A-Statistical Convergence

Saadia Yousuf

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
June 2016
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Mustafa Tümer
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Assoc. Prof. Dr. Hüseyin Aktuğlu
Supervisor

Examining Committee

1. Prof. Dr. Nazım Mahmudov

2. Prof. Dr. Mehmet Ali Özarslan

3. Assoc. Prof. Dr. Hüseyin Aktuğlu

ABSTRACT

The present study is conducted to study a new type convergence, called A-statistical convergence. In the beginning of the study, the concept of infinite, non-negative regular matrices is introduced. Some basic properties of regular and conservative matrices are studied. These matrices play an important role in the theory of A-statistical convergence. Every non-negative regular matrix defines a density function. These density functions are then used to define some new type of convergences such as, statistical convergence, lacunary statistical convergence and lambda statistical convergence. A-statistical convergence is the extension of the other statistical type convergences. Statistical convergence, Lacunary statistical convergence and Lambda statistical convergences can be considered as the special cases of A-statistical convergence produced by different non-negative regular matrices.

ÖZ

Bu çalışmada, yeni yakınsaklık türlerinden biri olan A-istatistiksel yakınsaklık ele alınmıştır. Öncelikle negative olmayan, sonsuz, regular ve konservatif matrisler üzerinde durulmuş ve böyle matrislerin temel özellikleri incelenmiştir. Yeni tip yakınsamalarda yoğunluk fonksiyonları temel rol oynamaktadır. Bu anlamda bakıldığında her sonsuz, regular ve konservatif matrisin bir yoğunluk fonksiyonu tanımlaması bu anlamda önem arz etmektedir. Lacunary, lamda ve istatistiksel yakınsaklık türleri değişik matricler tarafından üretilen yakınsamalar olup bu türlü yakınsamalarda A-istatistiksel yakınsama sınıfına girmektedir.

DEDICATION

TO MY LOVELY DAUGHTER BARIRAH

ACKNOWLEDGMENT

Firstly I would like to thank Almighty Allah for everything He has blessed me with. I am really thankful to my lovely parents who worked very hard to make me a person who I am today.

Most importantly I would love to express my special thanks to my supervisor Assoc. Prof. Dr. HUSEYIN AKTUGLU who supported and helped me so much throughout my thesis work. He is a wonderful and positive person who always encouraged me to work harder and gave me the strength to write this thesis. It was not possible to achieve the target without him, Sir thank you so much.

Now last but not the least, I would love to thank the most important person of my life, my lovely husband MUHAMMAD ABID, who was always there to support and help. He worked tirelessly with me to finish my target. He looked after our baby while I was busy with my work and he was always there for the moral support whenever I was badly in the need of encouragement.

I would like to thank my little doll BARIRAH ABID for her cooperation. At the end I am very grateful to my mathematics department. Indeed our department has an awesome staff. I thank all my teachers. It was a great experience to study in EMU. Dear EMU you will be missed.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

This thesis is about a new type of convergence i.e. A-statistical convergence. In the thesis we will see that the study of A-statistical convergence helps to introduce some other new type of convergences like statistical convergence, lacunary statistical convergence and lambda statistical convergence. The thesis includes the concept of infinite matrices and density functions. (See [1] to [4],[10] to [15])

The chapter 2 of the thesis is about some specific non-negative infinite matrices. This chapter includes the definitions of non-negative infinite matrices. Regular and conservative matrices, which are non-negative infinite matrices, are discussed in detail in the chapter. One can find the definitions of regular and conservative matrices and some examples related to them.

The Chapter 3 is about the density functions. In this chapter the idea of density functions is discussed in detail. It includes examples and some basic properties of density functions. It also includes some definitions of different sequences like lacunary sequence and lambda sequence. One can understand that any non-negative regular matrix gives a density function. This idea of obtaining density functions from different non-negative regular matrices is explained in detail with the help of examples. These density functions are very important in the study of new type of convergences, which can be understood from the next chapter.

Chapter 4 is the main chapter of the thesis. It explains the concept of A-statistical convergence. In the beginning the concept of A-density is explained with examples, on the basis of chapter 3. Later A-statistical convergence is defined and it is explained in detail with the help of some properties and theorems. The chapter includes the definitions of statistical convergence, lacunary statistical convergence and lambda statistical convergence and their relation with A-statistical convergence. From the study of this chapter one can find out that these new type of convergences (statistical convergence, lacunary statistical and lambda statistical convergence) are special classes of A-statistical convergence.

Chapter 2

INFINITE MATRICES

The main purpose of this chapter is to discuss basic definitions and properties of regular and conservative matrices. As it is well known infinite regular matrices play a vital role in the theory of new type convergences such as statistical convergence, A-statistical convergence, lacunary statistical convergence and lambda statistical convergence. In this chapter we mainly focus on infinite matrices which are conservative and regular. Another important tool in the theory of new type convergences is the density functions. In the present chapter the relation between regular matrices and density functions will also be discussed.

Definition 2.1: An infinite matrix, $A = (a_{nk})$ is the matrix which has infinitely many rows and columns.

Fact: In the case of infinite matrices addition and scalar multiplications are defined component wise. More precisely, let $A = (a_{nk})$ and $B = (b_{nk})$ be two infinite matrices then

i) $A + B = (a_{nk} + b_{nk})$

ii) $\lambda A = (\lambda a_{nk})$ for any scalar λ).

Definition 2.2: An infinite matrix, ($A = (a_{nk})$ whose element are non-negative (i.e. $(a_{nk}) \geq 0$), is called a non-negative, infinite matrix.

Example 2.1: The following matrix,

$$F=\begin{pmatrix}2 & 1 & 0 & 0 & \dots\\ 0 & 2 & 1 & 0 & \dots\\ 0 & 0 & 0 & 2 & 1..\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix}$$

is a non-negative, infinite matrix.

Definition 2.3: Let A= (a_{nk}) be an infinite matrix, for any sequence $y=(y_k)$ the A transform of y is defined as

$$A(y)=\left(\sum_{k=1}^{\infty} a_{nk}\, y_k\right)_n ,$$

provided that series converges for all n.

Example 2.2: Consider the matrix

$$F=\begin{pmatrix}2 & 1 & 0 & 0 & \dots\\ 0 & 2 & 1 & 0 & \dots\\ 0 & 0 & 0 & 2 & 1..\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix}$$

and the sequence $y=(y_n)=(y_1,y_2,\ldots,y_n,\ldots)$ then

$$Fy=\begin{bmatrix}2 & 1 & 0 & 0 & \cdots\\ 0 & 2 & 1 & 0 & \cdots\\ 0 & 0 & 0 & 2 & 1\cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\\ \vdots\end{bmatrix}=\begin{bmatrix}2y_1+y_2\\ 2y_2+y_3\\ \vdots\\ 2y_n+y_{n+1}\\ \vdots\end{bmatrix}.$$

If we take $y_n=(1+\frac{1}{n})$ then

$$Fy = \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 1\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 2 \\ 3/2 \\ \vdots \\ 1+\frac{1}{n} \\ \vdots \end{bmatrix} = \begin{bmatrix} 11/2 \\ 13/3 \\ \vdots \\ 3+\frac{3n+2}{n(n+1)} \\ \vdots \end{bmatrix}.$$

It should be mentioned that both $y = (1+\frac{1}{n})$ and $Fy = 3+\frac{3n+2}{n(n+1)}$ are convergent but converges to different limits.

Example 2.3: Consider the matrix

$$Fy = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and the sequence $y = (y_n) = (y_1, y_2, \ldots, y_n, \ldots)$ then

$$Fy = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ (1/2)y_2 + y_3 \\ \vdots \\ (1/n)y_n + y_{n+1} \\ \vdots \end{bmatrix}.$$

If we take $y_n = (1+\frac{1}{n})$ then

$$Fy = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 2 \\ 3/2 \\ \vdots \\ 1+\frac{1}{n} \\ \vdots \end{bmatrix} = \begin{bmatrix} 7/2 \\ 25/12 \\ \vdots \\ 1+\frac{2n^2+2n+1}{n^2(n+1)} \\ \vdots \end{bmatrix}.$$

It should be mentioned that both $y_n = (1+\frac{1}{n})$ and $Fy = 1+\frac{n^2+2n+1}{n^2(n+1)}$ are convergent and they converges to the same limit.

These two examples show that some infinite matrices preserve the limit of a sequence but some of them does not. This observation rises the following questions "under which conditions matrix transformation of a convergent sequences is again convergent" and "under which conditions matrix transformation preserves limit of the convergent sequences". In the present part of this chapter we shall discuss these two cases.

Definition 2.4: An infinite matrix A is said to be conservative, if for any convergent sequence y=(y_k),

$$A(y) = \left(\sum_{k-1}^{\infty} a_{nk}\, y_k\right)_n$$

is also convergent but the limit may change**.**The space of all conservative matricesis denoted by M_{Con} .

An infinite matrix is conservative if it satisfies the conditions of Kojima-Schur Theoremstated below.

Theorem 2.1: (Kojima-Schur) An infinite matrix A= (a_{nk}) n, k=1, 2, is conservative if and only if,

i) $\lim_{n\to\infty} a_{nk} = \alpha_{k,}$ for each k = 0, 1……, where α_k is a real number for each k .

ii) $\lim_{n\to\infty} \sum_{k-0}^{\infty} a_{nk} = \alpha$, for some $\alpha \in IR$.

iii) $sup \sum_{k-0}^{\infty} |a_{n,k}| \leq H < \infty \, or \, some \, H > 0$

Example 2.4: The following infinite matrix

$$F_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & .... \\ 0 & 1 & 1 & 0 & .... \\ 0 & 0 & 0 & 1 & 1.. \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is a conservative matrix and it is obvious that,

i) $\lim_{n\to\infty} a_{nk} = 0$, for each k = 0, 1……,

ii) $\lim_{n\to\infty} \sum_{k-0}^{\infty} a_{nk} = 2$ and

iii) $sup \sum_{k-0}^{\infty} |a_{n,k}| \leq 2 < \infty$.

Therefore it satisfies the given conditions of Kojima-Schur Theorem.

Example 2.5: Let $C_1 = (c_{nk})$ denotes the Cesáro matrix of order one where

$$c_{nk} = \begin{cases} \frac{1}{n} & if\; 1 \leq k \leq n \\ 0 & otherwise \end{cases}$$

or

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

then it is obvious that C_1 is conservative and

i) $\lim_{n\to\infty} a_{nk} = 0$, for each k = 0, 1……,.

ii) $\lim_{n\to\infty} \sum_{k-0}^{\infty} a_{nk} = 1,$

iii) $sup \sum_{k-0}^{\infty} |a_{n,k}| \leq 1 < \infty.$

Example 2.6: Let A be a matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

then A is not conservative matrix. Because ii) and iii) of Kojima-Schur Theoremdoes not hold. Moreover, for the convergent sequence $y = (1,1,1,\cdots)$ the A transform of y

$$Ay = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ \vdots \end{bmatrix}$$

is not convergent.

Lemma 2.1: Let E be a conservative matrix and m be a positive integer then E^m is also conservative.

Proof: Let $y = (y_k)$ be a sequence converging to L. If $m = 1$ then

$$E^m = E,$$

and it is conservative. Now let us suppose that it is true for $m = k$, that is

$$E^k y \to L_1$$

(L_1 may or may not be equal to L) then check it for $m = k + 1$

$$E^{k+1} y = E.E^k y \longrightarrow L_2$$

since E is conservative and $E^k y \to L_1$.

Therefore $\forall m \in \boldsymbol{N},\ E^m$ is also conservative whenever E is conservative.

Lemma 2.2: Let $A = (a_{nk})$ and $E = (e_{nk})$ be two conservative matrices, then

i) $E + A$ is also conservative.

ii) EA and AE are conservative.

iii) λE where λ is any scalar, is conservative.

Proof:

i) Let $y = (y_k)$ be a sequence converging to L. Then $Ay = L_1$ (L_1 may or may not be equal to L) and $Ey = L_2$. (L_2 may or may not be equal to L) then

$$(E + A)y = Ey + Ay \to L_1 + L_2$$

so $E + A$ is conservative.

ii) We have $(EA)y = E(Ay) \to L_2$, (because Ay is convergent sequence and E is conservative).

Thus EA is conservative. Similarly, we can easily show that, EA is also conservative.

iii)By the definition**, w**e have $(\lambda E)y = \lambda(Ey)$.

Since E is conservative and $y \to L_1$, we get $Ey \to L_2$ for some L_2. Hence, $\lambda Ey \to \lambda L_2$ which means that, λE is conservative.

Lemma 2.3: Let $E_1, E_2, ..., E_n$ be n conservative matrices then $E_1 + E_2 + ... + E_n$ is also conservative.

Proof: Let $y = (y_k)$ be a sequence converging to L. Then assume that

$E_i y \to L_i$, $1 \le i \le n$. Then,

$$(E_1 + E_2 + \cdots + E_n)y = E_1 y + E_2 y + \cdots E_n y \to L_1 + L_2 + \cdots + L_n.$$

Therefore, $E_1 + E_2 + ... + E_n$ is conservative.

Definition 2.5: An infinite matrix E is called regular if the convergence of the sequence $y = (y_k)$ implies the convergence of E_y and it preserves the limit. i.e, if

$$\lim_{n \to \infty} y_n = L$$

then

$$\lim_{n \to \infty} (Ey)_n = L.$$

The space of all regular matrices will be denoted by M_{Reg}.

An infinite matrix is regular if it satisfies the conditions of Silverman-Teoblitz Theorem stated below.

Theorem 2.2: (Silverman, Teoplitz)

A matrix $E = (e_{n,k})$ is regular if and only if

i) $\lim_{n\to\infty} e_{n,k} = 0$ For each k = 0, 1, 2…

ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} e_{n,k} = 1$

iii) $Sup_n \sum_{k=0}^{\infty} |e_{n,k}| \leq H < \infty$ for some H > 0.

Remark: Every regular matrix is conservative (*i.e.* $M_{\text{Reg}} \subset M_{Con}$). But converse inclusion does not hold.

Example 2.7: Let $A = (a_{n_k})$ be an infinite matrix defined as

$$A = (a_{n_k}) = \begin{cases} 1 & if\ k = n \\ \frac{1}{n} & if\ k = n+1 \\ 0 & otherwise \end{cases}$$

or equivalently,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

then A is a non-negative, regular matrix.

Example 2.8: The Cesáro matrix of order one is a regular matrix

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

then it is obvious that C_1 is conservative because,

i) $\lim_{n\to\infty} a_{nk} = 0$, for each k = 0, 1……,.

ii) $\lim_{n\to\infty} \sum_{k-0}^{\infty} a_{nk} = 1$,

iii) $sup \sum_{k-0}^{\infty} |a_{n,k}| \leq 1 < \infty$.

Example 2.9: Let A be the matrix such that

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & . & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

then A is also non-negative and regular matrix because

then it is obvious that C_1 is conservative and

i) $\lim_{n\to\infty} a_{nk} = 0$, for each k = 0, 1… .

ii) $\lim_{n\to\infty} \sum_{k-0}^{\infty} a_{nk} = 1$,

iii) $sup \sum_{k-0}^{\infty} |a_{n,k}| \leq 1 < \infty$.

Lemma 2.4: Let E be a non-negative regular matrix and m be a positive integer then E^m is again regular.

Proof: Assume that y_n is an arbitrary convergent sequence(say $y_n \to L$), we need to show that $E^m y \to L$.

If $m = 1$ then $E^1 y_n = Ey_n \to L$.

Now suppose that it is true for $m = k$ that is,

$$E^k y_n \to L.$$

Take $m = k + 1$, and use the fact that $E^k y_n$ is a sequence converging to L, we have,

$$E^{k+1} y_n = E\left(E^k y_n\right) \to L$$

which completes the proof.

Lemma 2.5: Let $A = (a_{nk})$ and $E = (e_{nk})$ be two regular matrices then,

i) $\frac{1}{2}(E + A)$ is regular

ii) EA is regular

iii) AE is regular

Proof: Assume that y_n is an arbitrary sequence converging to L then

i) $\frac{1}{2}(E + A)(y_n) = \left(\frac{1}{2}E\right)y_n + \left(\frac{1}{2}A\right)y_n$, since A and E are regular matrices $\left(\frac{1}{2}E\right)y_n + \left(\frac{1}{2}A\right)y_n = \frac{1}{2}\left(Ey_n\right) + \frac{1}{2}(Ey_n) \to \frac{L+L}{2} = L$, which completes the proof.

ii) $(EA)(y_n) = E\left(Ay_n\right)$, since A is regular and $y_n \to L$ we have $Ay_n \to L$.

On the other hand, E is regular and $Ay_n \to L$ implies that $(EA)(y_n) \to L$.

iii) $(AE)(y_n) = A(Ey_n)$, since E is regular and $y_n \to L$ we have $Ey_n \to L$

On the other hand, A is regular and $Ey_n \to L$ implies that

$(AE)(y_n) \to L$.

Lemma 2.6: Let $E_1, E_2, ..., E_n$ be n regular matrices then

i) $\frac{1}{n}(E_1 + E_2 + ... + E_n)$ is also regular.

ii) $E_1 E_2 \cdots E_n$ is also regular.

Proof: Let $y = (y_k)$ be a sequence converging to L.

i) $$\left[\frac{1}{n}(E_1 + E_2 + \cdots + E_n)\right] y = \frac{1}{n}(E_1 y + E_2 y + \cdots E_n y)$$

$$= \frac{1}{n}(L + L + \cdots + L)$$

$$= \frac{nL}{n} = L$$

ii) By the definition,

$$(E_1 E_2 \dots E_n) y = (E_1 E_2 \dots)\ (E_n y).$$

But since $E_n y$ is a sequence converging to L and all E_i, $1 \le i \le n$, are regular. Hence

$(E_1 E_2 \dots)(E_n y) \longrightarrow L$.

In this part we shall focus on the necessary and sufficient conditions for matrices that maps zero sequences to zero sequences. Of course by the zero sequences we mean the sequence which converges to zero.

Definition 2.6: An infinite matrix E is called zero preserving matrix if for every sequence $y \in c_0,\ Ey \in c_0$. The set of all zero preserving matrices will be denoted by M_0.

The necessary and sufficient condition for a matrix E to be a zero preserving matrix is given in the following theorem.

Theorem 2.3: (See [1] to [4]) A matrix $E = (e_{n,k})$ preserves zero limits if and only if

i) $\lim_{n\to\infty} e_{n,k} = 0,$ for each k = 0, 1, 2…

ii) $Sup_n \sum_{k=0}^{\infty} |e_{n,k}| \le H < \infty,$ for some H > 0.

Remark: Every regular matrix E is zero preserving matrix.

Example 2.10: The following matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

is neither regular nor conservative but it zero preserving matrix. In fact let $y = (y_n)$ be a zero sequence (i.e. $y_n \to 0$) then,

$$Ey = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_1 \\ 2y_2 \\ y_3 \\ 2y_4 \\ \vdots \end{bmatrix}$$

it is obvious that $Ey \to 0$.

Lemma 2.7: Let $A=(a_{nk})$ and $E=(e_{nk})$ be two elements of M_0 then,

i) $(E+A)\in M_0$,

ii) $EA\in M_0$,

iii) $AE\in M_0$,

iv) $\lambda E\in M_0$ for any scalar λ.

Proof: Let $y=(y_k)$ be a sequence converging to 0 then

(i) We have, $(E+A)y=Ey+Ay=0+0=0$.

Hence $(E+A)\in M_0$.

(ii) We have $(EA)y=E(Ay)$

Since Ay is a sequence converging to 0 and $E\epsilon\ M_0$ so $(EA)y\longrightarrow 0$. Thus,

$EA\in M_0$.

(iii) We have $(AE)y=A(Ey)$

Since Ey is a sequence converging to 0 and $A\epsilon\ M_0$ so $(AE)y\longrightarrow 0$.

(iv) We have $(\lambda E)y=\lambda(Ey)$ and $\lambda(Ey)\to 0$.

Lemma 2.8: Let $E_1,E_2,....,E_n$ be elements of M_0 then,

i) $E_1+E_2+....+E_n\in M_0$.

ii) $\lambda_1E_1+\lambda_2E_2+....+\lambda_nE_n\in M_0$, where $\lambda_1,\lambda_2,....,\lambda_n$ are scalars.

iii) $E_1E_2....E_n\in M_0$

iv) $E_1^{\ n}\in M_0$

Proof: Let $y = (y_k)$ be a sequence converging to zero then

(i) By the definition, $(E_1 + E_2 + ... + E_n)y = E_1 y + E_2 y + ... + E_n y$

$= 0 + 0 + \cdots + 0 = 0$

therefore

$$E_1 + E_2 + ... + E_n \in M_0.$$

(ii) Using definition we have,

$(\lambda_1 E_1 + \lambda_2 E_2 + ... + \lambda_n E_n)y$

$= \lambda_1(E_1 y) + \lambda_2(E_2 y) + ... + \lambda_n(E_n y) = \lambda_1(0) + \lambda_2(0) + ... + \lambda_n(0)$

$= 0$.

Thus $\lambda_1 E_1 + \lambda_2 E_2 + ... + \lambda_n E_n \in M_0$.

(iii) We have $(E_1 E_2 .... E_n)y = (E_1 E_2 ...)E_n y$

Since $E_n y \to 0$ andfor all E_i, $1 \le i \le n$,

We get, $E_1 E_2 \cdots E_n \in M_0$.

(iv) If $n = 1$ then $E_1 y = 0$ because $E_1 \in M_0$.

Let us suppose that it is true for $n = k$. Then we have

$$E_1{}^k y = 0.$$

Now check it for $n = k + 1$

$E_1{}^{k+1} y = E_1 . E^k y \to 0$ since E_1 and E^k belongs to M_0.

Definition 2.7: An infinite matrix E is a multiplicative with multiplier if for every sequence $y \in c$,

$$\lim_{n\to\infty}(Ey)_n = \lambda \lim_{n\to\infty} y_n.$$

The space of all multiplicative matrices with multiplier λ, will be denoted by M_λ.

Theorem 2.4: (See [1] to [4]) A matrix $E = (e_{n,k})$ is multiplicative with multiplier λ if and only if

i) $\lim_{n\to\infty} e_{n,k} = 0,$ For each k = 0, 1, 2…

ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} e_{n,k} = \lambda,$

iii) $Sup_n \sum_{k=0}^{\infty} |e_{n,k}| \le H < \infty,$ for some H > 0.

Remark: It is obvious that $M_1 = M_{reg}$.

Example 2.11: Let E be the matrix, $E = (e_{nk})$ where

$$(e_{nk}) = \begin{cases} 3 & if \quad n = k \\ 1/n & if \quad n = k+1 \\ 0 & otherwise \end{cases}$$

Or equivalently,

$$E = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0\cdots \\ 0 & 3 & 1/2 & 0 & 0 & 0\cdots \\ 0 & 0 & 3 & 1/3 & 0 & 0\cdots \\ 0 & 0 & 0 & 3 & 1/4 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Since,

i) $\lim_{n\to\infty} e_{n,k} = 0,$ For each k = 0, 1, 2…

ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} e_{n,k} = 3,$

iii) $Sup_n \sum_{k=0}^{\infty} |e_{n,k}| \le 4 < \infty.$

E is a multiplicative matrix with the multiplier λ=3. On the other hand let $y = (y_n)$ be a convergent sequence converging to L, then

$$Ey = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0\cdots \\ 0 & 3 & 1/2 & 0 & 0 & 0\cdots \\ 0 & 0 & 3 & 1/3 & 0 & 0\cdots \\ 0 & 0 & 0 & 3 & 1/4 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \\ \vdots \end{bmatrix} = \begin{bmatrix} 3y_1 + y_2 \\ 3y_2 + (y_3/2) \\ 3y_3 + (y_4/3) \\ \vdots \\ 3y_n + (y_{n+1}/n) \\ \vdots \end{bmatrix} = 3(y_n) + (y_{n+1}/n) \to 3L.$$

Example 2.12: Let E be a matrix such that

$$E = \begin{pmatrix} 2 & 0 & 0 & . & . & . \\ 0 & 1 & 1 & 0 & . & . \\ 0 & 0 & 1 & 1 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

Since,

i) $\lim_{n\to\infty} e_{n,k} = 0,$ For each k = 0, 1, 2…

ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} e_{n,k} = 2,$

iii) $Sup_n \sum_{k=0}^{\infty} |e_{n,k}| \le 2 < \infty.$

E is a multiplicative matrix with the multiplier $= 2$. Now take $y = (y_n) = \left(3 - \frac{1}{n}\right)$,

which converges to 3, then

$$Ey = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 2 \\ 5/2 \\ 8/3 \\ \vdots \\ 3-\frac{1}{n} \\ \vdots \end{bmatrix} = \begin{bmatrix} 4 \\ 31/6 \\ 65/12 \\ \vdots \\ 6-\frac{2n+1}{n(n+1)} \\ \vdots \end{bmatrix}.$$

In other words, $Ey = \left(6 - \frac{2n+1}{n(n+1)}\right) \to 6 = 2\left(\lim_{n\to\infty} y\right).$

Lemma 2.9: Let $E_1, E_2, \ldots., E_n$ be elements of M_λ then,

i) $\frac{1}{n}\left(E_1 + E_2 + \ldots. + E_n\right) \in M_\lambda$.

ii) $E_1 E_2 \in M_{\lambda^2}$

iii) $E_1^{\ n} \in M_{\lambda^n}$

Proof: Consider a sequence $y := (y_k)$ which is convergent to L then

(i) $(E_1 + E_2 + \ldots. + E_n)y = E_1 y + E_2 y + \ldots\ldots. + E_n y$

$\lambda L + \lambda L + \ldots. + \lambda L = n\lambda L$ since each E_i for $i = 1, \ldots n$ belongs to M_λ.

This implies that, $\frac{1}{n}(E_1 + E_2 + \cdots + E_n) \in M_\lambda$.

(ii) We have

$$(E_1 E_2)y = E_1(E_2 y).$$

Since

$$E_2 y \longrightarrow \lambda L \text{ and } E_1 \in M_\lambda \text{ so}$$

$E_1 E_2 \in M_{\lambda^2}$.

(iii) If $n=1$ then

$$E_1^n y = E_1 y$$

and

$$E_1 y \to \lambda L$$

since $E_1 \in M_\lambda$.

Now assume that it is true for $n=k$ that is

$$E_1^k y \to \lambda^k L \, .$$

Now check it for $k+1$,

$$E_1^{k+1} y = E_1(E_1^k y)$$

${E_1}^k y$ is a sequence converging to $\lambda^k L$ and $E_1 \in M_\lambda$ so

$$E_1^{k+1} y \to \lambda^{k+1} L$$

hence $E_1^n y \to \lambda^n L$ and $E_1^n \in M_{\lambda^n}$.

As a consequence of Kojima-Schur Theorem, Silverman-Teoblitz Theorem, Theorem 2.3 and Theorem 2.4, we can state the following lemma.

Lemma 2.10: For the spaces M_{reg}, M_{con}, M_λ and M_0 we have,

i) $M_{reg} \subset M_{con}$,

ii) $M_{reg} = M_1$,

iii) $M_\lambda \subset M_0$ for all λ.

Lemma 2.11: If $A \in M_{reg}$ and $B \in M_0$ then

i) $AB \in M_0$,

ii) $BA \in M_0$.

Proof: Let $y = (y_k)$ be a convergent sequence converging to L.

(i) We have $(AB)y = A(By)$

Since $By \to 0$ and A is regular so

$$(AB)y \to 0.$$

Hence $AB \in M_0$.

(ii) We have, $(BA)y = B(Ay)$

$Ay \to L$ as A is regular and $B \in M_0$ so

$$B(Ay) \to 0$$

and

$$BA \in M_0.$$

Lemma 2.12: If $A \in M_{reg}$ and $B \in M_\lambda$ then

i) $AB \in M_\lambda$,

ii) $BA \in M_\lambda$,

iii) $\lambda A \in M_\lambda$, for all λ

iv) $\frac{1}{\lambda} B \in M_{reg}$, for all $\lambda. \neq 0$.

Proof: Consider a sequence $y = (y_k)$ converging to L then,

(i) $(AB)y = A(By)$ and $By \to \lambda L$ as $B \in M_\lambda$

and since A is regular so

$$A(By) \to \lambda L.$$

Hence $AB \in M_\lambda$.

(ii) we have $(BA)(y) = B(Ay)$ and $Ay \to L$ as A is regular

since $B \in M_\lambda$ so

$$B(Ay) \to \lambda L .$$

Hence $BA \in M_\lambda$.

(iii) let λ be any scalar then

$(\lambda A)(y) = \lambda(Ay) = \lambda L$ since A is regular.

Hence for any λ, $\lambda A \in M_\lambda$.

(iv) $(\frac{1}{\lambda}B)(y) = \frac{1}{\lambda}(By) = \frac{1}{\lambda}(\lambda L)$ since $B \in M_\lambda$ and $\lambda \neq 0$, we get

$(\frac{1}{\lambda}B)(y) \to L.$

Hence $= \frac{1}{\lambda}B$ is regular.

Lemma 2.13: If $A \in M_0$ and $B \in M_\lambda$ then

i) $AB \in M_0$,

ii) $BA \in M_0$.

Proof: Let $y = (y_k)$ be a sequence converging to 0 then,

(i) $(AB)y = A(By)$ and $By \to \lambda 0 = 0$ as $B \in M_\lambda$ but

$A \in M_0$ implies that,

$$A(By) \to 0 .$$

Hence $AB \in M_0$.

(ii) $(BA)y = B(Ay)$ and $Ay \to 0$, $= B(0)$ since $A \in M_0$

$= 0$

Hence $BA \in M_0$

Chapter 3

DENSITIES

The main purpose of this chapter is to introduce definitions and basic properties of density functions. As it is well known, density functions play a vital role in the study of new type of convergences such as statistical convergence and all types of A-statistical convergences. In fact, A-statistical convergence is based on a density function, which is obtained from a non-negative regular matrix. In other words, different non-negative regular matrices give us different density functions. In this chapter we will try to underline two points, firstly, what is a density functions and well known properties of density functions and secondly, the relation between density functions and non-negative regular matrices.

Definition 3.1: Let A, E⊆ $\boldsymbol{N}$**,** then the symmetric difference of these two sets is defined as,

$$A\Delta E = (A/E) \cup (E/A)$$

and it is denoted by A~E.

Note: A and E have a relation "~" if their symmetric difference is finite, i.e. A~E if and only if $A\ \Delta\ E$ is finite.

Definition 3.2: *(See [3]* A function $\delta = 2^{\boldsymbol{N}} \rightarrow [0,1]$ will be called a lower asymptotic density or just density if it satisfies these four axioms d.1 if A~E then $\delta(A) = \delta(E)$

d.2 if A$\cap E = \phi$ then $\delta(A) + \delta(E) \leq \delta(A \cup E)$

d.3 $\forall$ A, E, $\delta(A) + \delta(E) \leq 1 + \delta(A \cap E)$

d.4 $\delta(\boldsymbol{N}) = 1$

Definition 3.3: *(See [3]* For any lower density δ we can define upper density $\bar{\delta}$ related with δ by

$$\bar{\delta}_E = 1 - \delta(\boldsymbol{N}/E)$$

where E ⊆ $\boldsymbol{N}$.

Proposition 3.1: *(See [3]* Let δ be a lower density and $\bar{\delta}$ be an upper density. For sets A and E, where A, E⊆ $\boldsymbol{N}$, we have

i) A$\subseteq E \Longrightarrow \delta(A) \leq \delta(E)$

ii) A$\subseteq E \Longrightarrow \bar{\delta}(A) \leq \bar{\delta}(E)$

iii) For all A, E⊆ $\boldsymbol{N}$, $\bar{\delta}(A) + \bar{\delta}(E) \geq \bar{\delta}(A \cup E)$

iv) $\delta(\emptyset) = \bar{\delta}(\emptyset) = 0$

v) $\bar{\delta}(\boldsymbol{N}) = 1$

vi) A~E $\Longrightarrow \bar{\delta}(A) = \bar{\delta}(E)$

vii) $\delta(E) \leq \bar{\delta}(E)$

Proof :

(i) Using $A \cap (E/A) = \emptyset$, then by using (d.2) and A$\subseteq E$, we have,

$$\delta(A) + \delta(E/A) \leq \delta(A \cup (E/A)) = \delta(E)$$

Thus

$$\delta(A) \leq \delta(E).$$

ii) From the assumption we have

$$\boldsymbol{N}/E \leq \boldsymbol{N}/A$$

By using (i) we get

$$\delta(\boldsymbol{N}/A) \geq \delta(\boldsymbol{N}/E)$$

$$1 - \delta(\boldsymbol{N}/A) \leq 1 - \delta(\boldsymbol{N}/E)$$

Then from the definition of the upper density we get

$$\bar{\delta}(A) \leq \bar{\delta}(E).$$

iii) Using the definition of the upper density

$$\bar{\delta}(A) = 1 - \delta(\boldsymbol{N}/A)$$

and

$$\bar{\delta}(E) = 1 - \delta(\boldsymbol{N}/E)$$

Therefore,

$$\bar{\delta}(A) + \bar{\delta}(E) = 2 - \delta(\boldsymbol{N}/A) - \delta(\boldsymbol{N}/E)$$

$$= 2 - (\delta(\boldsymbol{N}/A) + (\delta(\boldsymbol{N}/E)$$

$$\geq 2 - (1 + \delta(\boldsymbol{N}/A) \cap (\boldsymbol{N}/E)).$$

From $\delta((\boldsymbol{N}/A) \cap (\boldsymbol{N}/E)) = \delta((\boldsymbol{N}/(A \cup E))$ we get

$$\bar{\delta}(A) + \bar{\delta}(E) \geq 1 - \delta((\boldsymbol{N}/(A \cup E)) = \bar{\delta}(A \cup E)$$

iv) Take $A = \emptyset$ and $E = \boldsymbol{N}$ then from (d.2) we get

$$\delta(\emptyset) + \delta(\boldsymbol{N}) \leq \delta(\boldsymbol{N} \cup \emptyset) = \delta(\boldsymbol{N})$$

which gives $\delta(\emptyset) = 0$. The equation $\bar{\delta}(\emptyset)$ is a direct result of the definition of the upper density and (d.4).

v) From the definition of the upper density we get

$$\bar{\delta}(\boldsymbol{N}) = 1 - \delta(\boldsymbol{N}/\boldsymbol{N}) = 1 - \delta(\emptyset) = 1.$$

vi) Suppose that $A \sim E$ then we have

$$(\boldsymbol{N}/A)\Delta(\boldsymbol{N}/E) = ((\boldsymbol{N}/A)/(\boldsymbol{N}/E) \cup ((\boldsymbol{N}/E)/(\boldsymbol{N}/A))$$

$$(E/A) \cup (A/E) = A\Delta E$$

which implies that

$$\delta(\boldsymbol{N}/A) = \delta(\boldsymbol{N}/E)$$

Hence

$$\bar{\delta}(A) = \bar{\delta}(E).$$

vii) Consider two sets E and $\boldsymbol{N}/E$ and apply (d.2) on these sets we get,

$$\delta(\boldsymbol{N}/E) + \delta(E) \leq \delta((\boldsymbol{N}/E) \cup E) = \delta(\boldsymbol{N}) = 1$$

So,

$$\delta(E) \leq 1 - \delta((\boldsymbol{N}/E) = \bar{\delta}(E).$$

Definition 3.4: *(See [3]* A set of natural numbers E is said to have natural density with respect to δ, if

$$\delta(E) = \bar{\delta}(E).$$

Lemma 3.1: *(See [3]* Let$\alpha_\delta = \{E \subset \boldsymbol{N}: \delta(E)$ exists$\}$ and $\alpha_\delta^0 = \{E \subseteq \boldsymbol{N}: \delta(E)\}$

i) If $E \sim \boldsymbol{N}$then $E \in \alpha_\delta$ and $\delta(E) = 1$

ii) If $E \sim \phi$then $E \in {\alpha^0}_\delta$ and $\delta(E) = 0$

Proof (i) Since $E \sim \boldsymbol{N}$then by (d.1)

$$\delta(E) = \delta(\boldsymbol{N}) = 1$$

using

$$\delta(E) \leq \bar{\delta}(E)$$

we have

$$\delta(E) = \bar{\delta}(E) = 1$$

which means that

$$E \in \alpha_\delta \; and \; \delta(E) = 1$$

(ii) $E \sim \phi$ From Proposition 1, (vi) we get

$$\bar{\delta}(E) = \bar{\delta}(\phi) = 0$$

and

$$\delta(E) \leq \bar{\delta}(E) = 0$$

Thus we have

$$\delta(E) = \bar{\delta}(E) = 0$$

$$\delta(E) = 0 \; so \; E \in {\alpha^0}_\delta.$$

Lemma 3.2: *(See [3]* Let K be a finite subset of $\boldsymbol{N}$ then the density of the set K is zero. That is

$$\delta(K) = 0.$$

Proof: Let K be a finite set then, $K \sim \phi$ and $\delta(K) = 0$.

The following example shows that, density function is not countably additive

Example 3.1: Take $E_i = \{i\}$ where $i = 1, 2, 3, \dots \dots$, we have

$$E_i \in {\alpha^0}_\delta \subset \alpha_\delta \,, i = 1, 2, 3, \dots \dots \text{ and } E_i \cap E_j = \emptyset \qquad (i \neq j)$$

but

$$\bigcup_{i=1}^{\infty} E_i = \boldsymbol{N} \text{and } \delta(\boldsymbol{N}) = 1 \neq \sum_{i=1}^{\infty} \delta(E_i) = 0$$

Example 3.2: Density function is often used for the function

$$\delta(E) = \lim_{n\to\infty} inf \frac{|E(n)|}{n}$$

where, $|E(n)|$ is the cardinality of E.

$$E(n) = E \cap \{1,2,3 \ldots . n\}$$

$\mathcal{X}_k$ is a sequence of 0's and 1's and it denotes the characteristic sequence of E.

Example 3.3: The upperdensity $\bar{\delta}(E)$ corresponding to $\delta(E)$ is used for the function

$$\bar{\delta}(E) = \lim_{n\to\infty} sup \frac{|E(n)|}{n}$$

From definition 2.4. Any subset E of natural numbers is said to have density if

$$\delta(E) = \bar{\delta}(E)$$

It means that $\lim_{n\to\infty} inf = \lim_{n\to\infty} sup$ which means that the limit exists. Hence the density of the set E is defined as

$$\delta(E) = \lim_{n\to\infty} \frac{|E(n)|}{n}.$$

Example 3.4 Let E be a finite subset of natural numbers then

$$\delta(E) = \lim_{n\to\infty} \frac{|E(n)|}{n} = 0$$

since $|E(n)|$ is finite number.

Example 3.5: Let $E = \{3e : e \in \boldsymbol{N}\}$ and $A = \{3e + 1 : e \in \boldsymbol{N}\}$, then these sets have density 1/3.

Lemma 3.3: (See [1] to [6], [10] to [15]) Let $E = \{\lambda(n) : n \in \boldsymbol{N}\}$ then

$$\delta(E) = \lim_{n\to\infty} \frac{n}{\lambda(n)}.$$

Example 3.6: Let $E = \{e \in \boldsymbol{N} : e = n^2\}$ then

$$\delta(E) = \lim_{n\to\infty} \frac{|E(n)|}{n}$$

$$= \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence the density of the sets like $E = \{e \in \boldsymbol{N} : e = n^2\}$ is zero.

Example 3.7: *(See [6], [13], [14] and [19])* The Cesáro matrix of order one $C_1 = (C_{n,k})$ is a non-negative regular matrix.

$$C_{n,k} = \begin{cases} \frac{1}{n} & if \quad 1 \le k \le n \\ 0 & otherwise \end{cases}$$

then $\frac{E(n)}{n}$ is the n^th^ term of the sequence $C_1\mathcal{X}_E$ hence

$$\delta(E) = \lim\,(C_1.\mathcal{X}_E)_n.$$

The function $\delta(E)$ satisfies the four axioms of the density i.e. (d.1) to (d.4).

The Cesáro matrix of order one is a non-negative regular matrix and we obtained a density function $\delta(E)$ from it.

According to Kolk [9],one can extend this idea to any non-negative regular matrix. For every non-negative regular matrix we obtain a different density function.

Definition 3.5: *(See [3] and [4])* Let A be a non-negative regular matrix then δ_A is defined by

$$\delta_A(E) = \lim_{n\to\infty} (A.\mathcal{X}_E)_n$$

$\delta_A$satisfies (d.1) to (d.4) so it is a density function. And moreover

$$\bar{\delta}_A(E) = \lim_{n\to\infty} \sup(A.\mathcal{X}_E)_n.$$

Example 3.8: Let A be a non-negative regular matrix such that

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

And let $k = \{2\boldsymbol{N}\}$ then the A-density of the set k is defined as

$$\delta_A(2\boldsymbol{N}) = \lim_{n\to\infty} (A\mathcal{X}_{2\boldsymbol{N}})$$

$$= \lim_{n\to\infty} (\tfrac{1}{2})_n = \tfrac{1}{2}.$$

Example 3.9: Let B be a non-negative regular matrix such that

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . & . \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

and let $k = \{2\boldsymbol{N} + \boldsymbol{1}\}$ then the A-density of the set k is defined as

$$\delta_A(2\boldsymbol{N} + 1) = \lim_{n\to\infty} (B\mathcal{X}_{2\boldsymbol{N}+1})$$

$$= \lim_{n\to\infty} (1)_n = 1.$$

Definition 3.6: *(See [21])* A lacunary sequence $\theta = \{k_r\}$, is an increasing integer sequence such that $k_0 = 0$, and $h_r = k_r - k_r - 1 \rightarrow \infty$ as $r \rightarrow \infty$. In this case $I_r = (k_{r-1}, k_r]$.

Example 3.10: Let $\theta = \{k_r\} = 2^r - 1$ be an increasing sequence then,

$k_r = 2^r - 1, \quad k_0 = 2^0 - 1 = 1 - 1 = 0$

and

$h_r = k_r - k_{r-1} = (2^r - 1) - (2^{r-1} - 1) = 2^r - 1 - 2^{r-1} + 1$

$$= 2^r - 2^{r-1} = 2^{r-1}(2 - 1) = 2^{r-1}$$

since $h_r \longrightarrow \infty$ as $r \longrightarrow \infty$,

example satisfies the conditions of a lacunary sequence mentioned above in the

definition. Hence $\theta = 2^r - 1$

is a lacunary sequence.

Example 3.11: Let $\theta = \{k_r\} = r! - 1$be an increasing sequence then

$k_r = r! - 1, \quad k_0 = 0! - 1 = 0$

and

$h_r = k_r - k_{r-1} = (r! - 1) - [(r - 1)! - 1] = r! - 1 - (r - 1)! + 1$

$= r! - (r - 1)! = (r - 1)!\,.\,(r - 1)$

since $h_r \longrightarrow \infty$ as $r \longrightarrow \infty$,

$$\theta = r! - 1$$

is a lacunary sequence.

Definition 3.7: *(See [21])* Let θ be a lacunary sequence, then

$$C_\theta(r,k) = \begin{cases} \dfrac{1}{h_r} & if \quad k \in I_r \\ 0 & otherwise \end{cases}$$

is a non-negative regular matrix. Therefore

$$\delta_{C_\theta}(K) \lim_{r\to\infty} (C_\theta \mathcal{X}_k)_r = \lim_{r\to\infty} \sum_{k\in K\cap I_r} \frac{1}{h_r}$$

is a density function.

Example 3.12: Let $\theta = \{k_r\} = \{2^r - 1\}$ be a lacunary sequence, then the characteristic function of the set $K = \{k \in N : k = 2^r \text{ for some } r\}$ is defined as

$$\mathcal{X}_K(k) = \begin{cases} 1 & if \quad k = 2^r \\ 0 & if \quad K \neq 2^r \end{cases}$$

where $I_r = (k_{r-1}, k_r] = (2^{r-1} - 1, 2^r - 1]$. We can define $\delta_{C_\theta}(K)$for K, as

$$\delta_\theta(K) = \lim_{r\to\infty} \frac{\left|\left\{k \in \left(2^{r-1} - 1, 2^r - 1\right] : k \in K\right\}\right|}{h_r} = \lim_{r\to\infty} \frac{\left|\left\{k \in \left(2^{r-1} - 1,\ 2^r - 1\right] : k \in K\right\}\right|}{2^r - 2^{r-1}}$$

$$= \lim_{r\to\infty} \frac{\left|\left\{k \in \left(2^{r-1} - 1,\ 2^r - 1\right] : k = 2^r \ for\, some\, r\right\}\right|}{2^r - 2^{r-1}}$$

$$= \lim_{r\to\infty} \frac{1}{2^r - 2^{r-1}} = 0.$$

Definition 3.8: *(See [21] and [22])* A λ-sequence is a sequence (λ_r) of the positive, non-decreasing numbers, such that,

i) $\lambda_r \to \infty$, as $r \to \infty$, $\lambda_1 = 1$

ii) $\lambda_{r+1} \leq \lambda_r + 1$

iii) $M_r = [r - \lambda_r + 1, r]$.

Example 3.13: Let$\lambda_r = r$ be a non-decreasing sequence then,

(i) $\lambda_1 = 1$ and $\lambda_r \to \infty$ as $r \to \infty$,

(ii) $\lambda_{r+1} = r + 1 = \lambda_r + 1$

Since $\lambda_r = r$ satisfies the above mentioned conditions of a lambda sequence, it is lambda sequence.

Example 3.14: Let $\lambda_r = \left|[\sqrt{r}]\right|$ be a non-decreasing sequence then,

(i) $\lambda_r \to \infty$ as $r \to \infty$

$\lambda_1 = \left|[\sqrt{1}]\right| = 1$

(ii) $\lambda_{r+1} = \left|[\sqrt{r+1}\right|] \leq \left|[\sqrt{r}]\right| + 1 = \lambda_r + 1$is a lambda sequence.

Definition 3.9: Letλ_r be a lambda sequence then

$$A_\lambda = \begin{cases} \dfrac{1}{\lambda_r} & if \quad r \in M_r \\ 0 & otherwise \end{cases}$$

is a non-negative regular matrix. Therefore,

$$\delta_{A_k}(K) = \lim_{r\to\infty} (A_\lambda \mathcal{X}_k)_r$$

$$= \lim_{r\to\infty} \sum_{k \in K \cap I_r} \frac{1}{\lambda_r}$$

is a density function.

Example 3.15: Let $\lambda_r = r$ be a $\lambda -$ sequence and $K = \{k \in K : k = m^2 \ \ for\, some \ \ m\}$

then

$$\mathcal{X}_K = \begin{cases} 1 & if \quad k = m^2 \\ 0 & if \quad k \neq m^2 \end{cases}$$

and $M_r = [r - \lambda_r + 1, r] = [1, r]$.We can define $\delta_\lambda(K)$for K, as

$$\delta_\lambda(K) = \lim_{r\to\infty} \frac{|\{k \in [1,r] : k \in K\}|}{\lambda_r} = \lim_{r\to\infty} \frac{|\{k \in [1,r] : k = m^2,\ for\ some\ m\}|}{r}$$

$$\leq \lim_{r\to\infty} \frac{\sqrt{r}}{r} = 0\ .$$

Therefore,

$$\delta_\lambda(K) = 0.$$

Lemma 3.4: *(See [3])* Let A be a non-negative regular matrix, K be a subset of$N$ such that $\delta_A(K)$exists then,for any submatrix A_μ of A,

$\delta_A(K) = \delta_{A_\mu}(K)$.

Proof. Recall that, $\delta_A(K)$ is defined as

$$\delta_A(K) = \lim_{n\to\infty} (A\mathcal{X}_K),$$

where $(A\mathcal{X}_K)$ is a convergent sequence. Let the sequence $(A\mathcal{X}_K)$is convergent to L. Now assume that $A_\mu$is a submatrix of A then,

$$\delta_{A_\mu}(K) = \lim_{n\to\infty} (A_\mu \mathcal{X}_K)$$

where $(A_\mu \mathcal{X}_K)$is a subsequence of $(A\mathcal{X}_K)$since every subsequence of a convergent sequence is convergent and converges to the same limit we get,

$$\lim_{n\to\infty} (A\mathcal{X}_K) = L = \lim_{n\to\infty} (A_\mu \mathcal{X}_K).$$

Hence

$$\delta_A(K) = \delta_{A_\mu}(K).$$

Example 3.16 Let A be a non-negative regular matrix such that

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and consider the following submatrix of A,

$$A_\mu = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \end{pmatrix}$$

then for the subset$K = \{2\boldsymbol{N}\}$

$$\delta_A(2\boldsymbol{N}) = \delta_{A_\mu}(2\boldsymbol{N}) = \tfrac{1}{2}.$$

Chapter 4

A-STATISTICAL CONVERGENCE

In the previous chapters, we have discussed the density functions and non-negative regular matrices in detail. We studied the natural density, which plays an important role for statistical convergence and it can be obtained from the Cesáro matrix of order one. Moreover, we also studied that replacing Cesáro matrix, with A, which is a non-negative regular matrix, then we get the idea of A-density. With the help of A-density, we can define A-statistical convergence. A-statistical convergence is used by many researchers in their studies *(See [5] to [7], [15] to [20])*.

Definition 4.1: *(See [3])* Let $A = (a_{n_k})$ be a non-negative regular matrix. Then the A-density

$$\delta_A : 2^{\boldsymbol{N}} \rightarrow [0,1]$$

with

$$\delta_A(K) = \lim_{n\rightarrow\infty} (A\mathcal{X}_k)_n = \lim_{n\rightarrow\infty} \sum_{k \epsilon K} a_{nk}$$

where $K \subseteq \boldsymbol{N}$ and $\mathcal{X}_K$ is the characteristic function of the set K defined as

$$\mathcal{X}_K(k) = \begin{cases} 0 & if\ k \notin K \\ 1 & if\ k \in K\ . \end{cases}$$

In this case we say that K has A-density, provided that limit exists.

Example 4.1: Let A be an infinite regular matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & . & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

and let $K = \{2\boldsymbol{N}\}$. Then the A-density of the set K is defined as

$$\delta_A(2\boldsymbol{N}) = \lim_{n\to\infty}(A\mathcal{X}_{2\boldsymbol{N}}).$$

It is obvious that, $\chi_{2N} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$ and,

$$A\chi_{2N} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0\cdots \\ . & . & . & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \vdots \\ \vdots \end{bmatrix},$$

therefore,

$$\delta_A(2\boldsymbol{N}) = \lim_{n\to\infty}(A\mathcal{X}_{2\boldsymbol{N}}) = 1/2.$$

Example 4.2: Let A be an infinite regular matrix.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . & . & . \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

and let $K = \{2\boldsymbol{N} + 1\}$. Then A-density of the set K is defined as

$$\delta_A(2\boldsymbol{N} + 1) = \lim_{n\to\infty} (A\mathcal{X}_{2\boldsymbol{N}+1}).$$

$$\chi_{2N+1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \text{ and,}$$

$$A\chi_{2N+1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2}\cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2}\cdots \\ . & . & . & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix},$$

therefore,

$$\delta_A(2\boldsymbol{N} + \boldsymbol{1}) = \lim_{n\to\infty} (A\mathcal{X}_{2\boldsymbol{N}+\boldsymbol{1}}) = 1.$$

Remark: If K is a finite subset of **N**, then for any non-negative, regular matrix A then

$$\delta_A(K) = 0.$$

Proof: Assume that $K = \{k_1, k_2, \ldots\ldots k_m\}$ and

$$\delta_A(K) = \lim_{n\to\infty} \sum_{k\in K} a_{n_k} = \lim_{n\to\infty} (a_{n_{k_1}} + a_{n_{k_2}} + \cdots + a_{n_{k_m}}).$$

Since A is a non-negative and regular we have,

$$\lim_{n\to\infty} a_{n_{k_i}} = 0, for\ i = 1,2, \dots, n.$$

Thus,

$$\delta_A(K) = 0.$$

Lemma 4.1: *(See [3])* We have the following relation for an existing $\delta_A(K)$,

$$\delta_A(K) = 1 - \delta_A(\boldsymbol{N} \backslash K).$$

Definition 4.2: A sequence y_k is said to be A-statistically convergent to L if $\forall\, \varepsilon > 0$, the set $K(\varepsilon) = \{k \in \boldsymbol{N} : |y_k - L| \geq \varepsilon\}$ has A-density zero. We write it as

$$St_A - lim\, y_k = L$$

i.e

$$St_A - lim\, y_k = L \Leftrightarrow \delta_A(\{k \in \boldsymbol{N} : |y_k - L| \geq \varepsilon\}) = 0.$$

Example 4.3: Let A be the following non-negative regular matrix

$$A = (a_{n_k}) = \begin{pmatrix} 1 & 0 & 0 & . & . & . & . & . & . \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . & . \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & . & . & . \\ \frac{1}{n} & 0 & \frac{1}{n} & 0 & \frac{1}{n} & 0 & \frac{1}{n} & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

and the sequence y_k is given as

$$y_k = \begin{cases} 1 & if\ k \in 2\boldsymbol{N}+1 \\ 0 & if\ k \in 2\boldsymbol{N} \end{cases}.$$

If $\varepsilon > 1$, then

$$\delta_A\big(K(\varepsilon)\big) = \delta_A(\{k \in \boldsymbol{N}: |y_k - L| \geq \varepsilon\}) = \delta_A(\emptyset) = 0.$$

On the other hand, if $0 \leq \varepsilon \leq 1$, then

$$\delta_A\big(K(\varepsilon)\big) = \delta_A(\{k \in \boldsymbol{N}: |y_k - L| \geq \varepsilon\})$$

$$= \delta_A(2\boldsymbol{N}) = \lim_{n\to\infty} \sum_{k\in 2\boldsymbol{N}} a_{n_k} = 0.$$

Therefore

$$St_A - lim\, y_k = 1.$$

Lemma 4.2: Every convergent sequence is A-statistically convergent.

Proof: Assume that y_k converges to L in the ordinary sense, then $\forall\ \varepsilon > 0$

$$\delta_A(K_\varepsilon) = \delta_A(\{k \in \boldsymbol{N}: |y_k - L| \geq \varepsilon\}) = 0$$

since K_ε is finite for all $\varepsilon > 0$.

Hence y_k is also A-statistically convergent to L.

Converse implication of the above lemma does not hold. Moreover, an A-statistically convergent sequence need not be bounded. These case will be illustrated in the following examples.

Example 4.4: Consider the following sequence

$$y_k = \begin{cases} 1 & if\ k \in 2\boldsymbol{N}+1 \\ k & if\ k \in 2\boldsymbol{N} \end{cases}.$$

and the matrix

$$A = (a_{n_k}) = \begin{pmatrix} 1 & 0 & 0 & . & . & . & . & . & . \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . & . \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & . & . & . \\ \frac{1}{n} & 0 & \frac{1}{n} & 0 & \frac{1}{n} & 0 & \frac{1}{n} & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

Following the same lines as we used in the above example, we see that

$$St_A - lim\ y_k = 1,$$

but the sequence y_k is not bounded.

Example 4.5: Consider the following sequence

$$y_k = \begin{cases} 1 & if\ k \in 2\boldsymbol{N}+1 \\ 0 & if\ k \in 2\boldsymbol{N} \end{cases}.$$

and the matrix,

$$A = (a_{nk}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

For all $\varepsilon > 1$ and $L = 1$ or $\mathrm{L} = 0$,

$$\delta_A\big(K(\varepsilon)\big) = \delta_A(\{k \in \boldsymbol{N}: |y_k - L| \geq \varepsilon\}) = \delta_A(\emptyset) = 0.$$

On the other hand, if $0 \le \varepsilon \le 1$, and $L = 1$, then

$$\delta_A\big(K(\varepsilon)\big) = \delta_A(\{k \in \boldsymbol{N}: |y_k - L| \ge \varepsilon\})$$

$$\le \delta_A(2\boldsymbol{N}) = \lim_{n\to\infty} \sum_{k\in 2\boldsymbol{N}} a_{n_k} = 1/2.$$

If $0 \le \varepsilon \le 1$, and $L = 0$, then

$$\delta_A\big(K(\varepsilon)\big) = \delta_A(\{k \in \boldsymbol{N}: |y_k - L| \ge \varepsilon\})$$

$$\le \delta_A(2\boldsymbol{N} + \boldsymbol{1}) = \lim_{n\to\infty} \sum_{k\in(2\boldsymbol{N}+\boldsymbol{1})} a_{n_k} = 1/2.$$

Therefore

$$St_A - lim\, y_k$$

does not exists.

Theorem 4.1: *(See [1])* Let A be a non-negative, infinite, regular matrix and let y_k and z_k be two sequences, if

$$St_A - \lim_{k\to\infty} y_k = L$$

and

$$St_A - \lim_{k\to\infty} z_k = Q$$

then

i. $St_A - \lim(y_k + z_k) = L + Q$

ii. $St_A - \lim(\lambda\, y_k) = \lambda L$

iii. $St_A - \lim(y_k z_k) = LQ$

iv. $St_A - \lim(\frac{y_k}{z_k}) = \frac{L}{Q}$, provided that $Q \neq 0$ and $\forall\, k,\ z_k \neq 0$.

Proof:

(i) By the assumption

$$\forall\, \varepsilon > 0,\ \ \delta_A(\{k: |y_k - L| \geq \varepsilon\}) = 0$$

and

$$\delta_A(\{k: |z_k - Q| \geq \varepsilon\}) = 0$$

we need to show that

$$\delta_A(\{k: |(y_k + z_k) - (L + Q)| \geq \varepsilon\}) = 0.$$

We know that,

$$\{k: |(y_k + z_k) - (L + Q)| \geq \varepsilon\} \subset \left\{k: |y_k - L| \geq \frac{\varepsilon}{2}\right\} \cup \left\{k: |z_k - Q| \geq \frac{\varepsilon}{2}\right\}$$

and

$$\delta_A\{k: |(y_k + z_k) - (L + Q)| \geq \varepsilon\} \leq \delta_A(\left\{k: |y_k - L| \geq \frac{\varepsilon}{2}\right\} + \delta_A(\left\{k: |z_k - Q| \geq \frac{\varepsilon}{2}\right\}$$

$$\leq 0 + 0 = 0.$$

So $st_A - \lim(y_k + z_k) = L + Q$.

(ii) The case $\lambda = 0$ is obvious. Let us assume that $\lambda \neq 0$, by the assumption

$$\forall\, \varepsilon > 0,\ \ \delta_A(\{k: |y_k - L| \geq \varepsilon\}) = 0.$$

We need to show that

$$\delta_A(\{k: |\lambda y_k - \lambda L| \geq \varepsilon\}) = 0.$$

Now we have

$$|\lambda y_k - \lambda L| = |\lambda(y_k - L)|$$

$$\leq |\lambda||y_k - L|$$

that is

$$|\lambda y_k - \lambda L| \leq |\lambda||y_k - L|.$$

Now

$$\delta_A(\{k: |\lambda y_k - \lambda L|\}) \leq |\lambda|\delta_A(\{k: |y_k - L|\}) = 0$$

hence

$$st_A - lim\lambda y_k = \lambda L.$$

iii) $St_A - \lim_{k\to\infty} y_k = L$ implies that there exists a subset B of $\boldsymbol{N}$ such that

$$\delta_A\,(B) = 1$$

and $\lim_{k\to\infty} y_k = L$ on B in the ordinary sense. Similarly, there exists a subset D of $\boldsymbol{N}$ such that

$$\delta_A\,(D) = 1$$

and $\lim_{k\to\infty} z_k = Q$ on D in the ordinary sense.

Now

$$\delta_A\,(D) = \ \delta_A\,(B) = 1$$

implies that

$$\delta_A\,(B \cap D) = 1$$

and for $(B \cap D)$ the sequence $(y_k z_k)$ converges to LQ in the ordinary sense.
Hence

$$St_A - \lim(y_k z_k) = LQ$$

iv) For

$$St_A - \lim_{k\to\infty} y_k = L$$

there exists a subset M of $\boldsymbol{N}$ such that

$$\delta_A(M) = 1$$

and on the set M the sequence y_k converges to L in the ordinary sense.

Similarly, for

$$St_A - \lim_{k\to\infty} z_k = Q$$

there exists P, subset of $\boldsymbol{N}$ such that

$$\delta_A(P) = 1$$

and on the set P the sequence $z_k$converges to Q in the ordinary sense.

But,

$$\delta_A(M) = \delta_A(P) = 1$$

implies that

$$\delta_A(M \cap P) = 1,$$

and on $(M \cap P)$ the sequence $(\frac{y_k}{z_k})$ converges to $\frac{L}{Q}$ in the ordinary sense.

Hence

$$St_A - \lim(\frac{y_k}{z_k}) = \frac{L}{Q}.$$

Lemma 4.3: *(See [3])* Let A be a non-negative, infinite, regular matrix and let y_k be a sequences, if $St_A - \lim_{k\to\infty} y_k = L$, then for any sequence of positive integers $\mu = \mu(n)$

,

$$St_{A(\mu)} - \lim_{k\to\infty} y_k = L,$$

where $A(\mu)$ is a submatrix of A.

Proof: By the definition of A-statistical convergence if

$$st_A - \lim_{k\to\infty} y_k = L$$

then

$$\delta_A(\{k: |y_k - L| \geq \varepsilon\}) = 0$$

where A-density is defined for any set K as

$$\delta_A(K) = \lim_{n\to\infty} (A\mathcal{X}_K)_n = 0$$

Now $(A\mathcal{X}_K)_n$is a sequence which is convergent to zero in the ordinary sense.

As we know that any subsequence of a convergent sequence is also convergent, hence $\left(A_\mu \mathcal{X}_K\right)_n$is a convergent subsequence of $(A\mathcal{X}_K)_n$. So

$$\delta_{A_\mu}(K) = \delta_A(K) = \lim_{n\to\infty} \left(A_\mu \mathcal{X}_K\right)_n = 0$$

and

$$st_{A_\mu} - \lim_{k\to\infty} y_k = L.$$

Definition 4.3: *(See [5], [6], [7] and [15])* A sequence y_k is said to be A-statistically divergent to $-\infty$ and denoted by

$$St_A - \lim_{k\to\infty} y_k = -\infty,$$

if for any $P \in \boldsymbol{R}$

$$\delta_A(\{k \in \boldsymbol{N}: y_k < P\}) = 1$$

Example 4.6: Consider the following matrix

$$A=(a_{n_k})=\begin{pmatrix} 1 & 0 & 0 & . & . & . & . & . & . \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . & . \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & . & . & . \\ \frac{1}{n} & 0 & \frac{1}{n} & 0 & \frac{1}{n} & 0 & \frac{1}{n} & 0 & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

and the sequence

$$y_k = \begin{cases} -k & if\ k \in 2\boldsymbol{N}+1 \\ 3 & if\ k \in 2\boldsymbol{N}. \end{cases}$$

Then, for any $P \in \boldsymbol{R}$,the set

$$\delta_A(\{k \in \boldsymbol{N}: y_k < P\}) = \delta_A(2\boldsymbol{N}+1) = 1.$$

Therefore,

$$St_A - \lim_{k\to\infty} y_k = -\infty.$$

Definition 4.4: *(See [5], [6], [7] and [15])* A sequence y_k is said to be A-statistically divergent to $\infty$and denoted by

$$St_A - \lim_{k\to\infty} y_k = \infty,$$

if for any $Q \in \boldsymbol{R}$

$$\delta_A(\{k \in \boldsymbol{N}: y_k > Q\}) = 1$$

Example 4.7: Consider the following matrix

$$A=\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0\cdots \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and the sequence

$$y_k = \begin{cases} k^2, & k \in 2\boldsymbol{N} \\ 3, & k \in 2\boldsymbol{N} + \boldsymbol{1}. \end{cases}$$

Then, for any $Q \in \boldsymbol{R}$, the set

$$\delta_A(\{k \in \boldsymbol{N}: y_k > Q\}) = \delta_A(2\boldsymbol{N}) = 1.$$

Therefore,

$$St_A - \lim_{k\to\infty} y_k = \infty.$$

Definition 4.5: *(See [5] to [7], [15] to [20])* A sequence $y \coloneqq (y_k)$ is said to be A-statistically bounded if ∃ a positive constant Q

such that

$\delta_A(\{k: |y_k| > Q\}) = 0.$

Example 4.8: Consider a sequence

$$y = (y_k) \coloneqq \begin{cases} k & k = m^3 \; for\ some\ m \\ 0 & otherwise \end{cases}$$

And let A be the Cesaro matrix of order one then for any positive number Q we have,

$$\delta_A(\{k: |y_k| \geq Q\}) = \lim_{m\to\infty} \frac{|\{k \in [1, m^3]: |y_k| \geq Q\}|}{m^3}$$
$$\leq \lim_{m\to\infty} \frac{m}{m^3} = 0$$

which implies that y_k is A-statistically bounded.

Definition 4.6: *(See [5] to [7], [15] to [20])* A sequence $y \coloneqq (y_k)$ is said to be A-statistical monotone increasing if ∃ $Q \subseteq \boldsymbol{N}$ which has A-density one. i.e.

$$\delta_A(Q) = 1$$

such that the sequence $y \coloneqq y_k$ is monotone increasing on Q in the ordinary sense.

Example 4.9: Let $y \coloneqq (y_k)$ be a sequence such that

$$y = (y_k) \coloneqq \begin{cases} k & k \in 2N+1 \\ 0 & otherwise \end{cases}$$

Now for any non-negative regular matrix

$$A = (a_{n_k}) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . & . \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

the set $K = 2\boldsymbol{N} + 1$ has a-density one. i.e.

$$\delta_A(2\boldsymbol{N} + 1) = 1$$

Hence on $K = 2\boldsymbol{N} + 1$ the above sequence $y \coloneqq (y_k) = (1,3,5, \dots)$ is monotone increasing in the ordinary sense.

Definition 4.7: *(See [5] to [7], [15] to [20])* A sequence $y \coloneqq (y_k)$ is said to be A-statistical monotone decreasing, if $\exists\ Q \subseteq \boldsymbol{N}$which has A-density one. i.e.

$$\delta_A(Q) = 1$$

such that the sequence $y \coloneqq y_k$ is monotone decreasing on Q in the ordinary sense.

Example 4.10: Let $y \coloneqq (y_k)$ be a sequence such that

$$y = (y_k) \coloneqq \begin{cases} -k & k \in 2N \\ 1 & otherwise \end{cases}$$

Now for the non-negative regular matrix

$$A=(a_{n_k})=\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . & . & . \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

the set $K = 2\boldsymbol{N}$ has A-density one. i.e.

$$\delta_A(2\boldsymbol{N}) = 1$$

hence on the set $K = 2\boldsymbol{N}$ the sequence $y := (y_k) = (-2, -4, -6, \dots)$ is monotone decreasing in the ordinary sense.

Definition 4.8: *(See [5] to [7], [15] to [20])* If a sequence $y := y_k$ is A-statistically monotone increasing or A-statistically monotone decreasing, then the sequence y_k is called a A-statistically monotone sequence.

Proposition 4.1: *(See [15] to [20])* Let $y := y_k$ be a sequence which is a monotone sequence in the ordinary sense then it is also A-statistical monotone.

Proof: Suppose $y := (y_k)$ be a monotone increasing sequence i.e. $\forall\ k \in \mathbf{N}$

$$y_k \leq y_{k+1}$$

Now choose $Q = \boldsymbol{N}$. As the A-density of $\boldsymbol{N}$ is 1. i.e. $\delta_A(\boldsymbol{N}) = 1$, we see that $y = (y_k)$ is A-statistical monotone increasing.

The above idea can be used to show that if $y := (y_k)$ is a monotone decreasing sequence in the ordinary sense then it is A-statistical monotone decreasing.

Example 4.11: Consider a sequence $y := y_k$ such that

$$y_k = \begin{cases} 1 & if\ k = n^2 \\ k & otherwise \end{cases}, n = 1, 2, \dots..$$

and let A be the Cesáro matrix of order one i.e. $A = C_1$ then $y \coloneqq y_k$ is A-statistical monotone increasing.

Remark: The converse of the above proposition is not true. For instance, in the above example, the sequence $y \coloneqq (y_k)$ is not monotone increasing but it is A-statistical monotone.

Theorem 4.2: *(See [5] to [7], [15] to [20])* Let $y = (y_k)$ be a sequence which is A-statistical monotone increasing or A-statistical monotone decreasing then the sets,

$$\{n \in \boldsymbol{N}: y_{n+1} < y_n\}$$

or

$$\{n \in \boldsymbol{N}: y_{n+1} > y_n\}$$

has A-density zero.

Proof: Let us suppose that $y = (y_k)$ is A- statistical monotone increasing, which implies that $\exists\ Q \subseteq \boldsymbol{N}$ with $\delta_A(Q) = 1$ such that (y_k) is monotone increasing on Q which means

$$y_k \le y_{k+1,} \forall k \in Q$$

hence

$$\{n \in \boldsymbol{N}: y_{n+1} < y_n\} \quad \subset \boldsymbol{N} - Q$$

Taking A-density of the above sets, we get this inequality

$$\delta_A(\{n \in \boldsymbol{N}: y_{n+1} < y_n\}) \le \delta_A(\boldsymbol{N} - H) = 0$$

which satisfies the theorem.

Remark: In general the converse of the above theorem is not true.

Theorem 4.3: *(See [5] to [7], [15] to [20])* let $y = (y_k)$ be a bounded and A-statistical monotone sequence, then $y = (y_k)$ is A-statistically convergent.

Proof: By the definition there exists a subset Q of $\boldsymbol{N}$ with density one such that $y = (y_k)$ is monotone increasing on Q. Let q_n be the sequence of elements of Q and let us suppose that q_n is a monotone increasing sequence of natural numbers. Then for the sequence $y = (y_k)$, (y_{q_n}) is the monotone increasing subsequence. Since $y := (y_k)$ is bounded so (y_{q_n}) is also bounded. Hence (y_{q_n}) converges to $sup\ y_{q_n}$. That means $\forall\ \varepsilon > 0$ there exists a positive number

$$q_{n^\circ} = q_{n^\circ}(\varepsilon) \in \boldsymbol{N}$$

such that

$$\left|y_{q_n} - sup\ y_{q_n}\right| < \varepsilon$$

valid for all $q_n > q_{n^\circ}$

Since ordinary sequence implies A-statistical convergence. So

$$y_{q_n} \longrightarrow sup\ y_{q_n}(A - st)$$

We see that

$$Q(n) := \{q \leq n: \left|y_q - sup\ y_q\right| \geq \varepsilon\}$$

$$= \{q \leq n: q \neq q_n\ \ and\ \left|y_q - sup\ y_q\right| \geq \varepsilon\}$$

$$\cup\ \{q \leq n: q = q_n\ \ and\ \left|y_q - sup\ y_q\right| \geq \varepsilon\}$$

$$= Q^1(n) \cup\ Q^2(n)$$

Since $Q^1(n)$ is a subset of $\boldsymbol{N} - Q$ and $y_{q_n} \longrightarrow L \quad (A - st)$, so the $\delta\left(Q^1(n)\right) = 0$ and $\left(Q^2(n)\right) = 0$. Hence $y_k \longrightarrow L\ \ (A - st)$.

Remark: Generally the boundedness of A-statistical monotone sequence is not necessary for the A-statistical convergence, it is sufficient though. For example, consider the Cesáro matrix of order 1 and a sequence $y = (y_k)$ as

$$y_k = \begin{cases} n & if\ k = n^2 \\ \dfrac{1}{k} & if\ k \neq n^2 \end{cases}$$

where $k \in \boldsymbol{N}$**.**

It is obvious that $y = (y_k)$ is unbounded, but it is statistically monotone decreasing and statistically convergent to zero.

Theorem 4.4: *(See [5] to [7], [15] to [20])* Let $y = (y_k)$ be an A-statistical monotone sequence then $y = (y_k)$ is A-statistical convergent if and only if it is A-statistically bounded.

***Proof* :** Let $y = (y_k)$ be an A-statistical monotone sequence which is A-statistically bounded then we need to show that it is A-statistically convergent.

Since $y = (y_k)$ is A-statistically montone sequence there exists a subset K_1 of $\boldsymbol{N}$ such that

$$\delta_A(K_1) = 1$$

and on the set K_1 the sequence $y = (y_k)$ is monotone increasing in the ordinary sense.

Now there exists a subset K_2 of $\boldsymbol{N}$ such that

$$\delta_A(K_2) = 1$$

and on K_2 the sequence $y = (y_k)$ is bounded in the ordinary sense.

Since A-densities of $K_1$and K_2 are both one,

$$\delta_A(K_1 \cap K_2\ ) = 1.$$

Hence on $(K_1 \cap K_2)$ the sequence $y = (y_k)$ is both monotone increasing and bounded in the ordinary sense. This implies that on $(K_1 \cap K_2)$ the sequence $y = (y_k)$ is convergent to any number L in the ordinary sense. Hence $y = (y_k)$ is A-statistically convergent to L.

The converse of the theorem can be proved by using similar methods used above.

Definition 4.9: *(See [1], [4] and [16])* A sequence $y = (y_k)$ is statistically convergent to L (any number) if for all $\varepsilon > 0$ the natural density of the set

$$k_\varepsilon := \{k \in \boldsymbol{N} : |y_k - L| \geq \varepsilon\}$$

is zero. That is

$$\delta(\{k : |y_k - L| \geq \varepsilon\}) = 0.$$

Example 4.12: Let y_k be a sequence defined as

$$y_k = \begin{cases} 1 & if\ k = n^2 \\ 0 & if\ k \neq n^2 \end{cases}$$

then

$$St - \lim_{k \to \infty} y_k = 0,$$

because,

$$\delta(\{k : |y_k - L| \geq \varepsilon\}) = \delta(\{k^2 : k \in \boldsymbol{N}\}) = 0.$$

Recall that in Definition 3.8, we have defined a density function δ_{A_λ}, on the basis of which we will define λ-statistical convergence.

Definition4.10: *(See [22])* A sequence $y \coloneqq y_k$ is λ-statistically convergent to L, if ∀ $\varepsilon > 0$

$$\lim_{r\to\infty}\frac{1}{\lambda_r}\{|k\in M_r:|y_k-L|\geq\varepsilon|\}=0.$$

where $M_r=[r-\lambda_r+1,r]$.

This is denoted by

$$St_\lambda-\lim y_k=L$$

We have studied previously in the chapter that if we choose A to be the Cesaro matrix of order one, then A-density is reduced to the natural density and the A-statistical convergence is reduced to the statistical convergence.

Recall the density function δ_{c_θ}, on the basis of this density function, we can define a special kind of convergence called lacunary statistical convergence.

Definition4.9: *(See [21])* A sequence y_k is said to be Lacunary statistical convergent to L, if for all $\varepsilon>0$

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:|y_k-L|\geq\varepsilon\}|=0.$$

where $I_r=(k_{r-1},k_r]$.

This is denoted by

$$St_\theta-\lim y_k=L.$$

Similarly if $A=A_\theta$, where θ is any lacunary sequence then we can define $\delta_{A_\theta}=\delta_\theta$ and then A-statistical convergence reduces to lacunary statistical convergence.

Remark 4.5: The ordinary convergence imply lacunary statistical convergence, i.e $\lim y=L\Rightarrow St_\theta-\lim y=L.$ Moreover if $A=A_\lambda$, then $\delta_{A_\lambda}=\delta_\lambda$ and A-statistical convergence is reduced to λ-statistical convergence.

Remark: The ordinary convergence imply λ-statistical convergence, i.e $\lim y=L\Rightarrow St_\lambda-\lim y=L$. If $\lambda_r=r$, then λ-statistical convergence is equivalent to statistical convergence.

Since A-statistical convergence includes, statistical convergence, λ-statistical convergence and lacunary statistical convergence, hence we can say that these new type of convergences are special cases of A-statistical convergence.

REFERENCES

[1] Balcerzak, K. D., and Komisarski A. (2007). Statistical convergence and ideal convergence for sequence of functions, *J. Math. Anal. Appl.* 328 715-729.

[2] Fast H. Sur la convergence statistique. (1951). *Colloq. Math*. 2 241- 244.

[3] Freedman A. R. and J. J. Sember. (1981). Densities and summability, *Pacific J. Math.* 95 293-305.

[4] Fridy J. A. On statistical convergence. (1985). *Analysis* 5 301-313.

[5] Fridy J. A. (1993). Statistical limit points, *Proc. Amer. Math. Soc.* 118 1187-1192.

[6] Fridy J. A., and H. I. Miller. (1991). A matrix characterization of statistical convergence, *Analysis* 11 59-66.

[7] Fridy J. A. and Orhan C. (1997). Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc*. 125 3625-3631.

[8] Gadjiev A. D. and Orhan C. (2002). Some approximation theorems via statistical convergence. *Rocky Mountain J. Math.* 32 129-138.

[9] Kolk E. (1991). The statistical convergence in Banach spaces. *Acta Et Commentationes Tartuensis* 928 41-52.

[10] Steinhaus H. (1951). Sur la convergence ordinarie et la convergence asymptotique. *Colloq. Math.* 2 73-74.

[11] Stoll M. (2004). *Introduction to Real Analysis*. 2nd Ed.

[12] Tripathy B.C. (1998). On statistically convergent sequences. *Bull. Cal. Math. Soc.* 90 259-262.

[13] Agnew R. P. (1932). On deferred Cesàro mean *Ann. of Math*. 33413.421.

[14] Armitage D. H. and Maddox I. J. (1989). A new type of Cesàro mean, *Analysis*, 9195.204.

[15] Buck R. C. (1953). Generalized asymptotic density, Amer. J. Math., 75335.346.

[16] J. A. Fridy. (1985). On statistical convergence. *Analysis*, 301. 313.

[17] Fridy J. A. and Miller H. I. (1991). A matrix characterization of statistical convergence. *Analysis*, 1159 66.

[18] Kaya E. Küçükaslan M. and Wagner R. (2013). On statistical convergence and statistical monotonicity. *Annales Univ. Sci. Budapest. Sect. Comp*, 39257.270.

[19] Osikiewicz J. A. (1997). Summability of Matrix Submethods and Spliced Sequences. *PH.D. Thesis* August 90 pp. which University, Kent State University.

[20] Salat T. (2001). On statistically convergent sequences of real numbers, *Math. Slovaca.*

[21] Fridy J.A. and Orhan C. (1993) Lacunary statistical convergence, *Pacific J. Math.* 160.

[22] O. Duman and Orhan C. (2004). μ-statistically convergent function sequences. *Czechoslovak Mathematical Journal.*