Some Results on Laguerre Type and Mittag-Leffler Type Functions

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ABSTRACT

This thesis includes four chapters. In the first chapter, we give general information and some preliminaries that is used throughout the thesis.

In Chapter 2, by defining a new class of 2D-Mittag-Leffler functions $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$, the two-dimensional fractional integral and two-dimensional fractional derivative properties are derived for them. Moreover, linear generating function for $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in terms of $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ is obtained. Also, the double Laplace transform of these classes are investigated. A general singular integral equation containing $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel is considered and the solution is obtained in terms of $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$. Lastly, we obtain the image of $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ under the action of Marichev-Saigo-Maeda integral operators and some consequences are also exhibited.

In Chapter 3, linear and mixed multilateral generating functions for the general class of 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ are derived. Furthermore, a finite summation formula for $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ is obtained. Moreover, series relation between $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ and product of confluent hypergeometric functions is derived with the help of two-dimensional fractional derivative operator.

In Chapter 4, new classes of bivariate Mittag-Leffler functions $\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y)$ and 2D-Konhauser-Laguerre polynomials ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ are introduced. Some of them associated with fractional calculus are given. Also, a convolution type integral equation with the polynomials ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ in the kernel is considered and the solution is obtained by means of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$. Furthermore, a double linear generating function is obtained for the polynomials ${}_{\kappa}L^{(\alpha,\beta)}_{n}(x,y)$ in terms of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$. Finally, some miscellaneous properties of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_{\kappa}L^{(\alpha,\beta)}_{n}(x,y)$ are exhibited.

Keywords: Mittag-Leffler functions, Laguerre and Konhauser polynomials, Laplace transform, fractional integrals and derivatives, generating functions, convolution integral equation, singular integral equation

ÖZ

Bu tez 4 bölümden oluşmaktadır. Birinci bölümde tez ile ilgili genel bilgiler ve tezde kullanılan tanımlar hakkında bilgiler verilmiştir.

İkinci bölümde, 2D-Mittag-Leffler fonksiyonları $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ ve 2D-Laguerre polinomları $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ tanımlanarak, yukarıda belirtilen sınıfların kesirli integral ve türevleri hesaplanmıştır. Buna ek olarak, 2D-Laguerre polinomları $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ için 2D-Mittag-Leffler fonksiyonlarını $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ içeren linear doğurucu fonksiyon elde edilmiştir. Ayrıca, bu sınıfların iki boyutlu Laplace dönüşümleri de hesaplanmıştır. Çekirdeğinde $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ bulunan tekil integral denklemi ele alınmış ve çözümü $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ cinsinden verilmiştir. Son olarak, $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ fonksiyonlarının Marichev-Saigo-Maeda integral operatörü altındaki görüntüleri elde edilmiş ve bazı sonuçlar gösterilmiştir.

Üçüncü bölümde, 2D-Laguerre polinomları olarak tanımlanan $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ için linear ve multi-linear doğurucu fonksiyonlar elde edilmiştir. Buna ek olarak, $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ polinomları için sonlu toplam formülü elde edilmiştir. Bunun yanında, kesirli türev operatörü kullanarak, $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ ve birbirine karışan hipergeometrik fonksiyon arasındaki seri ilişkisi gösterilmiştir.

Dördüncü bölümde, 2D-Konhauser-Laguerre polinomları ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ ve yeni tanımlanan iki değişkenli Mittag-Leffler fonksiyonları $\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y)$ ele alınarak, onların kesirli türev ve integrallerle ilgili bazı sonuçları hesaplanmıştır. Ayrıca çekirdeğinde ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ içeren konvolüsyon integral denklemi ele alınmış ve çözümü $\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y)$

cinsinden elde edilmiştir. Bunun yanında ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ polinomları için $\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y)$ içeren linear doğurucu fonksiyon elde edilmiştir. Son olarak ise, $\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y)$ fonksiyonları ve ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ polinomları ile ilgili bir takım özellikler gösterilmiştir.

Anahtar Kelimeler: Mittag-Leffler fonksiyonları, Laguerre ve Konhauser polinomları, Laplace dönüşümleri, kesirli integraller ve türevler, üreten fonksiyonlar, konvolüsyon integral denklemi, tekil integral denklemi

To My Beloved Family

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LIST OF SYMBOLS

$\mathfrak{E}_{\alpha}(x)$	Mittag-Leffler Function
$(\gamma)_n$	Pochammer Symbol
$\Gamma(\alpha)$	Gamma Function
$B(\alpha,\beta)$	Beta Function
$L_n^{\alpha}(x)$	Laguerre Polynomials
$L_n(x,y)$	Bivariate Laguerre Polynomials
${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$	2D-Laguerre-Konhauser Polynomials
$L_{n_1,\cdots,n_j}^{(\alpha)}(x_1,\cdots,x_j)$	Multivariate Laguerre Polynomials
${}_1F_1(a;b;x)$	Confluent Hypergeometric Functions
${}_2F_1(a,b;c;x)$	Gauss Hypergeometric Function
$S_{C:D;D'}^{A:B;B'}$	Double Hypergeometric Series
$E_{\rho_1,\cdots,\rho_j,\lambda}^{(\gamma_1,\cdots,\gamma_j)}(x_1,\cdots,x_j)$	Multivariate Mittag-Leffler Functions
$\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$	2D-Mittag-Leffler Functions
$\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$	2D-Laguerre Polynomials
$\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y)$	Bivariate Mittag-Leffler Functions
${}_x\mathbf{I}_{a^+}^{\alpha}$	Riemann-Liouville Fractional Integral Operator
${}_y\mathbf{I}_{b^+}^{\beta}{}_x\mathbf{I}_{a^+}^{\alpha}$	Two-Dimensional R-L Fractional Integral Operator
${}_x\mathbf{D}_{a^+}^{\alpha}$	Riemann-Liouville Fractional Derivativel Operator
${}_y\mathbf{D}_{b^+}^{\beta}\,{}_x\mathbf{D}_{a^+}^{\alpha}$	Two-Dimensional R-L Fractional Derivative Operator
$\mathscr{I}_{0^+}^{\lambda,\lambda',\mu,\mu',\nu}$	Left-Sided M-S-M Fractional Integration Operator
$\mathscr{I}_{0^-}^{\lambda,\lambda',\mu,\mu',\nu}$	Right-Sided M-S-M Fractional Integration Operator
$\mathbb{I}_{0^+}^{(\lambda,\mu,\nu)}$	Left-Sided Saigo Integal Operator

$\mathbb{I}_{0^-}^{(\lambda,\mu,\nu)}$	Right-Sided Saigo Integal Operator
$\mathbb{L}_2[f(x,t)]$	Two-Dimensional Laplace Transform
$\mathscr{E}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$	Double Integral Operator

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Chapter 1

INTRODUCTION

In 1903, Mittag-Leffler function [8] were introduced in the following form

$$\mathfrak{E}_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}. \tag{1.0.1}$$
$$(\alpha \in \mathbb{C},\ Re(\alpha) > 0,\ x \in \mathbb{C})$$

More generalized form of the above function (1.0.1) was introduced by Wiman ([33],[34]) as follows

$$\mathfrak{E}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}. \tag{1.0.2}$$
$$(\alpha,\ \beta \in \mathbb{C},\ Re(\alpha),\ Re(\beta) > 0,\ x \in \mathbb{C})$$

It is obvious that, by using (1.0.1) and (1.0.2), we have $\mathfrak{E}_{\alpha,1}(x) = \mathfrak{E}_{\alpha}(x)$. Also, the following functions $\mathfrak{E}_{1,1}(x) = e^x$, $\mathfrak{E}_{2,1}(x^2) = \cosh(x)$, $\mathfrak{E}_{2,1}(-x^2) = \cos(x)$ and $\mathfrak{E}_{2,2}(-x^2) = \sin(x)/x$, such that exponential, hyperbolic, and trigonometric functions, respectively, are the extension form of Mittag-Leffler functions (1.0.2).

In [18], Prabhakar introduced a further generalization form of (1.0.2) in the following way

$$\mathfrak{E}^{\gamma}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!}, \tag{1.0.3}$$
$$(\alpha,\ \beta,\ \gamma \in \mathbb{C},\ Re(\alpha),\ Re(\beta) > 0,\ x \in \mathbb{C})$$

where $(\gamma)_n$ is the Pochammer symbol [20] which is defined as

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1 & ;n=0,\ \gamma \neq 0 \\ \gamma(\gamma+1)\cdots(\gamma+n-1) & ;n=1,2,\cdots \end{cases} .$$

It is clear that, we have

$$\mathfrak{E}^1_{\alpha,\beta}(x) = \mathfrak{E}_{\alpha,\beta}(x) \text{ and } \mathfrak{E}^1_{\alpha,1}(x) = \mathfrak{E}_{\alpha}(x).$$

For $k=1$, we obtain $Z_n^\alpha(x,1) = L_n^\alpha(x)$ where $L_n^\alpha(x)$ is denoted by the classical Laguerre polynomial, that is

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!}\ {}_1F_1(-n;1+\alpha;x),$$

where

$${}_1F_1(-n;1+\alpha;x) = \sum_{k=0}^{n} \frac{(-n)_k}{(1+\alpha)_k}\frac{x^k}{k!}.$$

In [15], a class of polynomials $Z^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ were defined as follows

$$\begin{aligned} & Z^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right) \\ & = \frac{\Gamma\left(\rho_1 n_1+\cdots+\rho_j n_j+\alpha+1\right)}{n_1!\cdots n_j!} \\ & \times \sum_{k_1,\cdots,k_j=0}^{n_1,\cdots,n_j} \frac{(-n_1)_{k_1}\cdots\left(-n_j\right)_{k_j} x_1^{\rho_1 k_1}\cdots x_j^{\rho_j k_j}}{\Gamma\left(\rho_1 k_1+\cdots+\rho_j k_j+\alpha+1\right) k_1!\cdots k_j!}. \end{aligned} \tag{1.0.4}$$

$$\left(\alpha,\ \rho_1,\ \cdots,\ \rho_j\ \in\ \mathbb{C},\ Re(\rho_i)>0\ (i=1,\cdots,j)\right)$$

Note that, for the univariate case, we refer [18].

Clearly, setting $\rho_1 = \cdots = \rho_j = 1$ in (1.0.4) gives

$$\begin{aligned}&L^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j\right)\\ =\ &\frac{\Gamma\left(n_1+\cdots+n_j+\alpha+1\right)}{n_1!\cdots n_j!}\sum_{k_1,\cdots,k_j=0}^{n_1,\cdots,n_j}\frac{(-n_1)_{k_1}\cdots\left(-n_j\right)_{k_j}x_1^{k_1}\cdots x_j^{k_j}}{\Gamma\left(k_1+\cdots+k_j+\alpha+1\right)k_1!\cdots k_j!},\end{aligned}$$

where $L^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j\right)$ is the multivariate Laguerre polynomials (see [3]).

It is known that the multivariate Mittag-Leffler functions are defined by the multiple series as [23]

$$E^{(\gamma_1,\cdots,\gamma_j)}_{\rho_1,\cdots,\rho_j,\lambda}(x_1,\cdots,x_j) = \sum_{k_1,\cdots,k_j=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots\left(\gamma_j\right)_{k_j}x_1^{k_1}\cdots x_j^{k_j}}{\Gamma\left(\rho_1 k_1+\cdots+\rho_j k_j+\lambda\right)k_1!\cdots k_j!}. \tag{1.0.5}$$

$(\lambda,\rho_1,\cdots,\rho_j,\gamma_1,\cdots,\gamma_j \in \mathbb{C},\ Re(\rho_i) > 0\ (i = 1,\cdots,j))$

Note that the function in (1.0.5) is a special case of the generalized Lauricella series in several variables introduced and investigated by Srivastava and Daoust [29] (see also [27] and [30]). Also, when $j = 1, \rho_1 = \alpha, \lambda = \beta, \gamma_1 = \gamma$, the function (1.0.5) reduces to (1.0.3).

From (1.0.4) and (1.0.5), it is obvious that (see [15])

$$\begin{aligned}&Z^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)\\ =\ &\frac{\Gamma\left(\rho_1 n_1+\cdots+\rho_j n_j+\alpha+1\right)}{n_1!\cdots n_j!}E^{(-n_1,\cdots,-n_j)}_{\rho_1,\cdots,\rho_j,\alpha+1}(x_1^{\rho_1},\cdots,x_j^{\rho_j}).\end{aligned} \tag{1.0.6}$$

Clearly, setting $\rho_1 = \rho_2 = \cdots = \rho_j = 1$ in (1.0.6) gives

$$L^{(\alpha)}_{n_1,\cdots,n_j}\left(x_1,\cdots,x_j\right) = \frac{\Gamma\left(n_1+\cdots+n_j+\alpha+1\right)}{n_1!\cdots n_j!}E^{(-n_1,\cdots,-n_j)}_{1,\cdots,1,\alpha+1}(x_1,\cdots,x_j).$$

Motivated by the above results, in [16], a class of 2D-Mittag-Leffler functions were introduced in the following form

$$\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)}\frac{x^r}{r!}\frac{y^s}{s!}. \quad (1.0.7)$$

$$(\gamma,\ \kappa,\ \alpha,\ \beta,\ \lambda,\ \eta,\ \xi \ \in \ \mathbb{C},\ Re(\alpha+\eta)>0,\ Re(\beta)>0)$$

Remark 1.0.1 *According to the convergence conditions investigated by Srivastava and Daoust ([27], p. 155) for the generalized Lauricella series in two variables, the series in (1.0.7) are converges absolutely for* $Re(\alpha+\eta)>0$ *and* $Re(\beta)>0$.

Also, a new general class of 2D-Laguerre polynomials in [16] were introduced as follows

$$\begin{aligned}&\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y) \quad (1.0.8)\\ =\ &\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\\ &\times\sum_{r=0}^{n}\sum_{s=0}^{m}\frac{(-n)_r(-m)_s}{\Gamma(\alpha r+\beta s+\gamma+1)\Gamma(\eta s+\xi)}\frac{x^{\alpha k_1}}{r!}\frac{y^{\beta k_2}}{s!}.\end{aligned}$$

$$(\alpha,\ \beta,\ \gamma,\ \eta,\ \xi \ \in \ \mathbb{C},\ Re(\alpha),\ Re(\beta),\ Re(\eta),\ Re(\xi)>0,\ Re(\gamma)>-1)$$

Comparing (1.0.7) and (1.0.8), we get

$$\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y) = \frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\mathfrak{E}_{-n,-m}^{(\alpha,\beta,\eta,\xi,\lambda)}(x^{\alpha},y^{\beta}). \quad (1.0.9)$$

The following unifications and generalizations of Laguerre polynomials

$${}_1L_{n,\rho}(x,y) = n!\sum_{k=0}^{n}\frac{y^{n-k}x^{k-\rho}}{k!(n-k)!\Gamma(\rho+k+1)} \quad (1.0.10)$$

and

$$L_n^{(m)}(x,y) = (m+n)!\sum_{k=0}^{n}\frac{(-1)^k y^{n-k}x^k}{k!(n-k)!(m+k)!}. \tag{1.0.11}$$

were defined by Dattoli et al. in [5].

When $\rho = 0,\ x \to -x$ and $m = 0$ in (1.0.10) and (1.0.11), respectively, we get the classical bivariate Laguerre polynomials

$$L_n(x,y) = n!\sum_{m=0}^{n}\frac{(-1)^m y^{n-m}x^m}{(m!)^2(n-m)!}.$$

Clearly, we have

$$L_n^{(\rho)}(x) = \frac{n!x^\rho}{\Gamma(\rho+n+1)}\ {}_1L_{n,\rho}(-x,1),$$

$$L_n^{(0)}(x,y) = L_n(x,y),$$

$$L_n^{(m)}(x,y) = y^n L_n^{(m)}(\frac{x}{y}).$$

Very recently, a class of generalized 2D-Laguerre-Konhauser polynomials were introduced by Bin-Saad et. al [2], that is

$$\begin{aligned} {}_\kappa L_n^{(\alpha,\beta)}(x,y) &= n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{s+r}x^{r+\alpha}y^{\kappa s+\beta}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(\kappa s+\beta+1)}, \\ &(\alpha,\ \beta \ \in \ \mathbb{R},\ \kappa = 1,\ 2,\ \cdots) \end{aligned} \tag{1.0.12}$$

Particularly, we have

$$\begin{aligned}
(-1)^{\rho} y^{n+\rho}\ {}_1L_n^{(\rho,0)}\left(-\frac{x}{y},0\right) &= {}_1L_{n,\rho}(x,y),\\
\frac{(m+n)!}{n!} y^{n+m} x^{-m}\ {}_1L_n^{(m,0)}\left(\frac{x}{y},0\right) &= L_n^{(m)}(x,y),\\
\frac{\Gamma(\kappa n+\beta+1)}{n!} y^{-\beta}\ {}_\kappa L_n^{(0,\beta)}(0,y) &= Z_n^{\beta}(y;\kappa), \qquad (1.0.13)\\
\frac{\Gamma(n+\alpha+1)}{n!} x^{-\alpha}\ {}_1L_n^{(\alpha,0)}(x,0) &= L_n^{(\alpha)}(x). \qquad (1.0.14)
\end{aligned}$$

Note that, from (1.0.13) and (1.0.14), (1.0.12) can also be written as

$${}_\kappa L_n^{(\alpha,\beta)}(x,y) = n! \sum_{s=0}^{n} \frac{(-1)^s x^{s+\alpha} y^{\beta} Z_{n-s}^{\beta}(y;\kappa)}{s!\Gamma(\alpha+s+1)\Gamma(\kappa n-\kappa s+\beta+1)}$$

and

$${}_\kappa L_n^{(\alpha,\beta)}(x,y) = n! \sum_{s=0}^{n} \frac{(-1)^s x^{\alpha} y^{\kappa s+\beta} L_{n-s}^{\alpha}(x)}{s!\Gamma(\alpha+n-s+1)\Gamma(\kappa s+\beta+1)}.$$

Remark 1.0.2 *(see [17]) By proposing another set of polynomials* $\left\{{}_\kappa\mathscr{L}_n^{(\alpha,\beta)}(x,y)\right\}$, *by*

$${}_\kappa\mathscr{L}_n^{(\alpha,\beta)}(x,y) = L_n^{(\alpha)}(x) \sum_{s=0}^{n} Y_s^{(\beta)}(y;\kappa),$$

where

$$Y_n^{(\alpha)}(x;k) = \frac{1}{n!} \sum_{i=0}^{n} \frac{x^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \left(\frac{j+\alpha+1}{k}\right)_n,$$

clearly, we see that, two polynomial sets $\left\{{}_\kappa\mathscr{L}_n^{(\alpha,\beta)}(x,y)\right\}$ *and* $\left\{{}_\kappa L_n^{(\alpha,\beta)}(x,y)\right\}$ *are bi-orthonormal with respect to the weight function* $\omega(x) = e^{-x-y}$ *over the interval* $(0,\infty)\times(0,\infty)$. *Indeed, using the relations (see [4])*

$$\int_0^\infty e^{-x}x^\alpha L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = \begin{cases} 0 & n \neq m \\ \frac{\Gamma(n+\alpha+1)}{n!} & (m=n) \end{cases}$$

and

$$\begin{aligned} J_{n,m} &= \int_0^\infty e^{-x}x^\beta Z_n^\beta(x;k)Y_m^\beta(x;k)dx \\ &= \frac{\Gamma(kn+\beta+1)}{n!}\delta_{nm}. \end{aligned}$$

It can be easly seen that

$$\int_0^\infty \int_0^\infty e^{-x}e^{-y}\ {}_\kappa\mathscr{L}_n^{(\alpha,\beta)}(x,y)\,{}_\kappa L_m^{(\alpha,\beta)}(x,y)dydx = \delta_{nm},$$

where δ_{nm} is the Kronecker's delta.

Motivated essentially by the above results, the following bivariate Mittag-Leffler functions [17] were introduced as in the following form

$$\begin{aligned} \mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(x,y) &= \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)}\frac{x^r}{r!}\frac{y^{\kappa s}}{s!}. \qquad (1.0.15) \\ (\alpha,\ \beta,\ \gamma &\in \mathbb{C},\ Re(\alpha),\ Re(\beta),\ Re(\kappa)>0) \end{aligned}$$

Remark 1.0.3 *According to the convergence conditions investigated by Srivastava and Daoust ([28], p. 155) for the generalized Lauricella series in two variables, the series in (1.0.15) are converges absolutely for $Re(\kappa)>0$.*

Taking $x=0$ and $r=0$ in (1.0.15), we see that

$$\mathfrak{E}_{\alpha,\beta,\kappa}^{(\gamma)}(0,y) = \frac{1}{\Gamma(\alpha)}\mathfrak{E}_{\kappa,\beta}^{(\gamma)}(y^\kappa).$$

Also, the following relations hold true:

$$\frac{1}{\Gamma(\alpha)}\mathfrak{E}^{1}_{\kappa,\beta}(y^{\kappa})=\frac{1}{\Gamma(\alpha)}\mathfrak{E}_{\kappa,\beta}(y^{\kappa})$$

and

$$\frac{1}{\Gamma(\alpha)}\mathfrak{E}_{1,\beta}(y)=\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{y^{n}}{\Gamma(\beta+n)}.$$

Comparing (1.0.12) and (1.0.15), we get that

$${}_{\kappa}L_{n}^{(\alpha,\beta)}(x,y)=x^{\alpha}y^{\beta}\mathfrak{E}^{(-n)}_{\alpha+1,\beta+1,\kappa}(x,y). \tag{1.0.16}$$

Remark 1.0.4 *(see [17])By taking into account the inverse operator* $\hat{D}_{x}^{-n}$*, which is given by*

$$\hat{D}_{x}^{-n}f(x):=\frac{1}{(n-1)!}\int_{0}^{x}(x-t)^{n-1}f(t)dt,$$

we can rewrite $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ *in the operational representation:*

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)=x^{1-\alpha}y^{1-\beta}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{r!s!}\hat{D}_{x}^{-r}\hat{D}_{y}^{-\kappa s}\left\{\frac{x^{\alpha-1}y^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\right\},$$

which further yields the Rodrigues-type relation

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)=\left(1-\hat{D}_{x}^{-1}\hat{D}_{y}^{-\kappa}\right)^{-\gamma}\left\{\frac{x^{\alpha-1}y^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\right\}.$$

We recall the following extension, $S^{A:B;B'}_{C:D;D'}$, of the double hypergeometric series (see [28], p. 199) in the form

$$
S\left(\begin{array}{l} A:\ B;\ B' \\ \\ C:\ D;\ D' \end{array}\left|\begin{array}{l} x \\ \\ y \end{array}\right.\right) \tag{1.0.17}
$$

$$
\equiv S\left(\begin{array}{l} A:\ B;\ B' \\ \\ C:\ D;\ D' \end{array}\left|\begin{array}{l} [(a):\vartheta,\varphi]:\ [(b):\psi];\ [(b'):\psi']; \\ \qquad\qquad\qquad\qquad\qquad\qquad x,y \\ [(c):\delta,\varepsilon]:\ [(d):\eta];\ [(d'):\eta']; \end{array}\right.\right)
$$

$$
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{\prod_{j=1}^{A}\Gamma[a_j+m\vartheta_j+n\varphi_j]\prod_{j=1}^{B}\Gamma[b_j+m\psi_j]\prod_{j=1}^{B'}\Gamma[b'_j+n\psi'_j]}{\prod_{j=1}^{C}\Gamma[c_j+m\delta_j+n\varepsilon_j]\prod_{j=1}^{D}\Gamma[d_j+m\eta_j]\prod_{j=1}^{D'}\Gamma[d'_j+n\eta_j]}\right.
$$

$$
\left.\times\frac{x^m}{m!}\frac{y^n}{n!}\right],
$$

where the coefficients

$$
\begin{cases} \vartheta_1,...,\vartheta_A;\quad \varphi_1,...,\varphi_A;\quad \psi_1,...,\psi_B;\quad \psi'_1,...,\psi'_{B'};\quad \delta_1,...,\delta_C; \\ \varepsilon_1,...,\varepsilon_C;\quad \eta_1,...,\eta_D;\quad \eta'_1,...,\eta'_{D'}; \end{cases}
$$

are real and positive. Here, (a) denotes the sequence of A parameters $a_1, a_{2,...,} a_A$ with a similar manner for (b), (b'), etc. are real and positive, and (a) abbreviates the array of A parameters $a_1, \cdots a_A, (b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$
b_j^{(k)}, j = 1, \cdots, B^{(k)} \forall k \in \{1, \cdots n\},
$$

with similar interpretations for (c) and $(d^{(k)})$, $k = 1, \cdots, n$. This function is further investigated in [27],[30].

By considering the special cases of the above series (1.0.17), in this thesis we consider the following functions:

$$\Psi^*\left(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t_1;-y^{\beta}t_2\right)$$

$$:=S\begin{matrix}0:&0;&0\\ \\1:&1;&0\end{matrix}\left(\begin{matrix}-: & -; & -; & \\ & & & -x^{\alpha}t_1,-y^{\beta}t_2\\ [\gamma+1:\alpha,\beta]: & [\xi:\eta]; & -; & \end{matrix}\right).$$

and

$$_2\Psi_4^*\left((1,\lambda),(1,\omega):(\alpha,\beta,\gamma+1),(\eta,\xi),(1,\mu_1+1),(1,\mu_2+1);-x^{\alpha}t_1,-y^{\beta}t_2\right)$$

$$:=S\begin{matrix}0:&1;&1\\ \\1:&2;&0\end{matrix}\left(\begin{matrix}-: & [\omega:1]; & [\lambda:1]; & \\ & & & -x^{\alpha}t_1,-y^{\beta}t_2\\ [\gamma+1:\alpha,\beta]: & [\xi,\mu_2+1:\eta,1]; & [\mu_1+1:1]; & \end{matrix}\right).$$

Remark 1.0.5 *According to the absolute convergence of the functions* $\Psi^*\left(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t_1,-y^{\beta}t_2\right)$*, we need* $Re(\alpha+\eta)>-1$ *and* $Re(\beta)>-1$ *(see [29] and also see [27],[30]). Similarly, for the absolute convergence of* $_2\Psi_4^*\left((1,\lambda),(1,\omega):(\alpha,\beta,\gamma+1),(\eta,\xi),(1,\mu_1+1),(1,\mu_2+1);-x^{\alpha}t_1,-y^{\beta}t_2\right)$*, we need* $Re(\beta+\eta)>-2$ *and* $Re(\alpha)>-2$ *(see [29] and also see [27],[30]).*

Definition 1.0.6 *([1],[15]) The Riemann-Liouville fractional integral of order* $\alpha\in\mathbb{C}$*,* $Re(\alpha)>0$ *is introduced as*

$$_x\mathbf{I}_{a^+}^{\alpha}[f]=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}}\,,\quad x>a.$$

In a similar way, two-dimensional Riemann-Liouville fractional integral of a function $f(x,y)$*, such that* $(x,y)\in\mathbb{R}\times\mathbb{R}$ *is introduced in the following fom:*

$$_x\mathbf{I}_{a^+}^{\alpha}f(x,y)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t,y)dt\,,\quad (x>a,\ Re(\alpha)>0)$$

$$ {}_y\mathbf{I}^{\beta}_{b^+} f(x,y) = \frac{1}{\Gamma(\beta)} \int_b^y (y-\tau)^{\beta-1} f(x,\tau) d\tau, \quad (y > b,\ Re(\beta) > 0) $$

$$ \begin{aligned} & {}_y\mathbf{I}^{\beta}_{b^+ x}\mathbf{I}^{\alpha}_{a^+} f(x,y) \\ = & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_b^y \int_a^x (x-t)^{\alpha-1} (y-\tau)^{\beta-1} f(t,\tau) dt d\tau \ . \end{aligned} \tag{1.0.18} $$

$$ (x > a,\ y > b,\ Re(\beta) > 0,\ Re(\alpha) > 0) $$

Definition 1.0.7 *([1],[15])The Riemann-Liouville fractional derivative of order* $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ *is introduced as*

$$ {}_x\mathbf{D}^{\alpha}_{a^+}[f] = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{\alpha-n-1} f(t) dt, \ \ n = [Re(\alpha)] + 1,\ x > a, $$

where, $[Re(\alpha)]$ *is the integral part of* $Re(\alpha)$.

In a similar way, two-dimensional Riemann-Liouville fractional derivative of a function $f(x,y)$*, such that* $(x,y) \in \mathbb{R} \times \mathbb{R}$ *is introduced in the following fom:*

$$ {}_x\mathbf{D}^{\alpha}_{a^+} f(x,y) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t,y) dx \ , \ \ (n = [Re(\alpha)] + 1,\ x > a) $$

$$ {}_y\mathbf{D}^{\beta}_{b^+} f(x,y) = \left(\frac{d}{dy}\right)^m \frac{1}{\Gamma(m-\beta)} \int_b^y (y-\tau)^{m-\beta-1} f(x,\tau) d\tau \ , \ \ (m = [Re(\beta)] + 1,\ y > b) $$

$$ \begin{aligned} {}_y\mathbf{D}^{\beta}_{b^+ x}\mathbf{D}^{\alpha}_{a^+} f(x,y) = & \left(\frac{d}{dx}\right)^n \left(\frac{d}{dy}\right)^m \frac{1}{\Gamma(n-\alpha)} \frac{1}{\Gamma(m-\beta)} \\ & \times \int_b^y \int_a^x (x-t)^{n-\alpha-1} (y-\tau)^{m-\beta-1} f(t,\tau) dt d\tau. \end{aligned} $$

$$ (a = [Re(\alpha)] + 1,\ b = [Re(\beta)] + 1,\ x > a,\ y > b) $$

Chapter 2

SOME RESULTS ON 2D-MITTAG-LEFFLER FUNCTIONS SUGGESTED BY 2D-LAGUERRE POLYNOMIALS

In this chapter, we calculate fractional calculus properties of the 2D-Mittag-Leffler functions $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Also, we get linear generating function for $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ suggested by $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$. Furthermore, considering above mentioned classes, we investigate their two-dimensional Laplace transform. Furthermore, by considering a general singular integral equation with $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel, we reach the solution suggested by $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ (1.0.7). Finally, we obtain the image of 2D-Mittag-Leffler functions $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ under the action of the Marichev-Saigo-Maeda integral operators with the special cases, such as Saigo and Riemann Lioville fractional integral operators.

2.1 Two-dimensional Fractional Integrals and Derivatives

In this section, we investigate two-dimensional Riemann-Liouville fractional integral and derivative of the classes $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Let $Re(\alpha)$, $Re(\beta)>0$, *and* $Re(\mu)$, $Re(\lambda)>0$, $Re(\gamma)>-1$.

Theorem 2.1.1 *Let* $Re(\alpha+\eta)>0$, $Re(\alpha)>0$ *and* $(\beta)>0$. *Then, we have*

$$\begin{aligned} &{}_y\mathbf{I}_{0^+}^{\alpha}{}_x\mathbf{I}_{0^+}^{\beta}\left[x^{\lambda-1}y^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right] \\ &= x^{\beta+\lambda-1}y^{\alpha+\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi+\alpha,\lambda+\beta)}\left(x^{\alpha},x^{\beta}y^{\eta}\right). \end{aligned}$$

Proof. Because of the hypothesis of the above Theorem, we have a right to interchange of the order of series and two-dimensional Riemann-Liouville fractional integral operator, which yields

$$
\begin{aligned}
&{}_y\mathbf{I}^{\alpha}_{0^+}{}_x\mathbf{I}^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma,\kappa}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right]\\
&=\int_0^y\int_0^x\frac{(y-\tau)^{\alpha-1}(x-t)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}t^{\lambda-1}\tau^{\xi-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma,\kappa}\left(t^{\alpha},t^{\beta}\tau^{\eta}\right)dtd\tau\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\\
&\times\int_0^y(y-\tau)^{\alpha-1}\tau^{\eta s+\xi-1}d\tau\int_0^x(x-t)^{\beta-1}t^{\alpha r+\beta s+\lambda-1}dt\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s x^{\alpha r+\beta s+\beta+\lambda-1}y^{\eta s+\xi+\alpha-1}}{\Gamma(\alpha r+\beta s+\lambda+\beta)\Gamma(\eta s+\xi+\alpha)r!s!}\\
&=x^{\beta+\lambda-1}y^{\alpha+\xi-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi+\alpha,\lambda+\beta)}_{\gamma,\kappa}\left(x^{\alpha},x^{\beta}y^{\eta}\right).
\end{aligned}
$$

This completes the proof.

■

In a similar manner, we have the following Corollary:

Corollary 2.1.2 *Let* $Re(\alpha)>0$ *and* $Re(\beta)>0$*. Then, we have*

$$
\begin{aligned}
&{}_y\mathbf{I}^{\alpha}_{0^+}{}_x\mathbf{I}^{\beta}_{0^+}\left[x^{\gamma}y^{\xi-1}\mathfrak{L}^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right)\right]\\
&=\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\frac{\Gamma(\alpha+\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+\beta+1)}\\
&\times x^{\beta+\gamma}y^{\alpha+\xi-1}\mathfrak{L}^{(\alpha,\beta,\gamma+\beta,\eta,\alpha+\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right).
\end{aligned}
$$

Theorem 2.1.3 *Let* $Re(\alpha+\eta)>0$, $Re(\alpha)\geqslant 0$ *and* $(\beta)>0$*. Then, we have*

$$
\begin{aligned}
&{}_y\mathbf{D}^{\alpha}_{0^+}{}_x\mathbf{D}^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma,\kappa}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right]\\
&=\ x^{\lambda-\beta-1}y^{\xi-\alpha-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi-\alpha,\lambda-\beta)}_{\gamma,\kappa}\left(x^{\alpha},x^{\beta}y^{\eta}\right).
\end{aligned}
$$

Proof. Because of the hypothesis of the above Theorem, we have a right to interchange of the order of series and two-dimensional Riemann-Liouville fractional derivative operator, which yields

$$
\begin{aligned}
&{}_y\mathbf{D}^{\alpha}_{0^+\,x}\mathbf{D}^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma,\kappa}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right]\\
&={}_y\,\mathbf{D}^{\alpha}_{0^+\,x}\mathbf{D}^{\beta}_{0^+}\left[x^{\lambda-1}y^{\xi-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s x^{\alpha r}xy^{\eta s}}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\right]\\
&=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s\,{}_y\mathbf{D}^{\alpha}_{0^+\,x}\mathbf{D}^{\beta}_{0^+}[x^{\alpha r+\beta s+\lambda-1}y^{\eta s+\xi-1}]}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\\
&=(\frac{d}{dy})^m(\frac{d}{dx})^n\frac{1}{\Gamma(n-\beta)}\frac{1}{\Gamma(m-\alpha)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\\
&\times\int_0^y(y-\tau)^{m-\alpha-1}\tau^{\eta s+\xi-1}d\tau\int_0^x(x-\xi)^{n-\beta-1}\xi^{\alpha r+\beta s+\lambda-1}d\xi\\
&=x^{\lambda-\beta-1}y^{\xi-\alpha-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s x^{\alpha r+\beta s}y^{\eta s}}{\Gamma(\alpha r+\beta s+\lambda-\beta)\Gamma(\eta s+\xi-\alpha)k_1!k_2!}\\
&=x^{\lambda-\beta-1}y^{\xi-\alpha-1}\mathfrak{E}^{(\alpha,\beta,\eta,\xi-\alpha,\lambda-\beta)}_{\gamma,\kappa}\left(x^{\alpha},x^{\beta}y^{\eta}\right).
\end{aligned}
$$

Whence the result.

■

In a similar manner, we have the following Corollary:

Corollary 2.1.4 *Let Re*$(\alpha+\eta)>0$ *and Re*$(\beta)>0$. *Then, we have*

$$
\begin{aligned}
&{}_y\mathbf{D}^{\alpha}_{0^+\,x}\mathbf{D}^{\beta}_{0^+}[x^{\gamma}y^{\xi-1}\mathfrak{L}^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right)]\\
&=\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\frac{\Gamma(\xi-\alpha+\eta m)}{\Gamma(\alpha n+\beta m+\gamma-\beta+1)}\\
&\times x^{\gamma-\beta}y^{\xi-\alpha-1}\mathfrak{L}^{(\alpha,\beta,\gamma-\beta,\eta,\xi-\alpha)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right).
\end{aligned}
$$

2.2 Singular Two-Dimensional Equation

In this section, we derive two-dimensional Laplace transform of the classes $\mathfrak{E}^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma,\kappa}(x,y)$ and $\mathfrak{L}^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}(x,y)$. Next, we obtain two-dimensional integral involving the product of two $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ functions in the integrand. Lastly, the solution of two-dimensional integral equation with $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ suggested by $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ in the kernel is obtained.

As usual,

$$\mathbb{L}_2[f(x,t)] = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)dtdx \tag{2.2.1}$$

$$(x,\ t > 0 \; and \; p,\ s \in \mathbb{C})$$

denote the two-dimensional Laplace transform of f (see [11]).

Theorem 2.2.1 *Let* $Re(\omega),\ Re(\sigma),\ Re(\alpha+\eta) > 0,\ Re(\beta) > 0,\ Re(s_1),\ Re(s_2) > 0$ *and* $\left|\frac{\omega^\alpha}{s_1^\alpha}\right|,\ \left|\frac{\sigma^\beta}{s_1^\beta s_2^\eta}\right| < 1$, *such that*

$$\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}((\omega x)^\alpha,(\sigma^\beta x^\beta y^\eta))](s_1,s_2) = \frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}(1-\frac{\omega^\alpha}{s_1^\alpha})^{-\gamma}(1-\frac{\sigma^\beta}{s_1^\beta s_2^\eta})^{-\kappa}.$$

Proof. With the help of (2.2.1) and considering $\left|\frac{\omega^\alpha}{s_1^\alpha}\right| < 1$ and $\left|\frac{\sigma^\beta}{s_1^\beta s_2^\eta}\right| < 1$, we get

$$\begin{aligned}
&\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}((\omega x)^\alpha,(\sigma^\beta x^\beta y^\eta))](s_1,s_2)\\
&= \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(\gamma)_r(\kappa)_s\omega^{\alpha r}\sigma^{\beta s}}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\\
&\times \int_0^\infty x^{\alpha r+\beta s+\lambda-1}e^{-s_1 x}dx \int_0^\infty y^{\eta s+\xi-1}e^{-s_2 y}dy\\
&= \frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}\sum_{r=0}^\infty \frac{(\gamma)_r}{r!}(\frac{\omega^\alpha}{s_1^\alpha})^r \sum_{s=0}^\infty \frac{(\kappa)_s}{s!}(\frac{\sigma^\beta}{s_1^\beta s_2^\eta})^s = \frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}(1-\frac{\omega^\alpha}{s_1^\alpha})^{-\gamma}(1-\frac{\sigma^\beta}{s_1^\beta s_2^\eta})^{-\kappa}.
\end{aligned}$$

The proof is completed.

■

From Theorem 2.2.1 by setting $\lambda - 1 = \gamma$ and using equation (1.0.9) we get the following result:

Corollary 2.2.2 *Let* $Re(\omega)$, $Re(\sigma)$, $Re(\alpha)$, $Re(\beta)$, $Re(\lambda)$, $Re(s_1)$, $Re(s_2) > 0$ *and* $\left|\frac{\omega^\alpha}{s_1^\alpha}\right|$, $\left|\frac{\sigma^\beta}{s_1^\beta s_2^\eta}\right| < 1$, *such that*

$$\begin{aligned} & \mathbb{L}_2[t^\gamma \tau^{\xi-1}\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}((\omega t),(\sigma t \tau^{\frac{\eta}{\beta}}))](s_1,s_2) \\ = & \frac{1}{s_1^{\gamma+1}}\frac{1}{s_2^\xi}\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\eta m+\xi)}(1-\frac{\omega^\alpha}{s_1^\alpha})^n(1-\frac{\sigma^\beta}{s_1^\beta s_2^\eta})^m. \end{aligned}$$

In the following Theorem, by using Theorem 2.2.1, we obtain two-dimensional integral involving the product of two 2D-Mittag-Leffler functions $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ in the integrand.

Theorem 2.2.3 *If* ω, $\sigma \in \mathbb{C}$, $Re(\alpha+\eta) > 0$ *and* $Re(\beta) > 0$, *then*

$$\begin{aligned} & \int_0^y \int_0^x \Big[(x-t)^{\lambda-1}(y-\tau)^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha(x-t)^\alpha, \lambda_2^\beta(x-t)^\beta(y-\tau)^\eta) \\ & \times t^{\gamma-1}\tau^{\zeta-1}\mathfrak{E}_{\tau,\rho}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\omega^\alpha t^\alpha, \sigma^\beta t^\beta \tau^\eta)dtd\tau\Big] \\ & = x^{\lambda+\gamma}y^{\xi+\zeta}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\omega^\alpha x^\alpha, \sigma^\beta x^\beta y^\eta)\mathfrak{E}_{\tau,\rho}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\omega^\alpha x^\alpha, \sigma^\beta x^\beta y^\eta). \end{aligned}$$

Proof. By using the convolution theorem for two-dimensional Laplace transform, we obtain

$$\mathbb{L}_2\left[\int_0^y\int_0^x (x-t)^{\lambda-1}(y-\tau)^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\omega^\alpha(x-t)^\alpha,\sigma^\beta(x-t)^\beta(y-\tau)^\eta)t^{\gamma-1}\tau^{\zeta-1}\right.$$

$$\left.\times\mathfrak{E}_{\tau,\rho}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\omega^\alpha t^\alpha,\sigma^\beta t^\beta\tau^\eta)dtd\tau\right](s_1,s_2)$$

$$=\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\omega^\alpha x^\alpha,\sigma^\beta x^\beta y^\eta)]$$

$$\times\mathbb{L}_2[x^{\gamma-1}y^{\zeta-1}\mathfrak{E}_{\tau,\rho}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\omega^\alpha t^\alpha,\sigma^\beta t^\beta\tau^\eta)](s_1,s_2)$$

$$=\frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}\left(1-\frac{\omega^\alpha}{s_1^\alpha}\right)^{-\gamma_1}\left(1-\frac{\sigma^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_2}\frac{1}{s_1^\gamma}\frac{1}{s_2^\zeta}\left(1-\frac{\omega^\alpha}{s_1^\alpha}\right)^{-\gamma_3}\left(1-\frac{\sigma^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_4}.$$

For $Re(s_1),\ Re(s_2)>0$, we have

$$\begin{aligned}
&\mathbb{L}_2\left[\int_0^y\int_0^x (x-t)^{\lambda-1}(y-\tau)^{\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\omega^\alpha(x-t)^\alpha,\sigma^\beta(x-t)^\beta(y-\tau)^\eta)\right. \quad (2.2.2)\\
&\left.\times t^{\gamma-1}\tau^{\zeta-1}\mathfrak{E}_{\tau,\rho}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\omega^\alpha t^\alpha,\sigma^\beta t^\beta\tau^\eta)dtd\tau\right](s_1,s_2)\\
=\ &\frac{1}{s_1^{\lambda+\gamma}}\frac{1}{s_2^{\xi+\zeta}}\left(1-\frac{\omega^\alpha}{s_1^\alpha}\right)^{-\gamma_1}\left(1-\frac{\omega^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_2}\left(1-\frac{\omega^\alpha}{s_1^\alpha}\right)^{-\gamma_3}\left(1-\frac{\sigma^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_4}\\
=\ &\mathbb{L}_2\left[x^{\lambda+\gamma}y^{\xi+\zeta}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\omega^\alpha x^\alpha,\sigma^\beta x^\beta y^\eta)\mathfrak{E}_{\tau,\rho}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\omega^\alpha x^\alpha,\sigma^\beta x^\beta y^\eta)\right](s_1,s_2).
\end{aligned}$$

Finally, we take the inverse two-dimensional Laplace transform on both sides of (2.2.2) to complete the proof. ■

By letting $\lambda-1=\gamma$ in Theorem 2.2.3 and taking into account (1.0.9), we get the following Corollary:

Corollary 2.2.4 *For $\omega,\sigma\in\mathbb{C}$, $Re(\lambda)$, $Re(\xi)$, $Re(\gamma)$, $Re(\zeta)>0$, we have*

$$\int_0^y\int_0^x (x-t)^\gamma(y-\tau)^{\xi-1}\mathfrak{L}_{n_1,m_1}^{(\alpha,\beta,\gamma_1,\eta,\xi_1)}(\omega(x-t),\sigma(x-t)(y-\tau)^{\frac{\eta}{\beta}})$$

$$\times t^\gamma\tau^{\xi-1}\mathfrak{L}_{n_2,m_2}^{(\alpha,\beta,\gamma_2,\eta,\xi_2)}(\omega t,\sigma t\tau^{\frac{\eta}{\beta}})dtd\tau$$

$$=x^{\gamma_1+\gamma_2+1}y^{\xi_1+\xi_2-1}\mathfrak{L}_{n_1,m_1}^{(\alpha,\beta,\gamma_1,\eta,\xi_1)}(\omega x,\sigma xy^{\frac{\eta}{\beta}})\mathfrak{L}_{n_2,m_2}^{(\alpha,\beta,\gamma_2,\eta,\xi_2)}(\omega x,\sigma xy^{\frac{\eta}{\beta}}).$$

Note that two-dimensional fractional integral $({}_x\mathbf{I}_{0^+}^{\alpha_1}{}_y\mathbf{I}_{0^+}^{\alpha_2}\varphi)(x,y)$ can be written as a convolution of the form

$$
\begin{aligned}
({}_x\mathbf{I}_{0^+}^{\alpha_1}{}_y\mathbf{I}_{0^+}^{\alpha_2}\varphi)(x,y) &= \left[\varphi(x,y) * \frac{x_t^{\alpha_1-1}y_\tau^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\right]. \\
(Re(\alpha_1),\, Re(\alpha_2) &> 0)
\end{aligned}
$$

Therefore, using the double convolution theorem for two-dimensional Laplace transform of two-dimesional fractional integral ${}_x\mathbf{I}_{0^+}^{\alpha_1}{}_y\mathbf{I}_{0^+}^{\alpha_2}\varphi$, we reach the following result

$$
\mathbb{L}_2\left({}_x\mathbf{I}_{0^+}^{\alpha_1}{}_y\mathbf{I}_{0^+}^{\alpha_2}\varphi\right)(p,q) = p^{-\alpha_1}q^{-\alpha_2}\mathbb{L}_2(\varphi)(p,q),
$$

which is also true for sufficiently good function φ if $Re(\alpha_1),\, Re(\alpha_2) > 0$.
Let constract the double convolution equation as below:

$$
\begin{aligned}
&\int_0^y\int_0^x (x-t)^{\gamma-1}(y-\tau)^{\xi-1}\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}((\omega x)^\alpha,(\sigma^\beta x^\beta y^\eta))\Phi(t,\tau)dtd\tau \\
= \; &\Psi(x,y), \qquad (2.2.3)
\end{aligned}
$$

where $Re(\gamma) > -1$.

The solution of a singular two-dimensional integral equation (2.2.3) is given by the following Theorem:

Theorem 2.2.5 *The singular two-dimensional integral equation (2.2.3) admits a locally integrable solution*

$$
\begin{aligned}
\Phi(t,\tau) &= \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)} \\
&\times \int_0^y\int_0^x (x-t)^{\alpha_1-\gamma-2}(y-\tau)^{\alpha_2-\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}((\omega x)^\alpha,(\sigma^\beta x^\beta y^\eta)) \\
&\times [I_{0^+}^{-\alpha_1}I_{0^+}^{-\alpha_2}\Psi(t,\tau)]dtd\tau.
\end{aligned}
$$

Proof. We first apply two-dimensional Laplace transform on both sides of (2.2.3) and then use two-dimensional convolution theorem to obtain

$$\frac{1}{s_1^{\gamma+1}}\frac{1}{s_2^{\xi}}\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\eta m+\xi)}(1-\frac{\omega^{\alpha}}{s_1^{\alpha}})^n(1-\frac{\sigma^{\beta}}{s_1^{\beta}s_2^{\eta}})^m\mathbb{L}_2[\Phi(t,\tau)](s_1,s_2)$$
$$= \mathbb{L}_2[\Psi(t,\tau)](s_1,s_2).$$

Therefore, we get,

$$\mathbb{L}_2[\Phi(t,\tau)](s_1,s_2) = \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$
$$\times(s_1)^{\gamma-\alpha_1+1}(s_2)^{\xi-\alpha_2}(1-\frac{\omega^{\alpha}}{s_1^{\alpha}})^{-n}(1-\frac{\sigma^{\beta}}{s_1^{\beta}s_2^{\eta}})^{-m}\{s_1^{\alpha_1}s_2^{\alpha_2}\mathbb{L}_2[\Psi(t,\tau)](s_1,s_2)\}.$$

Finally, by taking the inverse two-dimensional Laplace transform on both sides and using Lemma 3.2 of [1] and Theorem 2.2.1, we obtain

$$\Phi(t,\tau) = \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$
$$\times\int_0^y\int_0^x (x-t)^{\alpha_1-\gamma-2}(y-\tau)^{\alpha_2-\xi-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}((\omega x)^{\alpha},(\sigma^{\beta}x^{\beta}y^{\eta}))$$
$$\times[I_{0^+}^{-\alpha_1}I_{0^+}^{-\alpha_2}\Psi(t,\tau)]dtd\tau,$$

which completes the proof. ■

2.3 Marichev-Saigo-Maeda Fractional Integration Operator of 2D-Mittag-Leffler Functions

In this section, the images of 2D-Mittag-Leffler functions $\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ under the actions of Marichev-Saigo-Maeda fractional integral operators are obtained. Also, some special cases of the theorems (and corollaries) are investigated, and concluding remarks involving Saigo integral operators and Riemann-Liouville integral operators are discussed.

Let $\lambda,\ \lambda',\ \mu,\ \mu',\ \nu\ \in\ \mathbb{C}$ and $x>0$, then the left and right sided Marichev-Saigo-Maeda type fractional integral operators are defined by the following equations:

$$\left(\mathscr{I}_{0^+}^{\lambda,\lambda',\mu,\mu',\nu} f\right)(x) \tag{2.3.1}$$

$$= \frac{x^{-\lambda}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{-\lambda'} F_3(\lambda,\lambda',\mu,\mu';\nu;1-\frac{t}{x};1-\frac{x}{t}) f(t)dt$$

$$(Re(\nu) > 0)$$

and

$$\left(\mathscr{I}_{0^-}^{\lambda,\lambda',\mu,\mu',\nu} f\right)(x) \tag{2.3.2}$$

$$= \frac{x^{-\lambda'}}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} t^{-\lambda} F_3(\lambda,\lambda',\mu,\mu';\nu;1-\frac{x}{t};1-\frac{t}{x}) f(t)dt,$$

$$(Re(\nu) > 0)$$

respectively, where the symbol $F_3(.)$, that is

$$F_3(\lambda,\lambda',\mu,\mu';\nu;x;y) = \sum_{m,n=0}^{\infty} \frac{(\lambda)_m(\lambda')_n(\mu)_m(\mu')_n}{(\nu)_{m+n}m!n!} x^m y^n \quad (\max\{|x|,|y|\}<1)$$

is called the 3rd Appell function (see p. 413 of [9]).

In particular,

$$F_3(\lambda,\nu-\lambda;\mu,\nu-\mu;\nu;x;y) = {}_2F_1(\lambda,\mu;\nu;x+y-xy), \tag{2.3.3}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function.

Also, a couple of reduction formulas, such that

$$F_3(\lambda,0,\mu,\mu';\nu;x;y) = F_3(\lambda,\lambda',\mu;\nu;x;y) = {}_2F_1(\lambda,\mu;\nu;x)$$

and

$$F_3(0,\lambda',\mu,\mu';\nu;x;y) = F_3(\lambda,\lambda',\mu';\nu;x;y) = {}_2F_1(\lambda',\mu';\nu;x)$$

are easily derived.

In [7], the operators in (2.3.1) and (2.3.2) were defined by Marichev as Mellin type convolution operators with a special function $F_3(.)$ in the kernel. With the help of the reduction formula in (2.3.3), the fractional integration operators (of Marichev-Saigo-Maeda type) given in (2.3.1) and (2.3.2) becomes the Saigo integral operators $\mathbb{I}_{0^+}^{(\lambda,\mu,\nu)}$ and $\mathbb{I}_{0^-}^{(\lambda,\mu,\nu)}$ ([21]) (see also [12] and [32]).

$$\left(\mathbb{I}_{0^+}^{(\lambda,\mu,\nu)}f\right)(x)=\frac{x^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}\,{}_2F_1(\lambda+\mu,-\nu;\lambda;1-\frac{t}{x})f(t)dt,\quad (Re(\lambda)>0) \tag{2.3.4}$$

and

$$\left(\mathbb{I}_{0^-}^{(\lambda,\mu,\nu)}f\right)(x)=\frac{1}{\Gamma(\lambda)}\int_x^\infty(t-x)^{\lambda-1}t^{-\lambda-\mu}\,{}_2F_1(\lambda+\mu,-\nu;\lambda;1-\frac{x}{t})f(t)dt,\quad (Re(\lambda)>0), \tag{2.3.5}$$

respectively.

Clearly, by using the above definitions, the following relations hold true (see p.338, Eqns. (2.9) and (2.10) in [26]):

$$\left(\mathscr{I}_{0^+}^{(\lambda,0,\mu,\mu',\eta)}\right)(x)=\left(\mathbb{I}_{0^+}^{(\nu,\lambda-\nu,-\mu)}\right)(x)\quad (\nu\in\mathbb{C}) \tag{2.3.6}$$

and

$$\left(\mathscr{I}_{0^-}^{(\lambda,0,\mu,\mu',\nu)}\right)(x)=\left(\mathbb{I}_{0^-}^{(\nu,\lambda-\nu,-\mu)}\right)(x).\quad (\nu\in\mathbb{C}) \tag{2.3.7}$$

Some properties of the operators in (2.3.1) and (2.3.2) were obtained by Saigo and Maeda (see [22]). Later on, Saigo and Maeda gave further relationship between the Mellin transforms and hypergeometric operators (or Saigo fractional integral operators).

Note that, if we take $\lambda = \lambda^{'} = 0$, (2.3.1) and (2.3.2) yield the following classical left and right Riemann-Liouville fractional integral operators [10], respectively;

$$(\mathbf{I}_{0+}^{\nu} f)(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt \quad x > 0$$

and

$$(\mathbf{I}_{0-}^{\nu} f)(x) = \frac{1}{\Gamma(\nu)} \int_x^0 (t-x)^{\nu-1} f(t) dt, \quad x < 0$$

where Γ is called the Gamma function and $\Gamma(\nu) > 0$.

2.3.1 Left-sided Marichev-Saigo-Maeda Fractional Integration Operator of 2D-Mittag-Leffler Functions

In this section, the main results are obtained by considering the following Lemma.

Lemma 2.3.1 *(p.394 of [22]) Let* $\lambda,\ \lambda^{'},\ \mu,\ \mu^{'},\ \nu\ \in\ \mathbb{C}$ *and* $Re(\nu) > 0,\ Re(\zeta) > \max\{0, Re(\lambda+\lambda^{'}+\mu-\nu), Re(\lambda^{'}-\mu^{'})\}.$ *Then the following relation holds*:

$$\begin{aligned} &\left(\mathscr{I}_{0+}^{\lambda,\lambda^{'},\mu,\mu^{'},\nu} x^{\zeta-1}\right)(x) \qquad (2.3.8) \\ = \ & \frac{\Gamma(\zeta)\Gamma(\zeta+\nu-\lambda-\lambda^{'}-\mu)\Gamma(\zeta+\mu^{'}-\lambda^{'})}{\Gamma(\zeta+\mu^{'})\Gamma(\zeta+\nu-\lambda-\lambda^{'})\Gamma(\zeta+\nu-\lambda^{'}-\mu)} x^{\zeta+\nu-\lambda-\lambda^{'}-1}. \end{aligned}$$

We now give the image of the 2D-Mittag-Leffler functions (1.0.7) under the action of the left-sided Marichev-Saigo-Maeda fractional integral given in (2.3.1).

Theorem 2.3.2 *Let the parameters* $\lambda,\ \lambda^{'},\ \mu,\ \mu^{'},\ \nu,\ \gamma,\ \kappa,\ \alpha,\ \beta,\ \eta,\ \xi,\ \lambda,\ \rho\ \in\ \mathbb{C}$

and $Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\eta) > 0,\ Re(\lambda) > 0,\ Re(\xi) > 0,\ Re(\sigma) > 0,\ Re(\varsigma) > 0,\ Re(\nu) > 0,\ Re(\rho) > \max\{0, Re(\lambda+\lambda'+\mu-\nu), Re(\lambda'-\mu')\}$. *Then the following relation is valid*

$$\left(\mathscr{I}_{0^+}^{\left(\lambda,\lambda',\mu,\mu',\nu\right)}\left[t^{\rho-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(ct^{\sigma},cx^{\varsigma})\right]\right)(x) = \frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-\lambda-\lambda'-1} \tag{2.3.9}$$

$$\times S_{4:0;1}^{3:1;1}\left(\begin{array}{ll} [(\rho,\rho+\nu-\lambda-\lambda'-\mu):\sigma,\varsigma]:[(\gamma):1];[(\kappa):1] & ; \\ [(\lambda,\rho+\mu',\rho+\nu-\lambda-\lambda',\rho+\nu-\lambda'-\mu):(\alpha,\sigma),(\beta,\xi)]:-;[(\xi:\eta)] & ; \\ cx^{\sigma},cx^{\varsigma} & \end{array}\right),$$

for all $x > 0$.

Proof. Let the left hand side of (2.3.9) be $\mathscr{J}$. Then using definition (1.0.7), we get

$$\mathscr{J} = \left(\mathscr{I}_{0^+}^{\left(\lambda,\lambda',\mu,\mu',\nu\right)}\left[t^{\rho-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s c^{r+s}t^{\sigma r+\varsigma s}}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!})\right]\right)(x).$$

Since the converge conditions satisfied, we change the order of integration and summation, which yields

$$\mathscr{J} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s c^{r+s}}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\left(\mathscr{I}_{0^+}^{\left(\lambda,\lambda',\mu,\mu',\nu\right)}\{t^{\rho+\sigma r+\varsigma s-1}\}\right)(x).$$

Now, using Lemma 2.3.1 and relation (2.3.8) with ρ replaced by $(\rho+\sigma r+\varsigma s)$, we get

$$
\begin{aligned}
\mathscr{I} \;=\;& \frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-\lambda-\lambda'-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(\gamma+r)\Gamma(\kappa+s)}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\\
&\times\frac{\Gamma(\rho+\sigma r+\varsigma s)\Gamma(\rho+\sigma r+\varsigma s+\nu-\lambda-\lambda'-\mu)}{\Gamma(\rho+\sigma r+\varsigma s+\mu')\Gamma(\rho+\sigma r+\varsigma s+\nu-\lambda-\lambda')}\\
&\times\frac{\Gamma(\rho+\sigma r+\varsigma s+\mu'-\lambda')}{\Gamma(\rho+\sigma r+\varsigma s+\nu-\lambda'-\mu)}\frac{c^{r+s}x^{\sigma r+\varsigma s}}{r!s!}.
\end{aligned}
$$

By using (1.0.17), we obtain desired result. ■

2.3.2 Right-sided Marichev-Saigo-Maeda Fractional Integration Operator of 2D- Mittag-Leffler Functions

In this part, we need to use next Lemma to obtain our results.

Lemma 2.3.3 *(p.394 of [21]) Let* $\lambda,\ \lambda',\ \mu,\ \mu',\ \nu\ \in\mathbb{C}$ *and* $Re(\nu)>0,\ Re(\rho)<1+\min\{Re\{-\mu\},\ Re(\lambda+\lambda'-\nu),\ Re(\lambda+\mu'-\nu)\}$. *Then the following relation holds:*

$$
\left(\mathscr{I}_{0^-}^{\lambda,\lambda',\mu,\mu',\nu}t^{\rho-1}\right)(x) \tag{2.3.10}
$$
$$
= \frac{\Gamma(1-\rho-\mu)\Gamma(1-\rho-\nu+\lambda+\lambda')\Gamma(1-\rho+\lambda+\mu'-\nu)}{\Gamma(1-\rho)\Gamma(1-\rho+\lambda+\lambda'+\mu'-\nu)\Gamma(1-\rho+\lambda-\mu)}x^{\rho+\nu-\lambda-\lambda'-1}.
$$

The image of the right-sided Marichev-Saigo-Maeda fractional integral (2.3.2) for the 2D-Mittag-Leffler functions (1.0.7) is given by the following Theorem:

Theorem 2.3.4 *Let the parameters* $\lambda,\ \lambda',\ \mu,\ \mu',\ \nu,\ \gamma,\ \kappa,\ \alpha,\ \beta,\ \eta,\ \xi,\ \lambda,\ \rho\ \in\mathbb{C}$ *and* $Re(\alpha)>0,\ Re(\beta)>0,\ Re(\eta)>0,\ Re(\lambda)>0,\ Re(\xi)>0,\ Re(\sigma)>0,\ Re(\varsigma)>0,\ Re(\nu)>0,\ Re(\rho)<1+\min\{Re\{-\mu\},\ Re(\lambda+\lambda'-\nu),\ Re(\lambda+\mu'-\nu)\}$. *Then the following relation is valid*

$$\left(\mathscr{I}_{0^-}^{\left(\lambda,\lambda',\mu,\mu',\nu\right)}\left[t^{\rho-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(\frac{c}{t^{\sigma}},\frac{c}{t^{\varsigma}})\right]\right)(x) \tag{2.3.11}$$
$$= \frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-\lambda-\lambda'-1}$$

$$\times S_{4:0;1}^{3:1;1}\left(\begin{array}{c}[(1-\rho-\mu,1-\rho-\nu+\lambda+\lambda',1-\rho+\lambda+\mu'-\nu):\sigma,\varsigma]:[(\gamma):1];[(\kappa):1] \quad ; \\ \\ [(\lambda,1-\rho,1-\rho+\lambda+\lambda'+\mu'-\nu,1-\rho+\lambda-\mu):(\alpha,\sigma),(\beta,\xi)]:-;[(\xi:\nu)] \quad ; \\ \\ cx^{\sigma},cx^{\varsigma}\end{array}\right),$$

for all $x > 0$.

Proof. Let us denote the left hand side of (2.3.11) as $\mathscr{J}$. Using the definition (1.0.7), we get

$$\mathscr{J} = \left(I_{0^-}^{\left(\lambda,\lambda',\mu,\mu',\nu\right)}\left[t^{\rho-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s c^{r+s}t^{\sigma r+\varsigma s}}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!})\right]\right)(x).$$

Since the converge conditions satisfied, we change the order of integration and summation, which yields

$$\mathscr{J} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_r(\kappa)_s c^{r+s}}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\left(\mathscr{I}_{0^-}^{\left(\lambda,\lambda',\mu,\mu',\nu\right)}\{t^{\rho+\sigma r+\varsigma s-1}\}\right)(x).$$

Now, applying Lemma 2.3.3 and using (2.3.10) with ρ replaced by $(\rho-\sigma r-\varsigma s)$, we obtain:

$$\begin{aligned}\mathscr{J} &= x^{\rho+\nu-\lambda-\lambda'-1}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(\gamma+r)\Gamma(\kappa+s)}{\Gamma(\alpha r+\beta s+\lambda)\Gamma(\eta s+\xi)r!s!}\\ &\times\frac{\Gamma(1-\rho+\sigma r+\varsigma s)\Gamma(1-\rho+\sigma r+\varsigma s-\nu+\lambda+\lambda')}{\Gamma(1-\rho+\sigma r+\varsigma s)\Gamma(1-\rho+\sigma r+\varsigma s+\lambda+\lambda'+\mu'-\nu)}\\ &\times\frac{\Gamma(1-\rho+\sigma r+\varsigma s+\lambda+\mu'-\nu)}{\Gamma(1-\rho+\sigma r+\varsigma s+\lambda-\mu)}\frac{c^{r+s}x^{\rho+\sigma r+\varsigma s+\nu-\lambda-\lambda'-1}}{r!s!}.\end{aligned}$$

So we complete the proof by using (1.0.17). ■

2.3.3 Special Cases

In the case $\lambda' = 0$ in (2.3.6), we obtain the left-sided Saigo fractional integral operators which is given in (2.3.4). Therefore, as a result of Theorem 2.3.2, we get the next assertation:

Corollary 2.3.5 *Let the parameters* $\lambda,\ \mu,\ \nu,\ \gamma,\ \kappa,\ \alpha,\ \beta,\ \eta,\ \xi,\ \lambda,\ \rho\ \in\ \mathbb{C}$ *and* $Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\eta) > 0,\ Re(\lambda) > 0,\ Re(\xi) > 0\ Re(\sigma) > 0,\ Re(\varsigma) > 0,\ Re(\nu) > 0,\ Re(\rho) > \max\{0, Re(\nu-\lambda-\mu)\}$. *Then the following relation holds*

$$\begin{aligned}&\left(\mathbb{I}_{0^+}^{(\nu,\lambda-\nu,-\mu)}\left[t^{\rho-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(ct^{\sigma},ct^{\varsigma})\right]\right)(x)\\ =\ &\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-\lambda-1}\\ &\times S_{3:0;1}^{2:1;1}\left(\begin{array}{cc}[(\rho,\rho+\nu-\lambda-\mu):\sigma,\varsigma]:[(\gamma):1];[(\kappa):1] & ;\\ [(\lambda,\rho+\nu-\lambda,\rho+\nu-\mu):(\alpha,\sigma),(\beta,\xi)]:-;[(\xi:\eta)] & ;\\ cx^{\sigma},cx^{\varsigma} & \end{array}\right),\end{aligned}$$

for all $x > 0$.

In the case $\lambda' = 0$ in (2.3.7), we obtain the right-sided Saigo fractional integral operators which is given in (2.3.5). Therefore, we get the following Corollary from Theorem 2.3.4:

Corollary 2.3.6 *Let the parameters* $\lambda,\ \mu,\ \nu,\ \gamma,\ \kappa,\ \alpha,\ \beta,\ \eta,\ \xi,\ \lambda,\ \rho\ \in\ \mathbb{C}$ *and* $Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\eta) > 0,\ Re(\lambda) > 0,\ Re(\xi) > 0,\ \ Re(\sigma) > 0,\ Re(\varsigma) > 0,\ Re(\nu) > 0,\ Re(\rho) < 1 + \min\{Re(-\mu),\ Re(\lambda - \nu)\}$. *Then the following relation holds*

$$\begin{aligned}
&\left(\mathbb{I}_{0^-}^{(\nu,\lambda-\nu,-\mu)}\left[t^{\rho-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}\left(\frac{c}{t^{\sigma}},\frac{c}{t^{\varsigma}}\right)\right]\right)(x)\\
=\ &\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-\lambda-1}\\
&\times S_{3:0;1}^{2:1;1}\left(\begin{array}{cc}
[(1-\rho-\mu,1-\rho-\nu+\lambda):\sigma,\varsigma]:[(\gamma):1];[(\kappa):1] & ;\\
\\
[(\lambda,1-\rho,1-\rho+\lambda-\mu):(\alpha,\sigma),(\beta,\xi)]:-;[(\xi:\eta)] & ;\\
\\
cx^{\sigma},cx^{\varsigma} &
\end{array}\right),
\end{aligned}$$

for all $x > 0$.

When $\lambda = \lambda' = 0$ in the Marichev-Saigo-Maeda operators in Theorem 2.3.2 , we obtain the left-sided Riemann Liouville operator. Therefore, setting $\lambda = \lambda' = 0$, Theorem 2.3.2 reduces to the following corollary:

Corollary 2.3.7 *Let the parameters* $\nu,\ \lambda,\ \gamma,\ \kappa,\ \alpha,\ \beta,\ \eta,\ \xi,\ \lambda,\ \rho\ \in \mathbb{C}$ *and* $Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\eta) > 0,\ Re(\lambda) > 0,\ Re(\xi) > 0,\ Re(\sigma) > 0,\ Re(\varsigma) > 0,\ Re(\nu) > 0,\ Re(\rho) > \max\{0,\ Re(\nu-\mu)\}$. *Then the following relation holds*

$$\left(\mathbf{I}_{0^+}^{(\nu)}\left[t^{\rho-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}(ct^{\sigma},ct^{\varsigma})\right]\right)(x)$$

$$= \frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-1}$$

$$\times S_{2:0;1}^{1:1;1}\left(\begin{array}{ll} [(\rho):\sigma,\varsigma]:[(\gamma):1];[(\kappa):1] & ; \\ [(\lambda,\rho+\nu):(\alpha,\sigma),(\beta,\xi)]:-;[(\xi:\eta)] & ; \\ cx^{\sigma},cx^{\varsigma} & \end{array}\right),$$

for all $x > 0$.

When $\lambda = \lambda' = 0$ in the Marichev-Saigo-Maeda operators in Theorem 2.3.4 , we obtain the right-sided Riemann Liouville operator. Therefore, setting $\lambda = \lambda' = 0$, Theorem 2.3.4 reduces to the following Corollary:

Corollary 2.3.8 *Let the parameters* $\nu,\ \lambda,\ \gamma,\ \kappa,\ \alpha,\ \beta,\ \eta,\ \xi,\ \lambda,\ \rho\ \in\ \mathbb{C}$ *and* $Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\eta) > 0,\ Re(\lambda) > 0,\ Re(\xi) > 0,\ Re(\sigma) > 0,\ Re(\varsigma) > 0,\ Re(\nu) > 0,\ Re(\rho) < 1 + \min\{Re(-\nu)\}$. *Then the following relation holds*

$$
\begin{aligned}
&\left(\mathbf{I}_{0^-}^{(\nu)}\left[t^{\rho-1}\mathfrak{E}_{\gamma,\kappa}^{(\alpha,\beta,\eta,\xi,\lambda)}\left(\frac{c}{t^{\sigma}},\frac{c}{t^{\varsigma}}\right)\right]\right)(x)\\
=\;&\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\kappa)}x^{\rho+\nu-1}\\
&\times S_{2:0;1}^{1:1;1}\left(\begin{array}{c}[(1-\rho-\nu):\sigma,\varsigma]:[(\gamma):1];[(\kappa):1] \quad ;\\ \\ [(\lambda,1-\rho):(\alpha,\sigma),(\beta,\xi)]:-;[(\xi:\eta)] \quad ;\\ \\ cx^{\sigma},cx^{\varsigma}\end{array}\right),
\end{aligned}
$$

for all $x>0$.

Chapter 3

SOME RESULTS ON 2D-LAGUERRE POLYNOMIALS

$\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$

In this chapter, we take into account the class of the 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Then, we get linear and mixed multilateral generating functions for the polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Furthermore, a finite summation formula for the mentioned classes is derived. Finally, a series relation between the 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ and a product of confluent hypergeometric functions is represented by using two-dimensional fractional derivative operator.

3.1 Linear Generating Function and a Summation Formula

The main idea of this section is to obtain a linear, mixed multilinear generating functions and a summation formula of 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$.

Theorem 3.1.1 *The following generating function*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}\,t^n\,k^m \\ =\;& e^{t_1+t_2}\,\Psi^*\left(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t,-y^{\beta}k\right). \end{aligned} \tag{3.1.1}$$

holds true for the polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$.

Proof. Direct calculations yield that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\;n!\;m!}\,t^n\,k^m$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{n,m}\frac{(-n)_{k_1}\;(-m)_{k_2}\,x^{\alpha k_1}\;y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\;\Gamma(\xi+\eta k_2)\;n!\;m!\;k_1!\;k_2!}\,t^n\,k^m$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{n,m}\frac{(-1)^{k_1+k_2}\,x^{\alpha k_1}\;y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\;\Gamma(\xi+\eta k_2)\;(n-k_1)!\;(m-k_2)!\;k_1!\;k_2!}\,t^n\,k^m.$$

Letting $n=n+k_1$ and $m=m+k_2$, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\;n!\;m!}\,t^n\,k^m$$
$$=\sum_{n,m=0}^{\infty}\frac{t^n\,k^m}{n!\;m!}\sum_{k_1,k_2=0}^{\infty}\frac{(-1)^{k_1+k_2}\,x^{\alpha k_1}\;y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\;\Gamma(\xi+\eta k_2)\;k_1!\;k_2!}\,t^{k_1}\,k^{k_2}$$
$$=e^{t+k}\,\Psi^*\left(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t,-y^{\beta}k\right).$$

Thus, we get the desired result. ■

In the following Theorem, by using the same technique which is considered in [14] and [15] (see also [31]), we obtain the mixed multilateral generating functions for the polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Let $(\gamma):=(\gamma_1,\gamma_2),\ (\lambda):=(\lambda_1,\lambda_2),\ (\eta):=(\eta_1,\eta_2),\ (\psi):=(\psi_1,\psi_2),\ (\rho):=(\rho_1,\rho_2)$ be complex $2-$ tuples. Considering the above Theorem, the following result holds true.

Theorem 3.1.2 *Let $\Omega_{(\eta)}(\xi_1,\xi_2,...,\xi_s)$ be an identically non-vanishing function of complex variables $\xi_1,\xi_2,...,\xi_s$ $(s\in\mathbb{N})$, and let*

$$\Lambda_{(\eta),(\psi)}(\xi_1,\xi_2,...,\xi_s;\varsigma_1,\varsigma_2) \tag{3.1.2}$$
$$:=\sum_{k_1,k_2=0}^{\infty}a_{k_1,k_2}\,\Omega_{\eta_1+\psi_1k_1,\eta_2+\psi_2k_2}(\xi_1,\xi_2,...,\xi_s)\;\varsigma_1^{k_1}\;\varsigma_2^{k_2},\ (a_{k_1,k_2}\neq 0).$$

Suppose also that

$$\Theta_{n_1,n_2;q_1,q_2}^{(\gamma),(\lambda),(\eta),(\psi),\alpha}(\xi_1,\xi_2,...,\xi_s;x_1,x_2;(\alpha,\beta,\xi);\varsigma_1,\varsigma_2) \tag{3.1.3}$$
$$= \sum_{k_1,k_2=0}^{\left[\frac{n}{q_1}\right],\left[\frac{m}{q_2}\right]} a_{k_1,k_2}\,\Omega_{n+\psi_1 k_1,m+\psi_2 k_2}(\xi_1,\xi_2,...,\xi_s)$$
$$\times \frac{\mathfrak{L}_{n-q_1k_1,m-q_2k_2}^{(\alpha,\beta,\gamma,\xi,\eta)}(x,y)\,\Gamma(\eta(m-q_2k_2)+\xi)}{\Gamma(\alpha(n-q_1k_1)+\beta(m-q_2k_2)+\gamma+1)(n-q_1k_1)!(m-q_2k_2)!}\,\varsigma_1^{k_1}\,\varsigma_2^{k_2}.$$
$$(q_1,\ q_2\ \in\ \mathbb{N})$$

Then,

$$\sum_{n_1,\cdots,n_j=0}^{\infty}\Theta_{n,m;q_1,q_2}^{(\gamma),(\lambda),(\eta),(\psi),\alpha}\left(\xi_1,\xi_2,...,\xi_s;x_1,x_2;(\alpha,\beta,\xi);\frac{\varsigma_1}{t_1^{q_1}},\frac{\varsigma_2}{t_2^{q_2}}\right)t^{\mathfrak{g}}t^{\mathfrak{h}} \tag{3.1.4}$$
$$= e^{t_1+t_2}\,\Lambda_{(\eta),(\psi)}(\xi_1,\xi_2,...,\xi_s;\varsigma_1\varsigma_2)\,\Psi^*\left(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t;-y^{\beta}k\right),$$

which provided that each member of equation (3.1.4) exists and $|t|<1$ *and* $|k|<1$.

Proof. Let say $\mathscr{F}$ for the left side of (3.1.4). Then, we substitute the polynomials

$$\Theta_{n_1,n_2;q_1,q_2}^{(\gamma),(\lambda),(\eta),(\psi),\alpha}(\xi_1,\xi_2,...,\xi_s;x_1,x_2;(\alpha,\beta,\xi);\varsigma_1,\varsigma_2)$$

from the definition (3.1.3) into the left-hand side of (3.1.4), and we get

$$\mathscr{F} = \sum_{n_1,n_2=0}^{\infty}\sum_{k_1,k_2=0}^{\left[\frac{n}{q_1}\right],\left[\frac{m}{q_2}\right]} a_{k_1,k_2}\,\Omega_{n+\psi_1k_1,m+\psi_2k_2}(\xi_1,\xi_2,\ldots,\xi_s)\,\varsigma_1^{k_1}\,\varsigma_2^{k_2}$$
$$\times \frac{\mathfrak{L}_{n-q_1k_1,m-q_2k_2}^{(\alpha,\beta,\gamma,\xi,\eta)}(x,y)\,\Gamma(\eta(m-q_2k_2)+\xi)}{\Gamma(\alpha(n-q_1k_1)+\beta(m-q_2k_2)+\gamma+1)(n-q_1k_1)!\,(m-q_2k_2)!}\,t^{n_1-q_1k_1}k^{n_2-q_2k_2}$$
$$= \sum_{k_1,\cdots,k_j=0}^{\infty} a_{k_1,k_2}n_2\Omega_{n+\psi_1k_1,m+\psi_2k_2}(\xi_1,\xi_2,\ldots,\xi_s)n_2\varsigma_1^{k_1}n_2\varsigma_2^{k_2}$$
$$\times \sum_{n_1,\cdots,n_j=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\xi,\eta)}(x,y)\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}\,t^n\,k^m.$$

Using Theorem 3.1.1 with $\gamma_1\to\gamma_1+\lambda_1k_1$ and $\gamma_2\to\gamma_2+\lambda_2k_2$, we get

$$\mathscr{F} = a_{k_1,k_2}\ \Omega_{n+\psi_1 k_1, m+\psi_2 k_2}(\xi_1,\xi_2,...,\xi_s)\ \varsigma_1^{k_1}\ \varsigma_2^{k_2}\ e^{t+k} \times \Psi^*\left(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t,-y^{\beta}k\right).$$

Thus, we get the result by using (3.1.2). ■

The following theorem devote an interesting summation formula for the 2D-Laguerre polynomials $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ by using the above generating function which deals in Eq.(3.1.1) and a technique used by Srivastava ([24] and [25]).

Theorem 3.1.3 *We have*

$$\begin{aligned}\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y) &= \Gamma(\alpha n+\beta m+\gamma+1) \\ &\times \sum_{r,s=0}^{n,m}(nr)\,(ms)\,\frac{\mathfrak{L}_{n-r,m-s}^{(\alpha,\beta,\gamma,\eta,\xi)}(t,k)\ \Gamma(\xi+\eta m)}{\Gamma(\alpha(n-r)+\beta(m-s)+\gamma+1)} \\ &\times \left(\frac{x^{\alpha}}{t^{\alpha}}\right)^{n}\left(\frac{y^{\beta}}{k^{\beta}}\right)^{m}\left(\frac{t^{\alpha}}{x^{\alpha}}-1\right)^{r}\left(\frac{k^{\beta}}{y^{\beta}}-1\right)^{s}.\end{aligned} \tag{3.1.5}$$

Proof. Setting $t_1 = [-t^{\alpha}]z_1$ and $t_2 = [-k^{\beta}]z_2$ in (1.0.2), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\ \Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\ n!\ m!}\ ([-t^{\alpha}]z_1)^n\ ([-s^{\beta}]z_2)^m \\ &= e^{[-t^{\alpha}]z_1+[-k^{\beta}]z_2}\ \Psi^*\left(-:(\alpha,\beta,\gamma+1),(-,\eta,\xi);-x^{\alpha}[-t^{\alpha}]z_1;-y^{\beta}[-s^{\beta}]z_2\right).\end{aligned} \tag{3.1.6}$$

Interchanging x by t and y by s,we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(t,s)\ \Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\ n!\ m!}\ ([-x^{\alpha}]z_1)^n\ ([-y^{\beta}]z_2)^m \\ &= e^{[-x^{\alpha}]z_1+[-y^{\beta}]z_2}\ \Psi^*\left((\alpha,\beta,\gamma+1),(\eta,\xi);-x^{\alpha}t_1;-y^{\beta}t_2\right).\end{aligned} \tag{3.1.7}$$

Comparing (3.1.6) and (3.1.7), we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}(-t^{\alpha}z_1)^n\,(-s^{\beta}z_2)^m$$

$$= e^{-t^{\alpha}z_1-s^{\beta}z_2+x^{\alpha}z_1+y^{\beta}z_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(t,k)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}(-x^{\alpha}z_1)^n\,(-y^{\beta}z_2)^m$$

$$= \sum_{n,m=0}^{\infty}\sum_{r,s=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(t,s)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!\,r!\,s!}$$
$$\times(-x^{\alpha}z_1)^n\,(-y^{\beta}z_2)^m\,(-t^{\alpha}z_1+x^{\alpha}z_1)^r\,(-s^{\beta}z_2+y^{\beta}z_2)^s$$

$$= \sum_{n,m=0}^{\infty}\sum_{r,s=0}^{n,m}\left[\frac{\mathfrak{L}_{n-r,m-s}^{(\alpha,\beta,\gamma,\eta,\xi)}(t,s)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha(n-r)+\beta(m-s)+\gamma+1)\,(n-r)!\,(m-s)!\,r!\,s!}\right.$$
$$\left.\times(-x^{\alpha}z_1)^{n-r}\,(-y^{\beta}z_2)^{m-s}\,(-t^{\alpha}z_1+x^{\alpha}z_1)^r\,(-s^{\beta}z_2+y^{\beta}z_2)^s\right]$$

$$= \sum_{n,m=0}^{\infty}\sum_{r,s=0}^{n,m}\left[(nr)\,(ms)\,\frac{\mathfrak{L}_{n-r,m-s}^{(\alpha,\beta,\gamma,\eta,\xi)}(t,s)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha(n-r)+\beta(m-s)+\gamma+1)}\right.$$
$$\left.\times(-x^{\alpha}z_1)^{n-r}\,(-y^{\beta}z_2)^{m-s}\,(-t^{\alpha}z_1+x^{\alpha}z_1)^r\,(-s^{\beta}z_2+y^{\beta}z_2)^s\right].$$

Finally, on comparing the coefficients $z_1^n z_2^m$ on both sides, we reach (3.1.5). ■

3.2 A Series Relation for $\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$

We need the following proposition to get the main theorem of this section.

Proposition 3.2.1 *The following relation holds*

$$\mathbf{D}_x^{\lambda}(x^{\mu-1}) = \frac{d^{\lambda}}{dx^{\lambda}}(x^{\mu-1}) = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}x^{\mu-\lambda-1},\ for\ \lambda\neq\mu \tag{3.2.1}$$

where $\mu\in\mathbb{C}$.

Theorem 3.2.2 *The following relationship holds true between 2D-Laguerre polynomials and the confluent hypergeometric functions:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)\,(\lambda)_n\,(\omega)_m}{\Gamma(\alpha n+\beta m+\gamma+1)\,(\mu_1+1)_n\,(\mu_2+1)_m\,n!\,m!}$$
$$\times {}_1F_1(\mu_1-\lambda+1,n+\mu_1+1;t_1)\;{}_1F_1(\mu_2-\omega+1,m+\mu_2+1;t_2)\;t_1^n\,t_2^m$$
$$= e^{t_1+t_2}\,\frac{\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)}{\Gamma(\lambda_1)\,\Gamma(\lambda_2)}$$
$$\times {}_2\Psi_4^*\Big((1,\lambda),(1,\omega):(\alpha,\beta,\gamma+1),(\eta,\xi),(1,\mu_1+1),(1,\mu_2+1);-x^\alpha t_1;-y^\beta t_2\Big).$$

Proof. Let rewrite (3.1.1) in the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}\,t_1^n\,t_2^m\,e^{-t_1-t_2}$$
$$= \Psi^*\Big(-:(\alpha,\beta,\gamma+1),(\eta,\xi);-x^\alpha t_1;-y^\beta t_2\Big)$$

and expand the exponentional function to a series to have

$$\sum_{n,m=0}^{\infty}\sum_{r,k=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}\frac{(-t_1)^r}{r!}\frac{(-t_2)^k}{k!}t_1^n\,t_2^m$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x)^{\alpha n}\,(-y)^{\beta m}}{\Gamma(\alpha n+\beta m+\gamma+1)\,\Gamma(\eta m+\xi)\,n!\,m!}t_1^n\,t_2^m.$$

Now, we multiply both sides by $t_1^{\lambda-1}$ and $t_2^{\omega-1}$ to obtain

$$\sum_{n,m=0}^{\infty}\sum_{r,k=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}\frac{(-1)^r}{r!}\frac{(-1)^k}{k!}t_1^{n+\lambda+r-1}\;t_2^{m+\omega+k-1}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x^\alpha t_1)^n\,(-y^\beta t_2)^m\,t_1^{\lambda-1}\,t_2^{\omega-1}}{\Gamma(\alpha n+\beta m+\gamma+1)\,\Gamma(\eta m+\xi)\,n!\,m!}.$$

Applying the operator $D_{t_1}^{\lambda-\mu_1-1}$ and $D_{t_2}^{\omega-\mu_2-1}$, we get

$$\sum_{n,m=0}^{\infty}\sum_{r,k=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)\,n!\,m!}\frac{(-1)^r}{r!}\frac{(-1)^k}{k!}$$
$$\times \mathbf{D}_{t_1}^{\lambda-\mu_1-1}[t_1^{n+\lambda+r-1}]\;\mathbf{D}_{t_2}^{\omega-\mu_2-1}[t_2^{m+\omega+k-1}]$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x^\alpha t_1)^n\,(-y^\beta t_2)^m\,t_2^{\omega-1}}{\Gamma(\alpha n+\beta m+\gamma+1)\,\Gamma(\eta m+\xi)\,n!\,m!}$$
$$\times \mathbf{D}_{t_1}^{\lambda-\mu_1-1}[t_1^{\lambda-1}]\;\mathbf{D}_{t_2}^{\omega-\mu_2-1}[t_2^{\omega-1}].$$

By using (3.2.1), we obtain

$$\begin{aligned}
&\sum_{n,m=0}^{\infty}\sum_{r,k=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)\,\Gamma(n+\lambda+r)\,\Gamma(m+\omega+k)}{\Gamma(\alpha n+\beta m+\gamma+1)\,\Gamma(n+\mu_1+r+1)\,\Gamma(m+\mu_2+k+1)\,n!\,m!}\\
&\times\frac{(-1)^r}{r!}\frac{(-1)^k}{k!}t_1^{n+r}\,t_2^{m+k}\\
=\;&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x^{\alpha})^n\,(-y^{\beta})^m\,\Gamma(n+\lambda)\,\Gamma(m+\omega)}{\Gamma(\alpha n+\beta m+\gamma+1)\,\Gamma(\eta m+\xi)\,\Gamma(n+\mu_1+1)\,\Gamma(m+\mu_2+1)\,n!\,m!}\,t_1^n\,t_2^m\\
=\;&{}_2\Psi_4^*\Big((1,\lambda),(1,\omega):(\alpha,\beta,\gamma+1),(\eta,\xi),(1,\mu_1+1),(1,\mu_2+1);-x^{\alpha}t_1;-y^{\beta}t_2\Big).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{n,m=0}^{\infty}\sum_{r,k=0}^{\infty}\Bigg[\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)\,\Gamma(\lambda)\,\Gamma(\omega)}{\Gamma(\alpha n+\beta m+\gamma+1)\,\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)}\\
&\times\frac{(n+\lambda)_r\,(m+\omega)_k\,(\lambda)_n\,(\omega)_m}{(n+\mu_1+1)_r\,(\mu_1+1)_n\,(m+\mu_2+1)_k\,(\mu_2+1)_m}\frac{(-t_1)^r\,t_1^n\,(-t_2)^k\,t_2^m}{n!\,m!\,r!\,k!}\Bigg]\\
&={}_2\Psi_4^*\Big((1,\lambda),(1,\omega):(\alpha,\beta,\gamma+1),(\eta,\xi),(1,\mu_1+1),(1,\mu_2+1);-x^{\alpha}t_1,-y^{\beta}t_2\Big).
\end{aligned}\tag{3.2.2}$$

For the convenience, let the left hand side of (3.2.2) be $\mathscr{S}$, that is

$$\begin{aligned}
\mathscr{S}=\;&\frac{\Gamma(\lambda)\,\Gamma(\omega)}{\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)\,(\lambda)_n\,(\omega)_m}{\Gamma(\alpha n+\beta m+\gamma+1)\,(\mu_1+1)_n\,(\mu_2+1)_m\,n!\,m!}t_1^n\,t_2^m\\
&\times\sum_{r=0}^{\infty}\frac{(n+\lambda)_r}{(m+\mu_1+1)_r\,r!}(-t_1)^r\sum_{k=0}^{\infty}\frac{(m+\omega)_k}{(n+\mu_2+1)_k\,k!}(-t_2)^k.\\
=\;&\frac{\Gamma(\lambda)\,\Gamma(\omega)}{\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)\,(\lambda)_n\,(\omega)_m}{\Gamma(\alpha n+\beta m+\gamma+1)\,(\mu_1+1)_n\,(\mu_2+1)_m\,n!\,m!}\\
&\times\,{}_1F_1(n+\lambda,n+\mu_1+1;-t_1)\,{}_1F_1(m+\omega,m+\mu_2+1;-t_2).
\end{aligned}$$

Finally, since ${}_1F_1(a;b;z)=e^z\,{}_1F_1(b-a;b;-z)$, we get

$$\begin{aligned}
\mathscr{S}\;=\;&\frac{\Gamma(\lambda)\,\Gamma(\omega)}{\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)}e^{-t_1-t_2}\\
&\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{L}_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)(\lambda)_n\,(\omega)_m}{\Gamma(\alpha n+\beta m+\gamma+1)\,(\mu_1+1)_n\,(\mu_2+1)_m\,n!\,m!}t_1^n\,t_2^m\\
&\times{}_1F_1(\mu_1-\lambda+1,n+\mu_1+1;t_1)\,{}_1F_1(\mu_2-\omega+1,m+\mu_2+1;t_2).
\end{aligned}\tag{3.2.3}$$

Comparing (3.2.2) and (3.2.3), we get the desired result. ■

Chapter 4

SOME RESULTS ON BIVARIATE MITTAG-LEFFLER FUNCTIONS WITH 2D-LAGUERRE-KONHAUSER POLYNOMIALS

In this chapter, first of all, we calculate two-dimensional Riemann-Liouville fractional integral and derivative of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$. After that, we consider a convolution integral equation with 2D-Laguerre-Konhauser Polynomials in the kernel and we obtain its solution by introducing a new family of bivariate Mittag-Leffler functions. Moreover, two-dimensional fractional integral operator which deals with bivariate Mittag-Leffler functions in the kernel is introduced. Finally, considering 2D-Laguerre-Konhauser Polynomials and bivariate Mittag-Leffler functions, we obtain a double linear generating function, Schläfli's contour integral representations and integral representation.

4.1 Fractional Calculus Approach

This section, devote to obtain the Riemann-Liouville double fractional integrals and derivatives of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$.

Theorem 4.1.1 *For* $\alpha,\ \beta,\ \gamma,\ \kappa\ \in\ \mathbb{C}$, $Re(\alpha)$, $Re(\beta)$, $Re(\kappa)$, $Re(\gamma)$, $Re(\lambda)$, $Re(\mu)$, $Re(\omega_1)$, $Re(\omega_2) > 0$, *we have*

$$
\begin{aligned}
& {}_x\mathbf{I}^{\lambda}_{a^+}{}_y\mathbf{I}^{\mu}_{b^+}\left[(x-a)^{\alpha-1}(y-b)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \; & (x-a)^{\alpha+\lambda-1}(y-b)^{\beta+\mu-1}\mathfrak{E}^{(\gamma)}_{\alpha+\lambda,\beta+\mu,\kappa}(\omega_1(x-a),\omega_2(y-b)).
\end{aligned}
$$

Proof. Because of the hypothesis of the above Theorem, we have a right to interchange of the order of series and two-dimensional Riemann-Liouville fractional integral operator, which yields

$$
\begin{aligned}
& {}_x\mathbf{I}^{\lambda}_{a^+}{}_y\mathbf{I}^{\mu}_{b^+}\left[(x-a)^{\alpha-1}(y-b)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \; & \int_a^x\int_b^y \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\frac{(y-\tau)^{\mu-1}}{\Gamma(\mu)}(t-a)^{\alpha-1}(\tau-b)^{\beta-1} \\
& \times\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(t-a),\omega_2(\tau-b))d\tau\, dt \\
= \; & \frac{1}{\Gamma(\lambda)}\frac{1}{\Gamma(\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)}\frac{\omega_1^r}{r!}\frac{\omega_2^{\kappa s}}{s!} \\
& \times\int_a^x (x-t)^{\lambda-1}(t-a)^{\alpha+r-1}dt\int_b^y (y-\tau)^{\mu-1}(\tau-b)^{\beta+\kappa s-1}d\tau \\
= \; & (x-a)^{\alpha+\lambda-1}(y-b)^{\beta+\mu-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+\lambda+r)\Gamma(\beta+\mu+\kappa s)} \\
& \times\frac{\omega_1^r(x-a)^r}{r!}\frac{\omega_2^{\kappa s}(y-b)^{\kappa s}}{s!} \\
= \; & (x-a)^{\alpha+\lambda-1}(y-b)^{\beta+\mu-1}\mathfrak{E}^{(\gamma)}_{\alpha+\lambda,\beta+\mu,\kappa}(\omega_1(x-a),\omega_2(y-b)).
\end{aligned}
$$

Thus, we get the desired result. ∎

Corollary 4.1.2 *As a consequence of (1.0.16) and Theorem 4.1.1, we have*

$$
\begin{aligned}
& {}_x\mathbf{I}^{\lambda}_{a^+}{}_y\mathbf{I}^{\mu}_{b^+}\left[{}_{\kappa}L_n^{(\alpha,\beta)}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \; & (x-a)^{\alpha+\lambda}(y-b)^{\beta+\mu}\;{}_{\kappa}L_n^{(\alpha+\lambda,\beta+\mu)}(\omega_1(x-a),\omega_2(y-b)),
\end{aligned}
$$

where α, β, γ, $\kappa\in\mathbb{C}$, $Re(\alpha)$, $Re(\beta)$, $Re(\kappa)$, $Re(\gamma)$, $Re(\lambda)$, $Re(\mu)$, $Re(\omega_1)$, $Re(\omega_2)>0$.

Theorem 4.1.3 *For α, β, γ, $\kappa \in \mathbb{C}$, $Re(\alpha)$, $Re(\beta)$, $Re(\kappa)$, $Re(\gamma)$, $Re(\omega_1)$, $Re(\omega_2) > 0$ and $Re(\lambda)$, $Re(\mu) \geqslant 0$, we have*

$$
\begin{aligned}
& {}_x\mathbf{D}^{\lambda}_{a^+}{}_y\mathbf{D}^{\mu}_{b^+}\left[(x-a)^{\alpha-1}(y-b)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \;& (x-a)^{\alpha-\lambda-1}(y-b)^{\beta-\mu-1}\mathfrak{E}^{(\gamma)}_{\alpha-\lambda,\beta-\mu,\kappa}(\omega_1(x-a),\omega_2(y-b)).
\end{aligned}
$$

Proof. Because of the hypothesis of the above Theorem, we have a right to interchange of the order of series and two-dimensional Riemann-Liouville fractional derivative operator, which yields

$$
\begin{aligned}
& {}_x\mathbf{D}^{\lambda}_{a^+}{}_y\mathbf{D}^{\mu}_{b^+}\left[(x-a)^{\alpha-1}(y-b)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \;& {}_x\mathbf{D}^{n}\mathbf{I}^{n-\lambda}_{a^+}\;{}_y\mathbf{D}^{m}\mathbf{I}^{m-\mu}_{b^+}\left[(x-a)^{\alpha-1}(y-b)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \;& \frac{1}{\Gamma(n-\lambda)}\frac{1}{\Gamma(m-\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)}\frac{\omega_1^r(x-a)^r}{r!}\frac{\omega_2^{\kappa s}(y-b)^{\kappa s}}{s!} \\
& \times{}_xD^n\int_a^x(x-t)^{n-\lambda-1}(t-a)^{\alpha+r-1}dt\;{}_yD^m\int_b^x(y-\tau)^{m-\mu-1}(\tau-b)^{\beta+\kappa s-1}d\tau \\
= \;& (x-a)^{\alpha-\lambda-1}(y-b)^{\beta-\mu-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha-\lambda+r)\Gamma(\beta-\mu+\kappa s)} \\
& \times\frac{\omega_1^r(x-a)^r}{r!}\frac{\omega_2^{\kappa s}(y-b)^{\kappa s}}{s!} \\
= \;& (x-a)^{\alpha-\lambda-1}(y-b)^{\beta-\mu-1}\mathfrak{E}^{(\gamma)}_{\alpha-\lambda,\beta-\mu,\kappa}(\omega_1(x-a),\omega_2(y-b)).
\end{aligned}
$$

Whence the result.

■

Corollary 4.1.4 *As a consequence of (1.0.16) and Theorem 4.1.3, we have*

$$
\begin{aligned}
& {}_x\mathbf{D}^{\lambda}_{a^+}{}_y\mathbf{D}^{\mu}_{b^+}\left[{}_\kappa L_n^{(\alpha,\beta)}(\omega_1(x-a),\omega_2(y-b))\right] \\
= \;& (x-a)^{\alpha-\lambda}(y-b)^{\beta-\mu}\;{}_\kappa L_n^{(\alpha-\lambda,\beta-\mu)}(\omega_1(x-a),\omega_2(y-b)),
\end{aligned}
$$

where $\alpha,\ \beta,\ \gamma,\ \kappa\ \in\ \mathbb{C},\ Re(\alpha),\ Re(\beta),\ Re(\kappa),\ Re(\gamma),\ Re(\omega_1),\ Re(\omega_2)>0$ *and* $Re(\lambda),\ Re(\mu)\geqslant 0$.

4.2 Convolution Type Integral Equation with 2D-Laguerre-Konhauser Polynomials in the Kernel

In this section, two-dimensional Laplace transform of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ are investigated. After that, an integral involving the product of two $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ bivariate Mittag-Leffler functions suggested by the generalized Laguerre-Konhauser polynomials ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ in the kernel is calculated. Moreover, a convolution type integral equation in terms of ${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)$ is introduced in the kernel.

Theorem 4.2.1 *Let* $\alpha,\ \beta,\ \gamma,\ \kappa\in\mathbb{C},\ Re(\alpha),\ Re(\beta),\ Re(\kappa),\ Re(\gamma),\ Re(\omega_1),\ Re(\omega_2),\ Re(p),\ Re(q)>0,\ \left|\frac{\omega_2^{\kappa}}{q^{\kappa}}\right|<1$ *and* $\left|\frac{\omega_1 q^{\kappa}}{p(q^{\kappa}-\omega_2^{\kappa})}\right|<1$. *Then there holds*

$$\mathbb{L}_2\left(x^{\alpha-1}y^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1 x,\omega_2 y)\right)(p,q)=\frac{1}{p^{\alpha}}\frac{1}{q^{\beta}}\left(1-\frac{\omega_2^{\kappa}p+\omega_1 q^{\kappa}}{pq^{\kappa}}\right)^{-\gamma}.$$

Proof. Since the hypothesis of the above Theorem, we interchange the order of series and two-dimensional fractional integral, that is

$$\begin{aligned}
&\mathbb{L}_2\left(x^{\alpha-1}y^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1 x,\omega_2 y)\right)(p,q)\\
=&\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}\omega_1^r\omega_2^{\kappa s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)r!s!}\int_0^{\infty}e^{-px}x^{\alpha+r-1}dx\int_0^{\infty}e^{-qy}y^{\beta+\kappa s-1}dy\\
=&\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)r!s!}\frac{\omega_1^r}{p^{\alpha}}\frac{\omega_2^{\kappa s}}{q^{\beta}}\int_0^{\infty}e^{-u}u^{\alpha+r-1}du\int_0^{\infty}e^{-v}v^{\beta+\kappa s-1}dv\\
=&\frac{1}{p^{\alpha}}\frac{1}{q^{\beta}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{r!s!}\left(\frac{\omega_1}{p}\right)^r\left(\frac{\omega_2}{q}\right)^{\kappa s}\\
=&\frac{1}{p^{\alpha}}\frac{1}{q^{\beta}}\sum_{r=0}^{\infty}\frac{(\gamma)_r}{r!}\left(\frac{\omega_1}{p}\right)^r\sum_{s=0}^{\infty}\frac{(\gamma+r)_s}{s!}\left(\frac{\omega_2}{q}\right)^{\kappa s}=\frac{1}{p^{\alpha}}\frac{1}{q^{\beta}}\left(1-\frac{\omega_2^{\kappa}p+\omega_1 q^{\kappa}}{pq^{\kappa}}\right)^{-\gamma}.
\end{aligned}$$

Whence the result. ■

In a similar way, we have the following Corollary:

Corollary 4.2.2 *For the polynomials ${}_\kappa L_n^{(\alpha,\beta)}(x,y)$, we have*

$$\mathbb{L}_2\left({}_\kappa L_n^{(\alpha,\beta)}(\omega_1 x,\omega_2 y)\right)(p,q)=\frac{1}{p^{\alpha+1}}\frac{1}{q^{\beta+1}}\left(\frac{pq^\kappa-(\omega_2^\kappa p+\omega_1 q^\kappa)}{pq^\kappa}\right)^n.$$

Note that two-dimensional fractional integral $({}_xI^\mu_{0^+}{}_yI^\lambda_{0^+}\varphi)(x,y)$ can be written as a convolution of the form

$$\begin{aligned}({}_x\mathbf{I}^\mu_{0^+}{}_y\mathbf{I}^\lambda_{0^+}\varphi)(x,y) &= \left[\varphi(x,y)*\frac{x_t^{\mu-1}y_\tau^{\lambda-1}}{\Gamma(\mu)\Gamma(\lambda)}\right].\\ (Re(\mu),Re(\lambda) &> 0)\end{aligned}$$

Therefore, using the double convolution theorem for two-dimensional Laplace transform of two-dimetional fractional integral ${}_xI^\mu_{0^+}{}_yI^\lambda_{0^+}\varphi$, we reach the following result

$$\mathbb{L}_2\left({}_x\mathbf{I}^\mu_{0^+}{}_y\mathbf{I}^\lambda_{0^+}\varphi\right)(p,q)=p^{-\mu}q^{-\lambda}\mathbb{L}_2(\varphi)(p,q),$$

which is also true for sufficiently good function φ if $Re(\mu)$, $Re(\lambda)<0$.

Now, we consider the following double convolution equation:

$$\int_0^x\int_0^y {}_\kappa L_n^{(\alpha,\beta)}(\omega_1(x-t),\omega_2(y-\tau))\Phi(t,\tau)d\tau\,dt=\Psi(x,y). \tag{4.2.1}$$

For the solution of the integral equation (4.2.1), we have the following Theorem:

Theorem 4.2.3 *The singular double integral Eq. (4.2.1) gives the solution as*

$$\begin{aligned}\Phi(x,y) &= \int_0^x\int_0^y (x-t)^{\mu-\alpha-2}(y-\tau)^{\lambda-\beta-2}\\ &\quad\times\mathfrak{E}^{(n)}_{\mu-\alpha-1,\lambda-\beta-1,\kappa}((x-t),(y-\tau))\,{}_x\mathbf{I}^{-\mu}_{0^+}\,{}_y\mathbf{I}^{-\lambda}_{0^+}\Psi(t,\tau)d\tau\,dt,\end{aligned}$$

which provided that ${}_x\mathbf{I}^{-\mu}_{0^+}\,{}_y\mathbf{I}^{-\lambda}_{0^+}\Psi$ exists for $Re(\mu)>Re(\alpha)$ and $Re(\lambda)>Re(\beta)$ and is *locally integrable for* $0 < x < \delta_1 < \infty$ *and* $0 < y < \delta_2 < \infty$*, respectively.*

Proof. Applying the two-dimensional Laplace transform on both sides of (4.2.1) and using the convolution theorem and Corollary 4.2.2, then we get

$$\frac{1}{p^{\alpha+1}}\frac{1}{q^{\beta+1}}\left(\frac{pq^{\kappa}-(\omega_2^{\kappa}p+\omega_1 q^{\kappa})}{pq^{\kappa}}\right)^n \mathbb{L}_2\left[\Phi(t,\tau)\right](p,q) = \mathbb{L}_2\left[\Psi(t,\tau)\right](p,q),$$

which gives

$$\mathbb{L}_2\left[\Phi(t,\tau)\right](p,q) = p^{\alpha-\mu+1}q^{\beta-\lambda+1}\left(\frac{pq^{\kappa}-(\omega_2^{\kappa}p+\omega_1 q^{\kappa})}{pq^{\kappa}}\right)^{-n}\left\{p^{\mu}q^{\lambda}\Psi(t,\tau)(p,q)\right\}. \tag{4.2.2}$$

Taking inverse double Laplace transform of (4.2.2), we get the desired result. ■

4.3 An Integral Operator Involving $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ in the Kernel

Let us consider the following double (fractional) integral operator:

$$\left(\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) = \int_c^y\int_a^x (x-t)^{\alpha-1}(y-\tau)^{\beta-1} \times\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}[\omega_1(x-t),\omega_2(y-\tau)]\varphi(t,\tau)dt\,d\tau. \tag{4.3.1}$$

$$(x > a,\ y > c)$$

When $\gamma = 0$, the integral operator $\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}$ coincides with the left-sided two-dimensional Riemann-Liouville fractional integral defined in (4.4.3), such that

$$\left(\varepsilon^{(0)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) = \left({}_y\mathbf{I}^{\beta}_{c^+}\,{}_x\mathbf{I}^{\alpha}_{a^+}\varphi\right)(x,y). \tag{4.3.2}$$

The transformation properties of $\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}$ in the space $L((a,b)\times(c,d))$ of Lebesgue measurable functions are given as

$$L((a,b)\times(c,d)) = \left\{f : \| f \|_1 := \int_a^b\int_c^d |f(x,y)|\,dy\,dx < \infty\right\}.$$

The following Theorem proves that $\varepsilon^{(\gamma)}_{\alpha,\beta;\omega_1,\omega_2;a^+,c^+}$ is bounded on the space L(a,b).

Theorem 4.3.1 *Let $\alpha,\beta,\gamma,\kappa,\omega_1,\omega_2 \in \mathbb{C}$ with $Re(\kappa) > 0$. The double integral operator $\varepsilon^{(\gamma)}_{(\alpha,\beta;\omega_1,\omega_2;a^+,c^+)}$ is bounded in the space $L((a,b)\times(c,d))$, i.e.*

$$\left\|\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right\|_1 \leq A\,\|\varphi\|_1\,,$$

where the constant A $(0 < A < \infty)$ is given by

$$\begin{aligned} A &= (b-a)^{Re(\alpha)}(d-c)^{Re(\beta)} \\ &\quad \times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{|(\gamma)_{r+s}|}{\{Re(\alpha)+r\}\,|\Gamma(\alpha+r)|\,\{Re(\beta)+\kappa s\}\,|\Gamma(\beta+\kappa s)|} \\ &\quad \times \frac{|\omega_1(b-a)|^r}{r!}\frac{|\omega_2(d-c)|^{\kappa s}}{s!} \\ &< \infty. \end{aligned} \tag{4.3.3}$$

Proof. By using the Fubini's Theorem, we get

$$\begin{aligned} &\left\|\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right\|_1 \\ \leq\;& \int_a^b\int_c^d |\varphi(t,\tau)| \\ &\times\left(\int_t^b\int_\tau^d (x-t)^{Re(\alpha)-1}(y-\tau)^{Re(\beta)-1}\left|\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))\right| dy\,dx\right) d\tau\,dt \\ =\;& \int_a^b\int_c^d |\varphi(t,\tau)|\left(\int_0^{b-t}\int_0^{d-\tau} u^{Re(\alpha)-1}v^{Re(\beta)-1}\left|\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1 u,\omega_2 v)\right| du\,dv\right) d\tau\,dt \\ \leq\;& \int_a^b\int_c^d |\varphi(t,\tau)|\left(\int_0^{b-a}\int_0^{d-c} u^{Re(\alpha)-1}v^{Re(\beta)-1}\left|\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1 u,\omega_2 v)\right| du\,dv\right) d\tau\,dt \\ \leq\;& \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{|(\gamma)_{r+s}|}{|\Gamma(\alpha+r)|\,|\Gamma(\beta+\kappa s)|}\frac{|\omega_1|^r}{r!}\frac{|\omega_2|^{\kappa s}}{s!} \\ &\times\int_0^{b-a}u^{Re(\alpha)+r-1}du\int_0^{d-c}v^{Re(\beta)+\kappa s-1}dv\,\|\varphi\|_1 \\ =\;& A\,\|\varphi\|_1\,. \end{aligned}$$

Hence, we get the desired result. ■

Remark 4.3.2 *The constant A is finite, because the series* $\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)}\frac{x^r}{r!}\frac{y^{\kappa s}}{s!}$

is absolutely convergent for all x and y and since Re$(\kappa) > 0$ *(see [27]).*

Now, let us show that the integral operator $\varepsilon^{(\gamma)}_{\alpha,\beta;\omega_1,\omega_2;a^+,c^+}$ is bounded in the space $C([a,b]\times[c,d])$ of continuous functions on $[a,b]\times[c,d]$ with a max norm, i.e.

$$\|h\|_C = \max_{\substack{a\le x\le b\\ c\le y\le d}} |h(x,y)|. \tag{4.3.4}$$

Theorem 4.3.3 *Let* $\alpha,\beta,\gamma,\kappa,\omega_1,\omega_2 \in \mathbb{C}$. *The double integral operator* $\varepsilon^{(\gamma)}_{(\alpha,\beta;\omega_1,\omega_2;a^+,c^+)}$ *is bounded in the space* $C([a,b]\times[c,d])$, *i.e.*

$$\left\|\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right\|_C \le A\,\|\varphi\|_C,$$

where A is given by (4.3.3).

Proof. From (4.3.1) and (4.3.4), for any $x\in[a,b]$, $y\in[c,d]$ and $\varphi\in C([a,b]\times[c,d])$, we get

$$\begin{aligned}
&\left|\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right| \\
=\;& \left|\int_c^y\int_a^x (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}[\omega_1(x-t),\omega_2(y-\tau)]\varphi(t,\tau)dt\,d\tau\right| \\
\le\;& \int_c^y\int_a^x \left|(x-t)^{\alpha-1}(y-\tau)^{\beta-1}\right|\left|\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}[\omega_1(x-t),\omega_2(y-\tau)]\right||\varphi(t,\tau)|\,dt\,d\tau \\
\le\;& \|\varphi\|_C\int_c^y\int_a^x (x-t)^{Re(\alpha)-1}(y-\tau)^{Re(\beta)-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}[\omega_1(x-t),\omega_2(y-\tau)]dt\,d\tau \\
=\;& \|\varphi\|_C\int_0^{y-c}\int_0^{x-a} u^{Re(\alpha)-1}v^{Re(\beta)-1}\left|\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}[\omega_1 u,\omega_2 v]\right|du\,dv \\
\le\;& \|\varphi\|_C\int_0^{d-c}\int_0^{b-a} u^{Re(\alpha)-1}v^{Re(\beta)-1}\left|\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}[\omega_1 u,\omega_2 v]\right|du\,dv \\
=\;& A\,\|\varphi\|_C,
\end{aligned}$$

where A is given in (4.3.3). ■

In the following Theorem, by using Theorem 4.2.1, we obtain two-dimensional integral involving the product of bivariate Mittag-Leffler functions $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ in the integrand.

Theorem 4.3.4 *Let* $\alpha,\ \beta,\ \kappa,\ \zeta,\ \sigma,\ \gamma,\ \eta\ \in\ \mathbb{C}$, $Re(\alpha)$, $Re(\beta)$, $Re(\kappa)$, $Re(\gamma)$, $Re(\sigma)$, $Re(\omega_1)$, $Re(\omega_2) > 0$. *Then*

$$\begin{aligned}
&\int_0^x\int_0^y (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))t^{\zeta-1}\tau^{\sigma-1}\\
&\times\mathfrak{E}^{(\eta)}_{\zeta,\sigma,\kappa}(\omega_1 t,\omega_2\tau)d\tau\, dt\\
=\ & x^{\alpha+\zeta-1}y^{\beta+\sigma-1}\mathfrak{E}^{(\gamma+\eta)}_{\alpha+\zeta,\beta+\sigma,\kappa}(\omega_1 x,\omega_2 y).
\end{aligned}$$

Proof. With the help of the double convolution theorem for two-dimensional Laplace transform, we get

$$\begin{aligned}
&\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))\right.\\
&\left.\times\, t^{\zeta-1}\tau^{\sigma-1}\mathfrak{E}^{(\eta)}_{\zeta,\sigma,\kappa}(\omega_1 t,\omega_2\tau)d\tau\, dt\right\}(p,q)\\
=\ &\mathbb{L}_2\left\{x^{\alpha-1}y^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1 x,\omega_2 y)\right\}(p,q)\mathbb{L}_2\left\{x^{\zeta-1}y^{\sigma-1}\mathfrak{E}^{(\eta)}_{\zeta,\sigma,\kappa}(\omega_1 x,\omega_2 y)\right\}(p,q).
\end{aligned}$$

By the Theorem 4.2.1, for $Re(p),\ Re(q) > 0$ and $\left|\frac{\omega_2^\kappa}{q^\kappa}\right| < 1$ and $\left|\frac{\omega_1 q^\kappa}{p\left(q^\kappa-\omega_2^\kappa\right)}\right| < 1$, we have

$$\begin{aligned}
&\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))\right.\\
&\left.\times\, t^{\zeta-1}\tau^{\sigma-1}\mathfrak{E}^{(\eta)}_{\zeta,\sigma,\kappa}(\omega_1 t,\omega_2\tau)d\tau\, dt\right\}(p,q)\\
=\ &\frac{1}{p^\alpha}\frac{1}{q^\beta}\left(1-\frac{\omega_2^\kappa p+\omega_1 q^\kappa}{pq^\kappa}\right)^{-\gamma}\frac{1}{p^\zeta}\frac{1}{q^\sigma}\left(1-\frac{\omega_2^\kappa p+\omega_1 q^\kappa}{pq^\kappa}\right)^{-\eta}\\
=\ &\frac{1}{p^{\alpha+\zeta}}\frac{1}{q^{\beta+\sigma}}\left(1-\frac{\omega_2^\kappa p+\omega_1 q^\kappa}{pq^\kappa}\right)^{-(\gamma+\eta)}\\
=\ &\mathbb{L}_2\left\{x^{\alpha+\zeta-1}y^{\beta+\sigma-1}\mathfrak{E}^{(\gamma+\eta)}_{\alpha+\zeta,\beta+\sigma,\kappa}(\omega_1 x,\omega_2 y)\right\}(p,q). \qquad (4.3.5)
\end{aligned}$$

Taking inverse Laplace on both sides of (4.3.5) which yields

$$\int_0^x \int_0^y (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))t^{\zeta-1}\tau^{\sigma-1}$$

$$\times\mathfrak{E}^{(\eta)}_{\zeta,\sigma,\kappa}(\omega_1 t,\omega_2\tau)d\tau\, dt$$

$$= x^{\alpha+\zeta-1}y^{\beta+\sigma-1}\mathfrak{E}^{(\gamma+\eta)}_{\alpha+\zeta,\beta+\sigma,\kappa}(\omega_1 x,\omega_2 y).$$

The proof is completed. ■

The following Theorem gives us the composition of two operators of (4.3.1) with different indices:

Theorem 4.3.5 *Let* $\alpha,\ \beta,\ \kappa,\ \gamma,\ \zeta,\ \eta,\ \sigma,\ \omega_1,\ \omega_2 \in \mathbb{C}$ *and* $Re(\gamma), Re(\alpha), Re(\beta), Re(\sigma)$, $Re(\zeta),\ Re(\eta), Re(\kappa) > 0$. *Then the relation*

$$\left(\varepsilon^{(\gamma)}_{\alpha,\beta;\omega_1,\omega_2;a^+,c^+}\varepsilon^{(\sigma)}_{\zeta,\eta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) = \left(\varepsilon^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) \quad (4.3.6)$$

is valid for any summable function $\varphi \in L((a,b)\times(c,d))$. *Particularly,*

$$\left(\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varepsilon^{(-\gamma)}_{\zeta,\eta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) = \left({}_yI^{\beta+\eta}_{c^+}\ {}_xI^{\alpha+\zeta}_{a^+}\varphi\right)(x,y). \quad (4.3.7)$$

Proof. By using (4.3.1), we get

$$
\begin{aligned}
&\left(\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varepsilon^{(\sigma)}_{\zeta,\eta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)\\
= &\int_c^y\int_a^x (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))\\
&\times\mathfrak{E}^{(\sigma)}_{(\zeta,\eta;\omega_1,\omega_2;0)}\varphi(t,\tau)dt\,d\tau\\
= &\int_c^y\int_a^x\int_c^\tau\int_a^t (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))\\
&\times(t-u)^{\zeta-1}(\tau-v)^{\eta-1}\mathfrak{E}^{(\sigma)}_{\zeta,\eta,\kappa}(\omega_1(t-u),\omega_2(\tau-v))\varphi(u,v)du\,dv\,dt\,d\tau\\
= &\int_c^y\int_a^x\int_v^y\int_u^x (x-t)^{\alpha-1}(y-\tau)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-t),\omega_2(y-\tau))\\
&\times(t-u)^{\zeta-1}(\tau-v)^{\eta-1}\mathfrak{E}^{(\sigma)}_{\zeta,\eta,\kappa}(\omega_1(t-u),\omega_2(\tau-v))\varphi(u,v)dt\,d\tau\,du\,dv\\
= &\int_c^y\int_a^x\int_0^{y-v}\int_0^{x-u} (x-k-u)^{\alpha-1}(y-l-v)^{\beta-1}\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(\omega_1(x-k-u),\omega_2(y-l-v))\\
&\times k^{\zeta-1}l^{\eta-1}\mathfrak{E}^{(\sigma)}_{\zeta,\eta,\kappa}(\omega_1k,\omega_2l)\varphi(u,v)dk\,dl\,du\,dv.
\end{aligned}
$$

By the Theorem 4.3.4, we get

$$
\begin{aligned}
&\left(\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varepsilon^{(\sigma)}_{\zeta,\eta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)\\
= &\int_c^y\int_a^x (x-u)^{\alpha+\zeta-1}(y-v)^{\beta+\eta-1}E^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta,\kappa}(\omega_1(x-u),\omega_2(y-v))\varphi(u,v)du\,dv\\
= &\left(\varepsilon^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y).
\end{aligned}
$$

Whence the result. ■

Note that, in the case $\sigma=-\gamma$, (4.3.6) coincides with (4.3.7) in accordance with (4.3.2).

Corollary 4.3.6 *For* $\alpha,\ \beta,\ \zeta,\ \eta,\ \omega_1,\ \omega_2\in\mathbb{C}$ *and* $Re(\alpha)$, $Re(\beta)$, $Re(\zeta)$, $Re(\eta)$, $Re(\kappa)>0$, *the following relation holds true on* $L((a,b)\times(c,d))$

$$\varepsilon_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varepsilon_{\zeta,\eta,\kappa;\omega_1,\omega_2;a^+,c^+}=\varepsilon^2_{\alpha+\zeta,\beta+\eta,\kappa;\omega_1,\omega_2;a^+,c^+}.$$

When $\gamma=0$, (4.3.6) reduces to the following Corollary:

Corollary 4.3.7 *For* $\alpha,\ \beta,\ \kappa,\ \zeta,\ \eta,\ \sigma,\ \omega_1,\ \omega_2\in\mathbb{C}$ *and* $Re(\alpha)$, $Re(\beta)$, $Re(\zeta)$, $Re(\eta)$, $Re(\gamma)$, $Re(\kappa)>0$. *Then the following relation holds true on* $L((a,b)\times(c,d))$

$$\varepsilon^{(0)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varepsilon^{(\sigma)}_{\zeta,\eta,\kappa;\omega_1,\omega_2;a^+,c^+}={}_y\mathbf{I}^{\beta}_{c^+x}\mathbf{I}^{\alpha}_{a^+}\varepsilon^{(\sigma)}_{\zeta,\eta,\kappa;\omega_1,\omega_2;a^+,c^+}=\varepsilon^{(\sigma)}_{\alpha+\zeta,\beta+\eta,\kappa;\omega_1,\omega_2;a^+,c^+}.$$

Corollary 4.3.8 *For* $\alpha,\ \beta,\ \kappa,\ \gamma,\ \eta,\ \omega_1,\ \omega_2\in\mathbb{C}$ *and* $Re(\gamma)$, $Re(\alpha)$, $Re(\beta)$, $Re(\mu)$, $Re(\lambda)$, $Re(\kappa)>0$. *Then we have the following composition relationships:*

$$\begin{aligned}\left({}_y\mathbf{I}^{\lambda}_{c^+}\ {}_x\mathbf{I}^{\mu}_{a^+}\varepsilon^{(\gamma)}_{\alpha,\beta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) &= \left(\varepsilon^{(\gamma)}_{\alpha+\mu,\beta+\lambda,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)\\ &= \left(\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+y}I^{\lambda}_{c^+}\ {}_xI^{\mu}_{a^+}\right)(x,y)\end{aligned}$$

and

$$\begin{aligned}\left({}_y\mathbf{D}^{\lambda}_{c^+}\ {}_x\mathbf{D}^{\mu}_{a^+}\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) &= \left(\varepsilon^{(\gamma)}_{\alpha-\mu,\beta-\lambda,\kappa;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)\\ &= \left(\varepsilon^{(\gamma)}_{\alpha,\beta,\kappa;\omega_1,\omega_2;a^+,c^+y}D^{\lambda}_{c^+}\ {}_xD^{\mu}_{a^+}\varphi\right)(x,y),\end{aligned}$$

where $\varphi(x,y)$ *is in* $L((a,b)\times(c,d))$.

Corollary 4.3.9 *If* $\alpha,\ \beta,\ \kappa,\ \zeta,\ \eta,\ \sigma,\ \omega_1,\ \omega_2\in\mathbb{C}$ *and* $Re(\alpha)$, $Re(\beta)$, $Re(\zeta)$, $Re(\eta)$, $Re(\gamma)$, $Re(\kappa)>0$, *then there holds the relation on* $L((a,b)\times(c,d))$ *as follows*

$${}_y\mathbf{I}^{\lambda}_{c^+}\ {}_x\mathbf{I}^{\mu}_{a^+y}\mathbf{I}^{\beta}_{c^+}\ {}_x\mathbf{I}^{\alpha}_{a^+}\varphi={}_y\mathbf{I}^{\lambda+\beta}_{c^+x}\mathbf{I}^{\mu+\alpha}_{a^+}\varphi.$$

4.4 Miscelenenous Properties of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_\kappa L^{(\alpha,\beta)}_n(x,y)$

This section which provides to get a double linear generating function for the polynomials ${}_\kappa L^{(\alpha,\beta)}_n(x,y)$ suggested by $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$. Then, Schläfli's contour integral representations of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_\kappa L^{(\alpha,\beta)}_n(x,y)$ are given. Finally, the integral representations of $\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$ and ${}_\kappa L^{(\alpha,\beta)}_n(x,y)$ are obtained.

Theorem 4.4.1 *For $|t| < 1$, α, β, $\sigma \in \mathbb{C}$, we have*

$$\sum_{n=0}^{\infty} \frac{(\sigma)_n}{n!} \ {}_{\kappa}L_n^{(\alpha,\beta)}(x,y)t^n = x^{\alpha} y^{\beta} (1-t)^{-\sigma} \mathfrak{E}^{(\sigma)}_{\alpha+1,\beta+1,\kappa} \left(\frac{xt}{t-1}, y\left(\frac{t}{t-1}\right)^{\frac{1}{\kappa}} \right).$$

Proof. Using the Cauchy product of the series, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(\sigma)_n}{n!} \ {}_{\kappa}L_n^{(\alpha,\beta)}(x,y)t^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{r=0}^{n-r} \frac{(\sigma)_n(-1)^{s+r}x^{\alpha+r}y^{\beta+\kappa s}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(\beta+\kappa s+1)} t^n \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{n} \frac{(\sigma)_{n+s}(-1)^{s+r}x^{\alpha+r}y^{\beta+\kappa s}}{s!r!(n-r)!\Gamma(\alpha+r+1)\Gamma(\beta+\kappa s+1)} t^{n+s} \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\sigma)_{n+r+s}(-1)^{s+r}x^{\alpha+r}y^{\beta+\kappa s}}{s!r!\Gamma(\alpha+r+1)\Gamma(\beta+\kappa s+1)} \frac{t^{n+r+s}}{n!}.
\end{aligned}$$

Since $(\sigma)_{n+r+s} = (\sigma)_{r+s}(\sigma+r+s)_n$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\sigma)_n}{n!} \ {}_{\kappa}L_n^{(\alpha,\beta)}(x,y)t^n \\
&= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{s+r}(\sigma)_{r+s}x^{\alpha+r}y^{\beta+\kappa s}t^{s+r}}{s!r!\Gamma(\alpha+r+1)\Gamma(\beta+\kappa s+1)} \sum_{n=0}^{\infty} (\sigma+r+s)_n \frac{t^n}{n!} \\
&= x^{\alpha}y^{\beta}(1-t)^{-\sigma} \\
&\quad \times \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\sigma)_{r+s}}{s!r!\Gamma(\alpha+r+1)\Gamma(\beta+\kappa s+1)} \left(\left(\frac{xt}{t-1}\right)^r, y^{\kappa s}\left(\frac{t}{t-1}\right)^s \right) \\
&= x^{\alpha}y^{\beta}(1-t)^{-\sigma}\mathfrak{E}^{(\sigma)}_{\alpha+1,\beta+1,\kappa}\left(\frac{xt}{t-1}, y\left(\frac{t}{t-1}\right)^{\frac{1}{\kappa}} \right),
\end{aligned}$$

where we interchange the order of summations since the uniform convergence of the series for $|t| < 1$. ■

Theorem 4.4.2 *For α, β, κ, $\gamma \in \mathbb{C}$, $Re(\alpha)$, $Re(\beta)$, $Re(\kappa) > 0$, $\left|\frac{y^{\kappa}}{\tau^{\kappa}}\right| < 1$ and $\left|\frac{x\tau^{\kappa}}{t(\tau^{\kappa}-y^{\kappa})}\right| < 1$, we have the following Schläfli's type integral representation:*

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y) = -\frac{1}{4\pi^2}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{t+\tau}t^{-\alpha}\tau^{-\beta}\left(\frac{t\tau^{\kappa}}{t(\tau^{\kappa}-y^{\kappa})-x\tau^{\kappa}}\right)^{\gamma} dt\, d\tau,$$

where the contour of integration is a Hankel's loop which starts at $-\infty$ on the real axis in the complex ζ-plane, encircles the origin once in the positive (counter clockwise) direction, and then returns to $-\infty$ (see, for details, [35], pp. 244-246).

Proof. By using the Hankel's formula in [6], that is

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{-\infty}^{(0_+)} e^{\zeta}\zeta^{-z}d\zeta \quad (|\arg(\zeta)|\leqq \pi), \tag{4.4.1}$$

we find that

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)=-\frac{1}{4\pi^2}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{t+\tau}t^{-\alpha}\tau^{-\beta}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{r!s!}\left(\frac{x}{t}\right)^r\left(\frac{y^\kappa}{\tau^\kappa}\right)^s dt\, d\tau.$$

Since $(\gamma)_{r+s}=(\gamma)_r(\gamma+r)_s$, we get

$$\begin{aligned}
&\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)\\
=\;& -\frac{1}{4\pi^2}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{t+\tau}t^{-\alpha}\tau^{-\beta}\sum_{r=0}^{\infty}\frac{(\gamma)_r}{r!}\left(\frac{x}{t}\right)^r\sum_{s=0}^{\infty}\frac{(\gamma+r)_s}{s!}\left(\frac{y^\kappa}{\tau^\kappa}\right)^s dt\, d\tau\\
=\;& -\frac{1}{4\pi^2}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{t+\tau}t^{-\alpha}\tau^{-\beta}\left(\frac{\tau^\kappa}{\tau^\kappa-y^\kappa}\right)^\gamma\sum_{r=0}^{\infty}\frac{(\gamma)_r}{r!}\left(\frac{x\tau^\kappa}{t(\tau^\kappa-y^\kappa)}\right)^r dt\, d\tau\\
=\;& -\frac{1}{4\pi^2}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{t+\tau}t^{-\alpha}\tau^{-\beta}\left(\frac{t\tau^\kappa}{t(\tau^\kappa-y^\kappa)-x\tau^\kappa}\right)^\gamma dt\, d\tau,
\end{aligned}$$

which is the desired result. ■

In a similar manner, we obtain the following Corollary:

Corollary 4.4.3 *Let* $\alpha,\ \beta\ \in\mathbb{R},\ \kappa=1,2,\cdots,\ \left|\frac{y^\kappa}{\tau^\kappa}\right|<1$ *and* $\left|\frac{x\tau^\kappa}{t(\tau^\kappa-y^\kappa)}\right|<1$. *Then,*

$$_{\kappa}L_n^{(\alpha,\beta)}(x,y)=-x^\alpha y^\beta\frac{1}{4\pi^2}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{t+\tau}t^{-\alpha}\tau^{-\beta}\left(\frac{t(\tau^\kappa-y^\kappa)-x\tau^\kappa}{t\tau^\kappa}\right)^n dt\, d\tau.$$

Now, we state a double integral representation for the product of $E^{(\gamma)}_{\alpha,\beta,\kappa}(x,y)$. In proving the following Theorem, we need the following integral representations:

$$\Gamma(z) = \int_0^\infty e^{-\tau}\tau^{z-1}d\tau, \quad (Re(z) > 0)$$

and

$$B(\alpha,\beta) \ : \ = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \ . \tag{4.4.2}$$

$$(\min\{Re(\alpha), Re(\beta)\} > 0)$$

Theorem 4.4.4 *The following integral representation holds true:*

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y).\mathfrak{E}^{(\sigma)}_{\lambda,\delta,\kappa}(x,y) = \frac{1}{16\pi^4}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\sigma)}\int_0^1 t^{\gamma-1}(1-t)^{\sigma-1}$$

$$\times \int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{\zeta+\omega+\eta+\xi}\zeta^{-\alpha}\omega^{-\beta}\eta^{-\lambda}\xi^{-\delta}$$

$$\times S^{1:0;0}_{0:0;0}\left(\begin{array}{c} [(\gamma+\sigma):1;1]:-;- \quad ; \\ \\ -:-;- \quad ; \\ \\ x\left[(1-t)\eta^{-1}+\zeta^{-1}\right], y^\kappa[(1-t)\xi^{-\kappa}+\omega^{-\kappa}] \end{array}\right) \tag{4.4.3}$$

$$\times d\zeta \ d\omega \ d\eta \ d\xi \ dt.$$

where $|\arg(\zeta)|, |\arg(\omega)|, |\arg(\eta)|, |\arg(\xi)| \le \pi$, $\min\{Re(\alpha), Re(\lambda)\} > 0$, $\min\{Re(\kappa), Re(\gamma), Re(\sigma)\} > 0$ *and* $S^{1:0;0}_{0:0;0}$ *denotes the double hypergeometric series (see [28], p.199).*

Proof. By the definition (1.0.15), we get

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y).\mathfrak{E}^{(\sigma)}_{\lambda,\mu,\kappa}(x,y)$$

$$= \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)}\frac{x^r}{r!}\frac{y^{\kappa s}}{s!}\right)\cdot\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(\sigma)_{p+q}}{\Gamma(\lambda+p)\Gamma(\mu+\kappa q)}\frac{x^p}{p!}\frac{y^{\kappa q}}{q!}\right)$$

$$= \frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\sigma)}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\Gamma(\gamma+\sigma+p+q)}{p!q!}x^p y^{\kappa s}\sum_{r=0}^{p}\sum_{s=0}^{q}\binom{p}{r}\binom{q}{s}$$

$$\times\frac{B(\gamma+r+s,\sigma+p-r+q-s)}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)\Gamma(\lambda+p-r)\Gamma(\mu+\kappa(q-s))}$$

$$(\min\{Re(\alpha),Re(\lambda)\}>0;\ \min\{Re(\kappa),Re(\gamma),Re(\sigma)\}>0)$$

in terms of Beta function $B(\alpha,\beta)$ defined by (4.4.2).

Now, by using the integral formulas (4.4.1) and (4.4.2), we obtain

$$\mathfrak{E}^{(\gamma)}_{\alpha,\beta,\kappa}(x,y).\mathfrak{E}^{(\sigma)}_{\lambda,\delta,\kappa}(x,y) = \frac{1}{16\pi^4}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\sigma)}\int_0^1 t^{\gamma-1}(1-t)^{\sigma-1}$$

$$\times\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{\zeta+\omega+\eta+\xi}\zeta^{-\alpha}\omega^{-\beta}\eta^{-\lambda}\xi^{-\delta}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\Gamma(\gamma+\sigma+p+q)}{p!q!}$$

$$\times\left(x\left[(1-t)\eta^{-1}+\zeta^{-1}\right]\right)^p\left(y^{\kappa}[(1-t)\xi^{-\kappa}+\omega^{-\kappa}]\right)^q d\zeta\ d\omega\ d\eta\ d\xi\ dt$$

$$= \frac{1}{16\pi^4}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\sigma)}\int_0^1 t^{\gamma-1}(1-t)^{\sigma-1}$$

$$\times\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{\zeta+\omega+\eta+\xi}\zeta^{-\alpha}\omega^{-\beta}\eta^{-\lambda}\xi^{-\delta}$$

$$\times S^{1:0;0}_{0:0;0}\left(\begin{array}{c} [(\gamma+\sigma):1;1]:-;- \quad ; \\ \\ -:-;- \qquad ; \\ \\ x\left[(1-t)\eta^{-1}+\zeta^{-1}\right], y^{\kappa}[(1-t)\xi^{-\kappa}+\omega^{-\kappa}] \end{array}\right) \tag{4.4.4}$$

$$\times d\zeta\, d\omega\, d\eta\, d\xi dt.$$

Because of the convergence conditions for the generalized bivariate Lauricella series which were investigated by Srivastava and Daoust in ([27], p. 155), relation (4.4.4) is absolutely converges for $\left|x\left[(1-t)\eta^{-1}+\zeta^{-1}\right]\right| < \frac{\mu}{\mu+\nu}$ and $|y^{\kappa}[(1-t)\xi^{-\kappa}+\omega^{-\kappa}]| < \frac{\nu}{\mu+\nu}$. ■

Similary, we get the following result:

Corollary 4.4.5 *The following integral representation holds true:*

$${}_{\kappa}L_n^{(\alpha,\beta)}(x,y)\,{}_{\kappa}L_m^{(\lambda,\mu)}(x,y) = \frac{1}{16\pi^4}x^{\alpha+\lambda}y^{\beta+\mu}$$

$$\times \int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)}\int_{-\infty}^{(0_+)} e^{\zeta+\omega+\eta+\xi}\zeta^{-\alpha-1}\omega^{-\beta-1}\eta^{-\lambda-1}\xi^{-\delta-1}$$

$$\times\left(\frac{\zeta-x}{\zeta}-\frac{y^{\kappa}}{\omega^{\kappa}}\right)^{n}\left(\frac{\eta-x}{\eta}-\frac{y^{\kappa}}{\xi^{\kappa}}\right)^{m} d\zeta\, d\omega\, d\eta\, d\xi,$$

where $|\arg(\zeta)|, |\arg(\omega)|, |\arg(\eta)|, |\arg(\xi)| \leq \pi$, $\min\{Re(\alpha+1), Re(\lambda+1)\} > 0$ *and* $\min\{Re(\kappa)\} > 0$.

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