# Numerical Approximation Methods using Multiplicative Calculus

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# ABSTRACT

In this thesis, the $2^{nd}$, $3^{rd}$ and $4^{th}$ order multiplicative Runge-Kutta Methods are developed in analogy to the classical Runge-Kutta Method. The error analysis is only carried out for the 4th order multiplicative Runge-Kutta method based on the convergence and stability analysis. The convergence behaviour of the developed multiplicative Runge-Kutta method is analysed by examining examples of initial value problems with closed form solutions, as well as problems without closed form solutions. The obtained results are also compared to the results obtained from the solutions of the classical Runge-Kutta method for the same examples. The error analysis shows that the solutions of the multiplicative Runge-Kutta methods give better results especially when the solution has an exponential nature. The modified quadratic Lorenz attractor is developed to examine the applicability of the proposed multiplicative Runge-Kutta method on the chaotic systems. The chaotic system is analysed numerically for its chaotic behaviour. Finally, the chaotic system is transformed into the corresponding system in terms of multiplicative calculus and the analysis are also done based on the rules of the multiplicative calculus. The results of the analysis show that the multiplicative Runge-Kutta method is also applicable to multiplicative chaotic systems.



**Keywords:** Multiplicative calculus, complex multiplicative calculus, Runge-Kutta, differential equations, numerical approximation, dynamical systems.

# ÖZ

Bu tezde, 2. ,3. ve 4. derece Runge-Kutta metodları temelinde çarpımsal analiz kuralları kullanılarak 2. ,3. ve 4. dereceden çarpımsal Runge-Kutta yöntemleri bulunmuş ve incelenmiştir. Bulunan yöntemlerin hata analizleri, yakınsaklık ve istikrarlılık analizleri temel alınarak yapılmıştır. Bulunan metodların yakınsaklık özellikleri, çözümleri bilinen ve bilinmeyen diferansiyel denklemler çözülerek gösterilmiştir. Çözümleri bilinen adi diferansiyel denklemler, çarpımsal Runge-Kutta ve Runge-Kutta yöntemleri kullanılarak çözülmüş ve hata analizleri yapılmıştır. Bu sonuçlara göre, özellikle çözümü eksponensiyel olan denklemlerde, çarpımsal Runge-Kutta metodunun bilinen Runge-Kutta metoduna göre daha iyi sonuçlar verdiği görülmüştür. Son olarak da çarpımsal Runge-Kutta metodlarının karmaşık sistemler üzerinde uygulanabildiğini göstermek için karmaşık bir sistem bulunmuş ve numerik olarak incelenmiştir. Daha sonra bulunan sistem çarpımsal analiz kurallarına göre düzenlenmiş ve çarpımsal Runge-Kutta yöntemleri kullanılarak çözülmüştür. Elde edilen sonuçlar bulunan yöntemlerin karmaşık sistemler üzerinde de kullanılabileceğini göstermiştir.

**Anahtar Kelimeler:** Çarpımsal analiz, kompleks çarpımsal analiz, Runge-Kutta, diferansiyel denklemler, numerik yakınsama, dinamik sistemler.

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# TABLE OF CONTENTS

# LIST OF TABLES

# LIST OF FIGURES

# Chapter 1

## INTRODUCTION

Micheal Grossman and Robert Katz can be considered as the first inventors of Multiplicative calculus. In 1972, they have defined nine different Non-Newtonian calculi in their book, called the Non-Newtonian Calculi [15]. After that, the bigeometric multiplicative calculus was elaborated by Micheal Grossman in [14]. Micheal Grossman was not the only one who emphasized bigeometric multiplicative calculus. Cordova-Lepe has also proposed the bigeometric multiplicative calculus by [10] using the name proportional calculus. Although, Volterra and Hostinsky proposed a kind of flavor of multiplicative calculus in [35], we can not date the invention of multiplicative calculus back to 1938. After many years, Bashirov et al presented a mathematical description of the geometric multiplicative calculus in [7]. By this research, Bashirov et al give a start to many studies in the field of multiplicative calculus. The multiplicative numerical approximation methods in [22, 23, 28, 21, 26, 24, 27, 8] have been proposed and discussed after his studies. Apart from the multiplicative Runge-Kutta Method [26], also the multiplicative finite difference method was invented by Riza et al in [28], and the multiplicative Adams Bashforth-Moulton methods where developed by Mısırlı and Gürefe [21]. Furthermore, multiplicative calculus has been used in biomedical image analysis [13] and modelling with differential equations [6]. As an example of the multiplicative numerical approximation methods, Aniszewska developed the bigeometric Runge-Kutta method for applications in dynamic systems in [1]. Riza and Eminağa also proposed the bigeometric Runge-Kutta method, based on the bigeometric Taylor theorem derived in [27].

The fact that multiplicative calculus can only be applied to purely positive-valued functions or purely negative-valued functions of real variable can be considered as a disadvantage. To overcome this restriction, multiplicative calculus was extended to the complex domain. Uzer presented the first studies of complex multiplicative calculus in [33]. Then, Bashirov and Riza gave a complete mathematical description of the complex geometric multiplicative calculus in [5, 4]. The fact that the derivative is a local property, suggests the extension to the complex domain. Thus, by transforming the functions into complex valued functions of real variable we can remove the restriction of using purely positive valued ore negative valued functions. Since the Cauchy Riemann conditions will be trivial in this way, we can use the real and the imaginary parts separate.

The basic definitions, theorems and rules of the multiplicative derivative, that will be used in this thesis, can be listed as follows.

The multiplicative derivative is defined by the formula

$$\lim_{h \to 0} \left( \frac{f(x+h)}{f(x)} \right)^{1/h} \tag{1.0.1}$$

which shows us the number times that $f(x)$ changes at the time moment $x$.

If $f(x)$ is a positive function on $A$ and its derivative at $x$ exists, then the relation between the classical and the multiplicative derivative can be written as

$$
\begin{aligned}
f^*(x) &= \lim_{h\longrightarrow 0} \frac{f(x+h)}{f(x)}^{1/h} \\
&= \lim_{h\longrightarrow 0} (1+\frac{f(x+h)-f(x)}{f(x)})^{\frac{f(x)}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} \\
&= e^{\frac{f'(x)}{f(x)}} = e^{(\ln\circ f)'(x)}
\end{aligned}
$$

If $c$ is a positive constant, $f$ and $g$ are *differentiable, $h$ is differentiable functions, some basic rules of *differentiation and the main theorems that will be used widely in this study can be summarized as follows:

$$(cf)^*(x) = f^*(x) \tag{1.0.2}$$

$$(fg)^*(x) = f^*(x)g^*(x) \tag{1.0.3}$$

$$\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)} \tag{1.0.4}$$

$$(f^h)^*(x) = f^*(x)^{h(x)}\cdot f(x)^{h'(x)} \tag{1.0.5}$$

$$(f\circ h)^*(x) = f^*(h(x))^{h'(x)} \tag{1.0.6}$$

**Theorem 1.0.1 (Multiplicative Taylor's Theorem for One Variable)** *Let $A$ be an open interval and let $f: A\rightarrow\mathbb{R}$ be $n+1$ times *differentiable on $A$. Then for any $x$, $x+h\in A$, there exists a number $\theta\in(0,1)$ such that*

$$f(x+h) = \prod_{m=0}^{n} (f^{*(m)}(x))^{\frac{h^m}{m!}} \cdot (f^{*(n+1)}(x+\theta h))^{\frac{h^{n+1}}{(n+1)!}}$$

**Theorem 1.0.2 (Multiplicative Chain Rule)** *Let $f$ be a function of two variables $y$ and $z$ with continuous partial *derivatives. If $y$ and $z$ are differentiable functions on $(a,b)$ such that $f(y(x),z(x))$ is defined for every $x\in(a,b)$, then*

$$\frac{d^*f(y(x),z(x))}{dx} = f^*_y(y(x),z(x))^{y'(x)} f^*_z(y(x),z(x))^{z'(x)}$$

**Theorem 1.0.3 (Multiplicative Taylor's Theorem for Two Variables)** *Let A be an open subset of* $\mathbb{R}^2$ *. Assume that the function* $f : A \to \mathbb{R}$ *has all partial *derivatives of order* $n+1$ *on A. Then for every* $(x,y)$*,* $(x+h,y+k) \in A$ *so that the line segment connecting these two points belongs to A, there exists a number* $\theta \in (0,1)$ *such that,*

$$f(x+h,y+k) = \prod_{m=0}^{n}\prod_{i=0}^{m} f^{*(m)}_{x^i y^{m-i}}(x,y)^{\frac{h^i k^{m-i}}{i!(m-i)!}} \cdot \prod_{i=0}^{n+1} f^{*(n+1)}_{x^i y^{n+1-i}}(x+\theta h, y+\theta k)^{\frac{h^i k^{n+1-i}}{i!(n+1-i)!}}$$

Based on the given rules, definitions and the theorems given above, the multiplicative Runge-Kutta methods will be derived in chapter 2, in order to solve the multiplicative initial value problems of the form:

$$y^*(x) = g(x,y), \qquad \text{with } y(x_0) = y_0 \tag{1.0.7}$$

Based on the classical Runge-Kutta methods, the most widely used methods of order 2, which can also be considered as the multiplicative Heun's method, order 3, and order 4 are discussed for positive-valued functions of real variable. Then chapter 2 will be ended by explaining how the positive valued functions of real variable can be extended to complex valued functions of real variable. The solution for the break downs of the multiplicative derivatives at the roots is also given in Section 2.4.

In chapter 3, the error analysis for the proposed method is presented. The error analysis is based on the convergence and stability analysis which are applied in analogy to the classical Runge-Kutta method as in [32]. The results of the analysis show that, the error for the multiplicative Runge-Kutta method becomes smaller compared to the classical Runge-Kutta method for the same step size.

In order to show the applicability of the $4^{th}$ order geometric multiplicative Runge-Kutta Method, the proposed method is applied to various examples. In section 4.1, first order multiplicative initial value problems are discussed. The first example is a multiplicative initial value problem with a known closed form solution, that does not involve an exponential or logarithmic function. The solutions of the multiplicative and the classical Runge-Kutta method are compared for a fixed step width $h$. The next example is a multiplicative initial value problem, where the exact solution contains a logarithmic function. To show the applicability of the proposed method in different fields, a biological example is given. Based on the Baranyi model for bacterial growth [3, 2] using differential equations, the multiplicative Runge-Kutta method will be applied on the bacterial growth in food modelled by Huang [18, 17, 16]. In section 4.2, a higher order multiplicative initial value problem is solved. A second order multiplicative initial value problem, with a well-known closed form solution, is used to compare the results of the classical and the multiplicative Runge-Kutta methods. Since the same example was also used in [28], the results of the multiplicative finite difference method and the multiplicative Runge-Kutta method are also compared. Both comparisions show the superiority of the multiplicative Runge-Kutta method.

In the last chapter, the applicability of the multiplicative Runge-Kutta methods on the chaotic systems is analyzed. The first example is the Rössler attractor. Then a new chaotic system called the Modified Quadratic Lorenz Attractor is defined based on the chaotic Lorenz attractor. The new chaotic system is analyzed numerically for the chaotic behaviour. Then, the new chaotic system is transformed to a multiplicative chaotic system to show the applicability of the multiplicative Runge-Kutta methods on the chaotic systems. Finally, we close with the conclusions in Chapter 6, summarising all results.

# Chapter 2

## MULTIPLICATIVE RUNGE-KUTTA METHODS FOR REAL-VALUED FUNCTIONS OF REAL VARIABLE

One important family of the iterative methods for approximating the solutions of the ordinary differential equations are the Runge-Kutta methods. These methods are used to solve the differential equations of the form:

$$y'(x) = g(x,y), \quad y(x_0) = y_0. \tag{2.0.1}$$

In the following, the multiplicative Runge-Kutta methods will be derived, based on the classical Runge-Kutta methods, in order to approximate the solutions of the multiplicative initial value problems of the form:

$$y^*(x) = g(x,y) \quad \text{with} \quad y(x_0) = y_0. \tag{2.0.2}$$

### 2.1 $2^{\text{nd}}$ order Multiplicative Runge-Kutta Method (MRK2)

The $2^{\text{nd}}$ order multiplicative Runge-Kutta method, known also as the Heun's method, can be derived explicitly in analogy to the $2^{\text{nd}}$ order classical Runge-Kutta method by making the following ansatz:

$$y(x+h) = y(x) \cdot g_0^{\alpha h} \cdot g_1^{\beta h}, \tag{2.1.1}$$

where

$$g_0 = g(x,y), \tag{2.1.2}$$

$$g_1 = g(x+ph, y \cdot g_0^{qh}). \tag{2.1.3}$$

The starting point is the Taylor expansion of $y(x+h)$. By using the multiplicative Taylor's theorem as given in [7], the multiplicative Taylor expansion of $y(x+h)$ up to order 2 can be written as

$$y(x+h) = y(x) \cdot y^*(x)^h \cdot y^{**}(x)^{h^2/2} \cdot \ldots \tag{2.1.4}$$

Substitution of the multiplicative differential equation (2.0.2) in the Taylor expansion yields

$$y(x+h) = y(x) \cdot g(x,y)^h \cdot y^{**}(x)^{h^2/2} \cdot \ldots \tag{2.1.5}$$

On the other hand $y^{**}(x)$ can also be written in terms of $g(x,y)$ as:

$$y^{**}(x) = (y^*(x))^* = (g(x,y))^*. \tag{2.1.6}$$

Since $y(x)$ is also a function depending on $x$, $g(x,y(x))^*$ can be calculated by using the multiplicative chain rule as in [7]. According to that, $g(x,y(x))^*$ can be written as:

$$y^{**}(x) = g^*(x,y) = \frac{d^*}{dx^*} g(x,y) = g_x^*(x,y) g_y^*(x,y)^{y'(x)} = g_x^*(x,y) g_y^*(x,y)^{y \ln g(x,y)}. \tag{2.1.7}$$

Then by substituting (2.1.7) in equation (2.1.5) second order multiplicative Taylor expansion becomes:

$$\begin{aligned} y(x+h) &= y(x) \cdot g(x,y)^h \cdot \left( g_x^*(x,y) g_y^*(x,y)^{y \ln g(x,y)} \right)^{h^2/2} \cdot \ldots \\ &= y(x) \cdot g(x,y)^h \cdot g_x^*(x,y)^{h^2/2} g_y^*(x,y)^{y \ln g(x,y) h^2/2} \cdot \ldots, \end{aligned} \tag{2.1.8}$$

where $g_x^*(x,y)$ and $g_y^*(x,y)$ denotes the partial multiplicative derivatives with respect to $x$ and $y$ respectively.

In order to find the constants $\alpha, \beta, p$ and $q$ we need to compare the Taylor expansion of equation (2.1.1) with the equation (2.1.8). Thus, the next step is to find the Taylor expansion of (2.1.1). Since the power of (2.1.1) includes one $h$, we will expand $g_1$ up to order one, in order to compare the Taylor expansions. By using the multiplicative chain rule, the Taylor expansion of $g_1$ becomes

$$g_1 = g(x,y) \cdot g_x^*(x,y)^{ph} \cdot g_y^*(x,y)^{yqh \ln g_0}. \tag{2.1.9}$$

Remembering that $g_0 = g(x,y)$, the Taylor expansion of $g_1$ will be

$$g_1 = g(x,y) \cdot g_x^*(x,y)^{ph} \cdot g_y^*(x,y)^{yqh \ln g(x,y)}. \tag{2.1.10}$$

Thus, by inserting (2.1.3) and (2.1.10) in (2.1.1), we obtain the Taylor expansion of the 2nd order Multiplicative Runge-Kutta method as

$$y(x+h) = y(x) \cdot g(x,y)^{\alpha h} \cdot \left( g(x,y) \cdot g_x^*(x,y)^{ph} \cdot g_y^*(x,y)^{yqh \ln g(x,y)} \right)^{\beta h}. \tag{2.1.11}$$

After rearranging the terms with respect to the orders of $h$, the Taylor expansion of $y(x+h)$ for the proposed method becomes

$$y(x+h) = y(x) \cdot g(x,y)^{(\alpha+\beta)h} \cdot g_x^*(x,y)^{\beta p h^2} \cdot g_y^*(x,y)^{y \ln g(x,y) \beta q h^2}. \tag{2.1.12}$$

Comparing the equations (2.1.8) and (2.1.12) for the powers of $g(x,y)$ and its partial derivatives results:

$$\alpha + \beta = 1, \tag{2.1.13}$$

$$\beta p = \frac{1}{2}, \tag{2.1.14}$$

$$\beta q = \frac{1}{2}. \tag{2.1.15}$$

Since the number of equations is less than the number of unknowns, we get infinitely many solutions for the equations (2.1.13)-(2.1.15). Moreover, it is obvious that $p = q$ and $\alpha + \beta = 1$, so that we can easily obtain the multiplicative Butcher Tableau in analogy to the regular Butcher Tableau [9] as:

$$\begin{array}{c|cc} 0 & & \\ p & q & \\ \hline & \alpha & \beta \end{array}$$

Among all possibilities, one possible choice of the parameters is:

$$\alpha = \frac{1}{2}, \beta = \frac{1}{2}, p = 1, \text{ and } q = 1. \tag{2.1.16}$$

Thus, the choice of the parameters results in the 2nd order Multiplicative Runge-Kutta method known also as the multiplicative Heun's method as:

$$y(x+h) = y(x) \cdot g_0^{\frac{h}{2}} \cdot g_1^{\frac{h}{2}}, \tag{2.1.17}$$

$$g_0 = g(x,y), \qquad \text{and} \tag{2.1.18}$$

$$g_1 = g(x+h, y \cdot g_0^h). \tag{2.1.19}$$

The choice of the parameters can also be done differently depending on the problem, to optimise the solutions.

## 2.2 3rd order Multiplicative Runge-Kutta Method (MRK3)

The 3rd order MRK-method can also be derived, in the same analogy of the 2nd order MRK-method, to solve the multiplicative differential equation (2.0.2).

In order to derive the 3rd order Multiplicative Runge-Kutta method we make the following ansatz.

$$
\begin{aligned}
y(x+h) &= y(x)\cdot g_0^{\alpha h}\cdot g_1^{\beta h}\cdot g_2^{\gamma h}, && (2.2.1)\\
g_0 &= g(x,y), && (2.2.2)\\
g_1 &= g(x+ph, y\cdot g_0^{qh}), && (2.2.3)\\
g_2 &= g(x+p_1h, y\cdot g_0^{q_1h}\cdot g_1^{q_2h}). && (2.2.4)
\end{aligned}
$$

The starting point for the derivation of the 3[rd] order method is also the same with the Heun's method. The first step is to find the Taylor expansion of $y(x+h)$ up to order 3 in $h$. Using the Taylor's theorem as it is defined in [7], the Taylor expansion will be:

$$y(x+h) = y(x)\cdot (y^*(x))^h \cdot (y^{**}(x))^{h^2/2}\cdot (y^{***}(x))^{h^3/3!}\cdot \ldots \qquad (2.2.5)$$

Remembering that,

$$
\begin{aligned}
y^*(x) &= g(x,y), && (2.2.6)\\
y^{**}(x) &= (y^*(x))^* = (g(x,y))^*, && (2.2.7)\\
y^{***}(x) &= (y^*(x))^{**} = (g(x,y))^{**}. && (2.2.8)
\end{aligned}
$$

Since $y(x)$ is a function of $x$, in order to find the 2[nd] and 3[rd] multiplicative derivatives of $y(x)$ we will apply the multiplicative chain rule.

$$
\begin{aligned}
y^{**}(x) &= g(x,y)^* = g_x^*(x,y)\cdot g_y^*(x,y)^{y\ln g}, && (2.2.9)\\
y^{***}(x) &= g(x,y)^{**} = \frac{d^*}{dx}\left(g(x,y)^*\right)\\
&= \frac{d^*}{dx}\left(g_x^*(x,y)\cdot g_y^*(x,y)^{y\ln g}\right)\\
&= g_{xx}^*(x,y)\cdot g_{xy}^*(x,y)^{y\ln g}\cdot g_{yx}^*(x,y)^{y\ln g}\cdot\\
&\quad g_{yy}^*(x,y)^{(y\ln g)^2}\cdot g_y^*(x,y)^{y(\ln g)^2}. && (2.2.10)
\end{aligned}
$$

In newtonian calculus, the partial derivatives are commutative. It can be easily shown that, the partial multiplicative derivatives are also commutative in multiplicative calculus.

$$g^*_{xy}(x,y) = \partial^*_y(\partial^*_x g(x,y)) = \partial^*_y \exp\left\{\frac{g_x(x,y)}{g(x,y)}\right\} = \exp\left\{\partial_y \ln\exp\left\{\frac{g_x(x,y)}{g(x,y)}\right\}\right\} =$$
$$= \exp\left\{\partial_y\left\{\frac{g_x(x,y)}{g(x,y)}\right\}\right\} = \exp\left\{\frac{g_{xy}(x,y)g(x,y) - g_y(x,y)g_x(x,y)}{g(x,y)^2}\right\}, \quad (2.2.11)$$

and

$$g^*_{yx}(x,y) = \partial^*_x(\partial^*_y g(x,y)) = \partial^*_x \exp\left\{\frac{g_y(x,y)}{g(x,y)}\right\} = \exp\left\{\partial_x \ln\exp\left\{\frac{g_y(x,y)}{g(x,y)}\right\}\right\} =$$
$$= \exp\left\{\partial_x\left\{\frac{g_y(x,y)}{g(x,y)}\right\}\right\} = \exp\left\{\frac{g_{yx}(x,y)g(x,y) - g_x(x,y)g_y(x,y)}{g(x,y)^2}\right\}. \quad (2.2.12)$$

Using the commutativity property of the partial derivatives, it is clear that the multiplicative partial derivatives are also commutative. Thus the equation (2.2.10) can be simplified to

$$\begin{aligned} y^{***}(x) &= (g(x,y))^{**} \\ &= g^*_{xx}(x,y) \cdot g^*_{xy}(x,y)^{2y\ln g} \cdot g^*_{yy}(x,y)^{(y\ln g)^2} \cdot g^*_y(x,y)^{y(\ln g)^2}. \quad (2.2.13) \end{aligned}$$

Then by substituting the corresponding partial multiplicative derivatives in equation (2.2.5), the multiplicative Taylor expansion of order 3 can be written as:

$$y(x+h) = y(x) \cdot g(x,y)^h \cdot (g^*_x(x,y) \cdot g^*_y(x,y)^{y\ln g})^{h^2/2} \cdot (g^*_{xx}(x,y) \cdot g^*_{xy}(x,y)^{2y\ln g} \cdot$$
$$\cdot g^*_{yy}(x,y)^{(y\ln g)^2} \cdot g^*_y(x,y)^{y(\ln g)^2})^{h^3/3!} \cdot \ldots \quad (2.2.14)$$

rewriting the terms results

$$y(x+h) = y(x) \cdot g(x,y)^h \cdot g_x^*(x,y)^{h^2/2} \cdot g_y^*(x,y)^{y \ln g h^2/2} \cdot g_{xx}^*(x,y)^{h^3/6} \cdot g_{xy}^*(x,y)^{y \ln g h^3/3} \cdot$$
$$\cdot g_{yy}^*(x,y)^{y^2 (\ln g)^2 h^3/6} \cdot g_y^*(x,y)^{y (\ln g)^2 h^3/6} \cdot \ldots \quad (2.2.15)$$

As we have found the multiplicative Taylor expansion of $y(x+h)$, the next step is to find the Taylor expansion of the ansatz (2.2.1). In order to find its Taylor expansion, we have to find the Taylor expansions of (2.2.3) and (2.2.4) and substitute in equation (2.2.1).

The multiplicative Taylor expansion of $g_1$ will be:

$$g_1 = g(x,y) \cdot g_x^*(x,y)^{ph} \cdot g_y^*(x,y)^{yqh \ln g} \cdot g_{xx}^*(x,y)^{(ph)^2/2} \cdot g_{xy}^*(x,y)^{pqy \ln g h^2} \cdot$$
$$\cdot g_{yy}^*(x,y)^{(yq \ln g)^2 h^2/2} \cdot g_y^*(x,y)^{q^2 y (\ln g)^2 h^2/2}, \quad (2.2.16)$$

and the multiplicative Taylor expansion $g_2$ will be as follows:

$$g_2 = g(x,y) \cdot g_x^*(x,y)^{p_1 h} \cdot g_y^*(x,y)^{(y \ln g q_1 + y \ln g_1 q_2) h} \cdot g_{xx}^*(x,y)^{p_1^2 h^2/2} \cdot$$
$$\cdot g_{xy}^*(x,y)^{p_1 h (q_1 h y \ln g + q_2 h y \ln g_1)} \cdot g_{yy}^*(x,y)^{(q_1 y \ln g + q_2 y \ln g_1)^2 h^2/2} \cdot$$
$$\cdot g_y^*(x,y)^{\left(q_1^2 y (\ln g)^2 + 2 q_1 q_2 y \ln g \ln g_1 + q_2^2 y (\ln g_1)^2\right) h^2/2}. \quad (2.2.17)$$

Since the expansion of $g_2$ depends on both of the functions $g$ and $g_1$, in order to get the expansion of $g_2$ depending only the function $g$, the next step is to substitude the Taylor expansion of $g_1$ which is given in the equation (2.2.16) into the last expression. As it can be seen from the equation (2.2.17) that, we need to substitude $\ln g_1$ instead of $g_1$. Although the multiplicative Taylor expansions consists of multiplications of the terms, because of the logarithm, the terms of $\ln g_1$ can be written as the summations of the terms. After a lengthy calculation, some of the terms will cancel and the simplified form of the Taylor expansion of $g_2$ can be written as

$$g_2 = g(x,y)\cdot g_x^*(x,y)^{p_1h}\cdot g_y^*(x,y)^{(y\ln g q_1+y\ln g q_2)h}\cdot g_{xx}^*(x,y)^{p_1^2h^2/2}\cdot$$
$$\cdot g_{xy}^*(x,y)^{p_1h(q_1hy\ln g+q_2hy\ln g)}\cdot g_{yy}^*(x,y)^{(q_1y\ln g+q_2y\ln g)^2h^2/2}\cdot$$
$$\cdot g_y^*(x,y)^{\left(q_1^2y(\ln g)^2+2q_1q_2y(\ln g)^2+q_2^2y(\ln g)^2\right)h^2/2}. \quad (2.2.18)$$

Then by inserting the corresponding Taylor expansions of $g_1$ and $g_2$ in (2.2.1) gives

$$y(x+h) = y(x)\cdot g(x,y)^{\alpha h}\cdot (g(x,y)\cdot g_x^*(x,y)^{ph}\cdot g_y^*(x,y)^{yqh\ln g}\cdot g_{xx}^*(x,y)^{(ph)^2/2}\cdot$$
$$\cdot g_{xy}^*(x,y)^{pqy\ln g h^2}\cdot g_{yy}^*(x,y)^{(yq\ln g)^2h^2/2}\cdot g_y^*(x,y)^{q^2y(\ln g)^2h^2/2})^{\beta h}\cdot$$
$$\cdot (g(x,y)\cdot g_x^*(x,y)^{p_1h}\cdot g_y^*(x,y)^{(y\ln g q_1+y\ln g q_2)h}\cdot g_{xx}^*(x,y)^{p_1^2h^2/2}\cdot$$
$$\cdot g_{xy}^*(x,y)^{p_1h(q_1hy\ln g+q_2hy\ln g)}\cdot g_{yy}^*(x,y)^{(q_1y\ln g+q_2y\ln g)^2h^2/2}\cdot$$
$$\cdot g_y^*(x,y)^{\left(q_1^2y(\ln g)^2+2q_1q_2y(\ln g)^2+q_2^2y(\ln g)^2\right)h^2/2})^{\gamma h}. \quad (2.2.19)$$

Rearranging the terms with respect to the orders of $h$ gives the Taylor expansion of the ansatz for the 3$^{\text{rd}}$ order multiplicative Runge-Kutta method as

$$y(x+h) = y(x)\cdot g(x,y)^{(\alpha+\beta+\gamma)h}\cdot g_x^*(x,y)^{(\beta p+\gamma p_1)h^2}\cdot g_y^*(x,y)^{y\ln g(\beta q+\gamma(q_1+q_2))h^2}\cdot$$
$$\cdot g_{xx}^*(x,y)^{(\beta p^2+\gamma p_1^2)h^3/2}\cdot g_{xy}^*(x,y)^{y\ln g(\beta pq+\gamma p_1(q_1+q_2))h^3}\cdot$$
$$\cdot g_{yy}^*(x,y)^{y^2(\ln g)^2(\beta q^2+\gamma(q_1^2+2q_1q_2+q_2^2))h^3/2}\cdot$$
$$\cdot g_y^*(x,y)^{y(\ln g)^2(\beta q^2+\gamma(q_1^2+2q_1q_2+q_2^2))h^3/2}. \quad (2.2.20)$$

Then, comparing the equations (2.2.15) and (2.2.20) for the powers of $g(x,y)$ and its multiplicative partial derivatives, for the constants results in the following equations

$$\begin{aligned}
\alpha+\beta+\gamma &= 1, \\
\beta p+\gamma p_1 &= \frac{1}{2}, \\
\beta q+\gamma q_1+\gamma q_2 &= \frac{1}{2}, \\
\beta p^2+\gamma p_1^2 &= \frac{1}{3}, \\
\beta pq+\gamma p_1 q_1+\gamma p_1 q_2 &= \frac{1}{3}, \\
\beta q^2+\gamma q_1^2+2\gamma q_1 q_2+\gamma q_2^2 &= \frac{1}{3}.
\end{aligned}$$

From the given set of equations, it can be easily seen that

$$p = q, \tag{2.2.21}$$

$$p_1 = q_1+q_2, \tag{2.2.22}$$

resulting in the following equations

$$\alpha+\beta+\gamma = 1, \tag{2.2.23}$$

$$\beta p+\gamma p_1 = \frac{1}{2}, \tag{2.2.24}$$

$$\beta p^2+\gamma p_1^2 = \frac{1}{3}. \tag{2.2.25}$$

It is clear that the different choices of $q$,$q_1$, and $q_2$ will determine the values of $p$ and $p_1$. According to this, the equations (2.2.23)-(2.2.25) can be solved with respect to $\alpha$, $\beta$, and $\gamma$, where we get

$$\alpha = -\frac{-6pp_1+3p+3p_1-2}{6pp_1}, \quad (2.2.26)$$
$$\beta = -\frac{3p_1-2}{6p(p-p_1)}, \quad (2.2.27)$$
$$\gamma = -\frac{2-3p}{6p_1(p-p_1)}. \quad (2.2.28)$$

Resulting in the multiplicative Butcher Tableau

$$\begin{array}{c|ccc} 0 & & & \\ p & q & & \\ p_1 & q_1 & q_2 & \\ \hline & \alpha & \beta & \gamma \end{array}$$

As in the Heun's method, the number of equations is less than the number of unknowns, resulting infinitely many solutions. One possible choice of the constants is to evaluate the functions $g_0$ at the beginning, $g_1$ in the middle, and $g_2$ at the end of the interval. Moreover, we give equal weights to the function evaluated at the left and right endpoint of the interval of length $h$, and double the weight for the value in the middle of the interval; which results in the constants $\alpha = \frac{1}{6}, \beta = \frac{2}{3}, \gamma = \frac{1}{6}, p = \frac{1}{2}, p_1 = 1, q = \frac{1}{2}, q_1 = -1, q_2 = 2$. According to the choice of the constants, the 3$^{\text{rd}}$ order Multiplicative Runge-Kutta method can be written as:

$$\begin{aligned} y(x+h) &= y(x)\cdot g_0^{\frac{h}{6}}\cdot g_1^{\frac{2h}{3}}\cdot g_2^{\frac{h}{6}}, \\ g_0 &= g(x,y), \\ g_1 &= g(x+\frac{h}{2}, y\cdot g_0^{\frac{h}{2}}), \\ g_2 &= g(x+h, y\cdot g_0^{-h}\cdot g_1^{2h}). \end{aligned}$$

## 2.3 4th order Multiplicative Runge-Kutta Method (MRK4)

The 4th order Runge-Kutta method is the most widely used method to solve the initial value problems. The multiplicative counterpart of the 4th order Runge-Kutta method can be derived in analogy to the above described 2nd and 3rd order multiplicative Runge-Kutta methods.

Consequently, we start by the following ansatz

$$y(x+h) = y(x)\cdot g_0^{\alpha h}\cdot g_1^{\beta h}\cdot g_2^{\gamma h}\cdot g_3^{\delta h}, \tag{2.3.1}$$

$$g_0 = g(x,y), \tag{2.3.2}$$

$$g_1 = g(x+ph, y\cdot g_0^{qh}), \tag{2.3.3}$$

$$g_2 = g(x+p_1h, y\cdot g_0^{q_1h}\cdot g_1^{q_2h}), \tag{2.3.4}$$

$$g_3 = g(x+p_2h, y\cdot g_0^{q_3h}\cdot g_1^{q_4h}\cdot g_2^{q_5h}). \tag{2.3.5}$$

Remembering that the multiplicative Taylor expansion of $y(x+h)$ is given in equation (2.2.15) as

$$y(x+h) = y(x)\cdot g(x,y)^h\cdot g_x^*(x,y)^{h^2/2}\cdot g_y^*(x,y)^{y\ln g h^2/2}\cdot g_{xx}^*(x,y)^{h^3/6}\cdot g_{xy}^*(x,y)^{y\ln g h^3/3}\cdot$$
$$\cdot g_{yy}^*(x,y)^{y^2(\ln g)^2h^3/6}\cdot g_y^*(x,y)^{y(\ln g)^2h^3/6}\cdot\ldots \tag{2.3.6}$$

we will find the Taylor expansion of $y(x+h)$ for the ansatz (2.3.19). The first step is again to find the Taylor expansions of $g_0, g_1, g_2$ and $g_3$, in order to substitute in the equation (2.3.19).

After a lengthy calculation, as we have already explained for the Taylor expansion of

$g_2$ for the 3rd order multiplicative Runge-Kutta method, the Taylor expansions of $g_1$, $g_2$ and $g_3$ can be found in the same way. The Taylor expansion of $g_1$ can be written as

$$g_1 = g(x,y) \cdot g_x^*(x,y)^{ph} \cdot g_y^*(x,y)^{yqh\ln g} \cdot g_{xx}^*(x,y)^{(ph)^2/2} \cdot g_{xy}^*(x,y)^{pqy\ln gh^2} \cdot$$
$$\cdot g_{yy}^*(x,y)^{(yq\ln g)^2h^2/2} \cdot g_y^*(x,y)^{q^2y(\ln g)^2h^2/2}, \quad (2.3.7)$$

while the Taylor expansion of $g_2$ will be

$$g_2 = g(x,y) \cdot g_x^*(x,y)^{p_1h} \cdot g_y^*(x,y)^{(y\ln gq_1+y\ln gq_2)h} \cdot g_{xx}^*(x,y)^{p_1^2h^2/2} \cdot$$
$$\cdot g_{xy}^*(x,y)^{p_1h(q_1hy\ln g+q_2hy\ln g)} \cdot g_{yy}^*(x,y)^{(q_1y\ln g+q_2y\ln g)^2h^2/2} \cdot$$
$$\cdot g_y^*(x,y)^{\left(q_1^2y(\ln g)^2+2q_1q_2y(\ln g)^2+q_2^2y(\ln g)^2\right)h^2/2}, \quad (2.3.8)$$

and the Taylor expansion of $g_3$ is

$$g_3 = g(x,y) \cdot g_x^*(x,y)^{p_2h} \cdot g_y^*(x,y)^{(y\ln gq_3+y\ln gq_4+y\ln gq_5)h} \cdot g_{xx}^*(x,y)^{p_2^2h^2/2} \cdot$$
$$\cdot g_{xy}^*(x,y)^{p_2h(q_3hy\ln g+q_4hy\ln g+q_5hy\ln g)} \cdot g_{yy}^*(x,y)^{(q_3y\ln g+q_4y\ln g+q_5y\ln g)^2h^2/2} \cdot$$
$$\cdot g_y^*(x,y)^{\left(q_3^2y(\ln g)^2+q_4^2y(\ln g)^2+q_5^2y(\ln g)^2+2q_3q_4y(\ln g)^2+2q_3q_5y(\ln g)^2+2q_4q_5y(\ln g)^2\right)h^2/2}. \quad (2.3.9)$$

Using the Taylor expansions of $g_1$, $g_2$ and $g_3$, obtained in the equations (2.3.7)-(2.3.9), the Taylor expansion of the ansatz (2.3.19) can be written as

$$y(x+h) = y(x) \cdot g(x,y)^{\alpha h} \cdot (g(x,y) \cdot g_x^*(x,y)^{ph} \cdot g_y^*(x,y)^{yqh\ln g} \cdot g_{xx}^*(x,y)^{(ph)^2/2} \cdot$$
$$\cdot g_{xy}^*(x,y)^{pqy\ln gh^2} \cdot g_{yy}^*(x,y)^{(yq\ln g)^2h^2/2} \cdot g_y^*(x,y)^{q^2y(\ln g)^2h^2/2})^{\beta h} \cdot$$
$$\cdot (g(x,y) \cdot g_x^*(x,y)^{p_1h} \cdot g_y^*(x,y)^{(y\ln gq_1+y\ln gq_2)h} \cdot g_{xx}^*(x,y)^{p_1^2h^2/2} \cdot$$
$$\cdot g_{xy}^*(x,y)^{p_1h(q_1hy\ln g+q_2hy\ln g)} \cdot g_{yy}^*(x,y)^{(q_1y\ln g+q_2y\ln g)^2h^2/2} \cdot$$

$$\cdot g_y^*(x,y)^{\left(q_1^2 y(\ln g)^2+2q_1q_2y(\ln g)^2+q_2^2y(\ln g)^2\right)h^2/2}\Big)^{\gamma h} \cdot \big(g(x,y)\cdot g_x^*(x,y)^{p_2h}\cdot$$

$$\cdot g_y^*(x,y)^{(y\ln g q_3+y\ln g q_4+y\ln g q_5)h}\cdot g_{xx}^*(x,y)^{p_2^2h^2/2}\cdot$$

$$\cdot g_{xy}^*(x,y)^{p_2h(q_3hy\ln g+q_4hy\ln g+q_5hy\ln g)}\cdot g_{yy}^*(x,y)^{(q_3y\ln g+q_4y\ln g+q_5y\ln g)^2h^2/2}\cdot$$

$$\cdot g_y^*(x,y)^{\left(q_3^2y(\ln g)^2+q_4^2y(\ln g)^2+q_5^2y(\ln g)^2+2q_3q_4y(\ln g)^2+q_3q_5y(\ln g)^2+q_4q_5y(\ln g)^2\right)h^2/2}\Big)^{\delta h}. \quad (2.3.10)$$

Rearrangement of the terms with respect to the orders of $h$ gives the Taylor expansion of the 4[th] order multiplicative Runge-Kutta method as

$$y(x+h) = y(x)\cdot g(x,y)^{(\alpha+\beta+\gamma+\delta)h}\cdot g_x^*(x,y)^{(\beta p+\gamma p_1+\delta p_2)h^2}\cdot$$

$$\cdot g_y^*(x,y)^{y\ln g(\beta q+\gamma(q_1+q_2)+\delta(q_3+q_4+q_5))h^2}\cdot g_{xx}^*(x,y)^{(\beta p^2+\gamma p_1^2+\delta p_2^2)h^3/2}\cdot$$

$$\cdot g_{xy}^*(x,y)^{y\ln g(\beta pq+\gamma p_1(q_1+q_2)+\delta p_2(q_3+q_4+q_5))h^3}\cdot$$

$$\cdot g_{yy}^*(x,y)^{y^2(\ln g)^2(\beta q^2+\gamma(q_1^2+2q_1q_2+q_2^2)+\delta(q_3^2+q_4^2+q_5^2+2q_3q_4+2q_3q_5+2q_4q_5))h^3/2}\cdot$$

$$\cdot g_y^*(x,y)^{y(\ln g)^2(\beta q^2+\gamma(q_1^2+2q_1q_2+q_2^2)+\delta(q_3^2+q_4^2+q_5^2+2q_3q_4+2q_3q_5+2q_4q_5))h^3/2}. \quad (2.3.11)$$

Comparing the equations (2.3.6) and (2.3.11) for the powers of $g(x,y)$ and the multiplicative partial derivatives of $g(x,y)$ results in the equations :

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 1,\\
\beta p+\gamma p_1+\delta p_2 &= \frac{1}{2},\\
\beta q+\gamma q_1+\gamma q_2+\delta q_3+\delta q_4+\delta q_5 &= \frac{1}{2},\\
\beta p^2+\gamma p_1^2+\delta p_2^2 &= \frac{1}{3}, \quad (2.3.12)\\
\beta pq+\gamma p_1q_1+\gamma p_1q_2+\delta p_2q_3+\delta p_2q_4+\delta p_2q_5 &= \frac{1}{3},\\
\beta q^2+\gamma(q_1+q_2)^2+\delta(q_3^2+q_4^2+q_5^2+2q_3q_4+2q_3q_5+2q_4q_5) &= \frac{1}{3}.
\end{aligned}$$

In analogy to the 3rd order MRK-method, the number of equations can be reduced since $p, p_1$ and $p_2$ can be written in terms of $q, q_1, q_2, q_3, q_4$ and $q_5$ as

$$p = q, \tag{2.3.13}$$

$$p_1 = q_1 + q_2, \tag{2.3.14}$$

$$p_2 = q_3 + q_4 + q_5. \tag{2.3.15}$$

Thus the set of equations (2.3.12) can be reduced to the following set of equations:

$$\alpha + \beta + \gamma + \delta = 1, \tag{2.3.16}$$

$$\beta p + \gamma p_1 + \delta p_2 = \frac{1}{2}, \tag{2.3.17}$$

$$\beta p^2 + \gamma p_1^2 + \delta p_2^2 = \frac{1}{3}. \tag{2.3.18}$$

The results that we have obtained, can be summarized in the multiplicative Butcher Tableau as

$$\begin{array}{c|cccc} 0 & & & & \\ p & q & & & \\ p_1 & q_1 & q_2 & & \\ p_2 & q_3 & q_4 & q_5 & \\ \hline & \alpha & \beta & \gamma & \delta \end{array}$$

As it is already defined in 2nd and 3rd order multiplicative Runge-Kutta methods, we get infinitely many solutions for the equation set (2.3.16)-(2.3.18) since the number of unknowns is more than the number of equations. One of the possible choices of the constants is $\alpha=\frac{1}{6},\beta=\frac{1}{3},\gamma=\frac{1}{3},\delta=\frac{1}{6},p=\frac{1}{2},p_1=\frac{1}{2},p_2=1,q=\frac{1}{2},q_2=\frac{1}{2},q_5=1,q_1=q_3=q_4=0$. Thus the 4$^{\text{th}}$ order multiplicative Runge-Kutta method can be written as

$$y(x+h) = y(x)\cdot g_0^{\frac{h}{6}}\cdot g_1^{\frac{h}{3}}\cdot g_2^{\frac{h}{3}}\cdot g_3^{\frac{h}{6}}, \tag{2.3.19}$$

$$g_0 = g(x,y), \tag{2.3.20}$$

$$g_1 = g(x+\frac{h}{2},y\cdot g_0^{\frac{h}{2}}), \tag{2.3.21}$$

$$g_2 = g(x+\frac{h}{2},y\cdot g_1^{\frac{h}{2}}), \tag{2.3.22}$$

$$g_3 = g(x+h,y\cdot g_2^{h}). \tag{2.3.23}$$

## 2.4 Extension to complex valued functions of real variable

Multiplicative calculus is restricted to positive valued functions of real variable, which can be listed as one of its drawbacks. To solve this problem we can extend it to the complex domain. Since the Cauchy-Riemann conditions must be satisfied in the complex domain, the rules for the derivatives of complex valued functions of complex variable are more complicated. Thus, it might seem that extending the multiplicative calculus to the complex domain will cause some difficulties. However, since we are interested in complex valued functions of real variable, the Multiplicative Cauchy-Riemann conditions will not be considered, and we will find the derivatives of the real and imaginary parts separately. As it is mentioned in [5], in the complex plain one can evaluate the multiplicative derivatives of the functions everywhere excluding the point $0+0i$.

Since the phase factor will be responsible for the sign change, we can extend the 4$^{\text{th}}$ order multiplicative Runge-Kutta Method to negative valued functions. After extending multiplicative calculus to the complex domain, the only remaining problem that we can not solve is that the Multiplicative derivative is not defined at the roots of the function, which can be easily seen from the definition of the multiplicative derivative. So, in order to get rid of this problem we can switch to Newtonian Calculus at these points. At the points where the multiplicative derivative is not defined, we get the value of the function and the corresponding ordinary derivative and then use the classical Runge-Kutta Method for a few steps until the multiplicative derivative becomes again reasonably large. Afterwards, we can continue by taking the value of the function and the multiplicative derivative as an input for the Multiplicative Runge-Kutta method. The results are reasonably good, and often even better than using the classical Runge-Kutta method alone.

Assuming that $g(x)$ is a decreasing function, satisfying the conditions $g(x_{i-1}) > 0$, and $g(x_{i+1}) < 0$, then there is a point $\xi \in [x_{i-1}, x_{i+1}]$ such that $g(\xi) = 0$ (see Figure 2.1). As we have mentioned above, at the point $\xi$, where $g(\xi) = 0$, the multiplicative derivative of $g(x)$ is not defined. Therefore, on the intervals $[x_0, x_{i-1}]$, and $[x_{i+1}, x_n]$, where the multiplicative derivative is defined, we will apply the Multiplicative Runge-Kutta method. However, for the interval $[x_{i-1}, x_{i+1}]$ we will apply the classical Runge-Kutta Method. In order to apply the classical Runge-Kutta method we will use the values of $g(x_{i-1})$ and $g^*(x_{i-1})$, which are calculated by the Multiplicative Runge-Kutta Method, as input for the classical Runge-Kutta Method at the point $x_{i-1}$, and vice versa for the point $x_{i+1}$.

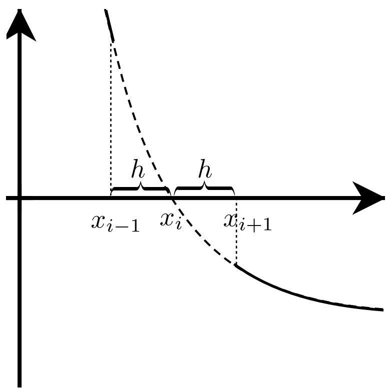



Figure 2.1: Bypass the roots where the multiplicative derivative becomes undefined. The dashed line denotes the region where the classical Runge-Kutta method is applied to prevent the multiplicative derivate to become infinite. The multiplicative Runge-Kutta method is applied in the region of the solid line.

# Chapter 3

## ERROR AND STABILITY ANALYSIS

### 3.1 Convergence Of One-Step Methods

In this section, the behaviour of the approximate solution $\eta(x;h)$ of the one-step method will be analyzed for convergence as $h \to 0$. Assuming that the function $g(x,y)$ is one time multiplicative differentiable on the open interval $(a,b)$ then the exact solution of the multiplicative initial value problem

$$y^* = g(x,y), \qquad y(x_0) = y_0. \tag{3.1.1}$$

is denoted by $y(x)$.

Let the one step method is defined by $\phi(x,y;h)$ as,

$$\begin{aligned} \eta_0 &:= y_0, \\ \text{for} \quad i = 0,1,\ldots, & \\ \eta_{i+1} &:= \eta_i \phi(x_i, \eta_i; h)^h, \\ x_{i+1} &:= x_i + h, \end{aligned} \tag{3.1.2}$$

The one step method defined as the system (3.1.2) generates the approximate solution $\eta(x;h)$:

$$\eta(x;h) := \eta_i, \qquad \text{if} \quad x = x_0 + ih. \tag{3.1.3}$$

for $x \in r_h := \{x_0 + ih \,|\, i = 0, 1, 2, \ldots\}$.

Let us choose the initial values $x$ and $y$ as fixed but arbitrary constants and denote $z(t)$ as the exact solution of the following multiplicative initial-value-problem

$$z^*(t) = g(t, z(t)), \qquad z(x) = y. \tag{3.1.4}$$

Then, for the exact solution $z(t)$ of the system (3.1.4), the multiplicative ratio function is represented as

$$\zeta(x, y; h) := \begin{cases} \left(\frac{z(x+h)}{y}\right)^h & \text{if } h \neq 0, \\ g(x, y) & \text{if } h = 0 \end{cases}, \tag{3.1.5}$$

for step size $h$, and the multiplicative ratio function for the approximate solution of the system (3.1.4), which is produced by $\phi$, is denoted by $\phi(x, y; h)$. The multiplicative ratio function corresponds to the difference quotient in Newtonian calculus.

For the ratio

$$\tau(x, y; h) := \frac{\zeta(x, y; h)}{\phi(x, y; h)}, \tag{3.1.6}$$

the magnitude shows how good the value $z(x+h)$ match the equation of the one-step method at $x + h$ .

At $(x, y)$, the *multiplicative local discretization error* is denoted by $\tau(x, y; h)$. For the one step method, we can require that

$$\lim_{h \to 0} \tau(x, y; h) = 1. \tag{3.1.7}$$

Choosing $x$ as fixed and letting $h \to 0$ where $h \in h_x := \left\{ \frac{(x-x_0)}{n} \,|\, n = 1,2,\ldots \right\}$, the point of interest is the behaviour of the *multiplicative global discretization error*

$$e(x;h) := \frac{\eta(x;h)}{y(x)}. \qquad (3.1.8)$$

Since, $\eta(x;h)$, is defined for $h \in h_x$, $e(x;h)$ is also only defined for the values of $h$ in $h_x$. Thus, we will study the convergence for

$$e(x;h_n), \qquad h_n := \frac{x-x_0}{n}, \qquad \text{as} \quad n \to \infty. \qquad (3.1.9)$$

For the values of $x$ in the closed interval $[a,b]$, and all of the functions $g(x,y)$ that are one time multiplicative differentiable on the open interval $(a,b)$, the one-step will be *convergent* if

$$\lim_{n\to\infty} e(x;h_n) = 1. \qquad (3.1.10)$$

For the functions $g(x,y)$ which are $p$-times multiplicative differentiable on the interval $(a,b)$, then we can say that the methods of order $p > 0$ are convergent, and satisfy

$$e(x;h_n) = O(e^{h_n^p}). \qquad (3.1.11)$$

Thus, the order of the multiplicative global discretization error is equal to the order of the multiplicative local discretization error.

**Lemma 3.1.1** *Let $\Xi_i$ be numbers satisfying the estimates given as*

$$|\Xi_{i+1}| \leq |\Xi_i|^{(1+\delta)}B, \qquad \delta > 0, \qquad B \geq 0, \qquad i = 0,1,2,..., \qquad (3.1.12)$$

*then*

$$|\Xi_n| \leq |\Xi_0|^{e^{n\delta}} B^{\frac{e^{n\delta}-1}{\delta}}. \qquad (3.1.13)$$

**Proof.** It is clear from the assumption that, since $0 < 1+\delta \leq e^{\delta}$ for $\delta > -1$, by mathematical induction we will obtain

$$
\begin{aligned}
|\Xi_1| &\leq |\Xi_0|^{(1+\delta)} \cdot B, & (3.1.14)\\
|\Xi_2| &\leq |\Xi_1|^{(1+\delta)} \cdot B & (3.1.15)\\
&\leq \left(|\Xi_0|^{(1+\delta)} \cdot B\right)^{(1+\delta)} \cdot B & (3.1.16)\\
&= |\Xi_0|^{(1+\delta)^2} \cdot B^{(1+\delta)} \cdot B & (3.1.17)\\
&= |\Xi_0|^{(1+\delta)^2} \cdot B^{1+(1+\delta)}, & (3.1.18)\\
|\Xi_3| &\leq |\Xi_2|^{(1+\delta)} \cdot B & (3.1.19)\\
&\leq \left(|\Xi_0|^{(1+\delta)^2} \cdot B^{1+(1+\delta)}\right)^{(1+\delta)} \cdot B & (3.1.20)\\
&= |\Xi_0|^{(1+\delta)^3} \cdot B^{(1+\delta)+(1+\delta)^2} \cdot B & (3.1.21)\\
&= |\Xi_0|^{(1+\delta)^3} \cdot B^{1+(1+\delta)+(1+\delta)^2}, & (3.1.22)\\
&\vdots & \\
|\Xi_{n-1}| &\leq |\Xi_0|^{(1+\delta)^{n-1}} \cdot B^{[1+(1+\delta)+(1+\delta)^2+\ldots+(1+\delta)^{n-2}]}, & (3.1.23)\\
|\Xi_n| &\leq |\Xi_0|^{(1+\delta)^n} B^{[1+(1+\delta)+(1+\delta)^2+\ldots+(1+\delta)^{n-1}]} & (3.1.24)\\
&= |\Xi_0|^{(1+\delta)^n} B^{\frac{(1+\delta)^n-1}{\delta}} & (3.1.25)\\
&\leq |\Xi_0|^{e^{n\delta}} B^{\frac{e^{n\delta}-1}{\delta}}, & (3.1.26)
\end{aligned}
$$

■

**Theorem 3.1.2** *Let us consider, the following multiplicative initial value problem, for the values of x and y, such that $x_0 \in [a,b]$, $y_0 \in \mathbb{R}$*

$$y^* = g(x,y), \quad y(x_0) = y_0, \tag{3.1.27}$$

*which has the exact solution $y(x)$. Assuming that $\phi$ is a continuous function on*

$$\Gamma := \left\{ (x,y,h) \,|\, a \leq x \leq b, \left|\frac{y}{y(x)}\right| \leq \gamma, 0 \leq |h| \leq h_0 \right\}, h_0 > 0, \gamma > 1, \tag{3.1.28}$$

*and M and N are positive constants such that*

$$\left|\frac{\phi(x,y_1;h)}{\phi(x,y_2;h)}\right| \le \left|\frac{y_1}{y_2}\right|^M, \tag{3.1.29}$$

*for all values of* $(x,y_i,h) \in \Gamma$, $i = 1,2$, *and*

$$|\tau(x,y(x);h)| = \left|\frac{\zeta(x,y(x);h)}{\phi(x,y(x);h)}\right| \le e^{N|h|^p}, \quad p > 0, \tag{3.1.30}$$

*for all* $x \in [a,b]$, $|h| \le h_0$.

*Then, for the multiplicative global discretization error* $e(x;h) = \dfrac{\eta(x;h)}{y(x)}$, *there exists an* $\overline{h}$, $0 < \overline{h} \le h_0$, *such that*

$$|e(x;h_n)| \le e^{|h_n|^p N \frac{e^{M|x-x_0|}-1}{M}}, \tag{3.1.31}$$

*for all values of* $x \in [a,b]$ *and* $h_n = \frac{x-x_0}{n}$, $n = 1,2,\ldots$, *where* $|h_n| \le \overline{h}$. *If* $\gamma = \infty$, *then* $\overline{h} = h_0$.

**Proof.** It is obvious that the function

$$\widetilde{\phi}(x,y;h) = \begin{cases} \phi(x,y;h) & \text{if } (x,y;h) \in \Gamma \\ \phi(x,y(x)\gamma;h) & \text{if } x \in [a,b],\ |h| \le h_0,\ y \ge y(x)\gamma \\ \phi(x,\frac{y(x)}{\gamma};h) & \text{if } x \in [a,b],\ |h| \le h_0,\ y \le \frac{y(x)}{\gamma} \end{cases} \tag{3.1.32}$$

is continuous on $\widetilde{\Gamma} := \{(x,y,h)\,|\, x \in [a,b],\ y \in \mathbb{R},\ |h| \ge h_0\}$ and satisfying the condition

$$\left|\frac{\widetilde{\phi}(x,y_1;h)}{\widetilde{\phi}(x,y_2;h)}\right| \le \left|\frac{y_1}{y_2}\right|^M, \tag{3.1.33}$$

for all values of $(x,y_i,h) \in \widetilde{\Gamma}, i = 1,2$, and since $\widetilde{\phi}(x,y(x);h) = \phi(x,y(x);h)$, the following condition is also satisfied

$$\left|\frac{\zeta(x,y(x);h)}{\widetilde{\phi}(x,y(x);h)}\right| \leq e^{N|h|^p}, \quad \text{for } x \in [a,b],\ |h| \leq h_0. \tag{3.1.34}$$

Assuming that the one-step method, generated by $\widetilde{\phi}$, provides the approximate values $\widetilde{\eta}_i := \widetilde{\eta}(x_i;h)$ for $y_i := y(x_i)$, $x_i := x_0 + ih$:

$$\widetilde{\eta}_{i+1} = \widetilde{\eta}_i \cdot \widetilde{\phi}(x_i, \widetilde{\eta}_i; h)^h. \tag{3.1.35}$$

Then with the help of

$$y_{i+1} = y_i \cdot \zeta(x_i, y_i; h)^h, \tag{3.1.36}$$

we obtain the recurrence formula, for the error $\widetilde{e_i} := \frac{\widetilde{\eta}_i}{y_i}$ as

$$\widetilde{e}_{i+1} = \widetilde{e}_i \cdot \left[\frac{\widetilde{\phi}(x_i, \widetilde{\eta}_i; h)}{\widetilde{\phi}(x_i, y_i; h)}\right]^h \cdot \left[\frac{\widetilde{\phi}(x_i, y_i; h)}{\zeta(x_i, y_i; h)}\right]^h. \tag{3.1.37}$$

Thus (3.1.33) and (3.1.34) implies that

$$\left|\frac{\widetilde{\phi}(x_i, \widetilde{\eta}_i; h)}{\widetilde{\phi}(x_i, y_i; h)}\right| \leq \left|\frac{\widetilde{\eta}_i}{y_i}\right|^M = |\widetilde{e_i}|^M \tag{3.1.38}$$

$$\left|\frac{\widetilde{\phi}(x_i, y_i; h)}{\zeta(x_i, y_i; h)}\right| \leq e^{N|h|^p} \tag{3.1.39}$$

and hence from (3.1.37) we get the recursive estimate

$$|\widetilde{e}_{i+1}| \leq |\widetilde{e_i}| \cdot |\widetilde{e_i}|^{|h|M} \cdot e^{N|h|^{p+1}}. \tag{3.1.40}$$

Since we are solving a multiplicative initial value problem, we will consider the initial values as exact, and start the iterations with $\widetilde{e}_0 = \dfrac{\widetilde{\eta}_0}{y_0} = 1$ as follows

$$|\widetilde{e}_0| = 1, \tag{3.1.41}$$

$$|\widetilde{e}_1| \leq |\widetilde{e}_0| \left[|\widetilde{e}_0|^M\right]^h \left[\left|e^{N|h|^p}\right|\right]^h \tag{3.1.42}$$

$$= \left|e^{N|h|^{p+1}}\right|, \tag{3.1.43}$$

$$
\begin{aligned}
|\widetilde{e}_2| &\leq |\widetilde{e}_1| \left[|\widetilde{e}_1|^M\right]^h \left[\left|e^{N|h|^p}\right|\right]^h && (3.1.44)\\
&\leq \left|e^{N|h|^{p+1}}\right| \left[\left|e^{N|h|^{p+1}}\right|^M\right]^h \left[\left|e^{N|h|^p}\right|\right]^h && (3.1.45)\\
&= \left|e^{N|h|^{p+1}}\right| \left[\left|e^{|h|MN|h|^{p+1}}\right|\right] \left[\left|e^{N|h|^{p+1}}\right|\right] && (3.1.46)\\
&= \left|e^{N|h|^{p+1}(1+(1+|h|M))}\right|, && (3.1.47)\\
|\widetilde{e}_3| &\leq |\widetilde{e}_2| \left[|\widetilde{e}_2|^M\right]^h \left[\left|e^{N|h|^p}\right|\right]^h && (3.1.48)\\
&\leq \left|e^{N|h|^{p+1}(1+(1+|h|M))}\right| \left[\left|e^{N|h|^{p+1}(1+(1+|h|M))}\right|^M\right]^h \left[\left|e^{N|h|^p}\right|\right]^h && (3.1.49)\\
&= \left|e^{(1+(1+|h|M))N|h|^{p+1}}\right| \left[\left|e^{|h|M(1+(1+|h|M))N|h|^{p+1}}\right|\right] \left[\left|e^{N|h|^{p+1}}\right|\right] && (3.1.50)\\
&= \left|e^{N|h|^{p+1}((1+(1+|h|M))+|h|M(1+(1+|h|M))+1)}\right| && (3.1.51)\\
&= \left|e^{N|h|^{p+1}(1+(1+|h|M)+(1+|h|M)^2)}\right|, && (3.1.52)\\
&\vdots\\
|\widetilde{e}_k| &\leq |\widetilde{e}_{k-1}| \left[|\widetilde{e}_{k-1}|^M\right]^h \left[\left|e^{N|h|^p}\right|\right]^h && (3.1.53)\\
&\leq \left|e^{N|h|^{p+1}(1+(1+|h|M)+(1+|h|M)^2+\cdots+(1+|h|M)^{k-1})}\right| && (3.1.54)\\
&= e^{N|h|^{p+1}\frac{(1+|h|M)^k-1}{|h|M}} && (3.1.55)\\
&= e^{N|h|^{p}\frac{(1+|h|M)^k-1}{M}} && (3.1.56)\\
&\leq e^{\frac{e^{k|h|M}-1}{M}N|h|^p}. && (3.1.57)
\end{aligned}
$$

Thus the recursive estimate in equation (3.1.40) simplifies to

$$|\widetilde{e}_k| \leq e^{N|h|^p\frac{e^{k|h|M}-1}{M}}. \tag{3.1.58}$$

Now let us choose $x$ in the closed interval $[a,b]$ satisfying $x \neq x_0$, as a fixed constant and $h := h_n = \frac{(x-x_0)}{n}$, $n > 0$ as an integer. Then, using $x_n = x_0 + nh = x$ and $k = n$, and also keeping in mind that $\widetilde{e}(x;h_n) = \widetilde{e}_n$, equation (3.1.58) can be written as

$$|\widetilde{e}(x;h_n)| \leq e^{N|h_n|^p\frac{e^{M|x-x_0|}-1}{M}}, \tag{3.1.59}$$

for all $x \in [a,b]$ and $h_n$ with $|h_n| \leq h_0$.

Since $|x-x_0| \leq |b-a|$ and $\gamma > 0$, there exists an $\overline{h}$, $0 < \overline{h} \leq h_0$, such that $|\widetilde{e}(x,;h_n)| \leq \gamma$ for all $x \in [a,b]$, $|h_n| \leq \overline{h}$, i.e., for the one-step method generated by $\Phi$,

$$\begin{aligned} \eta_0 &= y_0, \\ \eta_{i+1} &= \eta_i \phi(x_i, \eta_i; h), \end{aligned}$$

we have for $|h| \leq \overline{h}$, according to the definition of $\widetilde{\phi}$,

$$\widetilde{\eta}_i = \eta_i, \qquad \widetilde{e}_i = e_i, \qquad \textit{and} \qquad \widetilde{\phi}(x_i, \widetilde{\eta}_i; h) = \phi(x_i, \eta_i; h).$$

The assertion of the theorem,

$$|\widetilde{e}(x;h_n)| \leq e^{N|h_n|^p \frac{e^{M|x-x_0|}-1}{M}},$$

thus follows for all $x \in [a,b]$ and all $h_n = \frac{(x-x_0)}{n}$, $n = 1,2,\ldots$, with $|h_n| \leq \overline{h}$. ■

## 3.2 Stability Analysis

In this section, we will focus on the stability analysis of the multiplicative Runge-Kutta methods. The presence of the multiplicative Butcher Tableau allows us to analyze the stability of the $n$-th order Multiplicative Runge-Kutta method, but the analysis will be conducted exemplarily for the $4^{\text{th}}$ order Multiplicative Runge-Kutta method to be able to show explicitly its behaviour. In Newtonian calculus, the stability properties of the Runge-Kutta methods are analysed by the following basic test equation.

$$y'(x) = \lambda y(x), \qquad y(x_0) = y_0, \tag{3.2.1}$$

where $\lambda \in \mathbb{C}$. The behaviour of (3.2.1) was studied extensively by [11, 12, 19, 25]. The stability analysis of the Multiplicative Runge-Kutta methods can also be done based on this test equation. In order to do this, we rewrite the test equation in terms of the multiplicative calculus. We will consider the 4th order MRK method as denoted in (2.3.19) - (2.3.23). By (2.3.19) we obtain

$$y_{n+1} = y_n [g_0^{\alpha} \cdot g_1^{\beta} \cdot g_2^{\gamma} \cdot g_3^{\delta}]^h, \tag{3.2.2}$$

where

$$\alpha + \beta + \gamma + \delta = 1. \tag{3.2.3}$$

In analogy to [21] the multiplicative form of the basic test equation is given as

$$y^*(x) = e^{\lambda}, \qquad y(x_0) = y_0, \tag{3.2.4}$$

which has the analytic solution

$$y(x) = e^{\lambda(x - x_0)} y_0. \tag{3.2.5}$$

As $x \to \infty$ and $Re(\lambda) < 0$, the solution of the system approaches to zero. If the method also has the same behaviour, then we can say that the method is A-stable [11]. Since $y^*(x)$ is a constant function, equations (2.3.20)-(2.3.23) simplify to $g_0 = g_1 = g_2 = g_3 = e^{\lambda}$. Then, by (3.2.4) and (3.2.2), we obtain

$$\frac{y_{n+1}}{y_n} = e^z = R(z), \tag{3.2.6}$$

where $z = \lambda h$. $R(z)$ is the stability function of the proposed method. Then, the domain of stability is

$$s^* = \{z \in \mathbb{C} : |R(z)| < 1\}. \tag{3.2.7}$$

Consequently, by (3.2.7) we obtain

$$0 < e^{-|\lambda| h} < 1, \tag{3.2.8}$$

which leads to

$$0 < h < \infty. \tag{3.2.9}$$

Thus, the result shows that the proposed method is unconditionally stable. By (3.2.7), it can be seen that $Re(z) < 0$ where $|e^z| = e^{Re(z)}$. When $Re(z) < 0$ the left half plane will be the region of absolute stability, thus the method is A-stable. In Newtonian calculus, the explicit multistep methods can not be A-stable and the implicit multistep methods can be A-stable if the order is at most 2. Whereas in Multiplicative calculus both explicit and implicit methods are A-stable. One can say that a method is L-stable if the method is A-stable and $R(z) \to 0$ when $|z| \to \infty$ [12]. Since we have shown that the Multiplicative Runge-Kutta methods are A-stable and $e^z \to 0$ when $|z| \to \infty$, we can say that the proposed methods are L-stable by [12].

# Chapter 4

## APPLICATIONS OF THE MULTIPLICATIVE RUNGE-KUTTA METHOD

### 4.1 Solution of first order multiplicative differential equations

**Example 1.** (Square Root)

We want to discuss the following multiplicative initial value problem, where no exponential function or logarithm is involved in the exact solution.

$$y^{*}(x) = e^{\frac{1}{2y^{2}}}, \qquad y(0) = 1. \tag{4.1.1}$$

The corresponding Newtonian initial value problem for (4.1.1) becomes

$$y'(x) = \frac{1}{2y}, \qquad y(0) = 1, \tag{4.1.2}$$

where the exact solution of both Multiplicative and Newtonian initial value problems given in (4.1.1) and (4.1.2) is

$$y(x) = \sqrt{x+1}. \tag{4.1.3}$$

The Multiplicative initial value problem given in (4.1.1) is solved by using the $4^{\text{th}}$ order multiplicative Runge-Kutta method while the Newtonian initial value problem given in (4.1.2) is solved by the $4^{\text{th}}$ order Runge-Kutta method. The results of both methods are listed in the following table.

Table 4.1: Comparison of the multiplicative Runge-Kutta method and classical Runge-Kutta method. MRK4 and RK4 corresponds to $4^{th}$ order multiplicative Runge-Kutta method and $4^{th}$ order Runge-Kutta method respectively.

| $x$ | $y_{exact}$ | $y_{MRK4}$ | relative $err_{MRK4}$ in % | $y_{RK4}$ | relative $err_{RK4}$ in % |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 0 |
| 0.6 | 1.2649111 | 1.2649153 | $3.38 \times 10^{-6}$ | 1.2382302 | 0.021093074 |
| 1.2 | 1.4832397 | 1.483244 | $2.88 \times 10^{-6}$ | 1.4409643 | 0.028502049 |
| 1.8 | 1.6733201 | 1.673324 | $2.36 \times 10^{-6}$ | 1.6205072 | 0.031561693 |
| 2.4 | 1.8439089 | 1.8439125 | $1.97 \times 10^{-6}$ | 1.783364 | 0.032835088 |
| 3 | 2 | 2.0000034 | $1.69 \times 10^{-6}$ | 1.9334697 | 0.033265139 |

It is clear from Table 4.2 that the relative error of the $4^{th}$ order Multiplicative Runge-Kutta method is 4 orders less in magnitude compared to the $4^{th}$ order Runge-Kutta method. This is in well agreement to the error analysis presented in section 3.

Moreover, the basic operations used in multiplicative calculus are mainly multiplication, division, calculation of the exponential function and calculation of the logarithm function, while in Newtonian calculus, we just consider summation, subtraction and multiplication.

Letting the size of all numbers be $n$-bit, the computational complexities for the following arithmetic operations can be listed as:

Table 4.2: Computational complexities

| *Operation* | *Complexity* |
|---|---|
| addition and subtraction | O(n) |
| multiplication and division | $O(n^2)$ |
| exponential and logarithm | $O(n^{5/2})$ |

So, according to the computational complexities listed above it is clear that the $4^{th}$ order multiplicative Runge-Kutta method requires less number of operations compared to the $4^{th}$ order Runge-Kutta method. In order to be able to consider the $4^{th}$ order multiplicative Runge-Kutta method as a serious alternative to the $4^{th}$ order Runge-Kutta method, the performance of the proposed method has to be at least comparable. Here performance points out, higher accuracy, i.e. smaller errors, for the same computation time. Hence, we have measured the relative error as a function of the computation time by keeping the starting and the end point fixed with varying step size $h$. The comparison of the results for both methods are shown in figure 4.1.

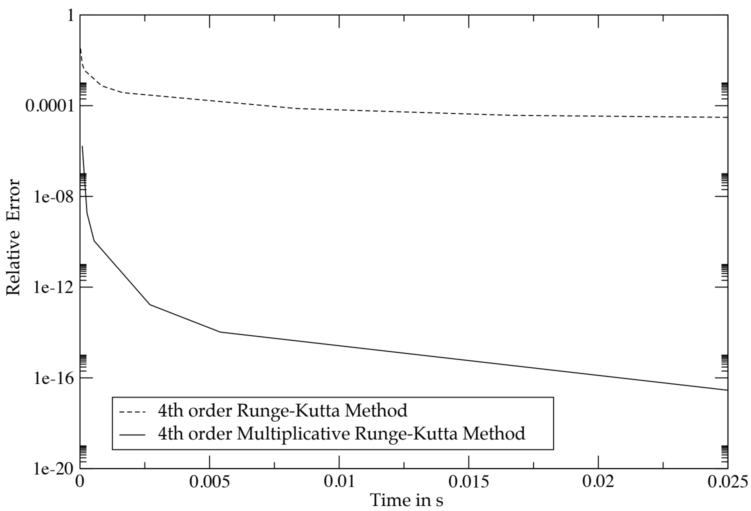



Figure 4.1: Comparison of the computation time and the relative error for the multiplicative initial value problem (4.1.1) and the initial value problem (4.1.2) for the same initial values $x_0 = 0$ and $y_0 = 1$ and fixed final values $x_n = 3$, $y_n = 2$ by varying $h$.

As it can be seen from figure 4.1, comparing the relative errors as function of the computation time shows that the multiplicative Runge-Kutta method is working more efficiently compared to the Runge-Kutta method, since there is a significant difference between the relative errors. The comparison has been carried out also for other sample problems with known closed form solutions. The results indicate that the 4$^{\text{th}}$ order multiplicative Runge-Kutta method is more powerful compared to the 4$^{\text{th}}$ order Runge-Kutta method.

**Example 2.** (Logarithmic Solution)

The solution of the first initial value problem did not contain an exponential or logarithmic function. For the second example, let us consider a function which has logarithmic solution. We will concentrate on the following multiplicative initial value problem

$$y^*(x) = e^{\frac{x-1}{xy}}, \qquad y(1) = 1. \tag{4.1.4}$$

The ordinary initial value problem corresponding to (4.1.4) can be written as:

$$y'(x) = 1 - \frac{1}{x}, \qquad y(1) = 1, \tag{4.1.5}$$

where the analytic solution of both initial value problems is obtained as

$$y(x) = x - \ln x. \tag{4.1.6}$$

The results of the 4$^{\text{th}}$ order multiplicative Runge-Kutta method and the 4$^{\text{th}}$ order multiplicative Runge-Kutta method as well as the relative errors of each method are shown in the following table.

Table 4.3: Comparison of the multiplicative Runge-Kutta method and the classical Runge-Kutta method. MRK4 and RK4 corresponds to $4^{th}$ order multiplicative Runge-Kutta method and $4^{th}$ order Runge-Kutta method respectively.

| $x$ | $y_{exact}$ | $y_{MRK4}$ | relative $err_{MRK4}$ in % | $y_{RK4}$ | relative $err_{RK4}$ in % |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 0 |
| 1.5 | 1.0945 | 1.0945 | $3.128 \times 10^{-3}$ | 1.2123 | 10.76 |
| 2 | 1.3069 | 1.3068 | $2.955 \times 10^{-3}$ | 1.4892 | 13.95 |
| 2.5 | 1.5837 | 1.5837 | $2.681 \times 10^{-3}$ | 1.8068 | 14.09 |
| 3 | 1.9014 | 1.9013 | $2.339 \times 10^{-3}$ | 2.1527 | 13.22 |

Comparison of the relative errors indicates that, using the multiplicative Runge-Kutta method produces more accurate results compared to the classical Runge-Kutta method.

**Example 3.** (Biological Example)

In order to prove that the newly defined method can also be applied to mathematical models in different fields, to get better results, we will discuss the model of the bacterial growth in food which was modelled by Huang [16, 17, 18]. We will focus on the Baranyi model [2, 3] for the bacterial growth in food. The model is defined by the differential equation

$$y'(t) = \mu_{max} \frac{1 - e^{y - y_{max}}}{1 + e^{-\alpha(t-\lambda)}}. \tag{4.1.7}$$

The corresponding multiplicative differential equation for (4.1.7) is:

$$y^*(t) = \exp\left\{ \frac{\mu_{max}}{y} \frac{1 - e^{y - y_{max}}}{1 + e^{-\alpha(t-\lambda)}} \right\}, \tag{4.1.8}$$

with the initial value $y_0 = y(0) = 7$.

Since there is no closed form solution available for the initial value problems (4.1.7) and (4.1.8), we have solved the initial value problems by the $4^{th}$ order multiplicative Runge-Kutta method and the $4^{th}$ order Runge-Kutta method for small $h$, where both solutions coincide. Then by increasing the step size $h$ we have checked which method deviates first from the solutions, that we have obtained for small $h$. As it is represented in figure 4.2, the $4^{th}$ order Runge-Kutta method deviates first. Then we have compared the solutions for the greatest $h$, where $4^{th}$ order multiplicative Runge-Kutta method still match with the solutions for small $h$, where the $4^{th}$ order Runge-Kutta method does not. Also in this case, the performance results for the $4^{th}$ order multiplicative Runge-Kutta method are better compared to $4^{th}$ order Runge-Kutta method.

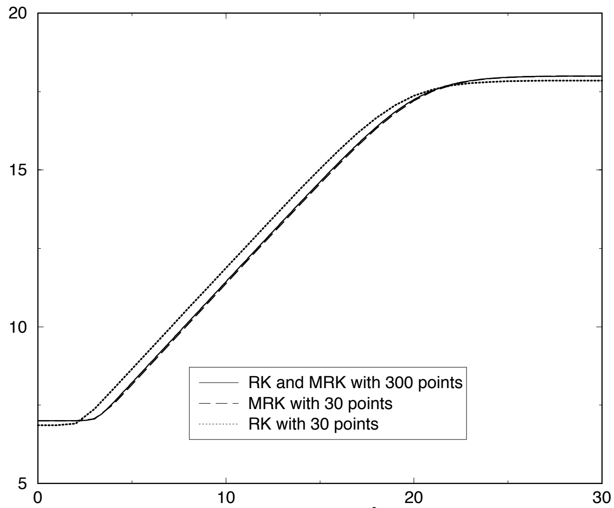



Figure 4.2: Solution for bacteria growth model, $\lambda = 3.21$, $\mu_{max} = 0.644$, $\alpha = 4$, $y_{max} = 18$.

It is clear from the figure 4.2 that, the numerical solutions of the differential equations (4.1.7) and (4.1.8) using the corresponding Runge-Kutta Methods are not distinguishable for $h = 0.1$. Furthermore, for bigger $h$ values, i.e. $h = 1$, the $4^{th}$ order Multiplicative Runge-Kutta method still mathches with the solution for $h = 0.1$, while the $4^{th}$ order Runge-Kutta method gives significantly different results (dotted line).

## 4.2 Solution of a second order multiplicative differential equation

**Example 1.** ($2^{\text{nd}}$ order Differential Equation)

Multiplicative Runge-Kutta methods are also applicable for solving the higher order multiplicative initial value problems. As an example, we will consider a $2^{\text{nd}}$ order multiplicative initial value problem, which has the following model

$$y^{**}(x) = f(x,y,y^*), \quad y(x_0) = y_0, \quad \text{and } y^*(x_0) = y_1. \tag{4.2.1}$$

In order to solve this type of multiplicative initial value problem, we need to solve the coupled system of first order multiplicative differential equations

$$y_0^*(x) = y_1(x), \tag{4.2.2}$$

$$y_1^*(x) = f(x,y_0,y_1). \tag{4.2.3}$$

Exemplarily, we will solve the multiplicative initial value problem for the $2^{\text{nd}}$ order multiplicative differential equation

$$y^{**}(x) = e. \tag{4.2.4}$$

The second order differential equation which corresponds to (4.2.4) is

$$y''(x) = \frac{y'(x)^2}{y(x)} + y(x), \tag{4.2.5}$$

where the exact solution of both differential equations (4.2.4) and (4.2) is

$$y(x) = \alpha \exp\left\{\frac{x^2}{2} + \beta x\right\}. \tag{4.2.6}$$

The same initial value problem, was also solved by using the multiplicative finite difference method, as an example for a multiplicative boundary value problem in [28]. To compare the solutions that we have obtained from the $4^{\text{th}}$ order Multiplicative Runge-Kutta method with the solutions of the multiplicative finite difference method, discussed in [28], we select $\alpha = 1$, $\beta = 1$, $x_0 = 1$, and $h = 0.25$. The choice of the constants results in the following initial conditions

$$y_0 = e^{3/2} \quad \text{and} \quad y_1 = e^2. \tag{4.2.7}$$

The results of both methods are summarized in the following table.

Table 4.4: Comparison of the multiplicative Runge-Kutta method and multiplicative Finite Difference method. MRK4 and MFD corresponds to 4th order multiplicative Runge-Kutta method and multiplicative finite difference method respectively.

| $x$ | $y_{exact}$ | $y_{MRK4}$ | relative $err_{MRK4}$in % | $y_{MFD}$ | relative $err_{MFD}$in % |
|---|---|---|---|---|---|
| 1 | 4.48168907 | 4.481689070 | 0 | 4.48168907 | 0 |
| 1.25 | 7.62360992 | 7.62360992 | $9.3 \times 10^{-15}$ | 7.62360991 | $3.5 \times 10^{-13}$ |
| 1.5 | 13.80457419 | 13.80457419 | $1.3 \times 10^{-14}$ | 13.80457418 | $5.3 \times 10^{-13}$ |
| 1.75 | 26.60901319 | 26.60901319 | $1.7 \times 10^{-14}$ | 26.60913187 | $1.8 \times 10^{-13}$ |

Table 4.4 shows the numerical approximation using the 4th order multiplicative Runge-Kutta method for (4.2.4) with the initial conditions (4.2.7) and the corresponding results for the multiplicative finite difference method from [28]. Numerical approximations for the 2nd order multiplicative differential equation (4.2.4), using both methods, shows that the 4th order multiplicative Runge-Kutta method gives slightly better results than the multiplicative Finite Difference method. The relative error for the 4th order multiplicative Runge-Kutta method is less by one order.

On the other hand, in order to compare the results of the proposed method with the results of the 4th order Runge-Kutta method, the ordinary differential equation (4.2) is solved with the corresponding initial values

$$y_0 = e^{3/2} \quad \text{and} \quad y_1 = 2e^{3/2}. \tag{4.2.8}$$

The results obtained from the solution of the second order multiplicative and ordinary differential equations (4.2.4) and as well as the corresponding relative errors are shown in table 4.5 below.

Table 4.5: Comparison of the multiplicative Runge-Kutta method and the classical Runge-Kutta method. MRK4 and RK4 corresponds to $4^{\text{th}}$ order multiplicative Runge-Kutta method and $4^{\text{th}}$ order Runge-Kutta method respectively.

| $x$ | $y_{exact}$ | $y_{MRK4}$ | relative $err_{MRK4}$in % | $y_{RK4}$ | relative $err_{RK4}$in % |
|---|---|---|---|---|---|
| 1 | 4.48168907 | 4.481689070 | 0 | 4.48168907 | 0 |
| 1.25 | 7.62360992 | 7.62360992 | $9.3 \times 10^{-15}$ | 7.61823131 | $7.1 \times 10^{-2}$ |
| 1.5 | 13.80457419 | 13.80457419 | $1.3 \times 10^{-14}$ | 13.77941017 | $1.8 \times 10^{-1}$ |
| 1.75 | 26.60901319 | 26.60901319 | $1.7 \times 10^{-14}$ | 26.51619718 | $3.5 \times 10^{-1}$ |

The comparison of the relative errors shows that the $4^{\text{th}}$ order Runge-Kutta method fails drastically, since the relative error differs by 13 orders in magnitude compared to its multiplicative counterpart. The results in table 4.4 and table 4.5 shows that both the $4^{\text{th}}$ order multiplicative Runge-Kutta method and the multiplicative finite difference method succeed to give proper results for this example, while the results of the $4^{\text{th}}$ order Runge-Kutta method are not that much accurate.

# Chapter 5

## APPLICATION OF THE MULTIPLICATIVE RUNGE-KUTTA METHODS IN CHAOS THEORY

### 5.1 Solution of a system of multiplicative differential equation

We want also to show that the method, developed for the universal applicability of the Multiplicative Runge-Kutta Method in section 2.4, works without major problems. Therefore we chose exemplarily the Rössler attractor [29, 30] to show that the MRK method can be extended to higher dimensions. Obviously in the Rössler attractor problem $x(t)$, $y(t)$ have roots and therefore the MRK method seems not be applicable. Using the extension proposed in section 2.4, the Rössler attractor problem becomes accessible also for the MRK method producing reasonable results. In the following we will give the general equations for the Rössler attractor problem in ordinary calculus and its multiplicative counterpart.

$$\dot{x}(t) = -y(t) - z(t), \tag{5.1.1}$$

$$\dot{y}(t) = x(t) + \alpha y(t), \tag{5.1.2}$$

$$\dot{z}(t) = \beta + (x(t) - \gamma) z(t). \tag{5.1.3}$$

The corresponding multiplicative counterparts of the equations (5.1.1)-(5.1.3) are then

$$x^*(t) = \exp\left\{-\frac{(y(t)+z(t))}{x(t)}\right\}, \quad (5.1.4)$$

$$y^*(t) = \exp\left\{\frac{x(t)+\alpha y(t)}{y(t)}\right\}, \quad (5.1.5)$$

$$z^*(t) = \exp\left\{\frac{\beta+(x(t)-\gamma)z(t)}{z(t)}\right\}. \quad (5.1.6)$$

Solving the system of multiplicative differential equations (5.1.4)-(5.1.6) using the 4th order MRK method for the parameters, $\alpha = \beta = 0.2$ and $\gamma = 8$ we get the result depicted in figure 5.1. The results of the classical Runge-Kutta and the multiplicative Runge- Kutta methods are comparable.

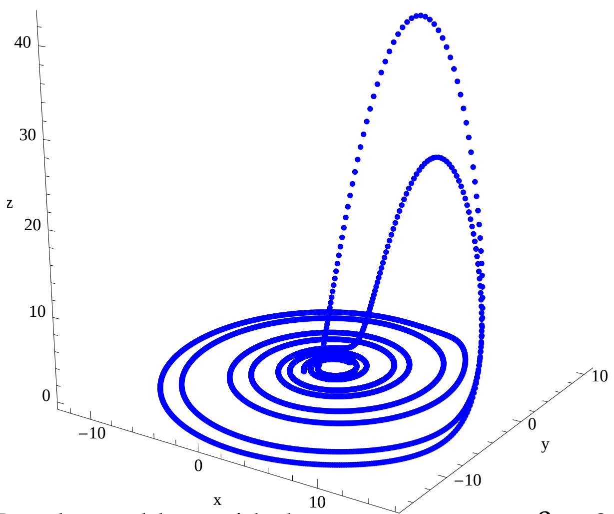



Figure 5.1: Rössler problem with the parameters $\alpha = \beta = 0.2$ and $\gamma = 8$.

For the solution of the coupled Multiplicative Runge-Kutta equation, all methods described in section 2.4 to remove the restrictions of geometric multiplicative calculus, were tested explicitly and the usage of the classical Runge-Kutta method for the transition region of the function gave the best results.

## 5.2 A Modified Quadratic Lorenz Attractor

In this section, we will concentrate on the applicability of the proposed method in chaotic systems. Instead of using a chaotic system which is already defined, we will define a new chaotic system. Firstly, we will analyze the new chaotic system for its chaotic behaviour in Newtonian calculus, afterwards we will define the system in Multiplicative calculus and examine its chaotic behaviour using the properties of Multiplicative calculus. Both systems, which are defined in Newtonian and Multiplicative calculus, will be solved by using the $4^{\text{th}}$ order Runge-Kutta method and the $4^{\text{th}}$ order Multiplicative Runge-Kutta method correspondingly.

### 5.2.1 Design of a new Chaotic System

The new Chaotic system will be derived from the Lorenz system, which is defined in [20] as

$$\frac{dx}{dt} = s(y-x), \tag{5.2.1}$$

$$\frac{dy}{dt} = x(r-z)-y, \tag{5.2.2}$$

$$\frac{dz}{dt} = xy-bz. \tag{5.2.3}$$

The chaotic system, named as the Modified Quadratic Lorenz attractor, is generated by the following simple three-dimensional system

$$\frac{dx}{dt} = s(yz-x), \tag{5.2.4}$$

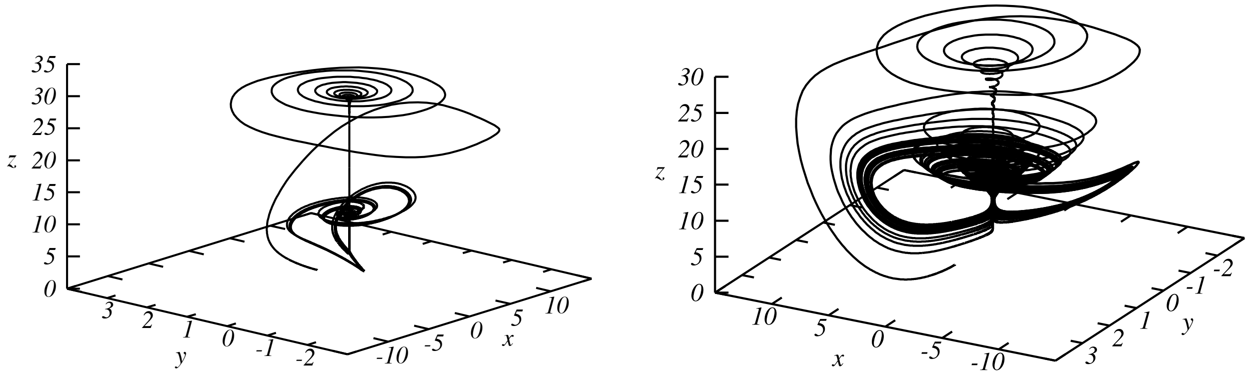
$$\frac{dy}{dt} = rx-xz, \tag{5.2.5}$$

$$\frac{dz}{dt} = (xy)^2-bz, \tag{5.2.6}$$

where $x$, $y$, and $z$ are variables and $s$, $r$, and $b$ are real parameters. In the new proposed chaotic system, all the equations have some differences compared to the original Lorenz system. In order to see the differences between the two systems we can compare the equations (5.2.1)-(5.2.3) and (5.2.4)-(5.2.6) one by one. Evidently, in the Lorenz system equation (5.2.1) is linear, whereas equation (5.2.4) is non-linear. Furthermore, equations (5.2.2) and (5.2.5) are both non-linear, where the $y$-dependence of equation (5.2.2) is eliminated in the equation (5.2.5). The most significant difference can be observed from the comparison of equations (5.2.3) and (5.2.6). In (5.2.6), the term $xy$ is squared compared to the equation (5.2.3).

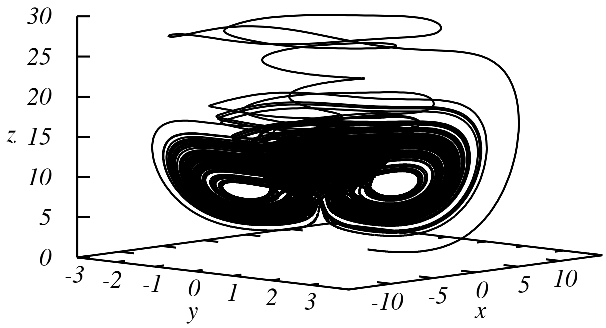
### 5.2.2 System Description

The initial values and the parameters of the system are chosen as $(1,1,1)$ and $s = 12$, $r = 8$ with varying $b$. The system is solved by using the 4$^{\text{th}}$ order Runge-Kutta method, for various $b$ values, resulting in the solutions given in the graphs of Figure 5.3. It can be observed from the graphs that, the behavior of the new chaotic system changes depending on the different values of $b$.

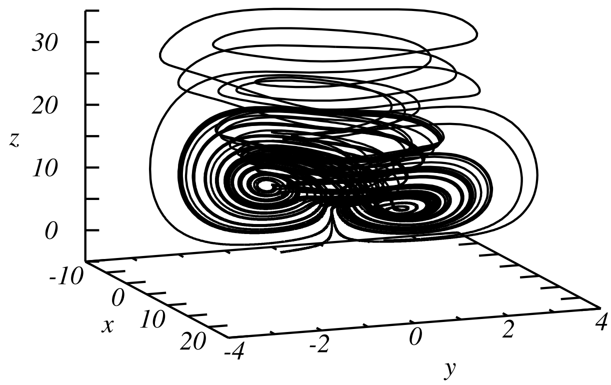



(a) $b = 0.1$

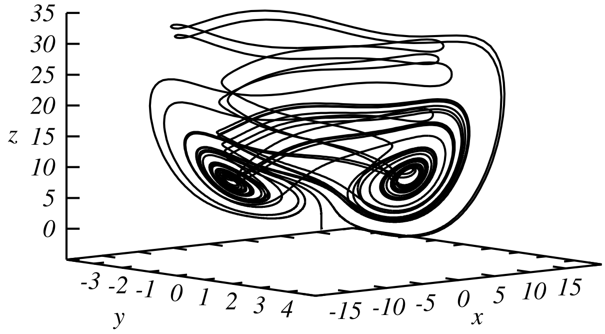
(b) $b = 0.5$




(a) $b = 2$




(b) $b = 4$




(c) $b = 10$

Figure 5.3: Simulation of the chaotic system when $s = 12$, $r = 8$ and various $b$ values

As a result of the calculations, as it is shown in Figure 5.3, the new system can be considered as a chaotic system for the parameters

$$s = 12, r = 8, \text{ and } b = 4. \tag{5.2.7}$$

Thus, from now on, the rest of the calculations will be done by using the parameters choosen above, to analyze the chaotic behaviour of the system.

### 5.2.3 System Analysis

The first step to analyze a chaotic system is to find the equilibrium points. In order to determine the equilibrium points of the proposed system (5.2.4)-(5.2.6), we need to solve the system

$$
\begin{aligned}
s(yz-x) &= 0, \\
rx-xz &= 0, \qquad (5.2.8) \\
(xy)^2-bz &= 0.
\end{aligned}
$$

Thus, the solution of the system (5.2.8) with respect to $x$, $y$, $z$ give the equilibria points as:

$$
\begin{aligned}
O &= (0,0,0), && (5.2.9) \\
E^+ &= \left(\sqrt[4]{br^3}, \sqrt[4]{\frac{b}{r}}, r\right), && (5.2.10) \\
E^- &= \left(-\sqrt[4]{br^3}, -\sqrt[4]{\frac{b}{r}}, r\right). && (5.2.11)
\end{aligned}
$$

Then, for the parameters chosen in (5.2.7), the calculated numerical values of the equilibria points are

$$
\begin{aligned}
O &= (0,0,0), && (5.2.12) \\
E^+ &= (6.73, 0.84, 8), && (5.2.13) \\
E^- &= (-6.73, -0.84, 8). && (5.2.14)
\end{aligned}
$$

In order to decide on the stability of the new proposed system, the eigenvalues of the Jacobian matrix must be analyzed.

Remembering that the Jacobian matrix of a $3 \times 3$ system can be written as

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}, \tag{5.2.15}$$

and considering the system (5.2.4)-(5.2.6) as

$$f_1 = \frac{dx}{dt} = s(yz - x), \tag{5.2.16}$$

$$f_2 = \frac{dy}{dt} = rx - xz, \tag{5.2.17}$$

$$f_3 = \frac{dz}{dt} = (xy)^2 - bz, \tag{5.2.18}$$

the Jacobian matrix for the system (5.2.16)-(5.2.18) can be easily obtained as

$$J = \begin{bmatrix} -s & sz & sy \\ r - z & 0 & -x \\ 2xy^2 & 2x^2y & -b \end{bmatrix}. \tag{5.2.19}$$

The expressions for the eigenvalues of the Jacobian matrix (5.2.19) are very long and complicated. As we are only interested in the numerical values of the eigenvalues at the equilibria points (5.2.12)-(5.2.14) for the given parameters (5.2.7), the eigenvalues corresponding to the equilibrium points $O, E^+$ and $E^-$ are stated in the table below:

Table 5.1: Eigenvalues of the Jacobian at the equilibrium points

| Equilibrium Point | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|
| $O$ | $-12$ | $-4$ | $0$ |
| $E^+$ | $2.65+23.87i$ | $2.65-23.87i$ | $-21.3$ |
| $E^-$ | $2.65+23.87i$ | $2.65-23.87i$ | $-21.3$ |

As the eigenvalues $\lambda_1$ and $\lambda_2$ for the equilibrium point $O$ are both negative, the system is unstable at this equilibrium point. The eigenvalues corresponding to the equilibrium point $E^-$ will be the same with the eigenvalues of $E^+$, because of the quadratic nature of the system. Since $\lambda_3$ is a negative real number and $\lambda_1$ and $\lambda_2$ are two complex conjugate eigenvalues with positive real parts, equilibrium points $E^+$ and $E^-$ are unstable according to [31].

### 5.2.4 Symmetry and Dissipativity

The System (5.2.4)-(5.2.6) has a natural symmetry and is invariant under the coordinate transformation $(x,y,z) \to (-x,-y,z)$ which persists for all values of the system parameters. So, system (5.2.4)-(5.2.6) has rotation symmetry about the $z-axis$. Let, $f_1 = \frac{dx}{dt}$ , $f_2 = \frac{dy}{dt}$ and $f_3 = \frac{dz}{dt}$ in the system (5.2.4)-(5.2.6). Then we get for the vector field

$$(\dot{x},\dot{y},\dot{z})^T = (f_1, f_2, f_3)^T . \qquad (5.2.20)$$

Consequently the divergence of the vector field $\mathbf{V}$ yields to:

$$\nabla \cdot (\dot{x},\dot{y},\dot{z})^T = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(s+b) = f. \qquad (5.2.21)$$

Note that $f = -(s+b) = -16$ is a negative value, so the system is a dissipative system and an exponential rate is:

$$\frac{dV}{dt} = fV \Longrightarrow V(t) = V_0 e^{ft} = V_0 e^{-16t}. \tag{5.2.22}$$

From (5.2.22), it can be seen that a volume element $V_0$ is contracted by the flow into a volume element $V_0 e^{-16t}$ at the time $t$ .

### 5.2.5 Lyapunov Exponent and Fractional Dimension

The Lyapunov exponents generally refer to the average exponential rates of divergence or convergence of nearby trajectories in the phase space. In order to define a system as a chaotic system, the system should have at least one positive Lyapunov exponent. Thus, to decide if the system defined by the equations (5.2.4)-(5.2.6) is chaotic or not, the Lyapunov exponents can be evaluated by the following formula

$$l = \lim_{N \to \infty} \sum_{n=1}^{N} \ln \frac{d_1}{d_0}. \tag{5.2.23}$$

According to the detailed numerical and theoretical analysis, the Lyapunov exponents of the system (5.2.4)-(5.2.6) are found to be $l_1 = 5.4162$, $l_2 = 2.1912$, and $l_3 = -19.2269$, which proves that the system is a chaotic system, since we have two positive Lyapunov exponents.

If we are dealing with a chaotic deterministic system, the Lyapunov dimension is generally a non-integer. For example, in a 3-dimensional chaotic system, where the Lyapunov exponents have the form $l_- < 0 < l_+$, the Lyapunov dimension is evaluated as,

$$D_L = 2 + \frac{l_+}{|l_-|}. \tag{5.2.24}$$

On the other hand, for an attractor, the condition $l_+ + l_- < 0$ must also hold, which is equivalent to $2 < D_L < 3$. Therefore, the Lyapunov dimension of a system is evaluated as:

$$D_L = j + \frac{\sum_{i=1}^{j} l_i}{|l_{j+1}|} = 2 + \frac{l_1 + l_2}{|l_3|}. \tag{5.2.25}$$

According to the given formula the Lyapunov dimension of the system is

$$D_L = j + \frac{\sum_{i=1}^{j} l_i}{|l_{j+1}|} = 2 + \frac{5.4162 + 2.1912}{|-19.2269|} = 2.3957. \tag{5.2.26}$$

Since the Lyapunov dimension is in the range $2 < D_L < 3$, the result is consistent with the findings in [34].

Equation (5.2.25) shows that the system (5.2.4)-(5.2.6) is a dissipative system, and the Lyapunov dimensions of the system are fractional. Having a strange attractor and positive Lyapunov exponent, it is obvious that the system is a 3D chaotic system.

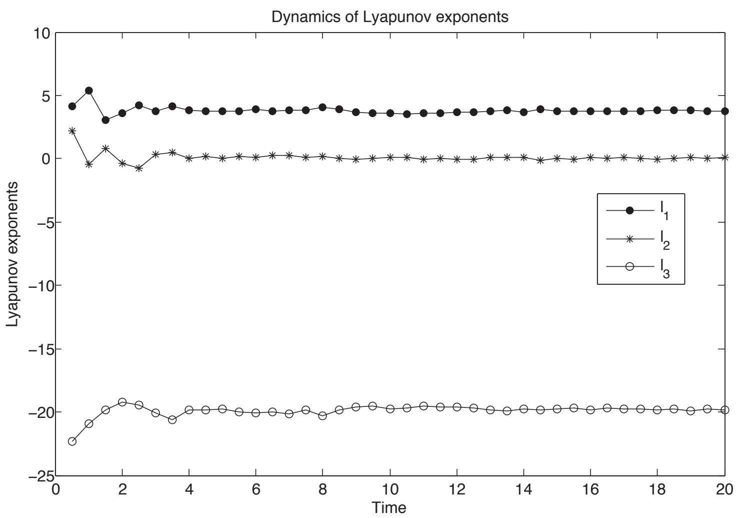
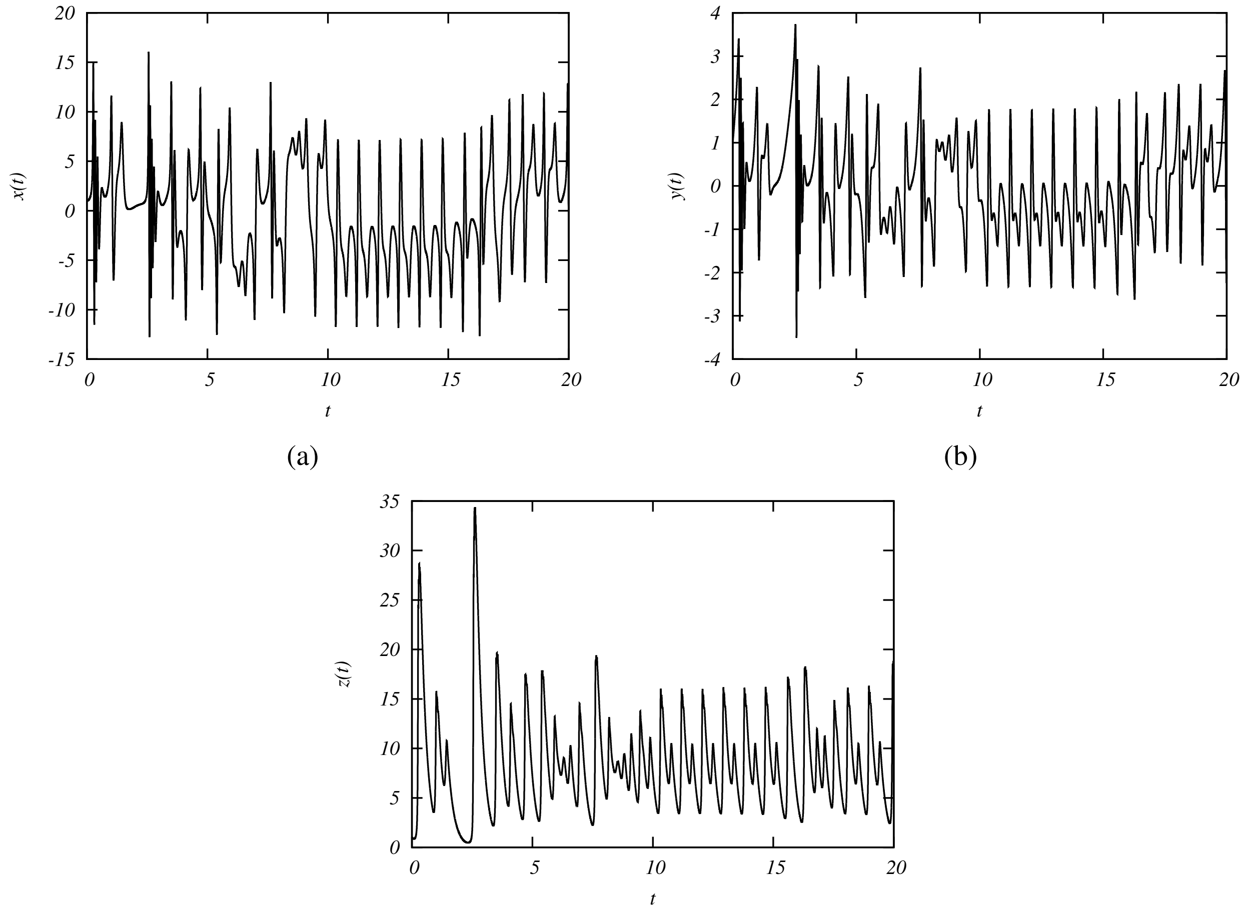


Figure 5.4: Plot of Lyapunov exponents

### 5.2.6 Numerical Simulations

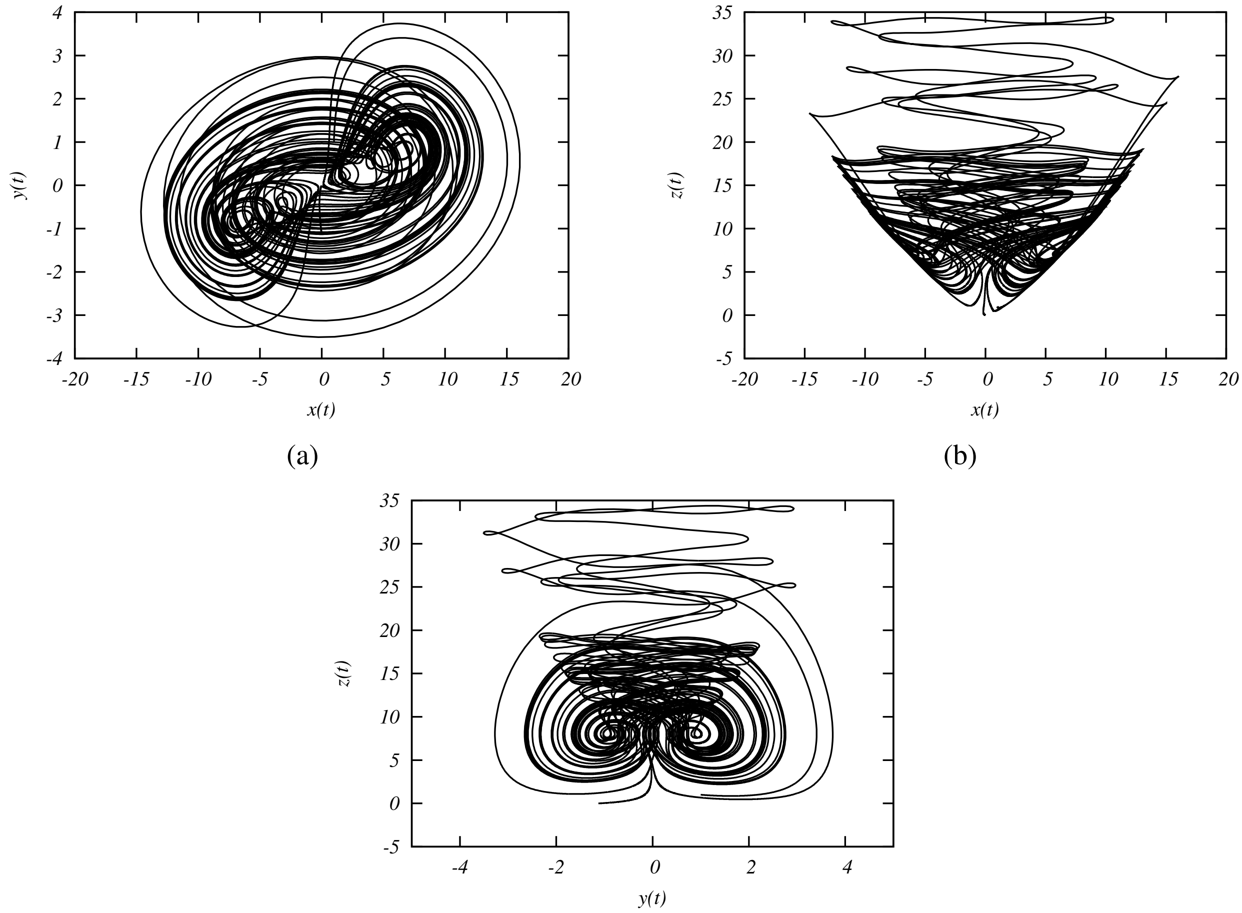
As we have mentioned before, the solutions of the chaotic systems are obtained by using the 4$^{th}$ order Runge-Kutta method. Thus, according to those solutions, the time series analysis of the system (5.2.4)-(5.2.6) with respect to $x(t), y(t), z(t)$ axes are listed seperately in the Figure 5.5.




(c)
Figure 5.5: Waveforms of $x(t)$, $y(t)$, $z(t)$ respectively

It is obvious from the figures 5.6a, 5.6b and 5.6c that the time series of $x(t)$, $y(t)$, and $z(t)$ are not periodic, which indicates that the system is a chaotic system.

The projections of the system (5.2.4)-(5.2.6), on the x-y plane, x-z plane and y-z plane are given in the Figure 5.6. All of the graphs shows the chaotic behaviour of the system. It appears that the new attractor exhibits an interesting complex chaotic dynamics behavior.




(c)

Figure 5.6: Projection of System (5.2.4)-(5.2.6) on the x-y plane, x-z plane, y-z plane respectively

## 5.3 Geometric Sense of the Modified Quadratic Lorenz System

As an application of the dynamical systems in multiplicative calculus, the newly formed chaotic system can also be written in the sense of geometric multiplicative calculus. After defining the system in multiplicative calculus, the analysis of the multiplicative Lorenz system will be done based on the rules of the multiplicative calculus.

### 5.3.1 The Modified Quadratic Lorenz Attractor in Multiplicative Calculus

#### 5.3.1.1 Numerical Simulations of the Multiplicative Chaotic System

In order to write the multiplicative counterpart of the chaotic system, we need to remember some properties of the multiplicative calculus. Remembering that addition, subtraction and multiplication of functions in Newtonian calculus can be written as multiplication, division and power respectively in multiplicative calculus, then the equation (5.2.4) can be written as

$$s(yz-x) \quad \rightarrow \quad \left(\frac{y^{\ln z}}{x}\right)^{\ln s}. \tag{5.3.1}$$

Thus the multiplicative counterparts of the equations (5.2.5) and (5.2.6) can also be written in the same way.

Then by using the given properties, the modified multiplicative quadratic Lorenz attractor corresponding to the system (5.2.4)-(5.2.6), can be written as

$$\begin{aligned} \frac{d^*x}{dt} &= \left(\frac{y^{\ln z}}{x}\right)^{s}, \\ \frac{d^*y}{dt} &= \frac{x^r}{x^{\ln z}}, \\ \frac{d^*z}{dt} &= \frac{x^{\ln x(\ln y)^2}}{z^b}. \end{aligned} \tag{5.3.2}$$

The powers of the functions are chosen suitably according to the rule $x^{\ln y} = y^{\ln x}$ and the constants $\ln s, \ln r$ and $\ln b$ are replaced by $s, r$ and $b$.

The analysis of the multiplicative chaotic system will be done in analogy to the Newtonian sense. Thus the first step is to find the equilibrium points of the proposed system. In order to get the equilibrium points of the system, we will solve the proposed system by the $4^{\text{th}}$ order multiplicative Runge-Kutta method. Thus the equilibrium points are obtained from the solution of the following system

$$\frac{d^*x}{dt} = 1, \tag{5.3.3}$$

$$\frac{d^*y}{dt} = 1, \tag{5.3.4}$$

$$\frac{d^*z}{dt} = 1, \tag{5.3.5}$$

which is equivalent to

$$\begin{aligned} \left(\frac{y^{\ln z}}{x}\right)^s &= 1, \\ \frac{x^r}{x^{\ln z}} &= 1, \\ \frac{x^{\ln x (\ln y)^2}}{z^b} &= 1. \end{aligned} \tag{5.3.6}$$

Then the equilibria of the system are found to be

$$E_1 = (1,1,1), \tag{5.3.7}$$

$$E_2 = \left(\exp\left(\sqrt[4]{b\,r^3}\right), \exp\left(\sqrt[4]{\frac{b}{r}}\right), \exp(r)\right), \tag{5.3.8}$$

$$E_3 = \left(\exp\left(-\sqrt[4]{b\,r^3}\right), \exp\left(-\sqrt[4]{\frac{b}{r}}\right), \exp(r)\right). \tag{5.3.9}$$

Remembering that the Jacobian matrix of a $3 \times 3$ multiplicative system is in the form

$$J = \begin{bmatrix} \ln\left(\frac{\partial^* f_1}{\partial x}\right) & \ln\left(\frac{\partial^* f_1}{\partial y}\right) & \ln\left(\frac{\partial^* f_1}{\partial z}\right) \\ \ln\left(\frac{\partial^* f_2}{\partial x}\right) & \ln\left(\frac{\partial^* f_2}{\partial y}\right) & \ln\left(\frac{\partial^* f_2}{\partial z}\right) \\ \ln\left(\frac{\partial^* f_3}{\partial x}\right) & \ln\left(\frac{\partial^* f_3}{\partial y}\right) & \ln\left(\frac{\partial^* f_3}{\partial z}\right) \end{bmatrix}. \tag{5.3.10}$$

If we denote the multiplicative chaotic system as:

$$
\begin{aligned}
f_1 &= \frac{d^*x}{dt} = \left(\frac{y^{\ln z}}{x}\right)^s, \\
f_2 &= \frac{d^*y}{dt} = \frac{x^r}{x^{\ln z}}, \\
f_3 &= \frac{d^*z}{dt} = \frac{x^{\ln x (\ln y)^2}}{z^b},
\end{aligned}
\tag{5.3.11}
$$

the corresponding Jacobian matrix for the system (5.3.11) will be

$$
J = \begin{bmatrix}
\ln\left(\exp\{-\frac{s}{x}\}\right) & \ln\left(\exp\{\frac{s\ln z}{y}\}\right) & \ln\left(\exp\{\frac{s\ln y}{z}\}\right) \\
\ln\left(\exp\{\frac{r-\ln z}{x}\}\right) & \ln\left(\exp\{0\}\right) & \ln\left(\exp\{-\frac{\ln x}{z}\}\right) \\
\ln\left(\exp\{\frac{2\ln x(\ln y)^2}{x}\}\right) & \ln\left(\exp\{\frac{2(\ln x)^2\ln y}{y}\}\right) & \ln\left(\exp\{-\frac{b}{z}\}\right)
\end{bmatrix}
\tag{5.3.12}
$$

$$
= \begin{bmatrix}
-\frac{s}{x} & \frac{s\ln z}{y} & \frac{s\ln y}{z} \\
\frac{r-\ln z}{x} & 0 & -\frac{\ln x}{z} \\
\frac{2\ln x(\ln y)^2}{x} & \frac{2(\ln x)^2\ln y}{y} & -\frac{b}{z}
\end{bmatrix}.
\tag{5.3.13}
$$

Thus, for the equilibrium point $E_1$ we obtain the Jacobian matrix as:

$$
J(E_1) = \begin{bmatrix}
-s & 0 & 0 \\
r & 0 & 0 \\
0 & 0 & -b
\end{bmatrix},
\tag{5.3.14}
$$

and for the equilibrium points $E_2$ and $E_3$ the Jacobian matrix will be

$$
J(E_{2,3}) = \begin{bmatrix}
-se^{cr} & sre^{c} & -cse^{-r} \\
0 & 0 & -cre^{-r} \\
4c^2re^{cr} & 4c^2re^{c} & -be^{-r}
\end{bmatrix},
\tag{5.3.15}
$$

where $c=\pm\sqrt[4]{\frac{b}{r}}$. Thus by using the equilibrium points given in the equations (5.3.7)-(5.3.9) the corresponding eigenvalues of the Jacobian matrices can be summarized in the following table.

Table 5.2: Eigenvalues of the Jacobian matrices of the multiplicative chaotic system at the equilibrium points

| Equilibrium Point | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|
| $E_1$ | $-12$ | $-4$ | $0$ |
| $E_2$ | $-0.0286149$ | $0.00644902+0.272747i$ | $0.00644902-0.272747i$ |
| $E_3$ | $-10017.4$ | $0.000690801+0.892422i$ | $0.000690801-0.892422i$ |

As we have discussed for the chaotic system defined by the equations (5.2.4)-(5.2.6), since the two eigenvalues of the first equilibrium point $E_1$ are negative real numbers, this shows that the system is unstable at this equilibrium point. On the other hand, one of the eigenvalues of the equilibrium points $E_2$ and $E_3$ is a negative real number and the other two are complex conjugate numbers with positive real parts, which proves that the system is again unstable at those equilibrium points.

Moreover, keeping in mind that the relation between the multiplicative and the ordinary derivative of the function $f(x)$ is

$$f^*(x)=e^{\frac{f'(x)}{f(x)}}, \tag{5.3.16}$$

the modified multiplicative quadratic Lorenz attractor can be expressed in terms of the additive derivatives as

$$\begin{aligned}
\frac{dx}{dt} &= xs\left(\ln(y)\ln(z)-\ln(x)\right),\\
\frac{dy}{dt} &= y(r\ln(x)-\ln(x)\ln(z)),\\
\frac{dz}{dt} &= z((\ln(x)\ln(y))^2-b\ln(z)).
\end{aligned}\tag{5.3.17}$$

In order to analyze the system defined by the additive derivatives, the first step is again to find the equilibrium points. Thus defining the system as

$$\begin{aligned}
xs\left(\ln(y)\ln(z)-\ln(x)\right) &= 0,\\
y(r\ln(x)-\ln(x)\ln(z)) &= 0,\\
z((\ln(x)\ln(y))^2-b\ln(z)) &= 0,
\end{aligned}\tag{5.3.18}$$

we will see that the equilibrium points of the system (5.3.18) are the same with the ones that we have obtained for the system (5.3.6), which are given in the Table 5.3.

Then by defining the system (5.3.17) as

$$\begin{aligned}
f_1 &= \frac{dx}{dt} = xs\left(\ln(y)\ln(z)-\ln(x)\right),\\
f_2 &= \frac{dy}{dt} = y(r\ln(x)-\ln(x)\ln(z)),\\
f_3 &= \frac{dz}{dt} = z((\ln(x)\ln(y))^2-b\ln(z)),
\end{aligned}\tag{5.3.19}$$

the Jacobian matrix of the system can be written as

$$J=\begin{bmatrix}
s\ln y\ln z-s\ln x-s & \frac{sx\ln z}{y} & \frac{sx\ln y}{z}\\
\frac{ry}{x}-\frac{y\ln z}{x} & r\ln(x)-\ln(x)\ln(z) & -\frac{y\ln x}{z}\\
\frac{2z\ln x(\ln y)^2}{x} & \frac{2z(\ln x)^2\ln y}{y} & (\ln(x)\ln(y))^2-b\ln z-b
\end{bmatrix}. \tag{5.3.20}$$

Finding the corresponding Jacobian matrices for the equilibrium points results in the following matrices:

$$J(E_1) = \begin{bmatrix} -s & 0 & 0 \\ r & 0 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{5.3.21}$$

and

$$J(E_{2,3}) = \begin{bmatrix} -s & e^{-c+cr}sr & e^{cr-r}sc \\ 0 & 0 & -e^{c-r}cr \\ 4c^2re^{-cr+r} & 4c^2re^{-c+r} & -b+4c^2r-br \end{bmatrix}, \tag{5.3.22}$$

where $c = \pm\sqrt[4]{\frac{b}{r}}$. In order to get the eigenvalues of the system we will solve the jacobian matrices for the values of $s, b$ and $r$.

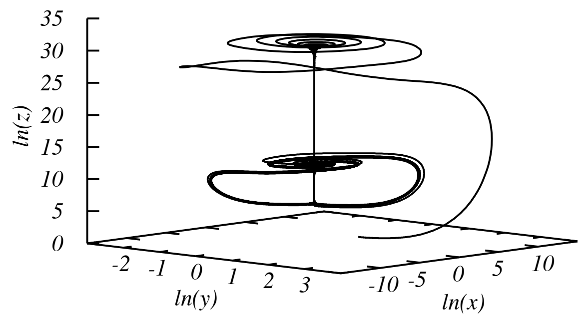
Thus the eigenvalues of the system (5.3.17) are listed in the following table.

Table 5.3: Eigenvalues of the Jacobian matrices of the multiplicative chaotic system, defined by additive derivatives, at the equilibrium points

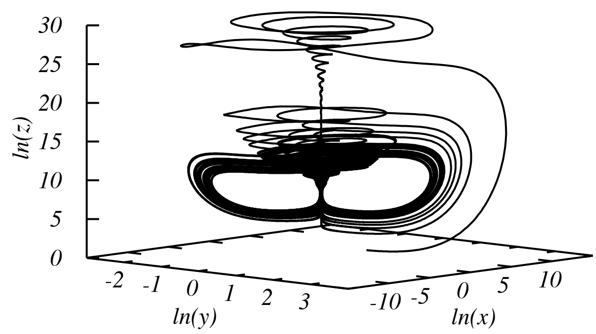
| Equilibrium Point | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|
| $E_1$ | $-12$ | $-4$ | $0$ |
| $E_2$ | $-21.3002$ | $2.65011+23.872i$ | $2.65011-23.872i$ |
| $E_3$ | $-21.3002$ | $2.65011+23.872i$ | $2.65011-23.872i$ |

As it is already explained for the previous chaotic systems, since the nonzero eigenvalues of the equilibrium point $E_1$ are negative real numbers, the system is unstable at this equilibrium point. If we consider the 2nd and the 3rd equilibrium points, $E_2$ and $E_3$, the corresponding eigenvalues are the same. Since, one of the eigenvalues is a negative real number and the other two are complex conjugate numbers with positive real parts, this shows that the system is again unstable at these equilibrium points.

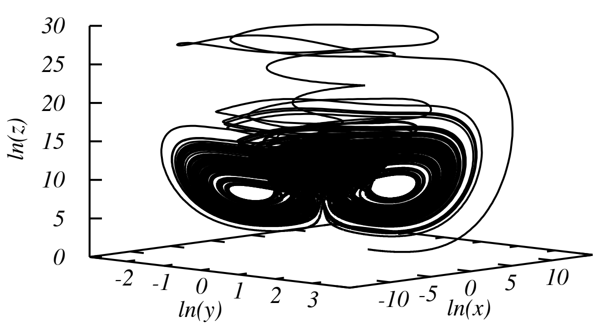
The multiplicative chaotic system (5.3.17), which is defined by additive derivatives, is solved by using the 4th order multiplicative Runge-Kutta method. Solutions shows that the system is chaotic for the same values of $s, r$ and $b$, as it is for the chaotic system defined by the equations (5.2.1)-(5.2.3). Keeping the values of $s$ and $r$ fixed, as $s = 12$ and $r = 8$, and using various $b$ values, the graphs of the solution of both system are shown in the following figure. It can be seen that the graphs of the multiplicative chaotic system is the same with the original system in logarithmic scale.



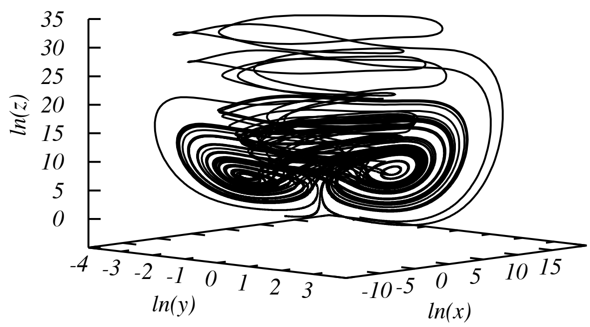

(a) $b = 0.1$



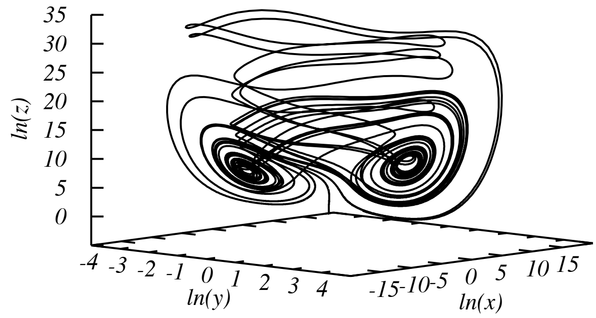

(b) $b = 0.5$




(a) $b = 2$




(b) $b = 4$




(c) $b = 10$

Figure 5.8: Simulation of the multiplicative chaotic system when $s = 12$, $r = 8$ and various $b$ values

#### 5.3.1.2 Lyapunov Exponents of the Multiplicative Chaotic System

As it is already explained in Section 5.2.5, the chaotic behaviour of a system can also be tested by the Lyapunov exponents. It was given in equation (5.3.23) that the Lyapunov exponents of a chaotic system can be evaluated by the formula:

$$l = \lim_{N \to \infty} \sum_{n=1}^{N} \ln \frac{d_1}{d_0}. \tag{5.3.23}$$

Then the Lyapunov exponents of a multiplicative chaotic system are also evaluated in analogy to the ordinary case. For the multiplicative chaotic systems, the formula used to evaluate the Lyapunov exponents can be written as

$$l = \lim_{N \to \infty} \sum_{n=1}^{N} \ln \frac{d_1^{(\ln)}}{d_0^{(\ln)}}. \tag{5.3.24}$$

It can be easily seen that the difference between the equations (5.3.23) and (5.3.24) is the evaluation of the distances. For the multiplicative chaotic systems the distances between the points are evaluated in logarithmic scale. Thus, the Lyapunov exponents of the multiplicative chaotic system (5.3.17) are found to be $l_1 = 8.1806, l_2 = 1.0684$ and $l_3 = -15.1893$. As it is explained before, a dynamical system can be considered as chaotic if the system has at least one positive Lyapunov exponent. The results of the Lyapunov exponents shows that the system (5.3.17) is a chaotic system. On the other hand, Lyapunov dimension of the system can be calculated as

$$D_L = j + \frac{\sum_{i=1}^{j} l_i}{|l_{j+1}|} = 2 + \frac{l_1 + l_2}{|l_3|} = 2.6089, \qquad (5.3.25)$$

Since the Lyapunov dimension of the multiplicative system is also in the range $2 < D_L < 3$ and there are positive Lyapunov exponents we can conclude that system is a chaotic system. The following graph shows all of the Lyapunov exponents of the system (5.3.17).

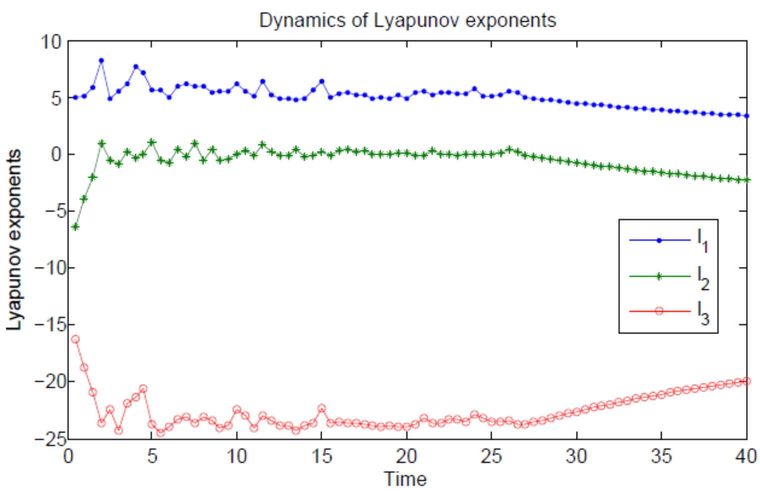


Figure 5.9: Plot of Lyapunov exponents of the Multiplicative Chaotic System

Comparison of the results obtained for both multiplicative chaotic systems, where one of the systems is defined by multiplicative derivatives which is the system (5.3.2) and the other is defined by additive derivatives which corresponds to the system (5.3.17), shows that they are the same systems defined in different calculi. On the other hand we may conclude that chaotic behaviour is a general property which is not very sensitive to the type of the calculus used.

# Chapter 6

## CONCLUSION

In this thesis the multiplicative Runge-Kutta methods of $2^{nd}$, $3^{rd}$ and $4^{th}$ order are described in order to solve the multiplicative initial value problem

$$y^*(x) = g(x,y), \text{ with } y(x_0) = y_0. \tag{6.0.1}$$

The derivation of all methods was carried out in detail. The Butcher tableaus corresponding to each method are also presented. Several methods to overcome the limitations of Multiplicative Calculus are presented to ensure the universal applicability of the Multiplicative Runge-Kutta methods. Then the convergence, error and stability analysis of the multiplicative one-step methods were discussed in detail. Furthermore, several problems are solved by using the $4^{th}$ order Multiplicative Runge-Kutta method. The results obtained from the solution of the problems by the Multiplicative Runge-Kutta method are then compared with the results obtained from the classical Runge-Kutta method and the Multiplicative Finite Difference Method. Comparison of the results show that Multiplicative Runge-Kutta method gives better results than the classical Runge-Kutta method and the Multiplicative Finite Difference Method, for the same step size $h$. The methods were also compared with respect to the computation time, where the errors of the Multiplicative Runge-Kutta method were smaller compared to the classical Runge-Kutta method for the same computation time.

Finally, a new chaotic system was defined, which is called the Modified Quadratic Lorenz Attractor. The system is analyzed numerically and theoretically to prove the chaotic behaviour. In order to show the applicability of the proposed methods on chaotic systems, the Modified Quadratic Lorenz Attractor is transformed to a multiplicative system. Then the multiplicative chaotic system is also analyzed for the chaotic behaviour. The results of the analysis proved that the multiplicative Runge-Kutta methods are also applicable to the chaotic systems.

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