Generic Spherically Symmetric Thin-shells in General Relativity

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ABSTRACT

We give a full investigation and assessment on the general spherically symmetric time-like thin shells in general relativity. In this main stream, we give the details of the Israel junction conditions which are used for gluing two distinct space-times on a hyper-surfaces including the case of time-like shells. We also study the general stability of thin-shells against a radial perturbation. Our results are fully analytic in closed forms.

Keywords: Thin-shells, General relativity, Spherically symmetric, Stability, Israel junction conditions.

ÖZ

Küresel simetrik genel görelilikle zaman-benzer ince kabuklar üzerinde araştırma ve değerlendirme yaptık. Bu ana akımda İsrail sınır koşulları detaylarının göz önünde bulundurulduğunda bunlar iki farklı uzay-zaman benzeri kabuklar dahil olmak üzere yüzeylerin yapıştırılmasında kullanılır. Ayrıca ince kabukların radyal pertürbasyonlara karşı genel kararlılığını da inceledik. Sonuçlarımız kapalı formlarda tamamen analitiktir.

Ana kelimeler: İnce-kabukler, Genel görelilik, Küresel Simetrik, Kararlı, İsrail sınır koşulları.

DEDICATION

To my family

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TABLE OF CONTENTS

Chapter 1

INTRODUCTION

Thin-shells in general relativity are objects connecting different space-times through a very thin surface of most probably physical matters. Such kind of shells, depends on their four-normal direction, can be time-like, space-like or null. A surface with a space-like / time-like four-normal vector is time-like / space-like surface and with null four-vector is a null surface. Although the technical details of these different types of thin-shells are more or less the same our concentration will be on the time-like thin-shells only.

Furthermore, the thin-shell under our investigation has spherically symmetric whose inside and outside space-times are both spherical solutions of the Einstein equations. Our approach is a generic and detailed one which considers the most general spherically symmetric space-times for the inside and outside of the shell.

Thin-shells cannot exist without being matched with the two incomplete manifolds presented inside and outside. There are certain conditions / rules which have to satisfy having an acceptable thin-shell. Due to the same rules one has to consider certain form of matter source on the surface. The 2+1-dimensional thin-shell requires an energy-momentum tensor whose energy density and the angular pressure are related via the equation of states. Such kind of relations provide the necessary mathematical tools and techniques to assess not only the static thin-shells but also the dynamical behavior of the shells.

To that end one has to construct the thin-shell based on the metrics of the bulks in either side and then apply a radial perturbation to the equilibrium shell and study the post perturbation motion of the radius of the shell. Limiting the perturbation to be only radial, however, provides advantages in the motion of the shell after the perturbation. For instance, the equation of motion is of the type of a one-dimensional particle moving under a one-dimensional potential. This allows us to assess the thin-shell's motion without solving analytically the equation. The general aspect of the motion is dictated by the potential itself.

Chapter 2

ISRAEL JUNCTION CONDITION

The Israel junction conditions, applying to both null and non-null hypersurfaces, is a regularity condition for the existence of smooth Lorentzian manifolds. No discontinuous happens in the metric. This relates the induced metric and extrinsic curvature to changes in the stress-energy tensor across a hypersurface.

Suppose we consider a (2+1)-dimensional hypersurface Σ that can be either time-like, space-like or null in a (3+1)-dimensional space-time (metric g_{AB}). The 4-normal $\vec{n}$ to these surfaces satisfy $\vec{n}.\vec{n} = \mp 1$ which is pointing to the positive direction with respect to the bulk space-time. Throughout the thesis, we consider the time-like surface $\vec{n}.\vec{n} = +1$.

For the technical convenience, we introduce the Gaussian normal coordinates in the vicinity of the surface Σ. Gaussian normal coordinate system in 4-dimensional space-time in which a hypersurface swept by the spherical shell divides into two regions is introduced starting from a certain coordinate system $x_{\pm}^{A}$ with a metric $g_{AB}^{\pm}(x_{\pm}^{A})$. In particular, assume the continuity of the four-dimensional coordinates $x_{\pm}^{A}$ across Σ, then $g_{AB}^{-} = g_{AB}^{+}$ is required. The surface Σ is parametrized by coordinates $x^{i} = (\tau, x^{2}, x^{3})$, where i runs from 1 to 3. Here τ is the proper-time variable that would be measured by an observer co-moving with the shell.

Consider a neighbourhood of Σ with a system of geodesics orthogonal to Σ. The neighborhood is chosen so that the geodesics do not intersect; that is, any point in the neighborhood is located on one and only one geodesic. In the Gaussian normal coordinate system, a geodesic in a neighbourhood of Σ which is orthogonal to Σ is taken as the third spatial coordinate denoted by w.

The metric g_{AB} has the form

$$ds^2 = g_{AB}dx^A dx^B = \epsilon dw^2 + \gamma_{\mu\nu}dx^\mu dx^\nu, \tag{2.1}$$

where $\epsilon = \vec{n}.\vec{n} = +1$ for a time-like hypersurface, w is constant, and

$$\gamma_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu , \tag{2.2}$$

is the induced metric on Σ or the first fundamental form.

The extrinsic curvature $K_{\mu\nu}$ in these coordinates (of the surfaces in which w is a constant) is defined as

$$K_{\mu\nu} = -\frac{1}{2}\frac{\partial\gamma_{\mu\nu}}{\partial w}. \tag{2.3}$$

Gauss-Codazzi equations connect the metric tensor of the bulk and the surface via the extrinsic curvature tensor of the shell is given by

$$R_{w\mu w\nu} = \frac{\partial \mathrm{K}_{\mu\nu}}{\partial w} + K_{\rho\nu}K_\mu^\rho , \tag{2.4}$$

$$R_{w\mu\nu\rho} = \nabla_\nu K_{\mu\rho} - \nabla_\rho K_{\mu\nu} , \tag{2.5}$$

and

$$R_{\lambda\mu\nu\rho} = {}^3R_{\lambda\mu\nu\rho} + \epsilon\left(K_{\mu\nu}K_{\lambda\rho} - K_{\mu\rho}K_{\lambda\nu}\right), \tag{2.6}$$

where ∇_ρ is the covariant derivative with respect to the three-dimensional metric $\gamma_{\mu\nu}$, ${}^3R_{\lambda\mu\nu\rho}$ Riemann tensor on hypersurface. From Eqs. (2.4), (2.5) and (2.6) one finds

($R_{AB} = g^{CD} R_{CADB}$), (where R_{CADB} is the Riemann curvature tensor) and of the scalar curvature ($R = g^{AB} R_{AB}$).

We can define

$$R_{ww} = \gamma^{\mu\nu} R_{w\mu w\nu}, \tag{2.7}$$

and from (2.4) we find

$$R_{ww} = \gamma^{\mu\nu} \frac{\partial K_{\mu\nu}}{\partial w} + Tr(K^2). \tag{2.8}$$

From (2.5) we find

$$R_{w\mu} = \gamma^{\rho\nu} R_{\rho w\nu\mu} = -\gamma^{\rho\nu} R_{w\rho\nu\mu}, \tag{2.9}$$

therefore

$$R_{w\mu} = \nabla_\mu K - \nabla_\nu K_\mu^\nu. \tag{2.10}$$

From (2.4) and (2.6) we find

$$R_{\mu\rho} = \gamma^{\lambda\nu} R_{\lambda\mu\nu\rho} + g^{ww} R_{w\mu w\rho}, \tag{2.11}$$

therefore

$$R_{\mu\rho} = {}^3R_{\mu\rho} + \epsilon \left[2K_\mu^\lambda K_{\lambda\rho} - KK_{\mu\rho} + \frac{\partial K_{\mu\rho}}{\partial w} \right], \tag{2.12}$$

where ${}^3R_{\mu\rho}$ Ricci tensor on hypersurface.

From (2.8) and (2.12) we find

$$R = \gamma^{\mu\rho} R_{\mu\rho} + g^{ww} R_{ww}, \tag{2.13}$$

therefore

$$R = \gamma^{\mu\rho} \left({}^3R_{\mu\rho} + \epsilon \left[2K_\mu^\lambda K_{\lambda\rho} - KK_{\mu\rho} + \frac{\partial K_{\mu\rho}}{\partial w} \right] \right) + \underbrace{g^{ww}}_{=1} \left(\gamma^{\mu\nu} \frac{\partial K_{\mu\nu}}{\partial w} + Tr(K^2) \right), \tag{2.14}$$

thus

$$R = {}^3R + \epsilon\left[3Tr(K^2) - K^2 + 2\gamma^{\mu\rho}\frac{\partial K_{\mu\rho}}{\partial w}\right], \tag{2.15}$$

in which $K = K_{\mu}^{\mu}$ and $Tr(K^2) = K^{\lambda\rho}K_{\lambda\rho}$.

Finally, we obtain

$$G_A^B = R_A^B - \frac{1}{2}\delta_A^B R = \kappa T_A^B, \tag{2.16}$$

where R_A^B is the Ricci curvature tensor, R is the Ricci scalar, ($\kappa = 8\pi G$, G is the gravitational constant) and T_A^B is the stress-energy tensor.

The field equations (on the hypersurface) have mixed components

$$G^w_{\ w} = R^w_w - \frac{1}{2}\delta^w_w R, \tag{2.17}$$

from (2.8) and (2.15) we find

$$G^w_{\ w} = -\frac{1}{2}\ {}^3R + \frac{1}{2}\epsilon[K^2 - Tr(K^2)] = \kappa T^w_{\ w}. \tag{2.18}$$

From the Einstein tensor, we have

$$G^w_{\ \mu} = R^w_\mu - \frac{1}{2}\delta^w_\mu R, \tag{2.19}$$

where $R^w_\mu = g^{ww}R_{w\mu}$, so that for $w \neq \mu$

$$G^w_{\ \mu} = R^w_\mu = g^{ww}R_{w\mu}, \tag{2.20}$$

so that from (2.10) we find

$$G^w_{\ \mu} = \epsilon\left[\nabla_\mu K - \nabla_\nu K^\nu_\mu\right] = \kappa T^w_{\ \mu}\ . \tag{2.21}$$

Einstein tensor is given

$$G^\mu_{\ \nu} = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R, \tag{2.22}$$

where

$$R^\mu_\nu = \gamma^{\mu\rho}R_{\nu\rho}, \tag{2.23}$$

so that from (2.12) we find

$$R^{\mu}_{\nu} = \gamma^{\mu\rho}\left[\ {}^{3}R_{\nu\rho} + \epsilon\left(2K^{\lambda}_{\nu}K_{\lambda\rho} - KK_{\nu\rho} + \frac{\partial K_{\nu\rho}}{\partial w}\right)\right], \tag{2.24}$$

therefore

$$R^{\mu}_{\nu} = \ {}^{3}R^{\mu}_{\nu} + \epsilon\left[2K^{\lambda}_{\nu}K^{\mu}_{\lambda} - KK^{\mu}_{\nu} + \gamma^{\mu\rho}\frac{\partial K_{\nu\rho}}{\partial w}\right]. \tag{2.25}$$

We can write

$$\gamma^{\mu\rho}\frac{\partial K_{\nu\rho}}{\partial w} = \frac{\partial}{\partial w}\left(\gamma^{\mu\rho}K_{\nu\rho}\right) - K_{\nu\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w}\,, \tag{2.26}$$

therefore

$$\gamma^{\mu\rho}\frac{\partial K_{\nu\rho}}{\partial w} = \frac{\partial K^{\mu}_{\nu}}{\partial w} - K_{\nu\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w}\,. \tag{2.27}$$

Now, we will find $\frac{\partial \gamma^{\mu\rho}}{\partial w}$. From

$$\gamma_{\nu\rho}\gamma^{\mu\rho} = \delta^{\mu}_{\nu}, \tag{2.28}$$

derivative with respect to w

$$\frac{\partial}{\partial w}\left[\gamma_{\nu\rho}\gamma^{\mu\rho}\right] = 0, \tag{2.29}$$

gives

$$\frac{\partial \gamma_{\nu\rho}}{\partial w}\gamma^{\mu\rho} + \gamma_{\nu\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w} = 0, \tag{2.30}$$

thus

$$\gamma_{\nu\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w} = -\gamma^{\mu\rho}\frac{\partial \gamma_{\nu\rho}}{\partial w}\,, \tag{2.31}$$

from (2.3), we obtain

$$\gamma_{\nu\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w} = 2\gamma^{\mu\rho}K_{\nu\rho} = 2K^{\mu}_{\nu}\,. \tag{2.32}$$

Multiply by $\gamma^{\nu\lambda}$ to obtain

$$\delta^{\lambda}_{\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w} = 2K^{\mu\lambda}, \tag{2.33}$$

now, put $\lambda = \rho$ to obtain

$$\frac{\partial \gamma^{\mu\rho}}{\partial w} = 2K^{\mu\rho}. \tag{2.34}$$

Upon substitution of (2.34) into (2.27), yields

$$\gamma^{\mu\rho}\frac{\partial K_{\nu\rho}}{\partial w} = \frac{\partial K_{\nu}^{\mu}}{\partial w} - 2K_{\nu\rho}K^{\mu\rho}, \tag{2.35}$$

and with (2.26) we obtain

$$R_{\nu}^{\mu} = {}^{3}R_{\nu}^{\mu} + \epsilon\left[2K_{\nu}^{\lambda}K_{\lambda}^{\mu} - KK_{\nu}^{\mu} + \frac{\partial K_{\nu}^{\mu}}{\partial w} - 2K_{\nu\rho}K^{\mu\rho}\right]. \tag{2.36}$$

From (2.15), we have

$$\delta_{\nu}^{\mu}R = \delta_{\nu}^{\mu}\,{}^{3}R + \epsilon\left[3\delta_{\nu}^{\mu}Tr(K^2) - \delta_{\nu}^{\mu}K^2 + 2\delta_{\nu}^{\mu}\gamma^{\mu\rho}\frac{\partial K_{\mu\rho}}{\partial w}\right], \tag{2.37}$$

where

$$\gamma^{\mu\rho}\frac{\partial K_{\mu\rho}}{\partial w} = \frac{\partial}{\partial w}\left(\gamma^{\mu\rho}K_{\mu\rho}\right) - K_{\mu\rho}\frac{\partial \gamma^{\mu\rho}}{\partial w}, \tag{2.38}$$

where $\gamma^{\mu\rho}K_{\mu\rho} = \mathrm{K}$. Substitute now (2.34) into (2.38) to obtain

$$\gamma^{\mu\rho}\frac{\partial K_{\mu\rho}}{\partial w} = \frac{\partial K}{\partial w} - 2K_{\mu\rho}K^{\mu\rho}, \tag{2.39}$$

where $K_{\mu\rho}K^{\mu\rho} = Tr(K^2)$, therefore we have

$$\gamma^{\mu\rho}\frac{\partial K_{\mu\rho}}{\partial w} = \frac{\partial K}{\partial w} - 2Tr(K^2), \tag{2.40}$$

so that from (2.37), we obtain

$$\delta_{\nu}^{\mu}R = \delta_{\nu}^{\mu}\,{}^{3}R + \epsilon\left[3\delta_{\nu}^{\mu}Tr(K^2) - \delta_{\nu}^{\mu}K^2 + 2\delta_{\nu}^{\mu}\frac{\partial K}{\partial w} - 4\delta_{\nu}^{\mu}Tr(K^2)\right]. \tag{2.41}$$

Now, substitute (2.36) and (2.41) into (2.22) to obtain

$$G^{\mu}_{\ \nu} =$$

$$^{3}R^{\mu}_{\nu} + \epsilon\left[2K^{\lambda}_{\nu}K^{\mu}_{\lambda} - KK^{\mu}_{\nu} + \frac{\partial K^{\mu}_{\nu}}{\partial w} - 2K_{\nu\rho}K^{\mu\rho}\right] -$$

$$\frac{1}{2}\delta^{\mu}_{\nu}\ ^{3}R - \epsilon\left[\frac{3}{2}\delta^{\mu}_{\nu}Tr(K^{2}) - \frac{1}{2}\delta^{\mu}_{\nu}K^{2} + \delta^{\mu}_{\nu}\frac{\partial K}{\partial w} - 2\delta^{\mu}_{\nu}Tr(K^{2})\right], \tag{2.42}$$

where $2K_{\nu\rho}K^{\mu\rho} = 2g_{\rho\lambda}K^{\lambda}_{\mu}K^{\mu\rho} = 2K^{\lambda}_{\mu}K^{\mu}_{\lambda}$, so that we obtain

$$G^{\mu}_{\ \nu} = \ ^{3}R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R + \epsilon\left(\frac{\partial K^{\mu}_{\nu}}{\partial w} - \delta^{\mu}_{\nu}\frac{\partial K}{\partial w}\right) + \epsilon\left(\frac{1}{2}\delta^{\mu}_{\nu}Tr(K^{2}) + \frac{1}{2}\delta^{\mu}_{\nu}K^{2} - KK^{\mu}_{\nu}\right) = \kappa T^{\mu}_{\ \nu}, \tag{2.43}$$

where $^{3}R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = \ ^{3}G^{\mu}_{\nu}$, and as a result

$$G^{\mu}_{\ \nu} = \ ^{3}G^{\mu}_{\nu} + \epsilon\left(\frac{\partial K^{\mu}_{\nu}}{\partial w} - \delta^{\mu}_{\nu}\frac{\partial K}{\partial w}\right) + \epsilon\left(\frac{1}{2}\delta^{\mu}_{\nu}Tr(K^{2}) + \frac{1}{2}\delta^{\mu}_{\nu}K^{2} - KK^{\mu}_{\nu}\right) = \kappa T^{\mu}_{\ \nu}\,. \tag{2.44}$$

Here the energy-momentum tensor is given by

$$T^{\mu}_{\nu} = T^{\mu\ -}_{\nu}\Theta(-w) + T^{\mu\ +}_{\nu}\Theta(w) + \delta(w)S^{\mu}_{\nu}\ , \tag{2.45}$$

with S^{μ}_{ν} being the energy-momentum tensor on the shell, and $T^{\mu\mp}_{\nu}$ are the energy-momentum tensors on both sides in the bulk.

If S^{A}_{B} involves a δ-function on Σ, we find

$$S^{A}_{\ B} = \lim_{\eta\to 0}\left(\int_{-\eta}^{\eta} T^{A}_{\ B}dw\right), \tag{2.46}$$

The Israel junction conditions can be obtained by the integration of the field equations (2.18), (2.21) and (2.44), to find

$$0 = \kappa S^{w}_{\ w}, \tag{2.47}$$

$$0 = \kappa S^{w}_{\ \mu}, \tag{2.48}$$

and

$$\epsilon\{[K^{\mu}_{\nu}] - \delta^{\mu}_{\nu}[K]\} = \kappa S^{\mu}_{\ \nu}\,. \tag{2.49}$$

This equation is called the Israel junction condition, where $[K^{\mu}_{\nu}] = K^{\mu+}_{\nu} - K^{\mu-}_{\nu}$.

We can rewrite (2.49) in the other form

$$[K_{\mu}^{\mu}] - \delta_{\mu}^{\mu}[K] = 8\pi G S^{\mu}{}_{\mu}, \tag{2.50}$$

where $\delta_{\mu}^{\mu} = 1 + 1 + 1 = 3$, therefore

$$[K] - 3[K] = 8\pi G S, \tag{2.51}$$

and

$$[K] = -4\pi G S, \tag{2.52}$$

so that from (2.49) we obtain

$$[K_{\nu}^{\mu}] = 8\pi G \left(S_{\nu}^{\mu} - \frac{1}{2}\delta_{\nu}^{\mu} S \right). \tag{2.53}$$

This equation also is called the Israel junction condition.

Chapter 3

THIN-SHELL FORMALISM IN GENERAL RELATIVITY

3.1 Thin-shell in 3+1-Dimensions

We consider standard general relativity, with the transition layer confined to a thin-shell. The bulk space-times (interior and exterior) on either side of the transition layer will be spherically symmetric and static but otherwise arbitrary. The thin-shell (transition layer) will be permitted to move freely in the bulk space-times, permitting a fully dynamic analysis.

To describe the geometry of the thin-shell, we use spherical coordinates (t, r, θ, φ) and we assume that the geometry is static, and spherically symmetric.

Consider two distinct space-time manifolds, an exterior M+, and an interior M−, that are to be joined together across a surface layer Σ (a spherical shell). Σ is called a singular hypersurface of order one, surface layer or thin-shell.

3.1.1 General Formalism

The metric for a thin-shell is given by the following line element:

$$ds_{\pm}^2 = -e^{2\Psi_{\pm}(r_{\pm})}\left[1-\frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right]dt_{\pm}^2 + \left[1-\frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right]^{-1} dr_{\pm}^2 + r_{\pm}^2 d\Omega_{\pm}^2, \qquad (3.1)$$

(we are using geometrized units c = G = 1), where $d\Omega_{\pm}^2 = d\theta_{\pm}^2 + sin\,\theta_{\pm}^2\ d\varphi_{\pm}^2$ the metric of the two-dimensional unit sphere with the two spherical polar coordinates θ and φ; $\pm$ refers to the exterior and interior geometry, respectively.

$\Psi(r)$ and $M(r)$ are non-negative functions from a given value of the radial coordinate, t is the time coordinate, and r is the space coordinate in the radial direction.

The covariant metric components $g^{\pm}_{\sigma\gamma}$:

$$g^{\pm}_{\sigma\gamma} = diag\left[-e^{2\Psi_{\pm}(r_{\pm})}\left[1-\frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right],\left[1-\frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right]^{-1}, r_{\pm}^2, r_{\pm}^2 \sin\theta_{\pm}^2\right]. \quad (3.2)$$

The contravariant metric tensor $g^{\sigma\gamma}_{\pm}$:

$$g^{\sigma\gamma}_{\pm} = diag\left[-e^{-2\Psi_{\pm}(r_{\pm})}\left[1-\frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right]^{-1},\left[1-\frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right], \frac{1}{r_{\pm}^2}, \frac{1}{r_{\pm}^2 \sin\theta_{\pm}^2}\right]. \quad (3.3)$$

To understand the physical meaning of the two metric functions, $\Psi_{\pm}(r_{\pm})$ and $M_{\pm}(r_{\pm})$, it is necessary to invoke the Einstein field equations.

3.2 Einstein Field Equations

We consider Einstein's equations in the form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\ , \quad (3.4)$$

where $G_{\mu\nu}$ is called the Einstein tensor, which can be obtained through a weary but straightforward calculation once the metric components $g_{\mu\nu}$ are given. $R_{\mu\nu}$ is the Ricci curvature tensor, $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, $T_{\mu\nu}$ is the stress-energy tensor, and G is the gravitational constant.

3.2.1 Components of The Einstein Tensor

To obtain the Components of the Einstein tensor for giving metric we should get to know the Christoffel symbols of the second kind:

$$\Gamma^{\sigma}_{\lambda\rho} = \frac{1}{2}g^{\sigma\gamma}\left[g_{\gamma\rho,\lambda} + g_{\lambda\gamma,\rho} - g_{\lambda\rho,\gamma}\right], \quad (3.5)$$

where $\Gamma^{\sigma}_{\lambda\rho}$ is called the connection coefficients or Christoffel symbols. If all the gradients of the metric tensor are zero, then all of the Christoffel symbols of the second kind are zero. The connection coefficients are symmetric, the symmetry of Christoffel symbols means that

$$\Gamma^{\sigma}_{\lambda\rho} = \Gamma^{\sigma}_{\rho\lambda}.$$

3.2.2 Non-zero Christoffel Symbols

From (3.5), when $\sigma = t,\ \lambda = r,\ \rho = t$, we have

$$\Gamma^{t}_{rt} = \tfrac{1}{2} g^{tt}\left[g_{tt,r} + \underbrace{g_{rt,t}}_{=0} - \underbrace{g_{rt,t}}_{=0}\right] = \tfrac{1}{2} g^{tt} g_{tt,r}, \tag{3.6}$$

from (3.2), we have

$$g_{tt,r} = \left[-e^{2\Psi_{\pm}(r_{\pm})}\left[1 - \frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right]\right]_{,r}, \tag{3.7.a}$$

therefore

$$g_{tt,r} = -\left[2\Psi' e^{2\Psi(r)}\left(1 - \frac{M}{r}\right) + e^{2\Psi(r)}\left(\frac{M - rM'}{r^2}\right)\right], \tag{3.7.b}$$

substitute (3.7.b) into (3.6) to obtain

$$\Gamma^{t}_{rt} = \Psi' + \left(\frac{M - rM'}{2r(r-M)}\right) = \Gamma^{t}_{tr}\ . \tag{3.8}$$

From Christoffel symbols (3.5), when $\sigma = \lambda = \rho = r$, we have

$$\Gamma^{r}_{rr} = \tfrac{1}{2} g^{rr}\left[g_{rr,r} + g_{rr,r} - g_{rr,r}\right] = \tfrac{1}{2} g^{rr} g_{rr,r}, \tag{3.9}$$

from (3.2), we have

$$g_{rr,r} = \left[1 - \frac{M_{\pm}(r_{\pm})}{r_{\pm}}\right]^{-1}_{,r} = -\frac{(M - rM')}{(r-M)^2}, \tag{3.10}$$

and substituting (3.10) into (3.9), we obtain

$$\Gamma^{r}_{rr} = -\frac{(M - rM')}{2r(r-M)}. \tag{3.11}$$

From (3.5), when $\sigma = r,\ \lambda = \rho = t$, we find

$$\Gamma^r_{tt} = \tfrac{1}{2} g^{rr} \left[\underbrace{g_{rt,t}}_{=0} + \underbrace{g_{tr,t}}_{=0} - g_{tt,r} \right] = -\tfrac{1}{2} g^{rr} g_{tt,r}, \tag{3.12}$$

substitute (3.7.b) into (3.12), we find

$$\Gamma^r_{tt} = \Psi' e^{2\Psi(r)} \left(\frac{r-M}{r}\right)^2 + \frac{e^{2\Psi(r)}}{2r^3} (r-M)(M - rM'). \tag{3.13}$$

From (3.5), when $\sigma = r,\ \lambda = \rho = \theta$, we find

$$\Gamma^r_{\theta\theta} = \tfrac{1}{2} g^{rr} \left[\underbrace{g_{r\theta,\theta}}_{=0} + \underbrace{g_{\theta r,\theta}}_{=0} - g_{\theta\theta,r} \right] = -\frac{1}{2} g^{rr} g_{\theta\theta,r}, \tag{3.14}$$

where

$$g_{\theta\theta,r} = (r^2)_{,r} = 2r, \tag{3.15}$$

substitute (3.15) into (3.14), we obtain

$$\Gamma^r_{\theta\theta} = -(r-M). \tag{3.16}$$

From Christoffel symbols (3.5), when $\sigma = r,\ \lambda = \rho = \varphi$, we find

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2} g^{rr} \left[\underbrace{g_{r\varphi,\varphi}}_{=0} + \underbrace{g_{\varphi r,\varphi}}_{=0} - g_{\varphi\varphi,r} \right] = -\frac{1}{2} g^{rr} g_{\varphi\varphi,r}, \tag{3.17}$$

where

$$g_{\varphi\varphi,r} = (r^2 \sin^2\theta)_{,r} = 2r \sin^2\theta, \tag{3.18}$$

and upon substituting (3.18) into (3.17), we obtain

$$\Gamma^r_{\varphi\varphi} = -(r-M)\sin^2\theta. \tag{3.19}$$

From Christoffel symbols (3.5), when$\sigma = \rho = \theta,\ \lambda = r$, we find

$$\Gamma^\theta_{r\theta} = \tfrac{1}{2} g^{\theta\theta} \left[g_{\theta\theta,r} + g_{r\theta,\theta} - g_{r\theta,\theta} \right] = \tfrac{1}{2} g^{\theta\theta} g_{\theta\theta,r}, \tag{3.20}$$

Substitute (3.15) into (3.20), to obtain

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r} = \Gamma^{\theta}_{\theta r}. \tag{3.21}$$

From Christoffel symbols (3.5), when$\sigma = \theta,\ \lambda = \rho = \varphi$, we find

$$\Gamma^{\theta}_{\varphi\varphi} = \tfrac{1}{2} g^{\theta\theta}\left[\underbrace{g_{\theta\varphi,\varphi}}_{=0} + \underbrace{g_{\varphi\theta,\varphi}}_{=0} - g_{\varphi\varphi,\theta}\right] = -\tfrac{1}{2} g^{\theta\theta} g_{\varphi\varphi,\theta}, \tag{3.22}$$

where

$$g_{\varphi\varphi,\theta} = (r^2 \sin^2\theta)_{,\theta} = 2r^2 \sin\theta \cos\theta, \tag{3.23}$$

Substitute (3.23) into (3.22), to obtain

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta. \tag{3.24}$$

From (3.5) when$\sigma = \varphi,\ \lambda = r,\ \rho = \varphi$, we find

$$\Gamma^{\varphi}_{r\varphi} = \tfrac{1}{2} g^{\varphi\varphi}\left[g_{\varphi\varphi,r} + g_{r\varphi,\varphi} - g_{r\varphi,\varphi}\right] = \tfrac{1}{2} g^{\varphi\varphi} g_{\varphi\varphi,r}, \tag{3.25}$$

Substitute (3.18) into (3.25), to obtain

$$\Gamma^{\varphi}_{r\varphi} = \frac{1}{r} = \Gamma^{\varphi}_{\varphi r}\ . \tag{3.26}$$

From (3.5) when$\sigma = \varphi,\ \lambda = \theta,\ \rho = \varphi$, we find

$$\Gamma^{\varphi}_{\theta\varphi} = \tfrac{1}{2} g^{\varphi\varphi}\left[g_{\varphi\varphi,\theta} + \underbrace{g_{\theta\varphi,\theta}}_{=0} - \underbrace{g_{\theta\varphi,\varphi}}_{=0}\right] = \tfrac{1}{2} g^{\varphi\varphi} g_{\varphi\varphi,\theta}, \tag{3.27}$$

Substitute (3.23) into (3.27), to obtain

$$\Gamma^{\varphi}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta} = \Gamma^{\varphi}_{\varphi\theta}\ . \tag{3.28}$$

3.3 Riemann Curvature Tensor

The left hand side of the Einstein field equations (3.4) represents the geometry of the space-time and are given as a nonlinear combination of the metric components $g_{\sigma\gamma}$ and their first and second derivatives. For completeness, we define the Riemann curvature tensor in terms of the Christoffel symbols:

$$R^{\sigma}_{\gamma\lambda\rho} = \Gamma^{\sigma}_{\gamma\rho,\lambda} - \Gamma^{\sigma}_{\gamma\lambda,\rho} + \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\gamma\rho} - \Gamma^{\sigma}_{\alpha\rho}\Gamma^{\alpha}_{\gamma\lambda}. \tag{3.29}$$

3.3.1 Non-zero Riemann Tensor Components

$$R^{t}{}_{rtr} = \underbrace{\Gamma^{t}_{rr,t}}_{=0} - \Gamma^{t}_{rt,r} + \underbrace{\Gamma^{t}_{tt}\Gamma^{t}_{rr}}_{=0} + \Gamma^{t}_{rt}\Gamma^{r}_{rr} + \underbrace{\Gamma^{t}_{\theta t}\Gamma^{\theta}_{rr}}_{=0} + \underbrace{\Gamma^{t}_{\varphi t}\Gamma^{\varphi}_{rr}}_{=0} - \Gamma^{t}_{tr}\Gamma^{t}_{rt} -$$

$$\underbrace{\Gamma^{t}_{rr}\Gamma^{r}_{rt}}_{=0} - \underbrace{\Gamma^{t}_{\theta r}\Gamma^{\theta}_{rt}}_{=0} - \underbrace{\Gamma^{t}_{\varphi r}\Gamma^{\varphi}_{rt}}_{=0}, \tag{3.30.a}$$

therefore

$$R^{t}{}_{rtr} = -\Gamma^{t}_{rt,r} + \Gamma^{t}_{rt}\Gamma^{r}_{rr} - \Gamma^{t}_{tr}\Gamma^{t}_{rt}, \tag{3.30.b}$$

from (3.8), we have

$$\Gamma^{t}_{rt,r} = \left[\Psi' + \left(\frac{M-rM'}{2r(r-M)}\right)\right]_{,r} = \Psi'' - \frac{M''}{2(r-M)} - \frac{(M-rM')(4r-2rM'-2M)}{4r^2(r-M)^2}. \tag{3.31}$$

Substitute (3.8), (3.11) and (3.31) into (3.30.b) to get

$$R^{t}{}_{rtr} = -\Psi'' - \Psi'^2 - \frac{3\Psi'(M-rM')}{2r(r-M)} + \frac{M''}{2r(r-M)} - \frac{(M-rM')^2}{2r^2(r-M)^2} + \frac{(M-rM')(4r-2rM'-2M)}{4r^2(r-M)^2} \tag{3.32}$$

From Riemann curvature tensor (3.29) we find

$$R^{t}{}_{\theta t\theta} = \underbrace{\Gamma^{t}_{\theta\theta,t}}_{=0} - \underbrace{\Gamma^{t}_{\theta t,\theta}}_{=0} + \underbrace{\Gamma^{t}_{tt}\Gamma^{t}_{\theta\theta}}_{=0} + \Gamma^{t}_{rt}\Gamma^{r}_{\theta\theta} + \underbrace{\Gamma^{t}_{\theta\theta}\Gamma^{\theta}_{\theta\theta}}_{=0} + \underbrace{\Gamma^{t}_{\varphi t}\Gamma^{\varphi}_{\theta\theta}}_{=0} - \underbrace{\Gamma^{t}_{t\theta}\Gamma^{t}_{\theta t}}_{=0} -$$

$$\underbrace{\Gamma^{t}_{r\theta}\Gamma^{r}_{\theta t}}_{=0} - \underbrace{\Gamma^{t}_{\theta\theta}\Gamma^{\theta}_{\theta t}}_{=0} - \underbrace{\Gamma^{t}_{\varphi\theta}\Gamma^{\varphi}_{\theta t}}_{=0}, \tag{3.33.a}$$

therefore

$$R^{t}{}_{\theta t\theta} = \Gamma^{t}_{rt}\Gamma^{r}_{\theta\theta}, \tag{3.33.b}$$

and substituting (3.8) and (3.16) into (3.33.b) one gets

$$R^{t}{}_{\theta t\theta} = -\Psi'(r-M) - \frac{M-rM'}{2r}. \tag{3.34}$$

From the Riemann curvature tensor (3.29) we find

$$R^{t}_{\ \varphi t\varphi} = \underbrace{\Gamma^{t}_{\varphi\varphi,t}}_{=0} - \underbrace{\Gamma^{t}_{\varphi t,\varphi}}_{=0} + \underbrace{\Gamma^{t}_{tt}\Gamma^{t}_{\varphi\varphi}}_{=0} + \Gamma^{t}_{rt}\Gamma^{r}_{\varphi\varphi} + \underbrace{\Gamma^{t}_{\theta t}\Gamma^{\theta}_{\varphi\varphi}}_{=0} + \underbrace{\Gamma^{t}_{\varphi t}\Gamma^{\varphi}_{\varphi\varphi}}_{=0} - \underbrace{\Gamma^{t}_{t\varphi}\Gamma^{t}_{\varphi t}}_{=0} -$$

$$\underbrace{\Gamma^{t}_{r\varphi}\Gamma^{r}_{\varphi t}}_{=0} - \underbrace{\Gamma^{t}_{\theta\varphi}\Gamma^{\theta}_{\varphi t}}_{=0} - \underbrace{\Gamma^{t}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi t}}_{=0}, \tag{3.35.a}$$

therefore

$$R^{t}_{\ \varphi t\varphi} = \Gamma^{t}_{rt}\Gamma^{r}_{\varphi\varphi} \ . \tag{3.35.b}$$

Substitute now (3.8) and (3.19) into (3.35.b) to get

$$R^{t}_{\ \varphi t\varphi} = -\Psi'(r-M)\sin^2\theta - \frac{(M-rM')}{2r}\sin^2\theta. \tag{3.36}$$

From the Riemann curvature tensor (3.29) we find

$$R^{r}_{\ ttr} = \underbrace{\Gamma^{r}_{tr,t}}_{=0} - \Gamma^{r}_{tt,r} + \Gamma^{r}_{tt}\Gamma^{t}_{tr} + \underbrace{\Gamma^{r}_{rt}\Gamma^{r}_{tr}}_{=o} + \underbrace{\Gamma^{r}_{\theta t}\Gamma^{\theta}_{tr}}_{=0} + \underbrace{\Gamma^{r}_{\varphi t}\Gamma^{\varphi}_{tr}}_{=0} - \underbrace{\Gamma^{r}_{tr}\Gamma^{t}_{tt}}_{=0} -$$

$$\Gamma^{r}_{rr}\Gamma^{r}_{tt} - \underbrace{\Gamma^{r}_{\theta r}\Gamma^{\theta}_{tt}}_{=0} - \underbrace{\Gamma^{r}_{\varphi r}\Gamma^{\varphi}_{tt}}_{=0}, \tag{3.37.a}$$

therefore

$$R^{r}_{\ ttr} = -\Gamma^{r}_{tt,r} + \Gamma^{r}_{tt}\Gamma^{t}_{tr} - \Gamma^{r}_{rr}\Gamma^{r}_{tt}, \tag{3.37.b}$$

and from (3.13) we have

$$\Gamma^{r}_{tt,r} = \left[\Psi' e^{2\Psi(r)}\left(\frac{r-M}{r}\right)^2 + \frac{e^{2\Psi(r)}}{2r^3}(r-M)(M-rM')\right]_{,r} . \tag{3.38.a}$$

As a result, we have

$$\Gamma^{r}_{tt,r} =$$

$$e^{2\Psi(r)}\Psi''\left(\frac{r-M}{r}\right)^2 + 2e^{2\Psi(r)}\Psi'^2\left(\frac{r-M}{r}\right)^2 +$$

$$e^{2\Psi(r)}\Psi'\left(\frac{2r^2(r-M)(1-M')-2r(r-M)^2}{r^4}\right) + e^{2\Psi(r)}\frac{\Psi'}{r^3}(r-M)(M-rM') +$$

$$\frac{e^{2\Psi(r)}}{2}\left(\frac{r^3(1-M')-3r^2(r-M)}{r^6}\right)(M-rM') - \frac{e^{2\Psi(r)}M''}{2r^2}(r-M) \ . \tag{3.38.b}$$

Substitute now (3.8), (3.11), (3.13), and (3.38.b) into (3.37.b) to get

$$R^{r}_{\ ttr} = -e^{2\Psi(r)}\Psi''\left(\frac{r-M}{r}\right)^2 - e^{2\Psi(r)}\Psi'^2\left(\frac{r-M}{r}\right)^2 + \frac{e^{2\Psi(r)}\Psi'}{2r^3}(r-M)(M-rM') - \frac{2\ e^{2\Psi(r)}\Psi'}{r^2}(r-M)(1-M') - \frac{e^{2\Psi(r)}}{2r^3}(M-rM')(1-M') + \frac{2e^{2\Psi(r)}\Psi'}{r}\left(\frac{r-M}{r}\right)^2 + \frac{3e^{2\Psi(r)}}{4r^4}(M-rM')(r-M) + \frac{e^{2\Psi(r)}\Psi''}{2r^2}(r-M) + \frac{e^{2\Psi(r)}}{2r^4}(M-rM')^2. \tag{3.39}$$

From the Riemann curvature tensor (3.29) we find

$$R^{r}_{\ \theta r\theta} = \Gamma^{r}_{\theta\theta,r} - \underbrace{\Gamma^{r}_{\theta r,\theta}}_{=0} + \underbrace{\Gamma^{r}_{tr}\Gamma^{t}_{\theta\theta}}_{=0} + \Gamma^{r}_{rr}\Gamma^{r}_{\theta\theta} + \underbrace{\Gamma^{r}_{\theta r}\Gamma^{\theta}_{\theta\theta}}_{=0} + \underbrace{\Gamma^{r}_{\varphi r}\Gamma^{\varphi}_{\theta\theta}}_{=0} - \underbrace{\Gamma^{r}_{t\theta}\Gamma^{t}_{\theta r}}_{=0} - \underbrace{\Gamma^{r}_{r\theta}\Gamma^{r}_{\theta r}}_{=o} - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} - \underbrace{\Gamma^{r}_{\varphi\theta}\Gamma^{\varphi}_{\theta r}}_{=0}, \tag{3.40.a}$$

therefore

$$R^{r}_{\ \theta r\theta} = \Gamma^{r}_{\theta\theta,r} + \Gamma^{r}_{rr}\Gamma^{r}_{\theta\theta} - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r}. \tag{3.40.b}$$

The derivative of (3.16) with respect to r is

$$\Gamma^{r}_{\theta\theta,r} = -(r-M)_{,r} = -1 + M', \tag{3.41}$$

and substituting (3.11), (3.16), (3.21), and (3.41) into (3.40.b) we get

$$R^{r}_{\ \theta r\theta} = -\ \frac{(M-rM')}{2r}. \tag{3.42}$$

From the Riemann curvature tensor (3.29) we find

$$R^{r}_{\ \varphi r\varphi} = \Gamma^{r}_{\varphi\varphi,r} - \underbrace{\Gamma^{r}_{\varphi r,r}}_{=0} + \underbrace{\Gamma^{r}_{tr}\Gamma^{t}_{\varphi\varphi}}_{=0} + \Gamma^{r}_{rr}\Gamma^{r}_{\varphi\varphi} + \underbrace{\Gamma^{r}_{\theta r}\Gamma^{\theta}_{\varphi\varphi}}_{=0} + \underbrace{\Gamma^{r}_{\varphi r}\Gamma^{\varphi}_{\varphi\varphi}}_{=0} - \underbrace{\Gamma^{r}_{t\varphi}\Gamma^{t}_{\varphi r}}_{=0} - \underbrace{\Gamma^{r}_{r\varphi}\Gamma^{r}_{\varphi r}}_{=o} - \underbrace{\Gamma^{r}_{\theta\varphi}\Gamma^{\theta}_{\varphi r}}_{=0} - \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi r}, \tag{3.43.a}$$

therefore

$$R^{r}_{\varphi r \varphi} = \Gamma^{r}_{\varphi\varphi,r} + \Gamma^{r}_{rr}\Gamma^{r}_{\varphi\varphi} - \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi r}. \tag{3.43.b}$$

The derivative of (3.19) with respect to r is

$$\Gamma^{r}_{\varphi\varphi,r} = -[(r-M)\sin^2\theta]_{,r} = (-1+M')\sin^2\theta, \tag{3.44}$$

and upon substituting (3.11), (3.19), (3.26), and (3.44) into (3.43.b) we get

$$R^{r}_{\varphi r \varphi} = -\left(\frac{M-rM'}{2r}\right)\sin^2\theta. \tag{3.45}$$

From the Riemann curvature tensor (3.29) we find

$$R^{\theta}_{tt\theta} = \underbrace{\Gamma^{\theta}_{t\theta,t}}_{=0} - \underbrace{\Gamma^{\theta}_{tt,\theta}}_{=0} + \underbrace{\Gamma^{\theta}_{tt}\Gamma^{t}_{t\theta}}_{=0} + \underbrace{\Gamma^{\theta}_{rt}\Gamma^{r}_{t\theta}}_{=0} + \underbrace{\Gamma^{\theta}_{\theta t}\Gamma^{\theta}_{t\theta}}_{=0} + \underbrace{\Gamma^{\theta}_{\varphi t}\Gamma^{\varphi}_{t\theta}}_{=0} - \underbrace{\Gamma^{\theta}_{t\theta}\Gamma^{t}_{tt}}_{=0} -$$

$$\Gamma^{\theta}_{r\theta}\Gamma^{r}_{tt} - \underbrace{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{tt}}_{=0} - \underbrace{\Gamma^{\theta}_{\varphi\theta}\Gamma^{\varphi}_{tt}}_{=0}, \tag{3.46.a}$$

therefore

$$R^{\theta}_{tt\theta} = -\Gamma^{\theta}_{r\theta}\Gamma^{r}_{tt}, \tag{3.46.b}$$

and substitute (3.13) and (3.21) into (3.46.b) to get

$$R^{\theta}_{tt\theta} = -\frac{\Psi' e^{2\Psi(r)}}{r}\left(\frac{r-M}{r}\right)^2 - \frac{e^{2\Psi(r)}}{2r^4}(r-M)(M-rM'). \tag{3.47}$$

From the Riemann curvature tensor (3.29) we have

$$R^{\theta}_{rr\theta} = \Gamma^{\theta}_{r\theta,r} - \underbrace{\Gamma^{\theta}_{rr,\theta}}_{=0} + \underbrace{\Gamma^{\theta}_{tr}\Gamma^{t}_{r\theta}}_{=0} + \underbrace{\Gamma^{\theta}_{rr}\Gamma^{r}_{r\theta}}_{=0} + \Gamma^{\theta}_{\theta r}\Gamma^{\theta}_{r\theta} + \underbrace{\Gamma^{\theta}_{\varphi r}\Gamma^{\varphi}_{r\theta}}_{=0} - \underbrace{\Gamma^{\theta}_{t\theta}\Gamma^{t}_{rr}}_{=0} -$$

$$\Gamma^{\theta}_{r\theta}\Gamma^{r}_{rr} - \underbrace{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{rr}}_{=0} - \underbrace{\Gamma^{\theta}_{\varphi\theta}\Gamma^{\varphi}_{rr}}_{=0}, \tag{3.48.a}$$

therefore

$$R^{\theta}_{rr\theta} = \Gamma^{\theta}_{r\theta,r} + \Gamma^{\theta}_{\theta r}\Gamma^{\theta}_{r\theta} - \Gamma^{\theta}_{r\theta}\Gamma^{r}_{rr}, \tag{3.48.b}$$

which from (3.21) gives

$$\Gamma^{\theta}_{r\theta,r} = \left(\frac{1}{r}\right)_{r} = -\frac{1}{r^2}, \tag{3.49}$$

and substitution of (3.11), (3.21), and (3.49) into (3.48.b) gives

$$R^{\theta}_{\ rr\theta} = \frac{1}{2r^2}\frac{(M-rM')}{(r-M)} . \tag{3.50}$$

From the Riemann curvature tensor (3.29) we have

$$R^{\theta}_{\ \varphi\theta\varphi} = \Gamma^{\theta}_{\varphi\varphi,\theta} - \underbrace{\Gamma^{\theta}_{\varphi\theta,\varphi}}_{=0} + \underbrace{\Gamma^{\theta}_{t\theta}\Gamma^{t}_{\varphi\varphi}}_{=0} + \Gamma^{\theta}_{r\theta}\Gamma^{r}_{\varphi\varphi} + \underbrace{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\varphi\varphi}}_{=0} + \underbrace{\Gamma^{\theta}_{\varphi\theta}\Gamma^{\varphi}_{\varphi\varphi}}_{=0} - \underbrace{\Gamma^{\theta}_{t\varphi}\Gamma^{t}_{\varphi\theta}}_{=0} -$$

$$\underbrace{\Gamma^{\theta}_{r\varphi}\Gamma^{r}_{\varphi\theta}}_{=0} - \underbrace{\Gamma^{\theta}_{\theta\varphi}\Gamma^{\theta}_{\varphi\theta}}_{=0} - \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta}, \tag{3.51.a}$$

so that

$$R^{\theta}_{\ \varphi\theta\varphi} = \Gamma^{\theta}_{\varphi\varphi,\theta} + \Gamma^{\theta}_{r\theta}\Gamma^{r}_{\varphi\varphi} - \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta}. \tag{3.51.b}$$

The derivative of (3.24) with respect to θ gives

$$\Gamma^{\theta}_{\varphi\varphi,\theta} = (-\sin\theta\cos\theta)_{,\theta} = -\cos^2\theta + \sin^2\theta, \tag{3.52}$$

which upon substitution of (3.52), (3.21), (3.19), (3.24) and (3.28) into (3.51.b) gives

$$R^{\theta}_{\varphi\theta\varphi} = \frac{M(r)}{r}\sin^2\theta. \tag{3.53}$$

From the Riemann curvature tensor (3.29) we have

$$R^{\varphi}_{\ tt\varphi} = \underbrace{\Gamma^{\varphi}_{t\varphi,t}}_{=0} - \underbrace{\Gamma^{\varphi}_{tt,\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{tt}\Gamma^{t}_{t\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{rt}\Gamma^{r}_{t\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{\theta t}\Gamma^{\theta}_{t\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{\varphi t}\Gamma^{\varphi}_{t\varphi}}_{=0} - \underbrace{\Gamma^{\varphi}_{t\varphi}\Gamma^{t}_{tt}}_{=0} -$$

$$\Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{tt} - \underbrace{\Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{tt}}_{=0} - \underbrace{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{tt}}_{=0}, \tag{3.54.a}$$

therefore

$$R^{\varphi}_{\ tt\varphi} = -\Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{tt} \ . \tag{3.54.b}$$

Substitute (3.13), and (3.26) into (3.54.b) to get

$$R^{\varphi}_{\ tt\varphi} = -\frac{\Psi' e^{2\Psi(r)}}{r}\left(\frac{r-M}{r}\right)^2 - \frac{e^{2\Psi(r)}}{2r^4}(r-M)(M-rM'). \tag{3.55}$$

From the Riemann curvature tensor we have

$$R^{\varphi}_{\ rr\varphi} = \Gamma^{\varphi}_{r\varphi,r} - \underbrace{\Gamma^{\varphi}_{rr,\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{tr}\Gamma^{t}_{r\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{rr}\Gamma^{r}_{r\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{\theta r}\Gamma^{\theta}_{r\varphi}}_{=0} + \Gamma^{\varphi}_{\varphi r}\Gamma^{\varphi}_{r\varphi} - \underbrace{\Gamma^{\varphi}_{t\varphi}\Gamma^{t}_{rr}}_{=0} -$$

$$\Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{rr} - \underbrace{\Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{rr}}_{=0} - \underbrace{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{rr}}_{=0}, \tag{3.56.a}$$

therefore

$$R^{\varphi}_{\ rr\varphi} = \Gamma^{\varphi}_{r\varphi,r} + \Gamma^{\varphi}_{\varphi r}\Gamma^{\varphi}_{r\varphi} - \Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{rr}, \tag{3.56.b}$$

From (3.26) we have

$$\Gamma^{\varphi}_{r\varphi,r} = \left(\frac{1}{r}\right)_{,r} = -\frac{1}{r^2}\,, \tag{3.57}$$

and substitute (3.26), (3.57) and (3.11) into (3.56.b) to get

$$R^{\varphi}_{\ rr\varphi} = \frac{1}{2r^2}\left(\frac{M-rM'}{r-M}\right). \tag{3.58}$$

From the Riemann curvature tensor (3.29) we have

$$R^{\varphi}_{\ \theta\theta\varphi} = \Gamma^{\varphi}_{\theta\varphi,\theta} - \underbrace{\Gamma^{\varphi}_{\theta\theta,\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{t\theta}\Gamma^{t}_{\theta\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{r\theta}\Gamma^{r}_{\theta\varphi}}_{=0} + \underbrace{\Gamma^{\varphi}_{\theta\theta}\Gamma^{\theta}_{\theta\varphi}}_{=0} + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\varphi}_{\theta\varphi} - \underbrace{\Gamma^{\varphi}_{t\varphi}\Gamma^{t}_{\theta\theta}}_{=0} -$$

$$\Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{\theta\theta} - \underbrace{\Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{\theta\theta}}_{=0} - \underbrace{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\theta\theta}}_{=0}, \tag{3.59.a}$$

therefore

$$R^{\varphi}_{\ \theta\theta\varphi} = \Gamma^{\varphi}_{\theta\varphi,\theta} + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\varphi}_{\theta\varphi} - \Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{\theta\theta}, \tag{3.59.b}$$

and from (3.28) we have

$$\Gamma^{\varphi}_{\theta\varphi,\theta} = \left(\frac{\cos\theta}{\sin\theta}\right)_{,\theta} = \frac{-\sin^2\theta-\cos^2\theta}{\sin^2\theta}\,. \tag{3.60}$$

Substituting (3.60), (3.28), (3.26), and (3.16) into (3.59.b) gives

$$R^{\varphi}_{\ \theta\theta\varphi} = -\frac{M(r)}{r}. \tag{3.61}$$

3.4 The Ricci Tensor

It is given by the contraction over the first and third index of the Riemann tensor:

$$R_{\gamma\rho} = R^{\sigma}_{\gamma\sigma\rho}. \tag{3.62.a}$$

where

$$R_{\gamma\rho} = daig\left[R_{tt}, R_{rr}, R_{\theta\theta}, R_{\varphi\varphi}\right]. \tag{3.62.b}$$

The Ricci tensor $R_{\gamma\rho}$ is symmetric.

The Ricci tensor R_{tt} is

$$R_{tt} = R^{\sigma}_{t\sigma t} = \underbrace{R^{t}_{ttt}}_{=0} + \underbrace{R^{r}_{trt}}_{=-R^{r}_{ttr}} + \underbrace{R^{\theta}_{t\theta t}}_{=-R^{\theta}_{tt\theta}} + R^{\varphi}_{t\varphi t}, \tag{3.63}$$

so that substituting (3.39), (3.47), and (3.55) into (3.63), we find

$$R_{tt} = e^{2\Psi(r)}\Psi''\left(\frac{r-M}{r}\right)^2 + e^{2\Psi(r)}\Psi'^2\left(\frac{r-M}{r}\right)^2 - e^{2\Psi(r)}\frac{\Psi'}{2r^3}(r-M)(M-rM') + 2e^{2\Psi(r)}\frac{\Psi'}{r^2}(r-M)(1-M') - e^{2\Psi(r)}\frac{M''}{2r^2}(r-M) - e^{2\Psi(r)}\frac{1}{2r^4}(r-M)(M-rM') + e^{2\Psi(r)}\frac{1}{2r^3}(M-rM')(1-M') - e^{2\Psi(r)}\frac{1}{2r^4}(M-rM')^2 . \tag{3.64}$$

The Ricci tensor R_{rr}

$$R_{rr} = R^{\sigma}_{r\sigma r} = R^{t}_{rtr} + \underbrace{R^{r}_{rrr}}_{=0} + R^{\theta}_{r\theta r} + R^{\varphi}_{r\varphi r}, \tag{3.65}$$

upon substituting (3.23), (3.50), and (3.58) into (3.65), we find

$$R_{rr} = -\Psi'' - \Psi'^2 + \frac{3\Psi'}{2r}\frac{(M-rM')}{(r-M)} + \frac{M''}{2(r-M)} + \frac{(M-rM')(4r-2M-2rM')}{4r^2(r-M)^2} - \frac{(M-rM')^2}{2r^2(r-M)^2} - \frac{1}{r^2}\frac{(M-rM')}{(r-M)} . \tag{3.66}$$

The Ricci tensor $R_{\theta\theta}$ is given by

$$R_{\theta\theta} = R^{\sigma}_{\theta\sigma\theta} = R^{t}_{\theta t\theta} + R^{r}_{\theta r\theta} + \underbrace{R^{\theta}_{\theta\theta\theta}}_{=0} + R^{\varphi}_{\theta\varphi\theta}, \tag{3.67}$$

and substituting (3.34), (3.42), and (3.61) into (3.67), we find

$$R_{\theta\theta} = -\Psi'(r-M) + M'. \tag{3.68}$$

The Ricci tensor $R_{\varphi\varphi}$

$$R_{\varphi\varphi} = R^{\sigma}_{\varphi\sigma\varphi} = R^{t}_{\varphi t\varphi} + R^{r}_{\varphi r\varphi} + R^{\theta}_{\varphi\theta\varphi} + \underbrace{R^{\varphi}_{\varphi\varphi\varphi}}_{=0}, \tag{3.69}$$

and substituting (3.36), (3.45), and (3.53) into (3.69), we find

$$R_{\varphi\varphi} = -\Psi'(r-M)\sin^2\theta + M'\sin^2\theta. \tag{3.70}$$

3.4.1 Ricci Scalar

The contraction of the Ricci tensor is called the Ricci scalar:

$$R = g^{\gamma\rho}R_{\gamma\rho}, \tag{3.71.a}$$

therefore

$$R = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi}\ . \tag{3.71.b}$$

From (3.3) we have

$$R = \frac{-re^{-2\Psi}}{(r-M)}R_{tt} + \frac{(r-M)}{r}R_{rr} + \frac{1}{r^2}R_{\theta\theta} + \frac{1}{r^2\sin^2\theta}R_{\varphi\varphi}, \tag{3.72}$$

and substituting (3.64), (3.66), (3.68) and (3.70) into (3.72), we find

$$R = -2\Psi''\left(\frac{r-M}{r}\right) - 2\Psi'^2\left(\frac{r-M}{r}\right) - \frac{\Psi'}{r^2}(M-rM') - \frac{2\Psi'}{r}(1-M') - \frac{2\Psi'}{r^2}(r-M) - \frac{1}{2r^3}(M-rM') + \frac{M''}{r} + \frac{2M'}{r^2} - \frac{(M-rM')(1-M')}{2r^2(r-M)} + \frac{(M-rM')(4r-2M-2rM')}{4r^3(r-M)}\ . \tag{3.73}$$

3.5 Mixed Form of Einstein Equations

$$G^{\sigma}_{\gamma} = R^{\sigma}_{\gamma} - \frac{1}{2}\delta^{\sigma}_{\gamma}R = 8\pi T^{\sigma}_{\gamma}, \tag{3.74}$$

where T_γ^σ is mixed form of the stress-energy tensor. Einstein equations must be solved for a perfect fluid[1], so the stress-energy tensor should have these components $T_\gamma^\sigma = diag[-\rho, P, P, P]$, where ρ is the mass-energy density and P is the hydrostatic pressure.

The tt-field equation of Einstein field equations is given by

$$G_t^t = \underbrace{R_t^t}_{=g^{tt}R_{tt}} - \frac{1}{2}\underbrace{\delta_t^t}_{=1} R = 8\pi T_t^t, \qquad (3.75.a)$$

therefore

$$g^{tt}R_{tt} - \frac{1}{2}R = -8\pi\rho \ . \qquad (3.75.b)$$

Substitute (3.64) and (3.73) into (3.75.b), to obtain

$$-\Psi''\left(\frac{r-M}{r}\right) - \Psi'^2\left(\frac{r-M}{r}\right) + \frac{\Psi'}{2r^2}(M - rM') - \frac{2\Psi'}{r}(1 - M') + \frac{1}{2r^3}(M - rM') +$$
$$\frac{(M-rM')^2}{2r^3(r-M)} + \frac{M''}{2r} - \frac{(M-rM')(1-M')}{2r^2(r-M)} + \Psi''\left(\frac{r-M}{r}\right) + \Psi'^2\left(\frac{r-M}{r}\right) + \frac{\Psi'}{2r^2}(M - rM') +$$
$$\frac{\Psi'}{r}(1 - M') + \frac{\Psi'}{r^2}(r - M) + \frac{1}{4r^3}(M - rM') - \frac{M''}{2r} + \frac{(M-rM')(1-M')}{4r^2(r-M)} -$$
$$\frac{(M-rM')(4r-2M-2rM')}{8r^3(r-M)} - \frac{M'}{r^2} = -8\pi\rho, \qquad (3.75.c)$$

therefore

$$G_t^t = -\frac{M'}{r^2} = -8\pi\rho, \qquad (3.76)$$

and by rearranging this equation, we find

$$\rho = \frac{M'}{8\pi r^2} \ . \qquad (3.77)$$

[1]A perfect fluid: is a fluid that can be completely characterized by its rest frame mass density ρ, and isotropic pressure P.

The rr-field equation of Einstein field equations is

$$G_r^r = \underbrace{R_r^r}_{=g^{rr}R_{rr}} - \frac{1}{2}\delta_r^r R = 8\pi T_r^r, \tag{3.78.a}$$

therefore

$$g^{rr}R_{rr} - \frac{1}{2}R = 8\pi P, \tag{3.78.b}$$

Substitute (3.66), (3.73) into (3.78.b), to obtain

$$-\Psi''\left(\frac{r-M}{r}\right) - \Psi'^2\left(\frac{r-M}{r}\right) - \frac{3\Psi'}{2r^2}(M - rM') + \frac{M''}{2r} + \frac{(M-rM')(4r-2M-2rM')}{4r^3(r-M)} - \frac{(M-rM')^2}{2r^3(r-M)} - \frac{(M-rM')}{r^3} + \Psi''\left(\frac{r-M}{r}\right) + \Psi'^2\left(\frac{r-M}{r}\right) + \frac{\Psi'}{2r^2}(M - rM') + \frac{\Psi'}{r}(1 - M') + \frac{\Psi'}{r^2}(r - M) + \frac{(M-rM')}{4r^3} - \frac{M''}{2r} + \frac{(M-rM')(1-M')}{4r^2(r-M)} - \frac{(M-rM')(4r-2M-2rM')}{8r^3(r-M)} - \frac{M'}{r^2} = 8\pi P, \tag{3.78.c}$$

thus

$$G_r^r = \frac{2\Psi'}{r^2}(r - M) - \frac{M'}{r^2} = 8\pi P, \tag{3.79}$$

and rearrange this equation to obtain

$$P = \frac{1}{8\pi r^2}[2\Psi'(r - M) - M']. \tag{3.80}$$

The $\theta\theta$-field equation of Einstein field equations is

$$G_\theta^\theta = \underbrace{R_\theta^\theta}_{=g^{\theta\theta}R_{\theta\theta}} - \frac{1}{2}\delta_\theta^\theta R = 8\pi T_\theta^\theta, \tag{3.81.a}$$

therefore

$$g^{\theta\theta}R_{\theta\theta} - \frac{1}{2}R = 8\pi P, \tag{3.81.b}$$

Substitute (3.68), (3.73) in (3.81.b), to obtain

$$-\frac{\Psi'}{r^2}(r-M)+\frac{M'}{r^2}+\Psi''\left(\frac{r-M}{r}\right)+\Psi'^2\left(\frac{r-M}{r}\right)+\frac{\Psi'}{2r^2}(M-rM')+$$

$$\frac{\Psi'}{r}(1-M')+\frac{\Psi'}{r^2}(r-M)+\frac{(M-rM')}{4r^3}-\frac{M''}{2r}+\frac{(M-rM')(1-M')}{4r^2(r-M)}-$$

$$\frac{(M-rM')(4r-2M-2rM')}{8r^3(r-M)}-\frac{M'}{r^2}=8\pi P, \quad (3.81.c)$$

therefore

$$G^{\theta}_{\theta}=\frac{\Psi''}{r}(r-M)+\frac{\Psi'^2}{r}(r-M)+\frac{\Psi'}{2r^2}(2r+M-3rM')-\frac{M''}{2r}=8\pi P, \quad (3.82)$$

and rearrange this equation to obtain

$$P=\frac{1}{16\pi r^2}[2r\Psi''(r-M)+2r\Psi'^2(r-M)+\Psi'(2r+M-3rM')-rM''].$$

(3.83)

3.6 Transition Layer

Now, consider a time-like 3-space Σ thin-shell which divides space-time into two distinct four-dimensional manifolds M+ and M−, located at $r=a(\tau)$

$$\Sigma:\ r=a(\tau)\ \Longrightarrow\quad dr=\frac{da}{d\tau}d\tau=\dot{a}d\tau,$$

note that τ is the proper time on the thin-shell hypersurface, and $a(\tau)$ is the shell's radius.

Substituting these in (3.1), we find

$$ds^2=-e^{2\Psi_{\pm}(a_{\pm})}\left[1-\frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right]dt^2_{\pm}+\left[1-\frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right]^{-1}\left(\frac{da_{\pm}}{d\tau}\right)^2 d\tau^2+a_{\pm}(\tau)^2 d\Omega^2_{\pm},$$

(3.84.a)

therefore

$$ds^2=d\tau^2_{\pm}\left[-e^{2\Psi_{\pm}(a_{\pm})}\left[1-\frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right]\dot{t}^2_{\pm}+\left[1-\frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right]^{-1}\dot{a}^2_{\pm}\right]+a_{\pm}(\tau)^2 d\Omega^2_{\pm} \quad (3.84.b)$$

Here we are defining the dot operation as the derivative with respect to τ

$$\dot{t} = \frac{dt}{d\tau} \ , \dot{a} = \frac{da}{d\tau} \ ,$$

and Σ is described by a line element on the shell:

$$ds_{\Sigma}^{2} = -d\tau^{2} + a(\tau)^{2} \mathrm{d}\Omega^{2}. \tag{3.85}$$

We compare (3.85) with (3.84.b) to obtain

$$-e^{2\Psi_{\pm}(a_{\pm})} \left[1 - \frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right] \dot{t}_{\pm}^{2} + \left[1 - \frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right]^{-1} \dot{a}_{\pm}^{2} = -1, \tag{3.86.a}$$

and rearrange the equation to yield

$$\dot{t}_{\pm}^{2} = e^{-2\Psi_{\pm}(a_{\pm})} \left[\frac{\left(1 - \frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right) + \dot{a}_{\pm}^{2}}{\left(1 - \frac{M_{\pm}(a_{\pm})}{a_{\pm}(\tau)}\right)^{2}}\right]. \tag{3.86.b}$$

Take the square root to get

$$\dot{t}_{\pm} = e^{-\Psi_{\pm}(a)} \frac{\sqrt{1 - \frac{M_{\pm}(a)}{a(\tau)} + \dot{a}^{2}}}{1 - \frac{M_{\pm}(a)}{a(\tau)}} \ . \tag{3.87}$$

3.7 Components of The Four-velocity

The four-velocity of the shell is given by

$$U_{\pm}^{\gamma} = \frac{dx^{\gamma}}{d\tau} = \left(\frac{dt}{d\tau}, \frac{da}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = (\ \dot{t}, \dot{a}, 0,0), \tag{3.88}$$

which upon considering (3.87) we find

$$U_{\pm}^{\gamma} = \left(e^{-\Psi_{\pm}(a)} \frac{\sqrt{1 - \frac{M_{\pm}(a)}{a(\tau)} + \dot{a}^{2}}}{1 - \frac{M_{\pm}(a)}{a(\tau)}} \ , \dot{a}, 0, 0\right). \tag{3.89}$$

3.8 The Unit Normal to The Junction Surface

Usually n_{γ} is the unit normal; the sign of $n_{\gamma} n^{\gamma} = \pm 1$ depends on whether the normal is time-like or space-like.

The unit normal n_{γ} is defined as

$$n_{\gamma} = \frac{\mathrm{d}\Sigma}{dx^{\gamma}} \frac{1}{\sqrt{*}}, \tag{3.90}$$

where the hypersurface Σ is described by the equation $\Sigma: \mathrm{r} - \mathrm{a}(\tau)$, and

$$* = g^{\sigma\gamma} \frac{\mathrm{d}\Sigma}{dx^{\sigma}} \frac{\mathrm{d}\Sigma}{dx^{\gamma}} \,, \tag{3.91}$$

and put $r = a$ in (3.3) to yield

$$g_{\pm}^{\sigma\gamma} = diag\left[-e^{-2\Psi_{\pm}(a_{\pm})}\left[1 - \frac{M_{\pm}(a_{\pm})}{a_{\pm}}\right]^{-1}, \left[1 - \frac{M_{\pm}(a_{\pm})}{a_{\pm}}\right], \frac{1}{a_{\pm}^2}, \frac{1}{a_{\pm}^2 \sin\theta_{\pm}^2}\right]. \tag{3.92}$$

Now, we should find $*$, from (3.91) to have:

$$* = g^{tt}\left(\frac{\mathrm{d}\Sigma}{dt}\right)^2 + g^{rr}\left(\frac{\mathrm{d}\Sigma}{dr}\right)^2 + g^{\theta\theta}\left(\frac{\mathrm{d}\Sigma}{d\theta}\right)^2 + g^{\varphi\varphi}\left(\frac{\mathrm{d}\Sigma}{d\varphi}\right)^2, \tag{3.93}$$

where

$$\frac{\mathrm{d}\Sigma}{dt} = \frac{d(r-a)}{dt} = -\frac{da}{dt} = -\frac{da}{d\tau}\frac{d\tau}{dt} = -\frac{\dot{a}}{\dot{t}}\,, \tag{3.94.a}$$

$$\frac{\mathrm{d}\Sigma}{dr} = \frac{d(r-a)}{dr} = 1, \tag{3.94.b}$$

and

$$\frac{\mathrm{d}\Sigma}{d\theta} = \frac{\mathrm{d}\Sigma}{d\varphi} = 0. \tag{3.94.c}$$

Substitute (3.94.a-c) and (3.92) into (3.93), to find

$$* = -e^{-2\Psi}\left(1 - \frac{M(a)}{a}\right)^{-1}\left(-\frac{\dot{a}}{\dot{t}}\right)^2 + \left(1 - \frac{M(a)}{a}\right), \tag{3.95}$$

and substitute $\dot{t}^2$ from (3.86.b) to obtain

$$* = \frac{\left(1 - \frac{M(a)}{a}\right)^2}{\left(1 - \frac{M(a)}{a}\right) + \dot{a}^2}. \tag{3.96}$$

Taking the square root for (3.96), we obtain

$$\sqrt{*} = \frac{\left(1 - \frac{M(a)}{a}\right)}{\sqrt{1 - \frac{M(a)}{a} + \dot{a}^2}}\,. \tag{3.97}$$

The contravariant unit normal vector is defined as:

$$n^{\gamma} = g^{\gamma\beta} n_{\beta} = g^{\gamma\beta} \frac{\mathrm{d}\Sigma}{dx^{\beta}} \frac{1}{\sqrt{*}} \ , \tag{3.98}$$

substitute now (3.97) into (3.98) to get the first component of unit normal vector as

$$n^{t} = g^{tt} \frac{\mathrm{d}\Sigma}{dt} \frac{1}{\sqrt{*}} = \left[-e^{-2\Psi} \left(1 - \frac{M(a)}{a}\right)^{-1} \right] \left[-\frac{\dot{a}}{\dot{t}} \right] \frac{\sqrt{1 - \frac{M(a)}{a} + \dot{a}^2}}{\left(1 - \frac{M(a)}{a}\right)},$$

therefore

$$n^{t} = \frac{\dot{a} e^{-\Psi}}{\left(1 - \frac{M(a)}{a}\right)} \, . \tag{3.99.a}$$

The second component of the unit normal vector is

$$n^{r} = g^{rr} \frac{\mathrm{d}\Sigma}{dr} \frac{1}{\sqrt{*}} = \left(1 - \frac{M(a)}{a}\right) (1) \frac{\sqrt{1 - \frac{M(a)}{a} + \dot{a}^2}}{\left(1 - \frac{M(a)}{a}\right)},$$

with

$$n^{r} = \sqrt{1 - \frac{M(a)}{a} + \dot{a}^2}. \tag{3.99.b}$$

The third component of the unit normal vector is

$$n^{\theta} = g^{\theta\theta} \frac{\mathrm{d}\Sigma}{d\theta} \frac{1}{\sqrt{*}} = 0. \tag{3.99.c}$$

and similarly the fourth component of the unit normal vector is

$$n^{\varphi} = g^{\varphi\varphi} \frac{\mathrm{d}\Sigma}{d\varphi} \frac{1}{\sqrt{*}} = 0. \tag{3.99.d}$$

The components of contravariant unit normal to the junction surface are

$$n_{\pm}^{\gamma} = \left[\frac{\dot{a} e^{-\Psi_{\pm}}}{\left(1 - \frac{M_{\pm}(a)}{a}\right)}, \sqrt{1 - \frac{M_{\pm}(a)}{a} + \dot{a}^2}, 0, 0 \right] . \tag{3.100.a}$$

The components of covariant unit normal to the junction surface are

$$n_{\gamma}^{\pm} = g_{\gamma\sigma}^{\pm} n_{\pm}^{\sigma} = [g_{tt}^{\pm} n_{\pm}^{t}, g_{rr}^{\pm} n_{\pm}^{r}, g_{\theta\theta}^{\pm} n_{\pm}^{\theta}, g_{\varphi\varphi}^{\pm} n_{\pm}^{\varphi}],$$

from (3.100.a) we find

$$n_{\gamma}^{\pm} = \left[-\dot{a} e^{\Psi_{\pm}}, \frac{\sqrt{1-\frac{M_{\pm}(a)}{a}+\dot{a}^2}}{\left(1-\frac{M_{\pm}(a)}{a}\right)}, 0,0 \right]. \tag{3.100.b}$$

From (3.100.a) and (3.100.b), it can be checked that

$$n_{\pm}^{\gamma} n_{\gamma}^{\pm} = +1, \tag{3.100.c}$$

which makes it space-like, indeed $n_{\pm}^{\gamma}$ is normal vector to a time-like hypersurface.

3.9 Extrinsic Curvature

The extrinsic curvature (second fundamental form) associated with the two sides of the shell is

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\sigma\rho}^{\gamma} \frac{\partial x^{\sigma}}{\partial \xi^i} \frac{\partial x^{\rho}}{\partial \xi^j} \right). \tag{3.101}$$

The discontinuity in the second fundamental form is defined as

$$\kappa_{ij} = K_{ij}^{+} - K_{ij}^{-} \;. \tag{3.102}$$

The components of the extrinsic curvature

$$K_{\tau\tau}^{\pm} = -n_t^{\pm} \left(\frac{\partial^2 t}{\partial \tau^2} + \underbrace{\Gamma_{tt}^{t}}_{=0} \left(\frac{\partial t}{\partial \tau}\right)^2 + 2\Gamma_{rt}^{t} \frac{\partial r}{\partial \tau} \frac{\partial t}{\partial \tau} + \underbrace{\Gamma_{rr}^{t}}_{=0} \left(\frac{\partial r}{\partial \tau}\right)^2 \right) - n_r^{\pm} \left(\frac{\partial^2 r}{\partial \tau^2} + \Gamma_{tt}^{r} \left(\frac{\partial t}{\partial \tau}\right)^2 + \right.$$

$$\left. \underbrace{\Gamma_{tr}^{r}}_{=0} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} + \Gamma_{rr}^{r} \left(\frac{\partial r}{\partial \tau}\right)^2 \right),$$

therefore

$$K_{\tau\tau}^{\pm} = -n_t^{\pm} \left(\frac{\partial^2 t}{\partial \tau^2} + 2\Gamma_{rt}^{t} \frac{\partial r}{\partial \tau} \frac{\partial t}{\partial \tau} \right) - n_r^{\pm} \left(\frac{\partial^2 r}{\partial \tau^2} + \Gamma_{tt}^{r} \left(\frac{\partial t}{\partial \tau}\right)^2 + \Gamma_{rr}^{r} \left(\frac{\partial r}{\partial \tau}\right)^2 \right). \tag{3.103}$$

Now, to find $\frac{\partial^2 t}{\partial \tau^2}$, we take derivative of (3.87) with respect to τ which gives

$$\frac{\partial^2 t}{\partial \tau^2} = \dot{a}\left[-\frac{\Psi' e^{-\Psi}}{\left(1-\frac{M}{a}\right)}\sqrt{1-\frac{M}{a}+\dot{a}^2} + \frac{e^{-\Psi}(M-aM')}{2a^2\left(1-\frac{M}{a}\right)}\frac{1}{\sqrt{1-\frac{M}{a}+\dot{a}^2}} + \frac{\ddot{a}e^{-\Psi}}{\left(1-\frac{M}{a}\right)\sqrt{1-\frac{M}{a}+\dot{a}^2}} - \right.$$

$$\left. \frac{e^{-\Psi}(M-aM')}{a^2\left(1-\frac{M}{a}\right)^2}\sqrt{1-\frac{M}{a}+\dot{a}^2}\right], \tag{3.104}$$

and by substituting (3.104), (3.8), (3.11), (3.13), (3.100.b) and (3.87) into (3.103), we find the first component of the extrinsic curvature

$$K_{\tau\tau}^{\pm} = -\left[\frac{\left(\ddot{a}+\frac{\left(M_{\pm}(a)-aM'_{\pm}(a)\right)}{2a^2}\right)}{\sqrt{1-\frac{M_{\pm}(a)}{a}+\dot{a}^2}} + \Psi'_{\pm}(a)\sqrt{1-\frac{M_{\pm}(a)}{a}+\dot{a}^2}\right]. \tag{3.105}$$

It follows that

$$K_{\tau}^{\tau\,\pm} = \underbrace{g_{\pm}^{\tau\tau}}_{=-1} K_{\tau\tau}^{\pm} = -K_{\tau\tau}^{\pm},$$

From (3.105), we find

$$K_{\tau}^{\tau\,\pm} = \frac{\left(\ddot{a}+\frac{\left(M_{\pm}(a)-aM'_{\pm}(a)\right)}{2a^2}\right)}{\sqrt{1-\frac{M_{\pm}(a)}{a}+\dot{a}^2}} + \Psi'_{\pm}(a)\sqrt{1-\frac{M_{\pm}(a)}{a}+\dot{a}^2}\,. \tag{3.106}$$

The second component of the extrinsic curvature is

$$K_{\theta\theta}^{\pm} = -n_t^{\pm}\left(\underbrace{\frac{\partial^2 t}{\partial\theta^2}}_{=0} + \underbrace{\Gamma_{\theta\theta}^{t}}_{=0}\left(\frac{d\theta}{d\theta}\right)^2\right) - n_r^{\pm}\left(\underbrace{\frac{\partial^2 r}{\partial\theta^2}}_{=0} + \Gamma_{\theta\theta}^{r}\underbrace{\left(\frac{d\theta}{d\theta}\right)^2}_{=1}\right),$$

therefore

$$K_{\theta\theta}^{\pm} = -n_r^{\pm}\Gamma_{\theta\theta}^{r}\,, \tag{3.107.a}$$

and by substituting (3.100.b) and (3.16) into (3.107.a), we obtain

$$K_{\theta\theta}^{\pm} = a\sqrt{1-\frac{M_{\pm}(a)}{a}+\dot{a}^2}\,. \tag{3.107.b}$$

We can also find

$$K_{\theta}^{\theta\,\pm} = g_{\pm}^{\theta\theta} K_{\theta\theta}^{\pm} = \frac{1}{a^2} K_{\theta\theta}^{\pm}.$$

From (3.107.b), it follows that

$$K_{\theta}^{\theta\,\pm} = \frac{1}{a}\sqrt{1 - \frac{M_{\pm}(a)}{a} + \dot{a}^2}\,. \qquad (3.108)$$

The third component of the extrinsic curvature

$$K_{\varphi\varphi}^{\pm} = -n_r^{\pm}\left(\Gamma_{\varphi\varphi}^{r}\left(\frac{d\varphi}{d\varphi}\right)^2\right), \qquad (3.109.a)$$

and upon substitution of (3.100.b) and (3.19) into (3.109.a), we find

$$K_{\varphi\varphi}^{\pm} = a\,\sin^2\theta\sqrt{1 - \frac{M_{\pm}(a)}{a} + \dot{a}^2}\,. \qquad (3.109.b)$$

We can also find

$$K_{\varphi}^{\varphi\,\pm} = g_{\pm}^{\varphi\varphi} K_{\varphi\varphi}^{\pm} = \frac{1}{a^2\sin^2\theta} K_{\varphi\varphi}^{\pm},$$

and from (3.109.b), we find

$$K_{\varphi}^{\varphi\,\pm} = \frac{1}{a}\sqrt{1 - \frac{M_{\pm}(a)}{a} + \dot{a}^2}\,. \qquad (3.110)$$

3.10 Lanczos Equation: Surface Stress-energy

The surface stress-energy tensor $S^i_{\ j}$ on Σ yields the surface energy density $S^{\tau}_{\tau} = -\sigma$ and, surface pressures $S^{\theta}_{\theta} = p = S^{\varphi}_{\varphi}$.

$$S^i_{\ j} = -\frac{1}{8\pi}\left(\kappa^i_{\ j} - \delta^i_{\ j}\kappa^k_{\ k}\right). \qquad (3.111)$$

This equation is called Lanczos equation, where $S^i_{\ j} = diag\,[-\sigma, p, p]$ and $\kappa^i_{\ j} = diag\,[\kappa^{\tau}_{\tau}, \kappa^{\theta}_{\theta}, \kappa^{\varphi}_{\varphi}]$.

From the Lanczos equation (3.111), we have

$$S^{\tau}_{\tau} = \frac{\kappa^{\theta}_{\theta}}{4\pi} \quad \Rightarrow \quad \sigma = -\frac{\kappa^{\theta}_{\theta}}{4\pi}, \qquad (3.112.a)$$

sine $\kappa^{\theta}_{\theta} = K_{\theta}^{\theta(+)} - K_{\theta}^{\theta(-)}$, we obtain

$$\sigma = -\frac{1}{4\pi}\left(K_{\theta}^{\theta(+)} - K_{\theta}^{\theta(-)}\right), \qquad (3.112.b)$$

and using (3.108), we obtain

$$\sigma(a) = -\frac{1}{4\pi a}\left[\sqrt{1 - \frac{M_{+}(a)}{a} + \dot{a}^2} - \sqrt{1 - \frac{M_{-}(a)}{a} + \dot{a}^2}\,\right]. \qquad (3.112.c)$$

Again from the Lanczos equation (3.111), we have

$$S_{\theta}^{\theta} = \frac{\kappa_{\tau}^{\tau} + \kappa_{\theta}^{\theta}}{8\pi} \quad \Rightarrow \quad p = \frac{\kappa_{\tau}^{\tau} + \kappa_{\theta}^{\theta}}{8\pi}, \qquad (3.113.a)$$

therefore

$$p = \frac{1}{8\pi}\left(K_{\tau}^{\tau(+)} - K_{\tau}^{\tau\,(-)} + K_{\theta}^{\theta\,(+)} - K_{\theta}^{\theta\,(-)}\right), \qquad (3.113.b)$$

and from (3.106) and (3.108), we obtain

$$p(a) = \frac{1}{8\pi a}\left[\frac{\left(1 + \dot{a}^2 + a\ddot{a} - \frac{\left(M_{+}(a) + a\mathrm{M}'_{+}(a)\right)}{2a}\right)}{\sqrt{1 - \frac{M_{+}(a)}{a} + \dot{a}^2}} + a\Psi'_{+}(a)\sqrt{1 - \frac{M_{+}(a)}{a} + \dot{a}^2} - \right.$$

$$\left. \frac{\left(1 + \dot{a}^2 + a\ddot{a} - \frac{\left(M_{-}(a) + a\mathrm{M}'_{-}(a)\right)}{2a}\right)}{\sqrt{1 - \frac{M_{-}(a)}{a} + \dot{a}^2}} - a\Psi'_{-}(a)\sqrt{1 - \frac{M_{-}(a)}{a} + \dot{a}^2}\right]. \qquad (3.113.c)$$

From the Lanczos equation (3.111), we can find

$$\sigma + 2p = \frac{\kappa_{\tau}^{\tau}}{4\pi}, \qquad (3.114.a)$$

since $\kappa_{\tau}^{\tau} = K_{\tau}^{\tau(+)} - K_{\tau}^{\tau\,(-)}$, we have

$$\sigma + 2p = \frac{1}{4\pi}\left(K_{\tau}^{\tau(+)} - K_{\tau}^{\tau\,(-)}\right). \qquad (3.114.b)$$

From (3.106) we obtain

$$\sigma + 2p = \frac{1}{4\pi}\left[\frac{\left(\ddot{a}+\frac{\left(M_{+}(a)-a\mathrm{M}'_{+}(a)\right)}{2a^{2}}\right)}{\sqrt{1-\frac{M_{+}(a)}{a}+\dot{a}^{2}}} + \Psi'_{+}(a)\sqrt{1-\frac{M_{+}(a)}{a}+\dot{a}^{2}} - \right.$$

$$\left.\frac{\left(\ddot{a}+\frac{\left(M_{-}(a)-a\mathrm{M}'_{-}(a)\right)}{2a^{2}}\right)}{\sqrt{1-\frac{M_{-}(a)}{a}+\dot{a}^{2}}} - \Psi'_{-}(a)\sqrt{1-\frac{M_{-}(a)}{a}+\dot{a}^{2}}\right]. \qquad (3.114.c)$$

3.10.1 Static[2] Space-time

Let us assume the shell is static. That means

$$a = constant = a_0 \quad \Rightarrow \quad \dot{a} = \ddot{a} = 0, \qquad (3.115)$$

using (3.115) in (3.112.c), (3.113.c) and (3.114.c), we find

from (3.112.c)

$$\sigma_0(a_0) = -\frac{1}{4\pi a_0}\left[\sqrt{1-\frac{M_{+}(a_0)}{a_0}} - \sqrt{1-\frac{M_{-}(a_0)}{a_0}}\right]. \qquad (3.116.a)$$

and from (3.113.c) we obtain

$$p_0(a_0) = \frac{1}{8\pi a_0}\left\{\frac{\left(1-\frac{\left(M_{+}(a_0)+a_0\mathrm{M}'_{+}(a_0)\right)}{2a}\right)}{\sqrt{1-\frac{M_{+}(a_0)}{a_0}}} + a_0\Psi'_{+}(a_0)\sqrt{1-\frac{M_{+}(a_0)}{a_0}} - \right.$$

$$\left.\frac{\left(1-\frac{\left(M_{-}(a_0)+a_0\mathrm{M}'_{-}(a_0)\right)}{2a_0}\right)}{\sqrt{1-\frac{M_{-}(a_0)}{a_0}}} - a_0\Psi'_{-}(a_0)\sqrt{1-\frac{M_{-}(a_0)}{a_0}}\right\}. \qquad (3.116.b)$$

Similarly from (3.114.c) we obtain

[2] A space-time is said to be static if it does not change over time and is also irrotational. It is a special case of a stationary spacetime: the geometry of a stationary spacetime does not change in time; however, it can rotate. Thus, the Kerr solution provides an example of a stationary spacetime that is not static; the non-rotating Schwarzschild solution is an example that is static

$$\sigma_0 + 2p_0 = \frac{1}{4\pi}\left\{\frac{\frac{\left(M_+(a_0)-a_0\mathrm{M}'_+(a_0)\right)}{2a_0^2}}{\sqrt{1-\frac{M_+(a_0)}{a_0}}} + \Psi'_+(a_0)\sqrt{1-\frac{M_+(a_0)}{a_0}} - \frac{\frac{\left(M_-(a_0)-a_0\mathrm{M}'_-(a_0)\right)}{2a_0^2}}{\sqrt{1-\frac{M_-(a_0)}{a_0}}} - \right.$$

$$\left.\Psi'_-(a_0)\sqrt{1-\frac{M_-(a_0)}{a_0}}\right\}. \qquad (3.116.c)$$

3.11 Conservation Identity

Now, we need to obtain the first contracted Gauss-Codazzi equation. We start with the Einstein field equations (3.4).

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\ .$$

Multiply both sides by $n^\mu n^\nu$, to find

$$G_{\mu\nu}\, n^\mu n^\nu = R_{\mu\nu}\, n^\mu n^\nu - \frac{1}{2}g_{\mu\nu}n^\nu n^\mu R, \qquad (3.117.a)$$

where $g_{\mu\nu}n^\nu = n_\mu$, to obtain

$$G_{\mu\nu}\, n^\mu n^\nu = R_{\mu\nu}\, n^\mu n^\nu - \frac{1}{2}n_\mu n^\mu R. \qquad (3.117.b)$$

But $n_\mu n^\mu = +1$, therefore

$$G_{\mu\nu}\, n^\mu n^\nu = R_{\mu\nu}\, n^\mu n^\nu - \frac{1}{2}R, \qquad (3.117.c)$$

where $n^\nu = g^{\nu\mu}n_\mu$, so that

$$G_{\mu\nu}\, n^\mu n^\nu = R_{\mu\nu}\, n^\mu g^{\nu\mu} n_\mu - \frac{1}{2}R, \qquad (3.117.d)$$

and snice $n^\mu n_\mu = +1$ and $R_{\mu\nu}\, g^{\nu\mu} = R$, we obtain

$$G_{\mu\nu}\, n^\mu n^\nu = \frac{1}{2}R. \qquad (3.118)$$

Now, we should make use of some basic equations. We start with the equation of Gauss

$$R_{\alpha\beta\gamma\delta}\frac{\partial x^{\alpha}}{\partial \xi^{i}}\frac{\partial x^{\beta}}{\partial \xi^{j}}\frac{\partial x^{\gamma}}{\partial \xi^{k}}\frac{\partial x^{\delta}}{\partial \xi^{l}} = R_{ijkl} - \Delta\big(K_{il}K_{jk} - K_{ik}K_{jl}\big), \qquad (3.119.a)$$

where $\Delta = u^{\alpha}u_{\alpha} = -1$, $\frac{\partial x^{\alpha}}{\partial \xi^{i}} = e_i^{\alpha}$, to find

$$R_{\alpha\beta\gamma\delta}\, e_i^{\alpha} e_j^{\beta} e_k^{\gamma} e_l^{\delta} = R_{ijkl} - \Delta\big(K_{il}K_{jk} - K_{ik}K_{jl}\big)\ . \qquad (3.119.b)$$

Multiply by g^{il}, to obtain

$$R_{\alpha\beta\gamma\delta}\, g^{il}\, e_i^{\alpha} e_j^{\beta} e_k^{\gamma} e_l^{\delta} = g^{il}R_{ijkl} - \Delta\big(g^{il}K_{il}K_{jk} - g^{il}K_{ik}K_{jl}\big), \qquad (3.120)$$

where $g^{il}\, e_i^{\alpha} e_l^{\delta} = g^{\alpha\delta}$, therefore

$$R_{\alpha\beta\gamma\delta}\, g^{\alpha\delta}\, e_j^{\beta} e_k^{\gamma} = R_{jk} - \Delta\big(KK_{jk} - K_{ik}K_j^{i}\big), \qquad (3.121.a)$$

where $R_{\alpha\beta\gamma\delta}\, g^{\alpha\delta} = R_{\beta\gamma}$, thus

$$R_{\beta\gamma}\, e_j^{\beta} e_k^{\gamma} = R_{jk} - \Delta\big(KK_{jk} - K_{ik}K_j^{i}\big)\ . \qquad (3.121.b)$$

Multiply by g^{jk} to obtain

$$R_{\beta\gamma}\ \underbrace{g^{jk}\, e_j^{\beta} e_k^{\gamma}}_{=g^{\beta\gamma}} = g^{jk}R_{jk} - \Delta\left(K\ \underbrace{K_{jk}\, g^{jk}}_{=K} - g^{jk}K_{ik}K_j^{i}\right), \qquad (3.122.a)$$

where $g^{jk}K_j^{i} = K^{ik}$, so that

$$R_{\beta\gamma}\, g^{\beta\gamma} = {}^{(3)}R - \Delta\big(K^2 - K_{ik}\, K^{ik}\big), \qquad (3.122.b)$$

where $R_{\beta\gamma}\, g^{\beta\gamma} = R$, and one obtains

$${}^{(4)}R = {}^{(3)}R - \Delta\big(K^2 - K_{ik}\, K^{ik}\big). \qquad (3.122.c)$$

This equation is called the first contracted Gauss equation. Here ${}^{(3)}R$ is Ricci scalar of 3-metric g^{ij}, K_{ik} is the extrinsic curvature 3-tensor of Σ.

From (3.122.c) put $k = j$ to obtain

$$R = \Delta\,{}^{(3)}R + K^2 - K_{ij}\, K^{ij}, \qquad (3.123)$$

substituting (3.123) into (3.118), we find

$$G_{\mu\nu}\, n^{\mu}n^{\nu} = \frac{1}{2}\left[\Delta\,{}^{(3)}R + K^2 - K_{ij}\, K^{ij}\right], \tag{3.124.a}$$

This equation is called the "Hamiltonian" constraint.

where $\Delta = -1$, therefore

$$G_{\mu\nu}\, n^{\mu}n^{\nu} = \frac{1}{2}\left[K^2 - K_{ij}\, K^{ij} - {}^{(3)}R\right]. \tag{3.124.b}$$

Again from Einstein equation (3.4)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu},$$

multiply by $e_i^{\mu}n^{\nu}$, to find

$$G_{\mu\nu}\, e_i^{\mu}n^{\nu} = R_{\mu\nu}\, e_i^{\mu}n^{\nu} - \frac{1}{2}R\, g_{\mu\nu}\, e_i^{\mu}\, n^{\nu}, \tag{3.125.a}$$

but since $g_{\mu\nu}n^{\nu} = n_{\mu}$, then

$$G_{\mu\nu}\, e_i^{\mu}n^{\nu} = R_{\mu\nu}\, e_i^{\mu}n^{\nu} - \frac{1}{2}R\, \underbrace{e_i^{\mu}n_{\mu}}_{=0}, \tag{3.125.b}$$

therefore

$$G_{\mu\nu}\, e_i^{\mu}n^{\nu} = R_{\mu\nu}\, e_i^{\mu}n^{\nu}. \tag{3.125.c}$$

Now, we start with the equation of Codazzi

$$R_{\alpha\beta\gamma\delta}n^{\alpha}\frac{\partial x^{\beta}}{\partial \xi^{j}}\frac{\partial x^{\gamma}}{\partial \xi^{k}}\frac{\partial x^{\delta}}{\partial \xi^{l}} = \nabla_l K_{jk} - \nabla_k K_{jl}\,, \tag{3.126.a}$$

to rewrite it in the form

$$R_{\alpha\beta\gamma\delta}\, n^{\alpha}e_j^{\beta}e_k^{\gamma}e_l^{\delta} = K_{jk;l} - K_{jl;k}\,, \tag{3.126.b}$$

Multiply by g^{jl} to obtain

$$R_{\alpha\beta\gamma\delta}\, n^{\alpha}e_j^{\beta}e_k^{\gamma}e_l^{\delta}g^{jl} = g^{jl}\left(K_{jk;l} - K_{jl;k}\right), \tag{3.126.c}$$

where $g^{jl} = g^{kl}\delta_k^j$, so that

$$R_{\alpha\beta\gamma\delta}\, n^{\alpha} e_j^{\beta} \underbrace{e_k^{\gamma} e_l^{\delta} g^{kl}}_{=g^{\gamma\delta}} \delta_k^j = K_{k;l}^{l} - K_{;k} \ , \tag{3.127.a}$$

but since $R_{\alpha\beta\gamma\delta}\, g^{\gamma\delta} = R_{\alpha\beta}$, then

$$R_{\alpha\beta}\, n^{\alpha} e_j^{\beta} \delta_k^j = K_{k;l}^{l} - K_{;k} \ , \tag{3.127.b}$$

where $e_j^{\beta} \delta_k^j = e_k^{\beta}$. As a result

$$R_{\alpha\beta}\, n^{\alpha} e_k^{\beta} = K_{k;l}^{l} - K_{;k}, \tag{3.127.c}$$

where $n^{\alpha} = n^{\beta} \delta_{\beta}^{\alpha}$, but $e_k^{\beta} \delta_{\beta}^{\alpha} = e_k^{\alpha}$, then

$$R_{\alpha\beta}\, n^{\beta} e_k^{\alpha} = K_{k;l}^{l} - K_{;k}, \tag{3.127.d}$$

and put $\alpha = \mu,\ \beta = \nu,\ k = i,\ l = j$ to obtain

$$R_{\mu\nu}\, n^{\nu} e_i^{\mu} = K_{i;j}^{j} - K_{;i} \ . \tag{3.128}$$

This equation is the second contracted Gauss-Codazzi equation.

Substituting (3.128) into (3.125.c), we find

$$G_{\mu\nu}\, n^{\nu} e_i^{\mu} = K_{i;j}^{j} - K_{;i} \ . \tag{3.129}$$

This is called The “ADM” constraint.

From Lanczos equation (3.111), we have

$$-8\pi S_j^i = K_j^i - \delta_j^i K_k^k, \tag{3.130.a}$$

and derivative of (3.130.a) with respect to i, yields

$$-8\pi S_{j;i}^i = K_{j;i}^i - \delta_j^i K_{;i} \ . \tag{3.130.b}$$

From (3.129) we have

$$K_{i;j}^{j} = G_{\mu\nu}\, n^{\nu} e_i^{\mu} + K_{;i} \ , \tag{3.131.a}$$

and change $i = j,\ j = i$ to obtain

$$K^i_{j;i} = G_{\mu\nu}\, n^\nu e^\mu_j + K_{;j}\ . \tag{3.131.b}$$

Substituting (3.131.b) into (3.130.b) gives

$$-8\pi S^i_{j;i} = G_{\mu\nu}\, n^\nu e^\mu_j + K_{;j} - \delta^i_j K_{;i}\ , \tag{3.132.a}$$

will $\delta^i_j K_{;i} = K_{;j}$, then it gives

$$-8\pi S^i_{j;i} = G_{\mu\nu}\, n^\nu e^\mu_j, \tag{3.132.b}$$

where $G_{\mu\nu} = 8\pi T_{\mu\nu}$, so that

$$-8\pi S^i_{j;i} = 8\pi T_{\mu\nu}\, n^\nu e^\mu_j, \tag{3.132.c}$$

so that

$$-S^i_{j;i} = \left[T_{\mu\nu}\, n^\nu e^\mu_j\right]^+_-. \tag{3.133}$$

This equation is the conservation identity.

A fundamental relation is the conservation identity. From right hand side of (3.133), we have

$$RHS = T_{\mu\nu}\, n^\nu e^\mu_j = T_{\mu\nu}\, n^\nu u^\mu, \tag{3.134.a}$$

where $e^\mu_j = u^\mu$, therefore

$$T_{\mu\nu}\, n^\nu u^\mu = T_{tt}\, n^t u^t + T_{rr}\, n^r u^r + \underbrace{T_{\theta\theta}\, n^\theta u^\theta}_{=0} + \underbrace{T_{\varphi\varphi}\, n^\varphi u^\varphi}_{=0}, \tag{3.134.b}$$

and from (3.89) and (3.100.a), we have

$$T_{\mu\nu}\, n^\nu u^\mu = \frac{\dot{a}\sqrt{1-\frac{M_\pm(a)}{a(\tau)}+\dot{a}^2}}{1-\frac{M_\pm(a)}{a(\tau)}}\left[T_{tt}\ \frac{e^{-2\Psi_\pm(a)}}{1-\frac{M_\pm(a)}{a(\tau)}} + T_{rr}\left(1-\frac{M_\pm(a)}{a(\tau)}\right)\right]. \tag{3.134.c}$$

From Einstein equations (3.4) we have

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\ , \tag{3.135.a}$$

multiply by $n^\nu u^\mu$ and rearrange the equation to obtain

$$T_{\mu\nu}\, n^{\nu}u^{\mu} = \frac{1}{8\pi} G_{\mu\nu}\, n^{\nu}u^{\mu}, \tag{3.135.b}$$

therefore

$$T_{\mu\nu}\, n^{\nu}u^{\mu} = \frac{1}{8\pi}\left[G_{tt}\, n^{t}u^{t} + G_{rr}\, n^{r}u^{r}\right] . \tag{3.135.c}$$

From (3.76), we have

$$G_{tt} = G_t^t g_{tt} = \frac{M'}{a^2} e^{2\Psi}\left(1 - \frac{M}{a}\right), \tag{3.136}$$

also from (3.79), we have

$$G_{rr} = G_r^r g_{rr} = \frac{2\Psi'}{a} - \frac{M'}{a(a-M)} , \tag{3.137}$$

and after substituting (3.136) and (3.137) into (3.135.c) we obtain

$$\left[T_{\mu\nu}\, n^{\nu}u^{\mu}\right]_{-}^{+} = \frac{\dot{a}}{4\pi a}\left[\Psi'_{+}\sqrt{1 - \frac{M_{+}(a)}{a(\tau)} + \dot{a}^2} - \Psi'_{-}\sqrt{1 - \frac{M_{-}(a)}{a(\tau)} + \dot{a}^2}\right]. \tag{3.138.a}$$

This equation can be rewritten as

$$\left[T_{\mu\nu}\, n^{\nu}u^{\mu}\right]_{-}^{+} = \dot{a}\Upsilon . \tag{3.138.b}$$

Let's introduce

$$\Upsilon = \frac{1}{4\pi a}\left[\Psi'_{+}\sqrt{1 - \frac{M_{+}(a)}{a(\tau)} + \dot{a}^2} - \Psi'_{-}\sqrt{1 - \frac{M_{-}(a)}{a(\tau)} + \dot{a}^2}\right] , \tag{3.139}$$

where Υ is related to the energy-momentum flux that impinges on the shell.

We have, from left hand side of (3.133)

$$-S^{i}_{j;i} = -\left[S^{\tau}_{j;\tau} + S^{\theta}_{j;\theta} + S^{\varphi}_{j;\varphi}\right] = -\left[S^{\tau}_{\tau;\tau} + S^{\theta}_{\tau;\theta} + S^{\varphi}_{\tau;\varphi}\right], \tag{3.140}$$

where

$$S^{\tau}_{\tau;\tau} = S^{\tau}_{\tau,\tau} + S^{\alpha}_{\tau}\Gamma^{\tau}_{\alpha\tau} - S^{\tau}_{\alpha}\Gamma^{\alpha}_{\tau\tau} = S^{\tau}_{\tau,\tau} , \tag{3.141.a}$$

therefore

$$S^{\tau}_{\tau;\tau} = -\frac{d\sigma}{d\tau} . \tag{3.141.b}$$

Also

$$S^{\theta}_{\tau;\theta} = S^{\theta}_{\tau,\theta} + S^{\alpha}_{\tau}\Gamma^{\theta}_{\alpha\theta} - S^{\theta}_{\alpha}\Gamma^{\alpha}_{\tau\theta} \,, \tag{3.142.a}$$

where $S^{\theta}_{\tau,\theta} = 0$, so that

$$S^{\theta}_{\tau;\theta} = -(\sigma + p)\Gamma^{\theta}_{\tau\theta} \,, \tag{3.142.b}$$

Now, to find $\Gamma^{\theta}_{\tau\theta}$, from (3.5) and (3.85) we have

$$\Gamma^{\theta}_{\tau\theta} = \tfrac{1}{2} g^{\theta\theta}\, g_{\theta\theta,\tau} = \frac{\dot{a}}{a} \,, \tag{3.143}$$

and substituting (3.143) into (3.142.b), we find

$$S^{\theta}_{\tau;\theta} = -(\sigma + p)\frac{\dot{a}}{a} \,. \tag{3.144}$$

We have, next

$$S^{\varphi}_{\tau;\varphi} = S^{\varphi}_{\tau,\varphi} + S^{\alpha}_{\tau}\Gamma^{\varphi}_{\alpha\varphi} - S^{\varphi}_{\alpha}\Gamma^{\alpha}_{\tau\varphi}, \tag{3.145.a}$$

since $S^{\varphi}_{\tau,\varphi} = 0$, then

$$S^{\varphi}_{\tau;\varphi} = -(\sigma + p)\Gamma^{\varphi}_{\tau\varphi}, \tag{3.145.b}$$

Now, we will find $\Gamma^{\varphi}_{\tau\varphi}$, similar to (3.5) and (3.85) we have

$$\Gamma^{\varphi}_{\tau\varphi} = \tfrac{1}{2} g^{\varphi\varphi}\, g_{\varphi\varphi,\tau} = \frac{\dot{a}}{a} \,, \tag{3.146}$$

and after substituting (3.146) into (3.145.b), we find

$$S^{\varphi}_{\tau;\varphi} = -(\sigma + p)\frac{\dot{a}}{a} \,. \tag{3.147}$$

Substitute (3.141.b), (3.144), and (3.147) into (3.140) to obtain

$$-S^{i}_{j;i} = \frac{d\sigma}{d\tau} + \frac{2\dot{a}}{a}(\sigma + p). \tag{3.148}$$

Substitute next (3.148) and (3.138.b) into (3.133), to obtain

$$\frac{d\sigma}{d\tau} + \frac{2\dot{a}}{a}(\sigma + p) = \Upsilon\dot{a}\ . \qquad (3.149)$$

The area of thin-shell is $A = 4\pi a^2$. Derivative of the area of thin-shell with respect to τ is

$$\frac{dA}{d\tau} = 8\pi a\dot{a}\ , \qquad (3.150.a)$$

This equation can be rewritten as

$$\frac{1}{A}\frac{dA}{d\tau} = 2\frac{\dot{a}}{a}\,, \qquad (3.150.b)$$

substituting (3.150.b) into (3.149), we obtain this relation

$$\frac{d\sigma}{d\tau} + \frac{1}{A}\frac{dA}{d\tau}(\sigma + p) = \Upsilon\dot{a}, \qquad (3.151.a)$$

or equivalently

$$\frac{d(\sigma A)}{d\tau} + p\frac{dA}{d\tau} = \Upsilon A\dot{a}. \qquad (3.151.b)$$

The first term $\frac{d(\sigma A)}{d\tau}$ is the change in the total energy $p\frac{dA}{d\tau}$,stands for the work done by the surface while $\Upsilon A\dot{a}$ can by considered as the work done due to an external work.

From (3.151.a) and (3.150.b) we have

$$\frac{d\sigma}{d\tau} = -\frac{2\dot{a}}{a}(\sigma + p) + \Upsilon\dot{a}\ , \qquad (3.152.a)$$

since we can write $\frac{d\sigma}{d\tau} = \frac{d\sigma}{da}\frac{da}{d\tau} = \dot{a}\frac{d\sigma}{da}$, we have

$$\dot{a}\frac{d\sigma}{da} = -\frac{2\dot{a}}{a}(\sigma + p) + \Upsilon\dot{a}, \qquad (3.152.b)$$

therefore

$$\sigma' = -\frac{2}{a}(\sigma + p) + \Upsilon\ , \qquad (3.153)$$

where a prime denotes differentiation with respect to a, $\sigma' = \frac{d\sigma}{da}$

Chapter 4

STABILITY OF THIN-SHELL IN 3+1-DIMENSIONAL STATIC SPHERICALLY SYSTEM BULK

4.1 Equation of Motion

It is useful to rearrange the surface energy density $\sigma(a)$ into the form $\frac{1}{2}\dot{a}^2 + V(a) = 0$.

From (3.112.c) we have

$$-4\pi a\sigma = \sqrt{1 - \frac{M_+(a)}{a} + \dot{a}^2} - \sqrt{1 - \frac{M_-(a)}{a} + \dot{a}^2}, \tag{4.1}$$

squaring the parties one obtains

$$(4\pi a\sigma)^2 = 1 - \frac{M_+}{a} + \dot{a}^2 - 2\sqrt{1 - \frac{M_+}{a} + \dot{a}^2}\sqrt{1 - \frac{M_-}{a} + \dot{a}^2} + 1 - \frac{M_-}{a} + \dot{a}^2, \tag{4.2}$$

which can be rearranged

$$(4\pi a\sigma)^2 + \left(\frac{M_+}{a} + \frac{M_-}{a}\right) - 2 - 2\dot{a}^2 = -2\sqrt{1 - \frac{M_+}{a} + \dot{a}^2}\sqrt{1 - \frac{M_-}{a} + \dot{a}^2}. \tag{4.3}$$

Now, put

$$Y = (4\pi a\sigma)^2 + \left(\frac{M_+}{a} + \frac{M_-}{a}\right) - 2, \tag{4.4}$$

$$X = \dot{a}^2, \tag{4.5}$$

and

$$\delta_1 = 1 - \frac{M_+}{a},\ \ \delta_2 = 1 - \frac{M_-}{a}\,, \tag{4.6}$$

and after substituting (4.5), (4.6) and (4.7) in (4.4), we find

$$Y - 2X = -2\sqrt{\delta_1 + X}\sqrt{\delta_2 + X}. \tag{4.7}$$

Squaring the parties and rearrange the equation to find

$$Y^2 - 4X(Y + \delta_1 + \delta_2) - 4\delta_1\delta_2 = 0\ . \tag{4.8}$$

Next, substitute (4.5), (4.6) and (4.7) into (4.8) to obtain

$$\left[(4\pi a\sigma)^2 + \left(\frac{M_+}{a} + \frac{M_-}{a}\right) - 2\right]^2 - 4\dot{a}^2(4\pi a\sigma)^2 - 4\left(1 - \frac{M_+}{a}\right)\left(1 - \frac{M_-}{a}\right) = 0, \tag{4.9}$$

so that

$$\left[(4\pi a\sigma)^2 + \frac{M_+}{a} + \frac{M_-}{a} - 2\right]^2 - 4\dot{a}^2(4\pi a\sigma)^2 - 4 + 4\left(\frac{M_+}{a} + \frac{M_-}{a}\right) - 4\frac{M_+M_-}{a^2} = 0. \tag{4.10}$$

Dividing by $-4(4\pi a\sigma)^2$, we find

$$\dot{a}^2 + \frac{1}{(4\pi a\sigma)^2} - \frac{1}{(4\pi a\sigma)^2}\left(\frac{M_+}{a} + \frac{M_-}{a}\right) + \frac{1}{(4\pi a\sigma)^2}\frac{M_+M_-}{a^2} - \frac{1}{4(4\pi a\sigma)^2}\left[(4\pi a\sigma)^2 + \frac{M_+}{a} + \frac{M_-}{a} - 2\right]^2 = 0, \tag{4.11}$$

therefore

$$\dot{a}^2 + \frac{1}{(4\pi a\sigma)^2} - \frac{1}{(4\pi a\sigma)^2}\left(\frac{M_+}{a} + \frac{M_-}{a}\right) + \frac{1}{(4\pi a\sigma)^2}\frac{M_+M_-}{a^2} - \frac{1}{4(4\pi a\sigma)^2}\left[((4\pi a\sigma)^2 - 2)^2 + 2((4\pi a\sigma)^2 - 2)\left(\frac{M_+}{a} + \frac{M_-}{a}\right) + \left(\frac{M_+}{a} + \frac{M_-}{a}\right)^2\right] = 0\ . \tag{4.12}$$

Rearrange this equation to obtain

$$\dot{a}^2 + 1 - \left(\frac{M_+ + M_-}{2a}\right) - (2\pi a\sigma)^2 - \frac{1}{(8\pi a\sigma)^2}\left(\frac{M_+}{a} - \frac{M_-}{a}\right)^2 = 0, \tag{4.13}$$

therefore

$$\frac{1}{2}\dot{a}^2 + \frac{1}{2}\left[1 - \left(\frac{M_+ + M_-}{2a}\right) - (2\pi a\sigma)^2 - \frac{1}{(8\pi a\sigma)^2}\left(\frac{M_+}{a} - \frac{M_-}{a}\right)^2\right] = 0. \tag{4.14}$$

This equation is the equation of motion, where $V(a)$ is the thin-shell potential.

$$V(a) = \frac{1}{2}\left[1 - \left(\frac{M_+ + M_-}{2a}\right) - (2\pi a\sigma)^2 - \frac{1}{(8\pi a\sigma)^2}\left(\frac{M_+}{a} - \frac{M_-}{a}\right)^2\right]. \tag{4.15}$$

The equation (4.14) can be rewritten as

$$\frac{1}{2}\dot{a}^2 + V(a) = 0. \tag{4.16}$$

This equation implies that $V(a)$ is not the type of external potential usually found in classical mechanics. The reason being that "the total energy" vanishes identically. Accordingly, every perturbation of the kinetic energy term in (4.16) will be compensated with a perturbation of the potential.

4.2 Stability of The Thin-shell

To analyze the stability of the thin-shell, we should consider $V(a)$, $V'(a)$ and $V''(a)$ around an assumed static solution, a_0.

Now, we will find $V(a)|_{a=a_0}$. From (4.15) we find

$$V(a)|_{a=a_0} = \frac{1}{2}\left[1 - \left(\frac{M_+ + M_-}{2a_0}\right) - (2\pi a_0 \sigma_0)^2 - \frac{1}{(8\pi a_0 \sigma_0)^2}\left(\frac{M_+}{a_0} - \frac{M_-}{a_0}\right)^2\right] = 0, \tag{4.17}$$

and we can rewrite (4.17) in the form

$$1 - \left(\frac{M_+ + M_-}{2a_0}\right) - (2\pi a_0 \sigma_0)^2 = \frac{1}{(8\pi a_0 \sigma_0)^2}\left(\frac{M_+}{a_0} - \frac{M_-}{a_0}\right)^2. \tag{4.18}$$

Multiply by $(8\pi a_0 \sigma_0)^2$ to obtain

$$(8\pi a_0 \sigma_0)^2 - (8\pi a_0 \sigma_0)^2\left(\frac{M_+ + M_-}{2a_0}\right) - (4\pi a_0 \sigma_0)^4 = \left(\frac{M_+}{a_0} - \frac{M_-}{a_0}\right)^2. \tag{4.19}$$

Substituting σ_0 from (3.116.a) we obtain

$$(8\pi a_0)^2 \frac{1}{(4\pi a_0)^2}\left(2 - \frac{M_+}{a_0} - 2\sqrt{1 - \frac{M_+}{a_0}}\sqrt{1 - \frac{M_-}{a_0}} - \frac{M_-}{a_0}\right) -$$

$$\left(\frac{8\pi a_0}{4\pi a_0}\right)^2\left(\frac{M_+ + M_-}{2a_0}\right)\left(2 - \frac{M_+}{a_0} - 2\sqrt{1 - \frac{M_+}{a_0}}\sqrt{1 - \frac{M_-}{a_0}} - \frac{M_-}{a_0}\right) - \left(\left(1 - \frac{M_+}{a_0}\right) -\right.$$

$$\left. 2\sqrt{1 - \frac{M_+}{a_0}}\sqrt{1 - \frac{M_-}{a_0}} + \left(1 - \frac{M_-}{a_0}\right)\right)^2 = \left(\frac{M_+}{a_0} - \frac{M_-}{a_0}\right)^2, \tag{4.20}$$

which can be rearranged

$$V(a)|_{a=a_0} = 0. \tag{4.21}$$

This equation implies the thin shell is in equilibrium at $a = a_0$.

Now, we will find the first derivative of the potential $\left.\frac{dV(a)}{da}\right|_{a=a_0}$. From (4.15) we can find

$$\left.\frac{dV(a)}{da}\right|_{a=a_0} = \frac{M_+ + M_-}{2{a_0}^2} - \frac{1}{2a_0}(M'_+ + M'_-) - 2a_0(2\pi\sigma_0)^2 - 2\sigma_0\sigma'_0(2\pi a_0)^2 + \frac{4}{a_0(8\pi a_0\sigma_0)^2}\left(\frac{M_+ - M_-}{a_0}\right)^2 + \frac{2\sigma'_0}{\sigma_0(8\pi a_0\sigma_0)^2}\left(\frac{M_+ - M_-}{a_0}\right)^2 - \frac{2}{(8\pi a_0\sigma_0)^2}\left(\frac{M_+ - M_-}{a_0}\right)\left(\frac{M'_+ + M'_-}{a_0}\right) = 0\ . \tag{4.22}$$

We can write this equation in the form

$$\frac{M_+ + M_-}{2{a_0}^2} - \frac{1}{2a_0}(M'_+ + M'_-) - 2a_0(2\pi\sigma_0)^2 - 2\sigma_0\sigma'_0(2\pi a_0)^2 = -\frac{4}{a_0(8\pi a_0\sigma_0)^2}\left(\frac{M_+ - M_-}{a_0}\right)^2 - \frac{2\sigma'_0}{\sigma_0(8\pi a_0\sigma_0)^2}\left(\frac{M_+ - M_-}{a_0}\right)^2 + \frac{2}{(8\pi a_0\sigma_0)^2}\left(\frac{M_+ - M_-}{a_0}\right)\left(\frac{M'_+ + M'_-}{a_0}\right). \tag{4.23}$$

Multiply by $\sigma_0(8\pi a_0\sigma_0)^2$ to find

$$\left.\frac{dV(a)}{da}\right|_{a=a_0} = 32\pi^2\sigma_0^3(M_+ + M_-) - 32\pi^2 a_0\,\sigma_0^3(M'_+ + M'_-) - \pi(8\pi a_0)^3\sigma_0^5 - 2\sigma_0^4\sigma'_0(4\pi a_0)^4 = -\frac{4\sigma_0}{a_0}\left(\frac{M_+ - M_-}{a_0}\right)^2 - 2\sigma'_0\left(\frac{M_+ - M_-}{a_0}\right)^2 + 2\sigma_0\left(\frac{M_+ - M_-}{a_0}\right)\left(\frac{M'_+ + M'_-}{a_0}\right). \tag{4.24}$$

From (3.116.a) we can find the first derivative of σ_0 with respect to a

$$\left.\frac{d\sigma_0}{da}\right|_{a=a_0} = \frac{1}{4\pi{a_0}^2}\left[\sqrt{1 - \frac{M_+}{a_0}} - \sqrt{1 - \frac{M_-}{a_0}}\right] - \frac{1}{8\pi a_0}\left[\frac{\frac{M_+}{{a_0}^2} - \frac{M'_+}{a_0}}{\sqrt{1 - \frac{M_+}{a_0}}} - \frac{\frac{M_-}{{a_0}^2} - \frac{M'_-}{a_0}}{\sqrt{1 - \frac{M_-}{a_0}}}\right]. \tag{4.25}$$

Substituting (3.116.a) and (4.25) into (4.24) and rearranging the equation it gives

$$\left.\frac{dV(a)}{da}\right|_{a=a_0} = 0\ . \tag{4.26}$$

The second derivative of the potential $\left.\frac{d^2V(a)}{da^2}\right|_{a=a_0}$, reads

$$\left.\frac{d^2V(a)}{da^2}\right|_{a=a_0} \neq 0 = \begin{cases} V''(a_0) > 0 \ implies\ stability \\ \\ V''(a_0) < 0 \ implies\ instability \end{cases} . \tag{4.27}$$

Example

Suppose the exterior space-time has the metric for a thin-shell

$$ds_+^2 = -r_+^{\alpha} dt_+^2 + dr_+^2 + r_+^2 d\Omega_+^2 \quad , \tag{4.28}$$

and the interior space-time

$$ds_-^2 = -dt_-^2 + dr_-^2 + r_-^2 d\Omega_-^2\ . \tag{4.29}$$

Contrast these equations with (3.1) to obtain

$$M_+(r) = M_-(r) = \Psi_-(r) = 0 \quad , \tag{4.30}$$

and

$$\Psi_+(r) = \frac{\alpha}{2} \ln r \quad . \tag{4.31}$$

So the stress-energy tensor should have these components

$$T_{\gamma}^{\sigma} = diag\left[0, \frac{\alpha}{r^2}, \frac{\alpha^2}{4r^2}, \frac{\alpha^2}{4r^2}\right] \qquad , \quad \alpha > 0\ ,$$

notice the mass-energy density $\rho = 0$. It is a black point solution to the Einstein's equation.

The surface energy density is given by (3.112.c)

$$\sigma(a) = \sigma_0(a_0) = 0\ . \tag{4.32}$$

The surface pressure is given by (3.113.c)

$$p(a) = \frac{\alpha}{16\pi a}\sqrt{1+\dot{a}^2}\ , \tag{4.33}$$

and at static space-time

$$p_0(a_0) = \frac{\alpha}{16\pi a_0}\ . \tag{4.34}$$

It is useful to rearrange the surface pressures into the form.

$$p(a) = p_0(a_0)\ , \tag{4.35}$$

so we have

$$\frac{1}{a}\sqrt{1+\dot{a}^2} = \frac{1}{a_0}\ . \tag{4.36}$$

Squaring the parties one obtains

$$1+\dot{a}^2 = \left(\frac{a}{a_0}\right)^2,$$

which can be rearranged

$$\frac{1}{2}\dot{a}^2 + \frac{1}{2}\left[1-\left(\frac{a}{a_0}\right)^2\right] = 0\ . \tag{4.37}$$

This equation is the equation of motion, where $V(a)$ is the thin-shell potential.

$$V(a) = \frac{1}{2}\left[1-\left(\frac{a}{a_0}\right)^2\right]. \tag{4.38}$$

Now, we should consider $V(a)$, $V'(a)$ and $V''(a)$ around an assumed static equilibrium, a_0. The closed forms of $V(a)|_{a=a_0}$, $\frac{dV(a)}{da}\Big|_{a=a_0}$ and $\frac{d^2V(a)}{da^2}\Big|_{a=a_0}$ are found to be

$$V(a)|_{a=a_0} = 0. \tag{4.39}$$

The first derivative of the potential $V'(a)$ is

$$\left.\frac{dV(a)}{da}\right|_{a=a_0} = -\frac{1}{a_0} \quad . \tag{4.40}$$

The second derivative of the potential $V''(a)$ is

$$\left.\frac{d^2V(a)}{da^2}\right|_{a=a_0} = -\left(\frac{1}{a_0}\right)^2 < 0 \quad . \tag{4.41}$$

As it is observed, both $V'(a_0)$ and $V''(a_0)$ are non-zero and also $V''(a_0) < 0$ which indicates the thin-shell is not stable.

Chapter 5

CONCLUSION

In this thesis, we have studied the Israel junction condition for a smooth joining of two metrics at a time-like hypersurface. However, we have constructed spherically symmetric thin-shell supported by two distinct space-time manifolds. We have used cut and paste procedure in order to build a class of 4-dimensional space-times, with 3-dimensional time-like transition layer. We have studied the unit normal to the junction surface and we have shown $\epsilon = +1$ which makes it space-like, indeed a time-like hypersurface.

We have considered the extrinsic curvature and the discontinuity in the second fundamental form (extrinsic curvature). We have analyzed the Lanczos equation.

We have also analyzed the equation of motion and performed a linearized stability analysis, after obtaining the static solution. The stability analysis, then, has been reduced to the study of the sign of the second derivative of an effective potential evaluated at the static solution, a_0

As an example, we presented the thin-shell with outside metric, a black point space-time and inside a flat space-time. With an equation of state $p = p_0$, and $\sigma = \sigma_0 = 0$, we showed that thin-shell is unstable.

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