Bigeometric Taylor Theorem and its Application to the Numerical Solution of Bigeometric Differential Equations

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ABSTRACT

Many studies in the field of Bigeometric Calculus are based on an approximation to the Bigeometric Taylor series, as the correct version is not known. The Bigeometric Taylor Series introduced in this research, is derived and proven explicitly. As an application of the Bigeometric Taylor Series, the Bigeometric Runge-Kutta method is derived in analogy to the classical Runge-Kutta method. The stability, as well as the convergence analysis is given explicitly for Bigeometric Runge-Kutta method. Application of the Bigeometric Runge-Kutta method to problems with known closed form solutions show the advantage of this method for a certain family of problems compared to the classical Runge-Kutta Method.

Keywords: Bigeometric calculus, Runge-Kutta, differential equations, numerical approximation, dynamical systems,electirical circuits.

ÖZ

Bigeometrik alanında yapılan birçok çalışmada Bigeometrik Taylor serisi doğru analiz edilmeden kullanılmıştır. Bu çalışmada Bigeometrik Taylor Serisinin ispatı açık olarak verilmiştir. Bigeometrik Taylor Serisinin bir uygulaması olarak, Bigeometric Runge-Kutta yöntemi nümerik analizde bilinen Runge-Kutta yöntemi baz alınarak çıkarılmıştır. Ayrıca Bigeometric Runge-Kutta yöntemi için yakınsak ve kararlılık testleri de analiz edilmiştir. Yöntem dinamik sistemler, bioloji ve elektrik devrelerinde uygulanmış ve Bigeometrik Runge Kutta ile elde edilen sonuçlar nümerik analizde bilinen Runge-Kutta yöntemi ile karşılaştırılmıştır.

Anahtar Kelimeler: Çarpımsal analiz,, Runge-Kutta, diferansiyel denklemler, nümerik yakınsama, dinamik sistemler, elektrik devreleri.

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INTRODUCTION

Each problem in Science and Engineering has its unique characteristics and properties, so apparently there is a significant number of problems where the solutions of the problems are of exponential nature. Grossmann and Katz proved that it is possible to produce infinitely many calculi independently [24]. They build a big family named Non-Newtonian Calculus, covering also the Newtonian or Leibnizian Calculus, the Geometric Multiplicative Calculus, the Bigeometric Multiplicative Calculus and infinitely many other calculi. The basic aim of the book [24] is to explain the requirements for the generation of new types of calculi. Moreover, Grossmann and Katz present nine specific non-Newtonian calculi, the general theory of non-Newtonian Calculus, and heuristic guides for the application.

The most popular representatives of the family of Non-Newtonian calculi are the Geometric Multiplicative Calculus and the Bigeometric Multiplicative Calculus. There are various applications of these two Non-Newtonian calculi available in the literature; Multiplicative Calculus, both geometric as well as bigeometric have a wide area of application, e.g. in modelling in finance and economics [10], numerical approximation methods in [30, 27, 26], biological image analysis in [21, 22], and application on literary texts in [6].

Geometric and Bigeometric calculus can applied either to purely positive valued functions or purely negative valued functions of a real variable. There are several approaches to extend Geometric Multiplicative Calculus to the complex domain. A first heuristic approach was presented by Uzer in [34], whereas the mathematically complete description of complex multiplicative calculus was given by Bashirov and Riza in [11, 7, 8, 31]. On the other hand, applications of Bigeometric Calculus found its way in the field of nonlinear dynamics by the group of Rybaczuk [33, 4, 3, 2, 12, 32].

In order to find potential application areas of Bigeometric Multiplicative Calulus, we have also studied mathematical models based on differential equations in biology. As in general closed form solutions for real world problems are not available, the systems of ordinary differential equations have to be solved using numerical methods. In general the 4^{th} order Runge-Kutta method applied to the numerical solution of these differential equations. As an example, the Modelling of Gene expression using differential equations [14], modelling Tumour growth [1], or modelling bacteria growth and cancer [19, 20] can be considered. In the present study, the Bigeometric Runge Kutta Method is applied to the mathematical model of tumor therapy using oncolytic virus[1].

As in the literature, it is not possible to find the proofs of all properties of the Bigeometric Derivative, After the introduction of a complete description of the Bigeometric Derivative, chapter 2 summarises all properties and proofs of the Bigeometric derivative analogous to [9]. In chapter 3, the relationship between the geometric multiplicative and bigeometric multiplicative derivative for higher order derivatives is stated and proven explicitly. As in all calculi, the fundamental theorem for numerical approximations is also in Bigeometric Calculus the Bigeometric Taylor expansion. Chapter 4 gives the derivation and proof Bigeometric Taylor theoremn based on the Geometric Taylor theorem given [9]. Although Aniszewska [2] introduced the Bigeometric Multiplicative Runge-Kutta Method using a different definition for the bigeometric derivative with a limited Bigeometric taylor expansion, an explicit derivation of the Bigeometric Runge-Kutta method as an application of the Bigeometric Taylor Theorem is given in Chapter 5. Furthermore, the convergence and stability analysis of Bigeometric Runge-Kutta method was given in Chapter 6. Consequently, the applicable area of Bigeometric- Runge-Kutta Method was introduced and tested exemplarily on several examples in Chapter 7. Finally, the thesis closes with the conclusion .

Chapter 2

BIGEOMETRIC CALCULUS AND IT'S PROPERTIES

An overview of multiplicative calculus stated in [9]. The proofs of differentiation rules and more information about multiplicative calculus are given in [9] and [10].Definition of bigeometric derivative of a function f stated in [9] as :

$$f^{\pi}(x) = \frac{d^{\pi} f(x)}{dx} = \lim_{h\to 0}\left(\frac{f\big((1+h)x\big)}{f(x)}\right)^{\frac{1}{h}}, \tag{2.0.1}$$

where f^{π} shows the bigeometric derivative if the limit in (2.0.1) exist.

By using the solution of main limit definition in (2.0.1), the connection of the ordinary and the Bigeometric derivative can be given as :

$$f^{\pi}(x) = \exp\left\{x\frac{f'(x)}{f(x)}\right\}. \tag{2.0.2}$$

The effect of that x-value seen, when the function has a unit. The multiplicative derivative yields as $f^{*} = e^{f'(x)/f(x)}$. Let's assume that x have the unit meter, and then the multiplicative derivative finally turns a unit $e^{1/m}$, which has no physical meaning. On the other hand, in the Bigeometric derivative we also have x value can have a unit, like the derivative itself, so at the end we get at least no unreasonable result.

The relation between first order Bigeometric derivative and the multiplicative derivative is derived in [30] as follows :

$$f^{\pi}(x) = \frac{d^{\pi} f(x)}{dx} = (f^{*}(x))^{x}. \tag{2.0.3}$$

If we extended equation (2.0.3) obviously exponential definitions can easly proved the below relation :

$$f^{\pi}(x) = \exp(x\,(ln \circ f(x))') = \exp\left(x\left[\frac{f'(x)}{f(x)}\right]\right) = (f^{*}(x))^{x} \tag{2.0.4}$$

The complete differentiation and integration rules for geometric-multiplicative calculus stated in [9]. However, no references presented all the properties of differentiation or integration rules for the Bigeometric calculus [23, 35, 15]. Therefore, all the properties of the Bigeometric derivatives stated in the following. The proofs are carried out analogously to [9] by using the relation between geometric-multiplicative and bigeometric derivative (2.0.4).

Bigeometric differentiation rules: Let $f(x)$, $g(x)$, and $h(x)$ be π-differentiable functions, and $c \in \mathbb{R}$.

1. Constant multiple rule:

$$(cf)^{\pi}(x) = (f)^{\pi}(x)$$

2. Product Rule :

$$(fg)^{\pi}(x) = f^{\pi}(x) g^{\pi}(x)$$

3. Quotient Rule :

$$\left(\frac{f}{g}\right)^{\pi}(x) = \frac{f^{\pi}(x)}{g^{\pi}(x)}$$

4. Power Rule :

$$(f^h)^\pi(x) = f^\pi(x)^{h(x)} f(x)^{xh'(x)}$$

5. Sum Rule :

$$(f+g)^\pi(x) = (f^\pi(x))^{\frac{f(x)}{f(x)+g(x)}} (g^\pi(x))^{\frac{g(x)}{f(x)+g(x)}}$$

6. Chain Rule:

For one variable :

$$(f \circ h)^\pi(x) = f^\pi(h(x))^{h'(x)}$$

For two variable:

$$f^\pi(y(x), z(x)) = (f_y^\pi(y(x), z(x)))^{y'(x)} (f_z^\pi(y(x), z(x)))^{z'(x)}$$

with $f_y^\pi(y(x), z(x))$ *denoting the partial Bigeometric derivative of* $f(y(x), z(x))$ *with respect to y ,and* $f_z^\pi(y(x), z(x))$ *denoting the partial Bigeometric derivative of* $f(y(x), z(x))$ *with respect to z respectively.*

Proofs of the Bigeometric differentiation rules:

1. Constant multiple Rule :

$$e^{x(\ln(cf(x)))'} = e^{x\left(\frac{1}{cf(x)}(cf(x)')\right)} = e^{x\frac{f'(x)}{f(x)}} = f^\pi(x)$$

2. Product Rule :

$$e^{x(\ln(f(x)g(x)))'} = e^{x\left[\frac{1}{f(x)g(x)}(f'(x)g(x)+f(x)g'(x))\right]} = e^{x\left(\frac{f'(x)g(x)}{f(x)g(x)}+\frac{f(x)g'(x)}{f(x)g(x)}\right)} = f^\pi(x)g^\pi(x)$$

3. Quotient Rule:

$$e^{x(\ln(\frac{f(x)}{g(x)}))'} = e^{x\left(\frac{g(x)}{f(x)}\left(\frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2}\right)\right)} = e^{x\frac{f'(x)}{f(x)}} e^{-x\frac{g'(x)}{g(x)}} = \frac{f^{\pi}(x)}{g^{\pi}(x)}$$

4. Power Rule :

$$e^{x\left(\ln(f(x))^{h(x)}\right)'} = e^{x\left(\frac{1}{f(x)^{h(x)}} f(x)^{h(x)}\left(h'(x)\ln f(x)+h(x)\frac{f'(x)}{f(x)}\right)\right)} =$$

$$= e^{x\left(h'(x)\ln f(x)+h(x)\frac{f'(x)}{f(x)}\right)} = \left(f(x)^{xh'(x)}\right)(f^{\pi}(x))^{h(x)}$$

5. Sum Rule :

$$e^{x(\ln(f(x)+g(x)))'} = e^{x\left(\frac{1}{f(x)+g(x)}(f'(x)+g'(x))\right)} = (f^{\pi}(x))^{\frac{f(x)}{f(x)+g(x)}}(g^{\pi}(x))^{\frac{g(x)}{f(x)+g(x)}}$$

6. Chain Rule:

For one variable :

$$e^{x\ln(f\circ h)'(x)} = e^{x\left(\frac{1}{f(h(x))}(f'(h(x))h'(x))\right)} = f^{\pi}(h(x))^{h'(x)}$$

For two variable :

$$e^{x(\ln[f(y(x),z(x))])'} = \exp\left\{x\frac{f_y'(y(x),z(x))y'(x)+f_z'(y(x),z(x))z'(x)}{f(y(x),z(x))}\right\} =$$

$$= \exp\left\{x\frac{f_y'(y(x),z(x))}{f(y(x),z(x))}y'(x)\right\}\exp\left\{x\frac{f_z'(y(x),z(x))}{f(y(x),z(x))}z'(x)\right\} =$$

$$= f_y^{\pi}(y(x),z(x))^{y'(x)} f_z^{\pi}(y(x),z(x))^{z'(x)}$$

Chapter 3

THE RELATION BETWEEN GEOMETRIC AND BIGEOMETRIC DERIVATIVE

As mentioned before it is obvious that we need to transform the bigeometric form to Multiplicative form for finding and proving the properties of Bigeometric Calculus . In other word we need to write bigeometric derivative in terms of the multiplicative derivative for finding the Bigeometric form properties. Let us calculate the Bigeometric derivatives in terms of the multiplicative derivatives up to order three. Using the power rule of the multiplicative derivative [9]. As a short revision for power rule :

$$f^{**} = ((f^{*})^{x})^{*} = (f^{**})^{x}(f^{*})^{x'} . \tag{3.0.1}$$

By using the same procedure in (3.0.1) we will write state $f^{\pi\pi}$ and $f^{\pi\pi\pi}$ as :

$$\begin{aligned} f^{\pi}(x) &= f^{*}(x)^{x} && (3.0.2)\\ f^{\pi\pi}(x) &= (f^{**}(x))^{x^{2}}(f^{*}(x))^{x} && (3.0.3)\\ f^{\pi\pi\pi}(x) &= (f^{***}(x))^{x^{3}}(f^{**}(x))^{3x^{2}}(f^{*}(x))^{x} && (3.0.4)\\ \vdots &= \vdots \end{aligned}$$

Let's give an idea for second order bigeometric derivative how its come :

$$f^{\pi\pi} = ((f^{\pi})^{*})^{x} = (((f^{*})^{x})^{*})^{x} = \left((f^{**})^{x}(f^{*})^{x'}\right)^{x} = (f^{**})^{x^{2}}(f^{*})^{x} \tag{3.0.5}$$

Analogously all higher multiplicative derivatives in terms of the bigeometric derivatives can be performed. In order to show higher order multiplicative derivatives in terms of bigeometric derivatives, we need to solve these equations (3.0.2) - (3.0.4) for the geometric-multiplicative derivatives. Hence, the higher order bigeometric derivatives derived as :

$$\begin{aligned} f^{*}(x) &= \left(f^{\pi}(x)\right)^{\frac{1}{x}} && (3.0.6) \\ f^{**}(x) &= \left(\frac{f^{\pi\pi}(x)}{f^{\pi}(x)}\right)^{\frac{1}{x^2}} && (3.0.7) \\ f^{***}(x) &= \left(\frac{f^{\pi\pi\pi}(x)\left(f^{\pi}(x)\right)^2}{\left(f^{\pi\pi}(x)\right)^3}\right)^{\frac{1}{x^3}} && (3.0.8) \\ \vdots &= \vdots \end{aligned}$$

Equations (3.0.6) - (3.0.8) suggest that the n-th order geometric-multiplicative derivative can be expressed in terms of the Bigeometric derivatives. (3.0.6) - (3.0.8) suggest that the n-th order geometric-multiplicative derivative can be expressed in terms of the Bigeometric derivatives using the unsigned Stirling Numbers first kind $s(n, j)$ [5] as following:

Theorem 3.0.1 (Relation between geometric and bigeometric multiplicative derivative) *The n-th geometric multiplicative derivative can be expressed as a product of bigeometric multiplicative derivatives up to order n as :*

$$f^{*(n)}(x) = \left(\prod_{j=1}^{n} \left(f^{\pi(j)}(x)\right)^{(-1)^{n-j}s(n,j)}\right)^{\frac{1}{x^n}}, \tag{3.0.9}$$

where $f^{*(n)}(x)$ denotes the n-th geometric-multiplicative derivative of $f(x)$ and $f^{\pi(j)}(x)$ denotes the j-th Bigeometric derivative of $f(x)$.

Proof. The proof for this relation can be carried out simply using mathematical induction. But First let us check the formula for the first non-trivial case $n = 2$.

$$f^{**}(x) = \left(f^{\pi}(x)^{(-1)^{2-1}s(2,1)} f^{\pi\pi}(x)^{(-1)^{2-2}s(2,2)}\right)^{1/x^2} = \left(\frac{f^{\pi\pi}(x)}{f^{\pi}(x)}\right)^{\frac{1}{x^2}} \tag{3.0.10}$$

Equations (3.0.10) are obviously identical.

Let (3.0.9) be true for n and check if it is true for $n+1$.

$$\begin{aligned} f^{*(n+1)}(x) &= \frac{d^*}{dx^*} f^{*(n)}(x) \\ &= \frac{d^*}{dx^*}\left(\prod_{j=1}^{n}(f^{\pi(j)}(x))^{\frac{(-1)^{n-j}s(n,j)}{x^n}}\right) \end{aligned}$$

With

$$\frac{d^* f(x)}{dx^*} = \left(\frac{d\pi f(x)}{dx^{\pi}}\right)^{1/x},$$

We can calculate the π-derivative of the product .

$$\begin{aligned} f^{*(n+1)}(x) &= \left[\frac{d^{\pi}}{dx^{\pi}}\left(\prod_{j=1}^{n}(f^{\pi(j)}(x))^{\frac{(-1)^{n-j}s(n,j)}{x^n}}\right)\right]^{1/x} \\ &= \left[\prod_{j=1}^{n}(f^{\pi(j+1)}(x))^{\frac{(-1)^{n-j}s(n,j)}{x^n}}(f^{\pi(j)}(x))^{\frac{x(-n)(-1)^{n-j}s(n,j)}{x^{n+1}}}\right]^{1/x} \\ &= \left[\prod_{j=1}^{n}(f^{\pi(j+1)}(x))^{\frac{(-1)^{n-j}s(n,j)}{x^n}}(f^{\pi(j)}(x))^{\frac{(-1)^{n+1-j}ns(n,j)}{x^{n}}}\right]^{1/x} \\ &= \left[\prod_{j=1}^{n}(f^{\pi(j+1)}(x))^{(-1)^{n-j}s(n,j)}(f^{\pi(j)}(x))^{(-1)^{n+1-j}ns(n,j)}\right]^{1/x^{n+1}} \\ &= \left[\prod_{j=1}^{n}(f^{\pi(j+1)}(x))^{(-1)^{n-j}s(n,j)}\prod_{j=1}^{n}(f^{\pi(j)}(x))^{(-1)^{n+1-j}ns(n,j)}\right]^{1/x^{n+1}} \end{aligned}$$

$$= \left[\prod_{j=2}^{n+1}(f^{\pi(j)}(x))^{(-1)^{n+1-j}s(n,j-1)}\prod_{j=1}^{n}(f^{\pi(j)}(x))^{(-1)^{n+1-j}ns(n,j)}\right]^{1/x^{n+1}}$$

$$= \left[\prod_{j=2}^{n}(f^{\pi(j)}(x))^{(-1)^{n+1-j}s(n,j-1)}\cdot(f^{\pi(n+1)}(x))^{(-1)^{n+1-(n+1}s(n,n)}\cdot\right.$$

$$\left.\cdot(f^{\pi(1)}(x))^{(-1)^{n}ns(n,1)}\cdot\prod_{j=2}^{n}(f^{\pi(j)}(x))^{(-1)^{n+1-j}ns(n,j)}\right]^{1/x^{n+1}}$$

$$= \left[\prod_{j=2}^{n}(f^{\pi(j)}(x))^{(-1)^{n+1-j}(s(n,j-1)+ns(n,j))}\cdot\right.$$

$$\left.\cdot(f^{\pi(n+1)}(x))^{(-1)^{n+1-(n+1}s(n,n)}\cdot(f^{\pi(1)}(x))^{(-1)^{n}ns(n,1)}\right]^{1/x^{n+1}}$$

Using the recurrence relation for the unsigned Stirling Number of first kind [5]

$$s(n+1,j) = ns(n,j)+s(n,j-1), \tag{3.0.11}$$

we have

$$s(n,j-1)+ns(n,j) = s(n+1,j), \tag{3.0.12}$$

and

$$ns(n,1) = s(n+1,1)-s(n,0) = s(n+1,1), \tag{3.0.13}$$

as

$$s(n+1,0) = 0$$

Finally with $s(n,n) = s(n+1,n+1)$ we obtain:

$$
\begin{aligned}
f^{*(n+1)}(x) &= \Bigg[\prod_{j=2}^{n}\big(f^{\pi(j)}(x)\big)^{(-1)^{n+1-j}(s(n,j-1)+ns(n,j))} \\
&\quad \times\big(f^{\pi(n+1)}(x)\big)^{(-1)^{n+1-(n+1}s(n,n)}\big(f^{\pi(1)}(x)\big)^{(-1)^{n}ns(n,1)}\Bigg]^{1/x^{n+1}} = \\
&= \Bigg[\prod_{j=2}^{n}\big(f^{\pi(j)}(x)\big)^{(-1)^{n+1-j}s(n+1,j)} \\
&\quad \times\big(f^{\pi(1)}(x)\big)^{(-1)^{n}s(n+1,1)}\cdot\big(f^{\pi(n+1)}(x)\big)^{(-1)^{n+1-(n+1}s(n+1,n+1)}\Bigg]^{1/x^{n+1}} \\
&= \Bigg[\prod_{j=1}^{n+1}\big(f^{\pi(j)}(x)\big)^{(-1)^{n+1-j}s(n+1,j)}\Bigg]^{1/x^{n+1}}
\end{aligned}
$$

which completes the proof. ■

Chapter 4

BIGEOMETRIC TAYLOR THEOREM

The Bigeometric Taylor theorem is not available any of the resources. The attempts of the autors in [2] and [30] show that finding the Bigeometric Taylor expansion is not that much easy. To show Bigeometric Taylor theorem we need the multiplicative form of Taylor theorem which is given in [9] as:

Theorem 4.0.2 (Multiplicative Taylor Theorem) *Let $f : B \to R$ where B denotes an open interval then the function f is $n+1$ times * differentiable on B. Then for any $x, x+h \in B$, there exists a number $\vartheta \in (0,1)$ such that*

$$f(x+h) = \prod_{m=0}^{n} \left(\left(f^{*(m)}(x) \right)^{\frac{h^m}{m!}} \right) \left(\left(f^{*(n+1)}(x+\vartheta h) \right)^{\frac{h^{n+1}}{(n+1)!}} \right) \qquad (4.0.1)$$

For showing bigeometric Taylor theorem we need to write n^{th} order geometric derivative in 4.0.1, which shown by $f^{*(m)}(x)$,in terms of bigeometric derivative. In other words, we need to use the relation between multiplicative and bigeometric derivative stated in theorem 3.0.1. This shows clearly, why finding the Bigeometric Taylor theorem is so difficult.The idea for derivation of Bigeometric Taylor Theorem is obviously seen by substituting the higher order geometric-multiplicative derivatives in the geometric multiplicative Taylor theorem After a serious simplification the Bigeometric Taylor theorem becomes visible .

$$f(x+h) = \prod_{m=0}^{\infty}\left(\left(f^{*(m)}(x)\right)^{\frac{h^m}{m!}}\right) = \prod_{m=0}^{\infty}\left(\prod_{j=1}^{m}(f^{\pi(j)}(x))^{(-1)^{m-j}s(m,j)/x^m}\right)^{\frac{h^m}{m!}} \quad (4.0.2)$$

Rearranging the terms with respect to the order of the bigeometric derivative we get:

$$f(x+h) = \prod_{m=0}^{\infty}\left(\prod_{j=m}^{\infty}(f^{\pi(m)}(x))^{(-1)^{m-j}s(m,j)/x^m}\right)^{\frac{h^m}{m!}} = \prod_{m=0}^{\infty}\left(\prod_{j=1}^{m}(f^{\pi(j)}(x))^{\frac{(-1)^{m-j}s(m,j)h^m}{x^m m!}}\right)$$

Rearranging the factors in terms of the orders of the Bigeometric derivatives we get :

$$f(x+h) = \prod_{m=0}^{\infty}\left(\prod_{j=m}^{\infty}(f^{\pi(m)}(x))^{\frac{(-1)^{m-j}s(j,m)h^j}{x^j j!}}\right) = \prod_{m=0}^{\infty}\left((f^{\pi(m)}(x))^{\sum_{j=m}^{\infty}\frac{(-1)^{m-j}s(j,m)h^j}{x^j j!}}\right). \quad (4.0.3)$$

With

$$\sum_{j=m}^{\infty}(-1)^{j-m}s(j,m)\frac{x^j}{j!} = \frac{(\ln(1+x))^m}{m!}, \quad (4.0.4)$$

Substituting (4.0.4) in (4.0.3), we obtain

$$f(x+h) = \prod_{m=0}^{\infty}\left((f^{\pi(m)}(x))^{\frac{\left(\ln\left(1+\frac{h}{x}\right)\right)^m}{m!}}\right). \quad (4.0.5)$$

Finally we can summerize the Bigeometric Taylor theorem as following.

$$f(x+h) = \prod_{i=0}^{\infty}\left(f^{\pi(i)}(x)\right)^{\frac{(\ln(1+\frac{h}{x}))^i}{i!}} \quad (4.0.6)$$

In [2] the Bigeometric Taylor theorem showed up to order 5 in h/x. The Expansion of the logarithms up to order 5 in h/x gives the same result of [2].

Theorem 4.0.3 (Bigeometric Taylor Theorem) *Let $f : B \to R$ where B denotes an* *open interval then the function f is $n+1$ times π differentiable on B. Then for any $x, x+h \in B$, there exists a number $\vartheta \in (0,1)$ such that*

$$f(x+h) = \prod_{i=0}^{n} \left(f^{\pi(i)}(x)\right)^{\frac{(\ln(1+\frac{h}{x}))^{i}}{i!}} \left(\left(f^{\pi(n+1)}(x+\vartheta h)\right)^{\frac{(\ln(1+\frac{h}{x}))^{n+1}}{(n+1)!}}\right). \tag{4.0.7}$$

Chapter 5

BIGEOMETRIC RUNGE-KUTTA METHOD

In this chapter, we derive the 2^{nd}, 3^{rd} and 4^{th} order Bigeometric Runge-Kutta method based on the ordinary Runge-Kutta method in the given sections respectively Sec. 5.1,5.2 and 5.3. Bigeometric Taylor theorem (4.0.7) formula becomes a critical tool for deriving the Bigeometric Runge-Kutta method. In the ordinary case, Runge-Kutta Method plays an important role for the numerical solution of initial value problems. Hence, Bigeometric Runge-Kutta method can applied to problems defined by the bigeometric initial value problem. Before starting to derivation of Bigeometric Runge-Kutta method let's define Bigeometric initial value problem as:

$$y^{\pi}(x) = \rho(x,y), \tag{5.0.1}$$

with the initial value

$$y(x_0) = y_0. \tag{5.0.2}$$

2^{nd} order Bigeometric Runge-Kutta Method is the simplest form to find an estimation of the solution with using (5.0.1) where the initial value (5.0.2). Because of this let's start with the derivation of 2^{nd} order Bigeometric Runge-Kutta Method

5.1 2^{nd} order Bigeometric Runge-Kutta Method(BRK2)

The second order bigeometric Runge-Kutta method can produce with the same idea in the ordinary case. The method also identified as Euler method. Let step-size $h > 0$ and define BRK2 method as follows :

$$y(x+h) = y(x)\rho_0^{a\ln(1+\frac{h}{x})}\rho_1^{b\ln(1+\frac{h}{x})} \tag{5.1.1}$$

with

$$\rho_0 = \rho(x,y), \tag{5.1.2}$$

$$\rho_1 = \rho((x+\ell h, y\rho_0^{\frac{\gamma h}{x}}) \tag{5.1.3}$$

The basic aim is to determine the unknown scalar values in (5.1.3) by using the Bigeometric Taylor expansion (4.0.7) for $y(x+h)$. We have to open our Bigeometric Taylor expansion up to order 2 given as:

$$y(x+h) = y(x)\left(y^{\pi}(x)\right)^{\ln(1+\frac{h}{x})}\left(y^{\pi\pi}(x)\right)^{\frac{1}{2!}\left[\ln(1+\frac{h}{x})\right]^2}\dots \tag{5.1.4}$$

For comparing the scalars we need to change $y^{\pi}(x)$ and $y^{\pi\pi}(x)$ in (5.1.4) as:

$$y^{\pi}(x) = \rho((x,y) \tag{5.1.5}$$

$$y^{\pi\pi}(x) = (y^{\pi}(x))^{\pi} = (\rho(x,y))^{\pi} \tag{5.1.6}$$

Application of chain rule stated in (6) to the function $\rho^{\pi}(x,y)$ we obtain :

$$y^{\pi\pi}(x) = (\rho(x,y))^{\pi} = \rho_x^{\pi}(x,y)\rho_y^{\pi}(x,y)^{y'} \tag{5.1.7}$$

substituting (5.1.6) and (5.1.7) in equation (5.1.4) we obtain :

$$y(x+h) = y(x)\left(\rho(x,y)\right)^{\ln(1+\frac{h}{x})}\left(\rho_x^{\pi}(x,y)\rho_y^{\pi}(x,y)^{y'}\right)^{\frac{1}{2!}\left[\ln(1+\frac{h}{x})\right]^2}\dots \tag{5.1.8}$$

By using the property $y^{\pi} = \rho(x,y) = exp\left(x\frac{y'}{y}\right)$ we will find first derivative in the bigeometric form as :

$$y' = \frac{y}{x} ln(\rho(x,y)) \tag{5.1.9}$$

Replacing y' in (5.1.8) our new form of bigeometric taylor theorem can stated as :

$$y(x+h) = y(x)\left(\rho(x,y)\right)^{\ln\left(1+\frac{h}{x}\right)} \left(\rho_x^{\pi}(x,y)\rho_y^{\pi}(x,y)^{\frac{y}{x}\ln(\rho(x,y))}\right)^{\frac{1}{2!}\left[\ln(1+\frac{h}{x}\right]^2} \cdots \tag{5.1.10}$$

For finding the unknown scalars in (5.1.1) we need to compare it with the equations (5.1.10) and we have to open ρ_1 from (5.1.3). Expansion of ρ_1 using Bigeometric Taylor Theorem with order two we obtain :

$$\rho_1 = \rho(x,y)\left[\rho^{\pi}(x,y)\right]^{\ln\left(1+\frac{h}{x}\right)} \tag{5.1.11}$$

Applying chain rule for $\rho^{\pi}(x,y)$ in equation (5.1.11)

$$\rho_1 = \rho(x,y)\left[\rho_x^{\pi}(x,y)^{\ell}\rho_y^{\pi}(x,y)^{y\frac{\gamma}{x}\ln(\rho_0)}\right]^{\ln\left(1+\frac{h}{x}\right)} \tag{5.1.12}$$

where the partial derivative of $y\rho_0^{\frac{\gamma h}{x}}$ stated as :

$$\frac{d}{dh}\left(y\rho_0^{\frac{\gamma h}{x}}\right) = y\rho_0^{\gamma h}\ln(\rho_0)\frac{\gamma}{x} = y\frac{\gamma}{x}\ln\rho_0 \tag{5.1.13}$$

with $h = 0$. Substituting $\rho_0 = \rho(x,y)$ (5.1.13), and (5.1.12) into (5.1.1) we obtain :

$$y(x+h) = y(x)\underbrace{\left[\rho(x,y)\right]}_{\rho_0}^{a\ln\left(1+\frac{h}{x}\right)} \underbrace{\left[\rho(x,y)\left[\rho_x^{\pi}(x,y)^{\ell}\rho_y^{\pi}(x,y)^{\frac{y\gamma}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\right]^{b\ln\left(1+\frac{h}{x}\right)}}_{\rho_1} \tag{5.1.14}$$

Simplifying (5.1.14) the Bigeometric Euler Method becomes

$$y(x+h) = y(x)\rho(x,y)^{(a+b)\ln(1+\frac{h}{x})}\rho_x^{\pi}(x,y)^{b\ell\left(\ln(1+\frac{h}{x})\right)^2}\rho_y^{\pi}(x,y)^{b\gamma\frac{y}{x}\ln(\rho(x,y))\left(\ln(1+\frac{h}{x})\right)^2} \tag{5.1.15}$$

Comparing the powers in Bigeometric Taylor theorem (5.1.10) and the calculated powers in the Bigeometric Euler (5.1.15) it is obvious that the parameters get the following relations :

$$a+b = 1 \tag{5.1.16}$$

$$\ell b = \frac{1}{2} \tag{5.1.17}$$

$$\gamma b = \frac{1}{2} \tag{5.1.18}$$

Since the number of equatons is less than the number of unknowns, obviously we will find infinitely many solutions of the given equations (5.1.16)-(5.1.18). Furthermore it's easily seen that $\ell = \gamma$ and $a+b=1$ must be satisfied. The relation of the parameters can represent in the Butcher tableau [13] as following :

<table>
<tr><td>0</td><td></td><td></td></tr>
<tr><td>ℓ</td><td>γ</td><td></td></tr>
<tr><td></td><td>a</td><td>b</td></tr>
</table>

We have numerous possibility for selection of unknowns a, b, p and γ. The parameters can arrange differently depending on the kind of the given problem. One possible selection of the parameters a, b, ℓ, and γ is :

$$a = b = \frac{1}{2}, \quad \text{and } \ell = \gamma = 1. \tag{5.1.19}$$

which is the parameters of classical Runge-Kutta method. Substituting parameters in Bigeometric Euler Method stated in (5.1.1)-(5.1.3) BRK2 methd turns to :

$$y(x+h) = y(x)\rho_0^{\frac{1}{2}\ln(1+\frac{h}{x})}\rho_1^{\frac{1}{2}\ln\left(1+\frac{h}{x}\right)}, \tag{5.1.20}$$

$$\rho_0 = \rho(x,y), \tag{5.1.21}$$

$$\rho_1 = \rho\left(x+h, y\rho_0^{\frac{h}{x}}\right). \tag{5.1.22}$$

5.2 3$^{\text{rd}}$ order Bigeometric Runge-Kutta Method(BRK3)

The derivation of 3^{rd} order Bigeometric Runge-Kutta method we apply the same procedure with Bigeometric Euler Method, i.e. the 2nd order Bigeometric Runge-Kutta method stated in section 5.1.

In this case $y(x+h)$ is open up to order 3 by using Bigeometric Taylor Theorem (4.0.7) and get

$$y(x+h) = y(x)\left(y^{\pi}(x)\right)^{\ln(1+\frac{h}{x})}\left(y^{\pi\pi}(x)\right)^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\left(y^{\pi\pi\pi}(x)\right)^{\frac{1}{3!}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}. \tag{5.2.1}$$

The derivation for the 3rd order Bigeometric Runge-Kutta method is :

$$y(x+h) = y(x)\rho_0^{a\ln(1+\frac{h}{x})}\rho_1^{b\ln\left(1+\frac{h}{x}\right)}\rho_2^{c\ln(1+\frac{h}{x})} \tag{5.2.2}$$

$$\rho_0 = \rho(x,y), \tag{5.2.3}$$

$$\rho_1 = \rho\left(x+\ell h, y\rho_0^{\frac{\gamma h}{x}}\right), \tag{5.2.4}$$

$$\rho_2 = \rho\left(x+\ell_1 h, y\rho_0^{\frac{\gamma_1 h}{x}}\rho_1^{\frac{\gamma_2 h}{x}}\right). \tag{5.2.5}$$

Now for comparing unknowns in (5.2.1) and (5.2.2) we have to change the Bigeometric Taylor expansion as :

$$y^{\pi}(x) = \rho(x,y) \tag{5.2.6}$$

$$y^{\pi\pi}(x) = (y^{\pi}(x))^{\pi} = \rho_x^{\pi}(x,y)\rho_y^{\pi}(x,y)^{y'(x)} \tag{5.2.7}$$

$$y^{\pi\pi\pi}(x) = [y^{\pi\pi}(x)]^{\pi} = \left[\rho_x^{\pi}(x,y)\rho_y^{\pi}(x,y)^{y'(x)}\right]^{\pi} \tag{5.2.8}$$

By applying chain rule satate in (6) to (5.2.8)

$$\begin{aligned} y^{\pi\pi\pi}(x) &= \rho_{xx}^{\pi}(x,y)\left[\rho_{xy}^{\pi}(x,y)\right]^{y'(x)}\left[\rho_{yx}^{\pi}(x,y)\right]^{y'(x)}\left[\rho_{yy}^{\pi}(x,y)\right]^{(y'(x))^2}\rho_y^{\pi}(x,y)^{y''(x)} \\ &= \rho_{xx}^{\pi}(x,y)\rho_{xy}^{\pi}(x,y)^{2y'(x)}\rho_{yy}^{\pi}(x,y)^{(y'(x))^2}\rho_y^{\pi}(x,y)^{y''(x)} \end{aligned} \tag{5.2.9}$$

Substituting (5.2.6),(5.2.7) and (5.2.9) in the 4 order Bigeometric Taylor formula stated in (5.2.1) we obtain :

$$\begin{aligned} y(x+h) = y(x)\,(\rho(x,y))^{\ln(1+\frac{h}{x})}\left(\rho_x^{\pi}(x,y)\rho_y^{\pi}(x,y)^{y'(x)}\right)^{\frac{1}{2!}\left[\ln(1+\frac{h}{x})\right]^2} \\ \left[\rho_{xx}^{\pi}(x,y)\rho_{xy}^{\pi}(x,y)^{2y'(x)}\rho_{yy}^{\pi}(x,y)^{(y'(x))^2}\rho_y^{\pi}(x,y)^{y''(x)}\right]^{\frac{1}{3!}\left[\ln\left(1+\frac{h}{z}\right)\right]^3} \end{aligned} \tag{5.2.10}$$

Replacing y' in (5.2.10) obviously the Bigeometric Taylor expansion turns to :

$$\begin{aligned} y(x+h) = y(x)\,(\rho(x,y))^{\ln\left(1+\frac{h}{x}\right)}\,(\rho_x^{\pi}(x,y))^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\left[\rho_y^{\pi}(x,y)^{\frac{y}{x}\ln(\rho(x,y))}\right]^{\frac{1}{2!}\left(\ln\left(1+\frac{h}{x}\right)\right)^2}\times \\ \times\left[\rho_{xx}^{\pi}(x,y)\rho_{xy}^{\pi}(x,y)^{2\frac{y}{x}\ln(\rho(x,y))}\rho_{yy}^{\pi}(x,y)^{\left(\frac{y}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{3!}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}\times \\ \times\left[\rho_y^{\pi}(x,y)^{y\left(\frac{1}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{3!}\left[\ln\left(1+\frac{h}{x}\right)\right]^3} \end{aligned} \tag{5.2.11}$$

Obviously we need to expanding ρ_1 and ρ_2 by using the Bigeometric Taylor theorem. Then we obtain :

$$\rho_1 = \rho(x,y)\left[\rho_x^\pi(x,y)^\ell \rho_y^\pi(x,y)^{y\frac{\gamma}{x}\ln(\rho(x,y))}\right]^{\ln(1+\frac{h}{x})}\cdot$$
$$\cdot\left[\rho_{xx}^\pi(x,y)^{\ell^2}\rho_{xy}^\pi(x,y)^{\ell y\frac{\gamma}{x}\ln(\rho(x,y))}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot$$
$$\cdot\left[\rho_{yy}^\pi(x,y)^{\left(y\frac{\gamma}{x}\ln(\rho(x,y))\right)^2}\rho_y^\pi(x,y)^{y\left(\frac{\gamma}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2} \tag{5.2.12}$$

$$\rho_2 = \rho(x,y)\left[\rho_x^\pi(x,y)^{\ell_1}\rho_y^\pi(x,y)^{(\gamma_1+\gamma_2)\frac{y}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\cdot$$
$$\cdot\left[\rho_{xx}^\pi(x,y)^{\ell_1^2}\rho_{xy}^\pi(x,y)^{(\ell_1\gamma_1+\ell_1\gamma_2)\frac{y}{x}\ln(\rho(x,y))}\right]^{\frac{1}{2!}\left[\ln(1+\frac{h}{x})\right]^2}\cdot$$
$$\cdot\left[\rho_{yy}^\pi(x,y)^{(\gamma_1+\gamma_2)^2\left(\frac{y}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{2!}\left[\ln(1+\frac{h}{x})\right]^2}\cdot$$
$$\cdot\left[\rho_y^\pi(x,y)^{y(\gamma_1+\gamma_2)^2\left(\frac{1}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2} \tag{5.2.13}$$

By substituting (5.2.12) and (5.2.13) into (5.2.11) we obtain :

$$y(x+h) = y(x)\rho(x,y)^{(a+b+c)\ln\left(1+\frac{h}{x}\right)}\cdot$$
$$\cdot\left[\rho_x^\pi(x,y)^{(\ell b+\ell_1 c)}\right]^{\left(\ln\left(1+\frac{h}{x}\right)\right)^2}\cdot$$
$$\cdot\left[\rho_y^\pi(x,y)^{\gamma b+(\gamma_1+\gamma_2)c}\right]^{\frac{y}{x}\ln(\rho(x,y))\left(\ln(1+\frac{h}{x})\right)^2}\cdot$$
$$\cdot\left[\rho_{xx}^\pi(x,y)^{\ell^2 b+\ell_1^2 c}\right]^{\frac{1}{2}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}\cdot$$
$$\cdot\left[\rho_{xy}^\pi(x,y)^{(b\ell\gamma+(\ell_1\gamma_1+\ell_1\gamma_2)c)\frac{y}{x}\ln\rho(x,y)}\right]^{\frac{1}{2}\left[\ln(1+\frac{h}{x})\right]^3}\cdot$$
$$\cdot\left[\rho_{yy}^\pi(x,y)^{\left(\gamma^2 b+(\gamma_1+\gamma_2)^2c\right).\left(\frac{y}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2}\left[\ln(1+\frac{h}{x})\right]^3}\cdot$$
$$\cdot\left[\rho_y^\pi(x,y)^{y\left(b\gamma^2+(\gamma_1+\gamma_2)^2c\right)\left(\frac{1}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2}\left[\ln(1+\frac{h}{x})\right]^3} \tag{5.2.14}$$

Now we will compare the powers of the Bigeometric derivatives in (5.2.14) with the ones in (5.2.11) then the relation of parameters stated as :

$$\begin{aligned}
a+b+c &= 1\\
\ell b+\ell_1 c+\ell_2 d &= \frac{1}{2}\\
\gamma b+\gamma_1 c+\gamma_2 c &= \frac{1}{2}\\
\ell^2 b+\ell_1^2 c &= \frac{1}{3}\\
b\ell\gamma+\ell_1\gamma_1 c+\ell_1\gamma_2 cd &= \frac{1}{3}\\
\gamma^2 b+\gamma_1^2 c+\gamma_2^2 c+2\gamma_1\gamma_2 c &= \frac{1}{3}\\
\gamma^2 b+\gamma_1^2 c+\gamma_2^2 c+2\gamma_1\gamma_2 c &= \frac{1}{3}
\end{aligned} \tag{5.2.15}$$

Then simplifying this finding in (5.2.15) we get analogously the parameters $a, b, c, \ell, q, \ell_1, \gamma_1$, and γ_2 as :

$$\ell = \gamma \tag{5.2.16}$$

$$\ell_1 = \gamma_1+\gamma_2 \tag{5.2.17}$$

and

$$a+b+c = 1 \tag{5.2.18}$$

$$\ell b+c\ell_1 = \frac{1}{2} \tag{5.2.19}$$

As in the case of the Bigeometric Euler Method, the number of equations is less than the number of unknowns; therefore we get again infinitely many solutions. We can summerize the results in the following Butcher tableau :

$$\begin{array}{c|ccc} 0 & & & \\ \ell & \gamma & & \\ \ell_1 & \gamma_1 & \gamma_2 & \\ \hline & a & b & c \end{array}$$

A reasonable selection of the parameters could be $a = c = \frac{1}{6}$, $b = \frac{2}{3}$, $\ell = \gamma = \frac{1}{2}$, $\ell_1 = 1$ $\gamma_1 = -1$ and $\gamma_2 = 2$. The function is evaluated at three positions, i.e. at x, $x + \ell h$ and $x + \ell_1 h$. Reasonably $\ell_1 = 1$ so that we evaluate the function at the beginning and the end of the interval $[x, x+h]$. We select $\ell = \frac{1}{2}$ to calculate the function also in the middle of the interval. The weights of the contributions of ρ_0, ρ_1, and ρ_2 are a, b, and c respectively. As $a+b+c = 1$, we give equal weights for the end points of the interval, and put the emphasis on midpoint of the interval and get therefore $a = \frac{1}{6}$, $b = \frac{2}{3}$, and $c = \frac{1}{6}$. Nevertheless, the parameters can be selected in the framework of the Butcher tableau for any problem independently to find the optimal solution. Finally we get for this selection of the parameters the 3rd order Bigeometric Runge-Kutta method

$$y(x+h) = y(x)\rho_0^{\frac{1}{6}\ln\left(1+\frac{h}{x}\right)}\rho_1^{\frac{2}{3}\ln\left(1+\frac{h}{x}\right)}\rho_2^{\frac{1}{6}\ln\left(1+\frac{h}{x}\right)}, \tag{5.2.20}$$

$$\rho_0 = \rho(x,y), \tag{5.2.21}$$

$$\rho_1 = \rho\left(x+\frac{h}{2}, y\rho_0^{\frac{h}{2x}}\right), \tag{5.2.22}$$

$$\rho_2 = \rho\left(x+h, y\rho_1^{-\frac{h}{x}}\rho_2^{\frac{2h}{x}}\right). \tag{5.2.23}$$

5.3 4^{th} order Bigeometric Runge-Kutta Method(BRK4)

We know that the main aim of 4th order Runge-Kutta method(RK4) performs the most accurate estimation for the initial value problems with a reasonable computational effort. Because of this property in many areas researchers, especially engineers and scientist, preferred to use the (RK4) method. The analysis for several problems showed that also in the framework of Bigeometric Calculus the 4th order Bigeometric Runge-Kutta method(BRK4) gives the most accurate approximation for the initial value problems. For proving BRK4 method we need Bigeometric Taylor expansion of $y(x+h)$ stated in equation (5.2.11) as :

$$\begin{aligned} y(x+h) = y(x)\left(\rho(x,y)\right)^{\ln\left(1+\frac{h}{x}\right)}\left(\rho_x^{\pi}(x,y)\right)^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\left[\rho_y^{\pi}(x,y)^{\frac{y}{x}\ln(\rho(x,y))}\right]^{\frac{1}{2!}\left(\ln\left(1+\frac{h}{x}\right)\right)^2}\cdot \\ \cdot\left[\rho_{xx}^{\pi}(x,y)\rho_{xy}^{\pi}(x,y)^{2\frac{y}{x}\ln(\rho(x,y))}\rho_{yy}^{\pi}(x,y)^{\left(\frac{y}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{3!}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}\cdot \\ \cdot\left[\rho_y^{\pi}(x,y)^{y\left(\frac{1}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{3!}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}\cdots \end{aligned} \quad (5.3.1)$$

By using same criteria in the 2nd and 3rd order case the 4th order Bigeometric Runge-Kutta method derived as :

$$y(x+h) = y(x)\rho_0^{a\ln\left(1+\frac{h}{x}\right)}\rho_1^{b\ln\left(1+\frac{h}{x}\right)}\rho_2^{c\ln\left(1+\frac{h}{x}\right)}\rho_3^{d\ln\left(1+\frac{h}{x}\right)} : \quad (5.3.2)$$

wihere ρ_0, ρ_1, ρ_2, and ρ_3 defined as following :

$$\rho_0 = \rho(x,y) \quad (5.3.3)$$

$$\rho_1 = \rho\left(x+\ell h, y\rho_0^{\frac{\gamma h}{x}}\right) \quad (5.3.4)$$

$$\rho_2 = \rho\left(x+\ell_1 h, y\rho_0^{\frac{\gamma_1 h}{x}}\rho_1^{\frac{\gamma_2 h}{x}}\right) \quad (5.3.5)$$

$$\rho_3 = \rho\left(x+\ell_2 h, y\rho_0^{\frac{\gamma_3 h}{x}}\rho_1^{\frac{\gamma_4 h}{x}}\rho_2^{\frac{\gamma_5 h}{x}}\right). \quad (5.3.6)$$

For determine the scalar values in (5.3.2)-(5.3.6) we need to use Bigeometric Taylor expansion for $y(x+h)$ up to order 4 stated in (5.3.1).Obviously we need to expand ρ_1, ρ_2, and ρ_3 by using the Bigeometric Taylor theorem. Then we get

$$\begin{aligned}\rho_1 = \rho(x,y) &\left[\rho_x^{\pi}(x,y)^{\ell}\rho_y^{\pi}(x,y)^{y\frac{\gamma}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\cdot\\ &\cdot\left[\rho_{xx}^{\pi}(x,y)^{\ell^2}\rho_{xy}^{\pi}(x,y)^{\ell y\frac{\gamma}{x}\ln\rho(x,y)}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot\\ &\cdot\left[\rho_{yy}^{\pi}(x,y)^{\left(y\frac{\gamma}{x}\ln\rho(x,y)\right)^2}\rho_y^{\pi}(x,y)^{y\left(\frac{\gamma}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\end{aligned} \tag{5.3.7}$$

$$\begin{aligned}\rho_2 = \rho(x,y) &\left[\rho_x^{\pi}(x,y)^{\ell_1}\rho_y^{\pi}(x,y)^{(\gamma_1+\gamma_2)\frac{y}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\cdot\\ &\cdot\left[\rho_{xx}^{\pi}(x,y)^{\ell_1^2}\rho_{xy}^{\pi}(x,y)^{(\ell_1\gamma_1+\ell_1\gamma_2)\frac{y}{x}\ln(\rho(x,y))}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot\\ &\cdot\left[\rho_{yy}^{\pi}(x,y)^{(\gamma_1+\gamma_2)^2\left(\frac{y}{x}\ln(\rho(x,y))\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot\\ &\cdot\left[\rho_y^{\pi}(x,y)^{y(\gamma_1+\gamma_2)^2\left(\frac{1}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\end{aligned} \tag{5.3.8}$$

$$\begin{aligned}\rho_3 = \rho(x,y) &\left[\rho_x^{\pi}(x,y)^{\ell_2}\rho_y^{\pi}(x,y)^{(\gamma_3+\gamma_4+\gamma_5)\frac{y}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\cdot\\ &\cdot\left[\rho_{xx}^{\pi}(x,y)^{\ell_2^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot\\ &\cdot\left[\rho_{xy}^{\pi}(x,y)^{(\ell_2\gamma_3+\ell_2\gamma_4+\ell_2\gamma_5)\frac{y}{x}\ln\rho(x,y)}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot\\ &\cdot\left[\rho_{yy}^{\pi}(x,y)^{(\gamma_3+\gamma_4+\gamma_5)^2\left(\frac{y}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\cdot\\ &\cdot\left[\rho_y^{\pi}(x,y)^{y(\gamma_3+\gamma_4+\gamma_5)^2\left(\frac{1}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2}\end{aligned} \tag{5.3.9}$$

By substituting (5.3.7) , (5.3.8) and (5.3.9) into (5.3.2) we get :

$$
\begin{aligned}
y(x+h) = y(x)\, & \underbrace{\rho(x,y)^{a\ln\left(1+\frac{h}{x}\right)}}_{\rho_0} \cdot \\
& \cdot \underbrace{\left[\rho(x,y)\rho_x^{\pi}(x,y)^{\ell}\rho_y^{\pi}(x,y)^{y\frac{\gamma}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)b\ln\left(1+\frac{h}{x}\right)}}_{\rho_1} \cdot \\
& \cdot \underbrace{\left[\rho_{xx}^{\pi}(x,y)^{\ell^2}\rho_{xy}^{\pi}(x,y)^{\ell y\frac{\gamma}{x}\ln\rho(x,y)}\rho_{yy}^{\pi}(x,y)^{\left(y\frac{\gamma}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2 b\ln\left(1+\frac{h}{x}\right)}}_{\rho_1} \cdot \\
& \cdot \underbrace{\left[\rho_y^{\pi}(x,y)^{y\left(\frac{\gamma}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2 b\ln\left(1+\frac{h}{x}\right)}}_{\rho_1} \cdot \\
& \cdot \underbrace{\left[\rho(x,y)\left[\rho_x^{\pi}(x,y)^{\ell_1}\rho_y^{\pi}(x,y)^{y\frac{(\gamma_1+\gamma_2)}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\right]^{c\ln\left(1+\frac{h}{x}\right)}}_{\rho_2} \cdot \\
& \cdot \underbrace{\left[\rho_{xx}^{\pi}(x,y)^{\ell_1^2}\rho_{xy}^{\pi}(x,y)^{y\frac{(\ell_1\gamma_1+\ell_1\gamma_2)}{x}\ln\rho(x,y)}\rho_{yy}^{\pi}(x,y)^{(\gamma_1+\gamma_2)^2\left(\frac{y}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2 c\ln\left(1+\frac{h}{x}\right)}}_{\rho_2} \cdot \\
& \cdot \underbrace{\left[\rho_y^{\pi}(x,y)^{y(\gamma_1+\gamma_2)^2\left(\frac{1}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2 c\ln\left(1+\frac{h}{x}\right)}}_{\rho_2} \cdot \\
& \cdot \underbrace{\left[\rho(x,y)\left[\rho_x^{\pi}(x,y)^{\ell_2}\rho_y^{\pi}(x,y)^{y\frac{(\gamma_3+\gamma_4+\gamma_5)}{x}\ln(\rho(x,y))}\right]^{\ln\left(1+\frac{h}{x}\right)}\right]^{d\ln\left(1+\frac{h}{x}\right)}}_{\rho_3} \cdot \\
& \cdot \underbrace{\left[\rho_{xx}^{\pi}(x,y)^{\ell_2^2}\rho_{xy}^{\pi}(x,y)^{y\frac{(\ell_2\gamma_3+\ell_2\gamma_4+\ell_2\gamma_5)}{x}\ln\rho(x,y)}\rho_{yy}^{\pi}(x,y)^{(\gamma_3+\gamma_4+\gamma_5)^2\left(\frac{y}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\left[\ln\left(1+\frac{h}{x}\right)\right]^2 d\ln\left(1+\frac{h}{x}\right)}}_{\rho_3} \cdot \\
& \cdot \underbrace{\left[\rho_y^{\pi}(x,y)^{y(\gamma_3+\gamma_4+\gamma_5)^2\left(\frac{1}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2!}\ln\left(1+\frac{h}{x}\right)^2 d\ln\left(1+\frac{h}{x}\right)}}_{\rho_3} \qquad (5.3.10)
\end{aligned}
$$

Simplifying (5.3.10) we get:

$$y(x+h) = y(x)\rho(x,y)^{(a+b+c+d)\ln\left(1+\frac{h}{x}\right)}.$$
$$\cdot\left[\rho_x^{\pi}(x,y)^{(\ell b+\ell_1 c+\ell_2 d)}\right]^{\left(\ln(1+\frac{h}{x})\right)^2}.$$
$$\cdot\left[\rho_y^{\pi}(x,y)^{[\gamma b+(\gamma_1+\gamma_2)c+(\gamma_3+\gamma_4+\gamma_5)d]}\right]^{\frac{y}{x}\ln(\rho(x,y))\left(\ln(1+\frac{h}{x})\right)^2}.$$
$$\cdot\left[\rho_{xx}^{\pi}(x,y)^{\ell^2 b+\ell_1^2 c+\ell_2^2 d}\right]^{\frac{1}{2}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}.$$
$$\cdot\left[\rho_{xy}^{\pi}(x,y)^{(b\ell\gamma+(\ell_1\gamma_1+\ell_1\gamma_2)c+(\ell_2\gamma_3+\ell_2\gamma_4+\ell_2\gamma_5)d)\frac{y}{x}\ln\rho(x,y)}\right]^{\frac{1}{2}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}.$$
$$\cdot\left[\rho_{yy}^{\pi}(x,y)^{\left(\gamma^2 b+(\gamma_1+\gamma_2)^2 c+(\gamma_3+\gamma_4+\gamma_5)d\right)\left(\frac{y}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2}\left[\ln\left(1+\frac{h}{x}\right)\right]^3}.$$
$$\cdot\left[\rho_{y}^{\pi}(x,y)^{y\left(b\gamma^2+(\gamma_1+\gamma_2)^2 c+(\gamma_3+\gamma_4+\gamma_5)^2 d\right)\left(\frac{1}{x}\ln\rho(x,y)\right)^2}\right]^{\frac{1}{2}\left[\ln\left(1+\frac{h}{x}\right)\right]^3} \qquad (5.3.11)$$

Now we will compare the powers of the Bigeometric derivatives with the ones in (5.3.1), then we catch the relation between parametersin BRK4 method as :

$$\begin{aligned}
a+b+c+d &= 1\\
\ell b+\ell_1 c+\ell_2 d &= \frac{1}{2}\\
\gamma b+\gamma_1 c+\gamma_2 c+\gamma_3 d+\gamma_4 d+\gamma_5 d &= \frac{1}{2}\\
\ell^2 b+\ell_1^2 c+\ell_2^2 d &= \frac{1}{3}\\
b\ell\gamma+\ell_1\gamma_1 c+\ell_1\gamma_2 c+\ell_2\gamma_3 d+\ell_2\gamma_4 d+\ell_2\gamma_5 d &= \frac{1}{3}\\
\gamma^2 b+\gamma_1^2 c+\gamma_2^2 c+2\gamma_1\gamma_2 c+\gamma_3^2 d+\gamma_4^2 d+\gamma_5^2 d+2\gamma_3\gamma_4 d+2\gamma_3\gamma_5 d+2\gamma_5\gamma_4 d &= \frac{1}{3}\\
\gamma^2 b+\gamma_1^2 c+\gamma_2^2 c+2\gamma_1\gamma_2 c+\gamma_3^2 d+\gamma_4^2 d+\gamma_5^2 d+2\gamma_3\gamma_4 d+2\gamma_3\gamma_5 d+\gamma_5\gamma_4 d &= \frac{1}{3}
\end{aligned} \qquad (5.3.12)$$

Simplifying (5.3.12) we obtain the relation of parameters as :

$$\ell = \gamma \tag{5.3.13}$$

$$\ell_1 = \gamma_1 + \gamma_2 \tag{5.3.14}$$

$$\ell_2 = \gamma_3 + \gamma_4 + \gamma_5 \tag{5.3.15}$$

and

$$a + b + c + d = 1 \tag{5.3.16}$$

$$b\ell + c\ell_1 + d\ell_2 = \frac{1}{2} \tag{5.3.17}$$

$$b\ell^2 + c\ell_1^2 + d\ell_2^2 = \frac{1}{3} \tag{5.3.18}$$

The results of (5.3.12) can be extended using the Bigeometric Butcher Tableau [13].

$$\begin{array}{c|cccc}
0 & & & & \\
\ell & \gamma & & & \\
\ell_1 & \gamma_1 & \gamma_2 & & \\
\ell_2 & \gamma_3 & \gamma_4 & \gamma_5 & \\
\hline
 & a & b & c & d
\end{array}.$$

Since the number of equations is less than the number of unknowns, the above system has infinitely many solutions. Depending on the nature of a problem, a suitable selection of the parameters can also be extended. The original parameters for the ordinary case of RK4 are stated below :

$$a = d = \frac{1}{6}, \tag{5.3.19}$$

$$b = c = \frac{1}{3}, \tag{5.3.20}$$

$$\ell = \ell_1 = \gamma = \gamma_2 = \frac{1}{2}, \tag{5.3.21}$$

$$\ell_2 = \gamma_5 = 1, \text{ and} \tag{5.3.22}$$

$$\gamma_1 = \gamma_3 = \gamma_4 = 0. \tag{5.3.23}$$

For selecting parameters in RK4, the main contribution to the approximation comes from the middle of the interval $[x, x+h]$. We evaluate f in the middle of the interval twice, both with a weight of $1/3$. Therefore for comparing two methods we get our parameters same with RK4. Substitution parametrs(5.3.19)-(5.3.23) into (5.3.2)-(5.3.6) we get the Bigeometric Runge-Kutta method as :

$$y(x+h) = y(x)\rho_0^{\frac{1}{6}\ln\left(1+\frac{h}{x}\right)}\rho_1^{\frac{1}{3}\ln\left(1+\frac{h}{x}\right)}\rho_2^{\frac{1}{3}\ln\left(1+\frac{h}{x}\right)}\rho_3^{\frac{1}{6}\ln\left(1+\frac{h}{x}\right)} \tag{5.3.24}$$

with

$$\rho_0 = \rho(x, y), \tag{5.3.25}$$

$$\rho_1 = \rho\left(x+\frac{h}{2}, y\rho_0^{\frac{h}{2x}}\right), \tag{5.3.26}$$

$$\rho_2 = \rho\left(x+\frac{h}{2}, y\rho_1^{\frac{h}{2x}}\right), \tag{5.3.27}$$

$$\rho_3 = \rho\left(x+h, y\rho_2^{\frac{h}{x}}\right). \tag{5.3.28}$$

Chapter 6

CONVERGENCE OF ONE-STEP METHODS

In this chapter, our aim is to test our BRK4 method for showing the completeness of the theory. This chapter consist of two sections. In the first section 6.1 we analyze the convergence analysis of the BRK4. Then second criteria in the numerical theory is stability. We test the stability of BRK4 in the section 6.2

6.1 Convergence Analysis

In this section the convergence property tested for seeing the behaviour of Bigeometric Runge-Kutta method. Let $\eta(x;h)$ is denote approximate solution of one step method where $h \to 0$. Suppose that f be a one time π-differentiable function on the interval (a,b) and $y(x)$ denote the exact solution of the initial-value problem :

$$y^{\pi} = \rho(x,y), \qquad y(x_0) = y_0. \tag{6.1.1}$$

Consider $\Phi(x,y;h)$ as a one-step method,

$$\begin{aligned} \eta_0 &:= y_0, \\ \text{for} \quad i = 0,1,\ldots : & \\ \eta_{i+1} &:= \eta_i \Phi(x_i,\eta_i;h)^{\log\left(1+\frac{h}{x}\right)}, \\ x_{i+1} &:= x_i + h, \end{aligned}$$

where $x \in R_h := \{x_0 + ih \,|\, i = 0, 1, 2, \ldots\}$ indicates the approximate solution of $\eta(x;h)$ as :

$$\eta(x;h) := \eta_i, \qquad \text{where} \quad x = x_0 + ih.$$

Assume that $z(t)$ as an exact solution of the initial-value problem in (6.1.1).Let x, y are arbitrary and fixed. The bigeometric derivative $z^{\pi}(t)$ defined as :

$$z^{\pi}(t) = \rho(t, z(t)), \qquad z(x) = y, \tag{6.1.2}$$

with initial values of x, y.

The bigeometric ratio function of the exact solution $z(t)$ in (6.1.2) for step size h denotes as :

$$\Delta(x,y;h) := \begin{cases} \left(\frac{z(x+h)}{y}\right)^{\log\left(1+\frac{h}{x}\right)} & \text{if } h \neq 0, \\ f(x,y) & \text{if } h = 0 \end{cases} \tag{6.1.3}$$

where $\Phi(x,y;h)$ is the bigeometric ratio function for step size h of the approximate solution of (6.1.2) indicated by Φ.

The magnitude ratio

$$\tau(x,y;h) := \frac{\Delta(x,y;h)}{\Phi(x,y;h)}$$

denotes how well the value $z(x+h)$ at $x+h$ fitted the equation of the one-step method.

Let $\tau(x,y;h)$ is the bigeometric local discretization error at the point (x,y).The main concept is to satisfy the following condition.

$$\lim_{h\to 0}\tau(x,y;h)=1$$

The curious part here is the role of the bigeometric global discretization error

$$e(x;h):=\frac{\eta(x;h)}{y(x)}$$

for $h\to 0$ and x is fixed where $h\in H_x:=\left\{\frac{(x-x_0)}{n}\,|\,n=1,2,\ldots\right\}$. Since $e(x;h)$ is only defined for $h\in H_x$, we have to check the convergence of

$$e(x;h_n),\qquad h_n:=\frac{x-x_0}{n},\qquad \text{as}\quad n\to\infty.$$

A one-step method in bigeometric calculus is *convergent* if it satisfied the condition (6.1)

$$\lim_{n\to\infty}e(x;h_n)=1$$

for all $x\in[a,b]$ and all functions f being one time π-differentiable on the interval (a,b).

Since f being p-times π-differentiable on (a,b), methods of order $p>0$ are convergent, and satisfy

$$e(x;h_n)=O\left(e^{\log\left(1+\frac{h_n}{x_n}\right)^p}\right).$$

The order of the bigeometric global discretization error is thus equal to the order of the bigeometric local discretization error. If the numbers ξ_i provide an estimates of the form :

$$|\xi_{i+1}| \leq |\xi_i|^{(1+\delta)}B, \qquad \delta > 0, \qquad B \geq 0, \qquad i = 0,1,2,...,$$

then we will shortly denotes below equation as :

$$|\xi_n| \leq |\xi_0|^{e^{n\delta}} B^{\frac{e^{n\delta}-1}{\delta}}$$

Proof. If we open (6.1) for $i = 0,1,2,...,$ we get :

$$\begin{aligned}
|\xi_1| &\leq |\xi_0|^{(1+\delta)}B \\
|\xi_2| &\leq |\xi_0|^{(1+\delta)^2}B^{1+(1+\delta)} \\
&\vdots \\
|\xi_n| &\leq |\xi_0|^{(1+\delta)^n}B^{[1+(1+\delta)+(1+\delta)^2+...+(1+\delta)^{n-1}]} \\
&= |\xi_0|^{(1+\delta)^n}B^{\frac{(1+\delta)^n-1}{\delta}} \\
&\leq |\xi_0|^{e^{n\delta}}B^{\frac{e^{n\delta}-1}{\delta}}
\end{aligned}$$

Consequently $0 < 1+\delta \leq e^{\delta}$ for $\delta > -1$. ■

Theorem 6.1.1 *Consider, $x_0 \in [a,b]$, $y_0 \in \mathbb{R}$, the initial-value problem*

$$y^{\pi} = f(x,y), \quad y(x_0) = y_0,$$

with the exact solution $y(x)$. Assume that the function Φ be continuous on

$$G := \left\{ (x,y,h) \,|\, a \leq x \leq b, \left| \frac{y}{y(x)} \right| \leq \gamma, 0 \leq |h| \leq h_0 \right\}, h_0 > 0, \gamma > 1,$$

and there exist positive constants M and N such that

$$\left| \frac{\Phi(x,y_1;h)}{\Phi(x,y_2;h)} \right| \leq \left| \frac{y_1}{y_2} \right|^M$$

for all $(x,y_i,h) \in G$, and $i = 1,2$ and

$$|\tau(x,y(x);h)| = \left|\frac{\Delta(x,y(x);h)}{\Phi(x,y(x);h)}\right| \leq e^{N|\log(1+\frac{h}{x})|^p}, \quad p > 0$$

for all $x \in [a,b]$, $|h| \leq h_0$. Then there exists an $\overline{h}$, $0 < \overline{h} \leq h_0$, such that for the bigeometric global discretization error $e(x;h) = \frac{\eta(x;h)}{y(x)}$,

$$|e(x;h_n)| \leq \exp\left\{N\left|\log\left(1+\frac{h_n}{x_n}\right)\right|^p \frac{1}{M}\left(e^{kM\left|\log\left(1+\frac{x-x_0}{nx}\right)\right|}-1\right)\right\}$$

for all $x \in [a,b]$ and all $h_n = \frac{x-x_0}{n}$, $n = 1,2,\ldots$, with $|h_n| \leq \overline{h}$. If $\gamma = \infty$, then $\overline{h} = h_0$.

Proof. The function

$$\widetilde{\Phi}(x,y;h) = \begin{cases} \Phi(x,y;h) & \text{if } (x,y;h) \in G \\ \Phi(x,y(x)\gamma;h) & \text{if } x \in [a,b],\ |h| \leq h_0,\ y \geq y(x)\gamma \\ \Phi(x,\frac{y(x)}{\gamma};h) & \text{if } x \in [a,b],\ |h| \leq h_0,\ y \leq \frac{y(x)}{\gamma} \end{cases}$$

is continuous on $\widetilde{G} := \{(x,y,h)\,|\,x \in [a,b],\ y \in \mathbb{R},\ |h| \geq h_0\}$ and satisfies the condition

$$\left|\frac{\widetilde{\Phi}(x,y_1;h)}{\widetilde{\Phi}(x,y_2;h)}\right| \leq \left|\frac{y_1}{y_2}\right|^M \tag{6.1.4}$$

for all $(x,y_i,h) \in \widetilde{G}, i = 1,2$, and because of $\widetilde{\Phi}(x,y(x);h) = \Phi(x,y(x);h)$, also the condition

$$\left|\frac{\Delta(x,y(x);h)}{\widetilde{\Phi}(x,y(x);h)}\right| \leq e^{N|log(1+\frac{h}{x})|^p}, \quad \text{for } x \in [a,b],\ |h| \leq h_0. \tag{6.1.5}$$

is satisfied.

For getting aproximate values $\widetilde{\eta}_i := \widetilde{\eta}(x_i;h)$ for $y_i := y(x_i)$, $x_i := x_0 + ih$, the one-step method generated by $\widetilde{\Phi}$ as :

$$\widetilde{\eta}_{i+1} = \widetilde{\eta}_i \widetilde{\Phi}(x_i, \widetilde{\eta}_i; h)^{\log\left(1+\frac{h}{x}\right)}.$$

In view of

$$y_{i+1} = y_i \Delta(x_i, y_i; h)^{\log\left(1+\frac{h}{x}\right)},$$

Calculate the error for $\widetilde{e}_i := \frac{\widetilde{\eta}_i}{y_i}$, the recurrence formula obtained as :

$$\widetilde{e}_{i+1} = \widetilde{e}_i \left[\frac{\widetilde{\Phi}(x_i, \widetilde{\eta}_i; h)}{\widetilde{\Phi}(x_i, y_i; h)}\right]^{\log\left(1+\frac{h}{x}\right)} \cdot \left[\frac{\widetilde{\Phi}(x_i, y_i; h)}{\Delta(x_i, y_i; h)}\right]^{\log\left(1+\frac{h}{x}\right)} \tag{6.1.6}$$

Simplifying (6.1.4), (6.1.5) it follows that

$$\begin{aligned}
\left|\frac{\widetilde{\Phi}(x_i, \widetilde{\eta}_i; h)}{\widetilde{\Phi}(x_i, y_i; h)}\right| &\leq \left|\frac{\widetilde{\eta}_i}{y_i}\right|^M = |\widetilde{e}_i|^M, \\
\left|\frac{\widetilde{\Phi}(x_i, y_i; h)}{\Delta(x_i, y_i; h)}\right| &\leq e^{N|\log\left(1+\frac{h}{x}\right)|^p},
\end{aligned}$$

Therefore from (6.1.6) the recursive estimation provided as:

$$|\widetilde{e}_{i+1}| \leq |\widetilde{e}_i|^{(1+|\log\left(1+\frac{h}{x}\right)|M)} e^{N|\log\left(1+\frac{h}{x}\right)|^{p+1}}.$$

Since we are related with an initial value problem, the initial values must be exact, and hence $\widetilde{e}_0 = \frac{\widetilde{\eta}_0}{y_0} = 1$, resulting in

$$|\widetilde{e}_k| \leq e^{N|\log\left(1+\frac{h}{x}\right)|^p \frac{e^{k\left|\log\left(1+\frac{h}{x}\right)\right|M-1}}{M}}. \tag{6.1.7}$$

Assume that $x \in [a,b]$ and fixed, with the conditions $x \neq x_0$ and $x \neq 0$. Let $h := h_n = \frac{(x-x_0)}{n}$ with an integer $n > 0$. Then it is obvious that $x_n = x_0 + nh = x$. Since $\widetilde{e}(x; h_n) =$

$\widetilde{e}_n$, from equation (6.1.7) with $k = n$,it follows that

$$|\widetilde{e}(x;h_n)| \leq \exp\left\{ N \left| \log\left(1+\frac{h_n}{x_n}\right)\right|^p \frac{1}{M}\left(e^{kM\left|\log\left(1+\frac{x-x_0}{nx}\right)\right|} - 1\right)\right\}$$

for all $x \in [a,b]$ and h_n with $|h_n| \leq h_0$. Since $|x - x_0| \leq |b-a|$ and $\gamma > 0$, there exists an $\overline{h}$, $0 < \overline{h} \leq h_0$, such that $|\widetilde{e}(x,;h_n)| \leq \gamma$ for all $x \in [a,b]$, $|h_n| \leq \overline{h}$, i.e., for the one-step method generated by Φ,

$$\begin{aligned} \eta_0 &= y_0, \\ \eta_{i+1} &= \eta_i \Phi(x_i, \eta_i; h), \end{aligned}$$

By using the definition of $\widetilde{\Phi}$ under the condition $|h| \leq \overline{h}$ we get :

$$\widetilde{\eta}_i = \eta_i, \qquad \widetilde{e}_i = e_i, \qquad and \qquad \widetilde{\Phi}(x_i, \widetilde{\eta}_i; h) = \Phi(x_i, \eta_i; h).$$

The claim of the theorem,

$$|\widetilde{e}(x;h_n)| \leq \exp\left\{ N \left| \log\left(1+\frac{h_n}{x_n}\right)\right|^p \frac{1}{M}\left(e^{kM\left|\log\left(1+\frac{x-x_0}{nx}\right)\right|} - 1\right)\right\}$$

thus follows for all $x \in [a,b]$ and all $h_n = \frac{(x-x_0)}{n}$, $n = 1,2,\ldots$, with $|h_n| \leq \overline{h}$. ■

6.2 Stability Analysis

The stability analysis of the Bigeometric Runge-Kutta methods was presented in this chapter. In Newtonian calculus, the stability properties of the ordinary Runge-Kutta methods are tested by the following equation.

$$y'(x) = \lambda y(x), \qquad y(x_0) = y_0 \tag{6.2.1}$$

where $\lambda \in \mathbb{C}$. The analysis of stability of (6.2.1) was tested extensively by [16, 18, 25, 28]. The stability test for the Bigeometric Runge-Kutta method can be done by applying same procedure in [26]. The bigeometric form of the (6.2.1) can be denoted as :

$$y^{\pi}(x) = e^{x\lambda}, \qquad y(x_0) = y_0, \tag{6.2.2}$$

The stability test function for 4$^{\text{th}}$ order BRK4 method with using the same process as in (5.3.24) - (5.3.28). The test equation obtained as :

$$y_{n+1} = y_n[\rho_0^a \cdot \rho_1^b \cdot \rho_2^c \cdot \rho_3^d]^{\log\left(1+\frac{h}{x}\right)} \tag{6.2.3}$$

where

$$a+b+c+d = 1. \tag{6.2.4}$$

which gives the analytic solution

$$y(x) = e^{\lambda(x-x_0)}y_0. \tag{6.2.5}$$

It is obvious that $x \to \infty$ and $Re(\lambda) < 0$, the solution of the system approaches to zero. We will decide that the method is A-stable [16] if the method has the same behaviour. Since $y^{\pi}(x)$ is a simple exponential function, equations (5.3.24) - (5.3.28) with (6.2.4) the equation transforms to

$$y_{n+1} = \exp\left\{\lambda x_n(a+b+c+d) + \lambda h\left(\frac{1}{2}(b+c)+d\right)\right\}^{\log\left(1+\frac{h}{x}\right)}, \tag{6.2.6}$$

We know that $a+b+c+d = 1$ and let assume that $\gamma = \frac{1}{2}(b+c)+d$ then y_{n+1} simplifies to

$$y_{n+1} = \exp\{\lambda x_n + \lambda h\gamma\}^{\log\left(1+\frac{h}{x}\right)} \tag{6.2.7}$$

At the end we get

$$y(x) = e^{2\lambda h\gamma \log\left(1+\frac{x-x_0}{x}\right)} y_0. \tag{6.2.8}$$

It is obvios that

$$\frac{y_{n+1}}{y_n} = e^z = R(z), \tag{6.2.9}$$

where $z = 2\lambda h\gamma \log\left(1+\frac{x-x_0}{x}\right)$. $R(z)$ denotes stability function of the proposed method.

The domain of stability is

$$S^* = \{z \in \mathbb{C} : |R(z)| < 1\}. \tag{6.2.10}$$

Consequently, by (6.2.10) we obtain

$$0 < e^{-2|\lambda|h\gamma \log\left(1+\frac{h}{x}\right)} < 1 \tag{6.2.11}$$

which leads to

$$0 < 2h\gamma \log\left(1+\frac{h}{x}\right) < \infty. \tag{6.2.12}$$

Consequently,

$$0 < \left|\frac{h^2}{x}\right| < \infty. \tag{6.2.13}$$

So, the result shows that the newly introduced method is unconditionally stable. By (6.2.10), it can be seen that $Re(z) < 0$ where $|e^z| = e^{Re(z)}$. When $Re(z) < 0$ the method is A-stable since the left half plane will be the region of absolute stability. In Newtonian calculus, the explicit multistep methods can not be A-stable and the implicit multistep methods can be said to be A-stable if the order is at most 2. On the other hand, In Bigeometric calculus both explicit and implicit methods are A-stable. A method is L-stable if the method is A-stable and $R(z) \to 0$ when $|z| \to \infty$ [18]. Since we have shown that the Bigeometric Runge-Kutta methods are A-stable and $e^z \to 0$ when $|z| \to \infty$, we can say that the proposed methods are L-stable by [18].

Chapter 7

APPLICATIONS OF THE BIGEOMETRIC RUNGE-KUTTA METHOD

In this chapter, the BRK4 method is applied to different examples in different areas. This chapter divided into section depending on the order of the differential equation. In section, 7.1 the order of the differential equation is one. In section, 7.2 include the modelling examples base on the order of the differential equation is two and three.

7.1 One Dimension example

Example 1. The important application for BRK4 is Gompertz function that an important mathematical model for a time series. You will find a lot of applications correlated with this function. As an example Gompertz function in [37] using for modeling the bacterial growth curve and in [36] used for modeling some cancer research. Now let's define this important function as:

$$y = a\exp(-b\exp(-cx)). \tag{7.1.1}$$

Let's select $a = 1$ and $b = c = -1$. Then we get :

$$y = \exp(\exp(x)). \tag{7.1.2}$$

For getting an initial value problem whose answer is this special function can denote as :

$$y'(x) = \exp\left(x + \exp(x)\right), \quad y(1) = 15.1543. \tag{7.1.3}$$

The equation (7.1.3) is in Newtonian form let's change the corresponding equation to

Bigeometric initial value problem by using (2.0.2) :

$$y^{\pi}(x) = \exp\left(x\exp(x)\right), \quad y(1) = 15.1543. \tag{7.1.4}$$

The numerical solutions of the Bigeometric initial value problem (7.1.4) and the ordinary initial value problem are summarised in table 7.1. The two method compared with their relative errors.

Table 7.1: Results of the 4th order Bigeometric Runge-Kutta and Runge-Kutta method with relative errors

x	y_{exact}	y_{BRK4}	y_{RK4}	relative error BRK4	relative error RK4
1	15.15426	15.15426	15.15426	0	0
1.5	88.3838	93.2806	89.7539	0.0554	0.0155
2	1618.1780	1804.2784	1724.6248	0.1150	0.0658
2.5	1.95339×10^5	2.32326×10^5	2.42767×10^5	0.1894	0.2428
3	5.28491×10^8	6.8233×10^8	9.1744×10^8	0.2911	0.7336
3.5	2.4091×10^{14}	3.4713×10^{14}	6.6619×10^{14}	0.4409	1.7653
4	5.1484×10^{23}	8.6352×10^{23}	2.3432×10^{24}	0.6773	3.5499

In given table 7.1 some steps RK4 has better results than BRK4 but for bigger $t-$values RK4 starts to give bad results as seen in the given table. Especially when $t = 4$ the relative error for RK4 is the percentage of 36 whereas BRK4 errors stay percentage of 6. As a result in this important and extreme example, we can conclude that the result of the BRK4 method is giving significantly better results compared to the ordinary

Runge-Kutta method.

Example 2. The third example is the best-known circuit in electrical engineering called as RC circuit. The circuit is for finding the voltage across to the capacitor, $y(t)$, for the depicted RC circuit Figure 7.1 in response to the applied voltage $x(t) = \frac{3}{5}e^{-2t}$ and initial condition $y(1) = 0.14$.

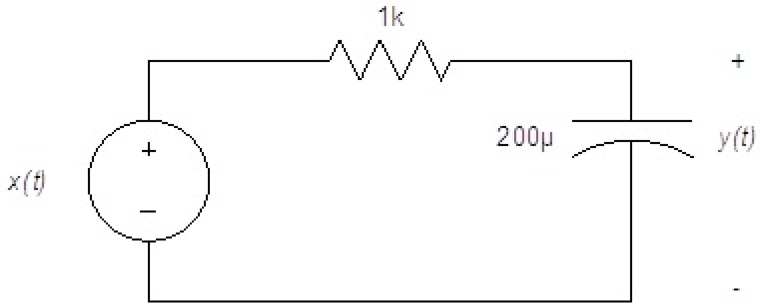


Figure 7.1: RC circuit for Example 1

Using Kirchhoff's Voltage Law, the behavior of the system Figure 7.1 can be described by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t) \tag{7.1.5}$$

From given component values $RC = 0.2s$. Substituting in Eq. (7.1.5)

$$\frac{dy(t)}{dt} + 5y(t) = 5x(t) \tag{7.1.6}$$

If we take $x(t) = \frac{3}{5}e^{-2t}$ the Eq. (7.1.6) changed to :

$$\frac{dy(t)}{dt} = 3e^{-2t} - 5y(t) \tag{7.1.7}$$

This representation needs to be transform into ordinary case to bigeometric case by using the properties (2.0.2). Where easily converted as :

$$f^{\pi}(x) = \exp\left\{t\,\frac{3e^{-2t} - 5y(t)}{y(t)}\right\} \tag{7.1.8}$$

The new method BRK4 tested against RK4 and the numerical results for both methods tabulated in Table 7.2 .

Table 7.2: Comparison of the numerical results of the both methods dependent on their relative errors

t	$y_{exact}(t)$	$y_{BRK4}(t)$	y_{RK4}	relative error BRK4	relative error RK4
1.5	0.0503402	0.050341072	0.049574276	1.8261×10^{-5}	0.0152140
2	0.0183610	0.018361359	0.018016422	1.7419×10^{-5}	0.0187689
2.5	0.0067417	0.006741771	0.006609736	1.4455×10^{-5}	0.0195705
3	0.0024791	0.002479088	0.002430097	1.2003×10^{-5}	0.0197498
3.5	0.0009119	0.000911916	0.000893861	1.0190×10^{-5}	0.0197898
4	0.0003355	0.000335468	0.000328823	8.8407×10^{-5}	0.0197987

In the table 7.2 the relative errors of RK4 and BRK4 are established and obviously BRK4 produces fewer errors compared with RK4. Moreover, relative errors of the BRK4 method are decreasing with depending on increasing step size (t) where the relative errors start to increase in the case of RK4. The test results for time computational analysis within fixed length of time against to relative errors presented in Fig.7.2

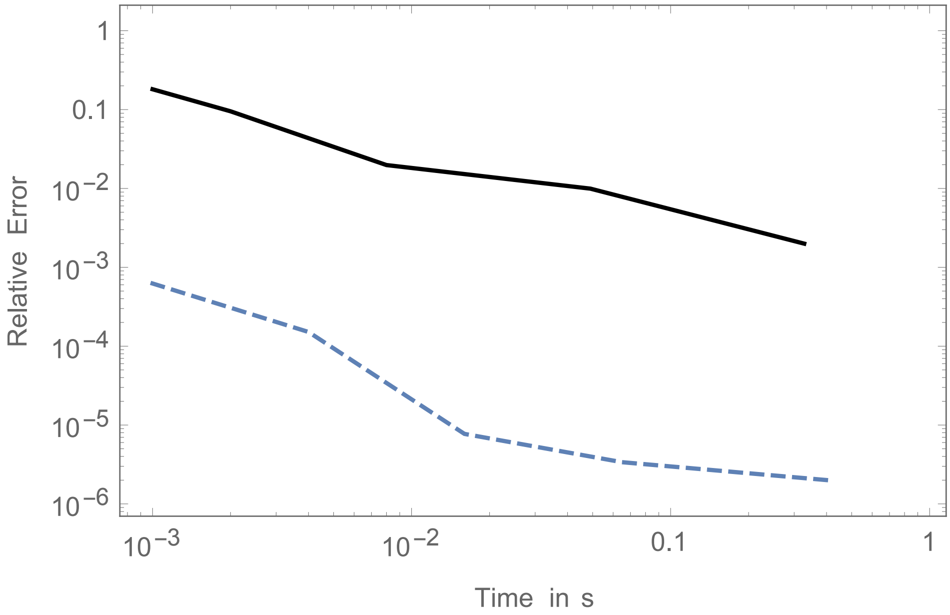


Figure 7.2: Time computation for Example 1

Obviously, the figure 7.2 underline that the relative errors for BRK4 are less than RK4 for the same execution period. However, this simple example is not sufficient to generalize the reliability of BRK4. Further examples of alternative configurations followed.

Example 3. The circuit configuration modified to introduce a more complex representation of nonlinear ordinary differential equation for showing the applicability of BRK4.

$$\frac{dy}{dt} + t\,(y(t))^2 = x(t) \tag{7.1.9}$$

where $x(t) = 3$ with initial condition $y(1) = 2.367$. The transformed bigeometric version of Eq. (7.1.9) can be extended as :

$$y^{\pi} = \exp\left\{t\left(-ty(t) - \frac{3}{y(t)}\right)\right\} \tag{7.1.10}$$

The method BRK4 tested against RK4 and the numerical results for both methods tabulated in Table 7.3 .

Table 7.3: Comparison of the numerical results of the both methods dependent on their relative errors

t	$y_{exact}(t)$	$y_{BRK4}(t)$	y_{RK4}	relative error BRK4	relative error RK4
1.5	1.6042283	1.6042835	1.5920433	0.00003438	0.0075956
2	1.3070175	1.3070394	1.2991601	0.00001673	0.0060117
2.5	1.1425094	1.1425188	1.1373069	8.26025×10^{-6}	0.0045535
3	1.0311113	1.0311164	1.0273429	4.94815×10^{-6}	0.0036547
3.5	0.9480675	0.9480707	0.9451631	3.40227×10^{-6}	0.0030635
4	0.8827656	0.8827679	0.8804331	2.51032×10^{-6}	0.0026423
4.5	0.8295645	0.8295661	0.8276354	1.93225×10^{-6}	0.0023254
5	0.7850882	0.7850894	0.7834570	1.53149×10^{-6}	0.0020777

Result in Table 7.3 indicates that BRK4 still produces more accurate results even for the case where a system represented by nonlinear ordinary differential equations.

7.2 Modelling Areas

In this section, we have three subsections as examples in biology 7.2.1,dynamical systems 7.2.2 and caotic circuits7.2.3 represented by the initial value problem. The first example is a real-world example where the ordinary Runge-Kutta method breaks down in certain situations, whereas the Bigeometric Runge-Kutta method gives accurate results. In second and third example, we try to show the application areas of the BRK4 method.

7.2.1 Application to Biological Modelling and its numerical results

Another important applicable area for BRK4 is modeling we can use BRK4 as a modeling tool. As an example, we select to show the applicability of BRK4 in the modeling field. Agarwal and Bhadauria [1] presented a mathematical model of tumor therapy with oncolytic virus. The introduced nonlinear model related with the system of ordinary differential equations. The size of the uninfected and infected tumor cell population modeling in given nonlinear model. By using the RK4 method, Agarwal and Bhadauria performed a stability analysis and compared the size of the uninfected and infected tumor cell population. For calculating the size of the uninfected and infected tumor cell population, $x(t)$ and $y(t)$ respectively, we used the BRK4 and RK4 methods. We perform the comparison only for infected tumor cell population denoted as $y(t)$, as the results for uninfected tumor cell population denoted as $x(t)$ are corresponding.

Oncolytic viruses permeate the tumor cells and replicate. Moreover, infected tumor cells denoted by $y(t)$ shows us infection of uninfected tumor cells with these oncolytic viruses. Anticancer proteins produced by oncolytic viruses as a result of infecting and penetrating in cancer cells. According to assumptions above Agarwal and Bhadauria [1] introduced nonlinear mathematical model as :

$$\frac{dx}{dt} = r_1 x\left(1-\frac{x+y}{K}\right)-\frac{bxy}{x+y+a} \tag{7.2.1}$$

$$\frac{dy}{dt} = r_2 y\left(1-\frac{x+y}{K}\right)+\frac{bxy}{x+y+a}-\alpha y \tag{7.2.2}$$

with initial conditions: $x(0)=x_0>0$ and $y(0)=y_0>0$. We have to denote the parameters appearing in this nonlinear model. The coefficients r_1 represents maximum per capita growth rates of uninfected cells and r_2 represents maximum per capita growth rates of infected cells. On the other hand K shows the carrying capacity, b shows the transmission rate, a shows the measure of the immune response of the individual to the viruses and α shows the rate of infected cell killing by the viruses. The selected parameters of the model should be nonnegative. The presented system transformed to bigeometric form. The bigeometric form can generate as :

$$x^{\pi}(t) = \exp\left[r_1 t\left(1-\frac{x+y}{K}\right)-\frac{tby}{x+y+a}\right] \tag{7.2.3}$$

$$y^{\pi}(t) = \exp\left[r_2 t\left(1-\frac{x+y}{K}\right)+\frac{tbx}{x+y+a}-\alpha t\right] \tag{7.2.4}$$

Comparing RK4 and BRK4 methods numerical approximations to the functions $x(t)$ and $y(t)$ was calculated. The results checked exemplarily for one set of parameters for what step size we get meaningful results. The time t is selected in the interval between 0 and 1000. Since nature of the equations (7.2.1)-(7.2.2) and (7.2.3)-(7.2.4) are strongly nonlinear, we take a small step size h in both cases. Therefore, we provided systematically changes in the step size and the number of points to be calculated.

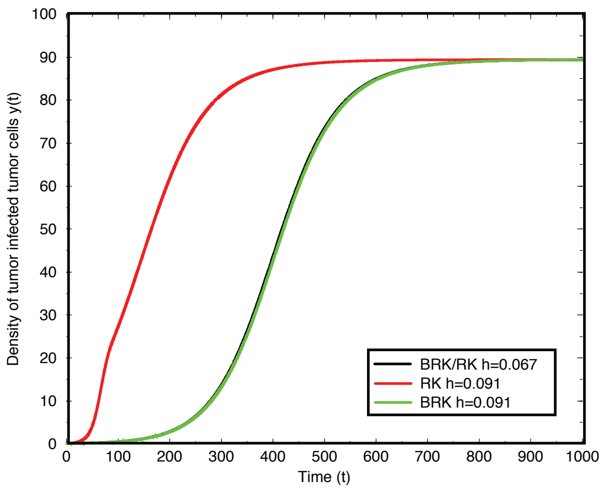


Figure 7.3: Density of infected tumor cells $y(t)$ as a function of time for the parameters $r_1 = 40$, $K = 100$, $r_2 = 2$, $a = 0.05$, $b = 0.02$, and $\alpha = 0.03$. The initial value $y(1) = 0.1$.

In the step-size $h = 0.067$ numerical results of BRK4 and RK4 gives relevant answers as an absolute difference of 10^{-10}. Because of, the exact answer is not available for this problem, we accepted the solution for $h = 0.067$ as the closed form of the exact answer. The step size increased both methods and realized that step size up to $h = 0.091$ the results of the BRK4 method is not significantly different from the ones for $h = 0.067$. On the other hand, RK4 method gives significantly different results as shown graphically in figure 7.3. Absolute eroor for two computation analyses presented in Table 7.4. Obviously the absolute difference for the BRK4 method is less than one, whereas the ordinary RK4 method changes significantly more for the same step size $h = 0.091$, i.e. up to 67.68.

Table 7.4: Density of infected tumor cells $y(t)$ as a function of time for the parameters $r_1 = 40$, $K = 100$, $r_2 = 2$, $a = 0.05$, $b = 0.02$, and $\alpha = 0.03$.The initial value $y(1) = 0.1$.

t	$y(t)$	$y_{BG}(t)$	$\lvert y(t) - y_{BG}(t)\rvert$	$y_{RK}(t)$	$\lvert y(t) - y_{RK}(t)\rvert$
1	0.1	0.1	0	0.1	0
100	0.534879	0.536288	0.00140927	27.2905	26.7556
200	2.84556	2.83341	0.0121523	61.8205	58.9749
300	13.5502	13.2125	0.337775	81.2303	67.6801
400	43.4665	42.5087	0.957788	87.0963	43.6298
500	73.3533	72.8296	0.523669	88.7399	15.3866
600	84.8726	84.7197	0.152969	89.2201	4.34745
700	88.1091	88.0662	0.0428714	89.3637	1.25459

For comparing the performance of these two methods, the absolute errors should be similar. Therefore, for satisfying the completeness we choose step size for RK4 method as $h = 0.0705$ and $h = 0.091$ for the BRK4 method. The results are tabulated in 7.5 and obviously the maximum absolute difference of RK4 method is nearly twice of the absolute difference compared to the BRK4. Moreover, the computation times are measured as 2.328 seconds for the RK4 method and 2.296 seconds for the BRK4 method. Hence, in this complicated mathematical model the introduced method BRK4 indicate a higher performance at a higher accuracy.

Table 7.5: Comparison of the absolute errors of the results from the calculations of the ordinary Runge-Kutta method with $h = 0.0705$ ($n = 14200$ points) and the Bigeometric Runge-Kutta Method for $h = 0.091$ ($n = 11000$ points).

t	$y(t)$	$y_{RK}(t)$	$\lvert y(t) - y_{RK}(t)\rvert$	$y_{BGRK}(t)$	$\lvert y(t) - y_{BGRK}(t)\rvert$
1	0.1	0.1	0	0.1	0
100	0.534876	0.577645	0.0427696	0.536288	0.00141207
200	2.84555	3.09541	0.249859	2.83341	0.0121377
300	13.5502	14.5747	1.02456	13.2125	0.337715
400	43.4664	45.341	1.87463	42.5087	0.957674
500	73.3532	74.3369	0.983688	72.8296	0.523608
600	84.8726	85.1577	0.285053	84.7197	0.152951
700	88.1091	88.1891	0.0800294	88.0662	0.0428665
800	89.0339	89.0574	0.0234774	89.0213	0.0125593
900	89.3078	89.3148	0.00703348	89.304	0.00376082
1000	89.3901	89.3923	0.00212228	89.389	0.00113462

Finally, we showed that the BRK4 method can be used to demonstrate approximate results for this model for a certain set of parameters. And we conclude that the BRK4 method produces better results than the RK4 method for larger step sizes. This claim that there is a strong suspicion for certain problems the Bigeometric Calculus can be a good base for the modelling and the numerical approximations of certain problems in science and engineering.

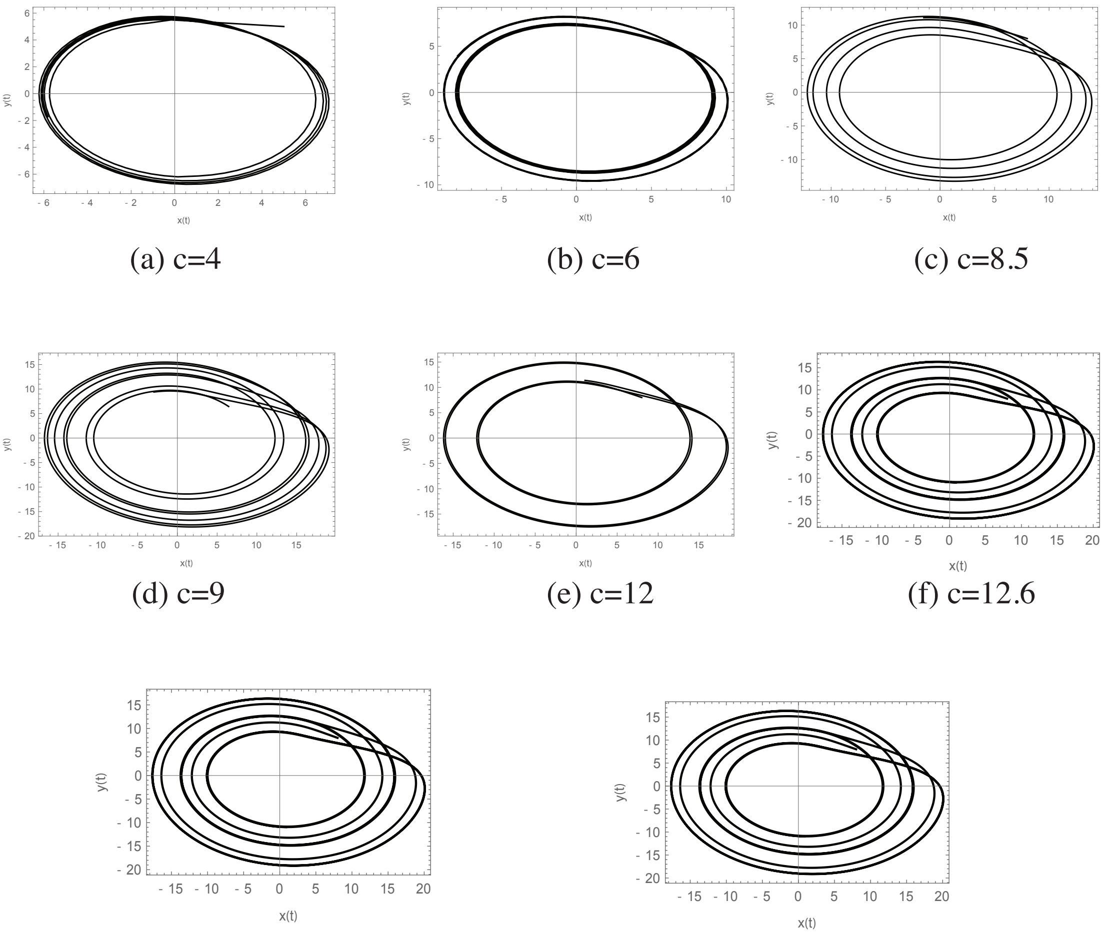
7.2.2 Bigeometric Rössler System

As a further application, to show the applicability of the Bigeometric Runge-Kutta method, we select an important dynamical system Rössler attractor. First of all we transformed the standard Rössler attractor to Bigeometric form and analyze the basic properties of the new dynamic system based on Rössler attractor. The analysis includes the orbits, the time series, and the solution of the coupled differential equations of the Bigeometric Rössler attractor carried out using the Bigeometric Runge-Kutta method.

The Bigeometric Rössler attractor stated as :

$$\begin{aligned} \frac{dx}{dt} &= \exp\left\{t\,\frac{-y-z}{x}\right\} \\ \frac{dy}{dt} &= \exp\left\{t\,\frac{x+ay}{y}\right\} \\ \frac{dz}{dt} &= \exp\left\{t\,\frac{b+z(x-c)}{z}\right\} \end{aligned} \tag{7.2.5}$$

The orbits in figure 7.4 calculated in BRK4 with fixed a and b parameters where parameters c was change for each calculation.



(a) c=4 (b) c=6 (c) c=8.5

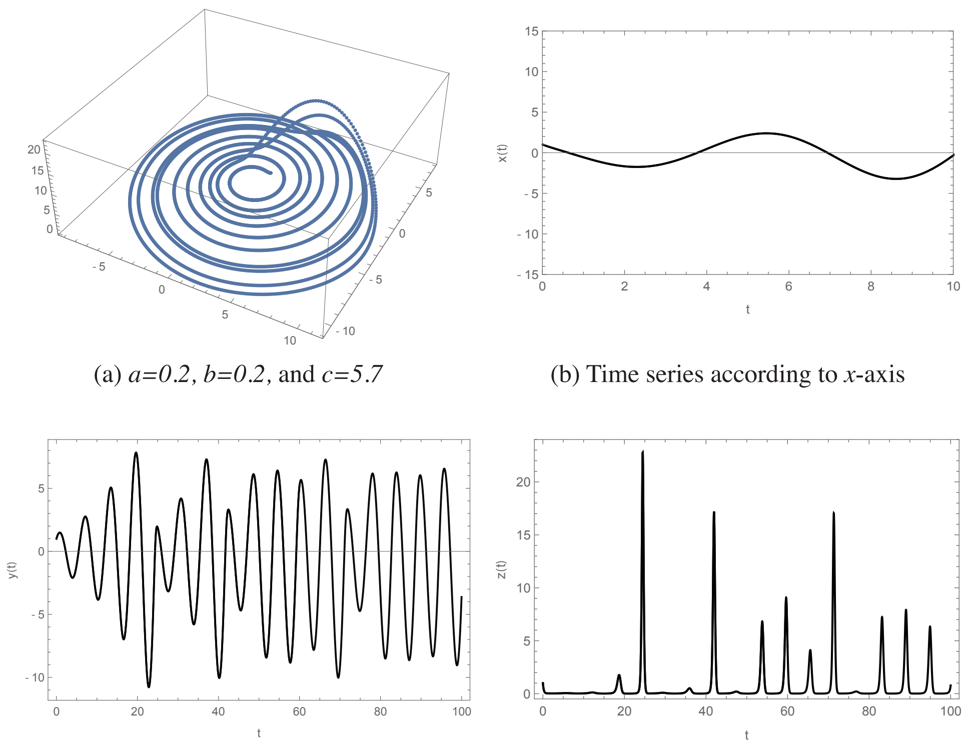
(d) c=9 (e) c=12 (f) c=12.6

(g) c=13 (h) c=18

Figure 7.4: Orbits of the Bigeometric Rössler Attractor for $a = 0.1, b = 0.1$ and varying c values.

The tests claim that the orbits of the Bigeometric Rössler Attractor exactly matches with orbits of the Rössler attractor in its original form as shown in the figure 7.4.

The time analysis of Bigeometric form of Rössler attractor is shown in figure 7.5.



(a) *a=0.2, b=0.2,* and *c=5.7*

(b) Time series according to *x*-axis

(c) Time series according to *y*-axis

(d) Time series according to *z*-axis

Figure 7.5: Bigeometric Rössler Attractor

The results showed that the time series of the Bigeometric Rössler Attractor exactly matches with time series of the Rössler attractor in its original form as shown in the figure 7.5.

7.2.3 Caotic Circuits

The application area for BRK4 can be generated by combining the dynamical systems and circuits. Chaotic behaviour in simple and complex systems has represented the importance of getting a significant behaviour for nonlinear systems in engineering applications. Now lets select a complicated system in engineering area for checking our method.The next application of BRK4 is a Gunn Oscillator system presented in [17]. The mathematical model of Gunn Oscillator extended in [17] as :

$$\frac{d^2q}{dt^2} - \beta q + \alpha q^3 = \varepsilon \left[V_s + \mu \frac{dq}{dt} - \lambda \left(\frac{dq}{dt} \right)^3 \right] \tag{7.2.6}$$

For solving equation (7.2.6) we have to subdivide the given system into two parts as :

$$\begin{aligned} p &= \frac{dq}{dt} \\ p' &= \varepsilon\left[V_s + \mu p - \lambda (p)^3\right] + \beta q - \alpha q^3 \end{aligned} \qquad (7.2.7)$$

The equation (7.2.7) is in Newtonian form, so next step is to transfer (7.2.7) into bigeometric form. For transformation, we need to use (2.0.2) and the second order bigeometric derivative. The second order bigeometric derivative definition determined as :

$$q^{\pi\pi} = \exp\left[t\left(\frac{\dot{q}}{q} - \frac{t(\dot{q})^2}{q^2} + \frac{t\ddot{q}}{q}\right)\right] \qquad (7.2.8)$$

With using same ideas in (7.2.7), require that subdiving equation (7.2.8) into two part. Before subdividing system we need to solve (7.2.8) for $\ddot{q}$ and substituting $\dot{q} = \frac{q}{t}\ln q^{\pi}$ into (7.2.6) we get second order bigeometric form of the differential equation (7.2.6) as :

$$q^{\pi\pi} = \exp\left[t^2\beta - \log(q^{\pi})^2 + \frac{t^2\varepsilon v_s}{q} - \frac{\left(t^3\alpha + \varepsilon\lambda \log\left(q^{\pi}\right)^3\right) q^2)}{t}\right] \left(q^{\pi}\right)^{1+t\varepsilon\mu} \qquad (7.2.9)$$

with

$$v_s = \delta\cos(\Omega_t t).$$

In order to solve (7.2.9) using the BRK4 method we employ the same idea as in the ordinary case (7.2.7) :

$$p = q^{\pi} \tag{7.2.10}$$

$$p^{\pi} = \exp\left[t^2\beta - \log(p)^2 + \frac{t^2\varepsilon\nu_s}{q} - \frac{\left(t^3\alpha + \varepsilon\lambda\log(p)^3\right)q^2}{t}\right](p)^{1+t\varepsilon\mu} \tag{7.2.11}$$

The numerical solution of (7.2.11) generated by BRK4 and tested with different values of the constants, amplitude and frequency of the sync signal given in [17]. Phase plane plot of the oscillator produced as in figure 7.6 -7.7 :

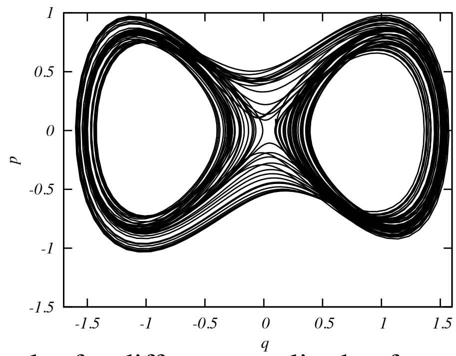


Figure 7.6: Phase plane plot for different amplitude of sync Signal for $\beta = 1$, $\alpha = 1$, $\varepsilon = 0.1$,$\delta = 0.5$,$\mu = 0.8$, $\lambda = 10$

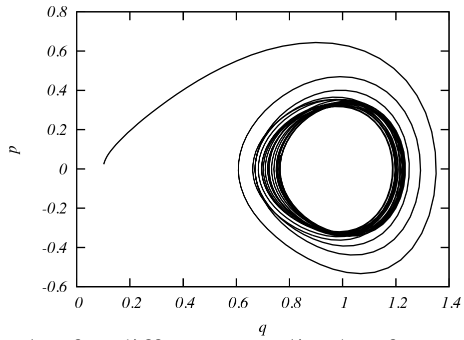


Figure 7.7: Phase plane plot for different amplitude of sync Signal for $\beta = 1$, $\alpha = 1$, $\varepsilon = 0.1$,$\delta = 0.1$,$\mu = 0.8$, $\lambda = 10$

In this application, BRK4 applied to analyze the chaotic motion of oscillator system that is more complicated. The method demonstrated same results as in [17]. The chaotic motion seen in (7.6) and (7.7).

Chapter 8

CONCLUSION

In this thesis, we have declared and derived the differentiation rules for the Bigeometric derivative explicitly. We introduced the Bigeometric Taylor theorem on the basis of the geometric multiplicative Taylor theorem by applying the relation between the geometric and Bigeometric multiplicative derivative. Moreover, we derived the Bigeometric Runge-Kutta by using the Bigeometric Taylor Theorem. The Bigeometric Runge-Kutta Method can applied in numerous examples for finding an estimation for the models that represented a type of ordinary differential equations. We tested Bigeometric Runge-Kutta method in different fields as biology, circuits design, and dynamical systems. We observed that the relative errors of the Bigeometric Runge-Kutta method stayed less that the ones of the ordinary Runge-Kutta method. In the case of the mathematical model of Agarwal and Bhadauria [1] we could observe that the Bigeometric Runge-Kutta method gave better results for larger step sizes h. The basic aim of this thesis was that the Bigeometric Runge-Kutta method is an applicable tool for the solution of initial value problems in different areas.

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