# Partial Approximate Controllability of Semilinear Control Systems

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# ABSTRACT

Most of the controllability concepts are for first ordered differential equations, while not all the control systems are of this kind; but by increasing the dimension of the state space, one can rewrite the control system in the form of first ordered differential equations. Therefore, it seems useful to define partial controllability concepts which maintain the original state space. In this thesis, a sufficient condition for partial approximate controllability of semilinear deterministic control systems is proved with a technique which is completely different from the methods using fixed point theorems. More specifically, the partial $S$-controllability has been weakened for partially observable semilinear stochastic systems and a sufficient condition is provided. The results obtained are demonstrated within examples.

**Keywords:** Controllability, approximate controllability, exact controllability, partial controllability, semilinear systems, stochastic systems.

# ÖZ

Kontrol edilebilirlik kavramlarının çoğu, birinci dereceden diferansiyel denklemlerı içeren kontrol sistemleri için formüle edilmiştir. Doğadaki bütün diferansiyel denklem sistemleri bu tür değildir, ama alanın boyutunu genişleterek bu formda yazılmış olabilir. Bu nedenle, orijinal alanı korumak kısmi kontrol edilebilirlik kavramları tanımlamak yararlı görünüyor. Bu tezde, yarı-lineer deterministik kontrol sistemlerinin kısmi yaklaşık kontrol edilebilirlik için yeterli bir koşul, sabit nokta teoremleri yöntemlerinden tamamen farklı bir teknik ile kanıtlanmıştır. Dahası, kısmen gözlemlenebilir yarı-lineer stokastik sistemleri için zayıflatılmış kısmi *S*-kontrol edilebilirlik incelenmiş ve bu kontrol edilebilirlik kavramı için yeterli bir koşul sağlanmıştır. Elde edilen sonuçlar, örneklerle gösterilmiştir.

**Anahtar Kelimeler:** Kontrol edilebilirlik, yaklaşık kontrol edilebilirlik, tam kontrol edilebilirlik, kısmi kontrol edilebilirlik, yarı-lineer sistemler, stokastik sistemler.

**To My Beloved Family**

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**"Seek knowledge from cradle to grave" Prophet Muhammad(PBUH)**

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# Chapter 1

## INTRODUCTION

In the world of control engineering, there are a lot of systems that need to be controlled. A control engineer is bound to design a controller to interact with these systems. Although, some systems cannot be easily controlled. Therefore, controllability as an important property of a control system, plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, or optimal control. To some researchers, controllability refers to the ability of a controller to modify the functionality of a system.

The concept of controllability was first introduced by the famous work of Kalman [48] for deterministic systems; where controllability was considered as a property of achieving every point in the state space from every initial state point for a finite time.

For linear deterministic systems of finite dimension, the well-known Kalman's rank condition ensures the controllability of the control systems; whereas for infinite dimensional control systems it is not as simple as before. Many infinite dimensional linear deterministic control systems can not be controlled in the way Kalman has mentioned [25, 26, 30, 40, 52, 92], etc. Therefore further studies in the field of controllability leads to a division of this concept into two main parts: exact (complete) controllability and approximate controllability. The exact controllability coincides with the controllability introduced by Kalman whereas the approximate controllability is a weaker concept.

Considering approximate controllability, many control systems of infinite dimensions, which are not exactly controllable, have the chance to be controlled with an arbitrarily small error.

Continuing the study of controllability theory for linear infinite dimensional control systems, Bashirov and Mahmudov have developed the concept of controllability by providing the resolvent conditions [17, 18, 19]. Afterwards having introduced the partial controllability concepts as in [13] and [20] and extending the basic controllability conditions to partial controllability concepts [11, 16], the study in control theory became more interesting and applicable.

Since the theorems mentioned for controllability are for systems of first-ordered differential equations while, most of the dynamical systems such as wave equations, delay equations and higher order differential equations are not in the desired form, but can be expressed in that form by increasing the dimension of the state space, the study for partial controllability was motivated. Therefore, partial controllability concepts are more suitable for them rather than the ordinary controllability concepts which are too strong in those cases. These concepts are discussed in more details in Section 3.3 of this thesis.

Thus, controllability theory for linear deterministic systems with infinite dimensions has been well developed. Moving ahead, controllability concepts for semilinear/nonlinear systems come next. The concepts of controllability for such systems are studied in various books by many researches [6, 14, 49, 50, 51, 58, 59, 60, 80, 81, 82, 83, 84, 85, 86], etc. The method in all the above mentioned researches are based on the fixed point theorems. An alternative method which has made the controllability possible for semilinear stochastic systems as well as all other deterministic and stochastic systems is the method introduced in [21]. The idea is to partition the given time interval $[0,T]$ into two parts $[0,T-\varepsilon]$ and $[T-\varepsilon,T]$. On the first part, an arbitrary control is chosen and the initial state is steered to some state at $T-\varepsilon$; on the second part a sequence of controls is chosen in a way that along the linear part of the system, the state at time $T-\varepsilon$ is steered arbitrarily close to target state at time $T$. Therefore the partial approximate controllability for semilinear systems is obtained considering the fact that, the linear part of the system is disturbed by its nonlinear part for a small value.

In nature, the majority of events occur accidentally or as in scientific way of saying, stochastically. Therefore, controllability of stochastic systems is of more importance. Extending the concepts of controllability from deterministic systems to stochastic systems, various researches have been done namely by Bashirov and Mahmudov. There are two different ways in order to extend the concepts of controllability from deterministic systems to stochastic systems, depending on the state space chosen. A space of random variables, mostly square integrable random variables measurable by the underlying Wiener processes, are chosen as the state space, in the first way. Filtration, as it is known, is an increasing family of $\sigma$-fields, therefore an increasing family of state spaces are obtained. This selection of state space leads to the approximate and exact controllability concepts. In the second way, the space of nonrandom values are selected and therefore, as time increases, the space does not change. Therefore, achieving or being close to constant random variables results the $C$- and $S$-controllability concepts.

Recently, partial controllability concepts have been established [14, 15, 22, 47]. The purpose of these concepts is that the conclusions are gained for first order deterministic/stochastic differential equations driven by different types of noises (white noises, wide band noises, coloured noises and their combinations), while most of the deterministic/stochastic systems are not first ordered but can be expressed in that way by inreasing the dimension of the state space. In these cases, the concepts of partial controllability are useful.

This dissertation is organized as follows: In Chapter 2, some basic preliminary concepts from functional analysis and stochastic calculus are presented. Those which are very essential and useful for the following chapters. Chapter 3 provides the controllability concepts for deterministic systems of both finite and infinite dimensions. Also a review on partial controllability of such systems have been mentioned. A new technique for controllability of semilinear systems have been introduced in Chapter 3. In Chapter 4, the controllability of stochastic systems have been over viewed which constructs the main part of this dissertation. Finally, Chapter 5 includes a brief statement of the achievements of this dissertation.

# Chapter 2

## PRELIMINARIES

In this chapter some basic and essential concepts and definitions from Functional Analysis and also Stochastic Calculus will be provided, those of which will be needed through out this research. The proofs of the theorems, lemmas and corollaries mentioned in this chapter are omitted, since they can be found in most of the books written in these areas such as [56] and [90]. The aim of this chapter is to enhance the reader with a short review of the above mentioned subjects for a better understanding of the forthcoming chapters.

## 2.1 Basic Concepts from Functional Analysis

Functional Analysis is a branch of mathematical analysis which deals with different vector spaces and operators acting on these spaces. Some of the main spaces which will be mentioned in this thesis are defined in the following section.

### 2.1.1 Abstract Spaces

**Definition 2.1.1** *A vector space $V$ is a mathematical structure defined over a scalar field $F$ with two binary operations; scalar multiplication and vector addition. The elements of $V$ must satisfy the below conditions for $\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V$ and $a,b \in F$.*

*(i) Closedness: $\boldsymbol{u}+\boldsymbol{v} \in V$ and $a\boldsymbol{u} \in V$;*

*(ii) Commutativity: $\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$;*

*(iii) Associativity in Addition:* $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$*;*

*(iv) Existence of Additive Identity:* $\exists 0 \in V$*, such that* $0+\boldsymbol{u}=\boldsymbol{u}$*;*

*(v) Existence of Additive Inverse:* $\exists(-\boldsymbol{u}) \in V$*, such that* $(-\boldsymbol{u})+\boldsymbol{u}=0$*;*

*(vi) Distributivity Laws:* $a(\boldsymbol{u}+\boldsymbol{v})=a\boldsymbol{u}+b\boldsymbol{v}$ *and* $(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u}$*;*

*(vii) Associativity in Multiplication:* $a(b\boldsymbol{u})=(ab)\boldsymbol{u}$*;*

*(viii) Property of Multiplication Identity:* $1\boldsymbol{u}=\boldsymbol{u}$*.*

**Definition 2.1.2** *A metric space is a nonempty set X with the distance between the elements given by a function* $d(x,y): X \times X \to \mathbb{R}$ $,x,y \in X$*. This function must satisfy the following axioms:*

*(i)* $\forall x,y \in X,\ \ d(x,y) \geqslant 0$*;*

*(ii)* $d(x,y)=0 \Leftrightarrow x=y$*;*

*(iii)* $\forall x,y \in X, d(x,y)=d(y,x)$*;*

*(iv)* $\forall x,y,z \in X,\ \ d(x,y) \leqslant d(x,z)+d(z,y)$*.*

**Definition 2.1.3** *A normed space is a vector space X with the length of the vectors measured by a function named norm of x, which is a real number and denoted by* $\|x\|$*. The norm satisfies the following axioms:*

*(i)* $\|x\| \geq 0$;

*(ii)* $\|x\| = 0 \Leftrightarrow x = 0$;

*(iii)* $\|ax\| = \mid a \mid \ \|x\|$;

*(iv)* $\forall x, y \in X,\ \|x+y\| \leq \|x\| + \|y\|$.

*For a normed space X, if the distance between the vectors are defined by* $d(x,y) = \|x-y\|$*, then X is a metric space as well.*

**Definition 2.1.4** *A Banach Space is a normed space where every Cauchy sequence is convergent (complete metric space).*

*In other words, a Banach space is a complete normed space.*

**Definition 2.1.5** *An inner product space is a nonempty set X with a relation between the elements, defined by the scalar function:*

$$\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$$

*which holds the properties below and called as the inner product of x and y. Also called the dot product or scalar product.*

*(i)* $\forall x \in X,\ \langle x, x \rangle \geq 0$;

*(ii)* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;

*(iii)* $\forall x, y \in X,\ \langle x, y \rangle = \langle y, x \rangle$;

*(iv)* $\forall x, y, z \in X$, $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

*(v)* $\forall x \in X$ *and* $\forall a \in \mathbb{R}$ , $\langle ax, y\rangle = a\langle x, y\rangle$.

*Every inner product space with a norm defined by* $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ *is a normed space.*

**Definition 2.1.6** *A complete inner product space is called a Hilbert space; i.e. a complete normed space with an inner product defined on its elements.*

*A separable space, is a Hilbert space with a countable dense subset.*

### 2.1.2 Operators

In this section we will have a brief review on the definition and properties of some operators which will be used through out this thesis.

**Definition 2.1.7** *Any mapping from a vector space* $X$ *to a vector space* $Y$ *is called an operator from* $X$ *to* $Y$.

*If the operator maps a vector space to the scalar field* $\mathbb{R}$*, then it's called a functional.*

The most widely used operators in this thesis, are the *linear*, *bounded* and *closed* operators which will be defined next.

**Definition 2.1.8** *For two vector spaces* $X$ *and* $Y$ *over a field* $\mathbb{F}$*, the operator* $\boldsymbol{T} : X \to Y$ *is said to be linear if* $\forall x_1, x_2 \in X$ *and* $\forall a, b \in \mathbb{F}$*:*

$$\boldsymbol{T}(ax_1 + bx_2) = a\boldsymbol{T}(x_1) + b\boldsymbol{T}(x_2).$$

**Definition 2.1.9** *Given two normed vector spaces* $X$ *and* $Y$*, a bounded linear operator* $\boldsymbol{T} : X \to Y$ *is an operator which satisfies the following relation for a real positive number* $n$ *and* $\forall x \in X$

$$\|\boldsymbol{T}x\|_Y \leq n\|x\|_X.$$

*The smallest value for n, which satisfies the above inequality is called as the operator norm of* $\boldsymbol{T}$ *and denoted by* $\|\boldsymbol{T}\|$.

*Mathematically:*

$$\|\boldsymbol{T}\| = sup_{\|x\|=1} \|\boldsymbol{T}x\|_Y.$$

*The collection of all linear bounded operators* $\boldsymbol{T} : X \to Y$ *are denoted by* $\mathscr{L}(X,Y)$*; which defines a Banach space considering the operator norm defined above.*

**Definition 2.1.10** *For a normed space X, the function* $g : [a,b] \to X$ *is said to be continuous at the point* $x_0 \in [a,b]$ *if as* $x \to x_0$*:*

$$\|g(x) - g(x_0)\|_X \longrightarrow 0.$$

*If a function is continuous at all the points of its domain, then it's called a continuous function.*

**Proposition 2.1.11** *Consider the linear operator* $\boldsymbol{T} : D(\boldsymbol{T}) \subset X \to Y$ *for two Banach spaces X and Y. Then the following statements hold:*

*i)* $\boldsymbol{T}$ *is bounded if and only if it is continuous, i.e.* $lim_{x \to x_0} \|\boldsymbol{T}x - \boldsymbol{T}x_0\| = 0$*;*

*ii) Continuity at one point implies continuity on all points of* $D(\boldsymbol{T})$.

Another important class of operators are *closed* operators which are defined on Banach spaces.

**Definition 2.1.12** *Let X and Y be two Banach spaces. Consider the following condition for the linear operator* $\boldsymbol{T} : D(\boldsymbol{T}) \subset X \longrightarrow Y$*, where the domain* $D(\boldsymbol{T})$ *is dense in X. If* $x_n \to x$ *and* $\boldsymbol{T}(x_n) \to y$*, then* $x \in D(\boldsymbol{T})$ *and* $\boldsymbol{T}x = y$*. If this condition holds, then* $\boldsymbol{T}$ *is* *called a closed linear operator.*

*Given a Banach space $X$, the collection of all linear bounded functionals on $X$ is denoted by $X^*$ and called the adjoint space of $X$. $X^*$ is again a Banach space. If $X$ is a Hilbert space then $X^* = X$.*

**Definition 2.1.13** *Consider the operator $\boldsymbol{M} \in \mathscr{L}(X,Y)$, where $X$ and $Y$ are two Banach spaces. Then there exits a unique operator $\boldsymbol{M}^* \in \mathscr{L}(Y^*,X^*)$ satisfying the following equation $\forall x \in X$ and $y^* \in Y^*$:*

$$(\boldsymbol{M}^* y^*)x = y^*(\boldsymbol{M}x).$$

*The operator $\boldsymbol{M}^*$ mentioned above is called the adjoint of operator $\boldsymbol{M}$.*

*Assume that in the definition above the Banach spaces $X$ and $Y$ are replaced by Hilbert spaces together with the inner product norm defined on them. In this case, the adjoint of operator $\boldsymbol{M} : X \longrightarrow Y$ is $\boldsymbol{M}^*$ such that:*

$$\forall x \in X,\ \forall y \in Y,\ \ \langle \boldsymbol{M}x, y\rangle = \langle x, \boldsymbol{M}^* y\rangle.$$

*The proof of existence and uniqueness of the above mentioned operator $\boldsymbol{M}^*$ is based on the Riesz representation theorem, which can be found in most of the books related to functional analysis.*

**Definition 2.1.14** *A given bounded operator $\boldsymbol{M}$ defined on a Hilbert space $X$, is said to be self-adjoint if:*

$$\boldsymbol{M} = \boldsymbol{M}^*$$

*or equivalently*

$$\forall x, y \in X, \quad \langle \boldsymbol{M}x, y \rangle = \langle x, \boldsymbol{M}y \rangle$$

*For a self-adjoint operator, the following classifications are available:*
*The operator* $\boldsymbol{M} \in \mathscr{L}(X)$ *is called:*

*(i)Nonnegative if* $\forall x \in X, \quad \langle \boldsymbol{M}x, x \rangle \geqslant 0.$

*(ii) Positive if* $\forall 0 \neq x \in X, \quad \langle \boldsymbol{M}x, x \rangle > 0.$

*(iii) Coercive if* $\exists \lambda > 0$ *such that* $\forall x \in X, \ \langle \boldsymbol{M}x, x \rangle \geqslant \lambda \|x\|^2.$

Next, the definition of a *projection* operator will be provided. For this reason first we need to review the concept of orthogonality in Hilbert spaces.

**Definition 2.1.15** *If* $\langle x, y \rangle = 0$ *for any given vectors* $x, y \in X$*, where* $X$ *is a Hilbert space, then x and y are called orthogonal.*
*Similarly, for a subspace* $N \subset X$*, the orthogonal complement of* $N$ *in* $X$ *is defined as the set below:*

$$N^{\perp} = \{x \in X \ \mid \ \langle x, n \rangle = 0, \ \ n \in N\}.$$

The following theorem gives us some useful relations in this respect.

**Theorem 2.1.16** ***Orthogonal Decomposition:*** *Consider the Hilbert space* $X$ *and its linear subspace* $N$*. The following relations hold:*

*(i)* $\ X = N^{\perp} \oplus N^{\perp\perp}$*;*

*(ii)* $\ X = N^{\perp} \oplus \overline{N}$*;*

*(iii) As a result of the previous parts* $\quad X = \overline{N} \Leftrightarrow N^{\perp} = \{0\}$.

**Definition 2.1.17** *According to the above definition and theorem, any vector x from the Hilbert space X, can be written in the form* $x = n + m$ *uniquely where* $n \in \overline{N}$ , $m \in N^{\perp}$. *The operator P which assigns a vector* $n \in N$ *to the vector* $x \in X$ *in the above relation, is said to be the projection operator from X onto N.*

*Mathematically,* $P : X \longrightarrow N$ *is a projection operator if and only if*

$$\forall x \in X \ , \ \forall n \in N \ : \ \langle x - Px, n \rangle = 0.$$

*It is clear that* $P \in \mathscr{L}(X, N)$, *it can also be checked that* $P = PP = P^2$ *and that P has a unit norm i.e.* $\|P\| = 1$.

Some of the properties of convergence in $\mathbb{R}^n$ can not be applied to Banach and Hilbert spaces. Therefore the concepts of *uniform*, *weak* and *strong* convergence of operators are defined below.

**Definition 2.1.18** *Consider the two Banach spaces X and Y. The sequence* $\{\boldsymbol{M}_n\} \in \mathscr{L}(X, Y)$ *is said to be convergent to* $\boldsymbol{M} \in \mathscr{L}(X, Y)$:

*(i) in uniform sense, if* $\|\boldsymbol{M}_n - \boldsymbol{M}\| \overset{n \to \infty}{\longrightarrow} 0$.

*(ii) in strong sense, if* $\forall x \in X, \ \|\boldsymbol{M}_n x - \boldsymbol{M} x\| \overset{n \to \infty}{\longrightarrow} 0$.

*(iii) in weak sense, if* $\forall x \in X, y^* \in Y^*, \ \langle (\boldsymbol{M}_n - \boldsymbol{M})x, y^* \rangle \overset{n \to \infty}{\longrightarrow} 0$.

*It's clear that* $i \rightarrow ii \rightarrow iii$ *but converse may not hold in general.*

*For the Euclidean spaces* $\mathbb{R}^n$ *and* $\mathbb{R}^m$, *the above statements are equivalent.*

In the sequel of this section, we ought to define another useful operator named the *resolvent* operator. For this reason, we need to define the concept of *semigroups* first. As we know, semigroups play an important role in solving a wide range of evolution equations. We will also go through a special class of semigroups named the $C_0$-*semigroups*. For further information the reference [79] would be helpful.

**Definition 2.1.19** *A set $S$ equipped with an associative binary operation $*$ $(i.e.\ * : S \times S \to S)$ constructs a semigroup, which doesn't necessarily need to have an identity nor an inverse element.*

*For bounded linear operators defined on a Banach space $X$, we say that the operator $\boldsymbol{M}$ has the semigroup property if it satisfies the following condition:*

$$\boldsymbol{M}(s+t) = \boldsymbol{M}(s)\boldsymbol{M}(t) \quad \forall s,t \in [0,\infty). \tag{2.1.1}$$

*Mostly the terms $s,t$ indicate the time. Therefore, $\boldsymbol{M}(0) = I$, since we have no transition at time zero.*

*A family of bounded linear operators satisfying equation (2.1.1), is called the semigroup of the indicated bounded linear operators.*

*From now on, whenever our variables are chosen from the time interval $\mathbb{R}^+ = [0,\infty)$, we will use the notation $\boldsymbol{M}_t$ instead of $\boldsymbol{M}(t)$.*

**Definition 2.1.20** *A strongly continuous semigroup of the operators $\boldsymbol{M} \in \mathscr{L}(X)$, is named as the $C_0$-semigroup of $\boldsymbol{M} \in \mathscr{L}(X)$.*

*Mathematically, a family $\boldsymbol{M} = \{\boldsymbol{M}_t \mid t \in [0,\infty)\}$ of $\boldsymbol{M} \in \mathscr{L}(X)$, which holds the following statements is a $C_0$-semigroup:*

*(i)* $\boldsymbol{M}_0 = I$;

*(ii)* $\boldsymbol{M}_{s+t} = \boldsymbol{M}_s \boldsymbol{M}_t \quad \forall s,t \in [0,\infty)$ *(Semigroup Property);*

*(iii)* $\forall x \in X,\ \ lim_{t\to 0^+} \boldsymbol{M}_t x \to x$ *(Strong continuity w.r.t the corresponding norm). The third condition can be replaced by* $lim_{t\to 0^+} \|\boldsymbol{M}_t x - x\| = 0$*; while replacing it with* $lim_{t\to 0^+} \|\boldsymbol{M}_t - I\| = 0$ *provides us another type of semigroups called the uniformly continuous semigroups.*

**Definition 2.1.21** *The infinitesimal generator of a semigroup* $\boldsymbol{M}$ *on a Banach space* $X$*, is a linear operator* $A$ *satisfying the following equation for* $x \in D(A)$*:*

$$Ax = lim_{t\to 0^+} A_t x = lim_{t\to 0^+} \frac{\boldsymbol{M}_t x - x}{t} = \frac{d}{dt} \boldsymbol{M}_t x \bigg|_{t=0}$$

*where* $D(A)$ *is the set of all* $x \in X$ *such that the above limit exists.*

**Theorem 2.1.22** *Assume that* $A$ *is a bounded linear operator on* $X$*. Then*

$$\boldsymbol{M} = \left\{ \boldsymbol{M}_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \ \middle| \ t \in [0,\infty) \right\}$$

*constructs a uniformly continuous semigroup.*

**Theorem 2.1.23** *Consider* $A$ *as the generator of the semigroup* $\boldsymbol{M}$ *defined on the Banach space* $X$*. Then* $\forall x \in X$*:*

*i)*

$$lim_{\gamma\to 0^+} \frac{1}{\gamma} \int_t^{t+\gamma} \boldsymbol{M}_s x ds = \boldsymbol{M}_s x.$$

*ii)*

$$\int_0^t \boldsymbol{M}_s x ds \in D(A) \quad and \quad A\left(\int_0^t \boldsymbol{M}_s x ds\right) = \boldsymbol{M}_t x - x.$$

*iii)* $\forall x \in D(A)$,

$$\boldsymbol{M}_t x - \boldsymbol{M}_s x = \int_s^t \boldsymbol{M}_r A x dr = \int_s^t A \boldsymbol{M}_r x dr.$$

**Theorem 2.1.24** *For a semigroup* $\boldsymbol{M}$, $\exists \lambda \in \mathbb{R}$ *and* $K \geqslant 1$ *s.t.:*

$$\|\boldsymbol{M}_t\| \leqslant K e^{\lambda t} \quad for \;\; t \in \mathbb{R}^+.$$

**Proposition 2.1.25** *Consider the* $C_0$*-semigroup* $\boldsymbol{M}_t$ *generated by the closed operator* $A \in \mathscr{L}(X)$, *where* $X$ *is a Banach space. Then,* $\boldsymbol{M}_t^*$ *is a semigroup on* $X^*$.

*If the Banach space is replaced with a Hilbert space, then the semigroup* $\boldsymbol{M}_t^*$ *on* $X$ *becomes a* $C_0$*-semigroup with the generator operator* $A^*$, *i.e.* $\boldsymbol{M}_t^* = e^{A^* t}$.

**Definition 2.1.26** *Consider the linear operator* $T \in \mathscr{L}(X)$, *where* $X$ *is a Banach space. The set of all complex numbers* $\alpha \in \mathbb{C}$, *for which the bounded operator* $\boldsymbol{R} = (T - \alpha I)^{-1}$ *exists, is called the Resolvent set of* $T$ *and is denoted by* $\rho(T)$.

*The operator* $\boldsymbol{R}(\alpha, T) = (T - \alpha I)^{-1}$, *is named as the resolvent operator of* $T$.

**Theorem 2.1.27** ***[Hille-Yosida Theorem]*** *An unbounded linear operator* $T$ *generates a* $C_0$*-semigroup if and only if:*

*i) It is closed,*

*ii) Has a dense domain* $D(T)$,

*iii)* $\alpha \in \rho(T), \ \forall \alpha > 0$,

*iv)* $\|\boldsymbol{R}(\alpha, T)\| \leqslant \frac{1}{\alpha}$.

The last part of this section is devoted to *evolution* equations; therefore the definitions and results related to them are provided below.

For the $C_0$-semigroup $\boldsymbol{M}_t = e^{At}$ on the Banach space $X$, with the infinitesimal generator $A$, consider the following linear system where $f \in L_1(0, T; X)$

$$\begin{cases} \frac{dx}{dt} = Ax_t + f(t), \ 0 < t \leq T, \\ x(0) = x_0 \in X. \end{cases} \tag{2.1.2}$$

**Definition 2.1.28** *The continuous function $x \in C(0, T; X)$ is considered as a:*

*(a) strong solution of the above linear system under the following circumstances:*

*i) for almost $\forall s \in [0, T]$, $x_s \in D(A)$;*

*ii) $x$ is strongly differentiable a.e. on $[0, T]$;*

*iii) the equation (2.1.2) holds for $x$ a.e considering $x(0) = x_0$.*

*(b) weak solution for the system (2.1.2), if $\langle x(\cdot), y^* \rangle$ is an absolutely continuous function for $\forall y^* \in D(A^*)$ on $[0, T]$ and:*

$$\langle x_s, y^* \rangle = \langle x_0, y^* \rangle + \int_0^s \Big( \langle x(r), A^* y^* \rangle + \langle f(r), y^* \rangle \Big) dr, \quad \forall s \in [0, T].$$

*(c) mild solution, when the statement below holds for $\forall s \in [0,T]$:*

$$x_s = e^{As}x_0 + \int_0^s e^{A(s-r)} f(r)dr.$$

The following proposition gives a useful relation in this respect.

**Proposition 2.1.29** *Consider the $C_0$-semigroup $\boldsymbol{M}_t$ with the generator $A$ on the Banach space $X$. Then the corresponding linear system* (2.1.2) *has a weak solution iff it has a mild solution.*

Existence and uniqueness of a mild solution for a *semilinear* system is mentioned in the following theorems:

**Theorem 2.1.30** *[61] Consider the semilinear system below where $f : [0,T] \times X \to X$:*

$$\begin{cases} \frac{dx}{dt} = Ax_t + f(t,x_t),\ 0 < t \leq T, \\ x(0) = x_0 \in X. \end{cases} \tag{2.1.3}$$

*$x \in C(0,T;X)$ is a unique mild solution for* (2.1.3) *if and only if the statements below are satisfied for $\forall x,y \in X$ and $t \in [0,T]$:*

*i) $\forall x \in X$, the function $f(\cdot,x)$ is strongly measurable;*

*ii) There exists an integrable function $M$ from $L_1(0,T;\mathbb{R})$ such that:*

$$\begin{cases} \|f(r,x) - f(r,y)\| \leq M(r)\|x-y\|, \\ \|f(r,0)\| \leq M(r). \end{cases}$$

**Theorem 2.1.31** *[61] For a given Banach space $X$, if the function $f$ mentioned in* (2.1.3) *is continuous at the time $r$, and satisfies the Lipschitz condition with respect to* *the second variable as below:*

$$\exists c \geqslant 0, \ \ \|f(r,x) - f(r,y)\| \leq c\|x-y\|,$$

*then the system* (2.1.3) *has a unique mild solution.*

The concept of semigroups of bounded linear operators can be generalized to a two-parameter case; which in this case will be named as *evolution operators*. A brief definition of *mild evolution operators* and a useful result for them is provided below. More and precise information can be found in Curtain and Pritchard [29].

Below we will use the following notation:

$$\Delta_T = \{(r,t) | 0 \leqslant r \leqslant T\}.$$

**Definition 2.1.32** *The function $\mathscr{V} : \Delta_T \to \mathscr{L}(X)$ where X is a Hilbert space, is said to be a mild evolution operator under the circumstances below:*

*i)* $\mathscr{V}_{t,t} = I, \ 0 \leqslant t \leqslant T$

*ii)* $\mathscr{V}_{t,r} = \mathscr{V}_{t,s}\mathscr{V}_{s,r}, \ 0 \leqslant r \leqslant s \leqslant t \leqslant T$ *(semigroup property)*

*iii)* $[\mathscr{V}_t] : [0,t] \to \mathscr{L}(X)$ *and* $[\mathscr{V}_r] : [r,T] \to \mathscr{L}(X)$ *are both weakly continuous for* $\forall t \in (0,T]$ *and* $\forall r \in [0,T]$

*iv)* $sup_{\Delta_T} \|\mathscr{V}_{t,r}\| < \infty$.

$\mathscr{E}(\Delta_T, \mathscr{L}(X))$ *denotes the class of all the mild evolution operators which operate from*

$\Delta_T$ *to* $\mathscr{L}(X)$.

If a strongly continuous semigroup $\mathscr{V}$ is written in a two-parameter form $\mathscr{V}_{t,r} = \mathscr{V}_{t-r}$, $0 \leqslant r \leqslant t \leqslant T$, then it can be considered as a mild evolution operator. Therefore, we can conclude that $\mathscr{C}_0(X) \subset \mathscr{E}(\Delta_T, \mathscr{L}(X))$.

**Proposition 2.1.33** *For the mild evolution operator* $\mathscr{V} : \Delta_T \to \mathscr{L}(X)$ *and the function* $f \in L_1(0,T;X)$ *:*

$$\phi_t = \int_0^t \mathscr{V}_{t,r} f_r dr, \;\; t \in [0,T],$$

*is weakly continuous.*

## 2.2 Basic Concepts from Stochastic Calculus

In this section we will go through some definitions and results from elementary stochastic calculus which will be needed in this thesis.

**Definition 2.2.1** *A triple* $(\Omega, \mathscr{F}, \boldsymbol{P})$ *is called a probability space where* $\Omega$ *is the sample space,* $\mathscr{F}$ *is a* $\sigma$*-algebra defined on the events of the sample space and* $\boldsymbol{P}$ *denotes the probability measure.*

For a better understanding of the above definition, the concepts of a $\sigma$-algebra and a probability measure are stated below:

**Definition 2.2.2** *Consider the set X and its power set* $2^X$*. The subset* $\mathscr{F} \subset 2^X$ *is said to be a* $\sigma$*-algebra (*$\sigma$*-field) under the following circumstances:*

*(i)* $\mathscr{F} \neq \varnothing$*;*

*(ii)* $\mathscr{F}$ *is closed under complementation; i.e. if* $A \in \mathscr{F}$ *then so is* $A^c$ *;*

*(iii) For a countable number of elements of $\mathscr{F}$ such as $A_1, A_2, \ldots$ their union is also in $\mathscr{F}$.*

*As a result of the third property, by using De Morgan's rule, a $\sigma$-algebra is closed under countable intersections as well. The pair $(X, \mathscr{F})$, is called a measurable space.*

**Definition 2.2.3** *A probability measure $\boldsymbol{P}$ over a measurable space $(\Omega, \mathscr{F})$ is a function $\boldsymbol{P} : \mathscr{F} \longrightarrow [0,1]$ which assigns a probability $\boldsymbol{P}(A)$ to every element of the sample space $\Omega$ and satisfies the following properties:*

*(i) $\boldsymbol{P}(\Omega) = 1$;*

*(ii) If $A_1, A_2, \ldots$ are mutually disjoint, then*

$$\boldsymbol{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \boldsymbol{P}(A_i).$$

**Definition 2.2.4** *Consider a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. The random variable X, is a measurable function $X : \Omega \to \mathbb{R}$; that is, foe every Borel set $A \subset \mathbb{R}$, $X^{-1}(A) \in \mathscr{F}$.*

**Definition 2.2.5** *For a given probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$, the mean value of the random variable X, also called the expected value of x, is defined by the integral below:*

$$E(X) = \int_{\Omega} X d\boldsymbol{P}.$$

*Using the above notation for mean value, the variance of a random variable can be defined as follows:*

$$Var(X) = E(X^2) - E(X)^2.$$

There are two widely used spaces of random variables over the $\sigma$-field $\mathscr{F}$; the space of *integrable r.v.* and the space of *square integrable r.v.* defined as below:

(a) The space of *integrable r.v.*:

$$L_1(\mathscr{F}) = \{X : \Omega \to \mathbb{R} | \sigma(X) \subseteq \mathscr{F},\ \mathbf{E} \mid X \mid < \infty\}.$$

(b) The space of *square integrable r.v.*:

$$L_2(\mathscr{F}) = \{X : \Omega \to \mathbb{R} | \sigma(X) \subseteq \mathscr{F},\ \mathbf{E}X^2 < \infty\}.$$

**Definition 2.2.6** *Let $\mathscr{F}_1$ be a sub-$\sigma$-field of $\mathscr{F}$. Conditional expectation of a random variable X with respect to the $\sigma$-algebra $\mathscr{F}_1$, is a random variable denoted by $E(X|\mathscr{F}_1)$ and satisfies:*

$$\int_A X d\boldsymbol{P} = \int_A E(X|\mathscr{F}_1) d\boldsymbol{P} \quad , \quad \forall A \in \mathscr{F}_1.$$

*Expectation is a particular case of conditional expectation when $\mathscr{F} = \{\emptyset, \Omega\}$;*

$$E(X) = E(X|\{\emptyset, \Omega\}).$$

**Definition 2.2.7** *A family of $\sigma$-fields $\{\mathscr{F}_\alpha\}$ is said to be independent, if the equality below holds:*

$$\forall \alpha_1, \cdots, \alpha_n \quad \forall A_{\alpha_i} \in \mathscr{F}_{\alpha_i},\ \boldsymbol{P}\big(\bigcap_{i=1}^{n} A_{\alpha_i}\big) = \prod_{i=1}^{n} \boldsymbol{P}(A_{\alpha_i}).$$

*The random variables are said to be independent, if the corresponding $\sigma$-fields are independent.*

Some useful properties of conditional expectations are listed below:

1) $\forall X, Y \in L_1(\Omega, \mathscr{F}, \mathbf{P})$, $E(aX + bY|\mathscr{F}) = aE(X|\mathscr{F}) + bE(Y|\mathscr{F})$ .

2) For a sub $\sigma$-algebra $\mathscr{F}'$ of $\mathscr{F}$

$$E(X|\mathscr{F}') = E(E(X|\mathscr{F})|\mathscr{F}') = E(E(X|\mathscr{F}')|\mathscr{F}).$$

3) If $X$ is $\mathscr{F}$-measurable, then: $E(X|\mathscr{F}) = X$ .

4) If $X$ is independent of $\mathscr{F}$, then: $E(X|\mathscr{F}) = E(X)$ .

5) If $X$ and $Y$ are two independent random variables, then:

$$E(XY) = E(X)E(Y) \quad \text{and} \quad cov(X,Y) = 0.$$

**Definition 2.2.8** *For a random variable $X$, it is said to be Gaussian and denoted by $X \sim \mathscr{N}(m, \sigma^2)$ if its density function is written as below for the mean value $m \in \mathbb{R}$ and the standard deviation $\sigma > 0$:*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad , \quad x \in \mathbb{R}.$$

*In the case when $m = 0$ and $\sigma = 1$, the random variable $X$ is called a standard Gaussian random variable and denoted by $X \sim \mathscr{N}(0,1)$. For a constant random variable, $X = m$ and $\sigma = 0$. In this case it is named as a degenerate Gaussian random variable and written as $X \sim \mathscr{N}(m,0)$.*

As far as a Gaussian random variable is defined, we can provide the definition of Gaussian systems which are of main importance in stochastic calculus.

**Definition 2.2.9** *A collection of random variables is called a Gaussian system if every linear combination of these random variables is a Gaussian random variable.*

*Note that all random variables chosen from a Gaussian system, are Gaussian themselves.*

**Definition 2.2.10** *A family of all random variables with respect to an argument is called a random process or a stochastic process.*

*Mostly the argument is chosen to be time. Throughout this chapter we will consider the time interval $T = [0, \infty)$.*

*As we know, random variables generate $\sigma$-fields; additionally, random processes generate filtrations. A process $X$ generates a filtration as below which is also called the natural filtration:*

$$\mathscr{F}_t^X = \sigma\big(X(r); 0 \leqslant r \leqslant t\big).$$

*Note that the natural filtration is the smallest filtration generated by $X$.*

*If the random process $X(r)$ results a Gaussian system, then it is said to be a Gaussian process. A random process $X$ satisfying $\boldsymbol{E}X(r)^2 < \infty$ , $\forall r \in T$ is called a second order process.*

For some random processes, the randomness does not change in time, i.e. the observations of the process in the time interval $(m,n)$ and $(m+h, n+h)$ are the same. Therefore, the distributions do not depend on the time when the process is being observed, but only depends on the time difference. For this reason we have to introduce another type of stochastic processes named as *stationary* processes.

**Definition 2.2.11** *A stochastic process is said to be stationary if the distributions of $X(m) - X(n)$ and $X(m+h) - X(n+h)$ are equal for all $m,n,h \in T$. In other words,* *for time intervals with the same width (i.e.* $[m,n]$ *and* $[m+h,n+h]$*), the increments of the random process are equally distributed.*

Another kind of stationary processes is *stationary in wide sense* as defined below.

**Definition 2.2.12** *A random process $X$ is called stationary in wide sense under the circumstances below:*

*i)* $\boldsymbol{E}\big(X(r+t)-X(t)\big)=0$*;*

*ii)* $\boldsymbol{E}\big(X(r+t)-X(t)\big)^2=\boldsymbol{E}\big(X(r)-X(0)\big)^2$

*where* $r, r+t \in T$.

**Definition 2.2.13** *Consider a filtration* $\{\mathscr{F}_t\}$*. An* $\mathscr{F}_t$*-measurable random process, is called an* $\mathscr{F}_t$*-adapted random process. In other words,* $X(t)$ *is* $\mathscr{F}_t$*-adapted if* $\sigma(X(t)) \subseteq \mathscr{F}_t$ *for all* $t \geqslant 0$.

Two of the most important random processes in stochastic calculus, are the *martingales* and *Wiener* processes, which will be defined next.

**Definition 2.2.14** *A random process $M(t)$ satisfying the following conditions is said to be a martingale if:*

*i)* $\boldsymbol{E}|M(t)|<\infty$*;*

*ii)* $\boldsymbol{E}(M(t)|\mathscr{F}_r)=M(r), \quad r<t$*;*

*iii)* $M$ *is* $\mathscr{F}_t$*-adapted* *where $r,t \in T$ and $\mathscr{F}_t$ is the corresponding filtration.*

*For a martingale with respect to the filtration $\mathscr{F}_t$, normally the notation $(M(t),\mathscr{F}_t)$ is used.*

**Definition 2.2.15** *Consider a certain probability space $(\Omega,\mathscr{F},\boldsymbol{P})$. For a random process $X$, if the sample parameter is fixed as $s = s_0$, then the function $X(\cdot,s_0)$ is said to be the path of the process $X$.*

**Definition 2.2.16** *A Wiener process or a standard process of Brownian motion is the random process $W : T\times\Omega \to \mathbb{R}$ which holds the properties below:*

*i)* $\boldsymbol{E}\big(W(r)-W(t)\big) = 0$;

*ii)* $\boldsymbol{E}\big(W(r)-W(t)\big)^2 = |r-t|$;

*iii)* $W(r_1)-W(r_0),\cdots,W(r_n)-W(r_{n-1})$ *are independent for* $0 \leqslant r_0 < r_1 < \cdots < r_n$;

*iv)* $W(r)-W(t)$ *is a Gaussian random variable;*

*v) The random process $W$ has continuous paths;*

*vi)* $W(0) = 0$.

**Theorem 2.2.17** ***Wiener**. For a given probability space, there exists an infinite number of independent Wiener processes.*

Some of the main properties of a Wiener process $W$ are listed below:

1) $W$ has independent and stationary increments.

2) $W$ is a Gaussian and also a second order process.

3) $\mathscr{F}_t^W$ and $W(r) - W(t)$ are independent where $r \geqslant t$.

4) $\left(W(r), \mathscr{F}_r^W\right)$ is a martingale.

5) $Cov(W(r), W(t)) = min(r,t)$.

6) The paths of $W$ are continuous but nowhere differentiable with infinite length over a bounded interval (w.p.1).

**Theorem 2.2.18** ***Levi***. *For $r \in T$, the random process $W(r)$ is a Wiener process under the circumstances below:*

*i) $W(0) = 0$;*

*ii) $\left(W(r), \mathscr{F}_r^W\right)$ is a martingale;*

*iii) $W$ has a continuous path w.p.1;*

*iv) $\boldsymbol{E}W(r) = 0$ and $\boldsymbol{E}W(r)^2 = r$.*

In the following part of this section, we ought to define the well-known *Ito integral*. For this we need to have a review on *Stieltjes* and also *Stochastic Stieltjes* integrals.

For more information see [9].

For given two functions $f, g : [0,t] \to \mathbb{R}$ where $f \in C[0,t]$ and $g \in BV[0,t]$ (a function of bounded variation), the integral $S = \int_0^t f(r)dg(r)$ exists and is named as the *Stieltjes* integral of $f$ with respect to $g$ on $[0,t]$.

The collection of all Stieltjes integrable functions on the interval $[0,t]$ with respect to $g$ is denoted by $S_g[0,t]$.

A special case of the Stieltjes integral is when $g(r) = r$. In this case we have the well-known *Riemann* integral $R = \int_0^t f(r)dr$.

**Definition 2.2.19** *Suppose the Stieltjes integral $\int_0^t X(r,s)dY(r,s)$ for the random process $X$ and $Y$, exists w.p.1. Then the integral on the left hand side in the equation below which is a random variable, is called a stochastic Stieltjes integral.*

$$\left[\int_0^t X(r)dY(r)\right](s) = \int_0^r X(r,s)dY(r,s)$$

When the random process $Y$ is Wiener, then the construction of stochastic Stieltjes integral is no longer possible. In this case by changing the mode of convergence from w.p.1 to $L_2(\mathscr{F})$ mode of convergence, one can define those stochastic integrals which can not be defined by stochastic Stieltjes integrals.

According to [12], the random processes which are contained in the set below are said to be *integrable in Ito sense* w.r.t the Wiener process $W$ over the interval $[0,r]$:

$$I_W[0,r] = \left\{ X : \Omega \times [0,r] \to \mathbb{R} \;\middle|\; X \text{ is } \mathscr{F}_s^W - \text{adapted, and } \int_0^r \mathbf{E}X(s)^2 ds < \infty \right\}.$$

The *Ito integral* of $X$ w.r.t the Wiener process $W$ is denoted as:

$$I = \int_0^r X(s)dW(s).$$

The following properties hold for the Ito integral, where $W$ is a Wiener process and $X, Y \in I_W[0,r]$:

1) $\int_0^r W(s)dW(s) = \frac{1}{2}\big(W(r)^2 - r\big)$

2) Expectation: $\mathbf{E}\int_0^r X(s)dW(s) = 0$ (Zero mean).

3) Isometry: $\mathbf{E}\big(\int_0^r X(s)dW(s)\big)^2 = \int_0^r \mathbf{E}X(s)^2 ds.$

4) Linearity: $\int_0^r \big(\alpha X(s) + \beta Y(s)\big)dW(s) = \alpha\int_0^r X(s)dW(s) + \beta\int_0^r Y(s)dW(s).$

5) Partitioning: $\int_0^r X(s)dW(s) = \int_0^t X(s)dW(s) + \int_t^r X(s)dW(s).$

6) $\int_0^r X(s)dW(s)$ is martingale with respect to $\mathscr{F}_r^W$. If $X$ is considered as non-random, then the integral $\int_0^r X(s)dW(s)$ would be a Gaussian random variable as well.

7) $\mathbf{E}\big(\int_0^r X(s)dW(s)\cdot\int_0^r Y(s)dW(s)\big) = \int_0^r \mathbf{E}X(s)Y(s)ds.$

8) $Cov\big(\int_0^m X(s)dW(s), \int_n^r Y(s)dW(s)\big) = \int_n^m \mathbf{E}X(s)Y(s)ds\ , 0 \leqslant n \leqslant m \leqslant r.$

**Definition 2.2.20** *Consider the representation below for the random process $X$:*

$$X(s) = X(0) + \int_0^s f(r)dr + \int_0^s g(r)dW(r).$$

*Then $X$ is said to be an Ito process and has the stochastic differential below:*

$$dX(s) = f(s)ds + g(s)dW(s).$$

# Chapter 3

## CONTROLLABILITY OF DETERMINISTIC SYSTEMS

In this chapter, main definitions and results of controllability theory will be provided. According to Kalman [48], controllability is a property of control systems so that every initial state can be steered to every state at terminal time moment. Later on researchers recognized that a detailed study in this concept needs a separation of this field into two main parts, i.e. exact (complete) controllability and approximate controllability. This was because many control systems are not exactly controllable while they are approximately controllable.

Throughout this chapter, both the exact and approximate controllability of deterministic and stochastic control systems will be discussed. One can find more detailed information on these systems in [14, 30, 52, 92].

### 3.1 Linear Deterministic Systems in Finite Dimensions

In the first section of this chapter, linear deterministic control systems and their controllability in finite dimensions will be discussed. The infinite dimensional control systems will be studied in the proceeding sections.

The general form of an initial value linear control system discussed in this thesis, is as follows:

$$\begin{cases} x_t' = Ax_t + Bu_t + f(t), \quad 0 < t \leq T, \\ x_0 = \eta \in X \end{cases} \tag{3.1.1}$$

Here, $A \in M_{n,n}$ and $B \in M_{n,m}$, where $M_{n,m}$ is a set of all $(n \times m)$-matrices.

Throughout this section, assume $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$.

A unique solution for the system (3.1.1) is given by:

$$x_t = e^{At}\eta + \int_0^t e^{A(t-s)}Bu(s)ds \ , \ \ t \in [0,T]. \tag{3.1.2}$$

The notation $x_t^{a,u} = b$ will be used to show that a control $u$ *transfers* a state $a$ to a state $b$ at time $t > 0$. It is also said that $a$ is *steered* to $b$ or $b$ is *attainable* from $a$.

For a control system given as (3.1.1), there is matrix, called the *controllability matrix* or the *controllability Gramian* defined as below:

$$Q_t = \int_0^t e^{As}BB^*e^{A^*s}ds \ , \ \ 0 \leq t \leq T, \tag{3.1.3}$$

where $A^*$ and $B^*$ are the transpose of the matrices $A$ and $B$ respectively.

**Proposition 3.1.1** *[92] Suppose $Q_t$ is an invertible matrix for some $t > 0$. Then,*

*i) $\forall a, b \in \mathbb{R}^n$ the control below transfers $a$ to $b$ at time $t$:*

$$\hat{u}(s) = -B^*e^{A^*(t-s)}Q_t^{-1}(e^{At}a - b), \ \ s \in [0,t]; \tag{3.1.4}$$

*ii) among all possible controls steering $a$ to $b$, the control $\hat{u}$ minimizes the integral $\int_0^t |u(s)|^2 ds$. Furthermore,*

$$\int_0^t |\hat{u}(s)|^2 ds = \langle Q_t^{-1}(e^{At}a - b), \ e^{At}a - b \rangle. \tag{3.1.5}$$

A useful condition for controllability of finite dimensional linear systems is provided in the next theorem. Consider that for arbitrary matrices $A \in M_{m,m}$ and $B \in M_{m,n}$, the matrix $[A|B]$ represents the matrix $[B, AB, \cdots, A^{m-1}B] \in M_{m,mn}$.

**Theorem 3.1.2** *[92] The statements below are equivalent:*

*i) An arbitrary state $b \in \mathbb{R}^n$ is reachable from* 0*.*

*ii) System* (3.1.1) *is controllable; that is, every point in $\mathbb{R}^n$ is attainable from every initial state $x_0$.*

*iii) $Q_t$ is invertible for all $t > 0$.*

*iv) rank $[A|B] = n$.*

*The last condition is called the Kalman rank condition.*

A necessary and sufficient condition for controllability of linear systems of finite dimensions, is the Kalman's rank condition, whereas it is not valid for infinite dimensional systems. The above mentioned controllability is well known as the *exact* controllability which was first introduced by Kalman (1960). Later, it was understood that many useful systems which are of infinite dimensions, are not exactly controllable but close to it. The concept of approximate controllability was then initiated.

**Example 1.** Consider the control system (3.1.1) in the two dimensional space $\mathbb{R}^2$ with matrix $A$ and vector $B$ as follows:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

One can easily calculate that:

$$rank[A:B] = rank \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} = 2 = dim\mathbb{R}^2.$$

Hence, according to Theorem 3.1.2 the system (3.1.1) is controllable.

**Example 2.** Consider the matrices $A$ and $B$ as follows:

$$A = \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

It is clear that,

$$rank[A:B] = rank \begin{bmatrix} 6 & 12 \\ 2 & 4 \end{bmatrix} = 1$$

For the control system (3.1.1) defined in $\mathbb{R}^2$,

$$rank[A:B] = 1 \neq dim\mathbb{R}^2 = 2.$$

So by Theorem 3.1.2, the system is not controllable.

## 3.2 Linear Deterministic Systems in Infinite Dimensions

In this section we will go through the exact and approximate controllability concepts of differential equations defined on infinite dimensional spaces. Similar to the finite case we have the following system considering that $X$ and $U$ are separable Hilbert spaces:

$$\begin{cases} x_t' = Ax_t + Bu_t + f(t), & 0 < t \leq T, \\ x_0 = \eta \in X_0 = X, & u \in U_{ad} = L_2(0,T;U) \end{cases} \tag{3.2.1}$$

where for the semigroup $e^{At}$, $A$ is the infinitesimal generator and $B \in \mathscr{L}(U,X)$ and $f$ is a function from $L_2(0,T;X)$.

The set $X_0$ is the set of initial states, which for deterministic systems is equal to $X$. The unique mild solution and the corresponding controllability operator for the above system are the same as (3.1.2) and (3.1.3) respectively, noting that $A$ and $B$ are operators in this case.

### 3.2.1 Exact Controllability

In order to provide a definition for exactly (completely) controllable systems, the set of *attainable values* must be defined first.

For this, considering a given control system, the set of *attainable values* at time $t$ is defined as below:

$$X_t^{\eta} = \{\hat{x}_t^{\eta,u} | u \in U_{ad}\}, \quad \eta \in X. \tag{3.2.2}$$

It is obvious that for deterministic systems, $X_t^{\eta} \subseteq X$.

**Definition 3.2.1** *Controllability of the system* (3.2.1) *for the time $T$, is said to be exact (complete), if $X_T^{\eta} = X$ for all $\eta \in X$.*

*Such systems are also said to be exactly controllable and throughout this thesis will be denoted as $E_T$-controllable.*

The *resolvent* of the operator $-Q_t$ is denoted by $R(\gamma, -Q_t)$ and is equal to:

$$R(\gamma, -Q_t) = (\gamma I + Q_t)^{-1}.$$

Here, $\gamma I + Q_t$ is coercive and so, for the operator $-Q_t$, the resolvent is well-defined for all positive $\gamma$.

**Theorem 3.2.2** *[12] The following statements are equivalent:*

*(1) The system* (3.2.1) *is $E_t$-controllable;*

*(2) $Q_t$ is coercive;*

*(3) $R(\gamma,-Q_t)$ converges uniformly as $\gamma \to 0^+$;*

*(4) $R(\gamma,-Q_t)$ converges strongly as $\gamma \to 0^+$;*

*(5) $R(\gamma,-Q_t)$ converges weakly as $\gamma \to 0^+$;*

*(6) $\gamma R(\gamma,-Q_t)$ converges uniformly to the zero operator as $\gamma \to 0^+$.*

*Condition (6) above, is called the* resolvent condition *for the system* (3.2.1) *to be exactly controllable.*

**Proof.** The equivalence relation $(1) \Leftrightarrow (2)$ is stated in many books such as [29]. Statement (2) also shows that, $Q_t$ is well-defined.

To prove $(2) \Rightarrow (3)$, suppose $Q_t$ is coercive. Then $\exists m > 0$ such that $\forall x \in X$ and $\forall \gamma \geq 0$ :

$$\langle x, (\gamma I + Q_t)x \rangle \geq (\gamma + m)\|x\|^2.$$

Hence $\|R(\gamma,-Q_t)\|$ is bounded as shown below:

$$\|R(\gamma,-Q_t)\| = \|(\gamma I + Q_t)^{-1}\| \leq \frac{1}{\gamma + m} \leq \frac{1}{m}.$$

Therefore,

$$
\begin{aligned}
\gamma\|R(\gamma,-Q_T)-Q_T^{-1}\| &= \|(\gamma I+Q_T)^{-1}-Q_T^{-1}\| \\
&= \|Q_t^{-1}(Q_t-\gamma I-Q_t)(\gamma I+Q_t)^{-1}\| \\
&\leq \gamma\|Q_t^{-1}\|\cdot\|(\gamma I+Q_t)^{-1}\| \\
&\leq \frac{\gamma}{m^2}.
\end{aligned}
$$

We conclude that, $R(\gamma,-Q_t)$ converges to $Q_t^{-1}$ in uniform topology as $\gamma$ approaches to $0^+$.

Proof of (3) $\Rightarrow$ (4) $\Rightarrow$ (5), considering the properties of convergence of operators is a straightforward result. Proof of (5) $\Rightarrow$ (6) is a result of boundedness of a weakly convergent sequence of operators.

Finally in order to prove the implication (6) $\Rightarrow$ (1), let

$$\gamma\|R(\gamma,-Q_T)\| = \gamma\|(\gamma I+Q_t)^{-1}\| \overset{\gamma\to 0^+}{\longrightarrow} 0.$$

Taking square root on the equation above, for a sufficiently small $\gamma_0>0$ we have:

$$\sqrt{\gamma}\|(\gamma_0 I+Q_t)^{-1/2}\| \leq \frac{1}{\sqrt{2}}.$$

Thus $\forall x\in X$, we obtain:

$$
\begin{aligned}
\|x\|^2 &= \left\|\Big(\gamma_0(\gamma_0 I+Q_t)^{-1}\Big)^{1/2}\Big(\gamma_0^{-1}(\gamma_0 I+Q_t)\Big)^{1/2}x\right\|^2 \\
&\leq \frac{1}{2}\left\|\Big(\gamma_0^{-1}(\gamma_0 I+Q_T)\Big)^{1/2}x\right\|^2 \\
&= \frac{1}{2}\langle\gamma_0^{-1}(\gamma_0 I+Q_t)x,x\rangle
\end{aligned}
$$

resulting

$$\langle\gamma_0^{-1}(\gamma_0 I+Q_t)x,x\rangle \geq 2\|x\|^2$$

which concludes that

$$\langle Q_t x, x \rangle \geq \gamma_0 \|x\|^2.$$

This completes the proof stating that $Q_t$ is coercive. ■

### 3.2.2 Approximate Controllability

For many infinite dimensional control systems, the concept of exact controllability is not applicable; therefore there is a need for a weaker concept named as the approximate controllability. Approximate controllability of linear deterministic systems will be introduced in this section.

**Definition 3.2.3** *Consider the attainable set* (3.2.2). *For the positive time $T$, the control system* (3.2.1) *is said to be approximately controllable, if $\forall \eta \in X$, we have $\overline{X_T^{\eta}} = X$.*

*Approximately controllable systems will be denoted by $A_T$-controllable.*

**Lemma 3.2.4** *Let $h \in X$ and $\gamma > 0$. Then there exists a unique optimal control $u^{\gamma} \in U_{ad}$ where the functional below achieves its minimum value subject to the system* (3.2.1)*:*

$$J(u) = \|x_T^u - h\|^2 + \gamma \int_0^T \|u_t\|^2 dt \tag{3.2.3}$$

*Moreover, for all $t \in [0, T]$,*

$$u_t^{\gamma} = -B^* e^{A^*(T-t)} R(\gamma, -Q_T)(e^{AT}\eta - h), \;\; almost \;\; everywhere \tag{3.2.4}$$

*and*

$$x_T^{u^{\gamma}} - h = \gamma R(\gamma, -Q_T)(e^{AT}\eta - h); \tag{3.2.5}$$

*where $R(\gamma, -Q_T)$ denotes the resolvent for the operator $-Q_T$.*

**Proof.** $u^\gamma \in U_{ad}$ is a unique optimal control for $J$. According to [17], an optimal solution $u^\gamma$ satisfying the equation below can be obtained.

$$u_t^\gamma = -\frac{1}{\gamma} B^* e^{A^*(T-t)} (x_T^{u^\gamma} - h), \ \ almost \ \ everywhere. \tag{3.2.6}$$

Substituting (3.2.6) in equation (3.2.1), we get

$$\begin{aligned} x_T^{u^\gamma} &= e^{AT}\eta + \frac{1}{\gamma}\int_0^T e^{A(T-s)} BB^* e^{A^*(T-s)} (x_T^{u^\gamma} - h)\, ds \\ &= e^{AT}\eta - \frac{1}{\gamma} Q_T (x_T^{u^\gamma} - h). \end{aligned}$$

Then,

$$\gamma x_T^{u^\gamma} = \gamma e^{AT}\eta - Q_T(x_T^{u^\gamma} - h). \tag{3.2.7}$$

Rewriting, we obtain

$$(\gamma I + Q_T) x_T^{u^\gamma} = \gamma e^{AT}\eta + Q_T h. \tag{3.2.8}$$

Since $(\gamma I + Q_T)^{-1}$ exists, this results

$$\begin{aligned} x_T^{u^\gamma} &= (\gamma I + Q_T)^{-1}\gamma e^{AT}\eta + (\gamma I + Q_T)^{-1}(\gamma I + Q_T - \gamma I)h \\ &= \gamma(\gamma I + Q_T)^{-1}(e^{AT}\eta - h) + h. \end{aligned}$$

Thus,

$$x_T^{u^\gamma} - h = \gamma R(\gamma, -Q_T)(e^{AT}\eta - h), \tag{3.2.9}$$

which proves (3.2.5). The equation (3.2.4) is obtained by substituting (3.2.5) into (3.2.6). ■

The following theorem introduces specific conditions for approximately controllable systems and clearly identifies them.

**Theorem 3.2.5** *The following statements are equivalent:*

*(1) The system* (3.2.1) *is $A_T$-controllable;*

*(2) $Q_T > 0$*

*(3) $\forall 0 \leq t \leq T$ satisfying $B^* e^{A^* t} x = 0$, implies $x = 0$;*

*(4) $\gamma R(\gamma, -Q_t) \overset{\gamma \to 0^+}{\longrightarrow} 0$ in strong operator topology;*

*(5) $\gamma R(\gamma, -Q_t) \overset{\gamma \to 0^+}{\longrightarrow} 0$ in weak operator topology.*

*Condition (4) is also known as the resolvent condition for $A_T$-controllable system* (3.2.1).

**Proof.** The implications $(1) \Leftrightarrow (2)$ and $(1) \Leftrightarrow (3)$ is mentioned and proved in many books such as [29]. In order to prove the implications $(1) \Leftrightarrow (4)$, assume that the control system (3.2.1) is approximately controllable on $U_{ad}$. According to Lemma 3.2.4, for an arbitrary $h \in X$, there is a sequence of controls, say, $\omega^m \in U_{ad}$ where as $m$ approaches to $\infty$:

$$\|x_T^{\omega^m} - h\| \to 0. \tag{3.2.10}$$

Furthermore, for a positive $\gamma$, we have:

$$\begin{aligned} \|x_T^{u^\gamma} - h\|^2 &\leq \|x_T^{u^\gamma} - h\|^2 + \gamma \int_0^T \|u_t^\gamma\|^2 \, dt \\ &\leq \|x_T^{\omega^m} - h\|^2 + \gamma \int_0^T \|\omega^m\|^2 \, dt. \end{aligned} \tag{3.2.11}$$

where the control $u^\gamma$ is such that the functional (3.2.4) takes on its minimum value. Now, consider an arbitrary positive $\varepsilon$, then for a sufficiently large $m$ we can gain:

$$\|x_T^{\omega^m} - h\| < \frac{\varepsilon}{\sqrt{2}}. \tag{3.2.12}$$

Moreover, for all values of $0 < \gamma < \delta$ selecting a sufficiently small $\delta$, we have:

$$\gamma \int_0^T \|\omega_t^m\|^2 \, dt \leq \frac{\varepsilon^2}{2}. \tag{3.2.13}$$

Therefore, substituting relations (3.2.12) and (3.2.13) in relation (3.2.11), we obtain $\|x_T^{u^\gamma} - h\|^2 \leq \varepsilon^2$ which results the convergence of $x_T^{u^\gamma}$ to $h$ as $\gamma \to 0^+$.

Now considering (3.2.5) the strong convergence of $\gamma R(\gamma, -Q_t) \overset{\gamma \to 0^+}{\longrightarrow} 0$ is satisfied.

In order to prove (4) $\Rightarrow$ (1), suppose (4) holds. For a sufficiently small $\gamma$ and an arbitrary $h \in X$, according to Lemma 3.2.4, a unique control $u^\gamma \in U_{ad}$ exists such that:

$$\|x_T^{u^\gamma} - h\| = \|\gamma R(\gamma, -Q_T)(e^{AT}\eta - h)\| \tag{3.2.14}$$

According to assumption (4) and equation above: $x_T^{u^\gamma} \overset{\gamma \to 0}{\longrightarrow} h$.

This shows that the system (3.2.1) is $A_T$-controllable.

Proving (4) $\Leftrightarrow$ (5), we know that (4) $\Rightarrow$ (5) is a fact in functional analysis; but to show the converse implication, suppose that we have the weak convergence. By the definition of weak convergence we have:

$$\forall x, y \in X, \langle \gamma R(\gamma, -Q_T)x, y \rangle \to 0 \quad \text{as} \quad \gamma \to 0^+.$$

In order to show strong convergence, we use the fact that $R(\gamma, -Q_T) \geqslant 0$, hence:

$$
\begin{aligned}
\|\gamma R(\gamma,-Q_T)x\|^2 &= \langle \gamma R(\gamma,-Q_T)x, \gamma R(\gamma,-Q_T)x\rangle \\
&\leq (\|\gamma R(\gamma,-Q_T)\|^2)^{\frac{1}{2}} \gamma \langle R(\gamma,-Q_T)x, x\rangle \\
&\leq \langle \gamma R(\gamma,-Q_T)x, x\rangle \to 0 \quad \text{as} \quad \gamma \to 0^+.
\end{aligned}
$$

Since $x$ was chosen arbitrarily, strong convergence of $\gamma R(\gamma,-Q_T)$ is satisfied. ■

**Example 1.** Consider two Hilbert spaces $X$ and $Y (X = Y = \ell_2)$; i.e. a space of numerical sequences $\{x_n\}$ which satisfy the condition $\sum_{n=1}^{\infty} x_n^2 < \infty$. The scalar product on these spaces is defined as below:

$$
\langle (x_n), (y_n)\rangle = \sum_{n=1}^{\infty} x_n y_n.
$$

It is well-known that the set $\Big\{e_1 = (1,0,0,\cdots),\ e_2 = (0,1,0,\cdots),\dots\Big\}$ constructs a basis for the spaces $X$ and $Y$. For the system (3.2.1), consider the corresponding linear differential equation below:

$$
y_t' = Ay_t + Bu_t,\ 0 < t \leq T,\ y_0 \in X. \tag{3.2.15}
$$

Let $A = 0$ so that $e^{At} \equiv I$ and consider $B$ as follows:

$$
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

In order to show that the system (3.2.1) is approximately controllable, it suffices to show that the corresponding linear equation (3.2.15) is approximately controllable. For this we will use Theorem 3.2.5, part 3.

It is obvious that:

$$\sum_{n=1}^{\infty} \langle Be_n, Be_n \rangle = B^2 \sum_{n=1}^{\infty} \langle e_n, e_n \rangle = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore $B$ is a Hilbert-Schmidt operator on $\ell_2$ which results $B = B^*$. Hence,

$$B^* e^{A^* t} x = 0 \Rightarrow Bx = 0 \Rightarrow x = 0$$

So, the system (3.2.1), with $A$ and $B$ defined as above, is approximately controllable.

To check exact controllability of the system, we must check whether the controllability operator is coercive or not.

Since $B = B^*$, we have:

$$Q_T = \int_0^T e^{As} BB^* e^{A^* s} ds = TB^2.$$

Therefore:

$$\langle Q_T e_n, e_n \rangle = T \langle B^2 e_n, e_n \rangle = \frac{T}{n^2} \overset{n \to \infty}{\longrightarrow} 0.$$

Which means that there is no positive value $c$ which satisfies the inequality $\langle Q_T e_n, e_n \rangle \geq c \|e_n\|^2$ which disproves the exact controllability of the system (3.2.1), since $Q_T$ is no longer coercive.

## 3.3 Partial Controllability of Linear Deterministic Systems

Now that controllability of deterministic systems have been defined, it's best to introduce partial controllability of such systems.

**Definition 3.3.1** *Let $H$ be a closed subspace of the separable Hilbert space $X$. Also let $L$ denote the operator which projects $X$ onto $H$. Then a deterministic system is called:*

*(i) L-partially exact controllable if $L(X_T^{\eta}) = H$ for the time $T$ and $\forall \eta \in X$; and shortly denoted by $LE_T$-controllable.*

*(ii) L-partially approximate controllable if $\overline{L(X_T^{\eta})} = H$ for the time $T$ and $\forall \eta \in X$; and shortly denoted by $LA_T$-controllable.*

The concepts defined above were first mentioned in [13, 20]. The motivation for the partial controllability concepts were the fact that the results on controllability were gained for first-order deterministic differential equations (systems in a standard form); while by increasing the state spaces' dimension, higher order differential equations can also be rewritten in the standard form. Therefore, if $L$ projects the enlarged space $X$ onto $H$, the $L$-partial controllability for the enlarged space $X$ will be the well-known ordinary controllability for the original system. Considering Theorems 3.2.2 and 3.2.5 and integrating the operator $L$ into them we have the following two theorems.

**Theorem 3.3.2** *The following statements are equivalent:*

*(i) The system* (3.2.1) *is $LE_T$-controllable.*

*(ii) $LQ_TL^*$ is coercive.*

*(iii) $\gamma R(\gamma, -LQ_tL^*) \overset{\gamma \to 0^+}{\longrightarrow} 0$ uniformly.*

Similarly we have the following theorem.

**Theorem 3.3.3** *The following statements are equivalent:*

*(i) The system* (3.2.1) *is $LA_T$-controllable.*

*(ii)* $LQ_T L^* > 0$ .

*(iii)* $\gamma R(\gamma, -LQ_T L^*) \overset{\gamma \to 0^+}{\longrightarrow} 0$ *strongly.*

Considering the projection operator $L$ as the identity operator, the results of Sections 3.1 and 3.2 will be achieved.

## 3.4 Semilinear Deterministic Systems

The study of controllability of semilinear systems in finite dimensional spaces, have been done by many researchers such as: [1, 2, 4, 52]. In all these studies, the research has been done by means of fixed point theorems. The concept of controllability of semilinear systems in infinite dimensional spaces has also been studied by some authors where they have established sufficient conditions for controllability of these systems in Banach spaces. Among various approaches, the fixed point theorems have been used the most. In these methods, the controllability problems are transformed into a fixed point problem in the given space.

Throughout this thesis we intend to demonstrate a different method, therefore in this section and in the proceeding chapters we will not mention the fixed-point theorems. Consider a general form of a semilinear control system as below:

$$\begin{cases} \frac{dx}{dt} = Ax_t + Bu_t + f(t, x_t, u_t), \ 0 \leq t \leq T, \\ x(0) = \eta \in X. \end{cases} \tag{3.4.1}$$

Here, similar to the system (3.2.1), the state and control processes are $x$ and $u$ respectively.

Consider the following assumptions:

(1) Consider the separable Hilbert spaces $X$ and $U$. Let $L$ be an operator projecting $X$ onto $H$ where $H$ is a closed subspace of $X$.

(2) $A$ and $B$ are considered the same as system (3.2.1).

(3) The set of admissible controls are $U_{ad} = PC(0,T;U)$.

(4) The nonlinear function $f : [0,T] \times X \times U \to X$ is such that:

- $f$ is bounded and continuous on $[0,T] \times X \times U$;

- $f$ satisfies the Lipschitz condition with respect to $x$.

Under the above conditions, for an arbitrary control $u \in U_{ad}$ and for $x_0 = \eta \in X$, considering the semilinear system (3.4.1), there exists a unique mild solution $x^{u,\eta}$ as below:

$$x_t^{u,\eta} = e^{At}\eta + \int_0^t e^{A(t-r)}(Bu_r + f(r, x_r^{u,\eta}, u_r))dr. \tag{3.4.2}$$

Let

$$D_T^\eta = \{x \in X | \exists u \in U_{ad} : x = x_T^{u,\eta}\}.$$

**Definition 3.4.1** *The semilinear system* (3.4.1) *is said to be $E_T$-controllable if $\forall \eta \in X$, $D_T^\eta = X$ and it is considered as $A_T$-controllable if $\forall \eta \in X$, $\overline{D_T^\eta} = X$, the closure of $D$ being denoted by $\overline{D}$.*

*Similarly, the system* (3.4.1) *is called L-partially exact controllable on $U_{ad}$ if $L(D_T^{\eta}) = H$ and L-partial approximate controllable if $\overline{L(D_T^{\eta})} = H$ for $\forall \eta \in X$.*

Before providing the method introduced in this thesis and the corresponding theorems, in order to motivate the partial concepts of controllability, consider the following example.

**Example 1.** Consider the nonlinear system below with the state space $X = R$.

$$X_t^{(n)} = f\left(t, x_t, x_t', \cdots, x_t^{(n-1)}, u_t\right) \tag{3.4.3}$$

One can rewrite the above system as the differential equation below:

$$\frac{dy}{dt} = Ay_t + F(t, y_t, u_t) \tag{3.4.4}$$

where:

$$y_t = \begin{bmatrix} x_t \\ x_t' \\ \vdots \\ x_t^{(n-2)} \\ x_t^{(n-1)} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$F(t, y, u) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t, x, x', \ldots, x^{(n-1)}, u) \end{bmatrix}.$$

The $n$-dimensional Euclidean space $\mathbb{R}^n$, is the state space for the system (3.4.4) and a subset of $\mathbb{R}^n$ constructs the corresponding attainable set. Hence, the concepts of controllability for the system (3.4.3) are weaker than the concepts for the system (3.4.4). Consider the projection operator below:

$$L = [\,1 \quad 0 \quad \cdots \quad 0 \quad 0\,] : \mathbb{R}^n \to \mathbb{R}.$$

Applying the operator $L$, the $L$-partial controllability concepts for the systems (3.4.4) and (3.4.3) coincide.

**Example 2.** Let $x$ be a real binary function with the variables $0 \leq \zeta \leq 1$ and $t \geq 0$. A semilinear wave equation has the form below:

$$\frac{\partial^2 x_{t,\zeta}}{\partial t^2} = \frac{\partial^2 x_{t,\zeta}}{\partial \zeta^2} + b_\zeta u_t + f(t, x_{t,\zeta}, \partial x_{t,\zeta}/\partial t, u_t), \tag{3.4.5}$$

The space of square integrable functions on $[0,1]$ ($L_2(0,1)$) is the state space of the above system. One can rewrite the system in the form of a $1^{st}$ order differential equation below:

$$\frac{dy}{dt} = Ay_t + Bu_t + F(t, y_t, u_t) \tag{3.4.6}$$

where

$$y_t = \begin{bmatrix} x_{t,\zeta} \\ \partial x_{t,\zeta}/\partial t \end{bmatrix}, \; A = \begin{bmatrix} 0 & I \\ d^2/d\zeta^2 & 0 \end{bmatrix}, \; F(t,y,u) = \begin{bmatrix} 0 \\ f(t, y_1, y_2, u) \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

and

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in L_2(0,1) \times L_2(0,1)$$

Expanding the state space of the system (3.4.5), the state space of the system (3.4.6) is gained as $L_2(0,1) \times L_2(0,1)$. The controllability concepts of the system (3.4.6) are too strong for (3.4.5), but considering a projection operator $L$ as below

$$L = [\, I \quad 0\,] : L_2(0,1) \times L_2(0,1) \to L_2(0,1),$$

the $L$-partial concepts of controllability coincide for the systems (3.4.5) and (3.4.6). More examples can be found in [21] where a new method has been established for partial controllability of semilinear systems. This method which is different from the well known fixed point theorems, will be used in this thesis. The purpose of this technique is to partition the given time interval $[0,T]$ into two parts; $[0,T-\varepsilon]$ and $[T-\varepsilon,T]$ for a positive $\varepsilon$. On the subinterval $[0,T-\varepsilon]$, any arbitrary control is chosen and the initial state is transferred to some state at time $T-\varepsilon$. Then on the second subinterval, i.e. $[T-\varepsilon,T]$, sequence of controls are chosen in a way that steer the state at time $T-\varepsilon$ along the linear part of the system arbitrarily close to the desired state at $T$.

Taking into account the fact that the linear part of the system is disturbed by its non-linear part for a small amount, in a small time interval, therefore partial approximate controllability of the system is obtained. In order to provide the theorem for $L$-partially approximate controllability of semilinear systems, some facts are needed which will be mentioned next.

For a positive $\varepsilon$, $0 < \varepsilon < T$, consider the following linear system corresponding to the semilinear system (3.4.1) as below:

$$\frac{dy}{dt} = Ay_t + Bv_t \ , \quad t \in (T-\varepsilon, T] \tag{3.4.7}$$

for which $v \in V_{ad}^{\varepsilon} = C(T-\varepsilon, T; U)$. The corresponding unique solution for the above equation is as follows:

$$y_t^{v, y_{T-\varepsilon}} = e^{A(t-T+\varepsilon)} y_{T-\varepsilon} + \int_{T-\varepsilon}^{t} e^{A(t-r)} B v_r dr \ , \quad t \in [T-\varepsilon, T], \ \ y_{T-\varepsilon} \in X \ . \tag{3.4.8}$$

Recall from Section 3.1, for the linear system (3.4.7), the controllability operator is defined by

$$Q_{\varepsilon} = \int_{T-\varepsilon}^{T} e^{A(T-t)} BB^* e^{A^*(T-t)} dt = \int_{0}^{\varepsilon} e^{At} BB^* e^{A^* t} dt \ .$$

Denote the $L$-partial controllability operator by $\tilde{Q}_{\varepsilon} = LQ_{\varepsilon}L^*$.

Consider the assumptions $(1)-(4)$ from the beginning of this section, and add another assumption which is a result of positiveness of $Q_{\varepsilon}$;

(5) $\forall 0 < \varepsilon \leqslant T, \ \ \tilde{Q}_{\varepsilon} > 0$

**Lemma 3.4.2** *[21] Under the above conditions, assumptions and notation, for any $h \in H$ and positive $\gamma$, there exists a unique optimal control $v^{\gamma} \in V_{ad}^{\varepsilon}$ where the functional below takes its minimal value on $V_{ad}^{\varepsilon}$ along the linear system in* (3.4.7)*:*

$$J^{\gamma}(v) = \left\| L y_t^{v, y_{T-\varepsilon}} - h \right\|^2 + \gamma \int_{T-\varepsilon}^{T} \|v_t\|^2 dt$$

*Furthermore, the following equations hold;*

$$v_t^{\gamma} = -\gamma^{-1} B^* e^{A^*(T-t)} L^* \left( L y_T^{v^{\gamma}, y_{T-\varepsilon}} - h \right) \ , \ \ t \in [T-\varepsilon, T] \tag{3.4.9}$$

*and*

$$Ly_T^{v^\gamma, y_{T-\varepsilon}} - h = \gamma R(\gamma, -\tilde{Q}_\varepsilon)\big(Le^{AT} y_{T-\varepsilon} - h\big). \tag{3.4.10}$$

**Lemma 3.4.3** *[21] Considering the conditions, assumptions and notation mentioned above, let the system* (3.4.7) *be LA-controllable on $V_{ad}^\varepsilon$. For $h \in X$, $y_{T-\varepsilon} \in X$ and the control $v^\gamma$ defined as in Lemma 3.4.2, we have:*

$$\left\|Ly_T^{v^\gamma, y_{T-\varepsilon}} - h\right\| \xrightarrow{\gamma \to 0} 0 \tag{3.4.11}$$

**Lemma 3.4.4** *[21] Assuming the above mentioned conditions, assumptions and notation, consider the LE-controllable system* (3.4.7) *on $V_{ad}^\varepsilon$. Then $\forall t \in [T-\varepsilon, T]$ and $0 < \gamma \leqslant \gamma_0$:*

$$\left\|v_t^\gamma\right\| \leqslant c_1\|y_{T-\varepsilon}\| + c_2\|h\| \tag{3.4.12}$$

*where $v^\gamma$ is a control defined as in Lemma 3.4.2 and the constants $c_1, c_2$ are non-negative.*

Considering the above Lemmas, we reach to the main theorem of this section where for partial $A$-controllability of semilinear systems, a sufficient condition is provided.

**Theorem 3.4.5** *[21] Considering the assumptions* $(1)-(5)$*, the semilinear system* (3.4.1) *is LA-controllable on $U_{ad}$.*

**Proof.** Take any given positive $\delta$, an initial state $\eta \in X$, $h \in H$, $0 < \varepsilon < T$ and $u \in C(0,T;U)$. Consider $x_t^{u,\eta}$ as the value of the mild solution of system (3.4.1) at time $t$ with respect to $u$ and $\eta$. Define the control $u^{\gamma,\varepsilon}$ as below:

$$u_t^{\gamma,\varepsilon} = \begin{cases} u_t, & 0 \leq t \leq T-\varepsilon, \\ -B^* e^{A^*(T-t)} L^* R(\gamma, -\tilde{Q}_\varepsilon)(Le^{AT} x_{T-\varepsilon}^{u,\eta} - h), & T-\varepsilon < t \leqslant T. \end{cases}$$

It's clear that for all positive $\gamma$ and $0 < \varepsilon < T$, $u^{\gamma,\varepsilon} \in U_{ad} = PC(0,T;U)$. Therefore considering (3.4.2), we can write:

$$x_T^{u^{\gamma,\varepsilon},\eta} = e^{A\varepsilon} x_{T-\varepsilon}^{u,\eta} + \int_{T-\varepsilon}^{T} e^{A(t-s)} (Bu_s^{\gamma,\varepsilon} + f(s, x_s^{u^{\gamma,\varepsilon},\eta}, u_s^{\gamma,\varepsilon}))ds.$$

Similarly considering equations (3.4.7) and (3.4.8), we have:

$$y_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} = e^{A\varepsilon} x_{T-\varepsilon}^{u,\eta} + \int_{T-\varepsilon}^{T} e^{A(T-s)} Bu_s^{\gamma,\varepsilon} ds\ .$$

Hence,

$$\left\| x_T^{u^{\gamma,\varepsilon},\eta} - y_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} \right\| \leqslant \int_{T-\varepsilon}^{T} \|e^{A(T-s)}\| \ \|f(s, x_s^{u^{\gamma,\varepsilon},\eta}, u_s^{\gamma,\varepsilon})\| \ ds.$$

Let $K = sup_{[0,T]\times X\times U} \|f(t,x,u)\|$ and $M = sup_{[0,T]} \|e^{At}\|$, then:

$$\left\| x_T^{u^{\gamma,\varepsilon},\eta} - y_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} \right\| \leqslant MK\varepsilon,$$

which results:

$$\begin{aligned} \|Lx_T^{u^{\gamma,\varepsilon},\eta} - h\| &\leqslant \left\| Lx_T^{u^{\gamma,\varepsilon},\eta} - Ly_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} \right\| + \left\| Ly_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} - h \right\| \\ &\leqslant MK\varepsilon + \left\| Ly_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} - h \right\|. \end{aligned}$$

Assumption (5) implies the $L$-partial approximate controllability of the linear system (3.4.7). Therefore, for a sufficiently small $\gamma > 0$, using Lemma 3.4.3 and considering $0 < \varepsilon < min\{T, \frac{\delta}{2MK}\}$ we have:

$$\left\| Ly_T^{u^{\gamma,\varepsilon},x_{T-\varepsilon}^{u,\eta}} - h \right\| < \frac{\delta}{2}$$

For the above values of $\gamma$ and $\varepsilon$, the control $u^{\gamma,\varepsilon}$ satisfies the following inequality:

$$\|Lx_T^{u^{\gamma,\varepsilon},\eta} - h\| < MK\frac{\delta}{2MK} + \frac{\delta}{2} = \delta$$

Since $\eta \in X$ and $h \in H$ were selected arbitrarily and $\delta > 0$, therefore the $L$-partial approximate controllability of the semilinear system (3.4.1) is achieved. ■

**Remark 3.4.6** *Note that if in the above theorem, the spaces $X$ and $U$ are finite dimensional $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, then we can drop the condition of boundedness of $f$ in assumption* $(4)$. *The reason is that in this case, the L-partial approximate and exact controllability coincide for the linear system. Therefore, considering Lemma 3.4.4 for a fix function $u \in PC(0,T;\mathbb{R}^n)$, the solution of system* (3.4.1) *over the compact interval $[0,T]$, is continuous and so bounded. For $l \geqslant 0$, assume $\|x_t^{u,\eta}\| \leqslant l$. Hence*

$$\|u_t^{\gamma,\varepsilon}\| \leqslant c_1 l + c_2\|h\| = r_1$$

*Recalling the Lipschitz condition in assumption* $(4)$, *we get*

$$\|x_t^{u^{\gamma,\varepsilon},\eta}\| \leqslant r_2 \tag{3.4.13}$$

*which implies the boundedness of all $x^{u^{\gamma,\varepsilon},\eta}$ for some $r_2 \geqslant 0$. So, the function $f$ can be restricted into the set $[0,T] \times B^m(r_2) \times B^n(r_1)$ which is compact and $B^m(r_2)$ and $B^n(r_1)$ are the well-known m and n dimensional closed balls with center at the origin and radius $r_2$ and $r_1$ respectively. Therefore, the boundedness condition of $f$ is a result of other conditions of $f$.*

To finalize this chapter, the features of Theorem 3.4.5, will be demonstrated on the following examples. The first example provides an $L$-partially approximate controllable system which may not be approximately controllable; and the second example demonstrates the partial controllability concepts on delay equations.

**Example 3.** Consider the system of differential equations

$$\begin{cases} \frac{dx}{dt} = y_t + bu_t, \quad x_0 \in \mathbb{R}, \\ \frac{dy}{dt} = f(t, x_t, y_t, u_t), \; y_0 \in \mathbb{R}, \end{cases} \tag{3.4.14}$$

where $u \in U_{ad} = PC(0,T;\mathbb{R})$ and $t \in [0,T]$. One can rewrite the above system in the form of the semilinear system below:

$$\frac{dz}{dt} = Az_t + Bu_t + F(t, z_t, u_t), \tag{3.4.15}$$

where

$$z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \; F(t,z,u) = \begin{bmatrix} 0 \\ f(t,x,y,u) \end{bmatrix} \tag{3.4.16}$$

and $z = \begin{bmatrix} x \\ y \end{bmatrix}$ is a vector in $\mathbb{R}^2$.

The corresponding system of equations for (3.4.15) is as follows

$$\begin{cases} x_t' = y_t, \;\; x_0 = k_1, \\ y_t' = 0, \; y_0 = k_2, \end{cases}$$

Solving the system above we gain the following results

$$x_t = k_2 t + k_1 \quad y_t = k_2.$$

As in matrix representation:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

resulting $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

Therefore, the controllability operator can be written as:

$$Q_t = \int_0^t e^{Ar} BB^* e^{A^* r}\, dr = b^2 t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } 0 < t \leq T.$$

Hence, $Q_t$ is not positive and the condition for approximate controllability of the linear part of the system (3.4.15), based on positiveness of $Q_t$, fails for this example. Consequently, all $A$-controllability results based on the $A$-controllability of the linear part of the system (3.4.15), fail.

In order to examine the $L$-partial $A$-controllability of the system (3.4.15) with respect to the first component of $z_t$, i.e. $x_t$, consider the projection operator $L = [\ 1 \quad 0\ ]$. Then for $t \in (0,T]$

$$\tilde{Q}_t = LQ_tL^* = b^2 t > 0.$$

Therefore, by Theorem 3.4.5 and Remark 3.4.6, if the function $f$ satisfies the Lipschitz condition and is continuous in $x$ and $y$, then the system (3.4.15) and consequently the system (3.4.14) is $L$-partially $A$-controllable.

**Example 4.** Consider the semilinear delay equation below on $[0,T]$

$$\begin{cases} \frac{dx}{dt} = Ax_t + \int_{-\varepsilon}^0 M_\theta x_{t+\theta}\, d\theta + Bu_t + f\left(t, x_t, \int_{-\varepsilon}^0 N_\theta x_{t+\theta}\, d\theta, u_t\right) \\ x_0 = \zeta,\ x_\theta = \eta_\theta,\ -\varepsilon \leq \theta \leq 0, \end{cases} \tag{3.4.17}$$

where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$, $M,N \in C(-\varepsilon, 0, \mathbb{R}^{m\times m})$, $\zeta \in \mathbb{R}^m$, $\eta \in L_2(-\varepsilon, 0; \mathbb{R}^m)$ and $u \in U_{ad} = PC(0,T;\mathbb{R}^n)$.

Define the function $\bar{x} : [0,T] \to L_2(-\varepsilon, 0; \mathbb{R}^m)$ as

$$[\bar{x}_t]_\theta = x_{t+\theta},\ 0 \le t \le T,\ -\varepsilon \le \theta \le 0. \tag{3.4.18}$$

Hence,

$$\frac{d\bar{x}}{dt} = (\frac{d}{d\theta})\bar{x}_t,\ \bar{x}_0 = \eta,\ 0 < t \le T. \tag{3.4.19}$$

The differential operator $\frac{d}{d\theta}$ generates a semigroup which will be denoted by $\mathscr{T}_t$. Define the integral operators $\Gamma_1$ and $\Gamma_2$ from $L_2(-\varepsilon, 0; \mathbb{R}^m)$ to $\mathbb{R}^m$ as below:

$$\Gamma_1 h = \int_{-\varepsilon}^{0} M_\theta h_\theta \, d\theta,\ \ \Gamma_2 h = \int_{-\varepsilon}^{0} N_\theta h_\theta \, d\theta,\ \ h \in L_2(-\varepsilon, 0; \mathbb{R}).$$

Assume

$$\tilde{A} = \begin{bmatrix} A & \Gamma_1 \\ 0 & \partial/\partial\theta \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ z_t = \begin{bmatrix} x_t \\ \bar{x}_t \end{bmatrix},\ F(t,z,u) = \begin{bmatrix} f(t,x,\Gamma_2\bar{x},u) \\ 0 \end{bmatrix},$$

Then we can rewrite the system (3.4.17) as below:

$$\frac{dz}{dt} = \tilde{A}z_t + \tilde{B}u_t + F(t,z_t,u_t),\ z_0 = \xi, \tag{3.4.20}$$

where

$$z = \begin{bmatrix} x \\ \bar{x} \end{bmatrix},\ \text{and } \xi = \begin{bmatrix} \zeta \\ \eta \end{bmatrix}$$

both belonging to $\mathbb{R}^m \times L_2(-\varepsilon, 0; \mathbb{R}^m)$.

Let $L = [\ I \quad 0\ ] : \mathbb{R}^m \times L_2(-\varepsilon, 0; \mathbb{R}^m) \to \mathbb{R}^m$, then the approximate controllability of (3.4.17) and $L$-partial approximate controllability of (3.4.20) coincide. According to Theorem 3.4.5, the system (3.4.20) is $LA$-controllable and consequently the system

(3.4.17) is approximately controllable if the corresponding controllability operator $\tilde{Q}_\delta$ is positive for all $0 < \delta \leqslant T$ and the continuous and bounded function $f$ satisfies the Lipschitz condition w.r.t its second and third variables. The controllability operator for (3.4.20) has been calculated in [20] as follows:

$$Q_\delta = \int_0^\delta \mathscr{K}_r BB^* \mathscr{K}_r^* dr$$

where $\mathscr{K}$ is a unique solution for the following equation:

$$\mathscr{K}_t = e^{At} + \int_0^{max(0,t-\varepsilon)} \int_{-\varepsilon}^0 e^{As} M_\theta \mathscr{K}_{t-s+\theta} d\theta ds \tag{3.4.21}$$

The case $n = m = 1$ has been studied in [14] where $A = a$, $B = b$ and $M_\theta \equiv 0$. Then the $L$-partial controllability operator is resulted as below:

$$\tilde{Q}_t = \frac{b^2(e^{2at} - 1)}{2a} > 0, \ \forall t > 0.$$

Therefore the approximate controllability of the system (3.4.17) for a bounded, continuous function satisfying Lipschitz condition is achieved.

Before providing the concept of controllability of stochastic systems, it's best to indicate the advantages of the method used in the above theorem comparing with the technique which uses the fixed point theorems:

1) Since the admissible control set is considered as piecewise continuous, therefore the larger space $L_2(0,T;U)$ is no longer needed.

2) No need for all the unusual inequalities mentioned in the methods using fixed point theorems.

3) The function $f$ satisfies the Lipschitz condition just with respect to $x$ and not $u$.

Apart from the above advantages, the only disadvantage of this method is that it can not be applied for the concept of exact controllability. For more details and examples, one can refer to [21].

# Chapter 4

## CONTROLLABILITY OF STOCHASTIC SYSTEMS

Through out the previous chapters, controllability of deterministic systems were introduced and discussed. Almost all the requirements for deterministic systems have been achieved. Further studies and researches in the field of controllability leads to the controllability of stochastic systems which will be discussed in the coming chapters. There is a number of researches done on stochastic controllability such as [13, 17, 18, 21, 32, 65, 66, 67, 68, 72, 75, 92], etc.

In the theory of stochastic controllability there are at least four types of controllability; exact, approximate and $C$- and $S$-controllability. In the present chapter these concepts will be introduced and discussed over partially observable stochastic systems.

### 4.1 Controllability of linear stochastic systems

Consider the general form of partially observable linear stochastic systems over $[0,T]$ as below:

$$\begin{cases} dx_t = (Ax_t + Bu_t + f(t))dt + dm_t, \\ dz_t = Cx_t dt + dn_t, \\ x_0 = \eta \in X_0,\ z_0 = 0,\ u \in U_{ad}\ . \end{cases} \tag{4.1.1}$$

A general solution for the above mentioned stochastic system has the following form:

$$x_t^{u,\eta} = e^{At}\eta + \int_0^t e^{A(t-s)}(Bu_s + f(s, x_s^{u,\eta}, u_s))ds + \int_0^t e^{A(t-s)}dm_s. \tag{4.1.2}$$

The class of finite dimensional partially observable stochastic systems are discussed in a number of researches such as [66] and [72]. A few properties and theorems have been stated there. In the present chapter, we will consider the infinite dimensional partially observable stochastic systems where in this case for the system (4.1.1), $A, B$ and $f$ are the same as for system (3.2.1) and $C \in \mathscr{L}(X, \mathbb{R}^k)$. Assume two independent and correlated standard Wiener processes $m$ and $n$ where the values appear in $X$ and $\mathbb{R}^k$ respectively. Also consider the corresponding observation process $z$.

In the theory of stochastic calculus we let $\mathscr{F}_t^{\eta,u}$ denote the smallest $\sigma$-field generated by the observations over $[0,T]$, corresponding to the control $u \in U_{ad}$ and the initial state $\eta \in X_0$. Also the conditional expectation $E(x_t^{\eta,u} | \mathscr{F}_t^{\eta,u})$ will be denoted by $\hat{x}_t^{\eta,u}$ (for a deterministic control system, since $\mathscr{F}_t^{\eta,u} = \{\Omega, \emptyset\}$, therefore $\hat{x}_t^{\eta,u} = x_t^{\eta,u}$). In stochastic calculus, $\hat{x}^{\eta,u}$ is a random process; for a completely observable system, $x^{\eta,u}$ will be used instead. Also, the set of attainable values is a collection of random variables.

Throughout this chapter, we will consider the following notation. The collection of all Gaussian random variables $\eta$ valued in $X$, which are independent of $(m,n)$ will construct the initial states set denoted by $X_0$. The set of admissible controls denoted by $U_{ad}$ will be the collection of all controls $u$ in the form below:

$$u_t = v_t + \int_0^t K_{t,s} dz_s^{\eta,u}, \quad 0 \leqslant t \leqslant T \ ,$$

where $K \in L_2(\Delta_T, \mathscr{L}_2(\mathbb{R}^k, U))$ with $\Delta_T$ and $v \in L_2(0,T;U)$ defined as in Chapter 2 and the observation process $z$ corresponding to $\eta \in X_0$ and $u \in U_{ad}$ is $z^{\eta,u}$. Since the filtrations $\{\mathscr{F}_t^{\eta,u}\}$ are independent on the control $u \in U_{ad}$, therefore for an initial value

$\eta$, we can consider $\mathscr{F}_t^{\eta} = \mathscr{F}_t^{\eta,u}$, $t \in [0,T]$.

The *innovation process* corresponding to the system (4.1.1) is defined as follows:

$$d\bar{z}_t^{\eta} = dz_t^{\eta,u} - C\hat{x}_t^{\eta,u}dt\ ,\ \ 0 < t \leqslant T,\ \ \bar{z}_0^{\eta} = 0$$

which is a Wiener process with respect to the filtration $\{\mathscr{F}_t^{\eta}\}$ and independent of $u \in U_{ad}$.

### 4.1.1 Exact controllability

In order to define the exact controllability of stochastic systems, we need to introduce a subspace of $L_2(\Omega, X)$ which is Gaussian. For this, being generated by constant random variables $\eta$, $m_t$ and $n_t$ over $[0,T]$, let $G_T^{\eta}(X)$ be a closed Gaussian subspace of $L_2(\Omega, X)$. Consider the set below which is also a subspace of $L_2(\Omega, X)$ and Gaussian:

$$\mathscr{G}_T^{\eta}(X) = \{h \in G_T^{\eta}(X) | h \text{ is } \mathscr{F}_T^{\eta}\text{-measurable }\} \tag{4.1.3}$$

**Definition 4.1.1** *Assume that L is an operator projecting X onto H (a closed subspace of X) and $L(X_T^{\eta}) = \{L\xi \mid \xi \in X_T^{\eta}\}$. For $\forall\eta \in X_0$, the stochastic control system* (4.1.1) *is said to be*

*i) exactly controllable, if $X_T^{\eta} = \mathscr{G}_T^{\eta}(X)$, at time $T$;*

*ii) L-partially exact controllable, if $L(X_T^{\eta}) = \mathscr{G}_T^{\eta}(H)$ at time $T$.*

The idea of the above definitions comes from [72] and [67] where the following theorem has been mentioned and proved:

**Theorem 4.1.2** *[67, 72] The stochastic system* (4.1.1) *is exactly controllable on* $[0,T]$ *if and only if one of the following conditions hold:*

1) *$Q_T$ is coercive;*

2) *$\gamma R(\gamma, -Q_T)$ converges to zero in uniform topology as $\gamma$ converges to zero;*

Consider the probabilistic type of the controllability operator (3.1.3) on $G_T(X)$ as below:

$$P_t h = \int_0^t e^{A(t-s)}BB^* e^{A^*(t-s)} E(h|\mathscr{F}_s^\eta) ds \tag{4.1.4}$$

**Theorem 4.1.3** *The stochastic system* (4.1.1) *is L-partially $E_T$-controllable if and only if $LP_T L^*$ is coercive.*

**Proof.** Proof can be found in [13]. In the case $L = I$, the well-known exact controllability is obtained and result is obvious. When $L = 0$, then $LP_T L^* = 0$ and consequently it is coercive and the system (4.1.1) is $LE_T$-controllable. ■

**Theorem 4.1.4** *[22] The stochastic system* (4.1.1) *is never L-partially $E_T$-controllable. Also the linear operator $LP_T L^*$ is never coercive unless $L = 0$.*

**Proof.** We will prove by contradiction. Let $L \neq 0$ and consider the contrary i.e. $LP_T L^*$ is coercive. According to the definition of coerciveness, $\exists c > 0$ such that $\forall h \in \mathscr{G}_T^\eta(H)$:

$$\langle LP_T L^* h, h\rangle_{\mathscr{G}_T^\eta(H)} \geqslant c\|h\|^2_{\mathscr{G}_T^\eta(H)}. \tag{4.1.5}$$

Consider the random variable $h \in \mathscr{G}_T^\eta(H)$ as below:

$$h = \int_0^T G_t d\bar{z}_t^\eta.$$

For a special choice of $G \in L_2(0, T; \mathscr{L}_2(\mathbb{R}^k, H))$, the random variable $h$ does not satisfy the inequality (4.1.5). Considering the above relations, we have:

$$LP_TL^*h = \int_0^T LQ_{T-t}L^*G_t d\bar{z}_t^{\eta},$$

resulting:

$$\begin{aligned}
\langle LP_TL^*h, h\rangle_{\mathscr{G}_T^{\eta}(H)} &= E\left\langle \int_0^T LQ_{T-t}L^*G_t d\bar{z}_t^{\eta},\ \int_0^T G_t d\bar{z}_t^{\eta} \right\rangle_H \\
&= \int_0^T \langle LQ_{T-t}L^*G_t,\ G_t\rangle_{\mathscr{L}_2(\mathbb{R}^k,H)} dt \\
&\leqslant \int_0^T \|Q_{T-t}\|_{\mathscr{L}(X)} \|L\|^2_{\mathscr{L}(X,H)} \|G_t\|^2_{\mathscr{L}_2(\mathbb{R}^k,H)} dt \\
&= \int_0^T \|Q_{T-t}\|_{\mathscr{L}(X)} \|G_t\|^2_{\mathscr{L}_2(\mathbb{R}^k,H)} dt.
\end{aligned}$$

Taking into account equation (3.1.3), one can see that as $t$ approaches to zero, $\|Q_t\|_{\mathscr{L}(H)}$ also approaches to zero. Therefore:

$$\exists \delta > 0 \text{ such that } \forall T-\delta < t \leqslant T\ , \|Q_{T-t}\|_{\mathscr{L}(X)} < \frac{c}{2}. \tag{4.1.6}$$

Now construct $G$ as below:

$$\|G_t\|_{\mathscr{L}_2(\mathbb{R}^k,H)} = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant T-\delta, \\ 1 & \text{if } T-\delta < t \leqslant T \end{cases}$$

then

$$\begin{aligned}
\|h\|^2_{\mathscr{G}_T^{\eta}(H)} &= E\left\| \int_{T-\delta}^T G_t d\bar{z}_t \right\|_H^2 \\
&= \int_{T-\delta}^T \|G_t\|^2_{\mathscr{L}_2(\mathbb{R}^k,H)} dt = \delta.
\end{aligned}$$

Considering equation (4.1.6) and the above equalities we get:

$$\begin{aligned}
\langle LP_TL^*h, h\rangle_{\mathscr{G}_T^{\eta}(H)} &\leqslant \int_{T-\delta}^T \|Q_{T-t}\|_{\mathscr{L}(X)} dt \\
&\leqslant \frac{c\delta}{2} < c\delta = c\|h\|^2_{\mathscr{G}_T^{\eta}(H)}.
\end{aligned}$$

which is in contradiction to (4.1.5). Therefore proof is completed. ■

There has been lots of works done on partial exact controllability of stochastic systems, but as stated in Theorem 4.1.4, stochastic systems are not $LE_T$-controllable unless $L = 0$. Therefore, defining $LE_T$-controllability for them is meaningless. The aim of bringing the definitions and theorems, is just to decline the works that have been done before on $LE_T$-controllability of stochastic systems. Theorem 4.1.4, also states that $LE_T$-controllability for nonlinear systems is also useless, since sufficient conditions for $LE_T$-controllability of nonlinear systems requires the $LE_T$-controllability of their linear part.

### 4.1.2 Approximate Controllability

Considering equation (4.1.3), a definition of approximate controllability is provided as below:

**Definition 4.1.5** $\forall \eta \in X_0$ *and for the time $T$, the stochastic system* (4.1.1) *is called*

*i) approximately controllable, if* $\overline{X_T^\eta} = \mathscr{G}_T^\eta(X)$;

*ii) L-partially approximate controllable, if* $\overline{L(X_T^\eta)} = \mathscr{G}_T^\eta(H)$.

Similar to Theorem 4.1.2 we have the following theorem for approximate controllability of the stochastic linear control system.

**Theorem 4.1.6** *The stochastic system* (4.1.1) *is approximately controllable on* $[0,T]$ *if and only if one of the following conditions hold:*

1) $Q_T > 0$;

2) $\gamma R(\gamma, -Q_T)$ *converges to zero in strong topology as* $\gamma$ *converges to zero;*

**Proof.** Proof can be found in [36, 66, 67, 72]. ■

And for partial approximate controllability the following theorem holds.

**Theorem 4.1.7** *[22] The following statements are equivalent:*

*(i) The control system* (4.1.1) *is L-partially $A_T$-controllable.*

*(ii)* $LQ_tL^* > 0 \quad \forall 0 < t \leqslant T$ .

*(iii)* $\gamma R(\gamma, -LQ_tL^*) \overset{\gamma \to 0^+}{\longrightarrow} 0 \;\; \forall 0 < t \leqslant T$ *in strong operator topology.*

### 4.1.3 *S*- and *C*-controllability

According to the previous sections and chapters, we concluded that stochastic systems are not $L$-partially exact controllable; whereas this fact motivates the researchers to introduce an analogue form for exact controllability which holds for linear stochastic systems, known as $C$- and $S$-controllability. The partial versions will also be discussed in this section. For more details one can refer to [11, 12, 16, 17, 18, 19, 20, 88].

In order to define the $S$- and $C$-controllability concepts, we need to introduce the sets below:

$$C_T^\eta = \bigcap_{\varepsilon>0,\; 0\leqslant p<1} C_{T,\varepsilon,p}^\eta \text{ and } S_T^\eta = \bigcap_{\varepsilon>0,\; 0\leqslant p<1} S_{T,\varepsilon,p}^\eta$$

where

$$\begin{aligned} C_{T,\varepsilon,p}^\eta = \{h \in H | \exists u \in U_{ad} \text{ such that } x_0^{\eta,u} = \eta, \\ \mathbf{P}(\|L\hat{x}_T^{\eta,u} - h\|^2 > \varepsilon) \leqslant 1 - p \text{ and } h = \mathbf{E}Lx_T^{\eta,u}\} \end{aligned}$$

and

$$S^{\eta}_{T,\varepsilon,p} = \{h \in H | \exists u \in U_{ad} \text{ such that } x_0^{\eta,u} = \eta,$$

$$\text{and } \mathbf{P}(\|L\hat{x}_T^{\eta,u} - h\|^2 > \varepsilon) \leqslant 1 - p\}.$$

Note that all the notation mentioned above are the same as defined at the beginning of the chapter.

**Definition 4.1.8** *For a time $T$, the stochastic system* (4.1.1) *is said to be*

*(i) L-partially C-controllable and written as $LC_T$-controllable if $\forall \eta \in X_0,\ C_T^{\eta} = H$;*

*(ii) C-controllable if for $H = X$ and $L = I$it is L-partially $C_T$-controllable;*

*(iii) L-partially S-controllable and written as $LS_T$-controllable if $\forall \eta \in X_0,\ S_T^{\eta} = H$;*

*(iv) S-controllable if for $H = X$ and $L = I$ it is $LS_T$-controllable.*

The following theorem provides a relation between $C_T$-, $S_T$-, $E_T$-, $A_T$-controllability of stochastic systems and their partial versions.

**Theorem 4.1.9** *[22] For $\forall t \in (0,T]$, the following relations hold for the system* (4.1.1)*:*

*(i) $Q_t$ is coercive if and only if the system is $C_T$-controllable;*

*(ii) $LQ_tL^*$ is coercive if and only if the system is L-partially $C_T$-controllable;*

*(iii) $Q_t > 0$ if and only if the system is $S_T$-controllable;*

*(iv) $LQ_tL^* > 0$ if and only if the system is $LS_T$-controllable.*

As it can be understood from the above theorem, the concepts of $LA_T$- and $LS_T$-controllability for partially observable linear systems are equivalent. As Theorem 4.1.4 states, $LE_T$-controllability concept fails for stochastic systems, but comparing the above theorem with Theorems 3.2.2 and 4.1.2, it can be seen that $LC_T$-controllability is an analogue for exact controllability of stochastic systems. The following two lemmas describe $LS_T$- and $LC_T$-controllability for stochastic systems.

**Lemma 4.1.10** *[22] For $h \in H$ and $\forall \eta \in X_0$, a stochastic control system (linear or nonlinear) is $LS_T$-controllable if and only if $\exists \{u^n\} \in U_{ad}$ so that one of the conditions below is satisfied:*

*(i) The sequence $\{L\hat{x}_T^{\eta,u^n}\} \to h$ in probability.*

*(ii) The sequence $\{L\hat{x}_T^{\eta,u^n}\} \to h$ almost sure.*

*(iii) The sequence $\{L\hat{x}_T^{\eta,u^n}\} \to h$ in distribution.*

**Proof.** According to the relations between different types of convergences, it suffices to prove one of the implications.

In order to prove the necessity of (i), suppose that the given stochastic system is $L$-partially $S_T$-controllable. Then by definition 4.1.8, $\forall \eta \in X_0,\ S_T^\eta = H$. For an arbitrary $\varepsilon > 0,\ \eta \in X_0$ and $h \in H$, consider a sequence $\{p_n\}$ where $p_n \to 1$. So, $\forall n = 1, 2, \cdots h \in S^\eta_{T,\varepsilon^2,p_n}$. Therefore by the definition of $S^\eta_{T,\varepsilon,p}$, there exists a control $u^n \in U_{ad}$ such that

$$\mathbf{P}(\|L\hat{x}_T^{\eta,u^n} - h\| > \varepsilon) < 1 - p_n \overset{n\to\infty}{\longrightarrow} 0,$$

which by the definition of convergence in probability, means that $L\hat{x}_T^{\eta,u^n}$ converges to $h$.

To prove the sufficiency of (i), assume that $L\hat{x}_T^{\eta,u^n}$ converges in probability to $h$. In order for the system to be $LS_T$-controllable we need to show that $S_T^\eta = H$. It is clear that $S_T^\eta \subseteq H$. To prove $H \subseteq S_T^\eta$, we must show that $H \subseteq S^\eta_{T,\varepsilon,p}$ for a fix $\varepsilon > 0$ and $0 \leqslant p < 1$. Therefore for an arbitrary $h \in H$, according to the assumption $L\hat{x}_T^{\eta,u^n}$ converges to $h$ in probability and there exists a sequence $\{u^n\} \in U_{ad}$ we have:

$$\mathbf{P}(\|L\hat{x}_T^{\eta,u^n} - h\| > \sqrt{\varepsilon}) \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, a sufficiently large $n$ exists such that

$$\mathbf{P}(\|L\hat{x}_T^{\eta,u^n} - h\|^2 > \varepsilon) < 1 - p,$$

which implies that $h \in S_{T,\varepsilon,p}^{\eta}$ and so $h \in S_T^{\eta}$. Hence $H = S_T^{\eta}$ and the system is $LS_T$-controllable.

The equivalency of (i) $\Leftrightarrow$ (ii) is a straightforward result of the fact that convergence in probability, is a result of almost sure convergence abut the reverse holds only for some subsequence.

Finally the equivalence relation (i) $\Leftrightarrow$ (iii), follows from the fact that convergence in probability to the nonrandom variable implies convergence in distribution and vice versa. $\blacksquare$

Similar results hold for $L$-partially $C_T$-controllability of stochastic systems.

**Lemma 4.1.11** *[22] For $h \in H$ and $\forall \eta \in X_0$, a stochastic control system (linear or nonlinear) is L-partially $C_T$-controllable if and only if one of the conditions below are satisfied:*

*(i) $\exists \{u^n\} \in U_{ad}$ such that $\forall n$ $\boldsymbol{E}LX_T^{\eta,u^n} = h$ and the sequence $\{L\hat{x}_T^{\eta,u^n}\} \to h$ in probability.*

*(ii) $\exists \{u^n\} \in U_{ad}$ such that $\forall n$ $\boldsymbol{E}LX_T^{\eta,u^n} = h$ and the sequence $\{L\hat{x}_T^{\eta,u^n}\} \to h$ almost sure.*

*(iii) $\exists \{u^n\} \in U_{ad}$ such that $\forall n$ $\boldsymbol{E}LX_T^{\eta,u^n} = h$ and the sequence $\{L\hat{x}_T^{\eta,u^n}\} \to h$ in distribution.*

**Proof.** Proof is similar to the proof of the previous lemma. $\blacksquare$

## 4.2 Controllability of semilinear stochastic systems

Having discussed controllability types for linear systems, we come up with controllability of semilinear systems. The definitions of approximate and exact controllability for linear systems, still hold for semilinear systems; but $S$- and $C$-controllability concepts are not satisfied for such systems. In this thesis we will go through only $LS_T$-controllability for such systems. There has been no research on $LC_T$-controllability for semilinear stochastic systems. As mentioned before, for the linear system (4.1.1), $LA_T$- and $LS_T$-controllability concepts coincide but they differ for semilinear systems.

As a result of previous discussions, $LA_T$-controllability results the $LS_T$-controllability for two reasons:

(1) The $L_2$-convergence results the convergence in probability and considering Lemma 4.1.10, $LS_T$-controllability can be written with respect to convergence in probability.

(2) In $LS_T$-controllability the convergence is required only to constant random variables, while $LA_T$-controllability requires the random variables to be as close as possible to every square integrable random variable.

According to the above reasons, we conclude that the sufficient conditions for $LA_T$- and $LS_T$-controllability are the same. There has been a number of researches done on sufficient conditions for $A_T$-controllability of semilinear stochastic systems with complete observations which are applicable to $LA_T$-controllability as well. Thus, the conditions are also applicable for $LS_T$-controllability. For partially observable stochastic systems, the study of controllability of semilinear systems is difficult for two main reasons: First, unlike the corresponding linear system, efficient filtering result does not exist whereas for linear systems Kalman filtering suits everything; Second, for completely observable semilinear stochastic systems, the method of fixed-point theorems can be applied, but for systems with partial observations, the fixed-point theorems can not be used any longer. Therefore, it's best to have an alternative method. This alternative method has been introduced and discussed in [22] where a controllability concept very similar to $LS_T$-controllability has been defined. The definitions and results mentioned in [22] are provided in this section.

**Definition 4.2.1** *[22] For $h \in H$ and $\forall \eta \in X_0$, and $\sigma \in (0,T)$, a stochastic partially observable control system is said to be:*

*(i) $LC_T^*$-controllable, if $\exists \{u^n\} \in U_{ad}$, so that $\forall n$, $\boldsymbol{E}Lx_T^{\eta,u^n} = h$ and*

$$\|\boldsymbol{E}(Lx^{\eta,u^n}|\mathscr{F}_{T-\sigma}^{\eta,u^n}) - h\| \overset{n-\infty}{\longrightarrow} 0 \ in \ \ probability;$$

*(ii) $C_T^*$-controllable, if for $L = I$ and $X = H$, it is $LC_T^*$-controllable;*

*(iii) $LS_T^*$-controllable, if $\exists \{u^n\} \in U_{ad}$, so that*

$$\|\boldsymbol{E}(Lx^{\eta,u^n}|\mathscr{F}_{T-\sigma}^{\eta,u^n}) - h\| \overset{n-\infty}{\longrightarrow} 0 \ in \ \ probability;$$

*(iv) $S_T^*$-controllable, if for $L = I$ and $X = H$, it is $LS_T^*$-controllable;*

Recalling Lemmas 4.1.10 and 4.1.11, it's clear that the above definitions are a weakened form of $LC_T$- and $LS_T$-controllability.

Consider the general form of a semilinear partially observable stochastic control system on the interval $[0,T]$ as below:

$$\begin{cases} dx_t = (Ax_t + Bu_t + f(t,x_t,u_t))dt + g(t,x_t,u_t)dm_t, \\ x_0 = \eta \in X_0,\ u \in U_{ad}. \end{cases} \tag{4.2.1}$$

The following assumptions will be considered for the above system throughout this section:

(1) $X$ and $U$ are considered as separable Hilbert spaces, and $L$ is assumed as an operator projecting $X$ onto its closed subspace $H$.

(2) For $t \geq 0$, $A$ is the infinitesimal generator of the strongly continuous semi-group $e^{At}$ and $B : U \to X$ is a linear operator.

(3) For a separable Hilbert space $Y$, $m$ is a $Y$-valued standard Wiener process with $covm_t = Mt$ which generates the continuous and complete filtration $\{\mathscr{F}_t^m\}$. Also:

$$X_0 = \{\eta \in L_2(\Omega, X) \mid \eta \in \mathscr{F}_t^0\}.$$

(4) The nonlinear functions $f : [0,T] \times X \times U \to X$ and $g : [0,T] \times X \times U \to \mathscr{L}_M(Y,X)$ satisfy the following conditions:

(i) $f$ and $g$ are continuous functions satisfying the Lipschitz condition with respect to x over $[0,T] \times X \times U$;

(ii) $\exists \varepsilon \in [0,T)$ so that the function $f$ over $[T-\varepsilon, T] \times X \times U$ is bounded.

(5)The set of admissible controls is as follows:

$$U_{ad} = \{u \in L_2([0,T] \times \Omega, U) | u \text{ is } \mathscr{F}_t^{\eta,u}\text{-adapted}\}$$

where for $u \in U_{ad}$ and $\eta \in X_0$, $\{\mathscr{F}_t^{\eta,u}\}$ is the filtration generated by the observation process $z : [0,T] \times \Omega \to Z$ corresponding to .

(6) $\forall 0 < t \leqslant T, LQ_tL^* > 0$.

(7) $\forall 0 \leqslant t \leqslant T, \mathscr{F}_t^{\eta,u} \subseteq \mathscr{F}_t^m$.

Under the assumptions (1)-(5), there exists a unique mild $\mathscr{F}_t^m$-adapted solution for the system (4.2.1) as below:

$$
\begin{aligned}
x_t = e^{At}\eta + \int_0^t e^{A(t-r)}(Bu_r + f(r,x_r,u_r))dr \\
+ \int_0^t e^{A(t-r)} g(r,x_r,u_r)dm_r.
\end{aligned}
$$

Considering the above mentioned conditions (1)-(7), we insist to show that the partially observable system (4.2.1) is $LS_T^*$-controllable. For this we will use the method introduced in [21] for deterministic systems.

For this, consider the linear deterministic equation below corresponding to (4.2.1):

$$
\frac{dy}{dt} = Ay_t + b\nu_t, \;\; 0 < \varepsilon < T, \;\; t \in [T-\varepsilon, T] \tag{4.2.2}
$$

where $\nu \in C(T-\varepsilon, T; U)$ and $y_{T-\varepsilon} = \eta \in X$. The mild solution of (4.2.2) has the form below:

$$
y_t^{\eta,\nu} = e^{A(t-T+\varepsilon)}\eta + \int_{T-\varepsilon}^t e^{A(t-r)}B\nu_r dr, \;\; t \in [T-\varepsilon, T].
$$

**Lemma 4.2.2** *[20] For a positive $\gamma$, $h \in H$, $\eta \in X$ and $\varepsilon \in (0,T)$, consider the conditions (1) and (2). Then there exists a unique $\nu^{\gamma,\varepsilon} \in C(T-\varepsilon, T; U)$ at which the* *functional below takes its minimum value along the linear system in* (4.2.2) *:*

$$J^{\gamma,\varepsilon}(v) = \|Ly_T^{\eta,v} - h\|^2 + \gamma \int_{T-\varepsilon}^{T} \|v_t\|^2 dt. \tag{4.2.3}$$

*Further more,*

$$v_t^{\gamma,\varepsilon} = -\gamma^{-1} B^* e^{A^*(T-t)} L^* (Ly_T^{\eta,v\gamma,\varepsilon} - h) \ t \in [T-\varepsilon, T] \tag{4.2.4}$$

*and*

$$Ly_T^{\eta,v\gamma,\varepsilon} - h = \gamma R(\gamma, -LQ_\varepsilon L^*)(Le^{AT}\eta - h). \tag{4.2.5}$$

We also have the following useful lemma.

**Lemma 4.2.3** *Consider the assumptions (1), (2) and (6). Recalling the previous lemma, the following convergence is satisfied for* $\eta \in L_2(\Omega, X)$*,* $h \in H$ *and* $\forall \varepsilon \in (0,T)$*:*

$$\|Ly_T^{\eta,v\gamma,\varepsilon} - h\| \xrightarrow{\gamma \to 0^+} 0 \text{ almost sure}. \tag{4.2.6}$$

**Proof.** By assumption (6), $LQ_tL^* > 0$. Hence according to Theorem 4.1.7, $\gamma R(\gamma, -LQ_tL^*)$ is convergent to zero in strong operator topology. Therefore, by equation (4.2.5), the desired almost sure convergence in (4.2.6) is satisfied. ■

The following Lemma can be found in many researches on controllability together with its proof. Therefore a brief proof is provided below.

**Lemma 4.2.4** *For every positive* $\gamma$ *and* $\delta \in (0,T)$*, considering the assumptions (1) and (2), we have:*

$$\|\gamma R(\gamma, -LQ_\delta L^*)\|_{\mathscr{L}(H)} \leqslant 1.$$

**Proof.** Consider an arbitrary non-zero $h \in H$. Let:

$$\kappa = \gamma R(\gamma, -LQ_\delta L^*)h = \gamma(\gamma I + LQ_\delta L^*)^{-1}h$$

Since $h \neq 0$ and $\gamma > 0$ , therefore $\kappa \neq 0$, resulting $(\gamma I + LQ_\delta L^*)\kappa = \gamma h$. Which implies $\gamma\kappa + LQ_\delta L^*\kappa = \gamma h$ and so $h = \kappa + \gamma^{-1}LQ_\delta L^*\kappa$.

Then:

$$\begin{aligned}\|\kappa\|^2 &\leqslant \|\kappa\|^2 + \gamma^{-1}\langle LQ_\delta L^*\kappa, \kappa\rangle = \langle \kappa, \kappa + \gamma^{-1}LQ_\delta L^*\kappa\rangle \\ &= \langle \kappa, h\rangle \leqslant \|\kappa\|\|h\|.\end{aligned}$$

Implying $\|\kappa\| \leqslant \|h\|$. Hence:

$$\|\gamma R(\gamma, -LQ_\delta L^*)\| = \|\gamma(\gamma I + LQ_\delta L^*)^{-1}h\| = \|\kappa\| \leqslant |h\|.$$

This proves the lemma. ■

Now that we have mentioned all the required theorems and lemmas, we can prove the main theorem of this thesis showing $LS_T^*$-controllability of semilinear stochastic system (4.2.1).

**Theorem 4.2.5** *[22] The partially observable semilinear stochastic system* $(x,y)$ *with the state in* (4.2.1) *is* $LS_T^*$*-controllable under the conditions (1)-(7).*

**Proof.** For a positive $\gamma$ consider the arbitrary values $h \in H$, $\eta \in X_0$, $\varepsilon \in (0,T)$ and $\delta \in (0,\varepsilon)$. Assume the control $u^{\gamma,\delta}$ as below:

$$u_t^{\gamma,\delta} = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant T-\delta, \\ -B^*e^{A^*(T-t)}L^*R(\gamma, -LQ_\delta L^*)(Le^{AT}\hat{x}_{T-\delta}^{\eta,0} - h) & \text{if } T-\delta < t \leqslant T \end{cases} \tag{4.2.7}$$

Since $\hat{x}_{T-\delta}^{\eta,0}$ is $\mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\delta}}$-measurable, therefore $u^{\gamma,\delta} \in Uad$. Also $h$ is considered as a nonrandom element of $H$. So we can write the solutions for the systems (4.2.1) and (4.2.2) as follows:

$$\begin{aligned} x_T^{\eta,u^{\gamma,\delta}} &= e^{A\delta} x_{T-\delta}^{\eta,0} \\ &+ \int_{T-\delta}^{T} e^{A(T-r)} \big(Bu_r^{\gamma,\delta} + f(r, x_r^{\eta,u^{\gamma,\delta}}, u_r^{\gamma,\delta})\big) dr \\ &+ \int_{T-\delta}^{T} e^{A(T-r)} g(r, x_r^{\eta,u^{\gamma,\delta}}, u_r^{\gamma,\delta}) dm_r. \end{aligned}$$

And for the system (4.2.2) with the initial value $y_{T-\delta} = \hat{x}_{T-\delta}^{\eta,0}$ we have:

$$y_T^{\hat{x}_{T-\delta}^{\eta,0}, u^{\gamma,\delta}} = e^{A\delta} \hat{x}_{T-\delta}^{\eta,0} + \int_{T-\delta}^{T} e^{A(T-r)} Bu_r^{\gamma,\delta} dr.$$

Recalling equation (4.2.7), $u^{\gamma,\delta} \in U_{ad}$ and so $y_T^{\hat{x}_{T-\delta}^{\eta,0}, u^{\gamma,\delta}}$ is $\mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\delta}}$-measurable. Hence we can calculate the difference below:

$$\begin{aligned} x_T^{\eta,u^{\gamma,\delta}} - y_T^{\hat{x}_{T-\delta}^{\eta,0}, u^{\gamma,\delta}} &= e^{A\delta} \big(x_{T-\delta}^{\eta,0} - \hat{x}_{T-\delta}^{\eta,0}\big) \\ &+ \int_{T-\delta}^{T} e^{A(T-r)} f\big(r, x_r^{\eta,u^{\gamma,\delta}}\big) dr \\ &+ \int_{T-\delta}^{T} e^{A(T-r)} g\big(r, x_r^{\eta,u^{\gamma,\delta}}, u_r^{\gamma}, \delta) dm_r. \end{aligned}$$

Considering condition (7) and taking conditional expectation on both sides of the above equation with respect to the filtration $\mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\delta}}$, we gain:

$$E\big(x_T^{\eta,u^{\gamma,\delta}} | \mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\delta}}\big) - y_T^{\hat{x}_{T-\delta}^{\eta,0}, u^{\gamma,\delta}} = E\big(\int_{T-\delta}^{T} e^{A(T-r)} f(r, x_r^{\eta,u^{\gamma,\delta}}, u_r^{\gamma,\delta}) dr | \mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\delta}}\big).$$

Now considering the smaller $\sigma$-field $\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}$, we get:

$$E\big(x_T^{\eta,u^{\gamma,\delta}} - y_T^{\hat{x}_{T-\delta}^{\eta,0}, u^{\gamma,\delta}} | \mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big) = E\big(\int_{T-\delta}^{T} e^{A(T-r)} f(r, x_r^{\eta,u^{\gamma,\delta}}, u_r^{\gamma,\delta}) dr | \mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\varepsilon}}\big).$$

Assuming $\alpha = sup_{[0,T]} \|e^{At}\|$ and $\beta = sup_{[0,T]\times X\times U} \|f(t,x,u)\|$ and applying Jensen's inequality, we have: $\big\|E\big(Lx_T^{\eta,u^{\gamma,\delta}}-Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\|$

$$\leqslant E\Big(\int_{T-\delta}^{T}\|L\|\|e^{A(T-r)}\|\big\|f(r,x_r^{\eta,u^{\gamma,\delta}},u_r^{\gamma,\delta})\big\|dr|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\Big)\leqslant\alpha\beta\delta.$$

Which results the following:

$$\begin{aligned}\big\|E\big(Lx_T^{\eta,u^{\gamma,\delta}}-h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\| &\leqslant \big\|E\big(Lx_T^{\eta,u^{\gamma,\delta}}-Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\| \\ &\quad+\big\|E\big(Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\| \\ &\leqslant \alpha\beta\delta+\big\|E\big(Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\|\end{aligned}$$

Taking both sides of the above inequality to power two we obtain:

$$E\big\|E\big(Lx_T^{\eta,u^{\gamma,\delta}}-h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\|^2\leqslant 2\alpha^2\beta^2\delta^2+2E\big\|E\big(Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\|^2.$$

By orthogonal projection property of conditional expectation and taking into account the fact that $Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}$ is $\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}$-measurable we gain:

$$E\big\|E\big(Lx_T^{\eta,u^{\gamma,\delta}}-h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma,\delta}}\big)\big\|^2\leqslant 2\alpha^2\beta^2\delta^2+2E\big\|Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h\big\|^2. \tag{4.2.8}$$

Since by Lemma 4.2.3, $E\big\|Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h\big\|^2$ converges to zero almost sure, so taking into consideration Lemmas 4.2.2 and 4.2.4 we can write:

$$\begin{aligned}\big\|Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h\big\|^2 &= \big\|\gamma R(\gamma,-LQ_\delta L^*)\big(Le^{AT}\hat{x}_{T-\delta}^{\eta,0}-h\big)\big\|^2 \\ &\leqslant \big(\|e^{AT}\|\|\hat{x}_{T-\delta}^{\eta,0}\|+\|h\|\big)^2\end{aligned} \tag{4.2.9}$$

which implies the following convergence for $\forall\delta\in(0,T)$:

$$E\big\|Ly_T^{\hat{x}_{T-\delta}^{\eta,0},u^{\gamma,\delta}}-h\big\|^2\xrightarrow{\gamma\to 0^+}0.$$

Hence, for a fixed $\varepsilon \in (0,T)$ a $\delta_n \in (0,\varepsilon)$ can be chosen such that: $\alpha^2\beta^2\delta_n^2 < \frac{1}{4n}$ and so for this $\delta_n$ there exists a positive $\gamma_n$ so that the inequality below holds:

$$E\left\|Ly_T^{\hat{x}_{T-\delta_n}^{\eta,0},u^{\gamma,\delta_n}} - h\right\|^2 < \frac{1}{4n}.$$

Now considering (4.2.8), we come to the conclusion

$$E\left\|E\left(Lx_T^{\eta,u^{\gamma_n,\delta_n}} - h|\mathscr{F}_{T-\varepsilon}^{\eta,u^{\gamma_n,\delta_n}}\right)\right\|^2 \leqslant \frac{1}{n} \tag{4.2.10}$$

Taking limit on both sides of the above inequality when $n \to +\infty$, the convergence in Definition 4.2.1 part (iii), is satisfied for the sequence of controls $\tilde{u}^n = u^{\gamma_n,\delta_n}$ in mean square convergence sense. Since, mean square convergence implies convergence in probability, so according to Definition 4.2.1, the $LS_T^*$-controllability of the system (4.2.1) is obtained which completes the proof. ■

It's best to note that if in the previous theorem, we choose $h$ randomly, then in equation (4.2.7), instead of $h$ we must write $E(h|\mathscr{F}_{T-\delta}^{\eta,u^{\gamma,\delta}})$ to make sure $u^{\gamma,\delta} \in U_{ad}$. In this case, in the equation (4.2.9), the estimation will be dependent on $\gamma$ and so the desired convergence will not be satisfied. Thus, in order to achieve $LS_T^*$-controllability, choosing a nonrandom $h$ is required.

The concepts of partial controllability for semilinear systems have been studied in [14] considering delay equations and higher order differential equations. At the end of this chapter some examples will be provided based on partially observable semilinear stochastic systems disturbed by coloured, wide band and shifted noises. For this, a brief definition of the mentioned noises are given first.

**Definition 4.2.6** *Consider a Wiener process $w$ and its derivative $w'$ which is a gener-*

*alized Gaussian random process with $E(w') = 0$ and $cov(w'_r, w'_s) = \delta(r-s)$, where $\delta$ is the Dirac delta function defined in [45, 57]. To give a short definition of the Dirac delta function, one can consider it as the generalized density function of the probability distribution $\boldsymbol{P}_\delta$ concentrated at the origin, on $\mathbb{R}$, that is:*

$$\boldsymbol{P}_\delta(\{0\}) = 1 \text{ and } \boldsymbol{P}_\delta(\mathbb{R} \setminus \{0\}) = 0.$$

*The generalized derivative $w'$ of a Wiener process is named Gaussian white noise process or shortly white noise.*

*Often, the representation below is used for a Wiener process $w$ in engineering:*

$$w_t = \int_0^t w'_r dr.$$

**Definition 4.2.7** *[12, 41] An $X$-valued random process $\tau : [0,T] \to X$ is called a wide band noise, if there exists a positive $\varepsilon$ so that for the nonzero autocovariance function $\Gamma$ we have:*

$$cov(\tau_r, \tau_s) = \begin{cases} \Gamma_{r,s}, & 0 \leqslant r-s < \varepsilon \\ 0, & r-s \geqslant \varepsilon. \end{cases} \tag{4.2.11}$$

*In other words, a wide band noise process is a random process which has a nonzero autocovariance function within a small time interval and zero outside that interval. Furthermore, if $E\tau_r = 0$ and $\Gamma_{r,s} = \Gamma_{r-s}$, then $\tau$ is called stationary in wide sense.*

To compare white and wide band noise processes, one can easily compute that, for a standard Wiener process $w$ and a random process

$$\tau_r^\varepsilon = \frac{w_{r+\varepsilon} - w_r}{\varepsilon}, \ 0 \leqslant r \leqslant T, \ \varepsilon > 0$$

the following results hold:

$$E\tau_r^\varepsilon = 0$$

and

$$\Gamma_{r,s} = cov(\tau_r^\varepsilon, \tau_s^\varepsilon) = \begin{cases} \frac{\varepsilon - r + s}{\varepsilon^2}, & 0 \leqslant r - s < \varepsilon \\ 0, & r - s \geqslant \varepsilon. \end{cases}$$

Therefore, the random process $\tau^\varepsilon$, which is an approximation to the white noise process $w'$ is a stationary in wide sense, wide band noise process. Hence, it can be realized that when $\varepsilon$ is sufficiently small which makes $\Gamma_{r,r}$ sufficiently large, the white noise process $w'$, is an ideal case of the wide band noise process $\tau^\varepsilon$.

The integral representation of the wide band noise random processes have been mentioned in many researches such as [10, 23, 24]. The representation used in this thesis will be as below for a $Y$-valued Wiener process with $0 < \varepsilon < T$, and $\varphi \in B_2([0,T] \times [-\varepsilon, 0], \mathscr{L}(Y,X))$ on $[0,T]$

$$\tau_r = \int_{max(0,r-\varepsilon)}^{r} \varphi_{r,\theta-r} dw_\theta, \; 0 \leqslant r \leqslant T \tag{4.2.12}$$

**Definition 4.2.8** *Another type of noise processes which are similar to wide band noise, but the estimation results for them are close to the results for white noise processes, are the so called coloured noise processes. An output of a linear system under a white noise disturbance is called a coloured noise. In other words, it is a solution of the linear stochastic differential equation below:*

$$d\psi_t = A\psi_t dt + \varphi_t dw_t, \quad \psi_0 = 0, \; 0 < t \leqslant T.$$

*The above differential equation has a mild solution in the form below:*

$$\psi_t = \int_0^t \mathscr{U}_{t,r}\varphi_r dw_r, \quad 0 \leqslant t \leqslant T, \tag{4.2.13}$$

*where* $\mathscr{U} = e^{At}$ *(strongly continuous semigroup generated by A).*

*Thus, an X-valued coloured noise is a random process* $\varphi$ *in the form* (4.2.13) *where* $\mathscr{U} \in \mathscr{E}(\Delta_T, \mathscr{L}(X))\Big($*i.e. the class of all mild evolution operators from* $\Delta_T$ *to* $\mathscr{L}(X)\Big)$*,* $\varphi \in B_\infty([0,T], \mathscr{L}(Y,X))$ *and w is a Y-valued Wiener process on* $[0,T]$.

**Example 1.** Assume $\psi$ and $\varphi$ are coloured noises generated by the following equations.

$$\begin{cases} d\psi_t = A_1\psi_t dt + dm_t^1, \quad 0 < t \leqslant T, \quad \psi_0 = 0, \\ d\varphi_t = A_2\varphi_t dt + dm_t^2, \quad 0 < t \leqslant T, \quad \varphi_0 = 0 \end{cases} \tag{4.2.14}$$

Consider a semilinear system driven by a coloured and white noise as below

$$\begin{cases} dx_t = (Ax_t + Bu_t + f(t,x_t,\psi_t,u_t))dt + g(t,x_t,u_t)dm_t, \\ dz_t = h(t,x_t,\varphi_t,u_t)dt + l(t,x_t,u_t)dn_t, \end{cases} \tag{4.2.15}$$

on $0 < t \leqslant T$, where $x_0 = \xi$ and $z_0 = 0$.

In the above equations, $A, A_1$ and $A_2$ are infinitesimal generators of strongly continuous semi-groups, the operator $B$ is bounded and linear and $m, m^1, m^2$ and $n$ denote standard Wiener processes. We wish to check the $LS_T^*$-controllability of the system (4.2.15).

For this, let

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \ \tilde{m}_t = \begin{bmatrix} m_t \\ m_t^1 \\ m_t^2 \end{bmatrix}.$$

Also assume $\tilde{x}_t = \begin{bmatrix} x_t \\ \psi_t \\ \varphi_t \end{bmatrix}$ and $\tilde{\xi} = \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix}$.

Then the system (4.2.15) can be expressed by means of the enlarged state $\tilde{x}$ on $0 < t \leqslant T$ as below:

$$\begin{cases} d\tilde{x}_t = (\tilde{A}\tilde{x}_t + \tilde{B}u_t + \tilde{f}(t,\tilde{x}_t,u_t))dt + \tilde{g}(t,\tilde{x}_t,u_t)d\tilde{m}_t, \\ dz_t^u = \tilde{h}(t,\tilde{x}_t,u_t)dt + \tilde{l}(t,\tilde{x}_t,u_t)dn_t, \end{cases} \tag{4.2.16}$$

where $\tilde{x}_0^u = \tilde{x}_0$, $\xi_0^u = 0$ together with the operators below:

$$\tilde{f}(t,\tilde{x},u) = \begin{bmatrix} f(t,x,\psi,u) \\ 0 \\ 0 \end{bmatrix}, \ \tilde{g}(t,\tilde{x},u) = \begin{bmatrix} g(t,x,u) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$\tilde{h}(t,\tilde{x},u) = h(t,x,\varphi,u) \quad \text{and} \quad \tilde{l}(t,\tilde{x},u) = l(t,x,u).$$

Now considering $L = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$ as the projection operator from the state space of system (4.2.16) to the state space of the system (4.2.15), it can be observed that the $S_T^*$-controllability for the system (4.2.15) driven by white and coloured noises is equivalent to the $LS_T^*$-controllability of the system (4.2.16) driven by just white noises. Hence, if all the conditions of Theorem 4.2.5 are satisfied (i.e. the continuous functions $f$ and $g$ satisfy the Lipschitz condition with respect to $x$ and $\psi$, and $\forall t \in (0,T]$, the controllability operator $Q_t$ is positive and $l$ and $h$ are continuous functions), then the $LS_T^*$-controllability of the system (4.2.16) is established which results the $S_T^*$-controllability of the system (4.2.15).

**Example 2.** As mentioned in [41], the noises appearing in engineering problems are mostly described by wide band noises. According to [23, 24], for a positive $\varepsilon$ and a positive $\delta$, two stationary wide band noises $\psi$ and $\varphi$ are represented as below:

$$\psi_t = \int_{max(0,t-\varepsilon)}^{t} \Psi_{\alpha-t} dm_{\alpha}^{1}, \quad \text{and} \quad \varphi_t = \int_{max(0,t-\delta)}^{t} \Phi_{\beta-t} dm_{\beta}^{2} \tag{4.2.17}$$

for which $m^1$ and $m^2$ are Wiener processes and the operator valued functions $\Phi$ and $\Psi$ are defined over the intervals $[-\delta,0]$ and $[-\varepsilon,0]$ respectively.

Consider the assumptions of Example 1 together with differentiability of $\Phi$ and $\Psi$ where $\Phi_{-\delta} = 0$ and $\Psi_{-\varepsilon} = 0$. Recalling the system (4.2.15), and supposing that the state spaces of $\varphi$ and $\psi$ belong to the Hilbert spaces $G$ and $F$, respectively, define the following random processes:

$$\begin{cases} \tilde{\psi} : [0,T] \times \Omega \to L_2(-\varepsilon,0;F), \\ d\psi_t = (-\frac{d}{d\alpha})\tilde{\psi}_t + \Psi^1 dm_t^1, \quad 0 < t \leqslant T, \quad \tilde{\psi}_0 = 0, \end{cases}$$

and

$$\begin{cases} \tilde{\varphi} : [0,T] \times \Omega \to L_2(-\delta,0;G), \\ d\varphi_t = (-\frac{d}{d\beta})\tilde{\varphi}_t + \Phi^1 dm_t^2, \quad 0 < t \leqslant T, \quad \tilde{\varphi}_0 = 0, \end{cases}$$

Then for the following linear and bounded operators

$$\Gamma_1 : L_2(-\varepsilon,0;F) \to F \quad \text{and} \quad \Gamma_2 : L_2(-\delta,0;G) \to G$$

we can define:

$$\psi_t = \Gamma_1 \tilde{\psi}_t = \int_{-\varepsilon}^{0} \tilde{\psi}_t d\alpha \quad \text{and} \quad \varphi_t = \Gamma_2 \tilde{\varphi}_t = \int_{-\delta}^{0} \tilde{\varphi}_t d\beta.$$

Let

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & -\frac{d}{d\alpha} & 0 \\ 0 & 0 & -\frac{d}{d\beta} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \ \tilde{m}_t = \begin{bmatrix} m_t \\ m_t^1 \\ m_t^2 \end{bmatrix}.$$

Denote

$$\tilde{x}_t = \begin{bmatrix} x_t \\ \tilde{\psi}_t \\ \tilde{\varphi}_t \end{bmatrix}, \ \tilde{\xi} = \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix}.$$

Then considering the matrix representations below, the system (4.2.15) can be rewritten in the form of system (4.2.16), by means of the enlarged state $\tilde{x}$;

$$\tilde{f}(t,(x,\tilde{\psi},\tilde{\varphi}),u) = \begin{bmatrix} f(t,x,\Gamma_1\tilde{\psi},u) \\ 0 \\ 0 \end{bmatrix}, \tilde{g}(t,(x,\tilde{\psi},\tilde{\varphi}),u) = \begin{bmatrix} g(t,x,u) & 0 & 0 \\ 0 & \Psi^1 & 0 \\ 0 & 0 & \Phi^1 \end{bmatrix},$$

$$\tilde{h}(t,(x,\tilde{\psi},\tilde{\varphi}),u) = h(t,x,\Gamma_2\tilde{\varphi},u) \quad \text{and} \quad \tilde{l}(t,(x,\tilde{\psi},\tilde{\varphi}),u) = l(t,x,u).$$

Similar to the previous example, considering $L$ as the projection operator from the state space of system (4.2.16) to the state space of system (4.2.15), the $LS_T^*$-controllability of the system driven by white noises coincides with the $S_T^*$-controllability of the system driven by wide band and white noises. Therefore, if the circumstances of Theorem 4.2.5 are satisfied, then the $S_T^*$-controllability of the system (4.2.15) is resulted, since the system (4.2.16) is $LS_T^*$-controllable.

The following example provides a system with a shifted white noise. According to [12], in applied point of view, in tracking of satellites, the pointwise shift of the state noise appears.

**Example 3.** [22] Consider a semilinear stochastic system on $(0,T]$, where a white noise, disturbing the state, is a pointwise delay of a white noise disturbing the observations.

$$\begin{cases} dx_t = (Ax_t + Bu_t + f(t,x_t,u_t))dt + dm_{t-\delta}, \\ dz_t = h(t,x_t,u_t)dt + dm_t, \end{cases} \tag{4.2.18}$$

in which $x_0 = \xi$ and $z_0 = 0$.

Suppose the assumptions of Theorem 4.2.5 for $A$, $B$, $m$ and $f$ are satisfied. Also let $h$ be a continuous function and $\delta \in (0,T)$. Considering the state space $X$ of the system (4.2.18), define the state $\tilde{x}$ with the state space $X \times L_2(-\delta,0;X)$ by the system below:

$$\begin{cases} d\tilde{x}_t = (\tilde{A}\tilde{x}_t + \tilde{B}u_t + \tilde{f}(t,\tilde{x}_t,u_t))dt + \tilde{I}dm_t, \quad 0 < t \leqslant T,\ \tilde{x}_0 = \xi, \\ dz_t = \tilde{h}(t,\tilde{x}_t,u_t)dt + dm_t, \quad 0 < t \leqslant T,\ z_0 = 0. \end{cases} \tag{4.2.19}$$

in which:

$$\tilde{A} = \begin{bmatrix} A & \Delta \\ 0 & \frac{d}{d\alpha} \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \tilde{I} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

and $\tilde{h}(t,\tilde{x},u) = h(t,L\tilde{x},u)$ where $L : X \times L_2(-\delta,0;X) \to X$ is a projection operator and for every $h \in [-\delta,0]$, $\Delta h = h_{-\delta}$(for a function $h$ assigns its value at $-\delta$).

One can calculate the strongly continuous semigroup generated by $\tilde{A}$ as below:

$$e^{\tilde{A}t} = \begin{bmatrix} e^{At} & \rho_t \\ 0 & \mathscr{T}_t \end{bmatrix}$$

where for $h \in L_2(-\delta,0;X)$, $\tau$ is a semigroup of right translation, defined as follows:

$$[\mathscr{T}_t h](\alpha) = \begin{cases} h_{\alpha - t}, & \alpha - t \geqslant -\delta, \\ 0, & \alpha - t < -\delta, \end{cases}$$

and

$$\rho_t h = \int_{-min(\delta,t)}^{0} e^{A(t+s)} h_s ds.$$

For the state processes of the systems (4.2.18) and (4.2.19), $x$ and $\tilde{x}$ respectively, it's clear that $x_t = L\tilde{x}_t$ and therefore, as in the previous examples, the $S_T^*$-controllability of the system (4.2.18) driven by shifted white noise and the $LS_T^*$-controllability of the system (4.2.19) driven by correlated white noises coincide. Hence, under certain conditions, the system (4.2.18) is $S_T^*$-controllable.

# Chapter 5

## CONCLUSION

To summarize, four main ideas are accomplished in this thesis: (1) For the partial approximate controllability of a given semilinear system, a sufficient condition is provided as in Theorem 3.4.5, which makes it possible to approximately control one or several components of the state space of the given system while the total of the state space is not approximately controllable. The given sufficient condition is suitable for the systems which by increasing the dimension of the state space, can be rewritten in the form of a first order differential equation. Stochastic systems driven by wide band noises are another kind of systems which satisfy the condition. (2) An alternative method is introduced which is very useful for the study of approximate controllability concepts specially for semilinear stochastic systems. Comparing this new method with the traditional method by fixed point theorems, it's less complicated and more applicable. In the alternative method, the linear and nonlinear parts are separated while in the method by fixed point theorems they are not. (3) For semilinear systems of stochastic type, with partial observations, an alternative controllability concept for $S$-controllability is given named as the $S^*$-controllability which is weaker than $S$-controllability. (4) The $S^*$-controllability for partially observable systems is defined and a sufficient condition for stochastic semilinear systems, driven by white noises is mentioned. By applying the result to systems driven by other types of noise processes, it has been recognized that the result does not depend to the nature of the disturbing noise process.

# REFERENCES


[1] Balachandran, K., & Dauer, J. (2002). Controllability of Nonlinear Systems in Banach Spaces: A Survey. *Journal of Optimization Theory and Applications*. 115, 7–28.

[2] Balachandran, K., & Dauer, J. (2013). Controllability of Nonlinear Systems via Fixed-Point Theorems. *Journal of Optimization Theory and Applications*. 53, 345–352.

[3] Balachandran, K., & Dauer, J. (1987). Controllability of Perturbed Nonlinear Delay Systems. *IEEE Transactions on Automatic Control*. 32, 172–174.

[4] Balachandran, K., & Dauer, J. (1998). Local Controllability of Semilinear Evolution Systems in Banach Spaces. *Indian Journal of Pure and Applied Mathematics*. 29, 311-320.

[5] Balachandran, K., Dauer, J., & Sangeetha, S. (2003). Controllability of Nonlinear Evolution Delay Integrodifferential Systems. *Applied Mathematics and Computation*. 139, 63–84.

[6] Balachandran, K., & Sakthivel, R. (2001). Controllability of Integrodifferential Systems in Banach Spaces. *Applied Mathematics and Computation*. 118, 63–71.

[7] Balakrishnan, A. V. (1976). *Applied Functional Analysis*. Springer-Verlag.

[8] Barcenas, D., Leiva, H., & Sivoli, Z. (2005). A Broad Class of Evolution Equations are Approximately Controllable, But Never Exactly Controllable. *IMA Journal of Mathematical Control and Information*. 22, 310–320.

[9] Bashirov, A. E. (2014). *Mathematical Analysis Fundamentals*. Elsevier Insights.

[10] Bashirov, A. E. (1988). On Linear Filtering Under Dependent Wide Band Noises. *Stochastics*. 23, 413–437.

[11] Bashirov, A. E. (1996). On Weakening of the Controllability Concepts. *In proceedings of the 35th conference on decision and control*. Kobe, Japan: IEEE. 640–645.

[12] Bashirov, A. E. (2003). *Partially Observable Linear Systems Under Dependent Noises*. Systems and Control: Foundations and Applications. Birkhäuser, Basel.

[13] Bashirov, A. E., Etikan, H., & Semi, N. (2010). Partial Controllability of Stochastic Linear Systems. *International Journal of Control*. 83, 2564–2572.

[14] Bashirov, A. E. & Jneid, M. (2013). On Partial Complete Controllability of Semilinear Systems. *Abstract and Applied Analysis*.

[15] Bashirov, A. E. & Jneid, M. (2014). Partial Complete Controllability of Deterministic Semilinear Systems. *TWMS Journal of Applied and Engineering Mathematics*. 4, 216–225.

[16] Bashirov, A. E., & Kerimov, K. R. (1997). On Controllability Conception for Stochastic Systems. *SIAM Journal of Control and Optimization*. 35, 384–398.

[17] Bashirov, A. E., & Mahmudov, N.I. (1999). Controllability of Linear Deterministic and Stochastic Systems. *Proceedings of the 38th Conference on Decision and Control. Phoenix, Arizona, USA*. 3196–3201.

[18] Bashirov, A. E., & Mahmudov, N.I. (1999). On Concepts of Controllability for Deterministic and Stochastic Systems. *SIAM Journal of Control and Optimization*. 37, 1808–1821.

[19] Bashirov, A. E., & Mahmudov, N.I. (1999). Some New Results in Theory of Controllability. *In proceedings of the 7th Mediterranean Conference on Control and Automation*. Haifa, Israel. 323–343,

[20] Bashirov, A. E., Mahmudov, N.I., Semi, N., & Etikan, H. (2007). Partial Controllability Concepts. *International Journal of Control*. 80, 1–7.

[21] Bashirov, A. E., & Ghahramanlou, N. (2014). On Partial Approximate Controlla-

bility of Semilinear Systems. *Cogent Engineering. Systems and Control*. 1, 1–13.

[22] Bashirov, A. E., & Ghahramanlou, N. (2015). On Partial *S*-Controllability of Semilinear Partially Observable Systems. *International Journal of Control*. Vol.88, 5, 969–982.

[23] Bashirov, A. E., & Ugural, S. (2002). Analyzing Wide Band Noise Processes with Application to Control and Filtering. *IEEE Transactions on Automatic control*. 47, 323–327.

[24] Bashirov, A. E., & Ugural, S. (2002). Representation of Systems Disturbed by Wide Band Noises. *Applied Mathematics Letters*. 15, 607–613.

[25] Bensoussan, A. (1992). *Stochastic Control of Partially Observable Systems*. Cambridge University Press, London.

[26] Bensoussan, A., Da Prato, G., Delfour, M. C., & Mitter, S. K. (1993). *Representation and Control of Infinite Dimensional Systems*. Vol.2. Systems and Control: Foundations and Applications. Birkhäuser, Boston.

[27] Brockett, R. W. (1970). *Finite Dimensional Linear Systems*. Wiley.

[28] Chukwu, E. N. (1991). Nonlinear Delay Systems Controllability. *Journal of*

*Mathematical Analysis and Applications*. 162, 564–576.

[29] Curtain, R. F., & Pritchard, A. J. (1978). *An Infinite Dimensional Linear Systems Theory. Lecture Notes on Control and Information Sciences*. Vol.8. Springer-Verlag.

[30] Curtain, R. F., & Zwart, H. J. (1995). *An Introduction to Infinite Dimensional Linear Systems Theory*. Berlin, Springer-Verlag.

[31] Dacka, C. (1981). On Controllability of Nonlinear Systems with Time-Variable Delays. *IEEE Transactions on Automatic Control*. 26, 956–959.

[32] Dauer, J., & Balachandran, K. (1997). Sample Controllability of General Nonlinear Stochastic Systems. *Libertas Mathematica*. 17, 143–153.

[33] Dauer, J., & Mahmudov, N.I. (2002). Approximate Controllability of Semilinear Functional Equations in Hilbert Spaces. *J. Mth. anal. Appl.*. 273, 310–327.

[34] Dauer, J., & Mahmudov, N.I. (2004). Controllability of Some Nonlinear Systems in Hilbert Spaces. *Journal of Optimization Theory and Applications*, Vol.123. 2, 319–329.

[35] Dauer, J., & Mahmudov, N.I. (2004). Controllability of Stochastic Semilinear

Functional Differential equations in Hilbert Spaces. *Journal of Mathematical Analysis and Applications*. Vol.290, 2, 373–394.

[36] Denker, A. (2002). *Controllability Concepts for Stochastic Control Systems*. PhD Thesis, Eastern Mediterranean University.

[37] Do, V. N. (1989). A Note on Approximate Controllability of Semilinear Systems. *Systems Control Letters*. 12, 365–371.

[38] Dubov, M. A. & Mordukhovich, B. S. (1978). Theory of Controllability of Linear Stochastic Systems. *Differential Equations*. 14, 1609–1612.

[39] Enrhardt, M., & Kliemann, W. (1982). Controllability of Stochastic Linear Systems. *Systems and Control Letters*. 2, 145–153.

[40] Fattorini, H. O. (1967). Some Remarks on Complete Controllability. *SIAM Journal of control*. 4, 686–694.

[41] Fleming, W. H., & Rishel, R. W. (1975). *Deterministic and Stochastic Optimal Control*. New York. Springer-Verlag.

[42] Ganesh, R., Sakthivel, R. & Mahmudov, N. (2014). Approximate Controllability of Fractional Functional Equations with Infinite Delay. *Topological Methods in*

*Nonlinear Analysis*. Vol.43, 2, 345–364.

[43] Ganesh, R., Sakthivel, R. & Mahmudov, N. (2013). Approximate Controllability of Fractional Integro-Differential Evolution Equations. *Journal of Applied Mathematics*.

[44] Ganesh, R. Sakthivel, R., Yong Ren, Anthoni, S. M. & Mahmudov, N. I. (2013). Controllability of Neutral Fractional Functional Equations with Impulses and Infinite Delay. *Abstract and Applied Analysis*.

[45] Hida, T. (1980). *Brownian Motion*. Springer-Verlag.

[46] Hille, E. (1965). What is a Semigroup?, in Studies in Real and Complex Analysis. Vol.3. *Mathematical Association of America*. Prentice-Hall, Englewood Cliffs.

[47] Jneid, M. (2014). *Partial Complete Controllability of Semilinear Control Systems*. PhD Thesis, Eastern Mediterranean University.

[48] Kalman, RE. (1960). A New Approach to Linear Filtering and Prediction Problems. *Transactions ASME. Series D (Journal of Basic Engineering)*. 82(1), 35-45.

[49] Klamka, J. (2001). Constrained Controllability of Semilinear Systems. *Nonlinear Analysis: Theory, Methods and Applications*. 47, 2939–2949.

[50] Klamka, J. (2008). Constrained Controllability of Semilinear Systems with Delayed Controls. *Bulletin of the Polish Academy of Sciences, Technical Sciences*. 56, 333–337.

[51] Klamka, J. (2002). Constrained Exact Controllability of Semilinear Systems. *Semilinear and Control Letters*. 47, 139–147.

[52] Klamka, J. (1991). *Controllability of Dynamical Systems*. Dordrecht, Kluwer Academic.

[53] Klamka, J. (2000). Shauder's Fixed Point Theorem in Nonlinear Controllability Problems. *Control and Cybernetics*. 29, 153–165.

[54] Klamka, J., & Socha, L. (1977). Some Remarks about Stochastic Controllability for Delayed Linear Systems. *IEEE Transactions on Automatic Control*. 22, 880–881.

[55] Klamka, J., & Socha, L. (1980). Some Remarks about Stochastic Controllability for Delayed Linear Systems. *International Journal of control*. 32, 561–566.

[56] Kreyszig, E. (1978). *Introductory Functional Analysis with Applications*. University of Windsor, Canada.

[57] Krylov, N. V. (1995). Introduction to the Theory of Diffusion Processes. *Amer. Math. Soc.*, Providence.

[58] Leiva, H., Merentes, N., & Sanchez, J. L. (2013). A Characterization of Semilinear Dense Range Operators and Applications. *Abstract Analysis and Applications*. 1–11.

[59] Leiva, H., Merentes, N., & Sanchez, J. L. (2012). Approximate Controllability of Semilinear Reaction Diffusion. *Mathematical Control and Related Fields*. 2, 171–182.

[60] Leiva, H., Merentes, N., & Sanchez, J. L. (2011). Interior Controllability of the n-dimensional Semilinear Heat Equation. *African Diaspora Journal of Mathematics*. 12, 1–12.

[61] Li, X., & Yong, J. *Optimal Control for Infinite Dimensional Systems. Systems and Control: Foundations and Applications*. Birkhäuser, Boston.

[62] Mahmudov, N.I. (2013). Approximate Controllability of Fractional Neutral Evolution Equations in Banach Spaces. *Abstract and Applied Analysis*.

[63] Mahmudov, N.I. (2013). Approximate Controllability of Fractional Sobolev-Type Evolution Equations in Banach Spaces. *Abstract and Applied Analysis*.

[64] Mahmudov, N.I. (2003). Approximate Controllability of Semilinear Deterministic and Stochastic Evolution Systems in Abstract Spaces. *SIAM Journal of Control and Optimization*. 42, 1604–1622.

[65] Mahmudov, N.I. (2013). Approximate Controllability of Some Nonlinear Systems in Banach Spaces. *Boundary Value Problems*. 1–13.

[66] Mahmudov, N.I. (2001). Controllability of Linear Stochastic Systems. *IEEE Transactions on Automatic Control*. Vol. 46, 5, 724–731.

[67] Mahmudov, N.I. (2001). Controllability of Linear Stochastic Systems in Hilbert Spaces. *Journal of Mathematical Analysis and Applications*. 259, 64–82.

[68] Mahmudov, N.I. (2003). Controllability of Semilinear Stochastic Systems in Hilbert Spaces. *Journal of Mathematical Analysis and Applications*. Vol.288, 1, 197–211.

[69] Mahmudov, N.I. (2013). Exact Null Controllability of Semilinear Evolution Systems. *Journal of Global Optimization*. Vol.56, 2, 317–326.

[70] Mahmudov, N.I. (2014). Existence and Approximate Controllability of Sobolev Type Fractiona Stochastic Evolution Equations. *Bulletin of the Polish Academy of Sciences-Technical Sciences*. Vol.62, 2, 205–215.

[71] Mahmudov, N.I., Capar, U., & Ustunel, AS. (2003). Controllability and Observability of Linear Stochastic Systems in Hilbert Spaces. *Stochastic Analysis and Related topics Viii*. Vol.53, 151–167.

[72] Mahmudov, N.I., & Denker, A. (2000)On Controllability of Linear Stochastic Systems. *International Journal of Control*. 73, 152–158.

[73] Mahmudov, N.I., & McKibben, M. A. (2013). Control, Stability, and Qualitative Theory of Dynamical Systems. *Abstract and Applied Analysis*.

[74] Mahmudov, N. I., & Zorlu, S. (2013). Approximate Controllability of Fractional Integro-Differential Equations Involving Nonlocal Initial Conditions. *Boundary Value Problems*.

[75] Mahmudov, N. I., & Zorlu, S. (2003). Controllability of Nonlinear Stochastic Systems. *International Journal of Control*. 76, No.2, 95–104.

[76] Mahmudov, N. I., & Zorlu, S. (2005). Controllability of Semilinear Stochastic Systems. *International Journal of Control*. 78, No.13, 997–1004.

[77] Mahmudov, N. I., & Zorlu, S. (2014). On the Approximate Controllability of Fractional Evolution Equations with Compact Analytic Semigroup. *Journal of Computational and Applied Mathematics*. 259, 194–204.

[78] Naito, K. (1987). Controllability of Semilinear Control Systems Dominated by the Linear Part. *SIAM J. Control Optim.*. 25, 715–722.

[79] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Vol.44. Springer-Verlag, New York.

[80] Sakthivel, R., Kim, J. H. & Mahmudov, N. I. (2006). On Controllability of Nonlinear Stochastic Systems. *Reports on Mathematical Physics*. Vol.58, 3, 433–443.

[81] Sakthivel, R., & Mahmudov, N. I. (2009). Controllability of Non-linear Impulsive Stochastic Systems. *International Journal of Control*. Vol.82, 5, 801–807.

[82] Sakthivel, R., Mahmudov, N. I. & Nieto, J. J. (2012). Controllability for A Class of Fractional-Order Neutral Evolution Control Systems. *Applied Mathematics and Computation*. Vol.218, 20.

[83] Sakthivel, R., Mahmudov, N. I. & Ren Yong (2011). Approximate Controllability of the Nonlinear Third-Order Dispersion Equation. *Applied Mathematics and Computation*. Vol.217, 21.

[84] Sakthivel, R., Nieto, J. J. & Mahmudov, N. I. (2010). Approximate Controllability of Nonlinear Deterministic and Stochastic Systems with Unbounded Delay.

*Taiwanese Journal of Mathematics*. Vol.14, 5, 1777–1797.

[85] Sakthivel, R., Ren Yong & Mahmudov, N. I. (2011). On the Approximate Controllability of Semilinear Fractional Differential Systems. *Computers and Mathematics With Applications*. Vol.62, 3.

[86] Sakthivel, R., Ren Yong & Mahmudov, N. I. (2010). Approximate Controllability of Second-Order Stochastic Differential Equations with Impulsive Effects. *Modern Physics Letters B*. Vol.24, 14, 1559–1572.

[87] Sukavanam, N., & Kumar, M. (2010). S-Controllability of an Abstract First Order Semilinear Control System. *Numerical Functional Analysis and Optimization*. 31, 1023–1034.

[88] Sunahara, Y., Kabeuchi, T., Asada, S., Aihara, S., & Kishino, K. (1974). On Stochastic Controllability for Nonlinear Systems. *IEEE Transactions on Automatic Control*. 19, 49–54.

[89] Vidyasagar, M. (1972). A Controllability Condition of Nonlinear Systems. *IEEE Transactions on Automatic Control*. 17, 569–570.

[90] Yosida, K. (1980). *Functional Analysis*. Springer, Berlin.

[91] Zabczyk, J. (1981). Controllability of Stochastic Linear System. *Systems and Control Letters*. 1, 25–31.

[92] Zabczyk, J. (1995). *Mathematical Control Theory: An Introduction. Systems and Control: Foundation and Applications*. Berlin, Birkhäuser.

[93] Zhou, H. X. (1983). Approximate Controllability for a Class of Semilinear Abstract Equations. *SIAM J. Control Optm.*. 21, 551–565.

[94] Zorlu, S. (2003). *Controllability of Stochastic Systems*. PhD Thesis, Eastern Mediterranean University.