

# **Equivalent and Dual Robotic Manipulators through Dual Transformations, Reciprocal Screws and Graph Theory**

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## ABSTRACT

Duality transformations have effective impacts on simplifying analysis and synthesis steps of systems, due to their additively topological richness in unification and generalization of theories. Duality between statics and kinematics of mechanical systems in general, and robotic manipulators in particular, aided the discoveries of novel dual structures which possess superiority among their topological types.

According to the previous hypothesis, this thesis addresses topological dualities in engineering systems as a concept, and the duality between different structures (geometrical wise) of robotic manipulators in vivid. The latter duality was found naturally leading to the commonly known reciprocity between actively coordinated systems provided by the theory of screws. The major contribution of this thesis is represented by generalizing the geometrical reciprocity problem of having a set of screws which each of its elements is reciprocal to all the elements of another set of screws, except one. Accordingly, this generalization breaks the confines of duality from existing only between special cases of serial and parallel manipulators, and extended its boundaries to combine a wide range of structures. Moreover, the geometrical meaning for Moore-Penrosians' pseudo inverses of Jacobians was clarified naturally by means of linear algebra. The latter resulted in a new insight for duality in robotic systems especially in terms of the usage of reciprocity leading to equivalency.

**Keywords:** Duality, Electrical Mechanical Analogs, Robotic Equivalents, Kinematics, Statics, Graph Theory, Screw Theory and Reciprocal Screws.

## ÖZ

Teorilerin birleşimi ve genelleştirilmesi alanlarındaki topolojik zenginliklerinden dolayı ikisel değişimlerin, sistemlerin basitleştirilmiş analizi ve sentezi aşamalarında etkili rolleri bulunmaktadır. Genelde mekanik sistemlerin ve özelde robotik işleticilerin statik ve kinematikleri arasındaki ikisellik, kendi topolojik türleri arasında üstünlüğe sahip olan yeni ikili yapıların keşiflerine katkı koymuştur.

Önceki hipoteze (kurama) göre, bu tez çalışması kavramsal olarak mühendislik sistemlerindeki topolojik ikiselliklere ve uygulamada robotik işleticilerin farklı yapılarındaki (geometrik yönden) ikiselliğe değinmektedir. Sonraki bahsedilen ikiselliğin, vida teorisi tarafından sağlanan aktif koordinasyonlu sistemler arasında gerçekleşen ve genelde bilinen karşılıklılığa doğal olarak yol açtığı bulunmuştur. Söz konusu tezin en önemli katkısı her bir elementi, başka bir set vidanın tüm elementleriyle biri dışında karşılıklı olan bir set vidanın geometrik karşılıklılık probleminin genelleştirilmesi ile temsil edilmiştir. Buna göre, bu genelleme seri ve paralel işleticilerin özel durumları arasında var olan ikiselliğin sınırlarını bozmuş ve sınırlarını geniş yelpazeli yapıları birleştirecek şekilde genişletmiştir. Buna ek olarak, Moore-Penrosian'ın Jacobians sözde ters matrisinin geometrik anlamı doğrusal cebir aracılığı ile açıklanmıştır. İkinci bahsedilen, robotik sistemlerdeki ikiselliğe ve özellikle denkliğe yol açan karşısallığın kullanımı bakımından yeni bir anlayış getirmiştir.

**Anahtar kelimeler:** İkisellik, Elektriksel Mekanik Analoglar, Robotik Eşdeğerler, Kinematik, Statik, Grafik Teorisi, Vida Teorisi ve Karşılıklı Vidalar

## **DEDICATION**

To my mother and father...  
Amina Jarikji and Dr. Hassan Harastani.

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# Chapter 1

## INTRODUCTION

### 1.1 Definition of Terms and Concepts:

#### 1.1.1 Duality Transformations:

Duality between physical or mathematical systems is considered as one of the most useful tools in the hands of scientists, two systems are considered dual if there is a one to one correspondence between some of their physical or mathematical properties [1].

A useful duality analysis can give its performer a topological insight in his way to design or analyze a system by observing its dual behavior. The theory of dual coding (DCT) as an example took a remarkable place in the development of linguistic based search engines [2]. Besides, many indispensable transformations such as Fourier transformation, Fresnel's vector and others were considered as dual transformations [3-4].

#### 1.1.2 Comparison between Serial and Parallel Manipulators:

Robots are classified in terms of their kinematic structures as: serial, parallel and hybrid (or composite) [26]. A robot is classified as serial if it possesses an open kinematic chain, in contrast; as parallel if it possesses a closed kinematic chain. Hence, the characteristics of serial and parallel manipulators exhibit remarkable differences as will be discussed hereafter.

### **1.1.3 Serial Manipulators:**

Serial manipulators are widely used in the industry due to their large workspace, simplicity in dynamic modelling, easy to solve forward kinematics and easy to control. Yet, they suffer from low accuracy, poor dynamic characteristics, low stiffness, difficulty solving inverse kinematics, and accumulated position errors. This is due to the open kinematic structure they possess. Therefore, the usage of serial manipulators in operations that requires highly precise procedures such as surgery and 3D printing, or operations that imply heavy load carrying actions such as flight or military equipment simulations is not possible due to aforementioned confrontations [28-31].

### **1.1.4 Parallel Manipulators:**

Parallel manipulators possess closed kinematic structures that allow the manipulators of this type to be characterized by: easy inverse kinematics, low position errors, high stiffness, very high dynamic characteristics, high speed and high acceleration. Yet, they suffer from very difficult forward kinematics, difficulty in calibration, complicated design procedure and small or complex workspace. Moreover, the number of passive (non-actuated) joints in a parallel manipulator is usually more than the actuated ones; in this manner the end effector (a platform usually) might possess an extra degree of freedom (DOF) at some position configuration called singular configurations. When a parallel manipulator is in one of its singular configurations, the moving lamina will be free to move with some direction under the influence of an external force, this phenomena known as Bricard-Borel phenomena requires high attention in the process of designing parallel manipulators [10]. The octahedron (known as General Stewart Gough Platform), is considered as one of the most successful designs for parallel manipulators. In fact, any small displacement for an

octahedron singular configuration breaks its singularity and return the platform into a nonsingular (safe) configuration [10].

### **1.1.5 Hybrid Manipulators:**

A robot is considered to have a hybrid structure if it possesses a general open kinematic structure in its outer loop, with some embedded closed kinematic structures within. These manipulators combine desired properties from both serial and parallel structures; especially in terms of the size of the workspace, accuracy and weight to load ratio [26].

In most general industrial purposes' serial manipulators, the base joint is designed as rotary to give the *Yaw* movement about the vertical axis of the structure and facilitate the interaction with its surrounding environment. Actually, in most hybrid designs of industrial manipulators, the base joint is left to be rotary and independent from any parallelism to assure maximum reachability for its end effector (tool), along with considering the base usually fixed to a table (or ground) as a fact, which gives the ability of implementing high torque actuators without the necessity of using tendons or other kinematic pairs to connect actuators with the moving joints with far from base links that can be found for instance in the wrist of a humanoid [26]. On the other hand, embedded parallel structures (parallel subsystems) are implemented as not to exceed 6 DOF to assure minimum singular configurations for the parallel structure, thus minimum Bricard-Borel phenomena. Both these facts will be dealt with carefully hereafter.

## **1.2 Organization of this thesis**

Chapter 1 (Introduction), is an introductory to some of the useful concepts regarding the work done in this thesis; it mainly discusses duality as a concept, and introduces some properties of types of manipulators in terms of their kinematic structure.

Chapter 2 (Literature Review), regarding that graph theory will be used to facilitate the formulation for dual transformations both in electrical and mechanical systems, the required fundamentals of dual graphs are first summarized, including planar connected graphs, tree analysis and dual graphs. Afterwards, commonly used duality in electrical circuits and the analogy between electrical and mechanical components will be summarized. Later in this chapter, the duality between statics and kinematics especially for robotic manipulators will be discussed; while fundamentals of screw theory were found to serve the topic of this work, only frequently used concepts in screw theory will be introduced to help moving on to the duality using Jacobian analysis of serial and parallel manipulators. By the end of this chapter, direct duality between serial and parallel manipulators via graph representations namely flow and potential line graphs is summarized with an example, as well as the common convention known as Denavit-Hartenberg (DH) will be briefed.

Chapter 3 (General Reciprocity and Serial to Hybrid Actuation), considers the general reciprocal screw problem for having a screw that is reciprocal to a set of screws except one, an original contribution for what we call general matrix linear system of reciprocity will be discussed in detail and demonstrated with an example. Moreover, the discussion on the geometrical meaning of the pseudo inverse of the Jacobian of a manipulator will be introduced for the first time in a novel way to the literature.



Discussions on the ability of transforming a sub-section of a serial manipulator with its reciprocal dual will be demonstrated via examples.

Chapter 4 (Equivalents in Electrical and Mechanical Systems), discusses the commonly used equivalents in electrical and robotic systems, and the ability of constructing a hybrid manipulator from an existing serial manipulator using the results obtained for general reciprocal screws in Chapter 3.

Chapter 5 (Conclusion and Future Work), discusses results of the work done in both chapters 3 and 4; where a new definition for structural equivalent manipulators will be introduced in a way that separates between dual and equivalents robotic manipulators. Moreover, promising future for the work done in this thesis will be discussed.

## Chapter 2

### LITERATURE REVIEW

#### 2.1 Graph Theory:

Graph theory is a part of discrete mathematics that aims to model relations between two or more interconnected objects of a system. Graph theory was established in the middle 18<sup>th</sup> century by the famous theoretician *Leonhard Euler*, where he was able to develop a systematic approach to solve one of the problems at that time known as the “*Seven Bridges of Königsberg*” using a graphical based theorem of what is currently called the “*Eulerean Graphs*” [20]. Later on, graph theory gained a remarkable popularity in modelling physical systems due to the simplicity that graphical representations offer to a physical or mathematical model [19]. Nowadays, the theory of graphs is deeply involved in many engineering applications. In fact, new graphical representations for engineering systems were developed in the past two decades that facilitated the way of modelling, designing and analyzing many engineering systems [11-12, 14-18].

##### 2.1.1 Defining the Graph:

A graph is defined as two sets as  $G = \{ V, E \}$ ,  $V$  is called the set of vertices and  $E$  is called the set of edges; where each edge in  $E$  represents a relation between two elements in  $V$ , i.e. given a graph  $G = \{ V, E \}$ , let  $V = \{ V_1, V_2 \dots V_n \}$ , and  $E = \{ E_1, E_2 \dots E_m \}$  where  $m, n \in \mathbb{N}$ ,  $\forall E_x$  where  $x \in [1, 2 \dots m]$ ,  $E_x$  represents a connection between the vertices  $V_y$  and  $V_z$  where  $y, z \in [1, 2 \dots n]$ .

A graph is said to be directed (oriented) if each of its edges represent a one way connection between two vertices of  $V$ , i.e. if  $E_x$  is an edge of an oriented graph represents a connection between  $V_y$  and  $V_z$  in the direction  $V_y \rightarrow V_z$ ,  $E_x = V_y V_z \neq V_z V_y$ . While in non-oriented graphs, the order of the vertices is commutative in any edge, i.e.  $V_y V_z = V_z V_y$ .

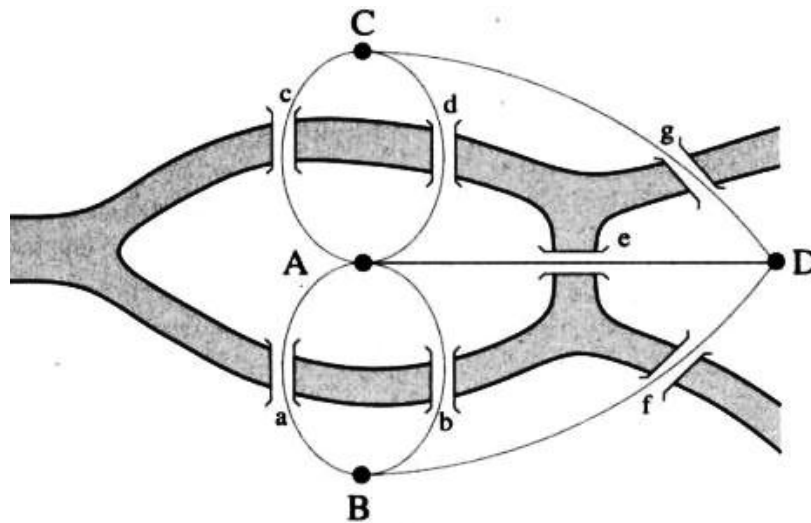


Figure 2.1: The Seven Bridges of Königsberg and its Graphical Representation [20]

### 2.1.2 Connected Planar Graphs:

In both electrical and mechanical circuits, the topological graphs that represent such systems are usually connected and planar [11-12, 16-20]. Connectivity of graphs requires the existence of a path from any vertex to any other vertex via the edges of the graph, where a path is defined by a sequence of edges that implies non repeating vertices, e.g. the graph in Figure 2.1 that represent a graphical representation of the Königsberg bridges problem is connected. Where a graph is considered a planar if it can be drawn on a plane without having any intersection between its edges [20], e.g. the aforementioned graph is also planar. Graphs possess both connected and planar properties will be further studied using what is known as tree analysis hereafter in this text.

### 2.1.3 Tree Analysis for Planar Connected Graphs:

Let a graph  $G = \{ V, E \}$  be a connected graph, and let  $n$  defines the number of elements of the set  $V$ . A tree is a connected sub-graph of  $G$  that contains all the vertices of  $V$  and exactly  $(n-1)$  number of edges from  $E$ .

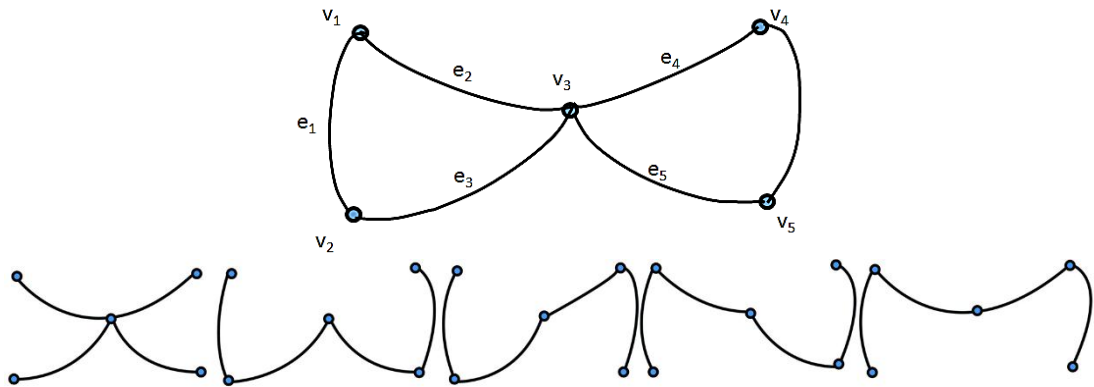


Figure 2.2: A Graph with All of Its Combinatorial Choices of Trees

After choosing a tree, the edges of the tree will be called as *branches*, where the remaining edges will be called as *co-branches* or simply *chords*.

### 2.1.4 Cycles and Cut-sets of Connected Planar Graphs:

A cycle is simply a loop, a closed path which all vertices occurs exactly once except of the first-last vertex which occurs exactly twice. Where a cut-set is defined by sequence of edges that removing them splits the graph into two connected sub-graphs [20]. As an example, consider the oriented graph given in Figure 2.3, one of its proper cut-sets (CS) is given by the sequence of the edges  $\{e_1 e_2 e_6 e_7 e_5\}$  and one of its proper cycles (C) is given by the sequence of the edges  $\{e_2 e_3 e_6\}$ .

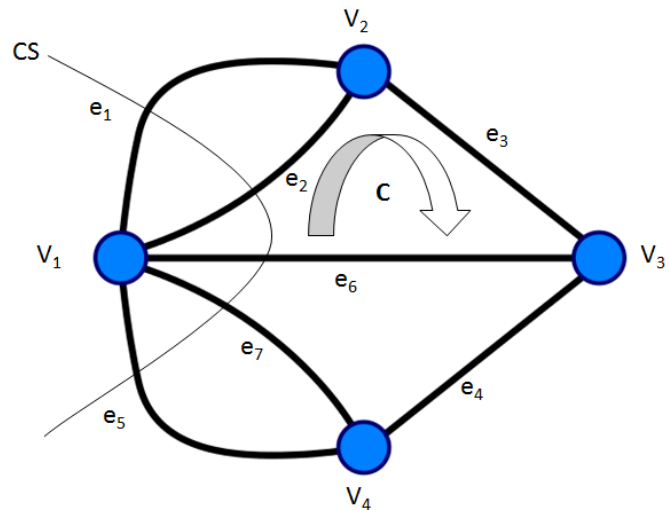


Figure 2.3: An Example of a Cut-set and a Cycle Assigned for a Graph

### 2.1.5 Fundamental Cut-sets and Fundamental Cycles:

Simply, a fundamental cut-set is defined as a proper cut-set that contains exactly one tree branch, while a fundamental cycle is defined as a proper cycle that contains exactly one chord, e.g. consider the oriented graph shown in Figure 2.4, its tree branches are represented by heavy lines  $\{e_2 e_4 e_6\}$ , where its chords are represented by dashed lines  $\{e_1 e_3 e_5\}$ . Three fundamental cycles are accompanied by its chords and three fundamental cut-sets are accompanied by its branches as follows:

Fundamental cut-sets:  $CS_1 = \{e_1 e_2 e_3\}$ ,  $CS_2 = \{e_3 e_4 e_5\}$  and  $CS_3 = \{e_5 e_6\}$ . And

Fundamental cycles:  $C_1 = \{e_1 e_2\}$ ,  $C_2 = \{e_2 e_3 e_4\}$ , and  $C_3 = \{e_4 e_5 e_6\}$ .

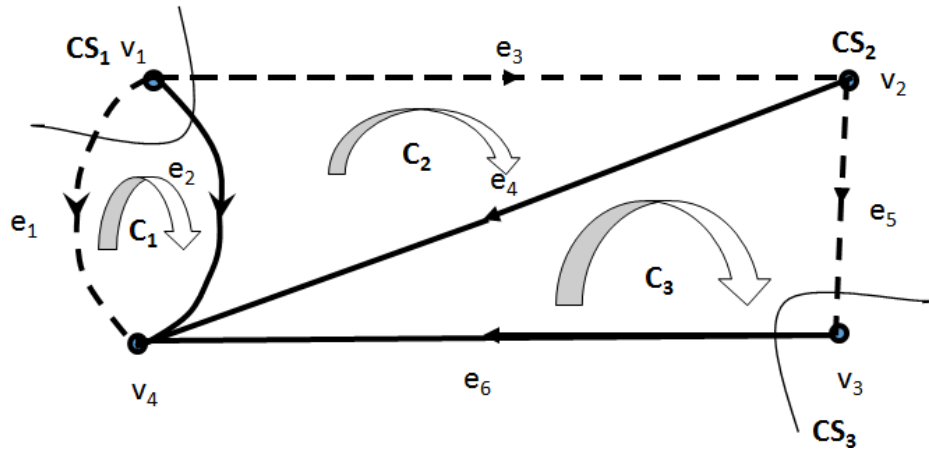


Figure 2.4: A Connected Graph with a Tree and the Corresponding Fundamental Cycles and Cut-sets

Fundamental cut-sets and cycles might be represented in matrices as follows:

- Matrix of fundamental cycles: by taking the direction each cycle along the direction of its corresponding chord; edges contained in the cycle will take values (+1) / (-1) if they are directed with / opposite the cycle direction, where the edges that are not contained in the cycle will take the value of zero in the matrix representation [19-20], e.g. the matrix of fundamental cycles of the graph shown in Figure 2.4 is given in equation (2.1).

$$C_f = \begin{matrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix} & \begin{matrix} (C1)e1 \\ (C2)e3 \\ (C3)e5 \end{matrix} \\ \begin{matrix} e1 & e2 & e3 & e4 & e5 & e6 \end{matrix} & \end{matrix} \quad (2.1)$$

- Matrix of fundamental cut-sets: by taking the direction of each cut-set along the direction of its corresponding branch; edges contained in the cut-set will take the values (+1) / (-1) if they are directed with / opposite of the cut-set direction, where the edges that are not contained in the cut-set will take the value of zero in the matrix representation [19-20], e.g. the matrix of fundamental cut-sets of the graph shown in Figure 2.4 is given in equation (2.2).

$$CSf = \begin{array}{cccccc} \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right] & \begin{array}{l} (CS1)e2 \\ (CS2)e4 \\ (CS3)e6 \end{array} \\ \begin{array}{cccccc} e1 & e2 & e3 & e4 & e5 & e6 \end{array} & \end{array} \quad (2.2)$$

## 2.2 Duality in Graph Theory:

### 2.2.1 Correspondence by Equations:

Two graphs are considered dual if there is one-one correspondence between all of the fundamental cut-sets / cycles of both graphs. This correspondence is from the type cut-set  $\rightarrow$  cycle / cycle  $\rightarrow$  cut-set can be noticed in the matrices as follows [19-20].

By arranging the columns of the fundamental cut-sets / cycles (matrices) as branches first and chords next, each matrix will be split into two matrices, one of each split will be a unity, where the other will carry more information about the structure of the graph. These meaningful matrices were found to exhibit duality and will be discussed next, e.g. the arranged matrices for equations (2.1) and (2.2) are found in equations (2.3) and (2.4) respectively.

$$Caf = \begin{array}{cccccc} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] & \begin{array}{l} (C1)e1 \\ (C2)e3 \\ (C3)e5 \end{array} \\ \begin{array}{cccccc} e1 & e3 & e5 & e2 & e4 & e6 \end{array} & \end{array} \quad (2.3)$$

$$CSaf = \begin{array}{cccccc} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right] & \begin{array}{l} (CS1)e2 \\ (CS2)e4 \\ (CS3)e6 \end{array} \\ \begin{array}{cccccc} e1 & e3 & e5 & e2 & e4 & e6 \end{array} & \end{array} \quad (2.4)$$

By representing the splits of each matrix in equations (2.3) and (2.4) by a matrix for each, the representation will take the form shown in equations (2.5) and (2.6) respectively.

$$Caf = [I \quad A] \quad (2.5)$$

$$CSaf = [B \quad I] \quad (2.6)$$

Where  $I$  is the unity in  $R^3$ , and the matrices  $A$  and  $B$  are given as:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (2.7)$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad (2.8)$$

It is easy to show that  $A = -B^T$ .

- The study of duality between fundamental cycles and fundamental cut-sets shows an ability of extracting the governing equations of the cycles using the cut-sets and vice versa.
- Given a graph  $G$ , a dual graph  $G^*$  is defined by the unique graph (in terms of isomorphism) that each fundamental cycle / cut-set in  $G$  corresponds to a fundamental cut-set / cycle in  $G^*$  and vice versa.

### 2.2.2 Direct Graphical Method for Finding Dual Graphs:

The dual graph  $G^*$  for a given connected planar graph  $G$  can be found using the simple following algorithm [20]:

- Place a vertex for  $G^*$  in the middle of each independent mesh of  $G$ .
- Place a vertex for  $G^*$  out of all the meshes of  $G$ .
- Cross the edges of  $G$  to connect between the vertices of  $G^*$  if the connection can be done by crossing only one edge of  $G$ .

Further illustration can be seen in Figure 2.5; this graphical method is found to be similar to the method given by Shai et al. in [11-12] and will further addressed in this text.



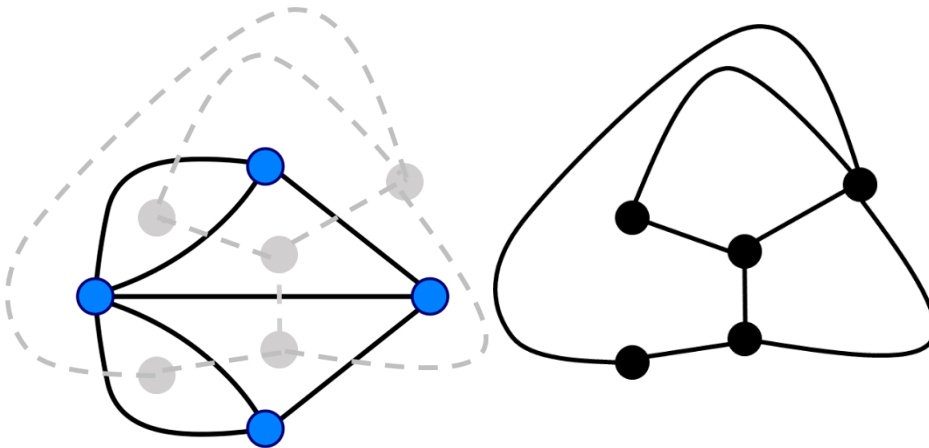


Figure 2.5: Illustration for Direct Graphical Method of Finding Dual Graphs

### 2.3 Duality in Electrical Circuits:

Duality in electrical systems has been vividly studied in the literature. Similarly to the duality in graph theory, there exist both an analytical approach and a direct graphical method to find dual circuits [21].

In the following, the duality between two electrical circuits will be demonstrated via an example.

Consider the two source free RLC circuits shown in Figure 2.6.

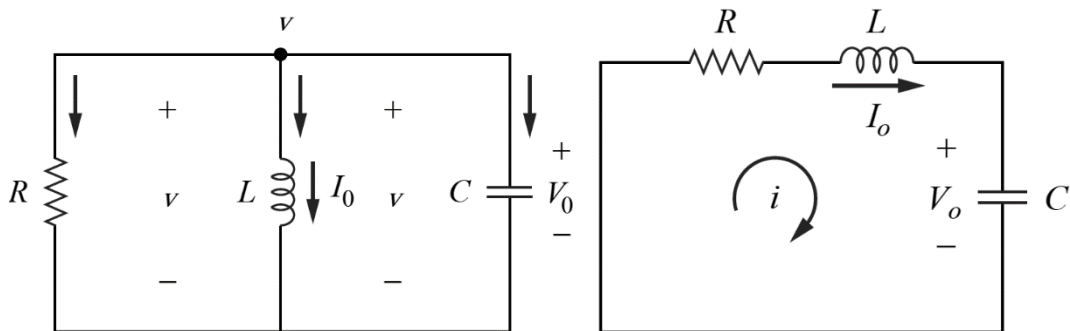


Figure 2.6: (a) Parallel RLC Source Free

(b) Serial RLC Source Free

The governing equation of voltages and currents with respect to the circuits' components for the circuits given in Figure 2.6 (a) and (b) are derived below.

For the circuit shown in Figure 2.6 (a); at  $t = 0$ , the current of the inductor is given as the following:

$$i(0) = \frac{1}{L} \int_{-\infty}^0 v dt = I_0 \quad (2.9)$$

Applying Kirchhoff Current law (KCL) at node  $v$  gives:

$$\frac{v}{R} + C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^t v dt = 0 \quad (2.10)$$

For the circuit shown in Figure 2.6 (b); at  $t = 0$ , the voltage across the capacitance is given as the following:

$$v(0) = \frac{1}{C} \int_{-\infty}^0 i dt = V_0 \quad (2.11)$$

Applying Kirchhoff Voltage Law (KVL) in loop  $i$  gives:

$$Ri + L \frac{di}{dt} + \frac{1}{C} \int_{-\infty}^t i dt = 0 \quad (2.12)$$

Simply, equations (2.9) and (2.10) will be identical to equations (2.11) and (2.12) respectively, by interchanging what is known as dual pairs that are summarized for dual electrical circuits in Table 2.1.

Table 2.1: Dual Pairs in Electrical Circuits [21]

Resistance $R$	Conductance $G$
Inductance $L$	Capacitance $C$
Voltage $v$	Current $i$
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thévenin	Norton

### 2.3.1 Direct Graphical Method for Finding Dual Circuits:

The graphical method to find dual circuits is very similar for the one seen in section 2.2.2 previously in this chapter in finding dual graphs. A direct algorithm depending on the dual pairs given in Table 2.1 is identical to the algorithm for dual graphs after adding the following [21]:

- The node (vertex) that was placed out of all the meshes of the circuit (graph) has to represent the ground node.
- The polarity of the voltage and current sources will be determined using the rule: A voltage source will be polarized from the ground to the dual node if it produces a positive sense current in the direction of the mesh, otherwise away from the node and towards the ground. The same polarization rule can be determined for current sources by interchanging the rule of the mesh current and the voltage source direction.

For further illustration, the aforementioned algorithm is demonstrated in Figure 2.7.

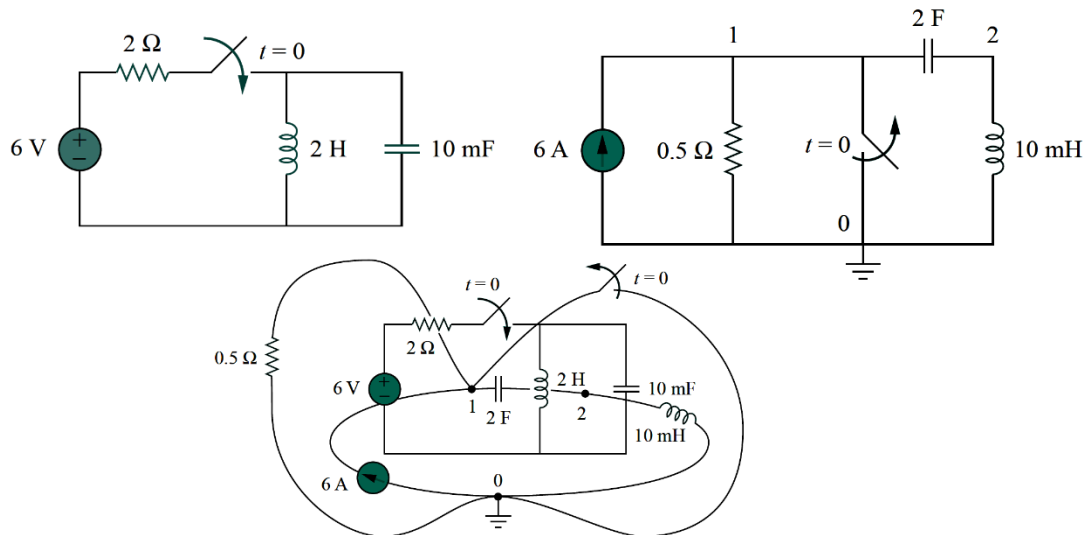


Figure 2.7: Demonstration for the Direct Graphical Method for Obtaining Dual Electrical Circuits [21]

## 2.4 Analogous Electrical / Mechanical Systems:

Originating a physical system as interconnection of components, then regarding the components that formulate electrical and mechanical circuits, one might find that the governing equations that relate the components of one system as similar to the governing equations governing another system [21]. This similarity has been witnessed by many researchers as an attempt to map theorems of electrical theory to be applied on mechanical domain [23].

Two types of analogy between electrical and mechanical systems were established, namely Mechanical-Electrical Analog (I) and Mechanical-Electrical Analog (II); these are more known as old and new analogies respectively. The new analogy as summarized in Table 2.2, studies the possibilities of applying electrical circuit theorems generally and Kirchhoff's current and voltage laws especially on mechanical systems. Indeed, some researchers were able to apply these theorems successfully and reached to the same results that can be found using typical approaches in mechanical systems such as Davis in [25].

Table 2.2: Electrical Mechanical Analog Pairs, Electrical / Mechanical Analog (II)  
[23]

Electrical Quantity / Equation	Mechanical Quantity / Equation
Voltage e	Force f
Current i	Velocity v
Resistance R	Friction B
Capacitance C	Compliance 1/K
Inductance L	Mass M
Transformer N <sub>1</sub> :N <sub>2</sub>	Lever L <sub>1</sub> :L <sub>2</sub>
$e = iR$	$f = vB$
$e = L \frac{di}{dt}$ $i = \frac{1}{L} \int e dt$	$f = M \frac{dv}{dt}$ $v = \frac{1}{M} \int f dt$
$e = \frac{1}{C} \int i dt$ $i = C \frac{de}{dt}$	$f = K \int v dt$ $v = \frac{1}{K} \frac{df}{dt}$
power = e . i	power = f . v
Transformer $\frac{e1}{e2} = \frac{N1}{N2} = \frac{i2}{i1}$	Lever $\frac{f1}{f2} = \frac{L2}{L1} = \frac{V2}{V1}$
Capacitor Energy $\frac{1}{2} Ce^2$	Spring Energy $\frac{1}{2} \frac{1}{K} f^2$
Inductor Energy $\frac{1}{2} Li^2$	Mass Energy $\frac{1}{2} Mv^2$
$\sum_{loop} \text{voltages} = 0$	$\sum_{object} \text{forces} = 0$
$\sum_{node} \text{currents} = 0$	$\sum_{loop} \text{velocities} = 0$ (Under equivalent distances)

As the duality in electrical circuits has been studied before in this chapter; the electrical quantities represented by voltage  $v$  and current  $i$  were found to be dual to each other in the dual circuits. These dual pairs, i.e.  $v$  and  $i$ , are mechanically analogous to the quantities force  $f$  and velocity  $v$  respectively.

This conclusion is consistent with the work done on the duality in mechanical systems between statics and velocity kinematics that will be studied in the next section.

## **2.5 Duality between Kinematics and Statics:**

The duality between kinematics and statics has been noticed by many researches in the literature. To serve the hereafter work of this thesis we select some of the topics that study the duality between statics and kinematics for robotic manipulators in accordance with the work done in this field in [5-14].

Statics and kinematics exhibits duality that was originated due what is known by the reciprocity (or orthogonality) between the representatives coordinates of velocity kinematics and statics. The theory of screws have been widely used to explain this duality and offered a solid platform in the way representing the coordinates of kinematics and statics that were named by the twist and wrench coordinates retrospectively.

The duality or more precisely the reciprocity between wrench and twist coordinates will be summarized in section 2.5.2 after defining the screw that represent these coordinates in the next section.

Moreover, Shai et al. in [11] developed two graphical representations of mechanisms namely by flow line graph (FLGR) and potential line graph (PLGR) that were shown

to be dual to help investigating dual mechanisms. Furthermore, another graphical representation has been developed by Shai in [12] namely topological and constraint graphs, which duality was found to hold in the position domain under special configurations.

For Davidson & Hunt in [10], they took the duality between statics and kinematics to the position domain and offered a systematic approach in finding what is known as instantaneously equivalent manipulators by means of reciprocal screws, this approach will be summarized in section 2.5.4 using screw theory, and will be used in the way finding hybrid structures in Chapter 3.

### **2.5.1 Definition of a Screw / Ray and Axis Coordinate Representations:**

The common screw may represent either the vector quantities of first order kinematics (angular and linear velocities), or the vector quantities of statics (forces and moments). The common notation of a screw is given by  $\$ = [ S ; S_o ]$ , where  $S$  holds the vector of the angular velocity in kinematics or the force in statics, while  $S_o$  represents the resultant linear velocity or resultant moment in kinematics and statics respectively.

In order to further explain the screw, we will study the general case of having a screw representation in kinematics as follows:

Assume a body is experiencing two movements, rotating about an axis  $\hat{s}$  which represents the direction of the screw  $\$$ , together with sliding along the direction of the screw  $\hat{s}$ ; assume further that the amount of translation is dependent on the amount of rotation with dependency parameter  $h$  which is called the pitch of the screw  $\$$  as shown in Figure 2.8.

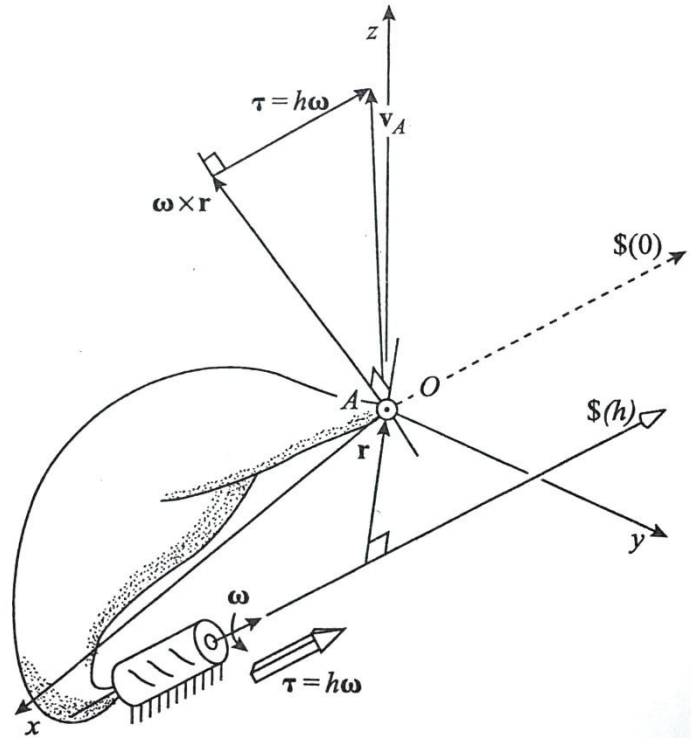


Figure 2.8: A Body *Screwing* in the Direction of  $\hat{S}$  [10]

Consider the point  $A$  in Figure 2.8;  $A$  is rotating with angular velocity that is equal to the angular velocity of the screw, i.e.  $\omega_A = \omega$ , while the linear velocity of  $A$  has two components, the first component is the resultant linear velocity from the rotation about  $\hat{S}$  with angular velocity  $\omega$  which is equal to the vector cross product between the distance vector  $r$  and the vector  $\omega$ , and the second component is equal to the amount of translation  $\tau = h\omega$  of the screw  $\hat{S}$ .

As the point  $A$  is at the origin of the reference coordinates, the screw  $\hat{S}$  is going to be exactly as  $\hat{S} = [\omega_A; v_A]$ , where  $v_A = r \times \omega + h\omega$ .

Now we can build a definition of the screw as:  $\hat{S} = [S; (r \times S + hS)]$  in kinematics and an identical definition in statics interchanging the role of the angular velocity / linear velocity in kinematics with the role of the force / moment in statics respectively.



The screw  $\$$  is a combination of two vectors that are three dimensional which gives the screw 6 dimensions that usually are expressed in terms of what known as Plücker coordinates as:  $\$ = [ L ; M ; N ; P ; Q ; R ]$ .

The screw representing kinematics is called a *twist*, where a screw representing statics is called by a *wrench*. The aforementioned duality between statics and kinematics is depending on the orthogonality (or reciprocity) between these coordinates and will be further investigated in this chapter.

The aforementioned representation of screws is known as Ray coordinate representation, while there exist another representation of screws known as Axis coordinate representation, the later takes the form:  $\$ = [S_o; S]$ ; these two representation are dual to each other and will be used to investigate reciprocity of screws in the next section.

### 2.5.2 Reciprocal Screw Axes:

A screw from kinematics (a twist) is reciprocal to a screw from statics (a wrench) if the force or the torque applied by the wrench can do no work on the twist and vice versa.

Given two screws  $\$_1$  and  $\$_2$ , the condition of reciprocity between two screws represented in the Plücker coordinates as  $\$_1 = [L_1; M_1; N_1; P_1; Q_1; R_1]$  and  $\$_2 = [L_2; M_2; N_2; P_2; Q_2; R_2]$  can be expressed by equation (1.13).

$$L_1 * P_2 + M_1 * Q_2 + N_1 * R_2 + P_1 * L_2 + Q_1 * M_2 + R_1 * N_2 = 0 \quad (1.13)$$

This reciprocity can be expressed as Euclidean product of the screws  $\$1$  and  $\$2$  equal to zero if one of the screws is represented in Ray coordinates and the other is represented in Axis coordinates as shown in equation (2.14).

$$[L_1 \quad M_1 \quad N_1 \quad P_1 \quad Q_1 \quad R_1] * \begin{bmatrix} P_2 \\ Q_2 \\ R_2 \\ L_2 \\ M_2 \\ N_2 \end{bmatrix} = 0 \quad (2.14)$$

For demonstration on the reciprocity between two screws, we take the following example:

Consider a body attached by two joints as in Figure 2.9, a revolute joint in the direction of  $\$1$  and a prismatic joint in the direction of  $\$2$  shown; the force applied by the wrench  $\$2$  can cause no angular velocity about the twist  $\$1$  and vice versa. Hence, the screws  $\$1$  and  $\$2$  are reciprocal.

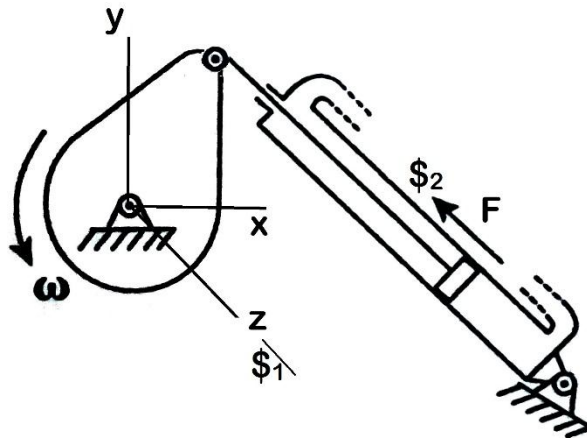


Figure 2.9: Reciprocal Screws, a Twist  $\$1$  and a Wrench  $\$2$  [10]

As we can see from Figure 2.9, there are infinitely many screws that are reciprocal to the screw  $\$1$ . As long as a screw is six dimensional, the uniqueness reciprocity is confined by reciprocal screw for 5 independent screws. This conclusion will be further

explained in Chapter 3 after mapping the problem of reciprocal screws into its linear system representation.

### **2.5.3 Duality between Serial and Parallel Manipulators:**

In both statics and velocity kinematics, the usage of what is known as the Jacobian matrix is very popular to find resultant velocities (angular and linear) and resultant forces and moments that are applied on the tool of the mechanism with respect to a reference frame and vice versa, i.e. the Jacobian matrix maps between the Cartesian space velocities and the joint space velocities (known as joint rates) of a robotic manipulator, and its transpose maps between the Cartesian space forces and moments with joint space forces and moments (required or performed by the manipulator actuators) [26-28].

Although the entries of the Jacobian matrix can be extracted by means of differentiating the function that defines position and orientation relations between a reference frame and an observation frame respectively, usually it is not easy to find the Jacobian matrix without the existence of a systematic approach to simplify the procedure [27].

For all of consistency with this work, generality and preserving maximum geometrical meaning assigned with the Jacobian, we will use what is known as the screw based Jacobian hereafter in this text.

The columns of a screw based Jacobian that relates between the velocity of the end effector (considered to be attached to the last screw) and the base coordinate frames of an  $n$  degree of freedom manipulator is simply formulated by the screws of the actuated joints with respect to the base coordinates as in equation (2.15).

$$J_n^0 = [ \$_1^0 \$_2^0 \dots \$_n^0 ] \quad (2.15)$$

The problem of finding the end effector velocities given the actuators velocities for a robotic manipulator, known as the forward velocity problem is expressed as:

$$[w ; v] = J_n^0 * [q_i] \quad (2.16)$$

Where  $[q_i]$  is the vector representing the joint velocities (joint rates).

On the other hand, the inverse problem represented by finding the actuators velocities, known (desired) velocities of the end effector with respect to the reference frame is known as the inverse velocity problem and expressed as:

$$[q_i] = (J_n^0)^{-1} * [w ; v] \quad (2.17)$$

In the same manner, dually in statics; the forward and inverse statics problems are expressed in equations (2.18) and (2.19) respectively [26-28].

$$[f ; m] = (J_n^{0T})^{-1} * [\tau_i] \quad (2.18)$$

$$[\tau_i] = J_n^{0T} * [f ; m] \quad (2.19)$$

Where  $f$ ,  $m$  and  $\tau_i$  represent the resultant force vector on the end effector, the resultant moment vector on the end effector and the vector of torques / forces applied by the actuators respectively.

It has been found in [5-10], that if the wrenches of a parallel manipulator are reciprocal to the twists in a serial manipulator and vice versa in a way that each wrench is reciprocal to all twists except one, the parallel and serial manipulators given will

exhibit duality in the sense of kinematics and statics. The dual pairs summarized in Table 2.3 are only true if the previous statement is true.

Table 2.3: Dual Pairs for Serial and Parallel Robotic Manipulators [9]

Parallel	Serial
Wrench Screw $\$ = [ f ; m ]$	Twist Screw $\$ = [ w ; v ]$
Twist Screw $\$ = [ w ; v ]$	Wrench Screw $\$ = [ f ; m ]$
Forward Statics Problem $[ f ; m ] = (J_n^0)^{-1} * [ \tau_i ]$	Forward Velocity Problem $[ w ; v ] = J_n^0 * [ q_i ]$
Forces / Torques	Angular / Linear Velocities
Angular / Linear Velocities	Forces / Torques
Inverse Velocity Problem $[ q_i ] = (J_n^0)^{-1} * [ w ; v ]$	Inverse Statics Problem $[ \tau_i ] = J_n^{0T} * [ f ; m ]$

The relation that leads to investigate further in the geometry of dual manipulators is the one related to the forward / inverse velocity problem with the forward / inverse statics problem for serial and parallel manipulators, and will be discussed later.

In accordance with the work done in [10], finding the Jacobian matrix of a manipulator (serial or parallel), and trying to reconstruct the screws from its transpose inverse leads to the dual for the given manipulator.

In the existing literature for finding dual manipulators by inverting the Jacobian matrix in [9-10], the method is confined to the existence of an inverse for the Jacobian; which in result confined the duality between manipulators that possess Jacobians that are square / reducible (or extendable) to square and invertible.

The following section will demonstrate an example of the duality between 3R serial planar and 3(RPR) parallel planar manipulators, then discuss the similarity in the obtained Jacobian matrix for one of the manipulators with the transposed version of the inverse of the Jacobian of the other.

#### 2.5.4 Dual 3DOF Planar Serial and Parallel Manipulators:

For consistency with the aforementioned methodology of finding dual manipulators by investigating the similarities between the Jacobian matrices, illustration of the methodology is implemented as follows:

Consider the serial 3R planar manipulator given in Figure 2.10 (a), the Jacobian of the manipulator expressed in terms of the screws given in equation (2.15) will be derived and inverted. Similarly, the Jacobian of the planar 3(RPR) manipulator given in Figure 2.10 (b) will be derived and compared with the result found in the previous step. Discussion will be followed.

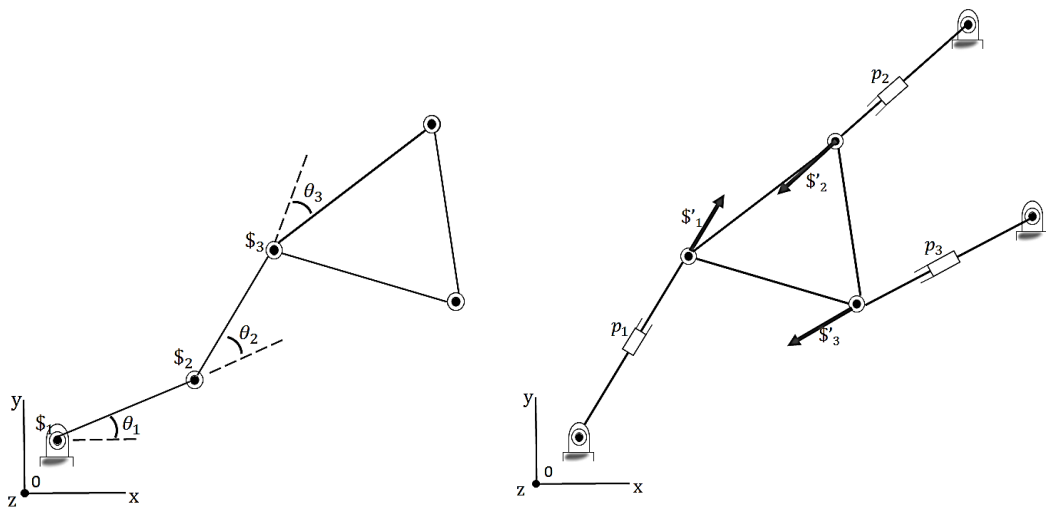


Figure 2.10: (a) 3R Planar Serial

(b) 3(RPR) Planar Parallel

To guarantee maximum generality, the Jacobian derived will be in terms of the orthogonal components of the distances between the origin of the reference coordinates and the screw direction lines as shown in Figure 2.11.

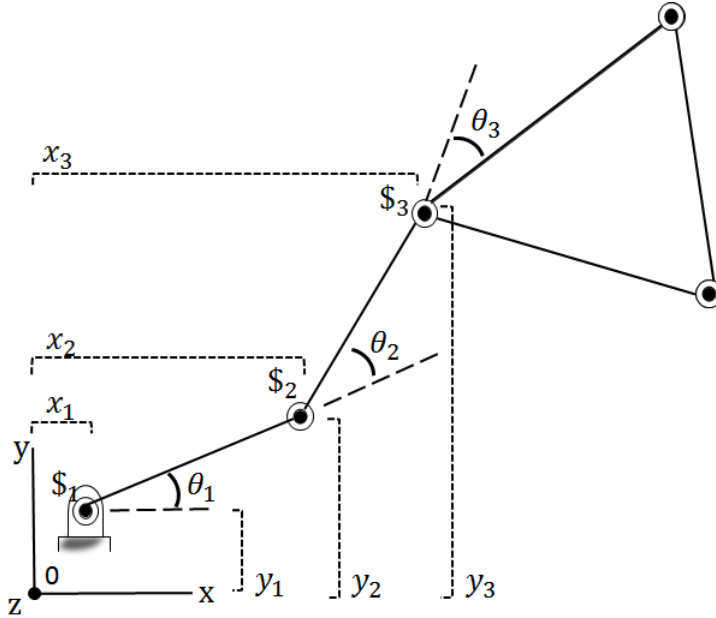


Figure 2.11: Planar 3R Manipulator Assigned with Screw Representation Parameters

The Jacobian with respect to the reference frame for the manipulator given in Figure 2.11 (a) is given by equation (2.20).

$$J_3^0 = [ \$1^0 \ \$2^0 \ \$3^0 ] \quad (2.20)$$

Away from the formulation of the governing equations of forward kinematics; the Jacobian can be derived by considering each screw in the Jacobian as the result of a unit length angular velocity (if the screw represents velocity kinematics) or a unit length force (if the screw represents statics).

For the screw  $\$1^0 = [ \hat{s}_1 ; S_{o,1} ]$ ,  $\hat{s}_1$  is the direction of the screw  $\$1^0$ , i.e.  $\hat{s}_1 = [0, 0, 1]^T$ ; where the vector  $S_{o,1} = r_1^0 \times \hat{s}_1$  and will be evaluated by representing the vector  $r_1^0$  by

its screw symmetric matrix with consistence with the methodology provided by Tsai

$$\text{in [26] as: } S_{o,1} = r_1^0 \times \hat{s}_1 = R_1 * \hat{s}_1, \text{ where: } R_1 = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix}.$$

As the manipulator is planar,  $z_l = 0$  and the resultant screw  $\$1^0$  is given by:

$$\$1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ y_1 \\ -x_1 \\ 0 \end{bmatrix}, \text{ and similarly, } \$2^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ y_2 \\ -x_2 \\ 0 \end{bmatrix}, \$3^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ y_3 \\ -x_3 \\ 0 \end{bmatrix}. \text{ Where } x_1, y_1, x_2, y_2, x_3 \text{ and } y_3$$

are the orthogonal components of the vectors  $r_1, r_2$  and  $r_3$  respectively.

Now we can construct the Jacobian of the planar 3R manipulator by combining the screws using equation (2.20) as follows:

$$J_3^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} L \\ M \\ N \\ P \\ Q \\ R \end{matrix} \quad (2.21)$$

The Jacobian matrix given in equation (2.21) is not square, but it can be reduced to square by deleting the zero rows, i.e. the rows of  $L, M$  and  $R$ ; without forgetting what they represent in future.

The reduced Jacobian will take the form:

$$J_3^0 = \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \end{bmatrix} \begin{matrix} N \\ P \\ Q \end{matrix} \quad (2.22)$$



The transpose inverse of the Jacobian given in equation (2.22) can be expressed as:

$$(J_3^{0T})^{-1} = \frac{\begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ y_3 - y_2 & y_1 - y_3 & y_2 - y_1 \end{bmatrix}}{\det J_3^0} \quad (2.23)$$

Where  $\det J_3^0 = x_1 y_2 - x_2 y_1 + x_1 y_3 + x_3 y_1 + x_2 y_3 - x_3 y_2$ .

Now, we must conduct similar work on the manipulator given in Figure 2.10 (b) in order to demonstrate duality; the Jacobian of the 3(RPR) parallel planar will be derived in terms of the orthogonal components of the distances between the origin of the reference coordinates and the screw direction lines as shown in Figure 2.12.

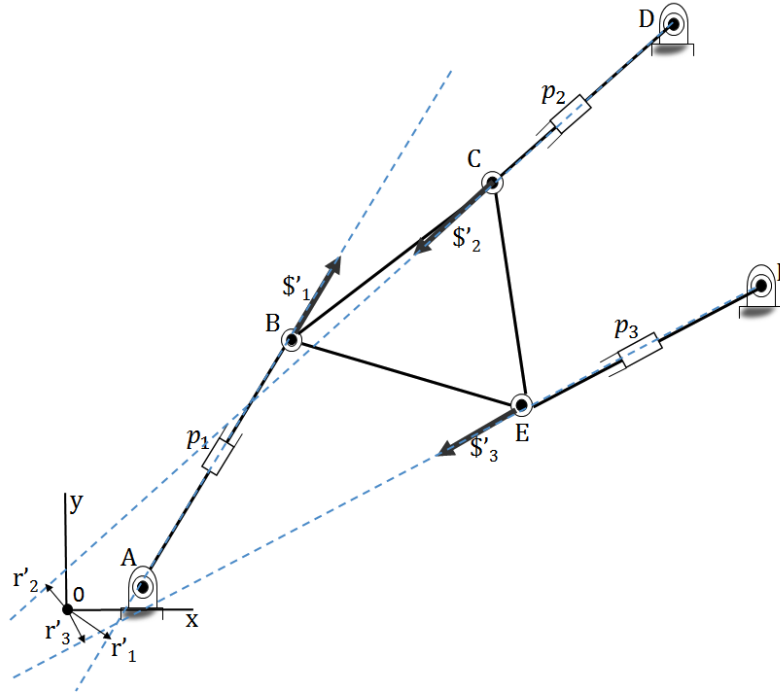


Figure 2.12: Planar 3(RPR) Manipulator Assigned with Screw Representation Parameters

For the screw  $\$s'_1 = [\hat{s}'_1{}^T ; S'_{o,1}{}^T]$ ,  $\hat{s}'_1$  is the direction of the screw  $\$s'_1$ , i.e.  $\hat{s}'_1 = [x_B - x_A, y_B - y_A, 0]^T / \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$ ; Where the vector  $S'_{o,1} = r'_1 \times \hat{s}'_1$

and will be evaluated by representing the vector  $r'_1$  by its screw symmetric matrix

$$\text{as } S'_{o,1} = r'_1 \times \hat{s}'_1 = R'_1 * \hat{s}'_1, \text{ where: } R'_1 = \begin{bmatrix} 0 & -z'_1 & y'_1 \\ z'_1 & 0 & -x'_1 \\ -y'_1 & x'_1 & 0 \end{bmatrix}.$$

If we express the values  $(x_B - x_A)/\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$  and  $(y_B - y_A)/\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$  by  $C_{1,1}$  and  $C_{1,2}$  respectively; the screw  $\$'_1$  can be

$$\text{expressed as: } \$'_1 = \begin{bmatrix} C_{1,1} \\ C_{1,2} \\ 0 \\ 0 \\ 0 \\ x'_1 C_{1,2} - y'_1 C_{1,1} \end{bmatrix}, \text{ and similarly: the screws } \$'_2 \text{ and } \$'_3 \text{ are given}$$

$$\text{by: } \$'_2 = \begin{bmatrix} C_{2,1} \\ C_{2,2} \\ 0 \\ 0 \\ 0 \\ x'_2 C_{2,2} - y'_2 C_{2,1} \end{bmatrix}, \$'_3 = \begin{bmatrix} C_{3,1} \\ C_{3,2} \\ 0 \\ 0 \\ 0 \\ x'_3 C_{3,2} - y'_3 C_{3,1} \end{bmatrix}.$$

Where:  $C_{2,1}$ ,  $C_{2,2}$ ,  $C_{3,1}$  and  $C_{3,2}$  represent  $(x_C - x_D)/\sqrt{(x_C - x_D)^2 + (y_C - y_D)^2}$ ,  $(y_C - y_D)/\sqrt{(x_C - x_D)^2 + (y_C - y_D)^2}$ ,  $(x_E - x_F)/\sqrt{(x_E - x_F)^2 + (y_E - y_F)^2}$ ,  $(y_E - y_F)/\sqrt{(x_E - x_F)^2 + (y_E - y_F)^2}$  respectively. And  $x'_2$ ,  $y'_2$ ,  $x'_3$  and  $y'_3$  are the orthogonal components of the distance vectors  $r'_2$  and  $r'_3$  respectively.

Now we can construct the Jacobian of the planar 3(RPR) manipulator by combining the screws using equation (2.20) as follows:

$$J_3^0 = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ x'_1 C_{1,2} - y'_1 C_{1,1} & x'_2 C_{2,2} - y'_2 C_{2,1} & x'_3 C_{3,2} - y'_3 C_{3,1} \end{bmatrix} \begin{matrix} L \\ M \\ N \\ P \\ Q \\ R \end{matrix} \quad (2.24)$$

Again, to obtain a square Jacobian, we may delete the zero rows from the Jacobian in equation (2.24), i.e. the rows in  $N, P$  and  $Q$  to obtain a squared Jacobian as given in equation (2.25).

$$J_3^0 \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ x'_1 C_{1,2} - y'_1 C_{1,1} & x'_2 C_{2,2} - y'_2 C_{2,1} & x'_3 C_{3,2} - y'_3 C_{3,1} \end{bmatrix} \begin{matrix} L \\ M \\ R \end{matrix} \quad (2.25)$$

The Jacobian given in equation (2.25) may be represented in Axis coordinate representation of the screw as given in 2.5.1 as:

$$J_3^0 = \begin{bmatrix} x'_1 C_{1,2} - y'_1 C_{1,1} & x'_2 C_{2,2} - y'_2 C_{2,1} & x'_3 C_{3,2} - y'_3 C_{3,1} \\ C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \end{bmatrix} \begin{matrix} R \\ L \\ M \end{matrix} \quad (2.26)$$

The similarity between the Jacobian found in equation (2.26) with the transposed version of the inverse Jacobian in equation (2.23) is an example of the dualities between serial and parallel manipulators summarized in Table 2.3.

Further illustration for dual to equivalent manipulators will be conducted in Chapter 4 for the 3R serial and 3(RPR) parallel planar manipulators.

### **2.5.5 Geometrical Meaning of the Inverse of Jacobian, Serial to Parallel Actuation [9-10]:**

As we have seen in the previous section, the Jacobian of dual serial and parallel manipulators were assigned with similarities, these similarities between one of the Jacobians and the transpose inverse of the other are considered to be an analytical proof of the duality between serial and parallel manipulators.

In this section we discuss the geometrical meaning of the inverse of the Jacobian matrix, and the way to generate dual manipulators by investigating the rows of the inverse. Moreover an example will be provided in extracting and reconstructing the screws from the inverse Jacobian geometrically.

It has been shown by Davidson, & Hunt in [10] that using the typical inverse of the Jacobian by means of transposing the cofactor matrix and dividing by the determinant, leads to the same condition of reciprocity given for reciprocal screws between each column of the cofactor matrix and all the columns of the Jacobian except one, i.e. the rows of the inverse will be reciprocal to the columns of a given Jacobian.

Further investigations on the validity and applicability of inverting the Jacobian matrix to construct dual manipulators will be discussed in Chapter 3.

For illustration, a reconstruction for the directions of the screws in the result given in equation (2.23) of the inverse Jacobian of planar 3R serial manipulator is shown in Figure 2.13.

The orthogonal components on  $x$  and  $y$  coordinates of the reconstructed screws will be the ones in the second and third row respectively according to the procedure given by Davidson and Hunt in [10].

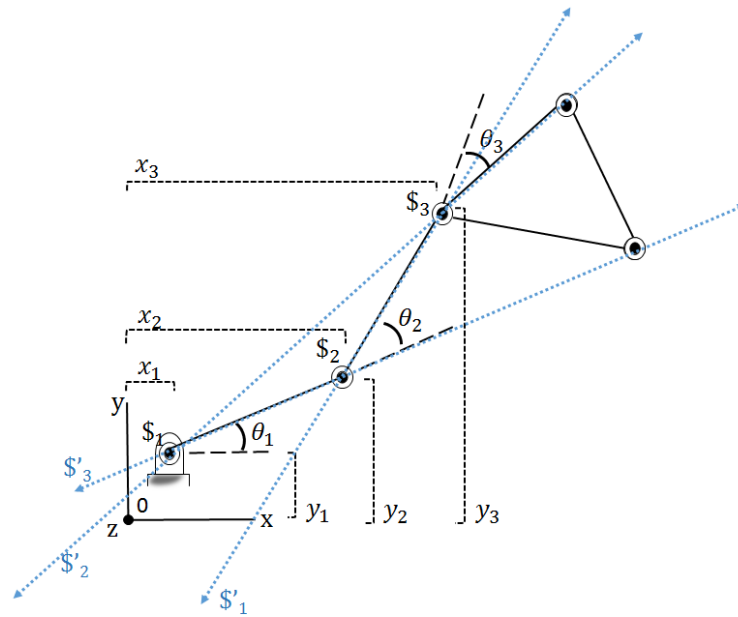


Figure 2.13: Original with Reconstructed Reciprocal Screws for 3R Manipulator

As we can see in Figure 2.13, the reconstructed screws are pairwise reciprocal, i.e.  $\$'_1$  is reciprocal to  $\$2$  and  $\$3$ ,  $\$'_2$  is reciprocal to  $\$1$  and  $\$3$  and  $\$'_3$  is reciprocal to  $\$1$  and  $\$2$ .

Hunt et al. in [10], discussed in the literature that dual manipulators are actually instantaneously equivalent, furthermore, mapping some of the workspace for one of them to the other we can have equivalent parallel actuated mechanism given a serial mechanism as vice versa. This result was only generalized for manipulators possessing a square Jacobian, i.e. 6 DOF general purpose manipulators. Further work will be conducted in turning serial manipulators into their equivalent hybrid manipulators in terms of the generalized reciprocity in Chapter 3 and Chapter 4 further in this text.

## 2.6 Duality in Mechanical Systems Using Graph Representation:

As it has been shown in section 2.3.1, there is a direct graphical approach in finding dualities in electrical circuits. Where also it has been shown in section 2.4 that there is

a remarkable analogy between electrical and mechanical quantities. Hence, a duality in mechanical systems using graphical procedures is not out of thought.

Shai et al. in [11-12], established a systematic approach to find dual mechanical systems by extending the theory of graph to represent kinematics and statics of some mechanical structures. Two graph representation were introduced, namely: flow line graph representation and potential line graph representation referred to as FLGR and PLGR respectively. Potential Line Graph is used to represent kinematics, while Flow Line Graph is used to represent statics.

These graphs can be used to give a direct graphical procedure in finding some dual serial / parallel manipulators in terms of duality in kinematics and statics.

The procedure of constructing graph representations for kinematic and static systems by means of these graphs is given in the following section.

### **2.6.1 Construction of Flow Line and Potential Line Representations [11]:**

Construction of Flow Line Graph for the statics of a parallel manipulator is given by the procedure:

- 1- A vertex is placed to represent each platform in a parallel manipulator (the moving laminas), and a ground vertex to represent the fixed base.
- 2- An edge is placed to represent a limb or an external force as follows:
  - Passive edge to represent the actuated or non-actuated limb.
  - Active edge that carries a flow source represents a force, directed from the ground vertex towards the non-reference vertex for an external force and vice versa for an internal force.

- 3- The linear component carried by each edge is a force, where the angular component carried by the edge is a moment.

While the Construction of Potential Line Graph for the kinematics of a serial manipulator is given by:

- 1- A vertex is placed to represent each link in a serial manipulator, and a ground vertex to represent the fixed base.
- 2- An edge is placed to represent a joint (kinematic pair) or an external velocity as follows:
  - Passive edge to represent an actuated joint (all joints of a serial manipulator are actuated, in general).
  - Active edge that carries a potential source represents an angular velocity, directed from the ground vertex towards the non-reference vertex for an external velocity and vice versa for an internal velocity.
- 3- The linear component carried by each edge is angular velocity that is measured relatively between the incident vertices.

For demonstration, the graphs shown in Figure 2.14 (b), (d) are proper isomorphic versions of the FLGR of the Stewart platform and the PLGR of the General Lobster arm shown in Figure 2.14 (a) and (c) respectively.

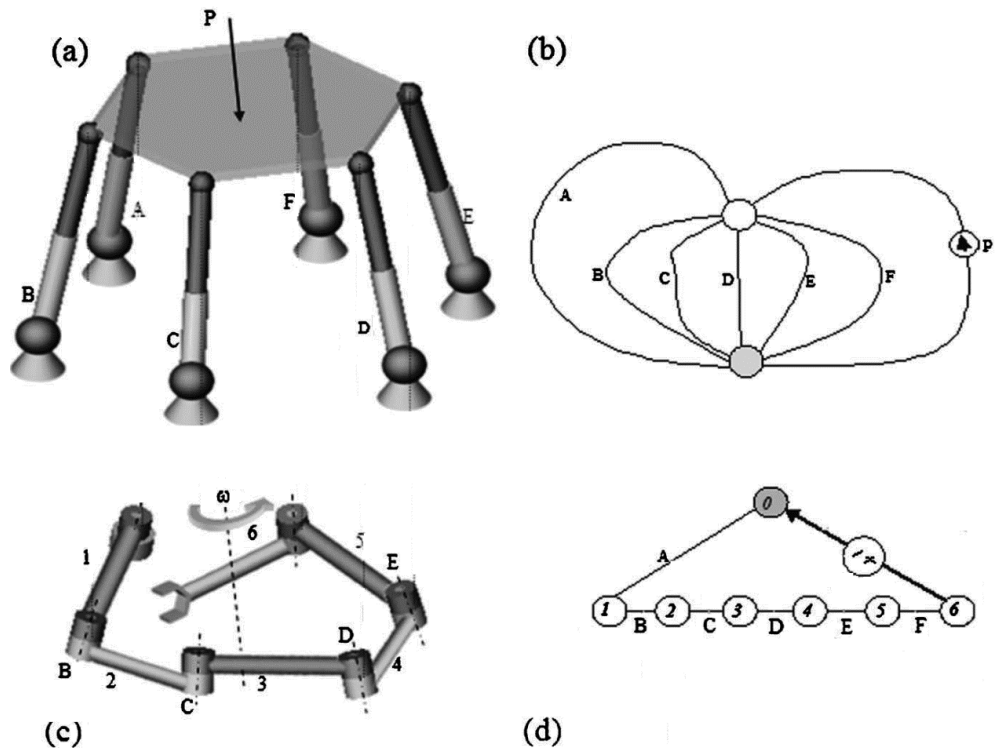


Figure 2.14: (a) Stewart Platform (b) FLGR of Stewart Platform  
 (c) General Lobster Arm (d) PLGR of the Generalized Lobster Arm [11]

The duality between statics and kinematics using PLGR and FLGR is conducted by building the corresponding graph representation ( $G$ ) for a given serial or parallel manipulator, i.e. FLGR or PLGR respectively, then finding the dual graph ( $G^d$ ) using the same graphical algorithm provided in section 2.3.1 for electrical circuits by replacing analogous pairs current / voltage sources by flow / potential sources; the corresponding manipulator for the dual graph ( $G^d$ ) will be the dual manipulator for the original manipulator represented by ( $G$ ).

A demonstration for the aforementioned graphical based method is shown in Figure 2.15.



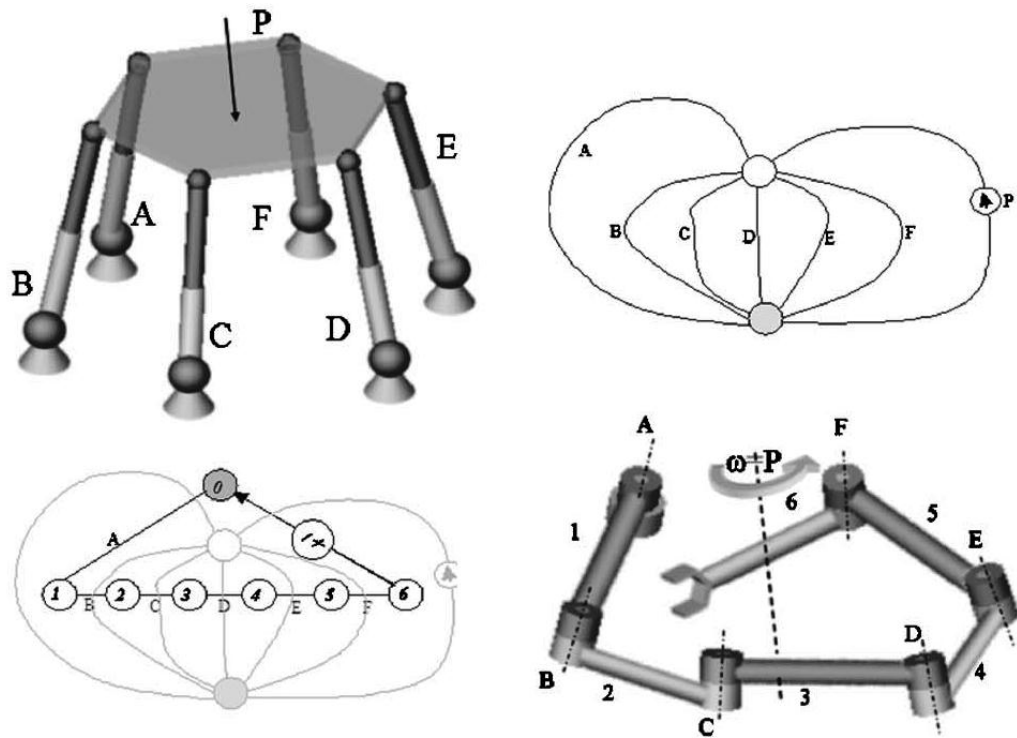


Figure 2.15: Demonstration of Finding Serial / Parallel Manipulators Using FLGR and PLGR [11]

## 2.7 Denavit-Hartenberg Convention:

Denavit-Hartenberg convention (DH) is one effective tool representing sequential coordinates of robotic manipulators. This convention facilitate the procedure of formulating forward and inverse kinematic equations for robotic manipulators in general, especially when the manipulators are fully in-serial.

Assigning DH coordinates has to be done via a systematic approach that relates between each two sequential joints by a matrix known by DH matrix, this mapping can be easily done after assigning proper DH coordinates using the following procedure [28]: (for simplicity we will use the terms old and new for each two sequential coordinate frames respectively) [28]:

- The convention starts with assigning z axis for each revolute joint about its rotational direction, and for each prismatic joint along its sliding direction.

- Then assigning x axes takes place regarding the following cases:
  - For the first joint (usually the base), the direction of its x axis is optional as long as it is orthogonal on the direction of its z axis.
  - For the remaining joints, the x axis should lie on the common normal between each two non-coincident z axes, or along the normal on the plane of two coincident z axes.
- Assigning y axes is accomplished using the common notation known by the “*right hand rule*” for orthogonal coordinates.
- Last frame coordinates (end effector) is taken parallel to its previous at the point of observation (the center) of the end effector.

Assigning DH coordinates characterizes each link with the following four parameters:

- $\theta_i$  represents the angle between two x axes measured about the old z axis, following the direction x old to x new.
- $d_i$  represents the distance between two sequential frames’ origins measured along the direction of the old z axis.
- $\alpha_i$  represents the angle between two sequential z axes measures about the new x axis following the direction z old to z new.
- $a_i$  represents the distance between two sequential frames’ origins measured along the direction of the new x axis.

For illustration, representation of the coordinates of joint (i +1) with respect to the coordinates of joint (i) is shown in Figure 2.16.

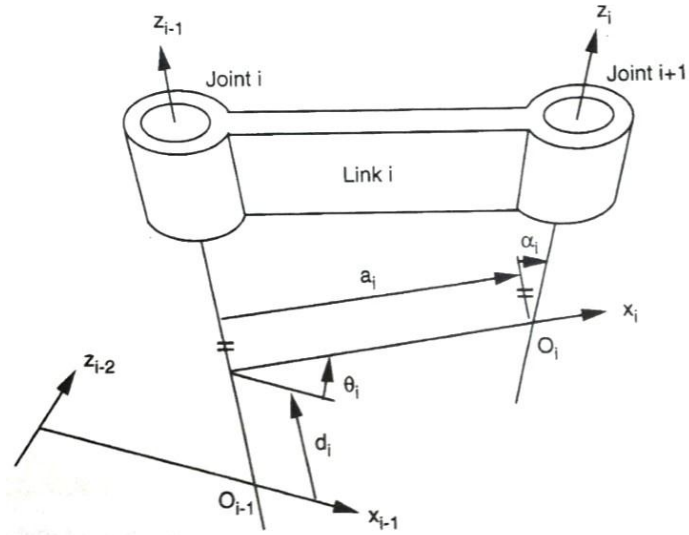


Figure 2.16: Assignment of DH Joint Coordinates and Link Parameters [26]

Substituting DH link parameters for each frame in the DH matrix given in equation (2.27) gives the matrix representation of the new frame with respect to the old frame for each sequential frames.

$$A_i^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.27)$$

For  $n$  degree of freedom serial manipulators, the matrix representation of the last frame with respect to the base frame is the usual matrix product between the matrices representing sequential coordinate frames as in equation (2.28).

$$T_n^0 = A_1^0 * A_2^1 * \dots * A_n^{n-1} \quad (2.28)$$

## Chapter 3

# GENERALIZED RECIPROCITY AND SERIAL TO HYBRID ACTUATION

### 3.1 Introduction:

It has been shown, in Chapter 2, that duality between parallel and serial manipulators is originated to the orthogonality (or reciprocity) between wrench and twist coordinates of statics and kinematics respectively, provided by the theory of screws. Moreover, this duality transformation was based on finding the inverse of the screw based Jacobian matrix of a given manipulator and reconstructing the resultant screws.

The method of inverting the Jacobian was confined with the existence of a square or reducible to square Jacobians [9-10]. Yet, Hunt et al. in [10], performed one example on inverting a deficient Jacobian by adding what he named by *Dummy* columns to the columns of the deficient Jacobian in order to invert it. The method of adding Dummy columns was done by inspection using the experience in the geometry of mechanics, and it was not generalized.

Other researchers such as Dai et al. in [33], provided a linear algebraic procedure in finding the reciprocal screws for a set of  $n$  screws by what he named *Augmenting and Shifting* the Jacobian matrix, these reciprocal screws are from the form reciprocal to all cannot be used to express duality between parallel and serial manipulators, yet they can be used to construct a basis for mobility analysis by observing the null space represented by these reciprocals.

In the following section, the reciprocity leading to duality will be further investigated by formulating a new general matrix equation and discussing its solutions.

### **3.2 Generalized Reciprocity by Means of Linear Algebra:**

The first contribution in the theory of screws was established in its version for statics by Poincot (1806), while the kinematic version was established by Chasles (1832). Robert Ball in (1873) combined and extended both versions in the nowadays known as screw theory [10].

What is worth mentioning to introduce and justify the work done and discussed in the current section, is that the foundation of what is known as *matrix* was not conceived till early 1850's. Indeed, the book written by Sir Robert Ball in (1900) named “*A Treatise on the Theory of Screws*” [15], did not mention the word matrix once. Yet, all the theorems and discussions were expressed in terms of the geometry of mechanics without the addressing what is known as the four fundamental subspaces assigned with the matrices representing linear systems.

In this work, the formulation and solutions of the linear system governing the problem of reciprocal screws is discussed in detail with an example. This formulation in despite of its simplicity is nowhere discussed before in the literature and is considered as an original contribution of this thesis.

Consider the  $n$  screw system given by the set of screws  $\$ = \{\$, \$_2 \dots \$_n\}$ . The problem of finding a set of screws  $\$' = \{\$, \$'_2 \dots \$'_n\}$  in such a way that each screw in  $\$'$  should be reciprocal to all screws in  $\$$  except one, can be expressed as next:

Say  $\$'_1$  is desired to be reciprocal to the screws  $\$2 \dots \$n$  while not reciprocal to  $\$1$ ; the system can be constructed by generalizing equation (2.14) as:

$$\begin{bmatrix} L_1 & M_1 & N_1 & P_1 & Q_1 & R_1 \\ L_2 & M_2 & N_2 & P_2 & Q_2 & R_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_n & M_n & N_n & P_n & Q_n & R_n \end{bmatrix} * \begin{bmatrix} P'_1 \\ Q'_1 \\ R'_1 \\ L'_1 \\ M'_1 \\ N'_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.1)$$

Where  $X_1$  can take any value except of zero in order not to have  $\$1$  and  $\$'_1$  reciprocal to each other.

Similarly, say  $\$'_2$  is desired to be reciprocal to the screws  $\$1, \$3 \dots \$n$  while not reciprocal to  $\$2$ ; the linear system for this problem is given by:

$$\begin{bmatrix} L_1 & M_1 & N_1 & P_1 & Q_1 & R_1 \\ L_2 & M_2 & N_2 & P_2 & Q_2 & R_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_n & M_n & N_n & P_n & Q_n & R_n \end{bmatrix} * \begin{bmatrix} P'_2 \\ Q'_2 \\ R'_2 \\ L'_2 \\ M'_2 \\ N'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ X_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.2)$$

Continuing this formulation by induction, then combining the equations leads to:

$$\begin{bmatrix} L_1 & M_1 & N_1 & P_1 & Q_1 & R_1 \\ L_2 & M_2 & N_2 & P_2 & Q_2 & R_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_n & M_n & N_n & P_n & Q_n & R_n \end{bmatrix} * \begin{bmatrix} P'_1 & P'_2 & P'_n \\ Q'_1 & Q'_2 & Q'_n \\ R'_1 & R'_2 & R'_n \\ L'_1 & L'_2 & \dots & L'_n \\ M'_1 & M'_2 & \dots & M'_n \\ N'_1 & N'_2 & \dots & N'_n \end{bmatrix} = \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & X_n \end{bmatrix} \quad (3.3)$$

It is obvious that the most left matrix in equation (3.3) is the transpose of the Jacobian matrix discussed in 2.5.4 expressed in Ray coordinate representation of screws and will be abbreviated by  $J^T$ , while the matrix of reciprocal screws is similar to a Jacobian

expressed in Axis coordinate representation of screws and will be abbreviated by  $J^r$ , where the screws in  $J^r$  does not possess unit lengths in this general case.

We can also discuss that the scalars  $X_1, X_2 \dots X_n$  can take random values except of zero, if we are interested in the direction lines of the reciprocal screws, where their values will effect only the lengths and not related to directions if all were selected positively. For simplicity all these scalars will be chosen to take the value  $X_1 = X_2 = \dots = X_n = 1$ .

Equation (3.3) now can be simplified to:

$$J^T * J^r = I \quad (3.4)$$

Where  $I$  is the unity  $n \times n$  matrix.

Equation (3.4) is known as the generalized matrix equation in linear algebra [32], where the generalized matrix equation with its solutions are given by equations (3.5) and (3.6) respectively.

$$A_{m \times n} * X_{n \times k} = B_{m \times k} \quad (3.5)$$

$$X_{n \times k} = A^+ * B + (I - A^+ * A) * Y, \text{ where } Y \in R^{n \times k} \text{ is arbitrary} \quad (3.6)$$

Now, the set of solutions for the non-normalized screws in  $J^r$  can be found by:

$$J^r = J^{T+} + (I - J^{T+} J^T) * Y, \text{ where } Y \in R^{n \times k} \text{ is arbitrary} \quad (3.7)$$

In the case  $(I - J^{T+} J^T) \neq 0$ , i.e.  $J^T$  has a non-empty null space, the values  $Y$  can take are restricted by the first assumption that the set  $\$' = \{ \$'_1, \$'_2 \dots \$'_n \}$  is a set of proper screws expressed in Axis coordinate representation.

In the special case of  $J$  is  $6 \times 6$  and invertible, the general solution given in equation (3.7) will be an alternative analytical proof of what was provided before in the literature for the geometrical meaning of the inverse of Jacobian provided in section 2.5.5.

Where the case that  $J$  is not  $6 \times 6$  will be further discussed in the following example:

Consider the 3R planar serial manipulator given in Figure 10 (a), the  $6 \times 3$  screw based Jacobian of the manipulator as given by equation (2.21) will be used to discuss the general matrix equation solutions of reciprocal screws given in equation (3.7) as:

$$J^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 0 & 0 & 0 \end{bmatrix}^{T^+} + \left( I_{6 \times 6} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 0 & 0 & 0 \end{bmatrix}^{T^+} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \\ 0 & 0 & 0 \end{bmatrix}^T \right) * Y$$

This solution can be simplified to:

$$J^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ y_3 - y_2 & y_1 - y_3 & y_2 - y_1 \\ 0 & 0 & 0 \end{bmatrix} / D + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} * Y, \quad (3.8)$$

Where:  $D = x_1y_2 - x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2$ .

Depending on  $Y$  value, we may have infinitely many options for constructing reciprocal screws.

For illustration, consider the planar 3R manipulator assigned with the constructed reciprocal screws in Figure 2.12, take  $\$'_1$  as an example, we can have another choice



for a screw that is reciprocal to  $\$2$  and  $\$3$  keeping the restriction that it is not reciprocal to  $\$1$  as shown in Figure 3.1, there exist a plane that all directions of the screws contained in it, can actually be reciprocal to  $\$2$  and  $\$3$  without being reciprocal to  $\$1$ ; this result is imbedded in the last column of the matrix that represents the null space of the system of reciprocal screws, i.e. the most right matrix in equation (3.8) .

So far, we discussed that constructed reciprocal screws are screws of pure force, while other combinations from the null space of the system of reciprocal screws give a possibility to apply a moment, in the direction of the constructed screws that is reciprocal to all original screws without breaking the aforementioned condition for pairwise reciprocity.

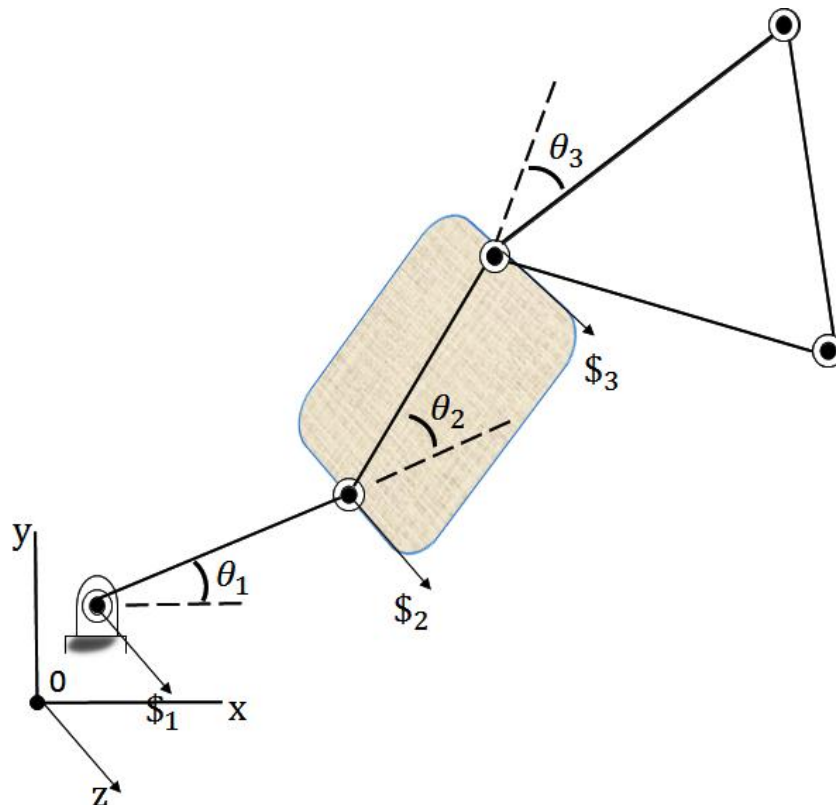


Figure 3.1: Plane of Reciprocal Screws on  $\$2$  and  $\$3$  not Reciprocal to  $\$1$

Combinations from the first two columns of the null space matrix given in equation (3.8) should be restricted to the form  $h*S$  which represents the depended translation or moment if the screw is representing kinematics or statics respectively, i.e. the values  $Y$  can take are not completely random.

The screws that have combinations from the type  $h*S$  represent joints that are called screw joints. In parallel actuated devices, the limbs correspond to screws that have such combinations are called wrench applicators, while the ones that have only translational or rotational movements are called force applicators and hinge applicators respectively [10].

Combining the obtained results, we can say that the parallel dual manipulator for the 3R serial planar can have limbs that are not restricted to just being force applicators, but they might be wrench applicators.

### **3.3 Geometrical Meaning of Pseudo-Inverse of the Jacobian:**

In this section, the geometrical meaning of the pseudo inverse of the Jacobian matrix will be discussed for the first time in consistence with the work done for the generalized matrix equation of reciprocal screws.

Initially, we will discuss two concepts: the screw that is reciprocal to itself and the pseudo inverse in abstract mathematics.

- A screw that represent pure force or pure couple, or there analogous quantities in velocity kinematics is said to be self-reciprocal (self-dual), i.e. representing the screw itself in both Ray and Axis coordinate representations, then performing Euclidean product between these representations will give zero as an outcome. On the other hand,

the screws that represent helical movements, i.e. screws with non-zero nor infinity pitches, are not self-reciprocal or self-dual [10].

- The common linear system  $Ax = b$ , assuming that it is solvable, i.e. the vector  $b$  belongs to the range of  $A$ , have only one solution that is pure from any linear combinations of the vectors of its null space, and this solution is given by  $A^+b$ . On the other hand, if the system  $Ax = b$  is not solvable, i.e. the vector  $b$  does not belong to the range of  $A$ , the vector given by  $A^+b$  is the solution for what is commonly known as the least square errors problem [32]. In this manner, we can conclude that using the pseudo inverse of the Jacobian to express reciprocity as shown early in this chapter, can find directions of screws that either are reciprocal to the screws in the original Jacobian, pure from all non-necessary combinations from the null space of the solution, or a set of screws that are close to reciprocal in terms of minimum square errors.

Consider a twist (screw in kinematics) that represents a pure rotation about the x axis of the reference frame at zero distance from the origin, i.e. a unit screw with Ray representation given as  $\$ = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T$ , the pseudo inverse of  $\$^T$  is given by  $\$^+ = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T$  and it should be thought of in Axis coordinates. Simply, we can re-represent  $\$^+$  in Ray coordinates as  $\$^+ = [0 \ 0 \ 0 \ 1 \ 0 \ 0]^T$ . Indeed,  $\$^+$  is the self-dual screw of  $\$$  pure from any linear combination of the directions of the other 4 screws that are reciprocal to  $\$$ .

### **3.4 Serial to Hybrid Actuation by Inverting Subsections:**

It has been shown early in section 2.5.5, that serial to parallel actuation and vice versa can be conceived by means of reciprocal screws for 6 DOF general purpose manipulators.

Consider the elbow manipulator shown in Figure 3.2, it is a serial manipulator with 6 revolute joints, hence 6 DOF.

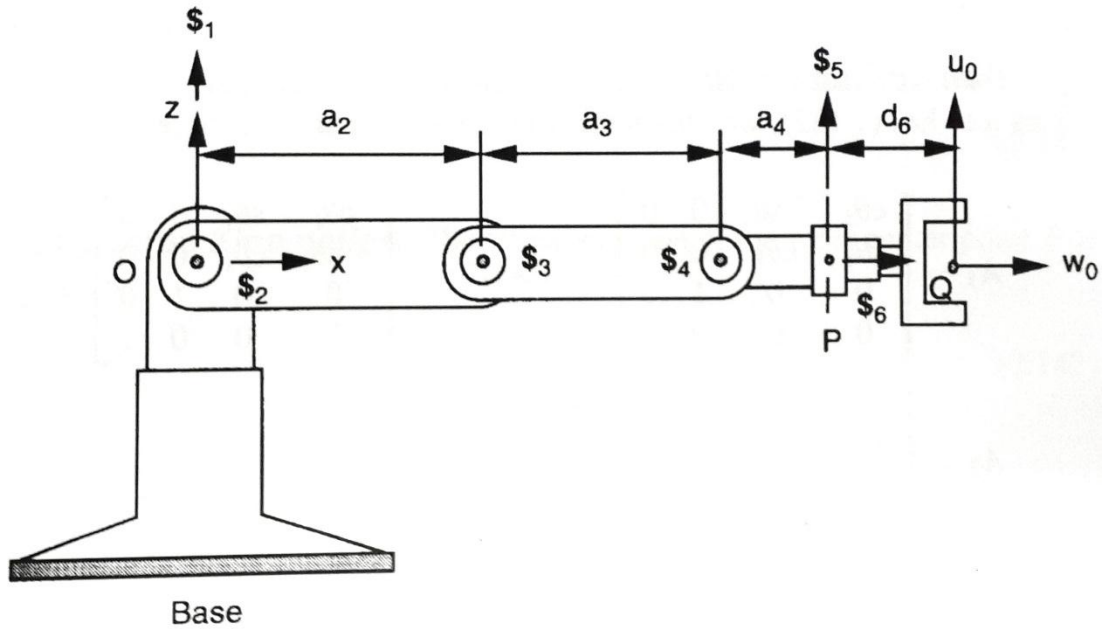


Figure 3.2: Elbow Manipulator Assigned with Screw Coordinates [26]

The screw based Jacobian of the elbow manipulator will be constructed; to simplify this work with a more systematic procedure, we will use the algorithm given by Tsai [26] for the screw based Jacobians, together with using Denavit-Hartenberg convention as in the following.

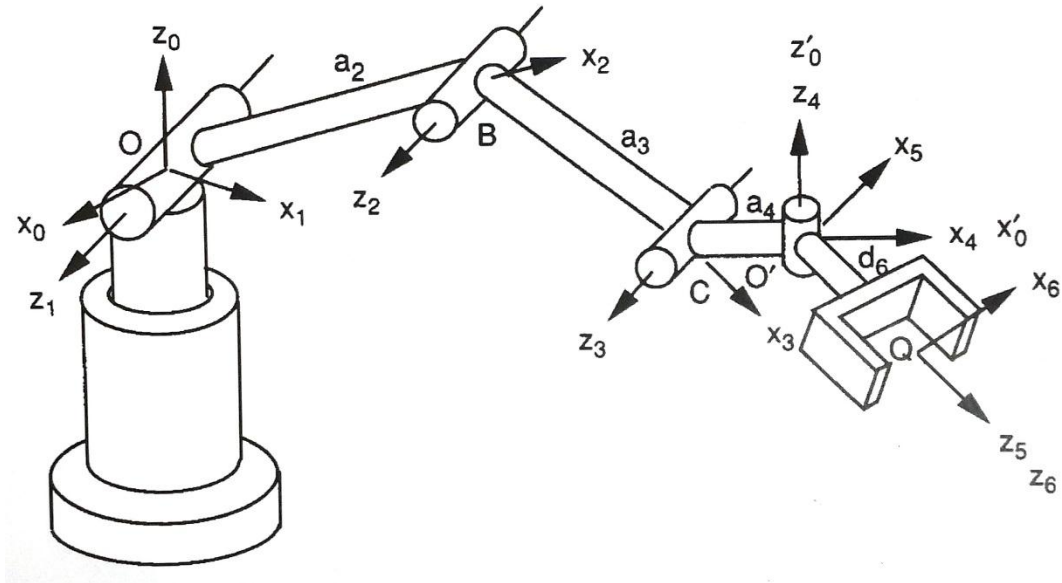


Figure 3.3: Elbow Manipulator Assigned with DH Coordinates [26]

The DH parameters assigned with the coordinates represented in Figure 3.3 are given in Table 3.1.

Table 3.1: DH Parameters for the Elbow Manipulator

Link / DH Parameter	$\theta_i$	$d_i$	$\alpha_i$	$a_i$
1	$\theta_1$	0	$\pi/2$	0
2	$\theta_2$	0	0	$a_2$
3	$\theta_3$	0	0	$a_3$
4	$\theta_4$	0	$-\pi/2$	$a_4$
5	$\theta_5$	0	$\pi/2$	0
6	$\theta_6$	$d_6$	0	0

The correspondent matrices that connect between the sequential coordinates of the links of the elbow manipulator using DH convention are given by:

$$A_1^0 = \begin{bmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ \sin \theta_1 & 0 & -\cos \theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2^1 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & a_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & a_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^2 = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & a_3 \sin \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & a_3 \cos \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4^3 = \begin{bmatrix} \cos \theta_4 & 0 & -\sin \theta_4 & a_4 \cos \theta_4 \\ \sin \theta_4 & 0 & \cos \theta_4 & a_4 \sin \theta_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4^3 = \begin{bmatrix} \cos \theta_5 & 0 & \sin \theta_5 & 0 \\ \sin \theta_5 & 0 & -\cos \theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6^5 = \begin{bmatrix} \cos \theta_6 & -\sin \theta_6 & 0 & 0 \\ \sin \theta_6 & \cos \theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding Jacobian of the elbow manipulator is given by equation (3.9). (The derivation of the screw based Jacobian by implementing Tsai algorithm can be found in Appendix (A).)

$$J^0 = \begin{bmatrix} 0 & s_1 & s_1 & s_1 & -s_{234}c_1 & c_5s_1 + c_{234}c_1s_5 \\ 0 & -c_1 & -c_1 & -c_1 & -s_{234}s_1 & c_{234}s_1s_5 - c_1c_5 \\ 1 & 0 & 0 & 0 & c_{234} & s_{234}s_5 \\ 0 & 0 & a_2c_1c_2 & c_1(a_3c_{23} + a_2c_2) & c_1(a_3c_{23} + a_2c_2) + a_4c_{234}c_1 & c_1(a_3c_{23} + a_2c_2) + a_4c_{234}c_1 \\ 0 & 0 & a_2c_2s_1 & s_1(a_3c_{23} + a_2c_2) & s_1(a_3c_{23} + a_2c_2) + a_4c_{234}s_1 & s_1(a_3c_{23} + a_2c_2) + a_4c_{234}s_1 \\ 0 & 0 & a_2s_2 & a_3s_{23} + a_2s_2 & a_3s_{23} + a_2s_2 + a_4s_{234} & a_3s_{23} + a_2s_2 + a_4s_{234} \end{bmatrix} \quad (3.9)$$

Where the  $s$  and  $c$  are abbreviations for  $\sin ()$  and  $\cos ()$  functions respectively, and the subscripts used below represent the sum of the angles of the corresponding subscripts, e.g.  $s_{234}$  represents  $\sin (\theta_2 + \theta_3 + \theta_3)$ .

To proceed further aiming to design an equivalent hybrid manipulator for the elbow manipulator by replacing a section, we may attempt to take a sequential section from the columns of the Jacobian, represent it in a matrix, then solve the generalized reciprocal system discussed early in this chapter.

Let the first three columns from  $J^0$  in equation (3.9) be represented in the matrix  $J_{1-3}$  as in equation (3.10).

$$J_{1-3} = \begin{bmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \\ 0 & 0 & a_2 c_1 c_2 \\ 0 & 0 & a_2 c_2 s_1 \\ 0 & 0 & a_2 s_2 \end{bmatrix} \quad (3.10)$$

The generalized matrix of reciprocal screws given in equation (3.4) has the following form and solution given in equations (3.11) and (3.12) respectively.

$$\begin{bmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \\ 0 & 0 & a_2 c_1 c_2 \\ 0 & 0 & a_2 c_2 s_1 \\ 0 & 0 & a_2 s_2 \end{bmatrix}^T * J_r^{1-3} = I \quad (3.11)$$

$$J_r^{1-3} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \\ 0 & -c_1 c_2 / a_2 & c_1 c_2 / a_2 \\ 0 & -s_1 c_2 / a_2 & s_1 c_2 / a_2 \\ 0 & -s_2 / a_2 & -s_2 / a_2 \end{bmatrix} \quad (3.12)$$

$$+ \begin{bmatrix} s_1^2 & s(2\theta_1)/2 & 0 & 0 & 0 & 0 \\ s(2\theta_1)/2 & c_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1^2 c_2^2 & s_1 c_1 c_2^2 & c_1 s_2 c_2 \\ 0 & 0 & 0 & s_1 c_1 c_2^2 & s_1^2 c_2^2 & s_1 s_2 c_2 \\ 0 & 0 & 0 & c_1 s_2 c_2 & s_1 s_2 c_2 & s_2^2 \end{bmatrix} * Y$$

Although the screws of the pseudo inverse of the Jacobian matrix  $J_r^{1-3}$  given in equation (2.12) are not normalized, yet it is possible to reconstruct the directions of the screw as follows.

Considering what the general solution of equation (3.4) represents, each of the columns of the transposed version of pseudo inverse Jacobian is a non-normalized screw represented in Axis coordinates. For consistency with the work done in [9-10], we will only reconstruct the screws from the solution given in equation (3.12) for the special

case  $Y = 0$ , other discussions for the general case will be similar to the work done in section 3.2.

The first column of  $J_{1-3}^{+T}$  is a pure moment (pure couple) about z axis, and will be represented by a hinge applicator. Where the second column represents a pure force along a vector direction that possesses orthogonal components on all of the x, y and z coordinates, yet it only intersects z axis (moment is zero about z axis). Finally, the last column represents a force applicator that is also possesses similar orthogonal components to its previous (symmetric), yet it intersects all of x, y and z axes (moment is zero about x, y and z axes). Figure 3.4, shows a proper sketch of the directions and the types of the constructed screws.

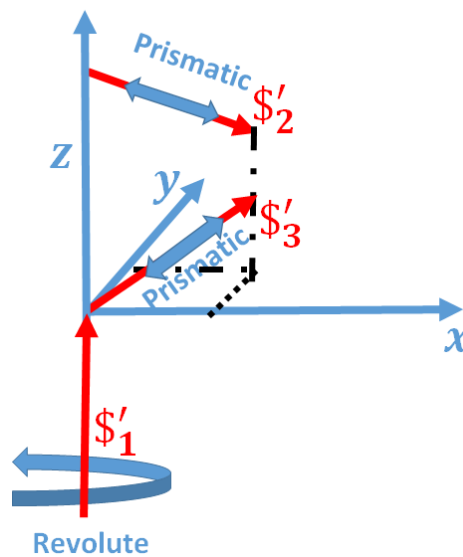


Figure 3.4: Sketch of the Constructed Reciprocal Screws from Pseudo Inverse of Jacobian  $J_r^{1-3}$

Another sequential section of the Jacobian  $J^0$  in equation (3.9) can be taken as the second, third and fourth columns of  $J^0$ , and will be represented in the matrix  $J_{2-4}$  as given in equation (3.13).



$$J_{2-4} = \begin{bmatrix} s_1 & s_1 & s_1 \\ -c_1 & -c_1 & -c_1 \\ 0 & 0 & 0 \\ 0 & a_2 c_1 c_2 & c_1 (a_3 c_{23} + a_2 c_2) \\ 0 & a_2 c_2 s_1 & s_1 (a_3 c_{23} + a_2 c_2) \\ 0 & a_2 s_2 & a_3 s_{23} + a_2 s_2 \end{bmatrix} \quad (3.13)$$

In equation (3.13),  $\theta_1$  plays the role of representing the screws with respect to the reference frame after a rotation about the base with an angle equal to  $\theta_1$ . For simplicity, we can substitute  $\theta_1$  with any possible value within its space limit, e.g. if  $\theta_1$  was substituted with the value  $\pi/2$ , the valued version for  $J_{2-4}$ , denoted by  $J_{2-4}^v$ , is given in equation (3.14).

$$J_{2-4}^v = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 c_2 & a_3 c_{23} + a_2 c_2 \\ 0 & a_2 s_2 & a_3 s_{23} + a_2 s_2 \end{bmatrix} \quad (3.14)$$

The matrix  $J_{2-4}^v$  given in equation (3.14) is very similar to the Jacobian matrix of a 3R serial planar manipulator given in equation (2.22). Indeed, the 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> joints of the elbow manipulator have the same structure of a 3R serial planar manipulator shown in Figure 2.10 (a). In this manner, the same results obtained previously in equation (3.8) with all discussions followed, are applicable to this case. Further discussions about a possible implementation of the aforementioned substructures will be carried on in Chapter 4.

## Chapter 4

# EQUIVALENCES IN ELECTRICAL AND MECHANICAL SYSTEMS

### 4.1 Introduction:

In many engineering applications, designers or analyzers tend to replace a section of a system with its equivalent, to improve certain characteristics of the overall system; or maybe to simplify analysis steps of the system without effecting input / output parameters. These equivalents are very popular in electrical systems and were vividly studied for one-phase and multi-phase systems [20-21] which is known by tear and reconstruction of circuits.

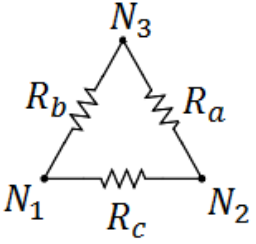
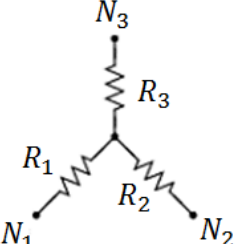
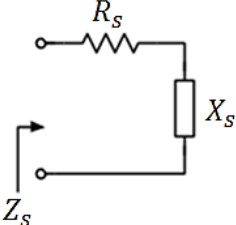
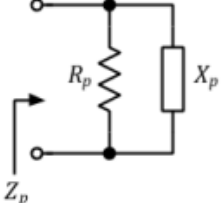
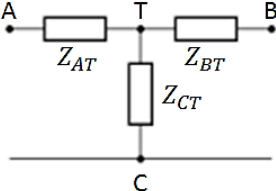
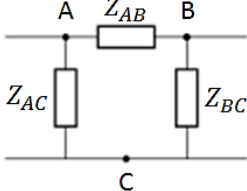
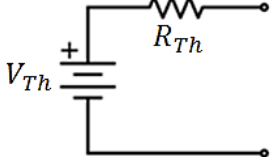
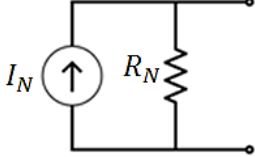
As has been studied in section 2.5.5 and will be discussed in this chapter, the dual mechanism of a given one can be found structurally equivalent under special conditions.

In this chapter, we seek to design structural equivalents for a serial subsection of a manipulator from its dual parallel structure, to improve the stiffness and accuracy of the serial manipulator by embedding a parallel actuated section, together with keeping some of the desired properties of the original serial manipulator. Running design improvements can simplify the work done on the contact between the end effector and the surrounding environment via the previous successful design of the serial manipulator.

One might think of a successful design of a fully in serial manipulator then try to locate the weaknesses in the given structure that might be further improved.

As initial study, we will demonstrate some validated equivalent subsystems for some robotic manipulators in the next section.

Table 4.1: Some Equivalent Pairs in Electrical Circuits

 <p style="text-align: center;">Delta Connection (<math>\Delta</math>)</p>	 <p style="text-align: center;">Star Connection (Y)</p>
 <p style="text-align: center;">Serial RX</p>	 <p style="text-align: center;">Parallel RX</p>
 <p style="text-align: center;">T Connection (T)</p>	 <p style="text-align: center;">Pi Connection (<math>\Pi</math>)</p>
 <p style="text-align: center;">Thévenin</p>	 <p style="text-align: center;">Norton</p>

Usually in electrical systems, equivalent pairs such as the pairs given in Table 4.1, are used to replace a section (subsystem) of the overall circuit. In this work, we try to adapt the aforementioned concept into its mechanically analog, i.e. replacing a subsystem of the mechanical structure in general and a robotic manipulator in particular is desired to improve or inherit the overall structure with some specific characteristics.

#### **4.2 Equivalent 6 SPS and 6 SRS Stewart Platforms:**

The typical 6 DOF Stewart platform consist of a base, a moving platform (lamina) and 6 actuated limbs. These limbs are usually from the form SPS (spherical - prismatic – spherical), where in each limb the spherical joints are passive and the prismatic joint is actuated.

Usually, the implementation of prismatic joints is found to be more difficult than revolute ones [26]. This is for the difficulty in finding sliding based actuators, i.e. actuated prismatic joints is usually conceived by hydraulic cylinders or an equivalent mechanisms which are more difficult to implement than rotational based actuators (motors). On the other hand, prismatic motion is usually easier to analyze and design by engineers; for this purpose, we may tend to design and analyze prismatic actuated limbs then map the governing equations from position analysis to be implemented using revolute actuators.

As demonstration, see the equivalent 6 SPS and 6 SRS Stewart platforms given in Figure 4.1 (a) and (b) respectively. The motion equations that are used to control the 6 SPS platform represented by the inverse kinematics will be derived and discussed in the following, then the equations will be further adapted to represent the motion equations of the 6 SRS platform.

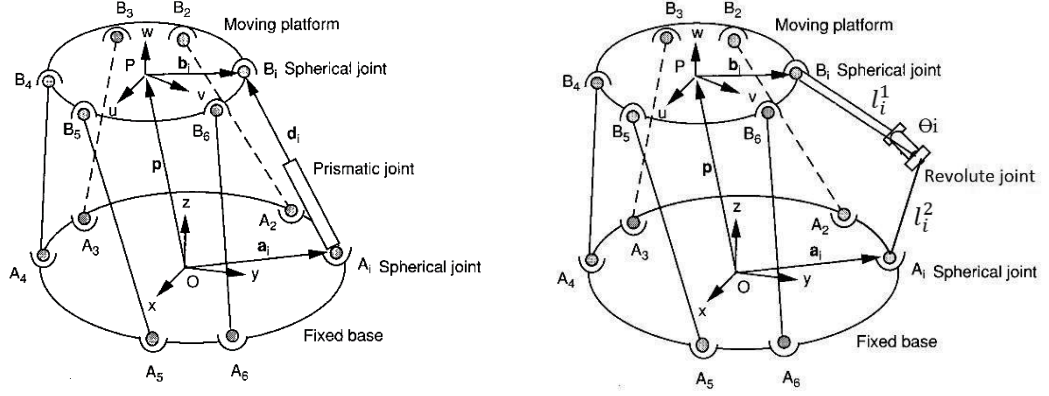


Figure 4.1: (a) 6 SPS Stewart Platform [26] (b) 6 SRS Stewart Platform

For the 6 SPS Stewart Platform shown in Figure 4.1 (a), the inverse kinematics (inverse position) equations are usually used to control the platform, these equations are derived from the linearly independent loops, corresponding to each limb as follows:

Suppose that the desired position and orientation of the moving platform is given with respect to the base; a usual representation of two orthogonal coordinate is conceived via a matrix (T) known as the general transformation matrix [26-31]; this transformation matrix consists of an orthogonal matrix that represents the projections of the observation frame coordinates onto the reference frame coordinates and will be represented as  $R_p^0$ , and a vector  $p_p^0$  that represents the position of the platform with respect to the base as follows:

$$T_p^0 = \begin{pmatrix} R_p^0 & p_p^0 \\ 0_{3 \times 3} & 1 \end{pmatrix} \quad (4.1)$$

The orthogonal matrix  $R_p^0$  expresses a rotation about an arbitrary axis; or by superposition: three rotations about x, y and z axes by angles  $\theta_x$ ,  $\theta_y$  and  $\theta_z$  respectively which are known as the *Roll-Pitch-Yaw* angles as in the following.

$$R_p^0 = R_z(\theta_z) * R_y(\theta_y) * R_x(\theta_x) \quad (4.2)$$

$$R_p^0 = \begin{pmatrix} \cos \theta_z & -\sin \theta_z & 0 \\ \sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \cos \theta_y & 0 & \sin \theta_y \\ 0 & 1 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x \\ 0 & \sin \theta_x & \cos \theta_x \end{pmatrix} \quad (4.3)$$

$$R_p^0 = \begin{pmatrix} \cos \theta_z \cos \theta_y & -\sin \theta_z \cos \theta_x + \cos \theta_z \sin \theta_y \sin \theta_x & \sin \theta_z \sin \theta_x + \cos \theta_z \sin \theta_y \cos \theta_x \\ \sin \theta_z \cos \theta_y & \cos \theta_z \cos \theta_x + \sin \theta_z \sin \theta_y \sin \theta_x & -\cos \theta_z \sin \theta_x + \sin \theta_z \sin \theta_y \cos \theta_x \\ -\sin \theta_y & \cos \theta_y \sin \theta_x & \cos \theta_y \cos \theta_x \end{pmatrix} \quad (4.4)$$

Where  $p_p^0$  the position vector is given by  $p_p^0 = [p_x \ p_y \ p_z]^T$ , supposed to be known (desired).

Let  $a_i$ 's and  $b_i$ 's represent the attachment vectors of each limb  $i$  on the base and the moving lamina respectively. Where  $a_i$ 's are represented with respect to the base and  $b_i$ 's are represented with respect to the moving lamina; and let  $d_i$ 's represent the length of each limb.

A vector loop equation can be written for each limb as [26]:

$$d_i + a_i = p + R_p^0 * b_i \quad (4.5)$$

Isolating  $d_i$  from equation (4.5) as the only unknown, gives the length of each actuated limb for a desired position and orientation of the lamina.

$$d_i = p + R_p^0 * b_i - a_i \quad (4.6)$$

The inverse kinematic problem for the 6 SPS Stewart platform is the solution of the system of equations in (4.6). Now we will map this solution to its equivalent 6 SRS Stewart platform shown in Figure 4.1 (b) as follows:

A virtual distance  $d_i$  can be added to the loop  $i$  in the 6 SRS Stewart platform as shown in Figure 4.2. Hence, the solution of the inverse kinematic problem for this equivalent manipulator can be expressed as in equation (4.8).

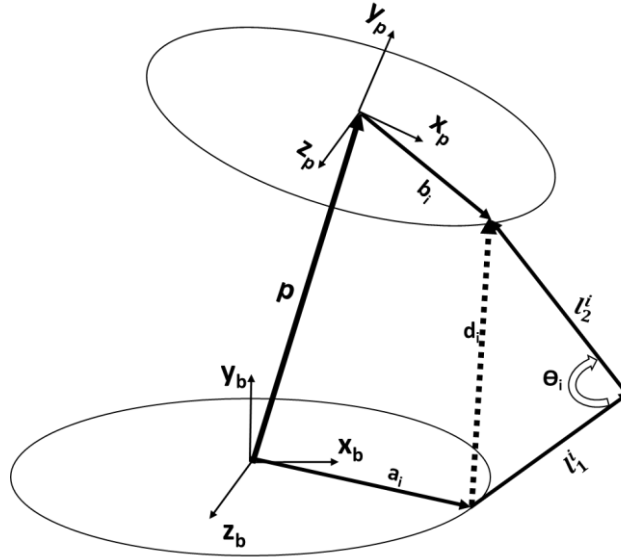


Figure 4.2: Illustration of Loop  $i$  of the SRS Stewart Platform

It appears to be possible to find the equations for the loop  $i$  directly by the cosine formula for the angles  $\theta_i$ 's as in equation (4.7).

$$\cos \theta_i = \frac{(l_1^i)^2 + (l_2^i)^2 - (d_i)^2}{2 * (l_1^i) * (l_2^i)} \quad (4.7)$$

While each  $\theta_i$  can be determined by the inverse of the cosine function as:

$$\theta_i = \cos^{-1} \frac{(l_1^i)^2 + (l_2^i)^2 - (d_i)^2}{2 * (l_1^i) * (l_2^i)} \quad (4.8)$$

Taking only the positive solution for equation (4.8), and mapping the interval of  $d_i$  to an interval for  $\theta_i$ ; the mathematical formulation for the equivalent Stewart platforms is completed.

It's worth mentioning that during the period of preparing this thesis, an implementation of another equivalent structure for the Stewart platform namely 6 RSS parallel manipulator, took place in collaboration between the Department of Electrical and Electronic Engineering and the Department of Mechanical Engineering in Eastern Mediterranean University. The designed structure was controlled by mapping equations (4.6) into its equivalents, and controlled by the means of inverse kinematics. A representative picture of the implemented Stewart platform is attached in Figure 4.3.

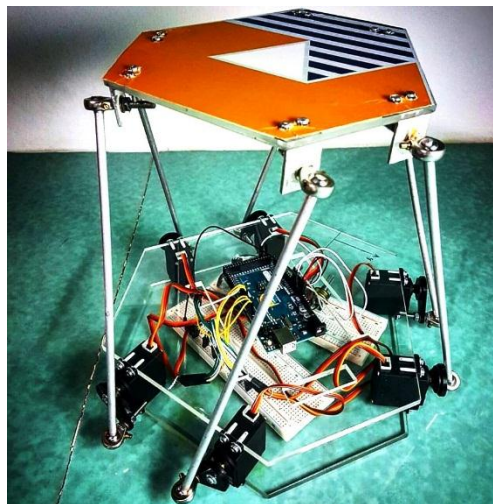


Figure 4.3: Implemented RSS Stewart Platform, EMU

### **4.3 Equivalent Manipulators by Active Support:**

In this section, we extend the concept of non-reciprocal screws that are usually found to be dual in the sense of parallel-serial duality, to support one of the joints of a serial manipulator by an additive parallel section to improve its stiffness and to increase its workload-tolerance.

Consider the 3R serial manipulator given in Figure 2.10 (a), assume further that we desire to support the base joint by some parallel structure without losing the overall simplicity for the serial design.



One possibility to perform this improvement regarding the aforementioned constraints is by attaching a force actuated limb to the first link that can do a work only on the base joint considering all position configurations of the manipulator. This can only be done by an attachment to the first link of the 3R manipulator as shown in Figure 4.4.

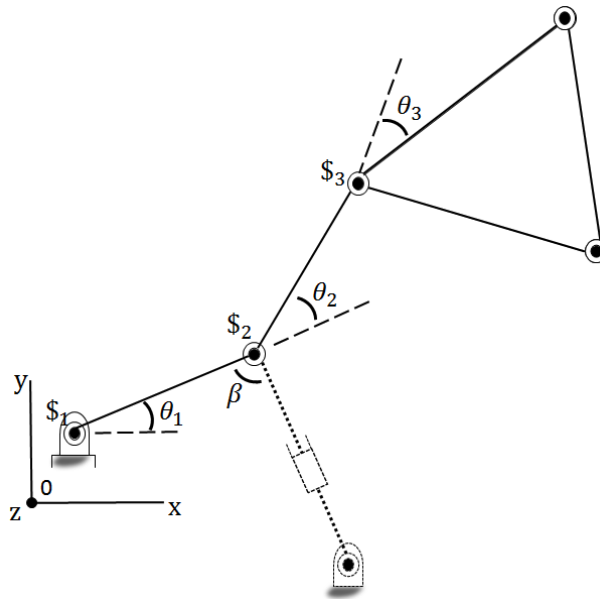


Figure 4.4: 3R Serial Planar Manipulator Connected with a Virtual Parallel Limb

If we express the angle of attachment by  $\beta$ , then the values of  $\beta$  are restricted by  $\beta \neq \{ 0, \pi \}$  in order not to have reciprocity (orthogonality) between the base and the additive screws, this is consistent with the physical axiom of transferring linear movement to rotational.

Consider the triangle given in Figure 4.5, the distance  $L$  is constant as both the additive limb and the base of the serial manipulator are fixed, and the constant offset angle is given by  $\theta_0$ ; using the trigonometric cosine expression, we can relate  $d_1$  with  $\theta_1$  as in equations (4.9) to (4.11).

$$\cos(\theta_1 + \theta_0) = \frac{a_{12}^2 + L^2 - d_1^2}{2a_{12} * L} \quad (4.9)$$

$$d_1 = \sqrt{a_{12}^2 + L^2 - 2a_{12} * L * \cos(\theta_1 + \theta_0)} \quad (4.10)$$

$$\theta_1 = \cos^{-1}\left(\frac{a_{12}^2 + L^2 - d_1^2}{2a_{12} * L}\right) - \theta_0 \quad (4.11)$$

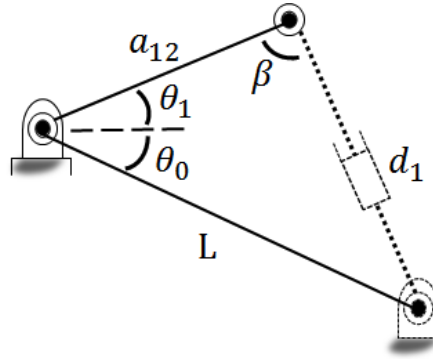


Figure 4.5: Triangle of Transformation from Revolute to Prismatic

Now, a possible equivalent implementation can be done by actuating the prismatic limb and keeping the base joint passive.

In the following, we may discuss performing forward and inverse kinematic analysis for the 3R serial manipulator neglecting the added parallel limb, then adapting the equations for the hybrid overall structure as follows.

First we assign Denavit-Hartenberg (DH) coordinates for the 3R manipulators in accordance with [28], as shown in Figure 4.6

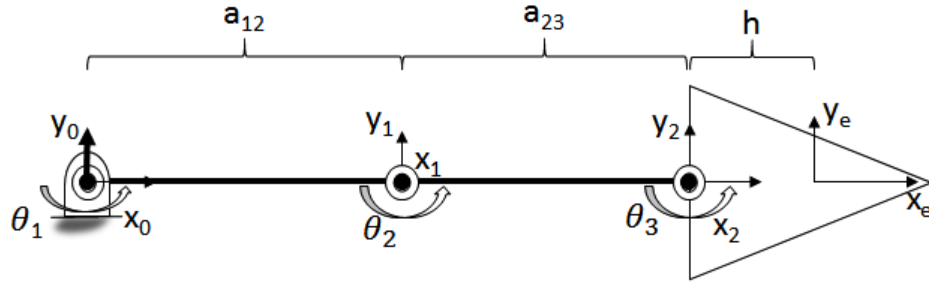


Figure 4.6: 3R Planar Manipulator with DH Coordinates

The transformation matrix between the first coordinate frame (the base) and the end effector are given in equation (4.12), from the outcome of substituting the correspondent DH parameters given in Table 4.2 into correspondent DH matrix representation and performing common matrix-matrix multiplications in a sequential manner as demonstrated below.

Table 4.2: DH Parameters for 3R Serial Planar Manipulator

Link / DH Parameter	$\theta_i$	$d_i$	$\alpha_i$	$a_i$
1	$\theta_1$	0	0	$a_{12}$
2	$\theta_2$	0	0	$a_{23}$
3	$\theta_3$	0	0	$h$

$$\begin{aligned}
 T_e^0 &= \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & a_{12} \cos \theta_1 \\ \sin \theta_1 & \cos \theta_1 & 0 & a_{12} \sin \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & a_{23} \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & a_{23} \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &* \begin{pmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & h \cos \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & h \sin \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) & -\sin(\theta_1 + \theta_2 + \theta_3) & 0 & a_{12} \cos \theta_1 + a_{23} \cos(\theta_1 + \theta_2) + h \cos(\theta_1 + \theta_2 + \theta_3) \\ \sin(\theta_1 + \theta_2 + \theta_3) & \cos(\theta_1 + \theta_2 + \theta_3) & 0 & a_{12} \sin \theta_1 + a_{23} \sin(\theta_1 + \theta_2) + h \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.12)
 \end{aligned}$$

Now by substituting  $\theta_1$  with its equivalent  $d_1$  from equation (4.11), we obtain a platform for both direct and inverse position problems for the hybrid manipulator given in Figure 4.4.

#### **4.4 Instantaneously Equivalent Serial / Parallel Manipulators:**

As has been shown in Chapter 2, there is a remarkable duality between a planar 3R serial manipulator and a planar 3(RPR) parallel manipulator shown in Figure 2.10 (a) and (b) respectively. This duality is not only in the sense of kinematics and statics, yet it is obvious in the position domain as well. Indeed, Murthy & Waldron in [5], showed that the dual serial and parallel manipulators have the same complexity while solving inverse and forward position problems respectively. Moreover, Davidson and Hunt in [10], suggested serial to parallel actuation as duality leads to equivalent as the method proposed in reconstructing the reciprocal screws from the inverse of the Jacobian matrix. In late (2015), Shai et al. in [12], showed that Stewart platform has very similar position configurations with an over constraint Bircard mechanism.

Up to today, there is no complete match between two dual manipulators in the position domain. Yet, the similarities between many of their position configurations and possessing the same degree of freedom was used to justify this equivalency.

For instance, picture a 3R serial manipulator attached with 3 force applicators along each of the directions of its reciprocal screws in the same plane of movement as shown in Figure 4.7.

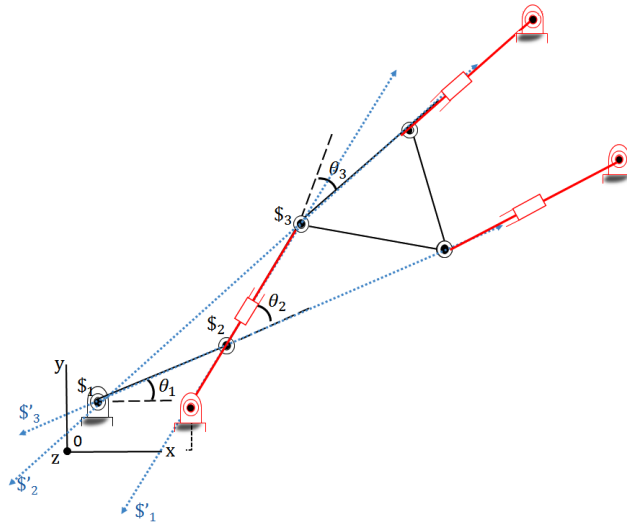


Figure 4.7: 3R Serial Planar with Its Dual 3(RPR) Parallel Planar at a Special Reciprocal Configuration

Although the similarity between the aforementioned dual manipulators is very obvious in terms of the workspace, degree of freedom, planarity and others; yet, they are only instantaneously equivalent, i.e. at the instant the sliding movement starts along one of its limbs, the direction of the screw of movement (the line) is reciprocal to two of the screws while non-reciprocal to the third, after an infinitesimal time, the reciprocity condition will be broken as shown in Figure 4.8.

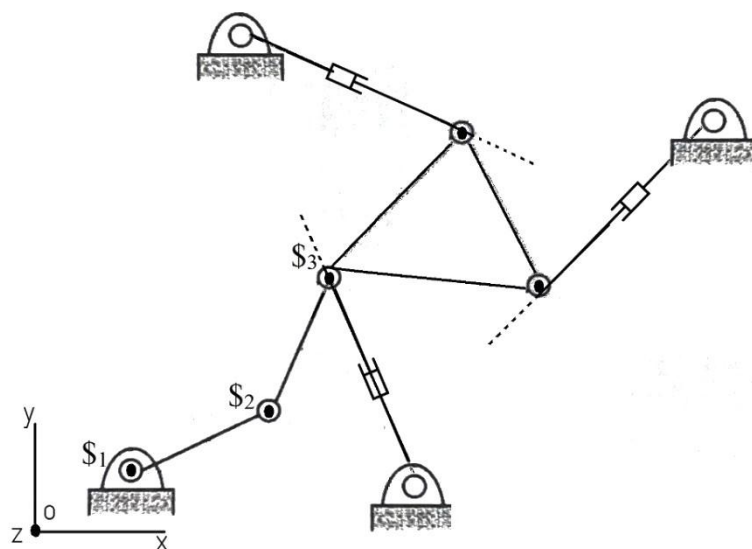


Figure 4.8: 3R Serial Planar and 3(RPR) Parallel Planar Manipulators at a Non-Reciprocal Position Configuration

In despite of the fact that two manipulators cannot be equivalent in terms of one to one in real time, they might be able to perform identical operations within very similar workspace limits; further discussion will be continued in Chapter 5.

#### **4.5 Serial to Hybrid Actuation:**

As discussed in the previous section, serial to parallel actuation via reciprocal screws is considered convenient for researchers that are interested in duality in the position domain in [5-7, 10, and 12]. This section proceeds further in the discussion established in Chapter 3 on the transformation from serial to hybrid actuation by replacing a subsection from a serial manipulator with its reciprocal-dual parallel section to inherit some of the properties of the overall manipulator.

Consider the elbow manipulator shown in Figure 3.3, a sequential serial sections of the manipulator will be replaced by their parallel dual, discussions will be followed.

A subsection represented by the first three joints and the links in between (the arm) from the Jacobian of the elbow manipulator was derived and inverted as given in equation 3.12, a proper sketch of the constructed screws are shown in Figure 3.4, an implementation of the subsection that has similar screws is shown in Figure 4.9.

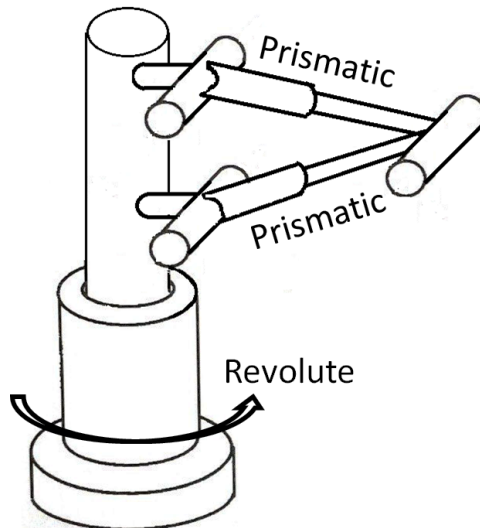


Figure 4.9: Equivalent Mechanism for the Arm of Elbow Manipulator

Figure 4.10 shows the elbow manipulator after replacing the arm with its dual structure provided in Figure 4.9

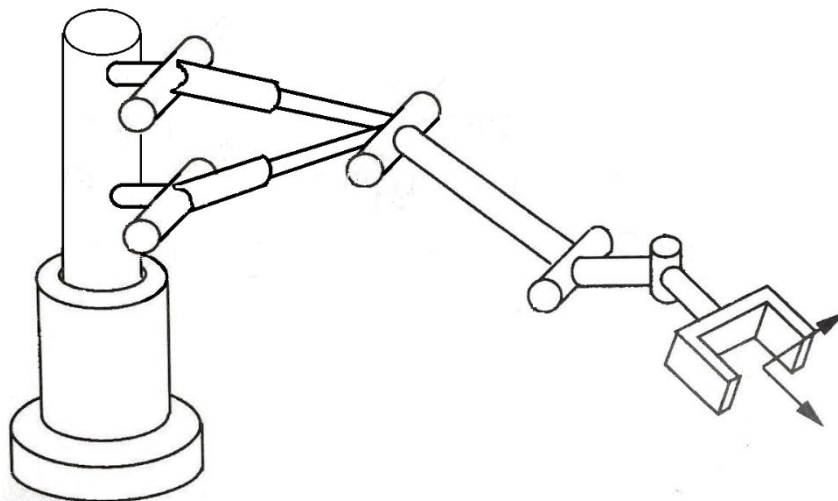


Figure 4.10: Sketch of Elbow Manipulator After Replacing the Arm with its Dual Structure

Where the subsection represented by the second, third and fourth joints and the links in between were found to be very similar to the 3 RPR parallel planar manipulator shown in Figure 2.10 (b). A replacement of the aforementioned sequence with its dual structure can be shown in Figure 4.11.

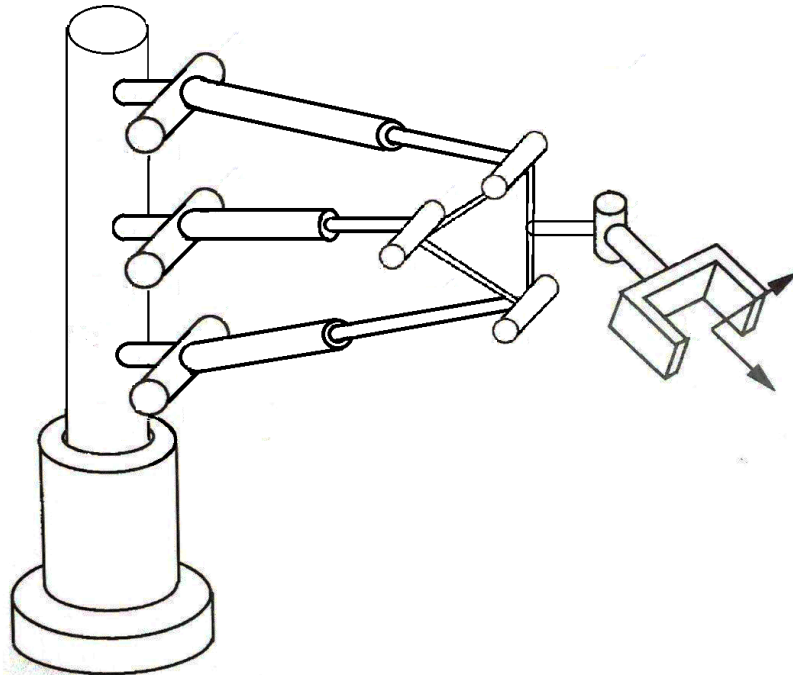


Figure 4.11: Sketch of Elbow Manipulator After Replacing the Sequence of the Joints 2, 3 and 4 with the Links in between with Their Dual Structure

It has been shown, that sequential subsection replacement of a serial manipulator by its reciprocal-dual, results in a hybrid-actuated mechanism that has instantaneous equivalence with the original manipulator in terms of the degree of freedom, type of geometrical displacement, purpose of usage and others.

This similarity or moreover, “*equivalency*” warrants further investigations as it leads to novel hybrid actuated mechanisms’ designs.



## Chapter 5

### CONCLUSION AND FUTURE WORK

#### 5.1 Conclusion

There exist a remarkable analogy between electrical and mechanical systems in terms of the similarity in assembly components (such as dampers and resistors, masses and capacitors etc...), and the similarity in the governing equations that relates analogous input / output parameters (voltages and currents with forces and velocities). On the other hand, some of the frequently used concepts in electrical systems such as equivalent, do not have mechanical analogs in the existing literature. In this thesis, we emphasized on the existence of equivalent structures that can be obtained both by inspection in the geometry of mechanics or by accepting instantaneously equivalent manipulators that are found dual in terms of common duality between twist and wrench coordinates in kinematics and statics respectively, as real time equivalents if they equip the design with some desired characteristics together with preserving some of the properties of the previous design such as degree of freedom, effective workspace and purpose of usage.

Furthermore, in this work, duality between statics and kinematics in terms of the reciprocity (orthogonality) between their corresponding coordinates was generalized by means of linear algebra, where a new insight on the geometrical meaning of the pseudo inverse of the Jacobian matrix has been discussed for the first time as an outcome of aforementioned general reciprocity solutions. The geometrical meaning of

pseudo inverse has been persuaded to aid the design of hybrid manipulators from existing serial manipulators by subsection replacement via reciprocal screws constructed from the pseudo inverse of sequential columns of serial manipulators' Jacobians, which correspond to sections of desired improvements.

## **5.2 Future Work**

A promising future for the work done in this thesis can be conceived by further adapting the improved structures and generalizing a scalar of comparison between old and improved designs from the outcome of serial to hybrid transformations, by thresholding the necessity of subsection replacement in terms of the new to old stiffness, new to old design complexity and new to old effective workspace for original and improved manipulators' designs.

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## **APPENDIX**



## Appendix A: Jacobian of Elbow manipulator (Matlab script)

```
%% Elbow manipulator Example, screw based jacobian
syms t d a1 a
syms a2 a3 a4 d6 % all constants
syms t1 t2 t3 t4 t5 t6 % all variables
A=trotz(t)*transl(a,0,d)*trotx(a1);
A1=subs(A,[a1 a d t],[pi/2 0 0 t1]);
A2=subs(A,[a1 a d t],[0 a2 0 t2]);
A3=subs(A,[a1 a d t],[0 a3 0 t3]);
A4=subs(A,[a1 a d t],[-pi/2 a4 0 t4]);
A5=subs(A,[a1 a d t],[pi/2 0 0 t5]);
A6=subs(A,[a1 a d t],[0 0 d6 t6]);
%% finding the reference wrt the base frame (to find
the duality easily)
A1=simplify(A1);
A12=simplify(A1*A2);
A123=simplify(A12*A3);
A1234=simplify(A123*A4);
A12345=simplify(A1234*A5);
A123456=simplify(A12345*A6);
%% finding ri for all frames:
r1=[0;0;0];
r2=[a2;0;0];
r3=[a3;0;0];
r4=[a4;0;0];
r5=[0;0;0];
%% Screw axis using algorithm page 194 Tsai
s1=[0; 0; 1]; so1=[0;0;0];
s2=A1(1:3,3); so2=simplify(so1+A1(1:3,1:3)*r1);
s3=A12(1:3,3); so3=simplify(so2+A12(1:3,1:3)*r2);
s4=A123(1:3,3); so4=simplify(so3+A123(1:3,1:3)*r3);
s5=A1234(1:3,3); so5=simplify(so4+A1234(1:3,1:3)*r4);
s6=A12345(1:3,3);
so6=simplify(so5+A12345(1:3,1:3)*r5);
J=[s1 s2 s3 s4 s5 s6; so1 so2 so3 so4 so5 so6]
```

## Appendix B: Hybrid Equivalents of Elbow Manipulator -3D Sketches



Figure B.1: Hybrid Equivalent Elbow Manipulator via Subsection Dual Replacement of the Arm

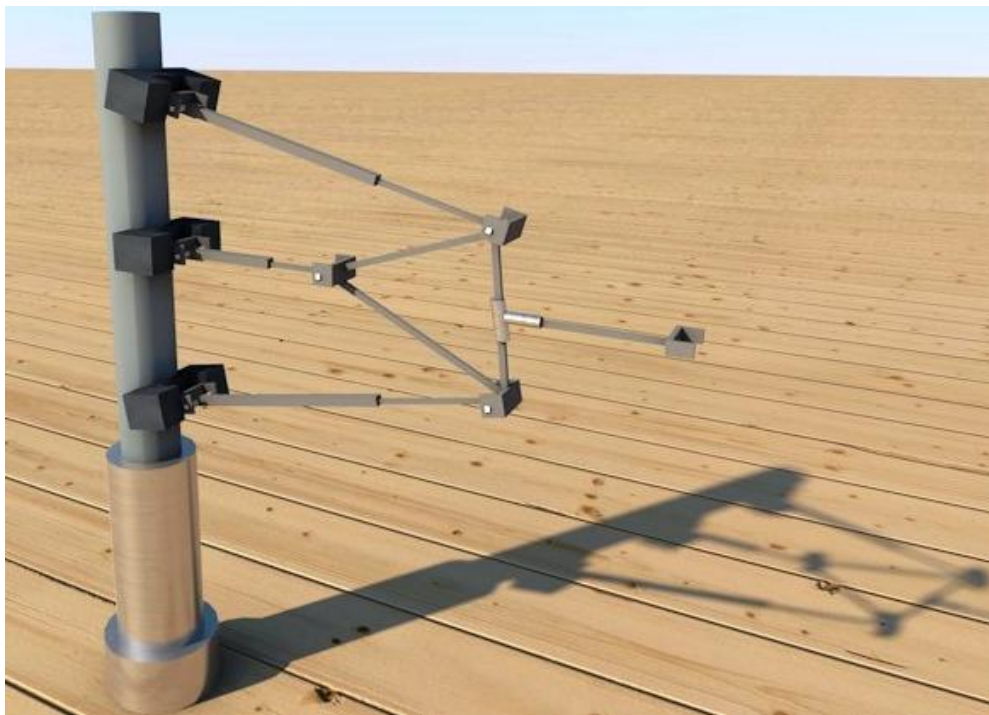


Figure B.1: Hybrid Equivalent Elbow Manipulator via Subsection Dual Replacement of the Second to the Fourth Joints and Connecting Links