Classical Orthogonal Polynomials and Differential Operators

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We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

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ABSTRACT

In this thesis we introduce the concept of classical orthogonal polynomials which are Hermite, Laguerre and Jacobi polynomials. We first provide the necessary overview on special functions. Then we give several properties of orthogonal polynomials in Chapter 2. In Chapter 3, we start to classical orthogonal polynomials firstly we obtain the orthogonality relation, Rodrigues formulas and we give the norm of the classical orthogonal polynomials. Finally we divide the examples of classical orthogonal polynomials into three chapters and for each of them we give the weight function, interval of the orthogonality, second order differential equation, hypergeometric representation.

Keywords: classical orthogonal polynomials, hypergeometric function, second order differential equation, Rodrigues formula.

ÖZ

Bu tezde klasik ortogonal polinomlar olan Hermite, Laguerre ve Jacobi polinomları tanımlanmıştır. İlk olarak özel fonksiyonlar hakkında ön bilgi verilmiştir. İkinci bölümde ortogonal polinomların bazı özellikleri çalışılmıştır. Üçüncü bölümde klasik ortogonal polinomlar tanımlanmış ve ilk olarak ortogonallik ilişkisi, Rodrigues formülü verilmiş ve klasik ortogonal polinomlar için norm hesabı yapılmıştır. Son olarak klasik ortogonal polinom örnekleri 3 bölüme ayrılmış ve herbiri için ağırlık fonksiyonları, ortogonallik aralığı, ikinci dereceden diferansiyel denklemi ve hipergeometrik gösterimi verilmiştir.

Anahtar Kelimeler: Klasik ortogonal polinomlar, hipergeometrik fonksiyon, ikinci dereceden diferansiyel denklem, Rodrigues formülü.

To My Husband and Family

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Chapter 1

INTRODUCTION

1.1 Mathematical Background

Definition 1.1 (Inner Product Space)

Let X be a vector space. The scalar valued function $<,>: X * X \to K$ where $K = R \; or \; C$ is called the inner product space, if it satisfies the following conditions and denoted by $(X, <>)$.

1) $\forall \, x, y, z \in X \quad < x + y, z >=< x, z > + < y, z >$,

2) $\forall \, x, y \in X \;\; and \; k \in K \;\; < kx, y >= k < x, y >$,

3) $\forall \, x, y \in X \;\; < x, y > = \overline{< y, x >}$,

4) $\forall x \in X \;\; < x, x > \geq 0 \quad or \quad < x, x >= 0 \Leftrightarrow x = 0.$

Example 1.2. $C[a, b]$ being the space of all real-valued continuous functions on a closed interval $[a, b]$ is an inner product space, whose inner product is defined by $< f, g >= \int_a^b f(x). h(x) dx$ where $f, g \in C[a, b]$.

Definition 1.3 $(X, <>)$ be an inner product space and x and y be any elements of X. We can say that x and y are orthogonal to each other if and only if $< x, y >= 0$.

Example 1.4 Let $f(x)$and $h(x)$ be two functions defined on $[a, b]$ .We can say that they are orthogonal on an interval [a,b] if their inner product is zero

$$\int_a^b f(x). h(x) dx = 0.$$

Definition1.5 (Hypergeometric Equation)

The second order differential equation

$A(x)p''(x) + B(x)p'(x) + \alpha p(x) = 0,$

is called hypergeometric equation where $A(x)$ has degree at most 2, $B(x)$ has degree at most 1 and α is a constant.

Theorem1.6 All the derivatives of the solutions of hypergeometric equation satisfy a hypergeometric equation.

Proof. For proof of this theorem, we first differentiate both sides with respect to the variable x;

$$A'(x)p''(x) + A(x)p'''(x) + B'(x)p'(x) + B(x)p''(x) + \alpha p'(x) = 0$$

$$A(x)p'''(x) + [A'(x) + B(x)]p''(x) + [B'(x) + \alpha]p'(x) = 0. \quad (1.1)$$

Now let us set $v_1(x) = p'(x)$ and substitute into the equation (1.1),

$A(x)v_1''(x)+B_1(x)v_1'(x) + \mu_1 v_1(x) = 0.$

This equation form a hypergeometric equation since $A(x)$ has degree at most 2 and $B_1(x)$ has degree 1, where $B_1(x) = A'(x) + B(x)$ and $\mu_1 = B'(x) + \alpha$.

If we differentiate hypergeometric equation m times again we obtain the generalized hypergeometric equation which has the following form;

$A(x)v_m''(x) + B_m(x)v_m'(x) + \mu_m v_m(x) = 0$, where $v_m(x) = p^{(m)}(x)$,

$B_m(x) = B(x) + mA'(x)$ and $\mu_m = \alpha + m\, B(x) + \frac{1}{2}m(m-1)A''(x).$

In Chapter 2 we are going to construct the polynomial solutions of hypergeometric equation corresponding to given α.

Definition1.7 When μ_m= 0 generalized hypergeometric equation has the particular solution $v_m(x) = constant.$

Since $v_m(x) = p^{(m)}(x)$, when $\alpha = \alpha_m$= $-m\,B'(x) - \frac{1}{2}m(m-1)A''(x),$

The equation of hypergeometric type has a particular solution of the form $p(x) = p_m(x)$ which is a polynomial of degree m. We shall call such solutions, polynomials of hypergeometric type.

Definition1.8 (Gamma Function) The Gamma function of y is defined as

$$\Gamma(y) = \int_0^\infty e^{-t}t^{y-1}dt \ , \qquad \forall y \in \mathbb{R}\{\dots,-2,-1,0\}. \qquad (1.2)$$

Some Properties of Gamma function

1. $\Gamma(y+1) = y\Gamma(y)$,
2. $\Gamma(y+1) = y!$,
3. $\Gamma(y+n) = (y)_n\Gamma(y)$,
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Theorem1.9 The following result

$$\int_{-\infty}^{\infty} e^{-x^2}d_x = \sqrt{\pi}$$

holds true.

Proof:

Since

$$\int_{-\infty}^{\infty} e^{-x^2}d_x = 2\int_0^\infty e^{-x^2}d_x$$

We get by letting $x^2 = t$ that

$$2\int_0^{\infty} e^{-x^2} d_x = \int_0^{\infty} t^{\frac{1}{2}} e^{-t} d_x = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} d_x = \sqrt{\pi}.$$

Definition1.10 (Beta Function) The beta function has the form;

$$\boldsymbol{B}(x,z) = \int_0^1 t^{x-1}(1-t)^{z-1}dt \quad where\ Re(x), Re(z) > 0\,, \tag{1.3}$$

and we can represent the beta function in terms of the gamma function as;

$$\boldsymbol{B}(x,z) = \frac{\Gamma(x)\Gamma(z)}{\Gamma(x+z)}. \tag{1.4}$$

Definition1.11 (Pochhammer Symbol) The $(y)_m$ notation will be used to denote the Pochhammer, where m is a non-negative integer and y is a real or complex number.

$$(y)_m = y(y+1)(y+2)\ldots(y+m-1)\,. \tag{1.5}$$

Some Properties of Pochhammer Symbol:

1. $(y)_m = \frac{\Gamma(y+m)}{\Gamma(y)}$,

2. $(-y)_m = (-1)^m (y-m+1)_m$,

3. $(y)_{2m} = 2^{2m}\left(\frac{y}{2}\right)_m \left(\frac{y+1}{2}\right)_m$,

4. $(y)_{m+n} = (y)_m (y+m)_n$,

5. $(y)_m = \frac{(-1)^m(-y)!}{(-y-m)!}$,

6. $(y)_{-m} = \frac{(-1)^m}{(1-y)_m}$,

7. $(n-k)! = \frac{(-1)^k n!}{(-n)_k}$,

8. $\binom{\boldsymbol{n}}{\boldsymbol{k}} = \frac{\boldsymbol{n}!}{\boldsymbol{k}!(\boldsymbol{n}-\boldsymbol{k})!} = \frac{(-\boldsymbol{1})^{\boldsymbol{m}}(-n)_k}{\boldsymbol{k}!}$ **.**

Definition 1.12 (Hypergeometric Functions) The generalized hypergeometric series, mFn, is defined to be;

$$ {}_mF_n(a_1, a_2 \dots a_m; b_1, b_2 \dots b_n; x) = \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l \dots (a_m)_l}{(b_1)_l (b_2)_l \dots (b_n)_l} \frac{x^l}{l!} \tag{1.6} $$

Properties of Hypergeometric Functions:

1. ${}_2F_1(a_1, a_2; b_1; 1) = \frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)}$,
2. ${}_2F_1(-n, a; b; 1) = \frac{(b-a)_n}{(b)_n}$,
3. ${}_1F_0(a; -; x) = (1-x)^{-a}$.

Definition1.13 (Differential Equations of Hypergeometric Functions) Hypergeometric functions which is defined as

$$ {}_2F_1(a_1, a_2; b; x) = \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l}{(b)_l} \frac{x^l}{l!} $$

has the differential equation as follows;

$$ x(1-x)p'' + [b - (a_1 + a_2 + 1)x]p' - a_1 a_2 p = 0 \tag{1.7} $$

Chapter 2

ORTHOGONAL POLYNOMIALS

Definition2.1. The set of infinite sequence of polynomials , $P_0(x)$, $P_1(x)$,.. where $P_n(x)$, has degree n , if any two polynomials in the set are orthogonal to each other ,then we can say that the set of polynomials form an orthogonal polynomials set.

To define the orthogonality of polynomials we need an orthogonality interval $[a, b]$ (this interval is not necessary to be finite) and also we need the weight function w(x) > 0.

There are two types of orthogonal polynomials:

- ✓ Continuous orthogonal polynomials
- ✓ Discrete orthogonal polynomials.

Definition2.2. If the weight function $w(x)$ is continuous or piecewise continuous or integrable then the polynomials form a continuous orthogonal polynomial set.

Orthogonality relation can be written in the form;

$$\int_a^b p_n(x)p_m(x)w(x)dx = \sigma_n\delta_{mn}. \tag{2.1}$$

where m,n ∈ {0,1,2...} and δ_{mn} is Kronecker delta defined by

$$\delta_{mn} \coloneqq \begin{cases} 0\,, & m \neq n \\ 1\,, & m = n \end{cases}, \qquad m,n \in \{0,1,2...\}.$$

Here we can define moments by using the weight function which exist on the interval [a,b].

$$\mu_n := \int_a^b w(x)x^n dx \ , \quad n = 0,1,2.\dots \qquad 0 < \mu_n < \infty.$$

Definition2.3. If the weight function $w(x)$ is a jump function, which means at the point x_0 the left and right limit exists but they are not equal, then the polynomials form a discrete orthogonal polynomial set.

Then the orthogonality relation can be written in the form;

$$\sum_{x\in X} p_m(x)p_n(x)w_x = \sigma_n \delta_{mn},$$

where m,n ∈ {0,1,2...N} it is possible N→ ∞.

Definition2.4. Now let us define σ_n which is called the **normalization function** ,

$$\sigma_n = \int_a^b \left(p_n(x)\right)^2 w(x)dx, \qquad (2.2)$$

where $n = 1,2, \dots$. for continuous orthogonal polynomials

or

$$\sigma_n = \sum_{x\in X} (p_n(x))^2 w_x, \qquad (2.3)$$

where $n = 1,2, \dots .N$ for discrete orthogonal polynomials where $N \to \infty$ is possible.

2.1 Properties of Orthogonal Polynomials

- Any polynomial $Q_n(x)$ which has degree n, can be written in terms of $p_0 , p_1, \dots , p_n$ and there exists coefficients γ_{in} such that

$$Q_n(x)$$

$$= \sum_{i=0}^{n} \gamma_{in} p_i(x). \tag{2.4}$$

The coefficients γ_{in}'s can be determined by using orthogonality property.

Finding γ_{in}'s;

$$\int_a^b Q_n(x) p_i(x) w(x) d_x = \int_a^b \sum\nolimits_{i=0}^{n} \gamma_{in} p_i(x)\, p_i(x)\, w(x) d_x$$

The integral is non-zero only when $i = i$ from the orthogonality property so,

write it as a;

$$\int_a^b Q_n(x) p_i(x) w(x) d_x = \gamma_{in} \int_a^b p_i^2(x) w(x)\, d_x$$

where

$$\int_a^b p_i^2(x) w(x) d_x = \sigma_n(x),$$

so,

$$\gamma_{in} = \frac{1}{\sigma_n(x)} \int_a^b Q_n(x) p_i(x) w(x) d_x \tag{2.5}$$

- $\{p_0(x), p_1(x), \ldots.\}$ be an orthogonal set of polynomials, each polynomial $p_n(x)$ is orthogonal to any polynomial of degree < n.
- If $\{p_0(x), p_1(x), \ldots.\}$ is a sequence of orthogonal polynomials on the interval $[a, b]$ with respect to the weight function $w(x)$, then the polynomial $p_n(x)$ has exactly n real simple zeros in the interval $[a, b]$.

Proof:

Let us write $p_n(x)$ as a $p_n(x) = (x - x_1)\ (x - x_2)\dots\ (x - x_l)$ and assume that zeros are not simple which means roots are repeteated.

If $l < n$ $p_n(x)$ will be $p_n(x) = (x - x_1)^k (x - x_2)^m \dots (x - x_l)^{\ i}$

At least one of the $k, m, i > 1$ since we have the repeated root

Now let define

$$q_k(x) = \begin{cases} 1\ if\ k = 0 \\ \prod_{b=1}^{k} (x - x_b)\ for\ 0 < k \le n \end{cases}.$$

The product of $p_n(x)$ with $q_k(x)$ will be;

$p_n(x) q_k(x) = (x - x_1)^{k+1} (x - x_2)^{m+1} \dots (x - x_l)^{\ i+1}$

If k, m, i are odd $\rightarrow k + 1, m + 1, i + 1$ are even which means sign will not change for $x \in (a, b)$ which implies

$\int_a^b p_n(x) q_k(x) w(x) dx \neq 0\ \ for\ k < n.$

This is the contradiction since the polynomials are orthogonal integral have to be 0 for $k < n$ from the above property. Which implies that $k = n$ so the polynomial $p_n(x)$ which has degree n, has n simple roots in the $[a, b]$.

- $\{p_0(x), p_1(x), \dots .\}$ be an othogonal set of polynomials, the polynomials in this set has a three term recurrence relation, that is,

$$x p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \delta_n p_{n-1}(x) \qquad n = 1,2, \dots \qquad (2.6)$$

where the coefficients α, β, δ depend on n.

Proof: We can write

$$xp_n(x) = \sum_{i=0}^{n+1} \gamma_{in} p_i(x),$$

and from (2.5)

$$\gamma_{in} = \frac{1}{\sigma_n(x)} \int_a^b p_i(x) x p_n(x) w(x) d_x,$$

$$xp_n(x) = \gamma_{on} p_0(x) + \gamma_{1n} p_1(x) + \cdots + \gamma_{n-1,n} p_{n-1}(x) + \gamma_{nn} p_n(x)$$
$$+ \gamma_{n+1,n} p_{n+1}(x).$$

Since $xp_i(x)$ has degree $i+1$, from the orthogonality property of $p_n(x)$ the coefficients $\gamma_{in} = 0$ when $i+1 < n$ which implies that;

$xp_n(x)$=$\gamma_{n-1,n} p_{n-1}(x) + \gamma_{nn} p_n(x) + \gamma_{n+1,n} p_{n+1}(x)$.

Let us compare the coefficients of $p_{n-1}(x), p_n(x), p_{n+1}(x)$ of the equations;

$\alpha_n = \gamma_{n+1,n}$,

$\beta_n = \gamma_{nn}$,

$\delta_n = \gamma_{n-1,n}$.

Write the γ_{in} one more with changing the index,

$$\gamma_{in} = \frac{1}{\sigma_n(x)} \int_a^b p_i(x) x p_n(x) w(x) d_x$$

$$\gamma_{ni} = \frac{1}{\sigma_i(x)} \int_a^b p_i(x) x p_n(x) w(x) d_x$$

$$\gamma_{in} \sigma_n(x) = \gamma_{ni} \sigma_i(x)$$

$$\gamma_{ni} = \frac{\gamma_{in} \sigma_n(x)}{\sigma_i(x)}$$

Turn back to ;

$\alpha_n = \gamma_{n+1,n}$ and take $n \to n-1$

$\alpha_{n-1} = \gamma_{n,n-1}$ define $n-1 = i$

$\alpha_i = \gamma_{ni} = \frac{\gamma_{in}\sigma_n(x)}{\sigma_i(x)}$

$\alpha_i = \frac{\sigma_n(x)}{\sigma_i(x)}\gamma_{in}$ $\rightarrow$ $\alpha_{n-1} = \frac{\sigma_n(x)}{\sigma_{n-1}(x)}\gamma_{n-1n}$ where $\delta_n = \gamma_{n-1,n}$

$$\alpha_{n-1} = \frac{\sigma_{n-1}(x)}{\sigma_n(x)}\delta_n$$

$$\delta_n = \alpha_{n-1}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}$$

Now we are going to use usual representation of $p_n(x)$ with the three term recurrence relation;

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + c_n p_{n-1}(x)$$

$$\begin{aligned} a_n x^{n+1} + a_{n-1}x^n + \cdots + a_0 \\ &= \alpha_n[a_{n+1}x^{n+1} + a_n x^n + \cdots + a_0] \\ &+ \beta_n[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0] + c_n[a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \\ &+ \cdots + a_0] \end{aligned}$$

Compare the coefficients of the terms x^{n+1} and x^n;

$a_n = \alpha_n a_{n+1}$

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad (2.7)$$

$a_{n-1} = \alpha_n a_n + \beta_n a_n \qquad a_{n-1} = a_n[\frac{a_n}{a_{n+1}} + \beta_n]$

$$\beta_n = \frac{a_{n-1}}{a_n} - \frac{a_n}{a_{n+1}}$$

Let $a_{n-1} = c_n$, $a_n = c_{n+1}$.

$$\beta_n = \frac{c_n}{a_n} - \frac{c_{n+1}}{a_{n+1}} \tag{2.8}$$

Since we have; $\delta_n = \quad \alpha_{n-1}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}$ where $\alpha_{n-1} = \frac{a_{n-1}}{a_n}$

$$\delta_n = \frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)} \tag{2.9}$$

- By using three-term recurrence relation we can create Christoffel-Darboux formula

$$\sum_{k=0}^{n} \frac{1}{\sigma_k(x)} p_k(x)p_k(y)$$

$$= \frac{1}{\sigma_n(x)}\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)} \tag{2.10}$$

Proof:

Write the three term recurrence relation with x and y terms;

$$xp_n(x) = \frac{a_n}{a_{n+1}}p_{n+1}(x) + \beta_n p_n(x) + \frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)$$

$$yp_n(y) = \frac{a_n}{a_{n+1}}p_{n+1}(y) + \beta_n p_n(y) + \frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(y)$$

Multiply the first equation with $p_n(y)$ and the second one with $p_n(x)$

$$xp_n(x)p_n(y) = \frac{a_n}{a_{n+1}}p_{n+1}(x)p_n(y) + \beta_n p_n(x)p_n(y)$$

$$+ \frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_n(y)$$

$$yp_n(y)p_n(x) = \frac{a_n}{a_{n+1}}p_{n+1}(y)p_n(x) + \beta_n p_n(y)p_n(x)$$

$$+\frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(y)p_n(x).$$

Subtract the equations;

$(x-y)p_n(x)p_n(y) = \frac{a_n}{a_{n+1}}[p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)] +$

$\frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}[p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)]$

$$p_n(x)p_n(y) = \frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)} +$$

$$\frac{a_{n-1}}{a_n}\frac{\sigma_n(x)}{\sigma_{n-1}(x)}\frac{p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)}{(x-y)}$$

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)} =$$

$$p_n(x)p_n(y)$$

$$-\frac{\sigma_n(x)}{\sigma_{n-1}(x)}\frac{a_{n-1}}{a_n}\frac{p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)}{(x-y)} \qquad (2.11)$$

We can get the second term into the RHS by taking $n \to n-1$ in (2.11) equation.

$$\frac{a_{n-1}}{a_n}\frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{(x-y)} = p_{n-1}(x)p_{n-1}(y) -$$

$$\frac{\sigma_{n-1}(x)}{\sigma_n(x)}\frac{a_{n-2}}{a_{n-1}}\frac{p_{n-2}(x)p_{n-1}(y) - p_{n-2}(y)p_{n-1}(x)}{(x-y)}$$

Put this into the equation with multiplying -1.

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)}$$

$$= p_n(x)p_n(y) - \frac{\sigma_n(x)}{\sigma_{n-1}(x)}[-p_{n-1}(x)p_{n-1}(y) +$$

$$\frac{\sigma_{n-1}(x)}{\sigma_n(x)}\frac{a_{n-2}}{a_{n-1}}\frac{p_{n-2}(x)p_{n-1}(y)-p_{n-2}(y)p_{n-1}(x)}{(x-y)}$$

Again we can find the last term of this equation by taking $n \to n-2$ in (2.11) equation

$$\frac{a_{n-2}}{a_{n-1}}\frac{p_{n-1}(x)p_{n-2}(y)-p_{n-1}(y)p_{n-2}(x)}{(x-y)}$$

$$= p_{n-2}(x)p_{n-2}(y)$$

$$-\frac{\sigma_{n-2}(x)}{\sigma_{n-3}(x)}\frac{a_{n-3}}{a_{n-2}}\frac{p_{n-3}(x)p_{n-2}(y)-p_{n-3}(y)p_{n-2}(x)}{(x-y)}$$

Put this equation to the (2.1.9)

$$\frac{a_{n-1}}{a_n}\frac{p_n(x)p_{n-1}(y)-p_n(y)p_{n-1}(x)}{(x-y)}$$

$$= p_{n-1}(x)p_{n-1}(y)+\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_{n-1}(y)-$$

$$\frac{\sigma_n(x)}{\sigma_{n-1}(x)}\frac{\sigma_{n-1}(x)}{\sigma_{n-2}(x)}\Bigg[-p_{n-2}(x)p_{n-2}(y)$$

$$+\frac{\sigma_{n-2}(x)}{\sigma_{n-3}(x)}\frac{a_{n-3}}{a_{n-2}}\frac{p_{n-3}(x)p_{n-2}(y)-p_{n-3}(y)p_{n-2}(x)}{(x-y)}\Bigg]$$

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}=$$

$$p_n(x)p_n(y)+\frac{\sigma_n(x)}{\sigma_{n-1}(x)}p_{n-1}(x)p_{n-1}(y)+\frac{\sigma_n(x)}{\sigma_{n-2}(x)}p_{n-2}(x)p_{n-2}(y)-$$

$$\frac{\sigma_n(x)}{\sigma_{n-3}(x)}\frac{a_{n-3}}{a_{n-2}}\frac{p_{n-3}(x)p_{n-2}(y)-p_{n-3}(y)p_{n-2}(x)}{(x-y)}$$

If we continue to iterate the equation we get;

$$\frac{a_n}{a_{n+1}}\frac{p_{n+1}(x)p_n(y)-p_{n+1}(y)p_n(x)}{(x-y)}$$

$$= p_n(x)p_n(y) + \frac{\sigma_n(x)}{\sigma_{n-1}(x)} p_{n-1}(x)p_{n-1}(y) + \frac{\sigma_n(x)}{\sigma_{n-2}(x)} p_{n-2}(x)p_{n-2}(y) + \cdots$$

$$= \frac{\sigma_n(x)}{\sigma_0(x)} p_0(x)p_0(y).$$

$$\frac{a_n}{a_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)} = \sum_{k=0}^{n} \frac{\sigma_n(x)}{\sigma_k(x)} p_k(x)p_k(y)$$

$$\sum_{k=0}^{n} \frac{1}{\sigma_k(x)} p_k(x)p_k(y) = \frac{1}{\sigma_n(x)} \frac{a_n}{a_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)}.$$

Chapter 3

CLASSICAL ORTHOGONAL POLYNOMIALS

Definition3.1. If the polynomial $p_n(x)$ satisfy the hypergeometric type differential equation which has form $A(x)p_n''(x) + B(x)p_n'(x) + \alpha_n p_n(x) = 0$ with the Pearson equation $\frac{d}{dx}[A(x).w(x)] = B(x).w(x)$ then we say that $p_n(x)$ form a classical orthogonal polynomials set.

To show it let us write the equation again with $p_n(x)$ and $p_m(x)$

$$A(x)p_n''(x) + B(x)p_n'(x) + \alpha_n p_n(x) = 0$$

$$A(x)p_m''(x) + B(x)p_m'(x) + \alpha_m p_m(x) = 0.$$

Multiply both equation with $w(x)$

$$A(x)w(x)p_n''(x) + B(x)w(x)p_n'(x) + \alpha_n w(x)p_n(x) = 0$$

$$A(x)w(x)p_m''(x) + B(x)w(x)p_m'(x) + \alpha_m w(x)p_m(x) = 0.$$

Since they satisfy the Pearson equation we can write them in the self adjoint form in such a way that,

$$[A(x)w(x)p_n'(x)]' + \alpha_n w(x)p_n(x) = 0 \qquad (3.1)$$

$$[A(x)w(x)p_m'(x)]' + \alpha_m w(x)p_m(x) = 0. \qquad (3.2)$$

Now multiply equation (3.1) with $p_m(x)$ and equation (3.2) with $p_n(x)$

$$[A(x)w(x)p_n'(x)]'p_m(x) + \alpha_n w(x)p_n(x)p_m(x) = 0$$

$$[A(x)w(x)p_m'(x)]'p_n(x) + \alpha_m w(x)p_m(x)p_n(x) = 0.$$

Subtract the equations;

$$[A(x)w(x)p_n'(x)]'p_m(x) - [A(x)w(x)p_m'(x)]'p_n(x)$$

$$= (\alpha_m - \alpha_n)w(x)p_m(x)p_n(x).$$

Apply the product rule for derivatives;

$$[A(x)w(x)]'p_n'(x)p_m(x) + A(x)w(x)p_n''(x)p_m(x) - [A(x)w(x)]'p_m'(x)p_n(x) -$$

$$A(x)w(x)p_m''(x)p_n(x) = (\alpha_m - \alpha_n)w(x)p_m(x)p_n(x)$$

$$[A(x)w(x)]'\left(p_n'(x)p_m(x) - p_m'(x)p_n(x)\right) + A(x)w(x)[p_n''(x)p_m(x)$$

$$- p_m''(x)p_n(x)$$

$$= (\alpha_m - \alpha_n)w(x)p_m(x)p_n(x).$$

Here we use the definition of Wronskian of two functions for simplify our equation;

$$\omega[u(x)v(x)] = det\begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix} = u(x)v'(x) - u'(x)v(x)$$

$$\frac{d}{dx}\omega[u(x)v(x)] = u(x)v''(x) + u'(x)v'(x) - u''(x)v(x) - u'(x)v'(x)$$

$$= u(x)v''(x) - u''(x)v(x).$$

So equation get the form;

$$\frac{d}{dx}\{ A(x)w(x)\ \omega[p_n(x)p_m(x)]\} = (\alpha_m - \alpha_n)w(x)p_m(x)p_n(x).$$

Now integrate both sides with respect to x from a to b

$$A(x)w(x)\ \omega[p_n(x)p_m(x)]|_a^b = (\alpha_m - \alpha_n)\int_a^b w(x)p_m(x)p_n(x)dx.$$

Since the $\omega[p_n(x)p_m(x)]$ is a polynomial in x instead of $\omega[p_n(x)p_m(x)]$ write x^k.

If function $w(x)$ satisfy the condition

$$\boldsymbol{A(x)w(x)x^k|_a^b = 0} \text{ ,for } \quad k = 0,1,2,\ldots \tag{3.3}$$

We get the orthogonality relation

$$\int_a^b w(x)p_m(x)p_n(x)d_x = 0 \tag{3.4}$$

for $\alpha_m \neq \alpha_n \quad \rightarrow \quad m \neq n.$

3.1 Properties of Classical Orthogonal Polynomials

- ✓ They satisfy an orthogonality relation.
- ✓ They have a Rodrigues formula, where the generalized Rodrigues formula is as follows for classical orthogonal polynomials.

$p_m(x) = \frac{K_m}{w(x)}\frac{d^m}{d_x^m}[w(x).A^m(x)].$

- ✓ They have a hypergeometric representation.
- ✓ $\{p_n'(x)\}$ form a system of orthogonal polynomial.
- ✓ They satisfy the hypergeometric type differential equation which has form $A(x)p_n''(x) + B(x)p_n'(x) + \alpha_n p_n(x) = 0.$

3.2 Examples for the Classical Orthogonal Polynomials

1. Hermite Polynomials: $H_n(x)$.

2. Laguerre Polynomials: $L_n^\alpha(x)$ where $\alpha > -1$.

3. Jacobi Polynomials: $P_n^{(\alpha,\beta)}(x)$ where $\alpha > -1$, $\beta > -1$.

3.3 Rodrigues Formula for Classical Orthogonal Polynomials

Definition3.2: Classical orthogonal polynomials can represented by using rodrigues formula, which is the formula that consists the nth term derivative of polynomials.

Obtain the Rodrigues formula:

Start from the differential equations;

$$A(x)p''(x) + B(x)p'(x) + \alpha_n p(x) = 0$$

$$A(x)v_m''(x) + B_m(x)v_m'(x) + \mu_m v_m(x) = 0.$$

Multiply first equation with $w(x)$ and second equation with $w_m(x)$

$$A(x)w(x)p''(x) + B(x)w(x)p'(x) + \alpha_n w(x)p(x) = 0$$

$$A(x)w_m(x)v_m''(x) + B_m(x)w_m(x)v_m'(x) + \mu_m w_m(x)v_m(x) = 0.$$

Then write the equations in the self adjoint form;

$$[A(x)w(x)p'(x)]' + \alpha_n w(x)p(x) = 0$$

$$[A(x)w_m(x)v_m'(x)]' + \mu_m w_m(x)v_m(x) = 0.$$

From the property of self adjoint form the equations must satisfy the following differential equation;

$$[A(x)w(x)]' = B(x)w(x) \tag{3.5}$$

$$[A(x)w_m(x)]' = B_m(x)w_m(x). \tag{3.6}$$

Now, let us construct the relation between $w(x)$ and $w_m(x)$;

Divide (3.6) with $w_m(x)$

$$\frac{[A(x)w_m(x)]'}{w_m(x)} = B_m(x),$$

use the fact that;

$B_m(x) = B(x) + mA'(x)$ and from (3.5) we get

$$B(x) = \frac{[A(x)w(x)]'}{w(x)}.$$

$[A(x)w(x)]' = A'(x)w(x) + A(x)w'(x)$

$[A(x)w_m(x)]' = A'(x)w_m(x) + w_m'(x)A(x)$

$$\frac{A'(x)w_m(x)}{w_m(x)} + \frac{w_m'(x)A(x)}{w_m(x)} = \frac{A'(x)w(x)}{w(x)} + \frac{A(x)w'(x)}{w(x)} + mA'(x)$$

$$\frac{w_m'(x)A(x)}{w_m(x)} = \frac{A(x)w'(x)}{w(x)} + mA'(x).$$

Divide both side with $A(x)$

$$\frac{w_m'(x)}{w_m(x)} = \frac{w'(x)}{w(x)} + m\frac{A'(x)}{A(x)}.$$

After integrating both sides with respect to x we find that;

$ln\, w_m(x) = ln\, w(x) + m\, ln\, A(x) \ \rightarrow m = 0,1,2 ...$

$$w_m(x) = w(x). A^m(x). \tag{3.7}$$

Let take $m = m + 1$ in (3.7)

$w_{m+1}(x) = w(x). A^{m+1}(x) = w(x). A^m(x). A(x)$

$$w_{m+1}(x) = w_m(x). A(x) \tag{3.8}$$

$[A(x)w_m(x)v_m'(x)]' + \mu_m w_m(x)v_m(x) = 0$

$w_m(x)v_m(x) = \frac{-1}{\mu_m}[\ w_{m+1}(x)v_{m+1}(x)]'$

We know that $v_m(x) = p^{(m)}(x) \ \rightarrow \ v_m'(x) = v_{m+1}(x)$

Let m=0 $w_0(x)v_0(x) = w(x)p(x) = \frac{-1}{\mu_0}[\ w_1(x)v_1(x)]'.$

We can find the term inside the derivative by taking m=1

$w_1(x)v_1(x) = \frac{-1}{\mu_1}[\ w_2(x)v_2(x)]'$

$w(x)p(x) = \frac{-1}{\mu_0}[\ w_1(x)v_1(x)]' = \frac{-1}{\mu_0}\frac{-1}{\mu_1}[\ w_2(x)v_2(x)]''.$

Iterate it up to m

$w(x)p(x) = \frac{-1}{\mu_0}\frac{-1}{\mu_1}\dots\frac{-1}{\mu_{m-1}}[w_m(x)v_m(x)]^{(m)}.$

Let us define

$$C_m = (-1)^m \prod_{k=0}^{m-1} \mu_k,$$

$w(x)p(x) = \frac{1}{C_m}[w_m(x)v_m(x)]^{(m)}$

If the polynomial $p(x)$ is degree $m \ \rightarrow \ p(x) = p_m(x)$

$w(x)p_m(x) = \frac{1}{C_m}[w_m(x)p_m^m(x)]^{(m)}$ since $p_m^m(x) = const$

$w(x)p_m(x) = \frac{p_m^m(x)}{C_m}[w_m(x)]^{(m)}$, use the fact that $w_m(x) = w(x).A^m(x)$

$$p_m(x) = \frac{p_m^m(x)}{w(x)C_m}[w(x).A^m(x)\ ]^{(m)}$$

$$p_m(x) = \frac{p_m^m(x)}{w(x)C_m}\frac{d^m}{d_x^m}[w(x).A^m(x)].$$

Let us combine the constants as a;

$$K_m = \frac{p_m^m(x)}{C_m}$$

$$p_m(x) = \frac{K_m}{w(x)}\frac{d^m}{d_x^m}[w(x).A^m(x)]. \tag{3.9}$$

Obtain the generalized Rodrigues Formula:

Theorem: Since the derivatives $p_n^{(m)}(x) = v_{mn}(x)$ are polynomials of degree $n - m$ and satisfy the equation $A(x)v''_{mn}(x) + B_m(x)v'_{mn}(x) + \mu_{mn}v_{mn}(x) = 0$ we can derive the Rodrigues formula for $y_n^{(m)}(x)$.

Proof: Write the equation into the self adjoint form;

$$[A(x)w_m(x)v'_{mn}(x)]' + \mu_{mn}w_m(x)v_{mn}(x) = 0.$$

By using the properties ; $v'_{mn}(x) = v_{mn+1}(x)$, $w_{m+1}(x) = w_m(x).A(x)$

$[w_{m+1}(x)v_{mn+1}(x)]' + \mu_{mn}w_m(x)v_{mn}(x) = 0$

$$w_m(x)v_{mn}(x) = \frac{-1}{\mu_{mn}}[w_{m+1}(x)v_{mn+1}(x)]'. \qquad (3.10)$$

In equation (3.10) let us take $m \to m + 1$

$w_{m+1}(x)v_{mn+1}(x) = \frac{-1}{\mu_{m+1n}}[w_{m+2}(x)v_{mn+2}(x)]'$.

Let us put it into equation (3.10)

$$w_m(x)v_{mn}(x) = \frac{-1}{\mu_{mn}}\frac{-1}{\mu_{m+1n}}[w_{m+2}(x)v_{mn+2}(x)]'' . \qquad (3.11)$$

In equation (3.10), let us take m instead of $m + 2$.

$w_{m+2}(x)v_{mn+2}(x) = \frac{-1}{\mu_{m+2n}}[w_{m+3}(x)v_{mn+3}(x)]'$.

Let us put it into equation (3.11)

$w_m(x)v_{mn}(x) = \frac{-1}{\mu_{mn}}\frac{-1}{\mu_{m+1n}}\frac{-1}{\mu_{m+2n}}[w_{m+3}(x)v_{mn+3}(x)]'''$ iterate upto $n - m$

$w_m(x)v_{mn}(x) = \frac{-1}{\mu_{mn}}\frac{-1}{\mu_{m+1n}}\frac{-1}{\mu_{m+2n}}\ldots\frac{-1}{\mu_{n-1n}}[w_n(x)v_n(x)]^{(n-m)}.$

Let us define

$$A_{mn} = (-1)^m \prod_{l=0}^{m-1} \mu_{ln} \ , \tag{3.12}$$

where $\mu_{ln} = \alpha_n - \alpha_l$

$$\mu_{ln} = -nB'(x) - \frac{1}{2}n(n-1)A''(x) - [-l\,B'(x) - \frac{1}{2}l(l-1)A''(x)]$$

$$= B'(x)[l-n] - \tfrac{1}{2}A''(x)[n^2 - n - l^2 + l]$$

$$\mu_{ln} = -(n-l)\left[B'^{(x)} + \tfrac{n+l-1}{2}A''(x)\right]. \tag{3.13}$$

$$A_{mn} = \frac{n!}{(n-m)!}\prod_{l=0}^{m-1}[B'(x) + \frac{n+l-1}{2}A''(x)]. \tag{3.14}$$

and since $v_n(x) = p_n^n(x)$ is constant;

$$w_m(x)v_{mn}(x) = \frac{A_{mn}p_n^n(x)}{A_{nn}}\frac{d^{n-m}}{d_x^{n-m}}[w_n(x)].$$

We know that $w_n(x) = \ w(x).A^n(x)$ and $K_n = \frac{p_n^n(x)}{A_{nn}}$

$$p_n^{(m)}(x) = v_{mn}(x) = \frac{A_{mnK_n}}{w(x).A^m(x)}\frac{d^{n-m}}{d_x^{n-m}}[w(x).A^n(x)]. \tag{3.15}$$

Applications of Generalized Rodrigues Formula:

1. Let take $m = 1$ in equation (3.15)

$$p_n'(x) = \frac{A_{1nK_n}}{w(x).A(x)}\frac{d^{n-1}}{d_x^{n-1}}[w(x).A^n(x)] = \frac{A_{1nK_n}}{w_1(x)}\frac{d^{n-1}}{d_x^{n-1}}[w(x).A^n(x)]$$

$w_m(x) = \ w(x).A^m(x) \quad\quad m = 1 \ w_1(x) = \ w(x).A(x) \quad\quad w(x) = \frac{w_1(x)}{A(x)}$

What is $\boldsymbol{A_{1n}}$? $A_{1n} = (-1)\ \mu_{ln} = -\alpha_n + \alpha_1 = -\alpha_n$

$$p_n'(x) = \frac{-\alpha_n K_n}{w_1(x)}\frac{d^{n-1}}{d_x^{n-1}}[A^{n-1}(x)w_1(x)]$$

multiply RHS $\frac{K_{n-1}}{K_{n-1}}$

$$p_n'(x) = \frac{-\alpha_n K_n}{w_1(x)} \frac{K_{n-1}}{K_{n-1}} \frac{d^{n-1}}{d_x^{n-1}} [A^{n-1}(x) w_1(x)],$$

since

$$p_m(x) = \frac{K_m}{w(x)} \frac{d^m}{d_x^m} [w(x). A^m(x)],$$

for $m = n - 1$

$$p_{n-1}(x) = \frac{K_{m-1}}{w(x)} \frac{d^{m-1}}{d_x^{m-1}} [w(x). A^{m-1}(x)].$$

We get;

$$p_n'(x) = \frac{-\alpha_n K_n}{K_{n-1}} p_{n-1}(x) \tag{3.16}$$

2. We know that polynomials have the form $p_n(x) = a_n x^n + c_n x^{n-1} + \cdots$ now we try to find the coefficinets a_n and c_n.

Proof: Start to take the derivative from $p_n(x) = a_n x^n + c_n x^{n-1} + \cdots$

$p_n'(x) = n a_n x^{n-1} + (n-1) c_n x^{n-2} + \cdots$

$p_n''(x) = n(n-1) a_n x^{n-2} + (n-1)(n-2) c_n x^{n-3} + \cdots$

$$p_n^{(n-1)}(x) = n!\, a_n x + (n-1)!\, c_n + \cdots \tag{3.17}$$

Turn back to equation;

$$p_n^{(m)}(x) = \frac{A_{mn} K_n}{w(x). A^m(x)} \frac{d^{n-m}}{d_x^{n-m}} [w(x). A^n(x)],$$

and take $m = n - 1$

$$p_n^{(n-1)}(x) = \frac{A_{n-1n} K_n}{w(x). A^{n-1}(x)} \frac{d}{d_x} [w(x). A^n(x)].$$

Use the facts;

$w_m(x) = w(x). A^m(x) \quad \rightarrow \quad w_{n-1}(x) = w(x). A^{n-1}(x)$

$$w_{n+1}(x) = w_n(x).A(x)$$

$$[A(x)w_m(x)]' = B_m(x)w_m(x).$$

$$p_n^{(n-1)}(x) = \frac{A_{n-1nK_n}}{w(x).A^{n-1}(x)}\frac{d}{d_x}[w(x).A^n(x)] = \frac{A_{n-1nK_n}}{w(x).A^{n-1}(x)}\frac{d}{d_x}[w_n(x)]$$

$$= \frac{A_{n-1nK_n}}{w_{n-1}(x)}\frac{d}{d_x}[w_{n-1}(x)A(x)] = \frac{A_{n-1nK_n}}{w_{n-1}(x)}[w_{n-1}(x)B_{n-1}(x)]$$

$$p_n^{(n-1)}(x) = A_{n-1n}K_nB_{n-1}(x). \tag{3.18}$$

Now let compare the equations (3.17) and (3.18)

$p_n^{(n-1)}(x) = n!\,a_nx + (n-1)!\,c_nx^{n-1} + \cdots$

$p_n^{(n-1)}(x) = A_{n-1n}K_nB_{n-1}(x)$

$$n!\,a_nx + (n-1)!\,c_n + \cdots = A_{n-1n}K_nB_{n-1}(x)\,. \tag{3.19}$$

a) for finding a_n take derivative with respect to x in equation (3.19).

$$n!\,a_n = A_{n-1n}K_nB'_{n-1}(x)$$

$$a_n = \frac{A_{n-1n}K_n}{n!}B'_{n-1}(x)$$

And since $B_n(x) = B(x) + nA'(x) \rightarrow B_{n-1}(x) = B(x) + (n-1)A'(x)$ take derivative $B'_{n-1}(x) = B'^{(x)} + (n-1)A''^{(x)}$.

And since

$$A_{mn} = \frac{n!}{(n-m)!}\prod_{l=0}^{m-1}[B'(x) + \frac{n+l-1}{2}A''(x)],$$

take $m = n-1$

$$A_{n-1n} = n!\prod_{l=0}^{n-2}[B'(x) + \frac{n+l-1}{2}A''(x)].$$

We get

$$a_n = K_n \prod_{l=0}^{n-1}[B'(x) + \frac{n+l-1}{2}A''(x)]. \quad (3.20)$$

b) For finding c_n take $x = 0$ in equation (3.18)

$$(n-1)!\, c_n = A_{n-1n}K_nB_{n-1}(0) = a_n n! \frac{B_{n-1}(0)}{B'_{n-1}(x)}$$

$$c_n = na_n \frac{B_{n-1}(0)}{B'_{n-1}(x)}\,. \quad (3.21)$$

Theorem 3.3: From the property that we mentioned in the preliminaries part; derivatives of the functions of hypergeometric type are all functions of hypergeometric type.

By using this property we can say that derivatives of the classical orthogonal polynomials $p_n^l(x)$ are also classical polynomials and they are orthogonal with weight function $w_n(x) = \ w(x). A^n(x)$ on the interval (a, b). Then we can write the orthogonality relation as a;

$$\int_a^b p_n^{(l)}(x)p_m^{(l)}(x)w_l(x)dx = \sigma_{ln}\delta_{mn}\ . \quad (3.22)$$

3.4 Finding the Normalization function for Classical Orthogonal Polynomials

For the polynomial $p_n^l(x) = v_{ln}$ we have the differential equation;

$A(x)v''_{ln}(x) + B_n(x)v'_{ln}(x) + \mu_{ln}v_{ln}(x) = 0$

$A(x)w_l(x)v''_{ln}(x) + B_n(x)w_l(x)v'_{ln}(x) + \mu_{mln}w_l(x)v_{ln}(x) = 0$

which has the self adjoint form;

$[A(x)w_l(x)v'_{ln}(x)]' + \mu_{ln}w_l(x)v_{ln}(x) = 0.$

Now use the facts;

$w_{n+1}(x) = w_n(x).A(x)$ and $p_n^l(x) = v_{ln}(x) \rightarrow v_{ln}'(x) = p_n^{l+1}(x)$

$[w_{l+1}(x)p_n^{l+1}(x)]' + \mu_{ln}w_l(x)p_n^l(x) = 0$ multiply the equation with $p_n^l(x)$

$[w_{l+1}(x)p_n^{l+1}(x)]'p_n^l(x) + \mu_{ln}w_l(x)[p_n^l(x)]^2 = 0$ integrate from a to b.

$$\int_a^b [w_{l+1}(x)p_n^{l+1}(x)]'p_n^l(x)d_x + \int_a^b \mu_{ln}w_l(x)[p_n^l(x)]^2 d_x = 0.$$

Use integration by parts for the first integral;

$u = p_n^l(x) \rightarrow d_u = p_n^{l+1}(x)d_x$

$d_v = [w_{l+1}(x)p_n^{l+1}(x)]\, d_x \rightarrow v = w_{l+1}(x)p_n^{l+1}(x)$

$$w_{l+1}(x)p_n^{l+1}(x)p_n^l(x)|_a^b - \int_a^b w_{l+1}(x)[p_n^{l+1}(x)]^2 d_x + \mu_{ln}\int_a^b w_l(x)[p_n^l(x)]^2 dx$$
$$= 0.$$

First term is going to be zero from the condition of orthogonality. From the orthogonality relation first integral is σ_{l+1n} and the second integral is σ_{ln}. So we get

$$-\sigma_{l+1n+}\mu_{ln}\sigma_{ln} = 0 \quad \rightarrow \quad \sigma_{l+1n} = \mu_{ln}\sigma_{ln} \tag{3.23}$$

Let iterate the equation (3.23)

For $l = 0$ $\sigma_{1n} = \mu_{0n}\sigma_{0n}$

For $l = 1$ $\sigma_{2n} = \mu_{1n}\sigma_{1n} = \mu_{1n}\mu_{0n}\sigma_{0n}$

For $l = 2$ $\sigma_{3n} = \mu_{2n}\sigma_{2n} = \mu_{2n}\mu_{1n}\mu_{0n}\sigma_{0n}$

For $l = n-1$ $\sigma_{nn} = \mu_{n-1n}\sigma_{n-1n} = \mu_{n-1n} \cdots \mu_{2n}\mu_{1n}\mu_{0n}\sigma_{0n}$ where $\sigma_{0n} = \sigma_n$

$$\sigma_{nn} = \prod_{k=0}^{n-1} \mu_{kn}\,\sigma_{0n}.$$

Let $m = 0$

$$\sigma_{mn} = \frac{\sigma_{nn}}{\prod_{k=m}^{n-1} \mu_{kn}} \,. \tag{3.24}$$

Turn back to equation (3.22) and take $m = n,\ l = n$

$$\int_a^b [p_n^{(n)}(x)]^2 w_n(x) dx = \sigma_{nn}$$

$$\int_a^b [p_n^{(n)}(x)]^2 w(x).A^n(x) dx = \sigma_{nn} \,. \tag{3.25}$$

Now turn back to Rodrigues formula ;

$$p_n^{(m)}(x) = \frac{A_{mnK_n}}{w(x).A^m(x)} \frac{d^{n-m}}{d_x^{n-m}} [w(x).A^n(x)],$$

and take $m = n$

$$p_n^{(n)}(x) = \frac{A_{nn}K_n}{w(x).A^n(x)} [w(x).A^n(x)].$$

$p_n^{(n)}(x) = A_{nn}K_n$, let us put it into equation (3.25)

$$\int_a^b [A_{nn}K_n]^2 w(x).A^n(x) dx = \sigma_{nn}. \tag{3.26}$$

By using

$$A_{mn} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} [B(x) + \frac{n+l-1}{2} A''(x)],$$

and letting $m = n$

$$A_{nn} = n! \prod_{l=0}^{n-1} [B(x) + \frac{n+l-1}{2} A''(x)].$$

Also from,

$$a_n = K_n \prod_{l=0}^{n-1} [B(x) + \frac{n+l-1}{2} A''(x)],$$

$$A_{nn} = n! \frac{a_n}{K_n} . \qquad (3.27)$$

Let put (3.27) into equation (3.26);

$$\int_a^b \left[n! \frac{a_n}{K_n} K_n\right]^2 w(x). A^n(x) dx = \sigma_{nn}$$

$$\int_a^b [n!\, a_n]^2 w(x). A^n(x) dx = \sigma_{nn} \qquad (3.28)$$

From $\sigma_{0n} = \sigma_n = \frac{\sigma_{nn}}{\prod_{k=0}^{n-1} \mu_{kn}}$ → $\sigma_n = \sigma_{nn} \frac{(-1)^n}{A_{nn}}$ → $\sigma_{nn} = \sigma_n \frac{A_{nn}}{(-1)^n}$ put this into equation (3.26)

$$\int_a^b [A_{nnK_n}]^2 w(x). A^n(x) dx = \sigma_n \frac{A_{nn}}{(-1)^n}$$

$$\sigma_n = (-1)^n A_{nn} K_n^2 \int_a^b w(x). A^n(x) dx \ . \qquad (3.29)$$

Chapter 4

HERMITE POLYNOMIALS

4.1 Finding the Generating Function for Hermite Polynomials

Let us start with the equation

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \ . \qquad (4.1)$$

From there we can find the form of the Hermite Polynomials,

→ Use the fact that ;

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$e^{2xt-t^2} = e^{2xt} e^{-t^2} = \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2x)^n t^n}{n!} \frac{(-1)^k t^{2k}}{k!}.$$

→ Now use the definition of Cauchy product of series;

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} A(k, n-2k),$$

so in the equation above get $n - 2k$ instead of n.

$$e^{2xt-t^2} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2x)^n t^n}{n!}\frac{(-1)^k t^{2k}}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(2x)^{n-2k} t^{n-2k}}{(n-2k)!}\frac{(-1)^k t^{2k}}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k (2x)^{n-2k} t^n}{(n-2k)!\,k!}$$

$$= \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n.$$

So we get the relation ;

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k (2x)^{n-2k}}{(n-2k)!\,k!} = \frac{H_n(x)}{n!}$$

$$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k (2x)^{n-2k} n!}{(n-2k)!\,k!}. \tag{4.2}$$

So ,

$H_n(x) = (2x)^n - \frac{n!(2x)^{n-2}}{(n-2)!2!} + \frac{n!(2x)^{n-4}}{(n-4)!3!} - \cdots$,

the highest degree of $H_n(x)$ is n.

And also we can represent the polynomial as follows;

$H_n(x) = 2^n x^n + \tau_{n-2}(x)$ Where $\tau_{n-2}(x)$ is a polynomial of degree $(n-2)$ in x.

→ If n is even the polynomial $H_n(x)$ is even,

If n is odd the polnomial $H_n(x)$ is odd polynomial.

$$H_n(-x) = \begin{cases} H_n(x) & if\ n\ is\ even \\ -H_n(x) & if\ n\ is\ odd \end{cases}$$

4.2 Computing $H_{2n+1}(0)$, $H_{2n}(0)$, $H'_{2n}(0)$, $H'_{2n+1}(0)$

Let us take $n = 2n$ and $x = 0$ in equation (4.1)

$$e^{(-t)^2} = \sum_{n=0}^{\infty} \frac{H_{2n}(0)}{(2n)!} t^{2n}, \tag{4.3}$$

$$e^{(-t)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}, \tag{4.4}$$

which is the even function.

Combine the equations (4.3) and (4.4). And we can seperate the equation (4.3) corresponding with even and odd functions.

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{H_{2n}(0)}{(2n)!} t^{2n} + \sum_{n=0}^{\infty} \frac{H_{2n+1}(0)}{(2n+1)!} t^{2n+1}$$

$H_{2n+1}(0) = 0$ since the RHS of equation is even.

$$\frac{(-1)^n t^{2n}}{n!} = \frac{H_{2n}(0)}{2n!} t^{2n}$$

$$H_{2n}(0) = \frac{(-1)^n (2n)!}{n!}.$$

From the third property of pochhammer symbol;

$(1)_{2n} = 2^{2n}(\frac{1}{2})_n(1)_n \quad \rightarrow \quad (2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n!$

$H_{2n}(0) = (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n.$

Let us take derivative with respect to x in equation (4.1)

$$2te^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} t^n,$$

and let $n = 2n\,, x = 0$

$$2te^{-t^2} = \sum_{n=0}^{\infty} \frac{H'_{2n}(0)}{(2n)!} t^{2n} \tag{4.5}$$

$$2t \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{2(-1)^n t^{2n+1}}{n!}, \tag{4.6}$$

which is the odd function.

Combine the equations (4.5) and (4.6). And we can seperate the equation (4.5) corresponding with even and odd functions.

$$\sum_{n=0}^{\infty} \frac{2(-1)^n t^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{H'_{2n}(0)}{(2n)!} t^{2n} + \sum_{n=0}^{\infty} \frac{H'_{2n+1}(0)}{(2n+1)!} t^{2n+1}$$

$H'_{2n}(0) = 0$, since the LHS of the equation is odd function.

$$\frac{2(-1)^n}{n!} = \frac{H'_{2n}(0)}{(2n+1)!}$$

$$H'_{2n+1}(0) = \frac{2(-1)^n (2n+1)!}{n!}.$$

From the third and fourth property of pocahammer symbol;

$$(1)_{2n+1} = (2)_{2n} = 2^{2n} (1)_n \left(\frac{3}{2}\right)_n$$

$$H'_{2n+1}(0) = 2(-1)^n 2^{2n} \left(\frac{3}{2}\right)_n$$

$$H'_{2n+1}(0) = (-1)^n 2^{2n+1} \left(\frac{3}{2}\right)_n.$$

4.3 Hypergeometric Representation of Hermite Polynomials

In equation (4.2) take $(2x)^n$ factor to the outside of the summation since it is not dependent on k.

$$H_n(x) = (2x)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{(n-2k)!\,k!}\left(\frac{1}{2x}\right)^{2k} = (2x)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{2^{2k}(n-2k)!\,k!}\left(\frac{-1}{x^2}\right)^k.$$

From the seventh and third property of pocahemmer symbol;

$$(n-2k)! = \frac{(-1)^{2k} n!}{(-n)_{2k}}$$

$$\frac{n!}{(n-2k)!} = (-n)_{2k} = 2^{2k}\left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k$$

$$H_n(x) = (2x)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{2^{2k}\left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k}{2^{2k} k!}\left(\frac{-1}{x^2}\right)^k$$

$$H_n(x) = (2x)^n \quad {}_2F_0\left(\frac{-n}{2}, \frac{-n+1}{2}; -; \frac{-1}{x^2}\right). \tag{4.7}$$

4.4 Recurrence Relations for Hermite Polynomials

Let say

$$H(x,t) = e^{2xt-t^2} \quad . \tag{4.8}$$

Take the partial derivative with respect to x and t.

$$\frac{\partial H}{\partial x} = 2te^{2xt-t^2},$$

$$\frac{\partial H}{\partial t} = 2(x-t)e^{2xt-t^2}.$$

If we combine the equations of partial derivatives we get the following equality;

$$(x-t)\frac{\partial H}{\partial x} - t\frac{\partial H}{\partial t} = 0 \tag{4.9}$$

Since

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

$$\frac{\partial H}{\partial x} = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n, \tag{4.10}$$

$$\frac{\partial H}{\partial t} = \sum_{n=1}^{\infty} n \frac{H_n(x)}{n!} t^{n-1}. \tag{4.11}$$

Put (4.10) and (4.11) into equation (4.9)

$$(x-t)\sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n - t\sum_{n=1}^{\infty} n \frac{H_n(x)}{n!} t^{n-1} = 0$$

$$x\sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n - \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^{n+1} - \sum_{n=1}^{\infty} n \frac{H_n(x)}{n!} t^n = 0.$$

For the last term we can start n from 0 since we have the factor n inside the summation.

For the second term take $n-1$ instead of n.

$$\sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^{n+1} = \sum_{n=1}^{\infty} \frac{H_{n-1}'(x)}{(n-1)!} t^n \frac{n}{n} = \sum_{n=0}^{\infty} n \frac{H_{n-1}'(x)}{n!} t^n$$

$$x\sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n - \sum_{n=0}^{\infty} n \frac{H_{n-1}'(x)}{n!} t^n - \sum_{n=0}^{\infty} n \frac{H_n(x)}{n!} t^n = 0$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} [xH_n'(x) - nH_{n-1}'(x) - nH_n(x)] = 0$$

$$xH_n'(x) - nH_{n-1}'(x) - nH_n(x) = 0\,. \tag{4.12}$$

$\rightarrow$ $e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$ take derivative with respect to x.

$$2te^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n$$

$$2t\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n$$

$$2\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^{n+1}=\sum_{n=0}^{\infty}\frac{H_n'(x)}{n!}t^n,$$

for the LHS of the equation take $n-1$ instead of n.

$$2\sum_{n=1}^{\infty}\frac{H_{n-1}(x)}{(n-1)!}t^n = H_0'(x)+\sum_{n=1}^{\infty}\frac{H_n'(x)}{n!}t^n$$

$$\sum_{n=1}^{\infty}t^n\left(\frac{H_{n-1}(x)}{(n-1)!}-\frac{H_n'(x)}{n!}\right)=0$$

$$2\frac{H_{n-1}(x)}{(n-1)!}-\frac{H_n'(x)}{n!}=0$$

$$2\frac{H_{n-1}(x)}{(n-1)!}-\frac{H_n'(x)}{n(n-1)!}=0$$

$$2nH_{n-1}(x)-H_n'(x)=0. \quad (4.13)$$

Let’s use the equation (4.12) in (4.13);

$xH_n'(x)-nH_{n-1}'(x)-nH_n(x)=0$

$2nH_{n-1}(x)-H_n'(x)=0 \quad \rightarrow \quad H_n'(x)=2nH_{n-1}(x)$

$2xnH_{n-1}(x)-nH_{n-1}'(x)-nH_n(x)=0.$

Divide both sides of the above equation with n,

$$H_n(x)=2xH_{n-1}(x)-H_{n-1}'(x). \quad (4.14)$$

Let us take $n+1$ instead of n in equation (4.14)

$$H_{n+1}(x)=2xH_n(x)-H_n'(x)\ ,$$

then differentiate both sides with respect to x.

$$H_{n+1}'(x)=2H_n(x)+2xH_n'(x)-H_n''(x)$$

$$2(n+1)H_n(x) = 2H_n(x) + 2xH_n'(x) - H_n''(x).$$

Collect all terms into the one side;

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0\ . \quad (4.15)$$

We reach the hypergeometric type equation with;

$A(x) = 1,$

$B(x) = -2x,$

$\theta_n = 2n.$

4.5 Orthogonality Relation for Hermite Polynomials

Write the hypergeometric equation for index m and n.

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$H_m''(x) - 2xH_m'(x) + 2mH_m(x) = 0.$$

Multiply both equations with $\boldsymbol{e^{-x^2}}$ and write the equations into the self adjoint form;

$$e^{-x^2}H_n''(x) - 2xe^{-x^2}H_n'(x) + 2ne^{-x^2}H_n(x) = 0$$

$$\left[e^{-x^2}H_n'(x)\right]' + 2ne^{-x^2}H_n(x) = 0 \quad (4.16)$$

$$e^{-x^2}H_m''(x) - 2xe^{-x^2}H_m'(x) + 2me^{-x^2}H_m(x) = 0$$

$$\left[e^{-x^2}H_m'(x)\right]' + 2me^{-x^2}H_m(x) = 0. \quad (4.17)$$

Multiply (4.16) with $H_m(x)$ and (4.17) with $H_n(x)$.

$$\left[e^{-x^2}H_n'(x)\right]'H_m(x) + 2ne^{-x^2}H_n(x)H_m(x) = 0$$

$$\left[e^{-x^2}H_m'(x)\right]'H_n(x) + 2me^{-x^2}H_m(x)H_n(x) = 0.$$

Subtract the equations;

$$\left[e^{-x^2}H_n'(x)\right]'H_m(x) - \left[e^{-x^2}H_m'(x)\right]'H_n(x) = 2(m-n)\, e^{-x^2}H_n(x)H_m(x)$$

Open the derivatives;

$$xe^{-x^2}H_n'(x)H_m(x) + e^{-x^2}H_n''(x)H_m(x) - 2xe^{-x^2}H_m'(x)H_n(x) -$$

$$e^{-x^2}H_m''(x)H_n(x) \ = 2(m-n)\ e^{-x^2}H_n(x)H_m(x).$$

$$[e^{-x^2}[H_n'(x)H_m(x) - H_m'(x)H_n(x)]]' = 2(m-n)\ e^{-x^2}H_n(x)H_m(x).$$

Integrate both sides from $-\infty$ to ∞,

$$e^{-x^2}[H_n'(x)H_m(x) - H_m'(x)H_n(x)]|_{-\infty}^{\infty} = 2(m-n)\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x).$$

The left hand sides going to be zero by the conditions of orthogonality which gives us the orthogonality of Hermite Polynomials,

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x) = 0,$$

where the orthogonality interval is $(-\infty, \infty)$ with weight function: $w(x) = e^{-x^2}$.

4.6 Rodrigues Formula for Hermite Polynomials

Now we can give the Rodrigues formula for Hermite polynomials;

where the $K_n = (-1)^n$.

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{d_x^n}(e^{-x^2})\,. \tag{4.18}$$

4.7 Derivative of Hermite Polynomials

From the equation

$$p_n'(x) = p_n'(x) = \frac{-\alpha_n K_n}{w(x).A(x)}\frac{d^{n-1}}{d_x^{n-1}}[w(x).A^n(x)],$$

we can easily obtain the derivative of Hermite.

First us let find what is α_n? By using $\alpha_m = -m\ B'(x) - \frac{1}{2}m(m-1)A''(x)$

$\alpha_n = -n(-2) = 2n.$

$$\frac{d}{dx}H_n(x) = \frac{-2n(-1)^n}{e^{-x^2}}\frac{d^{n-1}}{d_x^{n-1}}\left[e^{-x^2}\right] = \frac{2n(-1)^{n+1}}{e^{-x^2}}\frac{d^{n-1}}{d_x^{n-1}}\left[e^{-x^2}\right]$$

$$= \frac{2n(-1)^2(-1)^{n-1}}{e^{-x^2}}\frac{d^{n-1}}{d_x^{n-1}}\left[e^{-x^2}\right].$$

And since

$$H_{n-1}(x) = (-1)^{n-1}e^{x^2}\frac{d^{n-1}}{d_x^{n-1}}(e^{-x^2})$$

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x)\ . \tag{4.19}$$

4.8 Finding the Coefficinets $\boldsymbol{a_n}$ and $\boldsymbol{c_n}$ for Hermite Polynomials

✓ For $\boldsymbol{a_n}$ we have the formula

$$a_n = K_n\prod_{l=0}^{n-1}[B'(x) + \frac{n+l-1}{2}A''(x)],$$

$$a_n = (-1)^n\prod_{l=0}^{n-1}[-2] = 2^n$$

$$a_n = 2^n. \tag{4.20}$$

✓ For $\boldsymbol{c_n}$ we have the formula

$$c_n = na_n\frac{B_{n-1}(0)}{B'_{n-1}(x)},$$

where $B_n(x) = \ B(x) + nA'(x)$

$B_n(x) = -2x \quad B_{n-1}(0) = 0$

$B'_n(x) = -2 \quad B'_{n-1}(x) = -2$

$$c_n = n2^n0 \qquad \rightarrow \qquad c_n = 0. \tag{4.21}$$

4.9 Normalization Function for Hermite Polynomials

$$\sigma_n = \int_a^b \left(H_n(x)\right)^2 w(x)dx$$

$$\sigma_n = (-1)^n A_{nn} K_n^2 \int_a^b w(x).\, A^n(x) dx$$

$$A_{mn} = \frac{n!}{(n-m)!} \prod_{l=0}^{m-1} [B'(x) + \frac{n+l-1}{2} A''(x)$$

$$A_{nn} = n! \prod_{l=0}^{n-1} (-2) = n!\, (-2)^n$$

$$\sigma_n = (-1)^n (-2)^n n!\, (-1)^{2n} \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2^n n!\, \sqrt{\pi}$$

$$\sigma_n = 2^n n!\, \sqrt{\pi}\,. \tag{4.22}$$

By using the norm of Hermite Polynomials we can give the generalized form for the orthogonality which is equation (2.1)

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 2^n n!\, \sqrt{\pi} \delta_{mn}. \tag{4.23}$$

Chapter 5

LAGUERRE POLYNOMIALS

5.1 Rodrigues Formula and Hypergeometric Representation of Laguerre Polynomials

$$L_n^{\alpha}(x) = \frac{x^{-\alpha}e^{x}}{n!}\frac{d^n}{d_x^n}(e^{-x}x^{n+\alpha}). \tag{5.1}$$

Remember the leibniz formula fort the n-th derivative ;

$$\frac{d^n}{d_x^n}[f(x)h(x)] = \sum_{k=0}^{n}\binom{n}{k}[f(x)]^{(k)}$$

Apply this rule for the term $\frac{d^n}{d_x^n}(e^{-x}x^{n+\alpha})$ where $f(x) = e^{-x}, g(x) = x^{n+\alpha}$.

$$\frac{d^n}{d_x^n}[e^{-x}x^{n+\alpha}] = \sum_{k=0}^{n}\binom{n}{k}[e^{-x}]^{(k)}[x^{n+\alpha}]^{(n-k)}$$

$$= \sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}[(-1)^k e^{-x}(n+\alpha)(n+\alpha-1)(n+\alpha-2)\ldots(n$$

$$+\alpha-n+k+1)x^{n+\alpha-n+k}]$$

$$= \sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}[(-1)^k e^{-x}(n+\alpha)(n+\alpha-1)(n+\alpha-2)\ldots(\alpha$$

$$+k+1)x^{\alpha+k}].$$

Use the fact that;

$$\frac{(\alpha+1)_n}{(\alpha+1)_k}$$

$$= \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+1+k-1)(\alpha+1+k)(\alpha+1+k+1)\ldots(\alpha+1+n-1)}{(\alpha+1)(\alpha+2)\ldots(\alpha+1+k-1)},$$

where

$$\frac{(\alpha+1)_n}{(\alpha+1)_k} = (\alpha+k+1)\ldots(\alpha+n-1)(\alpha+n).$$

$$\frac{d^n}{d_x^n}(e^{-x}x^{n+\alpha}) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,k!} \frac{(\alpha+1)_n}{(\alpha+1)_k}[e^{-x}x^{\alpha+k}]$$

$$\frac{d^n}{d_x^n}(e^{-x}x^{n+\alpha}) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(\alpha+1)_n}{(\alpha+1)_k}[e^{-x}x^{\alpha+k}],$$

put it into equation (5.1)

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(\alpha+1)_n}{(\alpha+1)_k} \frac{x^{-\alpha}e^x}{n!}[e^{-x}x^{\alpha+k}] = \sum_{k=0}^{n} \frac{(-n)_k}{k!\,(\alpha+1)_k} x^k.$$

We reach the hypergeometric representation of Laguerre polynomials since

$$\sum_{k=0}^{n} \frac{(-n)_k}{k!\,(\alpha+1)_k} x^k = {}_1F_1(-n; (\alpha+1); x)$$

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x). \qquad (5.2)$$

5.2 Representation of Laguerre Polynomials with Gamma Functions

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{(\alpha+1)_n}{n!} \frac{x^k}{k!}$$

Since we have

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k},$$

$$\frac{(-n)_k}{n!} = \frac{(-1)^k}{(n-k)!}.$$

And from first property of pocahhammer symbol;

$$(k+1)_n = \frac{\Gamma(k+1+n)}{\Gamma(k+1)}$$

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)} \frac{\Gamma(1+\alpha)}{\Gamma(k+1+\alpha)} \frac{x^k}{k!}$$

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1+k)} \frac{(-x)^k}{(n-k)!\,k!} \ . \tag{5.3}$$

5.3 Generating Function for Laguerre Polynomials

Generating function of Laguerre polynomials has the form;

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \frac{1}{(1-z)^{\alpha+1}} e^{\frac{-xz}{1-z}}. \tag{5.4}$$

Proof:

Start from the LHS;

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \frac{(\alpha+1)_n}{(\alpha+1)_k} \frac{x^k z^n}{k!} = \sum_{k=0}^{\infty} \frac{(-x)^k}{(\alpha+1)_k k!} \sum_{n=k}^{\infty} \frac{(\alpha+1)_n z^n}{(n-k)!},$$

in the second summation let's take $n \to n+k$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \sum_{k=0}^{\infty} \frac{(-x)^k}{(\alpha+1)_k k!} \sum_{n=0}^{\infty} \frac{(\alpha+1)_{n+k} z^{n+k}}{n!}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-x)^k (\alpha+1)_{n+k} z^{n+k}}{(\alpha+1)_k k!\, n!}$$

Since ;

$$\frac{(\alpha+1)_{n+k}}{(\alpha+1)_k}$$

$$= \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+1+k-1)(\alpha+1+k)(\alpha+1+k+1)\ldots(\alpha+1+n+k-1)}{(\alpha+1)(\alpha+2)\ldots(\alpha+1+k-1)}$$

$$= (\alpha+1+k)(\alpha+1+k+1)\ldots(\alpha+1+n+k-1) = (\alpha+1+k)_n.$$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \sum_{k=0}^{\infty} \frac{(-x)^k z^k}{k!} \sum_{n=0}^{\infty} \frac{(\alpha+1+k)_n z^n}{n!}.$$

Use the fact that;

$$\sum_{n=0}^{\infty} \frac{(k)_n x^n}{n!} = \frac{1}{(1-x)^k},$$

so the second summation going to be ; $\frac{1}{(1-z)^{\alpha+1+k}}$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \sum_{k=0}^{\infty} \frac{(-x)^k z^k}{k!} \frac{1}{(1-z)^{\alpha+1+k}} = \frac{1}{(1-z)^{\alpha+1}} \sum_{k=0}^{\infty} \frac{(-xz)^k}{(1-z)^k k!}$$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \frac{1}{(1-z)^{\alpha+1}} e^{\frac{-xz}{1-z}}.$$

5.4 Recurrence Relations for Laguerre Polynomials

Let

$$H(x,z) = \frac{1}{(1-z)^{\alpha+1}} e^{\frac{-xz}{1-z}} = \sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n. \tag{5.5}$$

Take partial derivative with respect to z in (5.5);

$$\frac{\partial H}{\partial z} = (\alpha+1)(1-z)^{-\alpha-2} e^{\frac{-xz}{1-z}} + (1-z)^{-\alpha-1} \frac{-x}{(1-z)^2} e^{\frac{-xz}{1-z}}$$

$$= (\alpha+1)\frac{1}{(1-z)^{\alpha+2}} e^{\frac{-xz}{1-z}} + \frac{-x}{(1-z)^{\alpha+3}} e^{\frac{-xz}{1-z}} = \frac{1}{(1-z)^{\alpha+3}} e^{\frac{-xz}{1-z}} \left[\alpha+1-\frac{-x}{1-z}\right],$$

$$\frac{\partial H}{\partial z} = \frac{1}{(1-z)^2} \frac{1}{(1-z)^{\alpha+1}} e^{\frac{-xz}{1-z}} \left[(\alpha+1)(1-z)-x\right]$$

$$(1-z)^2 \frac{\partial H}{\partial z} = H(x,z)\left[(\alpha+1)(1-z)-x\right].$$

So we get the first recurrence relation;

$$(1-z)^2\frac{\partial H}{\partial z}+[x-(\alpha+1)(1-z)]H(x,z)=0\,. \tag{5.6}$$

Since

$$H(x,z)=\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n,$$

$$\frac{\partial H}{\partial z}=\sum_{n=1}^{\infty}nL_n^{\alpha}(x)z^{n-1}=\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n.$$

Put them into the recurrence relation;

$$(1-z)^2\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n+[x-(\alpha+1)(1-z)]\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n=0$$

$$(1-2z+z^2)\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n+[x-\alpha+\alpha z-1+z]\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n=0$$

$$\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n-2z\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n+z^2\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n+$$

$$x\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n-\alpha\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n-\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n+(\alpha+1)z\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n=0$$

$$\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n-2\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^{n+1}+\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^{n+2}+$$

$$x\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n-\alpha\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n-\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^n+(\alpha+1)\sum_{n=0}^{\infty}L_n^{\alpha}(x)z^{n+1}=0.$$

For the second summation; $n\to n-1$

$$\sum_{n=1}^{\infty}nL_n^{\alpha}(x)z^n=\sum_{n=0}^{\infty}nL_n^{\alpha}(x)z^n,$$

for the third summation; $n \to n-2$

$$\sum_{n=2}^{\infty}(n-1)L_{n-1}^{\alpha}(x)z^n = \sum_{n=1}^{\infty}(n-1)L_{n-1}^{\alpha}(x)z^n,$$

for the last summation; $n \to n-1$

$$\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)z^{n+1} = \sum_{n=1}^{\infty}L_{n-1}^{\alpha}(x)z^n.$$

$$\sum_{n=0}^{\infty}(n+1)L_{n+1}^{\alpha}(x)z^n - 2n\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)z^n + \sum_{n=1}^{\infty}(n-1)L_{n-1}^{\alpha}(x)z^n + x\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)z^n$$

$$- \ \alpha\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)z^n - \sum_{n=0}^{\infty}L_{n}^{\alpha}(x)z^n + (\alpha+1)\sum_{n=1}^{\infty}L_{n-1}^{\alpha}(x)z^n = 0$$

$$(n+1)L_{n+1}^{\alpha}(x) + (-2n + x - \alpha - 1)L_{n}^{\alpha}(x) + (n - 1 + \alpha + 1)L_{n-1}^{\alpha}(x) = 0.$$

We get the three term recurrence relation as;

$$(n+1)L_{n+1}^{\alpha}(x) + (x - 2n - \alpha - 1)L_{n}^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x) = 0. \qquad (5.7)$$

For $n = 1,2, \ldots$

Now turn back to equation;

$$H(x,z) = \frac{1}{(1-z)^{\alpha+1}}e^{\frac{-xz}{1-z}} = \sum_{n=0}^{\infty}L_{n}^{\alpha}(x)z^n,$$

take partial derivative with respect to x,

$$\frac{\partial H}{\partial x} = \frac{-z}{1-z}\frac{1}{(1-z)^{\alpha+1}}e^{\frac{-xz}{1-z}} = \frac{-z}{1-z}\ H(x,z)$$

$$(1-z)\frac{\partial H}{\partial x} = -z\,H(x,z)$$

$$(1-z)\frac{\partial H}{\partial x} + \ z\,H(x,z) = 0\,. \qquad (5.8)$$

Since

$$H(x,z) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n$$

$$\frac{\partial H}{\partial x} = \sum_{n=0}^{\infty} \frac{d}{d_x}[L_n^{\alpha}(x)] z^n.$$

Let put them into recurrence relation (5.8);

$$(1-z)\sum_{n=0}^{\infty} \frac{d}{d_x}[L_n^{\alpha}(x)] z^n + z\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = 0$$

$$\sum_{n=0}^{\infty} \frac{d}{d_x}[L_n^{\alpha}(x)] z^n - \sum_{n=0}^{\infty} \frac{d}{d_x}[L_n^{\alpha}(x)] z^{n+1} + \sum_{n=0}^{\infty} L_n^{\alpha}(x) z^{n+1} = 0.$$

For the first and second summation let $n \to n-1$

$$\sum_{n=0}^{\infty} \frac{d}{d_x}[L_n^{\alpha}(x)] z^{n+1} = \sum_{n=1}^{\infty} \frac{d}{d_x}[L_{n-1}^{\alpha}(x)] z^n,$$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^{n+1} = \sum_{n=1}^{\infty} L_{n-1}^{\alpha}(x) z^n,$$

$$\sum_{n=0}^{\infty} \frac{d}{d_x}[L_n^{\alpha}(x)] z^n - \sum_{n=1}^{\infty} \frac{d}{d_x}[L_{n-1}^{\alpha}(x)] z^n + \sum_{n=1}^{\infty} L_{n-1}^{\alpha}(x) z^n = 0.$$

So we get;

$$\frac{d}{d_x}[L_n^{\alpha}(x)] - \frac{d}{d_x}[L_{n-1}^{\alpha}(x)] + L_{n-1}^{\alpha}(x) = 0\,. \qquad (5.9)$$

Turn back to three term recurrence relation;

$$(n+1)L_{n+1}^{\alpha}(x) + (x-2n-\alpha-1)L_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x) = 0.$$

Write the second term again as a;

$$(x-2n-\alpha-1)L_n^\alpha(x) = xL_n^\alpha(x)-(n+1)L_n^\alpha(x)-(n+\alpha)L_n^\alpha(x)$$

$$(n+1)L_{n+1}^\alpha(x)+\ xL_n^\alpha(x)-(n+1)L_n^\alpha(x)-(n+\alpha)L_n^\alpha(x)+(n+\alpha)L_{n-1}^\alpha(x)$$

$$=0$$

$$xL_n^\alpha(x)+(n+1)[L_{n+1}^\alpha(x)-L_n^\alpha(x)]-(n+\alpha)[L_n^\alpha(x)-L_{n-1}^\alpha(x)]=0. \qquad (5.10)$$

Differentiate (5.10) with respect to x;

$$L_n^\alpha(x)+x\frac{d}{dx}[L_n^\alpha(x)]+(n+1)\left[\frac{d}{dx}L_{n+1}^\alpha(x)-\frac{d}{dx}L_n^\alpha(x)\right]-$$

$$(n+\alpha)\left[\frac{d}{dx}L_n^\alpha(x)-\frac{d}{dx}L_{n-1}^\alpha(x)\right]=0$$

$$L_n^\alpha(x)+x\frac{d}{dx}[L_n^\alpha(x)]-(n+1)L_n^\alpha(x)+(n+\alpha)L_{n-1}^\alpha(x)=0$$

$$x\frac{d}{dx}[L_n^\alpha(x)]=nL_n^\alpha(x)-(n+\alpha)L_{n-1}^\alpha(x). \qquad (5.11)$$

Differentiate (5.11) with respect to x;

$$\frac{d}{dx}[L_n^\alpha(x)]+x\frac{d^2}{d_x^2}[L_n^\alpha(x)]=n\frac{d}{dx}[L_n^\alpha(x)]-(n+\alpha)\frac{d}{dx}L_{n-1}^\alpha(x)$$

$$=(n+\alpha)\left[\frac{d}{dx}L_n^\alpha(x)-\frac{d}{dx}L_{n-1}^\alpha(x)\right]-\alpha\frac{d}{dx}L_n^\alpha(x)$$

$$=-(n+\alpha)L_{n-1}^\alpha(x)-\ \alpha\frac{d}{dx}L_n^\alpha(x)$$

$$\frac{d}{dx}[L_n^\alpha(x)]+x\frac{d^2}{d_x^2}[L_n^\alpha(x)]x=\frac{d}{dx}[L_n^\alpha(x)]-nL_n^\alpha(x)-\ \alpha\frac{d}{dx}L_n^\alpha(x)$$

$$x\frac{d^2}{d_x^2}[L_n^\alpha(x)]+(1+\alpha-x)\ \frac{d}{dx}[L_n^\alpha(x)]+\ nL_n^\alpha(x)=0.$$

We reach the hypergeometric type equation with;

$A(x)=x,$

$B(x) = 1 + \alpha - x,$

$\theta_n = n.$

5.5 Orthogonality Relation for Laguerre Polynomials

Write the hypergeometric equation for index m and n.

$$x\frac{d^2}{d_x^2}[L_n^\alpha(x)] + (1+\alpha-x)\,\frac{d}{dx}[L_n^\alpha(x)] + \,nL_n^\alpha(x) = 0$$

$$x\frac{d^2}{d_x^2}[L_m^\alpha(x)] + (1+\alpha-x)\,\frac{d}{dx}[L_m^\alpha(x)] + \,mL_m^\alpha(x) = 0.$$

Multiply both equation with $e^{-x}x^{\alpha}$.

$$e^{-x}x^{\alpha+1}\frac{d^2}{d_x^2}[L_n^\alpha(x)] + (1+\alpha-x)e^{-x}x^{\alpha}\,\frac{d}{dx}[L_n^\alpha(x)] + \,ne^{-x}x^{\alpha}L_n^\alpha(x) = 0$$

$$e^{-x}x^{\alpha+1}\frac{d^2}{d_x^2}[L_m^\alpha(x)] + (1+\alpha-x)e^{-x}x^{\alpha}\,\frac{d}{dx}[L_m^\alpha(x)] + \,me^{-x}x^{\alpha}L_m^\alpha(x) = 0.$$

Write the equations in the self adjoint form;

$$[e^{-x}x^{\alpha+1}\frac{d}{dx}[L_n^\alpha(x)]]' + \,ne^{-x}x^{\alpha}L_n^\alpha(x) = 0$$

$$[e^{-x}x^{\alpha+1}\frac{d}{dx}[L_m^\alpha(x)]]' + \,me^{-x}x^{\alpha}L_m^\alpha(x) = 0.$$

Multiply first equation with $L_m^\alpha(x)$ and the second one with $L_n^\alpha(x)$

$$[e^{-x}x^{\alpha+1}\frac{d}{dx}[L_n^\alpha(x)]]'\,L_m^\alpha(x) + \,ne^{-x}x^{\alpha}L_n^\alpha(x)\,L_m^\alpha(x) = 0$$

$$[e^{-x}x^{\alpha+1}\frac{d}{dx}[L_m^\alpha(x)]]'L_n^\alpha(x) + \,me^{-x}x^{\alpha}L_m^\alpha(x)L_n^\alpha(x) = 0.$$

Subtract the equations;

$$[e^{-x}x^{\alpha+1}\frac{d}{dx}[L_n^\alpha(x)]]'\,L_m^\alpha(x)-[e^{-x}x^{\alpha+1}\frac{d}{dx}[L_m^\alpha(x)]]'L_n^\alpha(x)$$

$$=(m-n)\,e^{-x}x^\alpha L_n^\alpha(x)\,L_m^\alpha(x)$$

$$\frac{d}{dx}[e^{-x}x^{\alpha+1}]\frac{d}{dx}[L_n^\alpha(x)]\,L_m^\alpha(x)+e^{-x}x^{\alpha+1}\frac{d^2}{d_x^2}[L_n^\alpha(x)]\,L_m^\alpha(x)$$

$$-\frac{d}{dx}[e^{-x}x^{\alpha+1}]\,\frac{d}{dx}[L_m^\alpha(x)]\,L_n^\alpha(x)-[e^{-x}x^{\alpha+1}]\frac{d^2}{d_x^2}[L_m^\alpha(x)]L_n^\alpha(x)$$

$$=(m-n)\,e^{-x}x^\alpha L_n^\alpha(x)\,L_m^\alpha(x)$$

$$\{[e^{-x}x^{\alpha+1}][\frac{d}{dx}[L_n^\alpha(x)]L_m^\alpha(x)-\frac{d}{dx}[L_m^\alpha(x)]\,L_n^\alpha(x)]\}'$$

$$=(m-n)\,e^{-x}x^\alpha L_n^\alpha(x)\,L_m^\alpha(x).$$

Integrate both sides from 0 to ∞.

$$[e^{-x}x^{\alpha+1}]\left[\frac{d}{dx}[L_n^\alpha(x)\right]L_m^\alpha(x)-\frac{d}{dx}[L_m^\alpha(x)]\,L_n^\alpha(x)]|_0^\infty$$

$$=(m-n)\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)\,L_m^\alpha(x)d_x.$$

The left hand side going to be zero by the conditions of orthogonality which gives us the orthogonality of Laguerre Polynomials.

$$\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)\,L_m^\alpha(x)d_x=0\,. \qquad (5.12)$$

Orthogonality interval: $[0,\infty)$

Weight function: $w(x)=e^{-x}x^\alpha$

Here we can give the rodrigues formula again for the Laguerre Polynomials;

Where the $K_n=\frac{1}{n!}$

$$L_n^\alpha(x)=\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{d_x^n}(e^{-x}x^{n+\alpha})\,. \qquad (5.13)$$

5.6 Derivative of Laguerre Polynomials

From the equation

$$p_n'(x) = p_n'(x) = \frac{-\alpha_n K_n}{w(x).A(x)} \frac{d^{n-1}}{d_x^{n-1}} [w(x).A^n(x)],$$

we can easily obtain the derivative of Laguerre.

First let find what is α_n? By using $\alpha_m = -m\, B'(x) - \frac{1}{2} m(m-1)A''(x)$

$\alpha_n = n.$

$$\frac{d}{dx} L_n^\alpha(x) = \frac{-n}{e^{-x} x^{1+\alpha} n!} \frac{d^{n-1}}{d_x^{n-1}} (e^{-x} x^{n+\alpha}) = \frac{-1}{e^{-x} x^{1+\alpha} (n-1)!} \frac{d^{n-1}}{d_x^{n-1}} (e^{-x} x^{n-1+\alpha+1}),$$

and since

$$L_{n-1}^{\alpha+1}(x) = \frac{x^{-\alpha+1} e^x}{(n-1)!} \frac{d^{n-1}}{d_x^{n-1}} (e^{-x} x^{n+\alpha}).$$

$$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x). \quad (5.14)$$

5.7 Finding the Coefficients a_n and c_n for Laguerre Polynomials

- ✓ For $\boldsymbol{a_n}$ we have the formula

$$a_n = K_n \prod_{l=0}^{n-1} [B'(x) + \frac{n+l-1}{2} A''(x)].$$

$$a_n = \frac{1}{n!} \prod_{l=0}^{n-1} (-1)$$

$$a_n = \frac{(-1)^n}{n!}. \quad (5.15)$$

- ✓ For $\boldsymbol{c_n}$ we have the formula

$$c_n = n a_n \frac{B_{n-1}(0)}{B_{n-1}'(x)},$$

where

$B_n(x) = B(x) + nA'(x)$. $\quad B_n(x) = 1 + \alpha - x + n$

$B_{n-1}(x) = 1 + \alpha - x + n - 1 = \alpha - x + n$

$B_{n-1}(0) = \alpha + n \qquad B'_{n-1}(x) = -1$

$$c_n = n\frac{(-1)^n}{n!}\frac{\alpha + n}{-1}$$

$$c_n = \frac{(-1)^{n-1}(\alpha + n)}{(n-1)!} \quad . \tag{5.16}$$

5.8 Normalization Function for Laguerre Polynomials

$$\sigma_n = (-1)^n A_{nn} K_n^2 \int_a^b w(x).A^n(x)dx$$

$$A_{mn} = \frac{n!}{(n-m)!}\prod_{l=0}^{m-1}[B'^{(x)} + \frac{n+l-1}{2}A''(x)]$$

$$A_{nn} = n!\prod_{l=0}^{n-1}(-1) = n!\,(-1)^n$$

$$\sigma_n = (-1)^{2n} n!\,(\frac{1}{n!})^2 \int_0^\infty e^{-x}x^{n+\alpha}dx$$

$$\sigma_n = \frac{\Gamma(n+\alpha+1)}{n!} \quad . \tag{5.17}$$

By using the norm of Laguerrre Polynomials we can give the generalized form for the orthogonality which is equation (2.1)

$$\int_0^\infty L_n^\alpha(x)L_m^\alpha(x)e^{-x}x^{n+\alpha}dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}. \tag{5.19}$$

Chapter 6

JACOBI POLYNOMIALS

6.1 Rodrigues Formula and Hypergeometric Representation of Jacobi Polynomials

Definition6.1: The rodrigues formula for Jacobi Polynomails are defined as

$$P_n^{(\alpha,\beta)}(x)$$
$$= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{d_x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]. \qquad (6.1)$$

There are 4 different hypergeometric representation for the Jacobi Polynomials.

→ To obtain the first representation start from the term $\frac{d^n}{d_x^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]$ and apply the leibnz formula for $f(x) = (1-x)^{\alpha+1}$ and $g(x) = (1+x)^{\beta+1}$

$$\frac{d^n}{d_x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] = \sum_{k=0}^{n}\binom{n}{k}[(1-x)^{\alpha+n}]^{(k)}[(1+x)^{\beta+n}]^{(n-k)}$$

$$= \sum_{k=0}^{n}\frac{n!\,(-1)^k}{k!\,(n-k)!}(\alpha+n)(\alpha+n-1)\ldots(\alpha+n-k+1)(1-x)^{\alpha+n-k} *$$

$$(\beta+n)(\beta+n-1)\ldots(\beta+n+k+1)(1+x)^{\beta+k}.$$

Now turn back to seventh property of pochhammer symbol;

$$\frac{n!\,(-1)^k}{k!\,(n-k)!} = (-n)_k,$$

and change the term;

$(\alpha+n)(\alpha+n-1)\dots(\alpha+n-k+1)=(-1)^k(-\alpha-n)_k.$

And change the term;

$$(\beta+n)(\beta+n-1)\dots(\beta+n+k+1)=\frac{(1+\beta)_n}{(1+\beta)_k}.$$

$$\frac{d^n}{d_x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]=\sum_{k=0}^{n}\frac{(-n)_k(-1)^k(-\alpha-n)_k(1+\beta)_n}{k!\,(1+\beta)_k}(1-x)^{\alpha+n-k}$$

$$*\,(1+x)^{\beta+k}\ . \qquad (6.2)$$

Let write (6.2) into the rodrigues formula and we get

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(-n)_k(-1)^k(-\alpha-n)_k(1+\beta)_n}{k!\,(1+\beta)_k}\frac{(-1)^n}{2^n n!}(1-x)^{n-k}(1+x)^k$$

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(-n)_k(-\alpha-n)_k(1+\beta)_n}{k!\,(1+\beta)_k}\frac{(x-1)^n}{2^n n!}\frac{(1+x)^k}{(x-1)^k}$$

$$P_n^{(\alpha,\beta)}(x)=\frac{(x-1)^n(1+\beta)_n}{2^n n!}\sum_{k=0}^{n}\frac{(-n)_k(-\alpha-n)_k}{k!\,(1+\beta)_k}\frac{(1+x)^k}{(x-1)^k} \qquad (6.3)$$

$$P_n^{(\alpha,\beta)}(x)=\left(\frac{x-1}{2}\right)^n\frac{(1+\beta)_n}{n!}\quad {}_2F_1\left(-n,-n-\alpha;\beta+1;\frac{1+x}{x-1}\right). \qquad (6.4)$$

→ To obtain the second representation in similar way start from the term $\frac{d^n}{d_x^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]$ and apply the leibnz formula for $f(x)=(1+x)^{\beta+1}$ and $g(x)=(1-x)^{\alpha+1}$.

$$\frac{d^n}{d_x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]=\sum_{k=0}^{n}\binom{n}{k}[(1-x)^{\alpha+n}]^{(n-k)}[(1+x)^{\beta+n}]^{(k)}$$

$$=\sum_{k=0}^{n}\frac{n!\,(-1)^{n-k}}{k!\,(n-k)!}(\alpha+n)(\alpha+n-1)\dots(\alpha+k+1)(1-x)^{\alpha+k}\,*$$

$$(\beta + n)(\beta + n - 1) \ldots (\beta + n - k + 1)(1 + x)^{\beta+n-k}.$$

Now turn back to seventh property of pochhammer symbol;

$$\frac{n!\,(-1)^k}{k!\,(n-k)!} = (-n)_k,$$

and the term;

$$(\alpha + n)(\alpha + n - 1) \ldots (\alpha + k + 1) = \frac{(1+\alpha)_n}{(1+\alpha)_k},$$

and the term;

$$(\beta + n)(\beta + n - 1) \ldots (\beta + n - k + 1) = (-1)^k(-\beta - n)_{k.}$$

$$\frac{d^n}{d_x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$

$$= \sum_{k=0}^{n} \frac{(-n)_k(-1)^{n-k}(1+\alpha)_n(-\beta-n)_k}{k!\,(1+\alpha)_k}(1-x)^{\alpha+k}$$

$$* \, (1+x)^{\beta+n-k} \quad . \qquad (6.5)$$

Let write (6.5) into the rodrigues formula and we get

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(-n)_k(-1)^{n-k}(-\beta-n)_k(1+\alpha)_n}{k!\,(1+\alpha)_k}\frac{(-1)^n}{2^n n!}(1-x)^k(1+x)^{n-k}$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(-n)_k(-\beta-n)_k(1+\alpha)_n}{k!\,(1+\alpha)_k}\frac{(1+x)^n}{2^n n!}\frac{(x-1)^k}{(1+x)^k}$$

$$P_n^{(\alpha,\beta)}(x) = (\frac{1+x}{2})^n \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n} \frac{(-n)_k(-\beta-n)_k}{k!\,(1+\alpha)_k}\frac{(x-1)^k}{(1+x)^k}$$

$$P_n^{(\alpha,\beta)}(x)$$

$$= \left(\frac{1+x}{2}\right)^n \frac{(1+\alpha)_n}{n!} \quad {}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{1+x}{x-1}\right). \qquad (6.6)$$

✓ For the third representation we are going to use equation (6.3) as a

$$P_n^{(\alpha,\beta)}(x) = \frac{(x-1)^n(1+\beta)_n}{2^n n!}\sum_{k=0}^{n}\frac{(-n)_k(-\alpha-n)_k}{k!\,(1+\beta)_k}\frac{(1+x)^k}{(x-1)^k}$$

$$P_n^{(\alpha,\beta)}(x) = \left[\frac{x-1}{2}\right]^n \frac{(1+\beta)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(-\alpha-n)_k}{k!\,(1+\beta)_k}\left(\frac{x+1}{x-1}\right)^k . \qquad (6.7)$$

Where

$$\left(\frac{x+1}{x-1}\right)^k = \left(1+\frac{2}{x-1}\right)^k .$$

and now we are going to use binomial expression

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k},$$

$$\left(1+\frac{2}{x-1}\right)^k = \sum_{l=0}^{k}\binom{k}{l}\left(\frac{2}{x-1}\right)^l .$$

let write this term into (6.7),

And also instead of terms let write

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!},$$

$$(-\alpha-n)_k = \frac{(-1)^k(n+\alpha)!}{(n+\alpha-k)!},$$

$$\frac{(1+\beta)_n}{(1+\beta)_k} = \frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+1)}{\Gamma(\beta+k+1)} = \frac{(\beta+n)!}{(\beta+k)!}.$$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^k n!}{(n-k)!}\frac{(-1)^k(n+\alpha)!}{n!\,(n+\alpha-k)!}$$

$$* \frac{(\beta+n)!}{(\beta+k)!\,(k-l)!\,l!}\left(\frac{2}{x-1}\right)^l$$

$$= \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(n+\alpha)!}{(n-k)!\,(n+\alpha-k)!}\frac{(\beta+n)!}{(\beta+k)!\,(k-l)!\,l!}\left(\frac{2}{x-1}\right)^l$$

$$= \left(\frac{x-1}{2}\right)^n \sum_{l=0}^{n} \sum_{k=l}^{n} \frac{(n+\alpha)!}{(n-k)!\,(n+\alpha-k)!} \frac{(\beta+n)!}{(\beta+k)!\,(k-l)!\,l!} \left(\frac{2}{x-1}\right)^l.$$

Let take $k \to k+l$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{l=0}^{n} \sum_{k=0}^{n-l} \frac{(n+\alpha)!}{(n-k-l)!\,(n+\alpha-k-l)!}$$

$$* \frac{(\beta+n)!}{(\beta+k+l)!\,k!\,l!} \left(\frac{2}{x-1}\right)^l,$$

and now let take $l \to n-l$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{l=0}^{n} \sum_{k=0}^{n} \frac{(n+\alpha)!}{(l-k)!\,(\alpha-k+l)!}$$

$$* \frac{(\beta+n)!}{(\beta+k+n-l)!\,k!\,(n-l)!} \left(\frac{2}{x-1}\right)^{n-l}.$$

Let's change the terms;

$$(n-l)! = \frac{(-1)^l n!}{(-n)_l},$$

$$(l-k)! = \frac{(-1)^k l!}{(-l)_k},$$

$$(n+\alpha)! = \Gamma(\alpha+1+n),$$

$$(n+\beta)! = \Gamma(\beta+1+n),$$

$$(\alpha-k+l)! = \Gamma(\alpha-k+l+1)$$

$$= \Gamma(\alpha+l+1)(\alpha+l+1)_{-k}$$

$$= \frac{\Gamma(\alpha-k+l+1)(-1)^k}{(-\alpha-l)_k},$$

$$(\beta+k+n-l)! = \Gamma(\beta+k+n-l+1) = \Gamma(\beta+n-l+1)(\beta+n-l+1)_k.$$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{l=0}^{n}\sum_{k=0}^{n} \frac{\Gamma(\alpha+1+n)(-n)_l}{(-1)^l n!\,\Gamma(\alpha+l+1)(-1)^k}$$

$$* \frac{(-l)_k \Gamma(\beta+1+n)(-\alpha-l)_k}{\Gamma(\beta+n-l+1)(\beta+n-l+1)_k k!\,(-1)^l l!}\left(\frac{2}{x-1}\right)^{n-l}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}{n!}$$

$$* \sum_{l=0}^{n} \frac{(-n)_l}{\Gamma(\alpha+l+1)\Gamma(\beta+n-l+1)l!}\left(\frac{1-x}{2}\right)^l \sum_{k=0}^{n} \frac{(-l)_k(-\alpha-l)_k}{(\beta+n-l+1)_k}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}{n!}$$

$$* \sum_{l=0}^{n} \frac{(-n)_l(\beta+n+\alpha+1)_l}{\Gamma(\alpha+1)(\alpha+1)_l\Gamma(\beta+n-l+1)(\beta+n-l+1)_l l!}\left(\frac{1-x}{2}\right)^l$$

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}{n!}$$

$$* \sum_{l=0}^{n} \frac{(-n)_l(\beta+n+\alpha+1)_l}{\Gamma(\alpha+1)(\alpha+1)_l\Gamma(\beta+n-l+1)(\beta+n-l+1)_l l!}\left(\frac{1-x}{2}\right)^l$$

$$P_n^{(\alpha,\beta)}(x)$$

$$= \frac{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}{n!} \sum_{l=0}^{n} \frac{(-n)_l(\beta+n+\alpha+1)_l}{\Gamma(\alpha+1)(\alpha+1)_l\Gamma(\beta+n+1)l!}\left(\frac{1-x}{2}\right)^l$$

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+1+n)}{n!\,\Gamma(\alpha+1)} \sum_{l=0}^{n} \frac{(-n)_l(\beta+n+\alpha+1)_l}{(\alpha+1)_l l!}\left(\frac{1-x}{2}\right)^l$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \quad {}_2F_1\left(-n, \beta+n+\alpha+1; \alpha+1; \tfrac{1-x}{2}\right). \qquad (6.8)$$

→ In a similar way we can get the last representation of the Jacobi Polynomials which is

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\beta+1)_n}{n!} \quad {}_2F_1\left(-n, \beta+n+\alpha+1; \beta+1; \frac{1+x}{2}\right). \qquad (6.9)$$

6.2 Symmetry Property of Jacobi Polynomials

From the equation

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(-n)_k(-1)^k(-\alpha-n)_k(1+\beta)_n}{k!\,(1+\beta)_k} \frac{(-1)^n}{2^n n!} (1-x)^{n-k}(1+x)^k$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(-n)_k(-\alpha-n)_k(1+\beta)_n}{k!\,(1+\beta)_k} \frac{1}{2^n n!} (x-1)^{n-k}(1+x)^k,$$

let's change the terms;

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!},$$

$$(-\alpha-n)_k = \frac{(-1)^k(n+\alpha)!}{(n+\alpha-k)!},$$

$$\frac{(1+\beta)_n}{(1+\beta)_k} = \frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+1)}{\Gamma(\beta+k+1)} = \frac{(\beta+n)!}{(\beta+k)!}.$$

We get;

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \frac{(-1)^k n!\,(-1)^k(n+\alpha)!\,(\beta+n)!}{k!\,(\beta+k)!\,(n+\alpha-k)!\,(n-k)!\,n!} (x-1)^{n-k}(1+x)^k$$

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \frac{(n+\alpha)!\,(\beta+n)!}{k!\,(\beta+k)!\,(n+\alpha-k)!\,(n-k)!} (x-1)^{n-k}(1+x)^k$$

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k} (x-1)^{n-k}(1+x)^k. \qquad (6.10)$$

Let take $x = -x$.

$$P_n^{(\alpha,\beta)}(-x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k} (-x-1)^{n-k}(1-x)^k$$

$$P_n^{(\alpha,\beta)}(-x) = \frac{(-1)^n}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k} (x+1)^{n-k}(x-1)^k\,. \qquad (6.11)$$

From (6.10) and (6.11) we get symmetry relation as a

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{6.12}$$

From this relation easily we can say that Jacobi polynomials are odd or even function depending on the degree n, of the polynomial.

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!},$$

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(\beta+1)_n}{n!}.$$

6.3 Generating Function of Jacobi Polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\beta+1)_n}{n!} \quad {}_2F_1\left(-n, \beta+n+\alpha+1; \beta+1; \frac{1+x}{2}\right)$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k!}\left[\frac{1-x}{2}\right]^k$$

$$= \sum_{k=0}^{n}\frac{(-1)^k(-n)_k(n+\alpha+\beta+1)_k(\alpha+1)_n}{(\alpha+1)_k k!\, n!}\left[\frac{x-1}{2}\right]^k$$

$$\frac{(-1)^k(-n)_k}{n!} = \frac{1}{(n-k)!}$$

$$(n+\alpha+\beta+1)_k = \frac{(\alpha+\beta+1)_{n+k}}{(\alpha+\beta+1)_n}$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\frac{(\alpha+\beta+1)_{n+k}(\alpha+1)_n}{(n-k)!\, k!\,(\alpha+1)_k(\alpha+\beta+1)_n}\left[\frac{x-1}{2}\right]^k.$$

Now let us check the term ;

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)t^n}{(\alpha+1)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha+\beta+1)_{n+k}}{(n-k)!\,k!\,(\alpha+1)_k}\left[\frac{x-1}{2}\right]^k t^n,$$

let $n \to n+k$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha+\beta+1)_{n+2k}}{n!\,k!\,(\alpha+1)_k}\left[\frac{x-1}{2}\right]^k t^{n+k}.$$

From the fourth property of pochhammer symbol;

$$(\alpha+\beta+1)_{n+2k} = (\alpha+\beta+1)_{2k}(\alpha+\beta+1+2k)_n.$$

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)t^n}{(\alpha+1)_n}$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha+\beta+1)_{2k}}{(\alpha+1)_k k!}\left[\frac{t(x-1)}{2}\right]^k \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1+2k)_n}{n!} t^n$$

$$= \frac{1}{(1-t)^{\alpha+\beta+1+2k}} \sum_{k=0}^{\infty} \frac{(\alpha+\beta+1)_{2k}}{(\alpha+1)_k k!}\left[\frac{t(x-1)}{2}\right]^k.$$

From the third property of pochhammer symbol;

$$(\alpha+\beta+1)_{2k} = 2^{2k}\left({}^{1}\!/_{2}\,(\beta+\alpha+1)\right)_k\left({}^{1}\!/_{2}\,(\beta+\alpha+2)\right)_k.$$

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)t^n}{(\alpha+1)_n}$$

$$= \frac{1}{(1-t)^{\alpha+\beta+1+2k}} \sum_{k=0}^{\infty} \frac{2^{2k}\left({}^{1}\!/_{2}\,(\beta+\alpha+1)\right)_k\left({}^{1}\!/_{2}\,(\beta+\alpha+2)\right)_k}{(\alpha+1)_k k!}\left[\frac{t(x-1)}{2}\right]^k$$

$$= \frac{1}{(1-t)^{\alpha+\beta+1}} \sum_{k=0}^{\infty} \frac{\left({}^{1}\!/_{2}\,(\beta+\alpha+1)\right)_k\left({}^{1}\!/_{2}\,(\beta+\alpha+2)\right)_k}{(\alpha+1)_k k!}\left[\frac{2^2 t(x-1)}{2(1-t)^2}\right]^k$$

$$= \frac{1}{(1-t)^{\alpha+\beta+1}} \sum_{k=0}^{\infty} \frac{\left({}^{1}\!/_{2}\,(\beta+\alpha+1)\right)_k\left({}^{1}\!/_{2}\,(\beta+\alpha+2)\right)_k}{(\alpha+1)_k k!}\left[\frac{2t(x-1)}{(1-t)^2}\right]^k$$

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x)t^n}{(\alpha+1)_n} = \frac{1}{(1-t)^{\alpha+\beta+1}} *$$

${}_2F_1(1/2\,(\beta+\alpha+1), 1/2\,(\beta+\alpha+2); \alpha+1; \frac{2t(x-1)}{(1-t)^2})$.

6.4 Orthogonality Relation for Jacobi Polynomials

Let $P_n^{(\alpha,\beta)}(x) = y_n(x)$.

Write the hypergeometric equation for index m and n.

$(1-x^2)\, y_n''(x) + [\beta-\alpha-(2+\alpha+\beta)x]y_n'(x) + n(1+n+\alpha+\beta)y_n(x) = 0$

$(1-x^2)\, y_m''(x) + [\beta-\alpha-(2+\alpha+\beta)x]y_m'(x) + m(1+m+\alpha+\beta)y_m(x) = 0.$

Multiply both equation with $(1-x)^\alpha(1+x)^\beta$.

$$(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n^{''(x)} + [\beta-\alpha-(2+\alpha+\beta)x](1-x)^\alpha(1+x)^\beta y_n'(x)$$
$$+ n(1+n+\alpha+\beta)(1-x)^\alpha(1+x)^\beta y_n(x) = 0$$
$$(1-x)^{\alpha+1}(1+x)^{\beta+1}y_m^{''(x)} + [\beta-\alpha-(2+\alpha+\beta)x](1-x)^\alpha(1+x)^\beta y_m'(x)$$
$$+ m(1+m+\alpha+\beta)(1-x)^\alpha(1+x)^\beta y_m(x) = 0.$$

Now write the equations into the self adjoint for

$$\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n'(x)\right]' + n(1+n+\alpha+\beta)(1-x)^\alpha(1+x)^\beta y_n(x)$$
$$= 0 \qquad (6.13)$$
$$\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}y_m'(x)\right]' + m(1+m+\alpha+\beta)(1-x)^\alpha(1+x)^\beta y_m(x)$$
$$= 0. \qquad (6.14)$$

Multiply (6.13) with $y_m(x)$ and (6.14) with $y_n(x)$

$$\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n'(x)\right]' y_m(x) + n(1+n+\alpha+\beta)(1-x)^\alpha *$$
$$(1+x)^\beta y_n(x)y_m(x) = 0 \qquad (6.15)$$

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}y_m{}'(x)]' y_n(x) + m(1+m+\alpha+\beta)(1-x)^{\alpha} * (1+x)^{\beta} y_m(x) y_n(x) = 0. \tag{6.16}$$

Subtract the equations (6.15) and (6.16)

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}y_n{}'(x)]' y_m(x)' - [(1-x)^{\alpha+1}(1+x)^{\beta+1}y_m{}'(x)]' y_n(x) = [m(1+m+\alpha+\beta) - \ n(1+n+\alpha+\beta)]\ (1-x)^{\alpha}(1+x)^{\beta} y_n(x) y_m(x)$$

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}(y_n' y_m - y_m{}' y_n]' = (m-n)(m+n+\alpha+\beta+1) * (1-x)^{\alpha}(1+x)^{\beta} y_n(x) y_m(x).$$

Integrate both sides from -1 to 1.

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}(y_n' y_m - y_m' y_n)]|_{-1}^{1} = (m-n)(m+n+\alpha+\beta+1)\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} y_n(x) y_m(x).$$

Sınce the left hand side is zero we get the orthogonality relation as

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x) = 0. \tag{6.17}$$

Orthogonality interval: $[-1,1]$

Weight function: $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$.

6.5 Finding the Coefficinets a_n and c_n for Jacobi Polynomials

- ✓ For $\boldsymbol{a_n}$ we have the formula

$$a_n = K_n \prod_{l=0}^{n-1}[B'(x) + \frac{n+l-1}{2}A''(x)].$$

$$a_n = \frac{1}{n!}\prod_{l=0}^{n-1}\left(-\alpha-\beta-2+\frac{n+l-1}{2}(-2)\right)$$

$$a_n = \frac{(-1)^n}{2^n n!}\prod_{l=0}^{n-1}(\alpha+\beta+n+l+1)$$

$$= \frac{(-1)^n}{2^n n!}(\alpha+\beta+n+1)(\alpha+\beta+n+2)\ldots(\alpha+\beta+n+1+n-1)$$

$$= (\alpha+\beta+n+1)_n$$

$$a_n = \frac{(\alpha+\beta+n+1)_n}{2^n n!} = \frac{\Gamma(\alpha+\beta+1+2n)}{2^n n!\,\Gamma(\alpha+\beta+1+n)}. \tag{6.18}$$

- ✓ For $\boldsymbol{c_n}$ we have the formula

$$c_n = n a_n \frac{B_{n-1}(0)}{B'_{n-1}(x)},$$

where

$B_n(x) = B(x) + nA'(x).$

$B_n(x) = \beta-\alpha-(\alpha+\beta+2)x+n(-2x) = \beta-\alpha-(\alpha+\beta+2+2n)x$

$B_{n-1}(x) = \beta-\alpha-(\alpha+\beta+2n)x \quad B_{n-1}(0) = \beta-\alpha$

$B_{n-1}^{\prime(x)} = -(\alpha+\beta+2n).$

$$c_n = n\frac{\Gamma(\alpha+\beta+1+2n)}{2^n n!\,\Gamma(\alpha+\beta+1+n)}\frac{\beta-\alpha}{(-1)(\alpha+\beta+2n)}$$

$$c_n = \frac{\Gamma(\alpha+\beta+2n)(\alpha-\beta)}{2^n(n-1)!\,\Gamma(\alpha+\beta+1+n)}. \tag{6.19}$$

6.6 Normalization Function for Jacobi Polynomials

$$\sigma_n = (-1)^n A_{nn} K_n^2 \int_a^b w(x).\,A^n(x)dx$$

$$A_{mn} = \frac{n!}{(n-m)!}\prod_{l=0}^{m-1}\left[B^{\prime(x)} + \frac{n+l-1}{2}A''(x)\right]$$

$$A_{nn} = n! \prod_{l=0}^{n-1} \left[-\alpha - \beta - 2 + \frac{n+l-1}{2}(-2) \right] = n!\,(-1)^n \prod_{l=0}^{n-1} [\alpha + \beta + n + l + 1]$$

$$A_{nn} = \; n!\,(-1)^n (\alpha + \beta + n + 1)_n.$$

Where $K_n = \frac{(-1)^n}{2^n n!}$.

$$\sigma_n = (-1)^n \, n!\,(-1)^n (\alpha + \beta + n + 1)_n \left[\frac{(-1)^n}{2^n n!} \right]^2 \int_{-1}^{1} (1+x)^{n+\beta} (1-x)^{n+\alpha} d_x$$

$$\sigma_n = \frac{(\alpha + \beta + n + 1)_n}{2^{2n} n!} \int_{-1}^{1} (1+x)^{n+\beta} (1-x)^{n+\alpha} d_x.$$

Let us calculate the integral

$$\int_{-1}^{1} (1+x)^{n+\beta} (1-x)^{n+\alpha} d_x$$

Let take $1 + x = 2t \quad d_x = 2d_t$

$x = -1 \quad t = 0, \; x = 1 \quad t = 1$

$$\int_{-1}^{1} (1+x)^{n+\beta} (1-x)^{n+\alpha} d_x = 2 \int_{0}^{1} (2t)^{n+\beta} (2-2t)^{n+\alpha} d_x$$

$$= 2^{1+\alpha+\beta+2n} \int_{0}^{1} (t)^{n+\beta} (1-t)^{n+\alpha} d_x$$

$$= 2^{1+\alpha+\beta+2n} \, \boldsymbol{B}(n + \beta + 1, n + \alpha + 1)$$

$$= 2^{1+\alpha+\beta+2n} \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(2n+\alpha+\beta+2)}. \qquad (6.20)$$

$$\sigma_n = 2^{1+\alpha+\beta+2n} \frac{\Gamma(\alpha+\beta+2n+1)}{2^{2n} n!\,\Gamma(n+\alpha+\beta+1)} \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(2n+\alpha+\beta+2)}$$

$$\sigma_n = 2^{1+\alpha+\beta} \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{n!\,\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)}\,. \qquad (6.21)$$

By using the norm of Jacobi Polynomials we can give the generalized form for the orthogonality which is equation (2.1)

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_{\substack{n\\m}}^{(\alpha,\beta)}(x) e^{-x} x^{n+\alpha} dx$$

$$= \ 2^{1+\alpha+\beta} \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{n!\,\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)} \delta_{mn}. \qquad (6.22)$$

6.7 Three Term Recurrence Relation for Jacobi Polynomials

Since the Jacobi Polynomials satisfy the relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \delta_n p_{n-1}(x),$$

with the coefficients

$$\alpha_n = \frac{a_n}{a_{n+1}},$$

$$\beta_n = \frac{c_n}{a_n} - \frac{c_{n+1}}{a_{n+1}},$$

$$\delta_n = \frac{a_{n-1}}{a_n} * \frac{\sigma_n(x)}{\sigma_{n-1}(x)}.$$

We can find the general relation for the Jacobi.

$$\alpha_n = \frac{\Gamma(\alpha+\beta+1+2n)}{2^n n!\,\Gamma(\alpha+\beta+1+n)} \frac{2^{n+1}(n+1)!\,\Gamma(\alpha+\beta+2+n)}{\Gamma(\alpha+\beta+3+2n)}$$

$$= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+2+\alpha+\beta)(2n+1+\alpha+\beta)}. \qquad (6.23)$$

$$\beta_n = \frac{\Gamma(\alpha+\beta+2n)(\alpha-\beta)}{2^n(n-1)!\,\Gamma(\alpha+\beta+1+n)} \frac{2^n n!\,\Gamma(\alpha+\beta+1+n)}{\Gamma(\alpha+\beta+1+2n)}$$

$$- \frac{\Gamma(\alpha+\beta+2n+2)(\alpha-\beta)}{2^{n+1} n!\,\Gamma(\alpha+\beta+2+n)} \frac{2^{n+1}(n+1)!\,\Gamma(\alpha+\beta+2+n)}{\Gamma(\alpha+\beta+3+2n)}$$

$$= \frac{(\alpha-\beta)}{\alpha+\beta+2n} - \frac{(n+1)(\alpha-\beta)}{\alpha+\beta+2n+2}$$

$$= \frac{n(\alpha-\beta)(\alpha+\beta+2n+2)-(n+1)(\alpha-\beta)(\alpha+\beta+2n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$$

$$\beta_n = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}. \tag{6.24}$$

$$\delta_n = \frac{\Gamma(\alpha+\beta-1+2n)}{2^{n-1}(n-1)!\,\Gamma(\alpha+\beta+n)} \frac{2^n n!\,\Gamma(\alpha+\beta+1+n)}{\Gamma(\alpha+\beta+1+2n)}$$

$$* \; \frac{2^{1+\alpha+\beta}\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{n!\,\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)} \; \frac{(n-1)!\,\Gamma(n+\alpha+\beta)(2n+\alpha+\beta-1)}{2^{1+\alpha+\beta}\Gamma(n+\beta)\Gamma(n+\alpha)}$$

$$= \frac{2\Gamma(\alpha+\beta+2n)(n+\beta)(n+\alpha)}{\Gamma(2n+\alpha+\beta+2)} = \frac{2(n+\beta)(n+\alpha)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}$$

$$\delta_n = \frac{2(n+\beta)(n+\alpha)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}\,. \tag{6.25}$$

Let's put (6.23) , (6.24) and (6.25) in three term recurrence relation.

$$xP_n^{(\alpha,\beta)}(x) = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+2+\alpha+\beta)(2n+1+\alpha+\beta)} P_{n+1}^{(\alpha,\beta)}(x) +$$

$$\frac{\beta^2-\alpha^2}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)} P_n^{(\alpha,\beta)}(x) +$$

$$\frac{2(n+\beta)(n+\alpha)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(x). \tag{6.26}$$

6.8 Derivative of Jacobi Polynomials

From the equation

$$p_n'(x) = p_n'(x) = \frac{-\alpha_n K_n}{w(x).A(x)} \frac{d^{n-1}}{d_x^{n-1}} [w(x).A^n(x)],$$

we can easily obtain the derivative of Laguerre.

First let find what is α_n?

By using $\alpha_m = -m\,B'(x) - \frac{1}{2}m(m-1)A''(x)$

$$\alpha_n = n(\alpha+\beta+2) - \frac{1}{2}n(n-1)(-2) = n(\alpha+\beta+n+1).$$

Where $K_n = \frac{(-1)^n}{2^n n!}$.

$$\frac{d}{d_x}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^{n+1}n(\alpha+\beta+n+1)}{2^n n!\,(1-x)^{\alpha+1}(1+x)^{\beta+1}}\frac{d^{n-1}}{d_x^{n-1}}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]$$

$$= \frac{(-1)^2(-1)^{n-1}(\alpha+\beta+n+1)}{2.2^{n-1}(n-1)!\,(1-x)^{\alpha+1}(1+x)^{\beta+1}}\frac{d^{n-1}}{d_x^{n-1}}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right],$$

Where

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!\,(1-x)^{\alpha+1}(1+x)^{\beta+1}}\frac{d^{n-1}}{d_x^{n-1}}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right].$$

$$\frac{d}{d_x}P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+n+1)}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \qquad (6.27)$$

CONCLUSION

The first and second chapters of this thesis constitute an introduction to orthogonal polynomials. They provide a survey of some general properties satisfied by any set of orthogonal polynomials. . An iterative process to produce a set of polynomials which are orthogonal to one another are given and then a number of properties satisfied by any set of orthogonal polynomials are described. The classical orthogonal polynomials arise when the weight function in the orthogonality condition has a particular form. These polynomials having a further set of properties and in particular satisfy a second order differential equation are studied in Chapter 3. Each subsequent chapter investigated the properties of a particular polynomial set starting from its differential equation. These are classical orthogonal polynomials named as Hermite, Laguerre and Jacobi polynomials. In these chapters, important characteristics of classical orthogonal polynomials such as the weight function, interval of the orthogonality, second order differential equation, Rodrigues formula, hypergeometric representation are given.

REFERENCES

[1] Andrews, G.E. & Askey, R. & Roy, R. (1999). Special Functions. Cambridge University Press, United States of America.

[2] Atakishiyev, N.M. & Rahman, M. & Suslov, S.K. (1995). On Classical Orthogonal Polynomials. Constructive Approximation Springer- Verlag New York Inc,

[3] Branquinho , A. (1996). A Note on Semi-classical Orthogonal Polynomials. Bull. Belg. Math. Soc. 3.

[4] Chihara , T.S. (1956). On Co-recursive Orthogonal Polynomials. Presented to the Society.

[5] Everitta , W.N. & Kwonb , K.H. & Littlejohnc , L.L. & Wellmanc, R. (2001). Orthogonal Polynomial Solutions of Linear Ordinary Differential Equations. Journal of Computational and Applied Mathematics.

[6] Gloden, R.F. (1997). Some Properties of the Orthogonal Polynomials of a Discrete Variable. Brussels Luxembourg, European Commission, Joint Research Center, Institute for Systems, Informatics and Safety 21020 Ispra (VA) Italy.

[7] Gradimir, V.M. (1991).Orthogonal Polynomial Systems and Some Applications. Mathematics Subject Classification.

[8] Hahn W. (1978). On Differential Equations for Orthogonal Polynomials. Technische Universitat Gr az, Austria.

[9] Koekoek, R. & Lesky, P.A. & Swarttouw, R.F. (2010). Hypergeometric Orthogonal Polynomials and Their q-Analogues. Springer-Verlag Berlin Heidelberg.

[10] Nikiforov, A.F. & Suslov, S.K. & Uvarov, V.B. (1991). Classical Orthogonal Polynomials of a Discrete Variable. Springer-Verlag Berlin Heidelberg.

[11] Rainville, E.D. (1960). Special Functions. The Macmillan Company ,New York.