n]_q!}$$

$$= m \sum_{n=1}^{\infty} \Big(\mathfrak{G}_{n,q}^{[l-1,\alpha]}\left(\frac{1}{m}, y \right) - \mathfrak{G}_{n,q}^{[l-1,\alpha]}(0,y) \Big) \frac{t^{n-1}}{[n]_q!} \sum_{k=0}^{\infty} \mathfrak{B}_{k,q}(mx,0) \frac{t^k}{m^k [k]_q!}$$

$$= m \sum_{n=0}^{\infty} \Big(\mathfrak{G}_{n+1,q}^{[l-1,\alpha]}\left(\frac{1}{m}, y \right) - \mathfrak{G}_{n+1,q}^{[l-1,\alpha]}(0,y) \Big) \frac{t^n}{[n+1]_q!} \sum_{k=0}^{\infty} \mathfrak{B}_{k,q}(mx,0) \frac{t^k}{m^k [k]_q!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{1}{m^{n-k-1}[k+1]_q} \Big(\mathfrak{G}_{k+1,q}^{[l-1,\alpha]}\left(\frac{1}{m}, y \right) - \mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(0,y) \Big)$$

$$\times \mathfrak{B}_{n-k,q}(mx,0) \frac{t^n}{[n]_q!}.$$

Finally, applying Lemma (3.20), leads to obtain the desired result. □

Corollary 3.22. *For $n \in \mathbb{N}_0$, and $m \in \mathbb{N}$ the following relations hold true between the generalized q-Genocchi and the generalized q-Bernoulli polynomials*

$$\begin{aligned}&\mathfrak{G}_{n,q}^{[l-1,1]}(x,y)= \\ &\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{1}{m^n [k+1]_q}\Big(\sum_{j=0}^{k-l+1}\left[\begin{array}{c} k+1 \\ j+l \end{array}\right]_q m^{j+l}2^l \frac{[j+l]_q!}{[j+2l]_q!}(x-1)_q^j \\ &-\sum_{j=0}^{k+1}\left[\begin{array}{c} k+1 \\ j \end{array}\right]_q m^j \sum_{i=0}^{l-1}\left[\begin{array}{c} j \\ i \end{array}\right]_q \mathfrak{G}_{j-i,q}^{[l-1,1]}(x,-1)) - m^{k+1}\mathfrak{G}_{k+1,q}^{[l-1,1]}(x,0)\Big) \\ &\times \mathfrak{B}_{n-k,q}(0,my),\end{aligned} \tag{3.5.9}$$

$$\begin{aligned}&\mathfrak{G}_{n,q}^{[l-1,1]}(x,y)= \\ &\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{1}{m^{n-k-1}[k+1]_q}\Big(\frac{2^m [k]_q!}{[k-l]_q!}\sum_{j=0}^{k-l}\left[\begin{array}{c} k-l \\ j \end{array}\right]_q \left(\frac{1}{m}-1\right)_q^{k-j-l} q^{\frac{j(j-1)}{2}} y^j \\ &-\sum_{j=0}^{k}\sum_{i=0}^{l-1}\left[\begin{array}{c} k \\ j \end{array}\right]_q \left[\begin{array}{c} j \\ i \end{array}\right]_q \left(\frac{1}{m}-1\right)_q^{k-j} \mathfrak{G}_{j-i,q}^{[l-1,\alpha]}(0,y) - \mathfrak{G}_{k+1,q}^{[l-1,\alpha]}(0,y)\Big) \qquad (3.5.10) \\ &\times \mathfrak{B}_{n-k,q}(mx,0).\end{aligned} \tag{3.5.11}$$

Proof. For $\alpha = 1$, substituting

$$\mathfrak{G}_{j,q}^{[l-1,0]}(x,-1) = (x-1)_q^j$$

and

$$\mathfrak{G}_{j,q}^{[l-1,0]}(0,y) = q^{\frac{j(j-1)}{2}} y^j$$

inside theorems (3.19) and (3.21) respectively, leads to obtain the desired results. □

Chapter 4

PROPERTIES AND RELATIONS INVOLVING GENERALIZED q-APOSTOL TYPE POLYNOMIALS

4.1 Introduction

Recently, Luo and Srivastava [23], [24] introduced and studied the generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x;\lambda)$ and the generalized Apostol-Euler polynomials $E_n^{\alpha}(x;\lambda)$. Kurt [62] gave the generalization of the Bernoulli polynomials $B_n^{[m-1,\alpha]}(x)$ of order α and studied their properties. They also studied these polynomials systematically, see [23]-[26], [30], [66]-[75]. There are numerous recent investigations on this subject by many other authors, see [20], [58], [62]-[66], [76]-[85]. More recently, Tremblay, Gaboury and Fug`ere further gave the definition of $B_n^{[m-1,\alpha]}(x;\lambda)$ and studied their properties, [63] . On the other hand, Mahmudov and Eini studied various two dimensional q-polynomials,[57], [86]. Motivated by these papers we define generalized Apostol type q-polynomials as follow.

Definition 4.1. Let $q,\alpha\in\mathbb{C},\ m\in\mathbb{N},\ 0<|q|<1$. The generalized q-Apostol-Bernoulli numbers $B_{n,q}^{[m-1,\alpha]}$and polynomials $B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ in x,y of order α are defined, in a suitable neighborhood of $t=0$, by means of the generating functions:

$$\left(\frac{t^m}{\lambda e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}=\sum_{n=0}^{\infty}B_{n,q}^{[m-1,\alpha]}(\lambda)\frac{t^n}{[n]_q!},$$

$$\left(\frac{t^m}{\lambda e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}e_q(tx)\,E_q(ty)=\sum_{n=0}^{\infty}B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!},\tag{4.1.1}$$

where $T_{m-1,q}(t)=\sum\limits_{k=0}^{m-1}\frac{t^k}{[k]q!}$.

Definition 4.2. Let $q,\alpha \in \mathbb{C},\ 0 < |q| < 1,\ m \in \mathbb{N}$. The generalized q-Apostol-Euler numbers $E_{n,q}^{[m-1,\alpha]}$ and polynomials $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ in x,y of order α are defined, in a suitable neighborhood of $t=0$, by means of the generating functions:

$$\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} = \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(\lambda)\frac{t^n}{[n]_q!}, \tag{4.1.2}$$

$$\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,E_q(ty) = \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!}. \tag{4.1.3}$$

Definition 4.3. Let $q,\alpha \in \mathbb{C},\ 0 < |q| < 1,\ m \in \mathbb{N}$. The generalized q-Apostol-Genocchi numbers $G_{n,q}^{[m-1,\alpha]}$ and polynomials $G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ in x,y of order α are defined, in a suitable neighborhood of $t=0$, by means of the generating functions:

$$\left(\frac{2^m t^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} = \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(\lambda)\frac{t^n}{[n]_q!}, \tag{4.1.4}$$

$$\left(\frac{2^m t^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,E_q(ty) = \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{t^n}{[n]_q!}. \tag{4.1.5}$$

Remark 4.4. Clearly, for $m=1$ we have

$$B_{n,q}^{[0,\alpha]}(x,y;\lambda) = B_{n,q}^{(\alpha)}(x,y;\lambda),$$

$$E_{n,q}^{[0,\alpha]}(x,y;\lambda) = E_{n,q}^{(\alpha)}(x,y;\lambda),$$

$$G_{n,q}^{[0,\alpha]}(x,y;\lambda) = G_{n,q}^{(\alpha)}(x,y;\lambda).$$

Also, for $m=1$ and $\lambda=1$ we have

$$B_{n,q}^{[0,\alpha]}(x,y;1) = B_{n,q}^{(\alpha)}(x,y),$$

$$E_{n,q}^{[0,\alpha]}(x,y;1) = E_{n,q}^{(\alpha)}(x,y),$$

$$G_{n,q}^{[0,\alpha]}(x,y;1) = G_{n,q}^{(\alpha)}(x,y).$$

Finally, for $x=y=0$ we have

$$B_{n,q}^{[m-1,\alpha]}(0,0;\lambda) = B_{n,q}^{[m-1,\alpha]}(\lambda),$$

$$E_{n,q}^{[m-1,\alpha]}(0,0;\lambda) = E_{n,q}^{[m-1,\alpha]}(\lambda),$$

$$G_{n,q}^{[m-1,\alpha]}(0,0;\lambda) = G_{n,q}^{[m-1,\alpha]}(\lambda).$$

4.2 Properties of the Apostol type q-polynomials

In this section, we show some basic properties of the generalized q-polynomials. We only prove the facts for one of them. Obviously, by applying the similar technique other ones can be proved.

Proposition 4.5. *The generalized q-polynomials $B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$, $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ and $G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ satisfy the following relations:*

$$B_{n,q}^{[m-1,\alpha+\beta]}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}^{[m-1,\alpha]}(x,0;\lambda) B_{n-k,q}^{[m-1,\beta]}(0,y;\lambda), \tag{4.2.1}$$

$$E_{n,q}^{[m-1,\alpha+\beta]}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(x,0;\lambda) E_{n-k,q}^{[m-1,\beta]}(0,y;\lambda),$$

$$G_{n,q}^{[m-1,\alpha+\beta]}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{[m-1,\alpha]}(x,0;\lambda) G_{n-k,q}^{[m-1,\beta]}(0,y;\lambda).$$

Proof. We only prove the second identity. By using definition (4.2), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha+\beta]}(x,y;\lambda) \frac{t^n}{[n]_q!} = \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha+\beta} e_q(tx) E_q(ty) \\
&= \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\alpha} e_q(tx) \left(\frac{2^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^{\beta} E_q(ty) \\
&= \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x,0;\lambda) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\beta]}(0,y;\lambda) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(x,0;\lambda) E_{n-k,q}^{[m-1,\beta]}(0,y;\lambda) \frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of the term $\frac{t^n}{[n]_q!}$ in both sides gives the result. $\square$

Corollary 4.6. *The generalized q-polynomials $B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$, $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ and $G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ satisfy the following relations:*

$$B_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}^{[m-1,\alpha]}(0,y;\lambda) x^{n-k}, \tag{4.2.2}$$

$$E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) x^{n-k},$$

$$G_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{[m-1,\alpha]}(0,y;\lambda) x^{n-k}.$$

Proposition 4.7. *The generalized q-polynomials $B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$, $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ and $G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ satisfy the following relations:*

$$\lambda B_{n,q}^{[m-1,\alpha]}(1,y;\lambda)-B_{n,q}^{[m-1,\alpha]}(0,y;\lambda)$$
$$=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q [k]_q B_{k,q}^{[m-1,\alpha]}(0,y;\lambda)B_{n-k,q}^{[0,-1]}(\lambda),\quad \textit{for}\quad n\geq 1,\quad (4.2.3)$$

$$\lambda E_{n,q}^{[m-1,\alpha]}(1,y;\lambda)+E_{n,q}^{[m-1,\alpha]}(0,y;\lambda)$$
$$=2\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)E_{n-k,q}^{[0,-1]}(\lambda),\quad (4.2.4)$$

$$\lambda G_{n,q}^{[m-1,\alpha]}(1,y;\lambda)+G_{n,q}^{[m-1,\alpha]}(0,y;\lambda)$$
$$=2\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q [k]_q G_{k,q}^{[m-1,\alpha]}(0,y;\lambda)G_{n-k,q}^{[0,-1]}(\lambda),\quad \textit{for}\quad n\geq 1.\quad (4.2.5)$$

Proof. We only prove (4.2.4). By using definition (4.2), and starting from the left hand side of the relation(4.2.4) we have:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\lambda E_{n,q}^{[m-1,\alpha]}(1,y;\lambda)+E_{n,q}^{[m-1,\alpha]}(0,y;\lambda)\right)\frac{t^n}{[n]_q!}\\
&=\lambda\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}e_q(t)\,E_q(ty)+\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\\
&=\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\,(\lambda e_q(t)+1)\\
&=2\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}E_q(ty)\left(\frac{2}{\lambda e_q(t)+1}\right)^{-1}\\
&=2\sum_{n=0}^{\infty}E_{n,q}^{[m-1,\alpha]}(0,y;\lambda)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}E_{n,q}^{[0,-1]}(\lambda)\frac{t^n}{[n]_q!}\\
&=2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)E_{n-k,q}^{[0,-1]}(\lambda)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of the term $\frac{t^n}{[n]_q!}$ in both sides gives the result. $\square$

4.3 q-analogue of the Luo-Srivastava addition theorem

In this section we state and prove a q-generalization of the Luo-Srivastava addition theorem.

Theorem 4.8. *The following relation holds between generalized q-Apostol-Euler and q-Apostol-Bernoulli polynomials:*

$$E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)=$$
$$\sum_{j=0}^{n}\frac{1}{[n+1]_q}\left[\begin{array}{c}n+1\\j\end{array}\right]_q\left(\lambda\sum_{k=0}^{n-j+1}\frac{1}{[n+1]_q}\left[\begin{array}{c}n-j+1\\k\end{array}\right]_q E_{k,q}^{[m-1,\alpha-1]}(0,y;\lambda)\right.$$
$$\left.-\,E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right)B_{j,q}(x,0;\lambda)+\frac{\lambda-1}{[n+1]_q}\left(\frac{2^m}{\lambda+1}\right)^{\alpha}B_{n+1,q}(x,0;\lambda). \tag{4.3.1}$$

Proof. We take aid of the following identity to prove identity (4.3.1):

$$\lambda\frac{t}{\lambda e_q(t)-1}e_q(tx)\,e_q(t)-\frac{t}{\lambda e_q(t)-1}e_q(tx)=\frac{te_q(tx)}{\lambda e_q(t)-1}(\lambda e_q(t)-1)=te_q(tx)\,.$$

Therefore, we can write:

$$\begin{aligned}&\lambda\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q B_{k,q}(x,0;\lambda)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}B_{n,q}(x,0;\lambda)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}x^n\frac{t^{n+1}}{[n+1]_q!}[n+1]_q\\&=\sum_{n=0}^{\infty}[n]_qx^{n-1}\frac{t^n}{[n]_q!},\end{aligned}$$

from that we can conclude:

$$\lambda\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q B_{k,q}(x,0;\lambda)-B_{n,q}(x,0;\lambda)=[n]_qx^{n-1},$$

that is

$$x^n=\frac{1}{[n+1]_q}\left(\lambda\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q B_{k,q}(x,0;\lambda)-B_{n+1,q}(x,0;\lambda)\right). \tag{4.3.2}$$

Substituting identity (4.3.2) into the right hand side of relation (4.2.2), we obtain:

$$
\begin{aligned}
E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \frac{1}{[n-k+1]_q} \\
&\times \left(\lambda \sum_{j=0}^{n-k+1} \left[\begin{array}{c} n-k+1 \\ j \end{array}\right]_q B_{j,q}(x,0;\lambda) - B_{n-k+1,q}(x,0;\lambda) \right) \\
&= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \frac{1}{[n-k+1]_q} \\
&\times \left(\lambda \sum_{j=0}^{n-k} \left[\begin{array}{c} n-k+1 \\ j \end{array}\right]_q B_{j,q}(x,0;\lambda) + (\lambda-1) B_{n-k+1,q}(x,0;\lambda) \right) \\
&= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \frac{\lambda}{[n-k+1]_q} \sum_{j=0}^{n-k} \left[\begin{array}{c} n-k+1 \\ j \end{array}\right]_q B_{j,q}(x,0;\lambda) \\
&+ \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \frac{\lambda-1}{[n-k+1]_q} B_{n-k+1,q}(x,0;\lambda) \\
&:= I_1 + I_2.
\end{aligned}
$$

Thus, from one hand we can write:

$$
\begin{aligned}
I_1 &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \frac{\lambda}{[n-k+1]_q} \sum_{j=0}^{n-k} \left[\begin{array}{c} n-k+1 \\ j \end{array}\right]_q B_{j,q}(x,0;\lambda) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{\lambda}{[n+1]_q} \left[\begin{array}{c} n+1 \\ n-k+1 \end{array}\right]_q \left[\begin{array}{c} n-k+1 \\ j \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) B_{j,q}(x,0;\lambda),
\end{aligned}
$$

According to the part (c) of the Proposition (2.6), we know that

$$
\left[\begin{array}{c} m \\ l \end{array}\right]_q \left[\begin{array}{c} l \\ n \end{array}\right]_q = \left[\begin{array}{c} m \\ n \end{array}\right]_q \left[\begin{array}{c} m-n \\ m-l \end{array}\right]_q, \qquad \text{for} \quad m \geq l \geq n,
$$

so, we may continue as

$$
\begin{aligned}
I_1 &= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{\lambda}{[n+1]_q} \left[\begin{array}{c} n+1 \\ j \end{array}\right]_q \left[\begin{array}{c} n-j+1 \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) B_{j,q}(x,0;\lambda) \\
&= \sum_{j=0}^{n} \frac{\lambda}{[n+1]_q} \left[\begin{array}{c} n+1 \\ j \end{array}\right]_q B_{j,q}(x,0;\lambda) \sum_{k=0}^{n-j} \left[\begin{array}{c} n-j+1 \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\
&= \sum_{j=0}^{n} \frac{\lambda}{[n+1]_q} \left[\begin{array}{c} n+1 \\ j \end{array}\right]_q B_{j,q}(x,0;\lambda) \left(E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda) - E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \right).
\end{aligned}
$$

From another hand for I_2, we can write:

$$
\begin{aligned}
I_2 &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \frac{\lambda-1}{[n-k+1]_q} B_{n-k+1,q}(x,0;\lambda) \\
&= \sum_{k=0}^{n} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_q \frac{\lambda-1}{[n+1]_q} B_{n-k+1,q}(x,0;\lambda) E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\
&= \sum_{k=0}^{n+1} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_q \frac{\lambda-1}{[n+1]_q} B_{n-k+1,q}(x,0;\lambda) E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\
&- \frac{\lambda-1}{[n+1]_q} B_{0,q}(x,0;\lambda) E_{k,q}^{[m-1,\alpha]}(0,y;\lambda),
\end{aligned}
$$

and as $B_{0,q}(x,0;\lambda)=0$, we have:

$$
\begin{aligned}
&= \sum_{k=0}^{n+1} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_q \frac{\lambda-1}{[n+1]_q} B_{n-k+1,q}(x,0;\lambda) E_{k,q}^{[m-1,\alpha]}(0,y;\lambda) \\
&= \sum_{j=0}^{n+1} \left[\begin{array}{c} n+1 \\ j \end{array} \right]_q \frac{\lambda-1}{[n+1]_q} B_{j,q}(x,0;\lambda) E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \\
&= \sum_{j=0}^{n} \left[\begin{array}{c} n+1 \\ j \end{array} \right]_q \frac{\lambda-1}{[n+1]_q} B_{j,q}(x,0;\lambda) E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \\
&+ \frac{\lambda-1}{[n+1]_q} B_{n+1,q}(x,0;\lambda) E_{0,q}^{[m-1,\alpha]}(0,y;\lambda).
\end{aligned}
$$

Adding I_2 to I_1 we obtain:

$$
\begin{aligned}
&E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = I_1 + I_2 = \\
&\sum_{j=0}^{n} \frac{\lambda}{[n+1]_q} \left[\begin{array}{c} n+1 \\ j \end{array} \right]_q B_{j,q}(x,0;\lambda) \left(E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda) - E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) \right) \\
&+ \sum_{j=0}^{n} \left[\begin{array}{c} n+1 \\ j \end{array} \right]_q \frac{\lambda-1}{[n+1]_q} B_{j,q}(x,0;\lambda) E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda) + \\
&\frac{\lambda-1}{[n+1]_q} B_{n+1,q}(x,0;\lambda) E_{0,q}^{[m-1,\alpha]}(0,y;\lambda).
\end{aligned}
$$

Consequently

$$E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)=\sum_{j=0}^{n}\frac{1}{[n+1]_q}\left[\begin{array}{c}n+1\\ j\end{array}\right]_q$$
$$\times\left(\lambda E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda)-E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)+(\lambda-1)E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right)B_{j,q}(x,0;\lambda)$$
$$+\frac{\lambda-1}{[n+1]_q}B_{n+1,q}(x,0;\lambda)E_{0,q}^{[m-1,\alpha]}(0,y;\lambda)$$
$$=\sum_{j=0}^{n}\frac{1}{[n+1]_q}\left[\begin{array}{c}n+1\\ j\end{array}\right]_q\left(\lambda E_{n-j+1,q}^{[m-1,\alpha]}(1,y;\lambda)-E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right)B_{j,q}(x,0;\lambda)$$
$$+\frac{(\lambda-1)}{[n+1]_q}B_{n+1,q}(x,0;\lambda)E_{0,q}^{[m-1,\alpha]}(0,y;\lambda)$$
$$=\sum_{j=0}^{n}\frac{1}{[n+1]_q}\left[\begin{array}{c}n+1\\ j\end{array}\right]_q$$
$$\left(\lambda\sum_{k=0}^{n-j+1}\left[\begin{array}{c}n-j+1\\ k\end{array}\right]_q E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)-E_{n-j+1,q}^{[m-1,\alpha]}(0,y;\lambda)\right)B_{j,q}(x,0;\lambda)$$
$$+\frac{(\lambda-1)}{[n+1]_q}\left(\frac{2^m}{\lambda+1}\right)^{\alpha}B_{n+1,q}(x,0;\lambda),$$

whence the result. □

Taking $m=1$ in Theorem 4.3.1 , we get a q-generalization of the Luo-Srivastava addition theorem [25].

Corollary 4.9. *The following relation holds between generalized q-Apostol-Euler and q-Apostol-Bernoulli polynomials:*

$$E_{n,q}^{(\alpha)}(x,y;\lambda)=\sum_{j=0}^{n}\frac{2}{[j+1]_q}\left[\begin{array}{c}n\\ j\end{array}\right]_q\left(E_{j+1,q}^{(\alpha-1)}(0,y;\lambda)\right.$$
$$\left.-E_{j+1,q}^{(\alpha)}(0,y;\lambda)\right)B_{n-j,q}(x,0;\lambda)+\frac{\lambda-1}{[n+1]_q}\left(\frac{2}{\lambda+1}\right)^{\alpha}B_{n+1,q}(x,0;\lambda). \tag{4.3.3}$$

Letting $q\uparrow 1$, we get the Luo-Srivastava addition theorem (see [77]):

$$E_{n}^{(\alpha)}(x+y;\lambda)=\sum_{j=0}^{n}\frac{2}{j+1}\binom{n}{j}\left(E_{j+1}^{(\alpha-1)}(y;\lambda)\right.$$
$$\left.-E_{j+1}^{(\alpha)}(y;\lambda)\right)B_{n-j,q}(x;\lambda)+\frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^{\alpha}B_{n+1}(x;\lambda).$$

Next theorem gives relationship between $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ and $G_{n,q}(x,0)$.

Theorem 4.10. *The following relation holds between generalized q-Apostol-Euler and q-Apostol-Genocchi polynomials:*

$$E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{[k+1]_q}\left(\lambda\sum_{j=k}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q\left[\begin{array}{c} j \\ k \end{array}\right]_q E_{n-j,q}^{[m-1,\alpha]}(0,y;\lambda)\right.$$
$$\left.+\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q E_{n-k,q}^{[m-1,\alpha]}(0,y;\lambda)\right)G_{k+1,q}(x,0).$$

Proof. The proof follows from the following identity:

$$\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)\,E_q(ty) =$$
$$\left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha} E_q(ty)\,\frac{2t}{e_q(t)+1}e_q(tx)\,\frac{e_q(t)+1}{2t}.$$

□

Theorem 4.11. *The following relation holds between generalized q-Apostol-Euler and q-Stirling polynomials $S_q(i,j)$of the second kind:*

$$E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n}\left[\begin{array}{c} n \\ n-j \end{array}\right]_q E_{n-j,q}^{[m-1,\alpha]}(0,y;\lambda)S_q(j,k)x_k(x).$$

Proof. The q-Stirling polynomials $S_q(n,k)$ of the second kind are defined by means of the following generating function:

$$x^n = \sum_{k=0}^{n} S_q(n,k)x_k(x), \tag{4.3.4}$$

where $x_k(x) = x(x-[1]_q)(x-[2]_q)\ldots(x-[k-1]_q)$, [87]. Replacing identity (4.3.4) in the right hand side of (4.2.2), we have:

$$\begin{aligned} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) &= \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q E_{k,q}^{[m-1,\alpha]}(0,y;\lambda)\sum_{k=0}^{n-k} S_q(n-k,k)x_k(x) \\ &= \sum_{k=0}^{n}\sum_{j=k}^{n}\left[\begin{array}{c} n \\ n-j \end{array}\right]_q E_{n-j,q}^{[m-1,\alpha]}(0,y;\lambda)S_q(j,k)x_k(x). \end{aligned}$$

□

Theorem 4.12. *The relationship*

$$E_{n,q}^{[m-1,\alpha]}(x,y;\lambda) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{n-2k}\left[\begin{array}{c} n \\ k \end{array}\right]_q\left[\begin{array}{c} n-2k \\ j \end{array}\right]_q\frac{[k]_q!}{[2]_q^n\,[k]_{q^2}!}E_{j,q}^{[m-1,\alpha]}\left(0,y;\lambda\right)H_{n-2k-j,q}\left(x\right)$$

holds between the polynomials $E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)$ *and the* q*-Hermite polynomials defined by, see [88],*

$$e_q\left(tx\right)E_{q^2}\left(-\frac{t^2}{[2]_q}\right)=\sum_{n=0}^{\infty}H_{n,q}\left(x\right)\frac{t^n}{[n]_q!}.$$

Proof. Indeed,

$$\begin{aligned}
&\sum_{n=0}^{\infty}E_{n,q}^{[m-1,\alpha]}\left(x,y;\lambda\right)\frac{t^n}{[n]_q!}=\left(\frac{2^m}{\lambda e_q\left(t\right)+T_{m-1,q}\left(t\right)}\right)^{\alpha}e_q\left(tx\right)E_q\left(ty\right)\\
&=\left(\frac{2^m}{\lambda e_q\left(t\right)+T_{m-1,q}\left(t\right)}\right)^{\alpha}E_q\left(ty\right)e_q\left(tx\right)E_{q^2}\left(-\frac{t^2}{[2]_q}\right)e_{q^2}\left(\frac{t^2}{[2]_q}\right)\\
&=\sum_{n=0}^{\infty}E_{n,q}^{[m-1,\alpha]}\left(0,y;\lambda\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}H_{n,q}\left(x\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{2n}}{[2]_q^n\,[n]_{q^2}!}\\
&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q E_{j,q}^{[m-1,\alpha]}\left(0,y;\lambda\right)H_{n-j,q}\left(x\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{2n}}{[2]_q^n\,[n]_{q^2}!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{[n]_q!}{[2]_q^n\,[k]_{q^2}!\,[n-2k]_q!}\sum_{j=0}^{n-2k}\left[\begin{array}{c} n-2k \\ j \end{array}\right]_q E_{j,q}^{[m-1,\alpha]}\left(0,y;\lambda\right)H_{n-2k-j,q}\left(x\right)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{n-2k}\left[\begin{array}{c} n \\ k \end{array}\right]_q\left[\begin{array}{c} n-2k \\ j \end{array}\right]_q\frac{[k]_q!}{[2]_q^n\,[k]_{q^2}!}E_{j,q}^{[m-1,\alpha]}\left(0,y;\lambda\right)H_{n-2k-j,q}\left(x\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

□

Chapter 5

A DETERMINANTAL REPRESENTATION FOR THE CLASS OF q-APPELL POLYNOMIALS

5.1 Introduction

Appell polynomials for the first time were defined by Appell in 1880, [34]. Inspired by the work of Thorne [89], Sheffer [35], and Varma [90], Al-Salam, in 1967, introduced the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$, and studied some of their properties [91]. According to his definition, the n-degree polynomials $A_{n,q}(x)$ are called q-Appell if they hold the following q-differential equation

$$D_{q,x}(A_{n,q}(x)) = [n]_q A_{n-1,q}(x), \quad n = 0, 1, 2, \ldots \tag{5.1.1}$$

Note to the fact that $A_{0,q}(x)$ is a non zero constant let say $A_{0,q}$. To begin with the relation(5.1.1) for $n = 1$, i. e.

$$D_{q,x}(A_{1,q}(x)) = [1]_q A_{0,q}(x) = A_{0,q}.$$

Using Jackson integral for the q-differential equation above, we get

$$A_{1,q}(x) = A_{0,q}x + A_{1,q},$$

where $A_{1,q}$ is an arbitrary constant. We can repeat the method above to obtain $A_{2,q}(x)$, as below by starting from the property(5.1.1) for q-Appell polynomials

$$D_{q,x}(A_{2,q}(x)) = [2]_q A_{1,q}x = [2]_q A_{0,q}x + [2]_q A_{1,q}.$$

Now take Jackson integral

$$A_{2,q}(x) = A_{0,q}x^2 + [2]_q A_{1,q} + A_{2,q},$$

where $A_{2,q}$ is an arbitrary constant.

By using induction on n and applying similar method to the methods used for finding $A_{1,q}(x)$, $A_{2,q}(x)$ and continuing taking Jackson integrals we have

$$A_{n-1,q}(x) = A_{n-1,q} + \left[\begin{array}{c} n-1 \\ 1 \end{array}\right]_q A_{n-2,q}x + \left[\begin{array}{c} n-1 \\ 2 \end{array}\right]_q A_{n-3,q}x^2 + \ldots + A_{0,q}x^{n-1}.$$

Considering the fact that for $n = 1, 2, 3, \ldots$, every $A_{n,q}(x)$ satisfies the relation (5.1.1), we can write

$$\begin{aligned} D_{q,x}(A_{n,q}(x)) = [n]_q A_{n-1,q} + [n]_q \left[\begin{array}{c} n-1 \\ 1 \end{array}\right]_q A_{n-2,q}x \\ + [n]_q \left[\begin{array}{c} n-1 \\ 2 \end{array}\right]_q A_{n-3,q}x^2 + \ldots + [n]_q A_{0,q}x^{n-1}. \end{aligned}$$

Now, taking the Jackson integral of the q-differential equation above can lead to

$$\begin{aligned} A_{n,q}(x) = A_{n,q} + [n]_q A_{n-1,q}x + \frac{[n]_q}{[2]_q} \left[\begin{array}{c} n-1 \\ 1 \end{array}\right]_q A_{n-2,q}x^2 \\ + \frac{[n]_q}{[3]_q} \left[\begin{array}{c} n-1 \\ 2 \end{array}\right]_q A_{n-3,q}x^3 + \ldots + \frac{[n]_q}{[n]_q} A_{0,q}x^n, \end{aligned}$$

where $A_{n,q}$ is an arbitrary constant. Since

$$\frac{[n]_q}{[i]_q} \left[\begin{array}{c} n-1 \\ i-1 \end{array}\right]_q = \left[\begin{array}{c} n \\ i \end{array}\right]_q,$$

so for $n = 0, 1, 2, \ldots$, we have

$$A_{n,q}(x) = A_{n,q} + [n]_q A_{n-1,q}x + \left[\begin{array}{c} n \\ 2 \end{array}\right]_q A_{n-2,q}x^2 + \left[\begin{array}{c} n \\ 3 \end{array}\right]_q A_{n-3,q}x^3 + \ldots + A_{0,q}x^n. \quad (5.1.2)$$

It is worthy of note that according to the discussion above there exists a one to one correspondence between the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$ and the numerical sequence $\{A_{n,q}\}_{n=0}^{\infty}$, $A_{n,q} \neq 0$. Moreover, every $A_{n,q}(x)$ can be obtained recursively from $A_{n-1,q}(x)$ for $n \geqslant 1$.

Also, q-Appell polynomials can be defined by means of generating function $A_q(t)$, as follows

$$A_q(x,t) := A_q(t)e_q(tx) = \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}, \quad 0 < q < 1, \tag{5.1.3}$$

where

$$A_q(t) := \sum_{n=0}^{\infty} A_{n,q}\frac{t^n}{[n]_q!}, \quad A_q(t) \neq 0, \tag{5.1.4}$$

is an analytic function at $t = 0$, $A_{n,q}(0) := A_{n,q}$, and $e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}$.

Based on different choices of $A_q(t)$, which is called the determining function for the set of $\{A_{n,q}(x)\}$, different families of q-Appell polynomials can be obtained. In the following we mention some of them:

a) Taking $A_q(t) = [1]_q = 1$ leads to obtain the family including all increasing integer powers of x starting from 0,

$$\{1, x, x^2, x^3, \ldots\}.$$

b) Taking $A_q(t) = \left(\frac{t^m}{e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}$, leads to obtain the family of generalized q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$, [86].

c) Taking $A_q(t) = \left(\frac{2^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}$, leads to obtain the family of generalized q-Euler polynomials $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$, [86].

d) Taking $A_q(t) = \left(\frac{2^m t^m}{e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}$, leads to obtain the family of generalized q-Genocchi polynomials $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,0)$, [86].

e) Taking $A_q(t) = \left(\frac{t^m}{\lambda e_q(t)-T_{m-1,q}(t)}\right)^{\alpha}$, leads to obtain the family of generalized q-Apostol Bernoulli polynomials $B_{n,q}^{[m-1,\alpha]}(x,0;\lambda)$ of order α, [92].

f) Taking $A_q(t) = \left(\frac{2^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}$, leads to obtain the family of q-Apostol-Euler polynomials $E_{n,q}^{[m-1,\alpha]}(x,0;\lambda)$ of order α, [92].

g) Taking $A_q(t) = \left(\frac{2^m t^m}{\lambda e_q(t)+T_{m-1,q}(t)}\right)^{\alpha}$, leads to obtain the family of q-Apostol-Genocchi polynomials $G_{n,q}^{[m-1,\alpha]}(x,0;\lambda)$ of order α, [92].

h) Taking $A_q(t) = H_q(t) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)} \frac{t^{2n}}{[2n]!!}$, leads to obtain the family of q-Hermite polynomials $H_{n,q}(x)$, [88].

Later, in 1982, Srivastava specified more characterizations of the family of q-Appell polynomials, [93]. Over the past decades, q-Appell polynomials have been studied from different aspects in [94], [95], using different methods such as operator algebra their properties are found in [96]. Also, recently, the q-difference equations satisfied by sequence of q-Appell polynomials have been derived by Mahmudov, [97]. In this paper, inspired by the Costabile et al.'s determinantal approach for defining Bernoulli polynomials as well as Appell polynomials, for the first time, we introduce a determinantal definition of the well known family of q-Appell polynomials, [98], [99]. This new determinantal definition, not only allows us to benefit from algebraic properties of determinant to prove the existing properties of q-Appell polynomials simpler, but also helps to find some new properties. Moreover, this approach unifies all different families of q-Appell polynomials some of which are mentioned in a)-h)and expresses them by using one single representation.

In the following sections, firstly we introduce the determinantal definition of q-Appell polynomials and then we show that this definition matches with the classical definitions. Next we prove some classical and new properties related to this family in the light of the new definition and by using the related algebraic approaches.

5.2 q-Appell polynomials from determinantal point of view

Assume that $P_{n,q}(x)$ is an n-degree q-polynomial defined as follows

$$\begin{cases} P_{0,q}(x) = \frac{1}{\beta_0} \\ P_{n,q}(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \dots & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \dots & \dots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \dots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix} \end{cases}, \tag{5.2.1}$$

where $\beta_0, \beta_1, \dots, \beta_n \in \mathbb{R}$, $\beta_0 \neq 0$, $n = 1, 2, 3, \dots$.

Then we can obtain the following results.

Lemma 5.1. *Suppose that $A_{n\times n}(x)$ is a matrix including elements $a_{ij}(x)$ which are first order q-differentiable functions of variable x. Then the q-derivative of $\det(A_{n\times n}(x))$ can be calculated by the following formula.*

$$\begin{aligned} D_{q,x}(\det(A_{n\times n}(x))) &= D_{q,x}(|a_{ij}(x)|) \\ &= \sum_{i=1}^{n} \begin{vmatrix} a_{11}(qx) & a_{12}(qx) & \dots & a_{1n}(qx) \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1}(qx) & a_{i-1,2}(qx) & \dots & a_{i-1,n}(qx) \\ D_{q,x}(a_{i1}(x)) & D_{q,x}(a_{i2}(x)) & \dots & D_{q,x}(a_{in}(x)) \\ a_{i+1,1}(x) & a_{i+1,2}(x) & \dots & a_{i+1,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix}. \end{aligned} \tag{5.2.2}$$

Proof. The proof can be done by induction on n. □

Theorem 5.2. *$P_{n,q}(x)$, satisfies the following identity*

$$D_{q,x}(P_{n,q}(x)) = [n]_q P_{n-1,q}(x), \quad n = 1, 2, \dots$$

Proof. Taking the q-derivative of determinant (5.2.1) with respect to x by using formula(5.2.2), given in Lemma 5.1 , we obtain

$$D_{q,x}(P_{n,q}(x)) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 0 & 1 & [2]_q x & \dots & [n]_q x^{n-1} \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_n \\ 0 & \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \dots & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}, \tag{5.2.3}$$

Expanding the determinant(5.2.3) above along with the first column, we have

$$D_{q,x}(P_{n,q}(x)) = \frac{(-1)^{n-1}}{(\beta_0)^n} \times$$

$$\begin{vmatrix} 1 & [2]_q x & \dots & \dots & [n-1]_q x^{n-2} & [n]_q x^{n-1} \\ \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \dots & \dots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\ 0 & \beta_0 & \dots & \dots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}. \tag{5.2.4}$$

Now, considering the fact that

$$\frac{[i-1]_q}{[j]_q} \begin{bmatrix} j \\ i-1 \end{bmatrix}_q = \frac{[i-1]_q [j]_q!}{[j]_q [i-1]_q! [j-i+1]_q} = \frac{[j-1]_q!}{[i-2]_q! [j-i+1]_q} = \begin{bmatrix} j-1 \\ i-2 \end{bmatrix}_q,$$

and multiplying the j^{th} column of the determinant(5.2.4) by $\frac{1}{[j]_q}$, as well as the i^{th} row by $[i-1]_q$ we obtain

$$D_{q,x}(P_{n,q}(x)) = \frac{(-1)^{n-1}}{(\beta_0)^n} \times \frac{[1]_q!}{[0]_q!} \times \frac{[2]_q}{[1]_q} \times ... \times \frac{[n]_q}{[n-1]_q} \times$$

$$\begin{vmatrix} 1 & x & ... & ... & x^{n-2} & x^{n-1} \\ \beta_0 & \beta_1 & ... & ... & \beta_{n-2} & \beta_{n-1} \\ 0 & \beta_0 & ... & ... & \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q \beta_{n-3} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & & \beta_0 & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q \beta_1 \end{vmatrix}, \tag{5.2.5}$$

which is exactly the desired result. □

Theorem 5.3. *The q-polynomials $P_{n,q}(x)$, defined in (5.2.3), can be expressed as*

$$P_{n,q}(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \alpha_{n-j} x^j, \tag{5.2.6}$$

where

$$\begin{cases} \alpha_0 = \frac{1}{\beta_0} \\ \alpha_j = \frac{(-1)^j}{(\beta_0)^{j+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & ... & ... & \beta_{j-1} & \beta_j \\ 0 & \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & ... & ... & \begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q \beta_{j-2} & \begin{bmatrix} j \\ 1 \end{bmatrix}_q \beta_{j-1} \\ 0 & 0 & \beta_0 & ... & ... & \begin{bmatrix} j-1 \\ 2 \end{bmatrix}_q \beta_{j-3} & \begin{bmatrix} j \\ 2 \end{bmatrix}_q \beta_{j-2} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & ... & ... & ... & 0 & \beta_0 & \begin{bmatrix} j \\ j-1 \end{bmatrix}_q \beta_1 \end{vmatrix} . \end{cases} \tag{5.2.7}$$

Proof. Expanding the determinant(5.2.1) along the first row, we obtain

$$P_{n,q}(x) = \frac{(-1)^{n+2}}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\ 0 & \beta_0 & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}$$

$$+ \frac{(-1)^{n+3}}{(\beta_0)^{n+1}} x \begin{vmatrix} \beta_0 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\ 0 & \beta_0 & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}$$

$$+ \ldots + \frac{(-1)^{2n+2}}{(\beta_0)^{n+1}} x^n \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} \\ 0 & \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 \end{vmatrix}. \quad (5.2.8)$$

Clearly, according to the given definition for α_i in relation (5.2.7), the first determinant leads to obtain α_n, which is the coefficient of x^0. Also, the last determinant, which is the determinant of an upper triangular $n \times n$ matrix, will lead to obtain the coefficient of x^n as follows

$$\alpha_0 = \frac{(-1)^{2n+2}}{(\beta_0)^{n+1}} (\beta_0)^n = \frac{1}{\beta_0}.$$

To calculate the coefficient of x^j for $0 < j < n$, consider the following determinant

$$= \frac{(-1)^n}{(\beta_0)^{n+1}}(-1)^{j+2}\times$$

$$\begin{vmatrix} \beta_0 & \beta_1 & \dots & \beta_{j-1} & \beta_{j+1} & \dots & \beta_n \\ 0 & \beta_0 & \dots & \begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q \beta_{j-2} & \begin{bmatrix} j+1 \\ 1 \end{bmatrix}_q \beta_j & \dots & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\ 0 & 0 & & \begin{bmatrix} j-1 \\ 2 \end{bmatrix}_q \beta_{j-3} & \begin{bmatrix} j+1 \\ 2 \end{bmatrix}_q \beta_{j-1} & \dots & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & & \beta_0 & \begin{bmatrix} j+1 \\ j-1 \end{bmatrix}_q \beta_2 & \dots & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-j-1} \\ \vdots & \ddots & & 0 & \begin{bmatrix} j+1 \\ j \end{bmatrix}_q \beta_1 & \dots & \begin{bmatrix} n \\ j \end{bmatrix}_q \beta_{n-j} \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 & 0 & \dots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}$$

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n+1}}(\beta_0)^j \begin{vmatrix} \begin{bmatrix} j+1 \\ j \end{bmatrix}_q \beta_1 & \dots & \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \beta_{n-j-1} & \begin{bmatrix} n \\ j \end{bmatrix}_q \beta_{n-j} \\ \beta_0 & & & \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}.$$

Now multiplying the first column of the last determinant by $\frac{1}{\begin{bmatrix} j+1 \\ j \end{bmatrix}_q}$, we obtain

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \times \frac{1}{\begin{bmatrix} j+1 \\ j \end{bmatrix}_q} \begin{vmatrix} \beta_1 & \begin{bmatrix} j+2 \\ j \end{bmatrix}_q \beta_2 & \dots & \begin{bmatrix} n \\ j \end{bmatrix}_q \beta_{n-j} \\ \frac{1}{\begin{bmatrix} j+1 \\ j \end{bmatrix}_q}\beta_0 & \begin{bmatrix} j+2 \\ j+1 \end{bmatrix}_q \beta_1 & \dots & \begin{bmatrix} n \\ j+1 \end{bmatrix}_q \beta_{n-j-1} \\ & \beta_0 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}.$$

Further similar calculations to get coefficients 1 for the first elements of each column in determinant above leads to

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \times \frac{1}{\left[\begin{array}{c} j+1 \\ j \end{array}\right]_q} \times \frac{1}{\left[\begin{array}{c} j+2 \\ j \end{array}\right]_q} \times ... \times \frac{1}{\left[\begin{array}{c} n-1 \\ j \end{array}\right]_q} \times \frac{1}{\left[\begin{array}{c} n \\ j \end{array}\right]_q} \times$$

$$\left| \begin{array}{ccccc} \beta_1 & \beta_2 & \cdots & \beta_{n-j-1} & \beta_{n-j} \\ \frac{1}{\left[\begin{array}{c} j+1 \\ j \end{array}\right]_q}\beta_0 & \frac{\left[\begin{array}{c} j+2 \\ j+1 \end{array}\right]_q}{\left[\begin{array}{c} j+2 \\ j \end{array}\right]_q}\beta_1 & \cdots & \frac{\left[\begin{array}{c} n-1 \\ j+1 \end{array}\right]_q}{\left[\begin{array}{c} n-1 \\ j \end{array}\right]_q}\beta_{n-j-2} & \frac{\left[\begin{array}{c} n \\ j+1 \end{array}\right]_q}{\left[\begin{array}{c} n \\ j \end{array}\right]_q}\beta_{n-j-1} \\ & \beta_0 & & & \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & \cdots & \beta_0 & \left[\begin{array}{c} n \\ n-1 \end{array}\right]_q \beta_1 \end{array} \right|.$$

In order to create coefficient 1 for the term β_0 placed in the second row of the above determinant, multiply this row by $\left[\begin{array}{c} j+1 \\ j \end{array}\right]$. As we are aware of the fact that

$$\frac{\left[\begin{array}{c} j+2 \\ j+1 \end{array}\right]_q}{\left[\begin{array}{c} j+2 \\ j \end{array}\right]_q} . \left[\begin{array}{c} j+1 \\ j \end{array}\right]_q = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]_q ,$$

and also

$$\frac{\left[\begin{array}{c} n \\ j+1 \end{array}\right]_q}{\left[\begin{array}{c} n \\ j \end{array}\right]_q} \left[\begin{array}{c} j+1 \\ j \end{array}\right]_q = \left[\begin{array}{c} n-j \\ 1 \end{array}\right]_q .$$

Thus we have

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \times \frac{1}{\left[\begin{array}{c} j+2 \\ j \end{array}\right]_q} \times ... \times \frac{1}{\left[\begin{array}{c} n-1 \\ j \end{array}\right]_q} \times \frac{1}{\left[\begin{array}{c} n \\ j \end{array}\right]_q}$$

$$\times \left| \begin{array}{ccccc} \beta_1 & \beta_2 & \cdots & \beta_{n-j-1} & \beta_{n-j} \\ \beta_0 & \left[\begin{array}{c} 2 \\ 1 \end{array}\right]_q \beta_1 & \cdots & \left[\begin{array}{c} n-j-1 \\ 1 \end{array}\right]_q \beta_{n-j-2} & \left[\begin{array}{c} n-j \\ 1 \end{array}\right]_q \beta_{n-j-1} \\ & \beta_0 & & & \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & \cdots & \beta_0 & \left[\begin{array}{c} n \\ n-1 \end{array}\right]_q \beta_1 \end{array} \right|.$$

We continue this method for each row. As the number of coefficients in

$$\frac{1}{\begin{bmatrix} j+1 \\ j \end{bmatrix}_q} \times \frac{1}{\begin{bmatrix} j+2 \\ j \end{bmatrix}_q} \times ... \times \frac{1}{\begin{bmatrix} n-1 \\ j \end{bmatrix}_q} \times \frac{1}{\begin{bmatrix} n \\ j \end{bmatrix}_q},$$

is $n-j$, so it is equal to the number of rows. Moreover, in each step one of the coefficients above will be cancelled by the corresponding inverse which will be multiplied later by each row. Therefore, we are sure that at the end we obtain

$$= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-j-1} & \beta_{n-j} \\ \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \cdots & \begin{bmatrix} n-j-1 \\ 1 \end{bmatrix}_q \beta_{n-j-2} & \begin{bmatrix} n-j \\ 1 \end{bmatrix}_q \beta_{n-j-1} \\ & \beta_0 & & & \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & \cdots & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1 \end{vmatrix}$$

$$= \alpha_{n-j},$$

whence the result. $\square$

Corollary 5.4. *The following identity holds true for the q-polynomials $P_{n,q}(x)$*

$$P_{n,q}(x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q P_{n-j,q}(0)x^j, \quad n = 0, 1, 2, ... \tag{5.2.9}$$

Proof. According to the definition(5.2.1), for $j = 0, 1, ..., n$, $P_{j,q}(x) = \alpha_j$, since $\square$

$$P_{j,q}(0) = \frac{(-1)^j}{(\beta_0)^{j+1}} \times$$

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{j-1} & \beta_j \\ 0 & \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \cdots & \cdots & \begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q \beta_{j-2} & \begin{bmatrix} j \\ 1 \end{bmatrix}_q \beta_{j-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \begin{bmatrix} j-1 \\ 2 \end{bmatrix}_q \beta_{j-3} & \begin{bmatrix} j \\ 2 \end{bmatrix}_q \beta_{j-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \begin{bmatrix} j \\ j-1 \end{bmatrix}_q \beta_1 \end{vmatrix}$$

$$= \frac{(-1)^j}{(\beta_0)^{j+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{j-1} & \beta_j \\ \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \cdots & \cdots & \begin{bmatrix} j-1 \\ 1 \end{bmatrix}_q \beta_{j-2} & \begin{bmatrix} j \\ 1 \end{bmatrix}_q \beta_{j-1} \\ 0 & \beta_0 & \cdots & \cdots & \begin{bmatrix} j-1 \\ 2 \end{bmatrix}_q \beta_{j-3} & \begin{bmatrix} j \\ 2 \end{bmatrix}_q \beta_{j-2} \\ & & \ddots & & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 0 & \beta_0 & \begin{bmatrix} j \\ j-1 \end{bmatrix}_q \beta_1 \end{vmatrix}$$

$$= \alpha_j.$$

Replacing $P_{n-j,q}(0)$, instead of α_{n-j} in relation (5.2.6), gives the expected result.

Corollary 5.5. *The following relations hold true for α_js in relation (5.2.6)*

$$\alpha_0 = \frac{1}{\beta_0}, \qquad (5.2.10)$$

$$\alpha_j = -\frac{1}{\beta_0} \sum_{i=0}^{j-1} \begin{bmatrix} j \\ i \end{bmatrix}_q \beta_{j-i}\, \alpha_i, \quad j = 1, 2, ..., n.$$

Proof. The proof is done by expanding α_j , defined in relation(5.2.7), along with the first row and also applying a similar technique to the proof of theorem 5.3. ☐

Theorem 5.6. *Suppose that $\{A_{n,q}(x)\}$ be the sequence of q-Appell polynomials with generating function $A_q(t)$, defined in the relations (5.1.3) and (5.1.4). If $B_{0,q}$, $B_{1,q}$, ..., $B_{n,q}$,* *with $B_{0,q} \neq 0$ are the coefficients of q-Taylor series expansion of the function $\frac{1}{A_q(t)}$ introduced in relation (2.2.15), then $for \quad n = 0, 1, 2, ...$ we have*

$$\begin{cases} A_{0,q}(x) = \frac{1}{B_{0,q}} \\ A_{n,q}(x) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times \\ \begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \cdots & \cdots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \end{vmatrix} \end{cases} \tag{5.2.11}$$

Proof. According to the relations (5.1.3) and (5.1.4), we have

$$A_q(t) = \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!} = A_{0,q} + A_{1,q}t + A_{2,q} \frac{t^2}{[2]_q!} + ... + A_{n,q} \frac{t^n}{[n]_q!} + ..., \tag{5.2.12}$$

and also

$$A_q(t)e_q(tx) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_q!} = A_{0,q}(x) + A_{1,q}(x)t + A_{2,q}(x) \frac{t^2}{[2]_q!} + ... + A_{n,q}(x) \frac{t^n}{[n]_q!} + ... \tag{5.2.13}$$

Let $B_q(t) = \frac{1}{A_q(t)}$. Thus, considering the hypothesis of the theorem and also noting the definition of q-Taylor series expansion of $B_q(t)$ at $a = 0$ given in relation (2.2.15) we have

$$B_q(t) = B_{0,q} + B_{1,q} \frac{t}{[1]_q!} + B_{2,q} \frac{t^2}{[2]_q!} + ... + B_{n,q}(x) \frac{t^n}{[n]_q!} + ... \tag{5.2.14}$$

By using Cauchy product rule for the series production $A_q(t)B_q(t)$, we obtain

$$\begin{aligned} 1 &= A_q(t)B_q(t) \\ &= \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,q} B_{n-k,q} \frac{t^n}{[n]_q!}. \end{aligned}$$

Consequently,

$$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q A_{k,q}B_{n-k,q}=\begin{cases} 1 & \text{for } n=0, \\ 0 & \text{for } n>0. \end{cases}$$

This means that

$$\begin{cases} B_{0,q}=\frac{1}{A_0} \\ B_{n,q}=-\frac{1}{A_0}(\sum_{k=1}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q A_{k,q}B_{n-k,q}), \quad n=1,2,3,.... \end{cases} \tag{5.2.15}$$

Now, multiply both sides of identity (5.2.13) by $B_q(t)=\frac{1}{A_q(t)}$, and then replace $e_q(tx)$ by its q-Taylor series expansion, i. e. $\sum_{k=0}^{\infty} x^n \frac{t^n}{[n]_q!}$. Therefore we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} x^n \frac{t^n}{[n]_q!} &= e_q(tx) \\ &= B_q(t)\sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

Using Cauchy product rule in the last part of relation above leads to

$$\sum_{k=0}^{\infty} x^n \frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q B_{n-k,q}A_{k,q}(x)\frac{t^n}{[n]_q!}. \tag{5.2.16}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides of equation(5.2.16), we have

$$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q B_{n-k,q}A_{k,q}(x)=x^n, \quad n=0,1,2,.... \tag{5.2.17}$$

Writing identity (5.2.17) for $n=0,1,2,...$ leads to obtain the following infinite system in the parameter $A_{n,q}(x)$

$$\begin{cases} B_{0,q}A_{0,q}(x)=1, \\ B_{1,q}A_{0,q}(x)+B_{0,q}A_{0,q}(x)=x, \\ B_{2,q}A_{0,q}(x)+\left[\begin{array}{c} 2 \\ 1 \end{array}\right]_q B_{1,q}A_{1,q}(x)+B_{0,q}A_{2,q}(x)=x^2, \\ \vdots \\ B_{n,q}A_{0,q}(x)+\left[\begin{array}{c} n \\ 1 \end{array}\right]_q B_{n-1,q}A_{1,q}(x)+\ldots+B_{0,q}A_{n,q}(x)=x^n, \\ \vdots \end{cases} \tag{5.2.18}$$

As it is clear the coefficient matrix of the infinite system (5.2.18) is lower triangular. So this property helps us to find $A_{n,q}(x)$ by applying Cramer's rule to only the first $n+1$ equations of this system. Hence we can obtain

$$A_{n,q}(x)=\frac{\begin{vmatrix} B_{0,q} & 0 & 0 & \cdots & 0 & 1 \\ B_{1,q} & B_{0,q} & 0 & \cdots & 0 & x \\ B_{2,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & B_{0,q} & \cdots & 0 & x^2 \\ \vdots & & & \ddots & & \vdots \\ B_{n-1,q} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \cdots & \cdots & B_{0,q} & x^{n-1} \\ B_{n,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} & \cdots & \cdots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} & x^n \end{vmatrix}}{\begin{vmatrix} B_{0,q} & 0 & 0 & \cdots & 0 & 0 \\ B_{1,q} & B_{0,q} & 0 & \cdots & 0 & 0 \\ B_{2,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & B_{0,q} & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ B_{n-1,q} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \cdots & \cdots & B_{0,q} & 0 \\ B_{n,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} & \cdots & \cdots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} & B_{0,q} \end{vmatrix}}$$

$$=\frac{1}{(B_{0,q})^{n+1}}\begin{vmatrix} B_{0,q} & 0 & 0 & \cdots & 0 & 1 \\ B_{1,q} & B_{0,q} & 0 & \cdots & 0 & x \\ B_{2,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & B_{0,q} & \cdots & 0 & x^2 \\ \vdots & & & \ddots & & \vdots \\ B_{n-1,q} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \cdots & \cdots & B_{0,q} & x^{n-1} \\ B_{n,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} & \cdots & \cdots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} & x^n \end{vmatrix}$$

Now, take the transpose of the last determinant and then interchange i^{th} row of the obtained determinant with $i+1^{th}$ row, $i=1,2,...,n$. This leads to obtain the desired result that is exactly relation(5.2.11).

Theorem 5.7. *The following facts are equivalent for the q-Appell polynomials:*

a) q-Appell polynomials can be expressed by considering the relations (5.1.1) and (5.1.2).

b) q-Appell polynomials can be expressed by considering the relations (5.1.3) and (5.1.4).

c) q-Appell polynomials can be expressed by considering the determinantal relation (5.2.11).

Proof. $(a \Rightarrow b)$ Suppose that relations (5.1.1) and (5.1.2) hold. Construct an infinite series $\sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!}$ form all constants $A_{n,q}$ used for defining $A_{n,q}(x)$ in relation (5.1.2). Now find the following Cauchy product

$$\begin{aligned}
&\sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!} e_q(tx) \\
&= \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \\
&\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k,q} x^k \frac{t^n}{[n]_q!}.
\end{aligned}$$

From relation (5.1.2) we know that

$$\sum_{k=0}^{n} A_{n-k,q} x^k = A_{n,q}(x),$$

So we find that

$$\sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!} e_q(tx) = A_{n,q}(x),$$

whence the result.

$(b \Rightarrow c)$ The proof follows directly from Theorem 5.6.

$(c \Rightarrow a)$ The proof follows from Theorems 5.2 and 5.6.

As the consequence of discussion above and particularly Theorem 5.7, we are allowed to

□

introduce the determinantal definition of q-Appell polynomials as follows

Definition 5.8. The family of q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$ are defined as

$$\begin{cases} A_{0,q}(x) = \frac{1}{B_{0,q}} \\ A_{n,q}(x) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times \\ \begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-1} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \end{vmatrix} \end{cases}, \tag{5.2.19}$$

where $B_{0,q}, B_{1,q}, B_{2,q}, \dots, B_{n,q} \in \mathbb{R},\ B_{0,q} \neq 0$ and $n = 1, 2, 3, \dots$.

5.3 Basic Properties of q-Appell polynomials from determinantal point of view

In this section by using Definition 5.8, we review the basic properties of q-Appell polynomials.

Theorem 5.9. *For q-Appell polynomials the following identities hold*

$$A_{n,q}(x) = \frac{1}{B_{0,q}}(x^n - \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-k,q} A_{k,q}(x)), \quad n = 1, 2, 3, \dots. \tag{5.3.1}$$

Proof. Start from expanding the determinant in the Definition 5.8 along with the $n+1^{th}$ row

$$A_{n,q}(x) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-1} \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & B_{0,q} & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{1,q} \end{vmatrix} + \frac{(-1)^{n+1}}{(B_{0,q})^{n+1}} B_{0,q} \times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-2} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-2,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q B_{n-4,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & & & B_{0,q} & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} \end{vmatrix}$$

$$= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_{0,q})^n} \times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-2} & x^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-2,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q B_{n-4,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & & & B_{0,q} & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} \end{vmatrix}$$

Now repeat the same method for the last determinant

$$= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_{0,q})^n} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} \times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^{n-3} & x^{n-2} \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & B_{n-3,q} & B_{n-2,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q B_{n-4,q} & \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q B_{n-3,q} \\ 0 & 0 & B_{0,q} & \dots & \begin{bmatrix} n-3 \\ 2 \end{bmatrix}_q B_{n-5,q} & \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q B_{n-4,q} \\ \vdots & & & & \vdots & \vdots \\ 0 & \dots & \dots & & B_{0,q} & \begin{bmatrix} n-2 \\ n-3 \end{bmatrix}_q B_{1,q} \end{vmatrix}$$

$$+\frac{(-1)^{n+2}}{(B_{0,q})^n} B_{0,q} \times$$

$$= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) + \frac{(-1)^{n-1}}{(B_{0,q})^n} \left(\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} \frac{(B_{0,q})^{n-1}}{(-1)^{n-2}} A_{n-2,q}(x) \right)$$

$$+\frac{(-1)^{n-2}}{(B_{0,q})^{n-1}} \times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-3} & x^{n} \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-3,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q B_{n-4,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-3 \\ 2 \end{bmatrix}_q B_{n-5,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & & 0 & B_{0,q} & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} \end{vmatrix}$$

$$= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} A_{n-2,q}(x) + \frac{(-1)^{n-2}}{(B_{0,q})^{n-1}} \times$$

$$\begin{vmatrix} 1 & x & x^2 & \dots & \dots & x^{n-3} & x^{n} \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-3,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q B_{n-4,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-3 \\ 2 \end{bmatrix}_q B_{n-5,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & & 0 & B_{0,q} & \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} \end{vmatrix}$$

Continue a similar method to arrive at

$$
\begin{aligned}
&= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} A_{n-2,q}(x) \\
&\quad - \ldots - \frac{1}{(B_{0,q})^2} \begin{vmatrix} 1 & x^n \\ B_{0,q} & B_{n,q} \end{vmatrix} \\
&= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} A_{n-2,q}(x) \\
&\quad - \ldots - \frac{1}{(B_{0,q})^2} (B_{n,q} - B_{0,q} x^n) \\
&= \frac{-1}{B_{0,q}} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q B_{2,q} A_{n-2,q}(x) - \ldots \\
&\quad - \frac{1}{B_{0,q}} B_{n,q} A_{0,q}(x) + \frac{1}{B_{0,q}} x^n \\
&= \frac{1}{B_{0,q}} (x^n - \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-k,q} A_{k,q}(x)).
\end{aligned}
$$

□

Corollary 5.10. *Powers of x can be expressed based on q-Appell polynomials as*

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-k,q} A_{k,q}(x), \quad n = 1, 2, 3, ... \tag{5.3.2}$$

Proof. The proof is the direct result of relation(5.3.1) in Theorem 5.9. □

Notation 5.11. Suppose $P_n(x)$ and $Q_n(x)$ are two polynomials of degree n. Let $P_n(x)$ be defined as in relation(5.2.1). Then for n=1,2,3,..., we have

$$(PQ)_n(x)(PQ)(x) := \frac{(-1)^n}{(\beta_0)^{n+1}} \times$$

$$
\begin{vmatrix}
Q_0(x) & Q_1(x) & Q_2(x) & \ldots & \ldots & Q_{n-1}(x) & Q_n(x) \\
\beta_0 & \beta_1 & \beta_2 & \ldots & \ldots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_1 & \ldots & \ldots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1} \\
0 & 0 & \beta_0 & \ldots & \ldots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2} \\
\vdots & & & \ddots & & \vdots & \vdots \\
\vdots & & & & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & \beta_0 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_1
\end{vmatrix}, \tag{5.3.3}
$$

Theorem 5.12. *Suppose that* $\{A_{n,q}(x)\}_{n=0}^{\infty}$ *and* $\{\tilde{A}_{n,q}(x)\}_{n=0}^{\infty}$ *are two families of* q*-Appell polynomials. Then*

a) For every α and $\beta \in \mathbb{R}$, $\{\alpha A_{n,q}(x) + \beta A\tilde{}_{n,q}(x)\}_{n}{}^{\infty}{}_{=0}$ is also a family of q-Appell polynomials.

b) $\{(A\tilde{A})_{n,q}(x)\}_{n=0}^{\infty}$ is also a family of q-Appell polynomials.

Proof. a) The proof is the direct consequence of linear properties of determinant.

b) According to the determinantal definition of q-Appell polynomials given in Theorem 5.6 relation(5.2.11) and also notation(5.3.3), we have

$$(A\tilde{A})_{n,q}(x) = A_{n,q}(\tilde{A}_{n,q}(x)) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times$$

$$\begin{vmatrix} \tilde{A}_{0,q}(x) & \tilde{A}_{1,q}(x) & \tilde{A}_{2,q}(x) & \cdots & \cdots & \tilde{A}_{n-1,q}(x) & \tilde{A}_{n,q}(x) \\ B_{0,q} & B_{1,q} & B_{2,q} & \cdots & \cdots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \cdots & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \end{vmatrix}.$$

Using formula(5.2.2) given in Lemma 5.1 we have

$$D_q((A\tilde{A})_{n,q}(x)) = \frac{(-1)^n}{(B_{0,q})^{n+1}}\times$$

$$\begin{vmatrix} D_q(\tilde{A}_{0,q}(x)) & D_q(\tilde{A}_{1,q}(x)) & D_q(\tilde{A}_{2,q}(x)) & ... & ... & D_q(\tilde{A}_{n-1,q}(x)) & D_q(\tilde{A}_{n,q}(x)) \\ B_{0,q} & B_{1,q} & B_{2,q} & ... & ... & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & ... & ... & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & ... & ... & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & ... & ... & ... & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \end{vmatrix}$$

Since $\{\tilde{A}_{n,q}(x)\}_{n=0}^{\infty}$ is a family of q-Appell polynomials, according to relation(5.1.1) we have

$$D_{q,x}(\tilde{A}_{n,q}(x)) = [n]_q \tilde{A}_{n-1,q}(x), \;\; n = 0, 1, 2, ...$$

Therefore we can continue as

$$D_q((A\tilde{A})_{n,q}(x)) = \frac{(-1)^n}{(B_{0,q})^{n+1}}\times$$

$$\begin{vmatrix} 0 & \tilde{A}_{0,q}(x) & [2]_q\tilde{A}_{1,q}(x) & ... & ... & [n-1]_q\tilde{A}_{n-2,q}(x) & [n]_q\tilde{A}_{n-1,q}(x) \\ B_{0,q} & B_{1,q} & B_{2,q} & ... & ... & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & ... & ... & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & ... & ... & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & ... & ... & ... & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}^q B_{1,q} \end{vmatrix}.$$

Now, expand the last determinant along with the first column as follows

$$= \frac{(-1)^n}{(B_{0,q})^{n+1}} \times -B_{0,q}\times$$

$$\begin{vmatrix} \tilde{A}_{0,q}(x) & [2]_q\tilde{A}_{1,q}(x) & \dots & \dots & [n-1]_q\tilde{A}_{n-2,q}(x) & [n]_q\tilde{A}_{n-1,q}(x) \\ B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \end{vmatrix}$$

$$= [n]_q(A\tilde{A})_{n-1,q}(x),$$

which means that $\{(A\tilde{A})_{n,q}(x)\}_{n=0}^{\infty}$ belongs to the family of q-Appell polynomials too.

□

Definition 5.13. 2D q-Appell polynomials $\{A_{n,q}(x,y)\}_{n=0}^{\infty}$, which was defined sooner in Section (3.2) by means of the relation (3.2.1), can be represented, also, as below

$$A_q(x,y,t) := A_q(t)e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} A_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{5.3.4}$$

or equivalently

$$\begin{cases} A_{0,q}(x,y) = \frac{1}{B_{0,q}} \\ A_{n,q}(x,y) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times \\ \begin{vmatrix} 1 & x+y & (x+y)_q^2 & \dots & \dots & (x+y)_q^{n-1} & (x+y)_q^n \\ B_{0,q} & B_{1,q} & B_{2,q} & \dots & \dots & B_{n-1,q} & B_{n,q} \\ 0 & B_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q B_{1,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q B_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q B_{n-1,q} \\ 0 & 0 & B_{0,q} & \dots & \dots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q B_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q B_{n-2,q} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & B_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q B_{1,q} \end{vmatrix} \end{cases}. \tag{5.3.5}$$

Remark 5.14. From the Definition 5.13, it is clear that

$$A_{n,q}(x,0) = A_{n,q}(x). \tag{5.3.6}$$

Theorem 5.15. *The following fact holds for 2D q-Appell polynomials* $\{A_{n,q}(x,y)\}_{n=0}^{\infty}$

$$A_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} A_{k,q}(x) y^{n-k}. \tag{5.3.7}$$

Proof. Proof is simple and based on properties of determinant. □

Corollary 5.16. *The following difference identity holds for q-Appell polynomials* $\{A_{n,q}(x)\}_{n=0}^{\infty}$

$$A_{n,q}(x,1) - A_{n,q}(x) = \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} A_{k,q}(x), \quad n = 0, 1, 2, ... \tag{5.3.8}$$

Proof. Using relations (5.3.6) and also (5.3.7) for $y = 1$ and $y = 0$ and replacing the results in the left side of relation(5.3.8) leads to reach to the right side of this relation. □

Theorem 5.17. *For every $t \in \mathbb{R}$, the following facts are equivalent for q-Appell polynomials* $\{A_{n,q}(x)\}_{n=0}^{\infty}$

a) $A_{n,q}(x,-y) = (-1)^n A_{n,q}(0,y)$,

b) $A_{n,q}(x) = (-1)^n A_{n,q}(0)$.

Proof. $(a \Rightarrow b)$ The proof is done using part (a) for $x = 0$.

$(b \Rightarrow a)$ We apply the relation(5.3.7) for the left hand side of part (a) as follows

$$\begin{aligned} A_{n,q}(x,-y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} A_{k,q}(x,0)(-y)^{n-k} \\ &= (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}} A_{k,q}(x,0)(-1)^k y^{n-k} \\ &= (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} A_{n-k,q}(x,0)(-1)^{n-k} y^k. \end{aligned}$$

Using part (b), we have

$$A_{n,q}(x,-y) = (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} A_{n-k,q}(0) x^k.$$

Now, using Definition 5.13 leads to obtain

$$A_{n,q}(x,-y) = (-1)^n A_{n-k,q}(0,y),$$

whence the result. $\square$

Lemma 5.18. *In relation(5.2.15) for the coefficients $A_{n,q}$ and $B_{n,q}$ we have*

$$A_{2n+1,q} = 0 \Leftrightarrow B_{2n+1,q} = 0, \quad n = 0,1,2,\dots. \tag{5.3.9}$$

Proof. ($\Rightarrow$)We have already known the following fact from relation(5.2.15) for $n = 0,1,2,\dots$ $\square$

$$\begin{cases} B_{1,q} = -\frac{1}{A_0} A_{1,q} B_{0,q}, \\ B_{2n+1,q} = -\frac{1}{A_0} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q A_{1,q} B_{2n,q} \\ + - \frac{1}{A_0} \left(\sum_{k=1}^{n} \left(\begin{bmatrix} 2n+1 \\ 2k \end{bmatrix}_q A_{2k,q} B_{2n-k+1,q} + \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q A_{2k+1,q} B_{2n-k+1,q} \right) \right). \end{cases}$$

Since $A_{2n+1,q} = 0$ for $n = 0,1,2,\dots$, then

$$\begin{cases} B_{1,q} = 0 \\ B_{2n+1,q} = -\frac{1}{A_0} \sum_{k=1}^{n} \begin{bmatrix} 2n+1 \\ 2k \\ q \end{bmatrix} A_{2k,q} B_{2n-k+1,q}, \quad n = 1,2,3,\dots. \end{cases}$$

Consequently, we should have $B_{2n+1,q} = 0$, for $n = 0,1,2,\dots$.

($\Leftarrow$) In a similar way to the above we can prove it.

Theorem 5.19. *The following facts are equivalent for q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$*

a) $A_{n,q}(-x) = (-1)^n A_{n,q}(x)$,

b) $B_{2n+1,q} = 0$, *for* $n = 0,1,2,\dots$.

Proof. According to Theorem 5.17, we know that

$$A_{n,q}(-x) = (-1)^n A_{n,q}(x) \Leftrightarrow A_{n,q}(t) = (-1)^n A_{n,q}(0)$$

So using Lemma5.18, we have

$$\Leftrightarrow A_{2n+1,q}(0) = (-1)^n A_{2n+1,q}(0) \Leftrightarrow A_{2n+1,q} = 0 \Leftrightarrow B_{2n+1,q} = 0.$$

$\square$

Theorem 5.20. *For every $n \geq 1$, q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$ satisfy the following identities*

$$\int_0^x A_{n,q}(t)d_qt = \frac{1}{[n+1]_q}\left(A_{n+1,q}(x) - A_{n,q}(0)\right) \tag{5.3.10}$$

$$\int_0^x A_{n,q}(t)d_qt = \frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} A_{n-k,q}(0) \tag{5.3.11}$$

Proof. Relation(5.3.10) is the direct result of property(5.1.1) for q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$. To prove equality(5.3.11), we start from relation(5.3.10) for $x = 1$ as follows

$$\int_0^1 A_{n,q}(t)d_qt = \frac{1}{[n+1]_q}\left(A_{n+1,q}(1) - A_{n,q}(0)\right).$$

Now, find $A_{n+1,q}(1)$ using relation(5.3.7) by assuming $x = 0$ and $y = 1$

$$A_{n+1,q}(1) = \sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} A_{n-k,q}(0).$$

Therefore, we obtain

$$\int_0^1 A_{n,q}(t)d_qt = \frac{1}{[n+1]_q}\sum_{k=0}^{n}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} A_{n-k,q}(0).$$

$\square$

5.4 Determinantal representation for Some q-Appell polynomials

5.4.1 q-Bernoulli polynomials

The q-Bernoulli polynomials $B_{n,q}(x)$ are defined by means of the generating function, [37]

$$B_q(x,t) := \frac{t}{e_q(t)-1} e_q(tx) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{5.4.1}$$

From this definition and also using Lemma (10) of [57], it is easy to achieve that

$$x^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{1}{[k+1]_q} B_{n-k,q}(x). \tag{5.4.2}$$

Using identity(5.4.2), we obtain

$B_{0,q}(x) = 1,$

$B_{1,q}(x) = x - \frac{1}{[2]_q},$

$B_{2,q}(x) = x^2 - x + \frac{q^2}{[2]_q[3]_q},$

$B_{3,q}(x) = x^3 - \frac{[3]_q}{[2]_q} x^2 - \frac{q^2[3]_q}{[4]_q} x + \frac{[3]_q - [2]_q}{[4]_q[2]_q} - \frac{q^2}{[2]_q^2},$

$\vdots$

Based on the discussion above and noting to the relation (5.2.11), given in Theorem 5.6, we obtain

$$\begin{aligned} B_{0,q} &= 1, \\ B_{n,q} &= \frac{1}{[n+1]_q}, \quad n = 1,2,3,.... \end{aligned} \tag{5.4.3}$$

5.4.2 Generalized q-Bernoulli polynomials

Based on the definition of $2D$ q-Bernoulli polynomials $\mathfrak{B}{[}_{n,q}{}^{m-1,\alpha]}$ three, (x, y) in chapter we have

$$\begin{aligned} \mathfrak{B}_{0,q}^{[m-1,1]}(x,0) &= [m]_q!, \\ \mathfrak{B}_{1,q}^{[m-1,1]}(x,0) &= [m]_q! \left(x - \frac{1}{[m+1]_q} \right), \\ \mathfrak{B}_{2,q}^{[m-1,1]}(x,0) &= x^2 - \frac{[2]_q[m]_q!}{[m+1]_q} x + \frac{[2]_q q^{m+1}[m]_q!}{[m+1]_q^2[m+2]_q}. \end{aligned}$$

Therefore, for the generalized q-Bernoulli numbers $B_{n,q}^{[m-1,1]}$ corresponding to $\mathfrak{B}_{n,q}^{[m-1,1]}(x,0)$ in the relation(5.3.5), given in Definition 5.13, we may write

$$B_{0,q} = \frac{1}{[m]_q!},$$
$$B_{n,q} = \frac{[n]_q}{[n+m]_q!}, \quad n = 1,2,3,...,m \in \mathbb{N}. \tag{5.4.4}$$

5.4.3 q-Euler polynomials

The q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function, [37], [86], [33]

$$E_q(x,t) := \frac{2}{e_q(t)+1} e_q(tx) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{5.4.5}$$

Based on this definition it is easy to see that

$$\begin{aligned}
&E_{0,q}(x) = 1,\\
&E_{1,q}(x) = x - \tfrac{1}{2},\\
&E_{2,q}(x) = x^2 - \tfrac{[2]_q}{2} x + \tfrac{[2]_q}{4} - \tfrac{1}{2},\\
&\vdots
\end{aligned}$$

Based on the discussion above and noting to the relation (5.2.11), given in Theorem 5.6, for the coefficients $B_{n,q}$ we obtain

$$B_{0,q} = 1 \tag{5.4.6}$$
$$B_{n,q} = \frac{1}{2}, \quad n = 1,2,\ldots.$$

5.4.4 q-Hermite polynomials

q-Hermite polynomials $H_{n,q}(x)$ are defined by means of the generating function, [88]

$$F_q(x,t) := F_q(t)\, e_q(tx) = \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!},$$
$$F_q(t) := \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)} \frac{t^{2n}}{[2n]_q!!}, \quad [2n]_q!! = [2n]_q [2n-2]_q \ldots [2]_q\,.$$

According to Theorem 10 in [88], we have

$$H_{0,q}(x) = 1,$$

$$H_{1,q}(x) = x,$$

$$H_{2,q}(x) = x^2 - 1,$$

$$H_{3,q}(x) = x^3 - [3]_q x,$$

$$H_{4,q}(x) = x^4 - \left(1 + q^2\right)[3]_q x^2 + [3]_q q^2.$$

Based on the discussion above and notting to the ralation(5.2.11), given in Theorem 5.6, for the coefficients $B_{n,q}$ we obtain

$$B_{2n,q} = 1, \tag{5.4.7}$$

$$B_{2n+1,q} = 0, \quad n = 1, 2, 3, ...$$

Chapter 6

THE q-UMBRAL PERSPECTIVE OF THE CLASS OF q-APPELL POLYNOMIALS

6.1 An Introduction to q-Umbral Calculus

1978, Roman and Rota viewed the classical umbral calculus from a new perspective and proposed an interesting approach based on a simple but innovative indication for the effect of linear functionals on polynomials, which Roman later called it the modern classical umbral calculus, [111], [112]. Using this new umbral calculus, they defined the sequence of Sheffer polynomials whose their characteristics proved that this new proposed family of polynomials is equivalent to the family of polynomials of type zero which was previously introduced by Sheffer, [35]. Roman, also, proposed a similar umbral approach under the area of nonclassical umbral calculus which is called q-umbral calculus, [112], [113], [114]. Inspired by his work, in the following, we recast the obtained results of umbral calculus for q-Appell polynomials.

Let $\mathbb{C}$ be the field of complex numbers and $\mathcal{F}$ set of all formal power q-series in the variable t over $\mathbb{C}$. In other words, $f(t)$ is an element of $\mathcal{F}$ if

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k, \tag{6.1.1}$$

where a_k is in $\mathbb{C}$.

Let $\mathcal{P}$ be the algebra of all polynomials in variable x over $\mathbb{C}$. Let $\mathcal{P}^*$ be the vector space of all linear functionals on $\mathcal{P}$. The action of a linear functional L on an arbitrary polynomial p(x) is denoted by $\langle L|p(x)\rangle$. We remind that the vector space addition and scalar multiplication operations on $\mathcal{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, and $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, for any constant $c \in \mathbb{C}$.The formal power q-series in (6.1.1) defines the following functional on $\mathcal{P}$

$$\langle f(x)|x^n\rangle = a_n, \quad \text{for all } n \geq 0. \tag{6.1.2}$$

Particularly, according to (6.1.1) and (6.1.2) we have

$$\langle t^k|x^n\rangle = [n]_q!\delta_{n,k} \quad n,k \geq 0, \tag{6.1.3}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Assume that $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{[k]_q!} t^k$. Since $\langle f_L(t)|x^n\rangle = \langle L|x^k\rangle$, so $f_L(t) = L$. Hence, it is clear that the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Therefore, $\mathcal{F}$ not only can be considered as the algebra of all formal power q-series in variable t, but also it is the vector space of all linear functionals on $\mathcal{P}$. This follows the fact that each member of $\mathcal{F}$ can be assumed as both a formal power $q$series and a linear functional. $\mathcal{F}$ is called the q-umbral algebra and studying its properties is called q-umbral calculus.

Remark 6.1. For the q-exponential function $e_q(t)$, it can be easily observed that $\langle e_q(yt)|x^n\rangle = y^n$ and consequently

$$\langle e_q(yt)|p(x)\rangle = p(y), \tag{6.1.4}$$

and

$$\langle e_q(yt) \pm 1|p(x)\rangle = p(y) \pm p(0). \tag{6.1.5}$$

Remark 6.2. For $f(t)$ in $\mathcal{F}$ we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{[k]_q!} t^k, \tag{6.1.6}$$

and for all polynomials $p(x)$ in $\mathcal{P}$ we have

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k | p(x) \rangle}{[k]_q!} x^k. \tag{6.1.7}$$

Proposition 6.3. *For $f(t)$ and $g(t) \in \mathcal{F}$ we have*

$$\langle f(t)g(t) | p(x) \rangle = \langle f(t) | g(t)p(x) \rangle. \tag{6.1.8}$$

Proposition 6.4. *For $f(t)$ and $g(t) \in \mathcal{F}$ we have*

$$\langle f(t)g(t) | x^n \rangle = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle f(t) | x^k \rangle \langle g(t) | x^{n-k} \rangle. \tag{6.1.9}$$

Proposition 6.5. *For $f_1(t), f_2(t), \dots, f_n(t) \in \mathcal{F}$ we have*

$$\begin{aligned} &\langle f(t)_1 f_2(t) \dots f_k(t) | x^n \rangle \\ &= \sum_{i_1+i_2+\dots+i_k=n} \begin{bmatrix} n \\ i_1, i_2, \dots, i_k \end{bmatrix}_q \langle f_1(t) | x_1^i \rangle \langle f_2(t) | x_2^i \rangle \dots \langle f_k(t) | x_k^i \rangle, \end{aligned} \tag{6.1.10}$$

where $\begin{bmatrix} n \\ i_1, i_2, \dots, i_k \end{bmatrix}_q = \frac{[n]_q!}{[i_1]_q![i_2]_q!\dots[i_k]_q!}$.

We use the notation t^k for the k-th q-derivative operator, D_q^k, on $\mathcal{P}$ as follows

$$t^k x^n = \begin{cases} \frac{[n]_q!}{[k]_q!} x^{n-k}, & k \le n, \\ 0, & k > n. \end{cases} \tag{6.1.11}$$

Consequently, using the notation above, each arbitrary function in the form of (6.1.1) can be considered as a linear operator on $\mathcal{P}$ defined by

$$f(t)x^n = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q a_k x^{n-k}. \tag{6.1.12}$$

Now, consider an arbitrary polynomials $p(x) \in \mathcal{P}$. Then, according to the relation (6.1.7) for its k-th q-derivative we have

$$D_q^k p(x) = p^{(k)}(x) = \sum_{j=k}^{\infty} \frac{\langle t^j | p(x) \rangle}{[j]_q!} [j]_q [j-1]_q \dots [j-k+1]_q x^{j-k}. \tag{6.1.13}$$

As the result of the fact above we obtain

$$t^k p(x) = D_q^k p(x) = p^{(k)}(x), \tag{6.1.14}$$

and, also,

$$p^{(k)}(0) = \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle. \tag{6.1.15}$$

The immediate conclusion of the relations (6.1.1), (6.1.2) and (6.1.12) is that each member of $\mathcal{F}$ plays three roles in the q-umbral calculus; a formal power q-series, a linear functional and a linear operator.

The order of a non-zero power q-series $f(t)$ in (6.1.1) is denoted by $O(f(t))$ and is defined as the smallest integer k for which the coefficient of t^k is non-zero, that is $a_k \neq 0$. A q-series $f(t)$ with $O(f(t)) = 0$ is called invertible and in case that $O(f(t)) = 1$ it is called a delta q-series.

Theorem 6.6. *Let $f(t)$ be a delta q-series and $g(t)$ be an invertible series. Then there exists a unique sequence $S_{n,q}(x)$ of q-polynomials satisfying the following conditions*

$$\langle g(t) f(t)^k | S_{n,q}(x) \rangle = [n]_q! \delta_{n,k}, \textit{for all } n, k \geq 0.$$

Definition 6.7. In Theorem (6.6), $\{S_{n,q}(x)\}_{n=0}^{\infty}$ is called the q-Sheffer sequence for the pair $(g(t), f(t))$. Moreover, the q-Sheffer sequences for $(g(t), t)$ is the q-Appell sequence for $g(t)$.

Theorem 6.8. *Let $A_{n,q}(x)$ be q-Appell for $g(t)$. Then*

a) (The Expansion Theorem) for any $h(t)$ in $\mathcal{F}$

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | A_{k,q}(x) \rangle}{[k]_q!} g(t) t^k,$$

b) (The Polynomial Expansion Theorem) for any $p(x)$ in $\mathcal{P}$ we have

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) t^k | p(x) \rangle}{[k]_q!} A_{k,q}(x).$$

Theorem 6.9. *The following facts are equivalent*

a) $A_{n,q}(x)$ *is* q*-Appell for* $g(t)$.

b) $tA_{n,q}(x) = [n]_q A_{n-1,q}(x)$, *where* $tA_{n,q}(x) = D_q(A_{n,q}(x))$.

c) For all $y \in \mathbb{C}$ $\quad \frac{1}{g(t)} e_q(tx) = \sum_{k=0}^{\infty} \frac{A_{k,q}(x)}{[k]_q!} t^k$.

d) $A_{n,q}(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle g^{-1}(t) | x^{n-k} \rangle x^k$.

e) $A_{n,q}(x) = g^{-1}(t) x^n$.

Remark 6.10. Based on different selections for $g(t)$ in part (c) of Theorem (6.9), we obtain various families of q-Appell polynomials. For instance, it is clear from Definitions (3.1), (3.2) and (3.3) that taking $g(t)$ as $\frac{e_q(t)-1}{t}$, $\frac{e_q(t)+1}{2}$ or $\frac{e_q(t)+1}{2t}$, leads to construct the families of q-Bernoulli, q-Euler or q-Genocchi polynomials, respectively.

Theorem 6.11. *(The Recurrence Formula for* q*-Appell Sequences) Suppose that* $A_{n,q}(x)$ *is* q*-Appell for* $g(t)$*. Then we have*

$$A_{n+1,q}(qx) = \Big[qx - q^n \frac{D_{q,t} g(t)}{g(qt)} \Big] A_{n,q}(x).$$

Proof. We prove this theorem in the light of the technique which is applied in the proof of Theorem 2 in [97]. Since $A_{n,q}(x)$ is q-Appell for $g(t)$ we can write

$$\frac{1}{g(t)} e_q(tqx) = \sum_{n=0}^{\infty} A_{n,q}(qx) \frac{t^n}{[n]_q!}. \tag{6.1.16}$$

Take $\frac{1}{g(t)} = A_q(t)$. According to (3.2.2), $A_q(t)$ is analytic. So, differentiating equation (6.1.16) and multiplying both sides of the obtained equality by t, we get

$$\sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = A_q(qt)e_q(tqx)\Big[t\frac{D_qA_q(t)}{A_q(qt)} + tqx\Big], \tag{6.1.17}$$

so it follows that

$$\sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n A_{n,q}(x)\frac{t^n}{[n]_q!}\Big[t\frac{D_qA_q(t)}{A_q(qt)} + tqx\Big]. \tag{6.1.18}$$

This means that

$$\sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty}\left[q^{n-1}A_{n-1,q}(x)\frac{D_qA_q(t)}{A_q(qt)} + qxA_{n-1,q}(x)\right]\frac{t^n}{[n]_q!}, \tag{6.1.19}$$

which is equivalent to write

$$\sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty}\left[q^{n-1}\frac{D_qA_q(t)}{A_q(qt)} + qx\right]A_{n-1,q}(x)\frac{t^n}{[n]_q!}. \tag{6.1.20}$$

Comparing both sides of identity(6.1.20), we have

$$A_{n,q}(qx) = \left[q^{n-1}\frac{D_qA_q(t)}{A_q(qt)} + qx\right]A_{n-1,q}(x), \tag{6.1.21}$$

whence the result. □

6.2 A q-Umbral Study on q-Genocchi numbers and polynomials, an example of q-Appell sequences

Over the past decades, many results have been derived using Umbral as well as q-Umbral methods for different members of the family of Appell and q-Appell polynomials. In this section, we look at the characteristics and properties of q-Genocchi numbers and polynomials, as an example of the family of q-Appell polynomials, from q-umbral point of view. Indeed, it is possible to derive similar results to the obtained results here for the q-Bernoulli and q-Euler polynomials. The interested readers may see, for instance [100]-[110].

6.2.1 Various results regarding q-Genocchi polynomials

According to the relation (b) of Theorem (6.9) for the sequence of q-Genocchi polynomials, $\{G_{n,q}(x)\}_{n=0}^{\infty}$, this family is q-Appell for $g(t) = \frac{e_q(t)+1}{2t}$. Therefore, relation (6.6) for the sequence of q-Genocchi polynomials, $\{G_{n,q}(x)\}$, can be expressed as follows

$$\left\langle \frac{e_q(t)+1}{2t} t^k | G_{n,q}(x) \right\rangle = [n]_q! \delta_{n,k}, \quad n, k \geq 0. \tag{6.2.1}$$

Remark 6.12. As direct corollaries of Theorems (6.9) and (6.11) we have

a) $tG_{n,q}(x) = D_q G_{n,q}(x) = [n]_q G_{n-1,q}(x)$,

b) $G_{n,q}(x) = \sum\limits_{k=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left\langle \frac{2t}{e_q(t)+1} \middle| x^{n-k} \right\rangle x^k$,

c) $G_{n,q}(x) = \frac{2t}{e_q(t)+1} x^n$,

d) $G_{n+1,q}(qx) = \left[qx - q^{n-1} \left(\frac{e_q(t)(t-1)+1}{2t^2} \right) \right] G_{n,q}(x)$.

Proposition 6.13. *For $n \in \mathbb{N}$ we have*

$$G_{0,q} = 1, \quad \sum_{k=1}^{n} \left[\begin{array}{c} n+1 \\ k+1 \end{array} \right]_q G_{n-k,q} = -[n+1]_q (1 + G_{n,q}).$$

Proof. According to the relations (2.11), (6.1.3) and (6.2.1) we can write

$$\left\langle \frac{e_q(t)+1}{2t} \middle| x^n \right\rangle = \frac{1}{2[n+1]_q} \left\langle \frac{e_q(t)+1}{t} \middle| t x^{n+1} \right\rangle = \frac{1}{2[n+1]_q}$$
$$= \frac{1}{2} \int_0^1 x^n d_q x.$$

Therefore, for an arbitrary polynomial $p(x) \in \mathcal{P}$ we can conclude

$$\left\langle \frac{e_q(t)+1}{2t} \middle| p(x) \right\rangle = \frac{1}{2} \left(\int_0^1 p(x) d_q x + p(0) \right). \tag{6.2.2}$$

Now, from one hand if we take $p(x) = G_{n,q}(x)$, then we have

$$\frac{1}{2}\Big(\int_0^1 G_{n,q}(x)d_qx + G_{n,q}(0)\Big) = \Big\langle \frac{e_q(t)+1}{2t}\Big| G_{n,q}(x)\Big\rangle$$
$$= \Big\langle 1\Big|\frac{e_q(t)+1}{2t}G_{n,q}(x)\Big\rangle = \Big\langle t^0\big|x^n\Big\rangle = [n]_q!\delta_{n,0}. \tag{6.2.3}$$

From another hand, considering the fact that

$$G_{n,q}(x) = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q G_{n-k,q}x^k, \tag{6.2.4}$$

we can conclude that

$$\int_0^1 G_{n,q}(x)d_qx = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q G_{n-k,q}\int_0^1 x^k d_qx$$
$$= \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{G_{n-k,q}(x)}{[k+1]_q}. \tag{6.2.5}$$

Comparing identity (6.2.3) with (6.2.5), we obtain

$$\int_0^1 G_{n,q}(x)d_qx = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{G_{n-k,q}(x)}{[k+1]_q} = \begin{cases} 2 - G_{0,q}(0) & n = 0 \\ -G_{0,q}(0) & n \neq 0 \end{cases}, \tag{6.2.6}$$

whence the result. □

Remark 6.14. According to part (b) of Theorem (6.8), for an arbitrary polynomial $p(x) \in \mathcal{P}$ we can write

$$p(x) = \sum_{k=0}^{\infty}\langle \frac{e_q(t)+1}{2t}t^k|p(x)\rangle\frac{G_{k,q}(x)}{[k]_q!} = \frac{1}{2}\sum_{k=0}^{\infty}\langle\frac{e_q(t)+1}{t}|t^kp(x)\rangle\frac{G_{k,q}(x)}{[k]_q!}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{G_{k,q}(x)}{[k]_q!}\Big(\int_0^1 t^kp(x)d_qx + t^kp(0)\Big).$$

Remark 6.15. We know that

$$\langle e_q(t)t^k|(x-1)_q^n\rangle = [n]_q!\delta_{n,k}.$$

Therefore, according to part(b) of Theorem (6.8), for $G_{n,q}(x)$ as a polynomial chosen from $\mathcal{P}$ we can obtain

$$G_{n,q}(x) = \sum_{k=0}^{n}\langle e_q(t)|t^kG_{n,q}(x)\rangle\frac{(x-1)_q^n}{[k]_q!} = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q G_{n-k,q}(1)(x-1)_q^n.$$

Proposition 6.16. *For $n \in \mathbb{N}$ we have*

$$(x-1)_q^n = \frac{1}{2}\Bigg(\sum_{k=0}^{n}\sum_{l=0}^{n-k}\left[\begin{array}{c} n \\ k \end{array}\right]_q \left[\begin{array}{c} n-k \\ l \end{array}\right]_q \frac{1}{[m+1]_q} G_{k,q}(x)(-1)^{n-k-l} q^{\frac{l(l-1)}{2}} + \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q G_{k,q}(x)\Bigg).$$

Proof. From the binomial relation(2.2.10), we obtain

$$(x-1)_q^n = \sum_{l=0}^{n}(-1)^{n-l} q^{\frac{l(l-1)}{2}} x^l. \tag{6.2.7}$$

Now, taking k-th q-derivative from both sides of identity(6.2.7), we have

$$t^k(x-1)_q^n = \sum l = k^n \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[n-k]_q!}(x-1)_q^{n-k} \tag{6.2.8}$$

According to part(b) of Theorem (6.8), we can write

$$\begin{aligned}
(x-1)_q^n &= \sum_{k=0}^{n} \frac{1}{[k]!}\Big\langle \frac{e_q(t)+1}{2t} t^k | (x-1)_q^n \Big\rangle G_{n,q}(x) \\
&= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q G_{n,q}(x) \Big\langle \frac{e_q(t)+1}{2t} | (x-1)_q^{n-k} \Big\rangle \\
&= \sum_{k=0}^{n} G_{n,q}(x) \Big(\int_0^1 (x-1)_q^{n-k} d_q x + 1 \Big).
\end{aligned}$$

Now, using relation (6.2.8) for the integral in the last relation above, will lead to obtain the desired result. □

Theorem 6.17. *Let $\mathcal{P}_n = \{p(x) \in \mathcal{P} | deg(p(x)) \leq n\}$. Then for an arbitrary $p(x) \in \mathcal{P}_n$ and a constant $c_{n,q}$, we may assume that $p(x) = \sum_{i=0}^{n} c_{i,q} G_{i,q}(x)$. Then for any constant k, the coefficient $c_{k,q}$ is equal to $\frac{1}{[k]_q!}\big\langle \frac{e_q(t)+1}{2t} | p^{(k)}(x) \big\rangle$, and it can be obtained from the following relation*

$$c_{k,q} = \frac{1}{2[k]_q!}\Big(\int_0^1 p^{(k)}(x) d_q x + p^{(k)}(0) \Big),$$

where $p^{(k)}(x) = D_q^k p(x)$.

Proof. For any polynomial $p(x) = \sum_{i=0}^{n} c_{i,q} G_{i,q}(x)$ in $\mathcal{P}_n$, we may write

$$\Big\langle \frac{e_q(t)+1}{2t} t^k | p(x) \Big\rangle = \sum_{i=0}^{n} c_{i,q} \Big\langle \frac{e_q(t)+1}{2t} t^k | G_{i,q}(x) \Big\rangle. \tag{6.2.9}$$

So, according to the relation (6.2.1), we obtain

$$= \sum_{i=0}^{n} c_{i,q}[i]_q! \delta_{i,k} = [k]_q! c_{k,q}, \tag{6.2.10}$$

which means that

$$c_{k,q} = \frac{1}{[k]_q!} \langle \frac{e_q(t)+1}{2t} t^k | p(x) \rangle. \tag{6.2.11}$$

According to the relation (6.1.14), this is equivalent to write

$$c_{k,q} = \frac{1}{[k]_q!} \langle \frac{e_q(t)+1}{2t} | t^k p(x) \rangle = \frac{1}{[k]_q!} \langle \frac{e_q(t)+1}{2t} | p^{(k)}(x) \rangle. \tag{6.2.12}$$

finally, using the relation (6.2.2), we obtain

$$c_{k,q} = \frac{1}{2[k]_q!} \Big(\int_0^1 p^{(k)}(x) d_q x + p^{(k)}(0) \Big). \tag{6.2.13}$$

□

6.2.2 Some results regarding q-Genocchi polynomials of higher order

Let $q \in \mathbb{C}$, $m \in \mathbb{N}$And $0 < | q| < 1$. The q-Genocchi Aolynomials $G^{[m]}_{n,q}(x)$ in x, of order m, in A Auitable neighborhood of $t = 0$, Are Aefined Ay means of the Aollowing Aenerating function, [97]

$$\Big(\frac{2t}{e_q(t)+1} \Big)^m e_q(tx) = \sum_{n=0}^{\infty} G^{[m]}_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{6.2.14}$$

In case that $x = 0$, $G^{[m]}_{n,q}(0) = G^{[m]}_{n,q}$ is called the n-th q-Genocchi number of order m.

From the above definition, it is clear that the class of q-Genocchi polynomials, $\{G^{[m]}_{n,q}(x)\}_{n=0}^{\infty}$, of order m is q-Appell for $g(t) = \Big(\frac{e_q(t)+1}{2t} \Big)^m$. Thus, according to the relation (6.6), for the sequence of q-Genocchi polynomials, $G^{[m]}_{n,q}(x)$, of order m, we can write

$$\Big\langle \Big(\frac{e_q(t)+1}{2t} \Big)^m t^k | G^{[m]}_{n,q}(x) \Big\rangle = [n]_q! \delta_{n,k}, \quad n, k \geq 0. \tag{6.2.15}$$

Lemma 6.18. *For any $n \in \mathbb{N}_0$, the following identity holds for the n-th q-Genocchi number of order m*

$$G^{[m]}_{n,q} = \sum_{i_1+i_2+\ldots+i_m=n} \left[\begin{array}{c} n \\ i_1, i_2, \ldots, i_m \end{array} \right]_q G_{i_1,q} G_{i_2,q} \ldots G_{i_m,q}$$

Proof. From one hand, according to the relation (6.2.14), it is obvious that

$$\left\langle \left(\frac{2t}{e_q(t)+1} \right)^m t^k | x^n \right\rangle = \sum_{k=0}^{\infty} \frac{G^{[m]}_{n,q}}{[k]_q!} \langle t^k | x^n \rangle = G^{[m]}_{n,q}. \tag{6.2.16}$$

From another hand, according to the Proposition (6.5), we have

$$G^{[m]}_{n,q} = \sum_{i_1+i_2+\ldots+i_m=n} \left[\begin{array}{c} n \\ i_1, i_2, \ldots, i_m \end{array} \right]_q \times \\ \langle \frac{2t}{e_q(t)+1} | x^{i_1} \rangle \langle \frac{2t}{e_q(t)+1} | x^{i_2} \rangle \ldots \langle \frac{2t}{e_q(t)+1} | x^{i_m} \rangle. \tag{6.2.17}$$

Based on the definition of q-Genocchi polynomials and also noting relation (6.1.3) for each $\langle \frac{2t}{e_q(t)+1} | x^{i_l} \rangle, \quad l \in \{1, 2, \ldots, m\}$ we can write

$$\langle \frac{2t}{e_q(t)+1} | x^{i_l} \rangle = \sum_{k=0}^{\infty} \frac{G_{i_l,q}}{[k]!} \langle t^k | x^{i_l} \rangle = G_{i_l,q}, \tag{6.2.18}$$

whence the result. □

Theorem 6.19. *For any $n \in \mathbb{N}_0$, the following identity holds for the n-th q-Genocchi polynomial of order m*

$$G^{[m]}_{n,q}(x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \langle \frac{e_q(t)+1}{2t} | G^{[m]}_{n-k,q}(x) \rangle G_{k,q}(x) \\ = \frac{1}{2^{m-1}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q G^{[m-1]}_{n-k,q} G_{k,q}(x).$$

Proof. According to the relation (6.2.14), it is clear that

$$G^{[m]}_{n,q}(x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q G^{[m]}_{n-k,q} x^k. \tag{6.2.19}$$

Therefore, we may assume that $G^{[m]}_{n,q}(x) = \sum_{k=0}^{n} c_{k,q} G_{k,q}(x)$ is a polynomial with degree n in $\mathcal{P}_n$. Since $G^{[m]}_{n,q}(x)$ is a q-Appell polynomial, according to part(b) of Theorem (6.9) for its k-th q-derivative we can write

$$D^k_q G^{[m]}_{n,q}(x) = [n]_q [n-1]_q \ldots [n-k+1]_q G^{[m]}_{n-k,q}(x) = \frac{[n]_q!}{[n-k]_q!} G^{[m]}_{n-k,q}(x). \tag{6.2.20}$$

Now, according to the relation (6.2.12), we may continue as

$$c_{k,q} = \frac{1}{[k]_q!}\Big\langle \frac{e_q(t)+1}{2t} \Big| t^k G^{[m]}_{n,q}(x)\Big\rangle = \frac{1}{[k]_q!}\Big\langle \frac{e_q(t)+1}{2t} \Big| D_q^k G^{[m]}_{n,q}(x)\Big\rangle$$
$$= \begin{bmatrix} n \\ k \end{bmatrix}_q \Big\langle \frac{e_q(t)+1}{2t} \Big| G^{[m]}_{n-k,q}(x)\Big\rangle. \tag{6.2.21}$$

According to part(e) of Theorem (6.9), it is clear that the q-Appell polynomial $G^{[m]}_{n-k,q}(x)$ is equal to $\left(\frac{e_q(t)+1}{2t}\right)^m x^{n-k}$. As the result of this fact and noting to the relation (6.1.15), we obtain from the last identity in (6.2.21)

$$c_{k,q} = \begin{bmatrix} n \\ k \end{bmatrix}_q \Big\langle t^0 \Big| \frac{2t}{e_q(t)+1}\Big(\frac{e_q(t)+1}{2t}\Big)^m x^{n-k}\Big\rangle = \frac{1}{2^{m-1}} \begin{bmatrix} n \\ k \end{bmatrix}_q G^{[m-1]}_{n-k,q}, \tag{6.2.22}$$

whence the result. □

Theorem 6.20. *For any arbitrary polynomial $p(x) \in \mathcal{P}_n$ the following identity holds*

$$p(x) = \sum_{k=0}^{n} \Big\langle \Big(\frac{e_q(t)+1}{2t}\Big)^m t^k \Big| p(x) \Big\rangle \frac{G^{[m]}_{k,q}(x)}{[k]_q!}.$$

Proof. Assume that $p(x) = \sum_{i=0}^{n} c_{i,q} G^{[m]}_{i,q}(x)$. Therefore, noting to the relation (6.2.15) for the q-Appell polynomial $G^{[m]}_{i,q}(x)$, we may conclude that

$$\Big\langle \Big(\frac{e_q(t)+1}{2t}\Big)^m t^k \Big| p(x) \Big\rangle = \sum_{i=0}^{n} c_{i,q} \Big\langle \Big(\frac{e_q(t)+1}{2t}\Big)^m t^k \Big| G^{[m]}_{i,q}(x) \Big\rangle \tag{6.2.23}$$
$$= \sum_{i=0}^{n} c_{i,q}[i]_q!\delta_{i,k} = c_{k,q}[k]_q!.$$

Thus,

$$c_{k,q} = \frac{1}{[k]_q!}\Big\langle \Big(\frac{e_q(t)+1}{2t}\Big)^m t^k \Big| p(x) \Big\rangle. \tag{6.2.24}$$

Substituting $c_{k,q}$ in the summation assumed in the beginning of the proof, leads to obtain the desired result. □

Theorem 6.21. *For any $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}$, the n-th q-Genocchi polynomial can be expressed based on the following relation*

$$G_{n,q}(x) = \sum_{k=0}^{m-1} \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \times \left\{ \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q \sum_{l=0}^{n+m-k} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q \times G_{n+m-k-l,q} \right\} G_{k,q}^{[m]}(x)$$

$$+ \sum_{k=m}^{n} \frac{\begin{bmatrix} n \\ k-m \end{bmatrix}_q}{2^m [k]_q! \begin{bmatrix} k \\ m \end{bmatrix}_q} \times \left\{ \sum_{i=0}^{m} \sum_{l=0}^{n-k+m} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q \times G_{n-k+m-l,q} \right\} G_{k,q}^{[m]}(x)$$

Proof. In Theorem (6.20), take $p(x)$ to be the n-th q-Genocchi polynomial $G_{n,q}(x)$, that is

$$G_{n,q}(x) = \sum_{k=0}^{n} c_{k,q} G_{k,q}^{[m]}(x), \tag{6.2.25}$$

where

$$c_{k,q} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)+1}{2t} \right)^m t^k \middle| G_{n,q}(x) \right\rangle. \tag{6.2.26}$$

Then, for $k < m$, we have

$$
\begin{aligned}
c_{k,q} &= \frac{1}{2^m[k]_q!}\Big\langle\frac{(e_q(t)+1)^m}{t^{m-k}}|G_{n,q}(x)\Big\rangle\\
&= \frac{1}{2^m[k]_q!}\times\frac{1}{[n+m-k]_q!\ldots[n+1]_q!}\Big\langle(e_q(t)+1)^m\Big(\frac{1}{t}\Big)^{m-k}|t^{m-k}G_{n+m-k,q}(x)\Big\rangle\\
&= \frac{[m]_q!}{2^m[k]_q![m-k]_q!}\times\frac{[m-k]_q!}{[n+m-k]_q!\ldots[n+1]_q!}\Big\langle(e_q(t)+1)^m|G_{n+m-k,q}(x)\Big\rangle\\
&= \frac{\begin{bmatrix} m\\ k\end{bmatrix}_q}{2^m}\times\frac{[m-k]_q!}{[m]_q![n+m-k]_q!\ldots[n+1]_q!}\Big\langle\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}_q(e_q(t))^m|G_{n+m-k,q}(x)\Big\rangle\\
&= \frac{\begin{bmatrix} m\\ k\end{bmatrix}_q}{2^m[m]_q!\begin{bmatrix} n+m-k\\ m-k\end{bmatrix}_q}\Big\langle\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}_q(e_q(t))^m|G_{n+m-k,q}(x)\Big\rangle.
\end{aligned}
$$

Applying relation (6.2.4) to $G_{n+m-k,q}(x)$, we may continue as

$$
c_{k,q} = \frac{\begin{bmatrix} m\\ k\end{bmatrix}_q}{2^m[m]_q!\begin{bmatrix} n+m-k\\ m-k\end{bmatrix}_q}\times
\Big\langle\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}_q(e_q(t))^m|\sum_{l=0}^{n+m-k}\begin{bmatrix} n+m-k\\ l\end{bmatrix}_q G_{n+m-k-l,q}x^l\Big\rangle.
$$

Using Proposition (6.5) and considering Remark (6.1), we obtain

$$
\begin{aligned}
c_{k,q} &= \frac{\begin{bmatrix} m\\ k\end{bmatrix}_q}{2^m[m]_q!\begin{bmatrix} n+m-k\\ m-k\end{bmatrix}_q}\times \\
&\sum_{i=0}^{m}\begin{bmatrix} m\\ i\end{bmatrix}_q\sum_{l=0}^{n+m-k}\sum_{l_1+l_2+\ldots+l_i=l}\begin{bmatrix} l\\ l_1,l_2,\ldots,l_i\end{bmatrix}_q\begin{bmatrix} n+m-k\\ l\end{bmatrix}_q G_{n+m-k-l,q}.
\end{aligned}
\tag{6.2.27}
$$

Now, assume that $k \geq m$. Then starting from the relation (6.2.26), we have

$$
\begin{aligned}
c_{k,q} &= \frac{1}{[k]_q!}\langle(\frac{e_q(t)+1}{2t})^m t^k|G_{n,q}(x)\rangle\\
&= \frac{1}{2^m[k]_q!}\langle(e_q(t)+1)^m|t^{k-m}G_{n,q}(x)\rangle\\
&= \frac{1}{2^m[k]_q!}.\frac{1}{[n+m-k]_q!\ldots[n+1]_q!}\Big\langle(e_q(t)+1)^m\Big|G_{n-k+m,q}(x)\Big\rangle\\
&= \frac{1}{2^m[k]_q!}.\frac{[n]_q![k-m]_q!}{[n-k-m]_q![k-m]_q!}\Big\langle(e_q(t)+1)^m\Big|G_{n-k+m,q}(x)\Big\rangle\\
&= \frac{[k-m]_q!}{2^m[k]_q!}\begin{bmatrix} n\\ k-m\end{bmatrix}_q\sum_{i=0}^{m}\Big\langle(e_q(t)+1)^i\Big|G_{n-k+m,q}(x)\Big\rangle.
\end{aligned}
$$

Finally, we obtain

$$c_{k,q} = \frac{\begin{bmatrix} n \\ k-m \end{bmatrix}_q}{2^m [k]_q! \begin{bmatrix} k \\ m \end{bmatrix}_q} \times \tag{6.2.28}$$
$$\sum_{i=0}^{m} \sum_{l=0}^{n-k+m} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q G_{n-k+m-l,q}.$$

Replacing identities (6.2.27) and (6.2.28) in the assumed sum in (6.2.25), completes the proof. □

Remark 6.22. According to the proof of Theorem (6.21), for any $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}$, the n-th q-Appell polynomial, $A_{n,q}(x)$, can be expressed based on the following relation

$$A_{n,q}(x) = \sum_{k=0}^{m-1} \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \times$$
$$\left\{ \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q \sum_{l=0}^{n+m-k} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q \times \right.$$
$$\left. A_{n+m-k-l,q} \right\} G_{k,q}^{[m]}(x)$$
$$+ \sum_{k=m}^{n} \frac{\begin{bmatrix} n \\ k-m \end{bmatrix}_q}{2^m [k]_q! \begin{bmatrix} k \\ m \end{bmatrix}_q} \times$$
$$\left\{ \sum_{i=0}^{m} \sum_{l=0}^{n-k+m} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q A_{n-k+m-l,q} \right\} G_{k,q}^{[m]}(x)$$

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