On Caputo Type Sequential Fractional Differential Equations

Hemn Pirot Hussein

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
January 2017
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Mustafa Tümer
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Sonuç Zorlu Oğurlu
Supervisor

Examining Committee

1. Prof. Dr. Nazim Mahmudov
2. Prof. Dr. Sonuç Zorlu Oğurlu
3. Prof. Dr. Mehmet Ali Özarslan

ABSTRACT

The goal of this thesis is to give basic information about fractional calculus, and fractional differential equations of different types and study the existence and uniqueness of certain type of fractional differential equation, namely the Caputo type sequential fractional differential equations. Fixed point theorems due to Banach, Krasnoselskii, and Leray-Schauder alternative criterion is applied to obtain the desired results. The results are well illustrated with the aid of examples.

Keywords: sequential fractional derivative, integral boundary conditions, fractional differential equation, fixed point theorems

ÖZ

Bu tezin amacı, kesirli kalkülüs ve farklı tipteki kesirli diferansiyel denklemleri hakkındaki temel bilgileri vermek ve belirli tip kesirli diferansiyel denklemin, yani Caputo tipi ardışık kesirli diferansiyel denklemlerin varlığını ve tekliğini incelemektir. İstenen sonuçları elde etmek için Banach, Krasnoselskii ve Leray-Schauder alternatif kriterlere göre sabit nokta teoremleri uygulanmaktadır. Sonuçlar, örnekler yardımıyla gösterilmiştir.

Anahtar Kelimeler: dizisel kesirli türev, integral sınır koşulları, kesirli diferansiyel denklemler, sabit nokta teoremleri

To My Family

ACKNOWLEDGMENT

I appreciate and very thanks full for this big chance God gave me to come to EMU and have this kind of teachers in EMU who advise me, support me and gave me all I need during my graduation study, especially my best teacher Prof. Dr. Sonuç Zorlu Oğurlu. I would like to tell her I am so happy to have you as my supervisor and thank you for all your support, patience and continuous guidance through the research, and for your corrections in the text. Indeed, she is a brilliant advisor.

Finally, I would like to extend my deep love and compassion to my parents and family members, actually without you I am nothing and I cannot do anything, God bless you all. I could continue and come to the end because of your encouragement, emotional supports that made me strong to achieve my aim.

TABLE OF CONTENTS

LIST OF ABBREVIATION

FLDE	First-Order Linear Differential Equations
RHS	Right Hand Side
LHS	Left Hand Side
BVP	Boundary Value Problem

Chapter 1

INTRODUCTION

The subject of fractional calculus that is, calculus of integrals and derivatives of any arbitrary real and complex order has gained considerable popularity and importance during the past three decades or so, due to mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

1.1 Brief History

Fractional calculus was introduced on September 30, 1695. On that day, Leibniz wrote a letter to L'Hôpital, raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order derivatives. L'Hôpital wanted to know the result for the derivative of order n = 1/2. Leibniz replied that "one day, useful consequences will be drawn" and, in fact, his vision became a reality. However, the study of non-integer order derivatives did not appear in the literature until 1819, when Lacroix presented a definition of fractional derivative based on the usual expression for the nth derivative of the power function (Lacroix 1819). Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., noninteger) differential operators were introduced: the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz (Hilfer 2000; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993) and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993) [10].

L'Hopital and Leibniz's primary investigation for fractional calculus was the initial research showed a creative mind in mathematics, Fourier, Laplace, Euler were among those who involved with fractional calculus and the mathematics results.

Caputo enhances the more established meaning of the Riemann-Liouville fractional derivative for purpose of utilizing integer order initial conditions to find a solution of these fractional order derivatives. In 1996 Kolowankar reformulate the Riemann-Liouville fractional derivative so as to enable him differentiate no-where differentiable fractal functions [2].

Chapter 2

REVIEW OF FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATION

2.1 The Gamma Function

It is known that the gamma function is basically attached to fractional calculus. The simplest clarification of the gamma function is just all-inclusive statement of the factorial for every real number. The gamma function is known by

$$\Gamma(h) = \int_0^{\infty} e^{-x} x^{h-1}\, dx \ , \qquad \forall h \in \mathbb{R} \,/\, \{\dots, -3, -2, -1, 0\} \tag{2.1}$$

Gamma function has same properties that make it more useful. For first we can see the equation given by,

$$\Gamma(h+1) = h\,\Gamma(h), \quad \text{when } h \in \mathbb{N}_+ , \qquad \Gamma(h) = (h-1)! \tag{2.2}$$

Also, we can describe gamma function as $Q(t)$, which later on will get to be distinctly appropriate for showing other types of the fractional integrals.

$$Q(x) = \frac{x_+^{\alpha+1}}{\Gamma(\alpha)} \ . \tag{2.3}$$

2.2 Beta Function

It is known that the Euler Integral and the Beta function are very useful in fractional calculus. The depiction answer is not only the act of different gamma functions but rather additionally profits a shape that is ordinarily similar to the fractional derivative/integral of few functions, primarily polynomials of the frame x^a and the Mittag- Leffer function. The Beta integral is characterized by (2.4) as

$$B(x,y) = \int_0^1 (1-u)^{x-1} u^{y-1} du\,, \qquad where\ \ x,y \in \mathbb{R}_+ \tag{2.4}$$

and

$$B(x\,,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

2.3 The Mittag-Leffler Function

Another huge function is Mittag-Leffler which has a pervasive use in the area of Fractional integral. In reality, the exponential function is exceptionally definite frame; the definition of the Mittag-Leffler is given by (2.5) [7].

$$E_{\varepsilon}(x) = \sum_{h=1}^{\infty} \frac{x^{k}}{\Gamma(\varepsilon h + 1)}\,, \qquad \varepsilon > 1 \tag{2.5}$$

Also, in the two point of view, ε *and* δ , the Mittag-Leffler function is represented in such a way that

$$E_{\varepsilon,\delta}(x) = \sum_{h=1}^{\infty} \frac{x^{k}}{\Gamma(\varepsilon h + \delta)}\,, \quad \varepsilon, \delta > 1. \tag{2.6}$$

2.4. Definition of Fractional Integral and Fractional Derivative

2.4.1 The Abel's Integral Equation

Let $0 < \alpha < 1$. The integral equation (2.7) is called as the Abel's equation

$$f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\, du, \quad t > 0\,. \tag{2.7}$$

2.4.2 The Fractional Integral

In the Introduction part we said that the possibility of plan for fractional integrals and derivatives was a characteristic improvement of the fractional exponent and derivatives and integer order integrals. In addition, the normal detailing for the Fractional integral can be gotten specifically from a conventional articulation of the rehashed integration of a function. This line is normally connected to the Riemann-Liouville approach. (2.8) proves the recipe regularly perceived to Cauchy for assessing the n^{th} reconciliation of the function $f(\mu)$ [8].

$$\int \dots \dots \int_0^s f(\mu)d\mu = \frac{1}{(n-1)!}\int_0^s (s-\mu)^{n-1} f(\mu)d\mu \quad . \tag{2.8}$$

Then n is restricted to be an integer. The main limitation is the usage of the factorial which in spirit has not any meaning for non-integer values. The gamma function is anyway a logical comprehensive of the factorial for all material. Therefore, by exchanging the factorial expression for the gamma function correspondingly, we can simplify equation (2.8) for all $\beta \in \mathbb{R}_+$, as shown in (2.9).

$$I_{a+}^{\beta} f(h) = \frac{1}{\Gamma(\beta)}\int_a^t (h-\mu)^{\beta-1} f(\mu)d\mu$$

$$I_{b-}^{\beta} f(h) = \frac{1}{\Gamma(\beta)}\int_x^b (\mu-h)^{\beta-1} f(\mu)\, d\mu \tag{2.9}$$

The former equation is known as left side integral and the latter as right side integral.

2. 4. 3 The Fractional Derivative

In view, the inversion of Abel's equation (2.7) can be used to expand (2.8) and (2.9). We land at the function $f(x)$ given in the interval $[a, b]$, each of expression (2.10) and (2.11) is called fractional derivative of order α where $0 < \alpha < 1$ are left sided and right sided respectively which are usually formed as Riemann-Liouville derivative.[8]

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-u)^{-\alpha} f(u)du \tag{2.10}$$

$$D_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (u-t)^{-\alpha} f(u)du \tag{2.11}$$

Definition 2.5. For $(k-1)$ times absolutely continuous function $g:[0,\infty)\rightarrow R$, the Caputo derivative fractional order q is characterized as

$$^{c}D^{\,q}g(x) = \frac{1}{\Gamma(n-q)} \int_0^x (x-t)^{n-q-1} g^{(n)}(t)dt, \tag{2.12}$$

with $\quad n-1<q<n,\ n=[q]+1$

where $[q]$ means the integral part of number q.

Definition 2.6. [2] The sequential fractional differential for a sufficiently even function $g\ (t)$ is defined as

$$D^{\delta}\, g(t) = D^{\delta_1}D^{\delta_2}\cdots\ D^{\delta_k}g(t), \tag{2.13}$$

where $\delta\ =(\delta_1,\cdots,\delta_k)$ is a multi-index.

Overall, the operator D^{δ} in (2.13) can either be Caputo or Riemann-Liouville or some other model of integer differential operator. For example,

$$^{c}D^{\,q}g(h) = D^{-(n-q)}\left(\frac{d}{dt}\right)^{n} g(h)\,, \qquad n-1<q<n \tag{2.14}$$

where $D^{-(n-q)}$ is fractional integral operator of order $n-q$. Here we complement that

$$D^{-p}f(h) = I^{p}f(h), \qquad q=n-p. \tag{2.15}$$

Definition 2.7. Give (X,d) a chance to be a metric space and let $k:\ X\ \rightarrow\ X$ be a mapping:

(i) A point $t\ \in\ X$ is named a fixed point of f if $t=\ k(t)$.

(ii) k is known as contraction if there exists a fixed constant $h<\ 1$ with the end goal that

$$d\big(k(t),k(y)\big) \le hd(t,y), \qquad for\ all\ t,y\ \in\ X. \tag{2. 16}$$

A contraction mapping is again perceived as Banach contraction. In the event that we substitute the inequality (2.16) with strict inequality and $k = 1$, at that point f is called strictly contractive. On the off chance that (2.16) holds for $k = 1$, then f is called non-expansive, and if (2.16) holds for fixed $k < \infty$, then f is named Lipschitz continuous.

Definition 2.8. Krasnoselskii's Fixed Point Theorem [1]. Let M be a closed, convex, non-empty set in a Banach Space $(X, \|.\|)$ and $P = X + Y$ be mapping with the end goal that:

- **(i)** X is continuous and compact,
- **(ii)** $Yx + Xy \in M$ for each $x, y \in M$.
- **(iii)** Y is a contraction mapping.

Consequently, P has a fixed point. A cautious analysis of the confirmation make realized that (ii) requires just to test that $Yx + Xy \in M$ after $x = Yx + Xy$. The confirmation moreover yields a framework for the presentation such that x is in M. [9]

Definition 2.9. Let X be a convex set in a real vector space and let $f : X \to \mathbb{R}$ be a function.

- f is called convex if:

$$f(xt_1 + (1-x)t_2) \leq xf(t_1) + (1-x)f(t_2), \qquad \forall t_1 t_2 \in X, \forall x \in [0,1]$$

- f is called strictly convex if:

$$f(xt_1 + (1-x)t_2) < xf(t_1) + (1-x)f(t_2), \qquad \forall t_1 \neq t_2 \in X, \forall x \in (0,1)$$

A function f is said to be strictly concave if f is strictly convex.

Another significant theorem of nonlinear functional analysis is the Leray-Schauder nonlinear Alternative substantiated in 1934. At present there exist a few sorts of

Leray-Schauder sort options demonstrated by different methods. We letter that, the standard Leray-Schauder nonlinear Alternative has few entries to ordinary differential equations too. Our uses of Leray-Schauder sort other options to the learning of complement issues represent another method for use of this traditional outcome.

Definition 2.10. Let $(K, \langle .,. \rangle)$ be a Hilbert Space and $Y \subset K$ a non-empty subset. Give $f: K \to K$ a chance to be a mapping. We say that f is compact on Y if $f(Y)$ is relatively compact and we say that *f* is absolutely continuous if f is continuous and for any bounded set $B \subset K$, $f(B)$ is relatively compact. We will indicate by ∂Y the boundary of Y. We will use similarly the accompanying traditional thought. We say that *f* is a completely continuous field, if f has a demonstration of form $f(x) = x - T(x)$, for all $x \in K$, where $T: K \to K$ is a completely continuous mapping [5].

Definition 2.11. (The Arzela-Ascoli Theorem) If a sequence $\{f_n\}_1^\infty$ in $C(X)$ is equicontinuous and bounded then it has a uniformly convergent subsequence, in this statement, [6]

- **a)** $K \subset C(X)$ is equicontinuous, funds for each $\mu > 0$ there exists $\varepsilon > 0$ (which depends just on μ) such that for $x, y \in X$: $d(x, y) < \varepsilon \Rightarrow |f(x) - f(y)| < \mu, \forall f \in F$, where d is the metric on X.
- **b)** $K \subset C(X)$ is bounded, funds that there exists a positive constant $M < \infty$ with the end goal that $|f(x)| \leq M$ for each $x \in X$ and each $f \in K$.

Chapter 3

EXISTENCE AND UNIQUENESS OF SOLUTION OF SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

3.1 Dynamic Energy Flow

We concentrate on the existence of solution for the sequential fractional in the differential equation of the form:

$$({}^{c}D^{\propto} + q\ {}^{c}D^{\propto-1})z(p) = h\big(p, z(p)\big), \qquad p \in [0,1] \quad , 2 \leq \propto \leq 3 \tag{3.1}$$

improved with the boundary conditions

$$z(0) = 0, \quad z'(0) = 0, \quad z(\zeta) = a\int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} z(s)ds, \qquad \beta > 0 \tag{3.2}$$

where ${}^{c}D^{\propto}$ mean the Caputo Fractional differential of request, $0 < \eta < \zeta < 1$, h is known to be a continuous function, and q , a are reasonably positive real numbers. At that point, we underline that the integral boundary condition (3.2) which can concur in the insight that importance of the indefinite function at the arbitrary position $\zeta \in (\eta, 1)$ is relative to Riemann-Liouville of the unknown function [1].

$$\int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} z(s)ds, \qquad \text{where} \qquad \zeta \in (\eta, 1). \tag{3.3}$$

In addition $\eta \in (0, \zeta)$ for $\beta = 1$ the integral boundary condition reduces to the standard type of a nonlocal integral condition,

$$z(\zeta) = a\int_0^{\eta} z(s)ds.$$

The substances of the article are agreed as follows a fundamental result that places the function for characterizing a fixed point problem equivalent to the given problem (3.1), (3.2). The results are based on Banach Contraction mapping principle rule, nonlinear option of Leray-Schauder alternative and Krasnoselskii's fixed point theorem [4][5].

Lemma 3.1. The integral solution of the linear equation for $f \in C([0,1],R)$

$$({}^{c}D^{\propto} + q\ {}^{c}D^{\propto-1})z(t) = f(p) \qquad p \in [0,1], \quad 2 \leq \propto \leq 3 \tag{3.4}$$

complemented with the boundary condition (3.2) is

$$z(p) = \frac{(qp - 1 + e^{-qp})}{\Delta}\left[a\int_0^{\eta}\frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s} e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}f(\tau)d\tau\right)dn\right)ds - \int_0^{\zeta} e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}f(\tau)d\tau\right)ds\right] + \int_0^{p} e^{-q(p-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}f(\tau)d\tau\right)ds. \tag{3.5}$$

Proof.

By using (F.L.D.E)

$$({}^{c}D^{\propto} + q\ {}^{c}D^{\propto-1})z(p) = f(p) \qquad 2 \leq \propto \leq 3$$

$$I^{\propto-1}\ ({}^{c}D^{\propto} + q\ {}^{c}D^{\propto-1})z(p) = I^{\propto-1}f(p)$$

$$(I^{\propto-1}\ {}^{c}D^{\propto})z(p) + q(I^{\propto-1}\ {}^{c}D^{\propto-1})z(p) = I^{\propto-1}f(p)$$

$$z'(p) + qz(p) = I^{\propto-1}\ f(p) - b_0 - b_1p \qquad \{\ (I^{\propto-1}\ {}^{c}D^{\propto-1})z(p) = z(p) +\ b_0 + b_1\}$$

Using (F.L.D.E)

$$I = e^{\int q dp} = e^{qp},$$

then multiplying both sides by I, we get

$$e^{qp}(z'(p) + q\ z(p)) = e^{qp}(I^{\propto-1}\ f(p) - b_0 - b_1p)$$

$$d\left(e^{qp}\ \cdot z(p)\right) = e^{qp}(I^{\propto-1}\ f(p) - b_0 - b_1p)$$

$$\int_0^s d\left(e^{qp} \cdot z(p)\right) = \int_0^s e^{qp} \left(I^{\propto-1} f(p) - b_0 - b_1 p\right) dp\,.$$

Multiplying both sides by e^{-qs}, we obtain

$$e^{qs}\, z(s) = \int_0^s e^{qp} \left(I^{\propto-1} f(p) - b_0 - b_1 p\right) dp.$$

Thus,

$$z(s) = b_2 e^{-qs} - b_0 e^{-qs} \int_0^s e^{qp} dp - b_1 e^{-qs} \int_0^s p e^{p} dp$$

$$+ e^{-qs} \int_0^s e^{qp} \left(I^{\propto-1} f(p)\right) dp$$

$$= b_2 e^{-qs} - \frac{b_0}{q}\left(1 - e^{-qs}\right) - \frac{b_1}{q^2}\left[qs - 1 + e^{-qs}\right]$$

$$+ e^{-qs} \int_0^s e^{qp} \left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} f(\tau) d\tau\right) ds\ .$$

$$= b_0 e^{-qs} + \frac{b_1}{q}\left(1 - e^{-qp}\right) + \frac{b_2}{q^2}\left[qp - 1 + e^{-qp}\right]$$

$$+ \int_0^s e^{q(p-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} f(\tau) d\tau\right) ds, \tag{3.6}$$

where b_0, b_1, b_2 are unidentified arbitrary constants. By the boundary condition (3.2) we get that $z(0) = 0,\ hence\ \ b_0 = 0, \ \ and \ \ \ z'(0) = 0, hence\ b_1 = 0.$

Also, by

$$z(\zeta) = a \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} z(s) ds, \qquad \beta > 0$$

and $z(\tau) = \frac{b_2}{q^2}\left[q\tau - 1 + e^{-q\tau}\right] + \int_0^p e^{-q(p-s)} \left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} f(\tau) d\tau\right) ds.$

Hence

$$a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}z(\tau)ds$$

$$=\frac{b_2}{q^2}\left[q\zeta-1+e^{-q\zeta}\right]+\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)ds,$$

$b_2 =$

$$\frac{q^2}{(q\zeta-1+e^{-q\zeta})}\left[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left[(q\tau-1+e^{-q\tau})+\int_0^{p}e^{-q(p-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)ds\right]-\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)ds\right].$$

Then replacing the expressions of b_0, b_1, b_2 in (3.6) yields the solution

$$z(p)=\frac{(qp-1+e^{-qp})}{\Delta}\left[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)dn\right)ds-\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)ds\right]+\int_0^{p}e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)ds.$$

We get the result, where

$$\Delta= q\zeta-1+e^{-q\zeta}-\frac{a}{\Gamma(\beta)}\left(\frac{q\eta^{\beta+1}}{\beta(\beta+1)}-\frac{\eta^{\beta}}{\beta}+\int_0^{\eta}(\eta-s)^{\beta-1}e^{-qs}ds\right)\neq 0\,. \tag{3.7}$$

3.2 Existence of Solutions

Let $C=C([0,1],\mathbb{R})$ be a Banach space then for all continuous functions from$[0,1]$ to R endowed with the *sup* norm characterized by$\|f\|=\sup\{|f(z)|, z\in[0,1]\}<\infty$.

To make less difficult the substantiations in the coming theorems, we start with the limits for the integrals emerging in the results [2].

Lemma 3.2. Let $\in C([0,1],R)$, then the following hold:

1. $\left|\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)dn\right)ds\right|\leq\frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q+e^{-q\eta}-1)\|f\|$
2. $\left|\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}f(\tau)d\tau\right)ds\right|\leq\frac{\zeta^{\propto-1}}{q\,\Gamma(\propto)}\left(1-e^{-q\zeta}\right)\|f\|,$

3. $\left|\int_0^t e^{-q(p-s)}\left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} f(\tau)d\tau\right)ds\right| \leq \frac{1}{q\Gamma(\propto)}(1-e^{-q})\|f\|$

Proof. For (1)

$$\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)} d\tau = \frac{-1}{(\propto-2)\Gamma(\propto-1)}(n-\tau)^{\alpha-1} \Big|_0^n = \frac{n^{\propto-1}}{\Gamma(\propto)}$$

$\int_0^s e^{-q(s-n)} \frac{n^{\propto-1}}{\Gamma(\propto)} dn \leq \frac{s^{\propto-1}}{\Gamma(\propto)} \int_0^s e^{-q(s-n)} dn = \frac{s^{\propto-1}}{q\Gamma(\propto)} e^{-q(s-n)} \Big|_0^s = \frac{s^{\propto-1}}{q\Gamma(\propto)}[1-e^{-qs}],$

hence we obtain

$$\left|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\frac{s^{\propto-1}}{q\Gamma(\propto)}[1-e^{-qs}]\right)ds\right|$$

$$\leq \|f\| \frac{1}{q\,\Gamma(\propto)\Gamma(\beta)} \int_0^\eta (\eta-s)^{\beta-1} s^{\propto-1}(1-e^{-qs})ds$$

$= \frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)} \int_0^\eta (1-e^{-qs})ds = \frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q + e^{-q\eta} - 1)\|f\|.$

For (2),

$$\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} f(\tau)d\tau\right) = \frac{-1}{\Gamma(\propto-1)} \int_0^s -(s-\tau)^{\alpha-2}\, d\tau$$

$$= \frac{-1}{(\propto-2)\Gamma(\propto-1)}(s-\tau)^{\alpha-1} \Big|_0^s = \frac{s^{\propto-1}}{\Gamma(\propto)},$$

then

$$\int_0^\zeta e^{-q(\zeta-s)} \frac{s^{\propto-1}}{\Gamma(\propto)} ds \leq \frac{\zeta^{\propto-1}}{\Gamma(\propto)} \int_0^\zeta e^{-q(\zeta-s)} ds$$

$$= \frac{\zeta^{\propto-1}}{\Gamma(\propto)} e^{-q\zeta} \int_0^\zeta e^{qs} ds = \frac{\zeta^{\propto-1}}{\Gamma(\propto)} e^{-q\zeta}[e^{qs} \Big|_0^\zeta$$

$= \frac{\zeta^{\propto-1}}{\Gamma(\propto)}(1-e^{-q\zeta})\|f\|.$

To prove (3),

$$\left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} f(\tau)d\tau\right)ds = \frac{-1}{\Gamma(\propto-1)} \int_0^s -(s-\tau)^{\alpha-2}\, d\tau$$

$$= \frac{-1}{(\propto-2)\Gamma(\propto-1)}(s-\tau)^{\alpha-1} \Big|_0^s = \frac{s^{\propto-1}}{\Gamma(\propto)},$$

then

$$\int_0^t e^{-q(p-s)} \frac{s^{\propto-1}}{\Gamma(\propto)} ds \le \frac{p^{\propto-1}}{\Gamma(\propto)} \int_0^p e^{-q(p-s)} ds$$

$$= \frac{p^{\propto-1}}{\Gamma(\propto)} e^{-qp} \int_0^p e^{qs} ds = \frac{p^{\propto-1}}{\Gamma(\propto)} e^{-qp} [e^{qs} \mid_0^p$$

$$= \frac{p^{\propto-1}}{\Gamma(\propto)} (1 - e^{-qp}) \|f\|.$$

As we have $\|z\| = \sup\{|z(p)|, p \in [0,1]\}$ we get

$$\int_0^p e^{-q(p-s)} ds \le \frac{1}{\Gamma(\propto)} (1 - e^{-q}) \|f\|.$$

What's more, the proof of nearness, we get

$$t = \sup_{p \in [0,1]} \left| \frac{(qp - 1 + e^{-qp})}{\Delta} \right| = \frac{1}{|\Delta|} (e^{-q} + q - 1) \tag{3.8}$$

and

$$\Lambda = t \left\{ |a| \frac{\eta^{\propto+\beta-2}}{q^2 \Gamma(\propto) \Gamma(\beta)} (\eta q + e^{-q\eta} - 1) + \frac{\zeta^{\propto-1}}{q \Gamma(\propto)} \left(1 - e^{-q,\zeta}\right) \right\} + \frac{1}{q\, \Gamma(\propto)} (1 - e^{-q}), \tag{3.9}$$

At this point we change issue (3.1) and (3.2) as

$$z = S(z), \qquad S: C \to C \tag{3.10}$$

$$(S_z)(p) = \frac{(qp - 1 + e^{-qp})}{\Delta}$$

$$\left[a \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} \left(\int_0^s e^{-q(s-n)} \left(\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) dn \right) ds \right.$$

$$\left. - \int_0^\zeta e^{-q(\zeta-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) ds \right]$$

$$+ \int_0^p e^{-q(p-s)} \left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) ds. \tag{3.11}$$

We note that the problem (3.1) and (3.2) has solution if the operator equation (3.10) has fixed point.

Theorem 3.2. Let $h: [0,1] \times R \to R$ be a continuous function then if the following the condition holds

(H_1) $| h(p,z) - h(p,y)| \leq L|z-y|, \qquad \forall p \in [0,1], \quad z, y \in R,$

where L is the Lipschitz constant then the boundary value problem (3.1) and (3.2) has a solution if $\Lambda < \frac{1}{L}$ where Λ is known by (3.9).

Proof. To begin with, we exhibit that the operator S, given by $z = S(z)$ maps C into itself. For that we see

$$\sup_{p \in [0,1]} |h(p,0)| = N < \infty.$$

Then for $z \in c$ we have

$$\|S_z\|$$

$$= \sup_{p \in [0,1]} \left| \frac{(qp - 1 + e^{-qp})}{\Delta} \right.$$

$$\cdot \left[a \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} \left(\int_0^s e^{-q(s-n)} \left(\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) dn \right) ds \right.$$

$$\left. - \int_0^{\zeta} e^{-q(\zeta - s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) ds \right]$$

$$\left. + \int_0^p e^{-q(p-s)} \left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) ds \right|$$

Then to get the formula of (H_1) we continue in the way that,

$$\|S_z\| \leq \sup_{p \in [0,1]} \left| \frac{(qp-1+e^{-qp})}{\Delta} \right| \left[|a| \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \left(\int_0^s e^{-q(s-n)} \left(\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)} \left| h(\tau, z(\tau)) - \right. \right. \right. \right.$$

$$\left. \left. \left. h(\tau,0) + h(\tau,0) \right| d\tau \right) dn \right) ds + \int_0^{\zeta} e^{-q(\zeta-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} \left| h(\tau, z(\tau)) - h(\tau,0) + \right. \right.$$

$$\left. \left. h(\tau,0) \right| d\tau \right) ds \Big] +$$

$$\int_0^p e^{-q(p-s)} \left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)} \left| h(\tau, z(\tau)) - h(\tau,0) + h(\tau,0) \right| d\tau \right) ds$$

Then for simplicity,

$\|S_z\|$

$$\leq \sup_{p\in[0,1]}\left|\frac{(qp-1+e^{-qp})}{\Delta}\right|\left[|a|\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}\right.\right.\right.$$

$$\left.\left.\cdot\left(\left|h(\tau,z(\tau))\ -h(\tau,0)\right|+|h(\tau,0)|\right)d\tau\right)dn\right)ds$$

$$\left.+\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}\left(\left|h(\tau,z(\tau))-h(\tau,0)\right|+|h(\tau,0)|\right)d\tau\right)ds\right]$$

$$+\int_0^{p}e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}\left(\left|h(\tau,z(\tau))-h(\tau,0)\right|+|h(\tau,0)|\right)d\tau\right)ds\,.$$

Hence by using $sup|h(\tau.0)|=N,$ and $|h(p,z)-h(p,y)|\leq L|z-y|$ formula we get

$$\|S_z\|\leq t\left[|a|\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}(L|z(p)-0|\right.\right.\right.$$

$$\left.\left.+N)d\tau\right)dn\right)ds$$

$$\left.+\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}(L|z(p)-0|+N)d\tau\right)ds\right]$$

$$+\int_0^{p}e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}(L|z(p)-0|+N)d\tau\right)ds\,.$$

$$\|S_z\|\leq(L\|z\|+N)\left\{t\left[|a|\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}d\tau\right)dn\right)ds\right.\right.$$

$$\left.+\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}d\tau\right)ds\right]$$

$$\left.+\int_0^{p}e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}d\tau\right)ds\right\}.$$

By using Lemma (3.2) we get

$$\|S_z\| \leq (L\|z\| + N)\left\{p\left[|a|\frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q + e^{-q\eta} - 1) + \frac{\zeta^{\propto-1}}{q\Gamma(\propto)}\left(1 - e^{-q\zeta}\right)\right]\right.$$

$$\left. + \frac{1}{q\,\Gamma(\propto)}(1 - e^{-q})\right\},$$

and by (3.9)

$$\|S_z\| = (L\|z\| + N)\Lambda < \infty.$$

Then we see that *S* maps *C* into itself, for $z, y \in C$ and $p \in [0,1]$ we get

$$\left\|S_z - S_y\right\| = \sup_{p\,\in\,[0,1]}\left|S_z(p) - S_y(p)\right|$$

$$\leq \sup_{p\,\in\,[0,1]}\left|\frac{(qp - 1 + e^{-qp})}{\Delta}\right|\left[|a|\int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s e^{-q(s-n)}\left(\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto - 1)}\right.\right.\right.$$

$$\left.\left.\cdot\left|h\big(\tau, z(\tau)\big) + h\big(\tau, y(\tau)\big)\right|d\tau\right)dn\right)ds$$

$$\left. + \int_0^\zeta e^{-q(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto - 1)}\left|h\big(\tau, z(\tau)\big) - h\big(\tau, y(\tau)\big)\right|d\tau\right)ds\right]$$

$$+ \int_0^p e^{-q(p-s)}\left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto - 1)}\left|h\big(\tau, z(\tau)\big) - h\big(\tau, y(\tau)\big)\right|d\tau\right)ds$$

$$\leq L\|z - y\|\left\{t\left[|a|\int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s e^{-q(s-n)}\left(\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto - 1)}d\tau\right)dn\right)ds\right.\right.$$

$$\left. + \int_0^\zeta e^{-q(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto - 1)}d\tau\right)ds\right]$$

$$\left. + \int_0^p e^{-q(p-s)}\left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto - 1)}d\tau\right)ds\right\}.$$

By Lemma (3.2)

$$\left\|S_z - S_y\right\| \leq L\|z - y\|\left\{p\left[|a|\frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q + e^{-q\eta} - 1) + \frac{\zeta^{\propto-1}}{q\,\Gamma(\propto)}\left(1 - e^{-q\zeta}\right)\right]\right.$$

$$\left. + \frac{1}{q\,\Gamma(\propto)}(1 - e^{-q})\right\}$$

$\|S_z - S_y\| = L\|z-y\|\Lambda,$

where Λ is given by (3.9) such that $\Lambda < \frac{1}{L}$, in this way S is a contraction along these lines. The desired result of theorem obtained by the contraction mapping principle.

Theorem 3.3. Let N be a convex, bounded, closed, and nonempty subset of a Banach space in Z and G_1, G_2 be the operators to such an extent that

1) $G_1 z + G_2 y \in N$ when $z, y \in N$.

2) G_1 is continuous and compact.

3) There exists $z \in N$ such that $z = G_1 z + G_2 z$, where G_2 is contraction mapping.

Theorem 3.4. Assume that $h: [0,1] \times R \to R$ is a jointly continuous function satisfying

(H_1) Assume that the accompanying assumption holds.

(H_2) When $|h(p,z)| \le \mu(p),\ \forall (p,z) \in [0,1] \times R \quad with\ \mu \in c([0,1], R)$ then at that point the BVP (3.1) and (3.2) has one solution on [0,1] if

$$t\left[|a|\frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q + e^{-q\eta} - 1) + \frac{\zeta^{\propto-1}}{q\,\Gamma(\propto)}\left(1 - e^{-q\zeta}\right)\right] < 1. \tag{3.12}$$

Proof. Let $\sup_{p\in[0,1]} |\mu(p)| = \|\mu\|$ we fix $r \ge \Lambda\|\mu\|$ (3.13)

at the point when Λ is known (3.9) and suppose $B_r = \{z \in c : \|z\| \le r\}$ describe the operator S_1 and S_2 on B_r as

$$(S_1 z)(p) = \int_0^p e^{-q\,(p-s)}\left(\int_0^p \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h\big(\tau, z(\tau)\big) d\tau\right) ds.$$

$$(S_2 z)(p) = \frac{(qp - 1 + e^{-qp})}{\Delta}$$

$$\left[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)dn\right)ds\right.$$

$$\left.-\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)ds\right]$$

where $z,y\in B_r$,

$$\|S_1z-S_2z\|\le \begin{matrix}sup\\ p\in[0,1]\end{matrix}\left|\frac{(qp-1+e^{-qp})}{\Delta}\right|$$

$$\left[|a|\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}|\mu(p)|d\tau\right)dn\right)ds\right.$$

$$\left.+\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}|\mu(p)|d\tau\right)ds\right]$$

$$+\int_0^{p}e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}|\mu(p)|d\tau\right)ds$$

$$\le\|\mu\|\left\{t\left[|a|\frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q+e^{-k\eta}-1)+\frac{\zeta^{\propto-1}}{q\,\Gamma(\propto)}\big(1-e^{-q\zeta}\big)\right]\right.$$

$$\left.+\frac{1}{q\,\Gamma(\propto)}(1-e^{-q})\right\}\le r,$$

So $S_1z+S_2y\in B_r$.

In understanding by condition (3.12) it can surely be made realized that S_2 is Contraction mapping. The continuity of h demonstrates that the operator S_1 is continuous, besides S_1 is uniformly bounded on B_r then

$$\|S_1z\|\le\frac{1}{q\,\Gamma(\propto)}(1-e^{-q})\|\mu\|$$

As of now, we demonstrate the compactness of S_1 by setting $\Omega=[0,1]\times B_r$, then defining $\sup_{p\in[0,1]}|f(p,z)|=N_r$. Thus we get

$$|(S_1z)(p_1)-(S_2z)(p_2)|$$

$$=\left|\int_0^{p_1} e^{-q(p_1-s)}\left(\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}h(u,z(u))du\right)ds\right.$$

$$\left.-\int_0^{p_2} e^{-q(p_2-s)}\left(\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}h(u,z(u))du\right)ds\right|$$

<u>Part1</u> $\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}|h(u,z(u))|du=\frac{N_r}{\Gamma(\propto-1)}\int_0^{s}(s-u)^{\alpha-2}du=\frac{N_r}{(\propto-1)\Gamma(\propto-1)}$

$$\left\{(s-u)^{\alpha-1}\Big|\begin{matrix}s\\0\end{matrix}\right\}=\frac{N_r}{\Gamma(\propto)}s^{\propto-1}$$

then

$\int_0^{p_1} e^{-q(p_1-s)}\left(\frac{N_r}{\Gamma(\propto)}s^{\propto-1}\right)ds=\frac{N_r}{\Gamma(\propto)}{p_1}^{\propto-1}e^{-qp_1}\left\{\frac{e^{qs}}{q}\Big|\begin{matrix}p_1\\0\end{matrix}\right\}=\frac{N_r}{q\Gamma(\propto)}[|{p_1}^{\propto-1}-{p_1}^{\propto-1}e^{-qp_1}|]$.

In the same way for Part2, we get

$$|(S_1z)(p_1)-(S_2z)(p_2)|\le\frac{N_r}{q\Gamma(\propto)}(|{p_1}^{\propto}-{p_1}^{\propto}e^{-qp_1}|+|{p_2}^{\propto}-{p_2}^{\propto}e^{-qp_2}|)$$

$$\le\frac{N_r}{q\Gamma(\propto)}(|{p_1}^{\propto}-{p_2}^{\propto}|+|{p_1}^{\propto}e^{-qp}-{p_2}^{\propto}e^{-qp_2}|),$$

which is independent of z and tends to zero as $p_2\to p_1$ so S_1 is relatively compact on B_r. By the Arzela-Ascoli theorem, S_1 is compact on B_r. Hence all the thought of Theorem (3.3) are fulfilled and the deduction of Theorem (3.3) indicates that the BVP (3.1) and (3.2) has unique solution on [0,1].

Remark 3.5. In the Theorem 3.4, we can interchange the role of the operators $S_1\ and\ S_2$ to get the second result we substitute (3.12) by this condition

$$\frac{1-e^{-q}}{\Gamma(\propto)}<1.$$

In the next theorem we prove the existence for the BVP (3.1) and (3.2) by means of Leray-Schauder nonlinear Alternative.

Lemma 3.3. (Leray-Schauder nonlinear Alternative)

When we have a Banach space of Q, G be a closed subset and convex when Q, N be an open subset of G and $0 \in N$, assume that $h: \overline{N} \to G$ is continuous and compact (when $h(\overline{N})$ is a generally compact subset of G) guide then [6]

1. h has a fixed point in $\overline{N}$.
2. (the boundary of N in G) is $\lambda \in (0,1)$and $n \in \partial N$ with $n \in \lambda h(n)$.

Theorem 3.7. Suppose that $h: [0,1] \times R \to R$ is jointly continuous function further, it is assumed that the following conditions hold:

(H_3) There exists a non-decreasing function $\varphi: R^+ \to R^+$ and a function $\emptyset \in$ (C$[0,1], R$) such that for all $(p, z) \in [0,1] \times R$, $|h(p,z)| \leq \emptyset(p)\varphi\|z\|$.

(H_4) There exists a constant $N > 0$ such that Let $\frac{N}{\varphi(N)\|\emptyset\|\Lambda} > 1$ where Λ is given by (3.9). Then the BVP (3.1) and (3.2) has at least one solution on [0,1].

Proof. Assume the operator $S: C \to C$ where

$$(S_z)(p) = \frac{(qp - 1 + e^{-qp})}{\Delta}$$

$$\left[a \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} \left(\int_0^{s} e^{-q(s-n)} \left(\int_0^{n} \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) dn \right) ds \right.$$

$$\left. - \int_0^{\zeta} e^{-q(\zeta-s)} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) ds \right]$$

$$+ \int_0^{p} e^{-q(p-s)} \left(\int_0^{p} \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} h(\tau, z(\tau)) d\tau \right) ds \, .$$

We demonstrate that *S* maps bounded sets into bounded sets in C$([0,1], R)$. For positive number *r*, let $B_r = \{z \in C([0,1], R) \colon \|z\| \leq r\}$ be a bounded set in $C([0,1], R)$, then

$$|(S_z)(p)| = \left|\frac{(qp-1+e^{-qp})}{\Delta}\left[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\right.\right.$$

$$\times\left(\int_0^{s} e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)dn\right)ds$$

$$\left.-\int_0^{\zeta} e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)ds\right]$$

$$\left.+\int_0^{p} e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)ds\right|$$

$$\leq \sup_{p\in[0,1]}\left|\frac{(qp-1+e^{-qp})}{\Delta}\right|\left[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\right.$$

$$\times\left(\int_0^{s} e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}(\emptyset(\tau)\varphi(\|z\|))d\tau\right)dn\,ds\right.$$

$$\left.+\int_0^{\zeta} e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}(\emptyset(\tau)\varphi(\|z\|))d\tau\right)ds\right]$$

$$+\int_0^{p} e^{-q(p-s)}\left(\int_0^{p}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}(\emptyset(\tau)\varphi(\|z\|))d\tau\right)ds$$

$$\leq \varphi(\|z\|)\|\emptyset\|\left\{t\left[|a|\frac{\eta^{\propto+\beta-2}}{q^2\Gamma(\propto)\Gamma(\beta)}(\eta q+e^{-q\eta}-1)+\frac{\zeta^{\propto-1}}{q\,\Gamma(\propto)}\big(1-e^{-q\zeta}\big)\right]\right.$$

$$\left.+\frac{1}{q\,\Gamma(\propto)}(1-e^{-q})\right\}$$

$$=\varphi(\|z\|)\|\emptyset\|\Lambda.$$

Consequently

$$\|S_z\| \leq \varphi(r)\|\emptyset\|\Lambda.$$

Next we prove that S maps bounded sets into equicontinuous sets of $C([0,1],R)$. Let $p_1, p_2 \in [0,1]$ with $p_1 < p_2$ and $z \in B_r$ where B_r is a bounded set of $C([0,1],R)$, then we set,

$$|(s_z)(p_2)-(s_z)(p_1)|$$

$$=\Bigg|\Bigg[\frac{(qp_2-1+e^{-qp_2})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}$$

$$\times\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)dn\right)ds$$

$$-\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)ds\Bigg]$$

$$+\int_0^{p_2}e^{-q(p_2-s)}\left(\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}h(u,z(u))du\right)ds\Bigg]$$

$$-\Bigg[\frac{(qp-1+e^{-qp})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}$$

$$\times\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)dn\right)ds$$

$$-\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)ds\Bigg]$$

$$+\int_0^{p_1}e^{-q(p_1-s)}\left(\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}h(u,z(u))du\right)ds\Bigg]\Bigg|$$

$$\le\Bigg(\Bigg|\frac{(q(p_2-p_1)+e^{-qp_2}-e^{-qp_1})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}$$

$$\times\left(\int_0^{s}e^{-q(s-n)}\left(\int_0^{n}\frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)dn\right)ds$$

$$-\int_0^{\zeta}e^{-q(\zeta-s)}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto-1)}h\big(\tau,z(\tau)\big)d\tau\right)ds\Bigg]\Bigg|$$

$$+\Bigg|\int_0^{p_1}\big(e^{-q(p_2-s)}-e^{-q(p_1-s)}\big)\left(\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}h\big(u,z(u)\big)du\right)ds$$

$$+\int_0^{p_2}e^{-q(p_2-s)}\left(\int_0^{s}\frac{(s-u)^{\alpha-2}}{\Gamma(\propto-1)}h(u,z(u))du\right)ds\Bigg|\Bigg)$$

$$\leq \left(\left| \frac{(q(p_2 - p_1) + e^{-qp_2} - e^{-qp_1})}{\Delta} \left[a \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} \right. \right. \right.$$

$$\times \left(\int_0^s e^{-q(s-n)} \left(\int_0^n \frac{(n-\tau)^{\alpha-2}}{\Gamma(\propto -1)} \varphi(r) \emptyset(\tau) d\tau \right) dn \right) ds$$

$$\left. \left. - \int_0^{\zeta} e^{-q(\zeta - s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\propto -1)} \varphi(r) \emptyset(\tau) d\tau \right) ds \right] \right|$$

$$+ \left| \int_0^{p_1} \left(e^{-q(p_2 - s)} - e^{-q(p_1 - s)} \right) \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\propto -1)} \varphi(r) \emptyset(u) du \right) ds \right.$$

$$\left. \left. + \int_0^{p_2} e^{-q(p_2 - s)} \left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\propto -1)} \varphi(r) \emptyset(u) du \right) ds \right| \right).$$

Clearly the correct hand side of the above imbalance tends to zero independent of $z \in B_r$ as $p_2 - p_1 \to 0$, as S satisfies the above assumptions, along these lines it takes after by the Arzela-Ascoli theorem that $S: C \to C$ is completely continuous. The result will come from the Leray-Schauder nonlinear Alternative (Lemma 3.3) when we have proved the boundedness of the set of all solution to equation $z = \lambda S_z$ for $\lambda \in [0,1]$. Give z a chance to be an answer. At that point for $p \in [0,1]$ and utilizing the calculations utilized as a part of demonstrating that S is bounded

$$|z(p)| = |\lambda S_z(p)|$$

$$\leq \varphi \|z\| \|\emptyset\| \left\{ t \left[|a| \frac{\eta^{\propto + \beta - 2}}{q^2 \Gamma(\propto) \Gamma(\beta)} (\eta q + e^{-q\eta} - 1) + \frac{\zeta^{\propto - 1}}{q\, \Gamma(\propto)} \left(1 - e^{-q\zeta}\right) \right] \right.$$

$$\left. + \frac{1}{q\, \Gamma(\propto)} (1 - e^{-q}) \right\}$$

$$= \varphi \|z\| \|\emptyset\| \Lambda.$$

Consequently,

$$\frac{\|z\|}{\varphi \|z\| \|\emptyset\| \Lambda} \leq 1.$$

In view of (H_4) there exists N with the end goal that $\|z\| \neq N$ let us set

$$U = \{z \in C([0,1], R): \|z\| < N\}.$$

Take note of that the operator $S: \overline{U} \to C([0,1], R)$ is continuous and completely continuous from the choice of U. For $\lambda \in (0,1)$ there is no $z \in \partial U$ which $z = \lambda S_z$. Thus, by the Leray-Schauder nonlinear Alternative (Lemma 3.3), we conclude that S has a fixed point $z \in \overline{U}$ which is a solution of problem (3.1) and (3.2). The completes the proof.

3.2 Examples

The following examples are concerned with the illustration of Theorem 3.2.

Examples 3.2.1 Consider the problem

$$\begin{cases} {}^cD^{\frac{3}{2}}(D+2)p(a) = \frac{L}{2}(\sqrt{a^2+1} + sina + p(a) + tan^{-1}p(a), & 0 \le a \le 1 \\ p(0) = 0, \quad p'(0) = 0, \qquad p\left(\frac{1}{2}\right) = \int_0^{\frac{1}{3}} p(s)ds & \end{cases} \tag{3.14}$$

Solution. Here $\propto = \frac{5}{2}$, $h\big(a, p(a)\big) = \frac{1}{2}\left(\sqrt{a^2+1} + sina + p(a) + tan^{-1}p(a)\right)$

$$q = 2, \quad b = 1, \qquad \eta = 1/3\ , \qquad \zeta = 1/2, \qquad \beta = 1$$

$$|h(a,p) - h(a,f)| \le \frac{L|p - f + tan^{-1}p - tan^{-1}f|}{2} \le L|p-f|.$$

Then $\Delta \approx 0{,}34681$, $t \approx 3{,}27365$, $\Lambda \approx 0{,}60751$ for $L < \frac{1}{\Lambda} \approx 1{,}64604$. Using Theorem 3.2, the problem (3.14) has a unique solution.

Example 4.2. Let us consider the problem (3.14) with

$$h\big(a, p(a)\big) = \frac{e^{-a}}{4\sqrt{1+a^2}}(p^2+1).$$

Solution. We check the conditions of Theorem 3.7 and the hypothesis (H_3) holds with

$\|\emptyset\| = \frac{1}{4}, \ \varphi(\|t\|) = 1 + \|t\|^2.$

By assumption(H_4), we get $N_1 < N < N_2$.

If $N_1 \approx 0{,}15555 \quad and \quad N_2 = 6{,}428612$, by applying Theorem 3.7, hence the problem (3.14) value $h\big(a, t(a)\big)$ has a solution on [0,1].

Chapter 4

CONCLUSION

We have accomplished particular existence and uniqueness results of Caputo type sequential fractional differential equation using nonlinear alternative of Leray-Schauder type with the Banach contraction mapping principle and Krasnoselskii's fixed point theorem. We see that few new solitary results take after by settling the components entangled in the known problem. For example, in the event that we pick $\beta = 1,$ then the results of this study identify with the Caputo type sequential of fractional differential equation and boundary condition of fractional differential equation have the form

$$p(0) = 0, \quad p'(0) = 0, \qquad p(\zeta) = a \int_0^{\eta} p(s)ds.$$

REFERENCES

[1] Alsaedi, A., Ntouyas, S. K., Agarwal, R. P., & Ahmad, B. (2015). On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Advances in Difference Equations*, *2015*(1), 1-12.

[2] Benchohra, M., Henderson, J., Ntouyas, S. K., & Ouahab, A. (2008). Existence results for fractional order functional differential equations with infinite delay. *Journal of Mathematical Analysis and Applications*, *338*(2), 1340-1350.

[3] Browder, F. (1976). *Nonlinear Operators and Nonlinear Equations of Evolution, Part 2* (Vol. 18). American Mathematical Society.

[4] Dhage, B. C. (1987). A fixed point theorem and applications to nonlinear integral equations. In *Proc. Internat. Symp. Nonlinear Anal. Appl. Bio-Math., Waltair, India* (pp. 53-59).

[5] Dhage, B. C., & Ntouyas, S. K. (2002). Existence Results For Nonlinear Functional Integral Equations Via A Fixed Point Theorem Of Krasnoselkii-Schaefer Type. *Nonlinear Studies*, *9*(3), 307-318.

[6] Green, J. W., & Valentine, F. A. (1961). On the Arzela-Ascoli theorem. *Mathematics Magazine*, *34*(4), 199-202.

[7] Kilbas, A. A., Saigo, M., & Saxena, R. K. (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transforms and Special Functions*, *15*(1), 31-49.

[8] Kober, H. (1940). On fractional integrals and derivatives. *The Quarterly Journal of Mathematics*, *11*, 193-211.

[9] Krasnoselski, M. A., & Rutickii, Y. B. (1961). *Convex functions and Orlicz spaces* (p. M3). Groningen: Noordhoff.

[10] Pinelis, I. (2002). L'Hospital type rules for monotonicity, with applications. *J. Ineq. Pure & Appl. Math*, *3*(1).