Accelerated Overrelaxation Method for the Solution of Discrete Laplace's equation on a Rectangle

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ABSTRACT

On a rectangle given the Dirichlet Laplace's equation, for its solution by finite differences there exist numerous direct methods and iterative methods. Examples of direct methods are block decomposition, block elimination, block cyclic reduction methods, discrete Fourier transform methods. Among the iterative methods, Successive Overrelaxation Methods, Accelerated Overrelaxation Method (AOM), are widely used methods.

In this thesis we studied the Accelerated Overrelaxation Method (AOR) for the numerical solution of discrete Laplace's equation on a rectangle obtained by 5-point difference scheme. Numerical results are given for different values of the two parameters, w and r and for mesh size h.

Keywords: Successive Overrelaxation Method (SOR), Accelerated Overrelaxation Method (AOR), Laplace's equation, 5-point scheme.

ÖZ

Dikdörtgen üzerinde, Dirichlet sınır koşullu Laplas denklemi verildiğinde sonlu farklar ile sayısal çözümü için birçok doğrudan ve tekrarlama yöntemleri mevcuttur. Doğrudan yöntemlere örnek olarak blok ayrıştırma, blok yok etme, blok döngüsel indirgeme, ayrık Fourier dönüşüm yöntemleri mercuttur. Tekrarlama yöntemleri arasında Successive Overrelaxation yöntemi ve Accelerated Overrelaxation yöntemi sıkça kullanılan metodlardır.

Bu tezde dikdörtgen üzerinde ayrık Laplace denkleminin 5-nokta sonlu fark şeması ile sayısal çözümü için Accelerated Overrelaxation yöntemi çalışılmıştır. Sayısal sonuçlar, w ve r parametrelerinin farklı değerleri için ve adım uzunluğu h için verildi.

Anahtar kelimeler: Successive Overrelaxation Yöntemi, Accelerated Overrelaxation Yöntemi, Laplace denklemi, 5-nokta şeması.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGUES

Chapter 1

INTRODUCTION

1.1 General Knowledge

One of the aim in mathematics is often to solve problems. The solution of a problem is usually done based on some assumptions. A well-defined problem is solved using some specific formula or method. In the fields of physics, chemistry, economics, let us say in sciences, solving a problem usually leads to the use of some equations. There exists various types of equations, arising from various fields of sciences. The type of equation to be considered in this study is the Laplace Equation.

Consider the following equation:

$$\begin{aligned} L(u) \equiv A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} \\ + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y). \end{aligned} \tag{1.1}$$

This is a linear second order partial differential equation with two independent variables x and y; one dependent variable u. The real functions A, B, C, D, E and F of variables x and y are called coefficients. Let R be the domain over which the solution is desired. The coefficients A, B, C, D, E and F are assumed to be twice differentiable with their second derivative continuous over R. From (1.1),

if $L(u) = 0$, $x, y \in R$ then equation (1.1) is called homogeneous equation. If $L(u) \equiv G(x,y) \neq 0$ with $x, y \in R$ then equation (1.1) is called nonhomogeneous equation. A quasi linear first order equation in two independent variables is an equation of the structure

$$P(x,y,u)\frac{\partial u}{\partial x}+Q(x,y,u)\frac{\partial u}{\partial y}=S(x,y,u). \qquad (1.2)$$

The general form of an almost-linear second order equation in two independent variables is

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\,\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=F(x,y,u,u_x,u_y). \quad (1.3)$$

In physical problems the time is a very important parameter. It is therefore common to replace one of the independent variables x or y by the variable t, to refer to the time. The following are some physical well known partial differential equations.

The one-dimensional heat equation

$$L(u)=\frac{\partial u}{\partial t}-\kappa^2\frac{\partial^2 u}{\partial x^2}=G(x,t)\ ;\ 0\le x\le L;\ t>0;\ u=u(x,t). \qquad (1.4)$$

The one-dimensional wave equation

$$L(u)=\frac{\partial^2 u}{\partial t^2}-\tau^2\frac{\partial^2 u}{\partial x^2}=G(x,t)0\le x\le L;\ t>0;\ u=u(x,t). \qquad (1.5)$$

Laplace’s equation

$$L(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ x,y\in R. \qquad (1.6)$$

Poisson equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = G(x,y),\ x,y \in R. \tag{1.7}$$

1.2 Type of Almost-Linear Equations of Two Independent Variables

Let L be the operator defined by

$$\begin{aligned} Lu \equiv {} & A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} \\ & + C(x,y)\frac{\partial^2 u}{\partial y^2} + M(x,y,u,u_x,u_y) = 0, \end{aligned} \tag{1.8}$$

the almost-linear equation in the real independent variables x, y. Let the coefficients A, B, C be real-valued function with continuous second derivatives on a region R of the xy -plane and assume that A, B, C do not vanish simultaneous. The function Δ defined on R by

$$\Delta(x,y) = B^2(x,y) - A(x,y)C(x,y). \tag{1.9}$$

Is called the discriminant of L. The discriminant (1.9) helps to classify the canonical form of the partial differential equation (1.8). The operator L is said to be

1. Hyperbolic at a point (x,y) if $\Delta(x,y) > 0.$
2. Parabolic at a point (x,y) if $\Delta(x,y) = 0.$
3. Elliptic at a point (x,y) if $\Delta(x,y) < 0.$

1.3 Elliptic Differential Equations and Boundary Value Problems

A problem in a class of boundary-value problems of interest in the applications is described as follows. Let R be a bounded region with boundary ∂R and let $\overline{R} = R + \partial R$, the union of R with its boundary ∂R, that is the closure of R let L be a linear second order self-adjoin partial differential operator which is elliptic on $\overline{R}$. A solution of

$$Lu = G(x, y) \text{ in } R, \tag{1.10}$$

is desired such that u is continuous on $\overline{R}$. Here $G(x, y)$ is continuous function on $\overline{R}$.

Dirichlet problem:

Let

$$u = f \text{ on } \partial R, \tag{1.11}$$

where f is a given continuous function on the boundary ∂R. This problem is called Dirichlet problem for the region R. Condition (1.11) is referred as Dirichlet boundary condition.

*N*eumann problem:

A problem of a somewhat different type is to determine a solution of (1.10) that satisfies

$$\frac{\partial u}{\partial n} = f \text{ on } \partial R, \tag{1.12}$$

where $\frac{\partial u}{\partial n}$ denotes the derivative in the direction of the exterior normal on ∂R. This problem is called Neumann problem and the condition (1.12) is called Neumann boundary condition.

Mixed (Robin) boundary problem:

A boundary condition of the form

$$a\frac{\partial u}{\partial n}+bu=f \text{ on } \partial R, \tag{1.13}$$

is a mixed boundary condition. It is assumed that the given function a,b and f are continuous on ∂R and a and b do not vanish simultaneous. The problem of determining a solution of equation (1.10) such that the solution has continuous fist derivatives on $\overline{R}$ and satisfies (1.13) on ∂R is called Mixed or Robin problem.

The type of boundary value problem which will be discuss in this study is the Dirichlet Poisson problem, and specifically the Dirichlet Laplacian problem. It is an elliptic partial differential equation. Its' applications are found in mechanical engineering; electromagnetism, theoretical physic and electrostatics. The most known form of Poisson equation is

$$\Delta\varphi=g. \tag{1.14}$$

In which the symbols are identified as follows: Δ is called the Laplace operator. The functions φ and g are either real or complex functions defined on a manifold. The function g is usually given and the function φ is the sought function. We are usually concerned by real functions and therefore the manifold used is the Euclidean space. When the manifold is the Euclidean space, the Laplace operator is denoted by ∇^2 and the Poisson equation given by (1.14) is defined as follows:

$$\nabla^2\varphi=g, \tag{1.15}$$

And it is expanded as follows in a three dimensional Cartesian coordinate system

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi(x,y,z)=g(x,y,z). \tag{1.16}$$

When the function g mentioned in (1.14), (1.15) and (1.16) is the zero function, then

(1.14); (1.15) and (1.16) are now called Laplace's equation and denoted by

$$\Delta\varphi = 0 \tag{1.17}$$

$$\nabla^2\varphi = 0 \tag{1.18}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi(x,y,z)=0, \tag{1.19}$$

respectively.

1.4 Objectives in the Thesis

There are various methods to solve the Laplace equation on a rectangle with Dirichlet boundary conditions. It can be solved using the green function, or by a numerical method to approach the solution. On a rectangle, given the Dirichlet Laplace's equation, for its solution by finite differences there exist numerous direct methods and iterative methods. Examples of direct methods are block decomposition, block elimination, block cyclic reduction methods, discrete Fourier transform methods. Among the iterative methods, Successive Overrelaxation Methods, Accelerated Overrelaxation Method (AOM), are commonly used method. In this work, we will focus on an iterative method called Accelerated Overrelaxtion Method (AOM) to approach numerically the solution of the discrete Laplace's equation, on a rectangle. In Chapter 2, we study derivation and convergence analysis of the (AOM) for weak diagonal dominant and irreducible matrices, for $L-$matrices and for consistently ordered matrices. The realization of the (AOM) for solving the Dirichlet Laplace problem on a rectangle is also studied. In Chapter 3 numerical result are given for a chosen test problem for various mesh size h and different values of the two parameters. In Chapter 4 concluding remarks are given based on the analysis made.

Chapter 2

ACCELERATED OVERRELAXATION METHOD (AOM) FOR THE NUMERICAL SOLUTION OF LINEAR SYSTEMS OF EQUATION

In this Chapter we study on an iterative method known as the Accelerated Overrelaxation Method (AOR) to obtain the solution of linear systems of equation. Successive Overrelaxation Method is a reduced form of this method when the parameter r is equal to the parameter w.

2.1 Construction of (AOR)

Let A be $n \times n$ real matrix whose diagonal entries are different from zero. Consider the linear system

$$Ax = b, \tag{2.1}$$

and the splitting of the matrix A as follow:

$$A = D - L_{\mathbf{A}} - U_{\mathbf{A}}, \tag{2.2}$$

where the matrices D, $L_{\mathbf{A}}$ and $U_{\mathbf{A}}$ are a diagonal, a lower triangular and an upper triangular matrix respectively. The numerical solution of equation (2.1) is tackle as follow, based on [1], we consider

$$Cx^{(m+1)} = Rx^{(m)} \; m = 0, 1, 2, ..., \tag{2.3}$$

where $C, R \in R^{n\times n}$ and C is nonsingular matrix. It is well know that the iteration (2.3) is convergent iteration if $\rho(C^{-1}R) < 1$, [2], page 214. The proposed scheme is of the form:

$$(\alpha_1 D + \alpha_2 L_{\mathbf{A}})x^{(m+1)} = (\alpha_3 D + \alpha_4 L_{\mathbf{A}} + \alpha_5 U_{\mathbf{A}})x^{(m)} + \alpha_6 b, \tag{2.4}$$

with $m = 0,1,\ldots,$ and $\alpha_i \,|\, i = 1(1)6$ are constants. The constants α_i are to be sought with the conditions that $(\alpha_1 \neq 0)$. The initial approximation $x^{(0)}$ to the solution, is arbitrary. Dividing both sides of the equation (2.4) by α_1 leads to

$$\left(D + \alpha_2' L_{\mathbf{A}}\right) x^{(m+1)} = \left(\alpha_3' D + \alpha_4' L_{\mathbf{A}} + \alpha_5' U_{\mathbf{A}}\right) x^{(m)} + \alpha_6' b, \qquad (2.5)$$

with the coefficients $\alpha_i' = \dfrac{\alpha_i}{\alpha_1}$, $i = 2(1)6$. The scheme defined by (2.5) is consistent with the equation (2.1) under the following conditions:

$$(1-\alpha_3')D + (\alpha_2' - \alpha_4')L_A - \alpha_5' U_A \equiv \alpha_6' A,\ \alpha_6' \neq 0. \qquad (2.6)$$

From (2.2), equations (2.6) yields a two parameters solution given by

$1-\alpha_3' = \alpha_6'$, $\alpha_2' - \alpha_4' = -\alpha_6'$ and $-\alpha_5' = -\alpha_6'$, with the parameters r and $w \neq 0$ as;

$$\alpha_2' = -r,\ \alpha_3' = 1-w,\ \alpha_4' = \mathrm{w} - r,\ \alpha_5' = w, \text{ and } \alpha_6' = w. \qquad (2.7)$$

Therefore (2.5) can be written as:

$$(I - rL)x^{(m+1)} = \left[(1-w)I + (w-r)L + wU\right]x^{(m)} + wc;\ m = 0,1,\ldots, \qquad (2.8)$$

where $L = D^{-1}L_{\mathbf{A}}$, $U = D^{-1}U_{\mathbf{A}}$, $c = D^{-1}b$ and I is $n \times n$ identity matrix. The scheme (2.8) is known as the Accelerated Overrelaxation Method (AOM). It is also called the $M_{r,w}$ and reduces to the following methods as given in Table 2.1; for some specific values of r and w.

Table 2.1: $M_{r,\,w}$ method for some specific values of r and w

$(r,\ w)$	Method
$(0,\ 1)$	$M_{0,\,1}$: Jacobi method
$(1,\ 1)$	$M_{1,\,1}$: Gauss- Seidel method
$(0,\ w)$	$M_{0,\,w}$: Simultaneous Overrelaxation method
$(w,\ w)$	$M_{w,\,w}$: Successive Overrelaxation method

At this point, r and w are called acceleration and overrelaxation parameters respectively. Recalling the scheme described by the equation (2.8), the iterative matrix is represented in that case by $L_{r,\,w}$ and

$$L_{r,\,w} = \left(I - rL\right)^{-1}\left[\left(1-w\right)I + \left(w - r\right)L + wU\right]. \tag{2.9}$$

Let $\rho\left(L_{r,\,w}\right)$ denote the spectral radius of $L_{r,\,w}$. When $r \neq 0$, the Accelerated Overrelaxation Method (AOM) is a form extrapolated Successive Overrelaxation Method (SOR) with the Overrelaxation parameter r and extrapolation parameter $s = \dfrac{w}{r}$. One can easily prove that $L_{r,\,w} = sL_{r,\,w} + \left(1-s\right)I$. Therefore if we consider v to be an eigenvector of $L_{r,\,w}$ $\left(r \neq 0\right)$ and we consider λ, to be the corresponding eigenvalue of $L_{r,\,w}$ then the following relation holds:

$$\lambda = sv + \left(1-s\right). \tag{2.10}$$

For the following sections, our aim will be to study the constraints and conditions on

r and w under which the $M_{r,w}$ method is convergent

2.2 Convergence Analysis for Irreducible Matrices with Weak Diagonal Dominance and L-matrices

Let $G(A)$ be the directed graph of A.

Definition 1: [2], page 126. If, to each ordered pair of disjoint point p_i, p_j in a directed graph $G(A)$ there exist a directed path $\overrightarrow{p_{i0,}\, p_{i1}}$, $\overrightarrow{p_{i1,}\, p_{i2}}, \ldots, \overrightarrow{p_{ir-1}, p_{ir}}$ with $io = i$, $ir = j$ then $G(A)$ is called strongly connected.

Theorem 1: [2], page 126. A matrix A is irreducible matrix if and only if $G(A)$ is connected. For an irreducible matrix A which has weak diagonal dominance the following theorem holds and can be proved.

Theorem 2: [1], page 151. Let A be an irreducible matrix which has weak diagonal element dominance, thus the $M_{r,w}$ -method is convergent for all $0 \le r \le 1$ and $0 < w \le 1$.

Proof: [1], page 151. Assuming for some eigenvalue λ of $L_{r,\mathrm{w}}$ that we have $|\lambda| \ge 1$. For this particular eigenvalue the following relationship holds,

$$\det\left(L_{r,w} - \lambda I\right) = 0. \tag{2.11}$$

From (2.9) and (2.11)

$$\det\left((I - rL)^{-1}[(1-w)I + (w - r)L + wU] - \lambda I\right) = 0$$

$$\Rightarrow \det\left((I - rL)^{-1}[(1-w)I + wL - rL + wU - \lambda I + r\lambda L]\right) = 0$$

$$\Rightarrow \det(I - rL)^{-1} \neq 0 \text{ thus } \Rightarrow \det[(1-w)I + wL - rL + wU - \lambda I + r\lambda L] = 0$$

$$\Rightarrow \det[-(\lambda + w - 1)I + (w - r + \lambda r)L + wU] = 0$$

$$\Rightarrow \det\left[-(\lambda + w - 1)\left[I - \frac{(w - r + \lambda r)}{(\lambda + w - 1)}L - \frac{w}{(\lambda + w - 1)}U\right]\right] = 0$$

$$\Rightarrow \det\left[I - \frac{r(\lambda - 1) + w}{\lambda + w - 1}L - \frac{w}{\lambda + w - 1}U\right] = 0. \qquad (2.12)$$

Let $Q = I - \frac{r(\lambda-1)+w}{\lambda+w-1}L - \frac{w}{\lambda+w-1}U$ we get

$$\det(Q) = 0. \qquad (2.13)$$

To prove that the coefficients of L and U in (2.12) satisfy $\left|\frac{r(\lambda-1)+w}{\lambda+w-1}\right| < 1$ and $\left|\frac{w}{\lambda+w-1}\right| < 1$ respectively, it is suffices to prove the following relations in order to prove the previous statement.

$$|\lambda - 1 + w| \geq |r(\lambda - 1) + w| \text{ and } |\lambda - 1 + w| \geq |w|. \qquad (2.14)$$

If the inverse of λ, say $\lambda^{-1} = qe^{i\theta}$ with the coefficients θ and q being real such that $0 < q \leq 1$, then the left side inequality in (2.14) is

$$|\lambda - 1 + w| \geq |r(\lambda - 1) + w|. \qquad (2.15)$$

Let z be a complex number. The polar representation of z is $z = r[\cos\theta + i\sin\theta]$ and in exponential form $z = re^{\theta}$, where $r = |z|$, $\theta = \arg(z)$. So for this eigenvalue λ we have $\lambda^{-1} = q^{i\theta} = q[\cos\theta + i\sin\theta]$, if $\lambda \in R$, $\lambda^{-1} = \frac{1}{\lambda}$ and

$$\lambda=(\lambda^{-1})^{-1}=(qe^{i\theta})^{-1}=\frac{1}{q}e^{-i\theta}=$$
$$\frac{1}{q}[\cos(-\theta)+i\ \sin(-\theta)]=\frac{1}{q}[\cos\theta-i\ \sin\theta]. \tag{2.16}$$

Substitute (2.16) into (2.15) we get

$$\left|(\frac{1}{q}\cos\theta-1+w)-i\ \frac{1}{q}\sin\theta\right|\geq\left|(\frac{r}{q}\cos\theta-r+w)-i\ \frac{r}{q}\sin\theta\right|$$

$$\Rightarrow\sqrt{(\frac{1}{q}\cos\theta-1+w)^2+(\frac{1}{q}\sin\theta)^2}\geq\sqrt{(\frac{r}{q}\cos\theta-r+w)^2+(\frac{r}{q}\sin\theta)^2}.$$

Squaring both sides of the inequality above lead us to

$$\Rightarrow\frac{1}{q^2}\cos^2\theta+\frac{2}{q}\cos\theta(w-1)+(w-1)^2+\frac{1}{q^2}\sin^2\theta\geq\frac{r^2}{q^2}\cos^2\theta$$
$$+\frac{r^2}{q^2}\sin\theta+\frac{2r}{q}\cos\theta(w-r)+(w-r)^2 \tag{2.17}$$

multiply both sides of (2.17) by q^2

$$\Rightarrow 1+2q\cos\theta(w-1)+q^2(w-1)^2\geq r^2+2rq\cos\theta(w-r)+q^2(w-r)^2$$

$$\Rightarrow(1-r^2)+2q\cos\theta(w-1)+q^2(w-1)^2-2rq\cos\theta\left(w-r\right)\geq 0$$

$$\Rightarrow(1-r^2)+2q\cos\theta[w-1-r(w-r)]+q^2(w-1)^2q^2\geq 0$$

$$\Rightarrow(1-r^2)+2q\cos\theta[-1+r^2]+2q\cos\theta w[1-r]+q^2[(w-1)^2-(w-r)^2]\geq 0$$

$$\Rightarrow(1-r^2)-2q\cos\theta[r^2-1]+2q\cos\theta w[1-r]+q^2[-2w+2wr-r^2+1]\geq 0$$

$$\left(1-r^2\right)+\left(1-r^2\right)q^2-\left(1-r^2\right)2q\cos\theta$$
$$+\left(1-r\right)2qw\ \cos\theta-\left(1-r\right)2q^2w>0, \tag{2.18}$$

which holds for $r=1$; For $r\neq 1$ (2.18) becomes

$$\left(1+r\right)+\left(1+r\right)q^2-\left(1-r^2\right)2q\cos\theta+\left(1-r\right)2qw\ \cos\theta-\left(1-r\right)2q^2w\geq 0. \tag{2.19}$$

Because of the nonnegativeness of (2.19) it holds for all real number θ if and only if it holds for the value $\cos\theta = 1$. Thus (2.19) yields:

$$(1-q)\left[(1+r)(1-q)+2qw\right] \geq 0. \tag{2.20}$$

Similarly second inequality in (2.14) is also equivalent to

$$1+q^2-2q(1-w)\cos\theta-2q^2w \geq 0. \tag{2.21}$$

This relation must be satisfied for all θ if it also hold for $\cos\theta = 1$. This leads us to the following inequality

$$(1-q)\left[(1-q)+2qw\right] \geq 0. \tag{2.22}$$

Because of the properties of the matrix A, which is irreducible with weak diagonal dominance. $D^{-1}A = I - L - U$ Satisfy the same properties. All these hold for the matrix Q because the coefficients of L and U satisfy (2.14). This means the matrix, Q is nonsingular, this contradicts to (2.13) and also contradicts to (2.11). Thus $\rho(L_{r,w}) < 1$. Let us consider the $M_{r,w}$ method with the following corresponding pairs $(r, w) = (0, 1)$, $(1, 1)$, $(0, w)$, and (w, w).

Corollary 1: [1], page 152. Gauss-Seidel, Jacobi, Successive Overrelaxation, and Simultaneous Overrelaxation (the last two method for $0 < w \leq 1$) converge, if a matrix A is irreducible with weak diagonal dominancy.

Definition 2: [1], page 152. An L-matrix is a matrix which elements $a_{ij} \,|\, i, j = 1(1)n$ satisfy the relationship $a_{ii} > 0 \,|\, i = 1(1)n$ and $a_{ij} \le 0 \,|\, i \ne j, i, j = 1(1)n$.

Theorem 3: [1], page 152. Let A be an L-matrix. $M_{r,w}$ method converges if and only if $M_{0,1}$ method converges and r and w satisfy $0 < r \le w \le 1 \;(w \ne 0)$

Proof: [1], page 152. It is clear that when the $M_{r,w}$ method converges so does the $M_{0,1}$ method. Let us assume that $\bar{\lambda} = \rho\left(L_{r,w}\right) \ge 1.$ Based on these assumptions we get

$(1-w)I + (w-r)L + wU \ge 0$ and also that

$$\left(I - rL\right)^{-1} = I + rL + r^2 L + \ldots + r^{N-1} L^{N-1} + \ldots \ge 0. \tag{2.23}$$

We therefore have for the iterative matrix that

$L_{r,w} = \left(I - rL\right)^{-1}\left[(1-w)I + (w-r)L + wU\right] \ge 0.$ Because the matrix $L_{r,w}$ is nonnegative, $\bar{\lambda}$ is an eigenvalue of $L_{r,w}$. Let $v \ne 0$ be the corresponding eigenvector, we then have $L_{r,w} v = \bar{\lambda} v$ which we get

$\left(\left(I - rL\right)^{-1}\left[(1-w)I + (w-r)L + wU\right]\right)v = \bar{\lambda}\, v$ multiplying by $\left(I - rL\right)$ result

$\Rightarrow \left((1-w)I + (w-r)L + wU\right)v = \left(I - rL\right)\bar{\lambda}\, v$

$\Rightarrow \left((w-r)L + wU\right)v = (I - rL)\overline{\lambda v} - (1-w)I\; v$

$\Rightarrow \left((w-r)L + wU\right)v = \overline{\lambda v} - rL\bar{\lambda}\, v - (1-w)v$

$\Rightarrow \left((w-r)L + wU + rL\bar{\lambda}\right)v = \left(\bar{\lambda} - 1 + w\right)v$

$\Rightarrow\left((w-r+r\bar{\lambda})L+wU\right)v=\left(\bar{\lambda}-1+w\right)v$ dividing by $w\neq 0$, we get

$$\left(\frac{w-r+r\bar{\lambda}}{w}L+U\right)v=\frac{\bar{\lambda}-1+w}{w}v. \tag{2.24}$$

This implies that $\frac{\left(\bar{\lambda}-1+w\right)}{w}$ is an eigenvalue of $\left(\frac{w-r+r\bar{\lambda}}{w}L+U\right)$, corresponding to the eigenvector v.

Therefore,

$$\frac{\bar{\lambda}-1+w}{w}\leq\rho\left(\frac{w-r+r\bar{\lambda}}{w}L+U\right). \tag{2.25}$$

On the other hand, it is clear that $\frac{w-r+r\bar{\lambda}}{w}\geq 1$, which implies that

$$0\leq\frac{w-r+r\bar{\lambda}}{w}L+U\leq\frac{w-r+r\bar{\lambda}}{w}\left(L+U\right)=\frac{w-r+r\bar{\lambda}}{w}L_{0,\,1}. \tag{2.26}$$

From the relationships (2.25) and (2.26) it can be deduced that $\bar{\lambda}-1+w\leq\left(w-r+r\bar{\lambda}\right)\rho\left(L_{0,\,1}\right)$ which leads to $\rho\left(L_{0,\,1}\right)\geq 1$. We have previously proved that when $\bar{\lambda}\geq 1$, then $\rho\left(L_{0,\,1}\right)\geq 1$, we directly obtain that $\rho\left(L_{0,\,1}\right)<1$ implies readily $\bar{\lambda}<1$ such a way that if the $M_{0,\,1}$ method is convergent then so does also the $M_{r,\,w}$ method.

2.3 Convergence Analysis for Consistently Ordered Matrices

Definition 3: [3], page 144. the matrix A of order n is consistently ordered if for some t there exist disjoint subsets $S_1,S_2,\ldots,S_t$ of $w=\{1,2,3,\ldots,N\}$ such that

$\sum_{k=1}^{t} S_k = w$ and such that if i and j are associated, then $j \in S_{k+1}$ if $j > i$ and $j \in S_{k-1}$ if $j < i$ where S_k is subset containing i.

Assume that A is consistently ordered matrix. This means that the determinant expression $\det\left(\alpha A_L + \alpha^{-1} A_U - \beta D\right)$ is independent of α, for $\alpha \neq 0$ and for all β. The following three Lemmas are necessary to understand what will follow in this Section.

Lemma 1: [1], page 153. Let A have nonvanishing diagonal elements and let A be a consistently ordered matrix. If $\mu \neq 0$ is an eigenvalue of $L_{0,\,1}$ with multiplicity p, this implies that $-\mu$ is also an eigenvalue of $L_{0,\,1}$ with the same multiplicity p.

Lemma 2: [1], page 153. Let A have nonvanishing diagonal elements and let A be a consistently ordered matrix. If μ is an eigenvalue of $L_{0,\,1}$ and v satisfies

$$(v - 1 + r)^2 = r^2 \mu^2 v\,, \tag{2.27}$$

then v is also an eigenvalue of $L_{r,\,r}$ and vice versa.

Lemma 3: [1], page 153. Consider the real β and γ to be the roots of the quadratic equation given by $\lambda^2 - \beta\lambda + \gamma = 0$ then β and γ are less than one in modulus if and only if the following relations hold:

$$\lambda^2 - \beta\lambda + \gamma = 0 \tag{2.28}$$

$$|\gamma| < 1,\; |\beta| < 1 + \gamma\,, \tag{2.29}$$

Proof: [2], page 172. Assume that r_1 and r_2 are real the roots of (2.28). Then if $|r_1|<1$ and $|r_2|<1$, lead to $|\lambda|<1$, because of that $\gamma = r_1 r_2$ and also $\beta = r_1 + r_2$ we have

$$1+\gamma-|\beta| = 1+r_1r_2-(r_1+r_2) = (1-r_1)(1-r_2) > 0, \tag{2.30}$$

if $r_1 + r_2 \geq 0$, and

$$1+\gamma-|\beta| = 1+r_1r_2+r_1+r_2 = (1+r_1)(1+r_2) > 0, \tag{2.31}$$

if $r_1 + r_2 < 0$. Otherwise $|\beta| < 1+\lambda$. On the other side, if the relation (2.29) holds, it follows that the relations (2.30) or (2.31) hold. If (2.30) holds, thus the real r_1 and r_2 are either all less than one or greater than one at the same time. But the case where there are all greater than one is impossible because $\lambda = r_1 r_2 < 1$ if we have $r_1 \leq -1$ or $r_2 \leq -1$ it follows that, since $r_1 + r_2 \geq 0$ we would obtain $r_1 \geq 1$ or $r_2 \geq 1$ impossible or simply absurd. This help us to conclude that $|r_1|<1$ and $|r_2|<1$. We use a similar argument when $r_1 + r_2$ is negative.

Theorem 4: [1], page 153. Let A have nonvanishing diagonal elements and let A be a consistently ordered matrix. If μ is an eigenvalue of the matrix $L_{0,\,1}$ and if λ satisfies

$$\left(\lambda-1+w\right)^2 = w\mu^2\left[r\left(\lambda-1\right)+w\right], \tag{2.32}$$

then λ is also an eigenvalue of the matrix $L_{r,\,w}$ and vice versa.

Theorem 5: [1], page 153. Let A have nonvanishing diagonal elements and let A be a consistently ordered matrix. If $L_{0,\,1}$ has real eigenvalue say $\mu_i \,|\, i = 1(1)N$, with the values $\underline{\mu} = \min_i |\mu_i|$ and $\bar{\mu} = \max_i |\mu_i|$, then the $M_{r,\,w}$ method is convergent if and only if the $M_{0,\,1}$ method is convergent and the parameters r and w has their values on I_r, I_w respectively, which are given as in Table 2.2 for $\mu = 0$, and in Table 2.3 for $\mu \neq 0$, with

$$\alpha(z) \equiv \frac{1}{wz}\left(\frac{1}{2}w^2 z - \frac{1}{2}w^2 + 2w - 2\right) \text{ and } \beta(z) \equiv \frac{1}{z}(wz - w + 2). \quad (2.33)$$

Table 2.2: Intervals I_r and I_w for $\mu = 0$

I_r	I_w
$\left(\alpha(\bar{\mu}^2),\ \beta(\underline{\mu}^2)\right)$	$(0,\ 2)$

Table 2.3: Intervals I_r and I_w for $\mu \neq 0$

I_r	I_w
$\left(\beta(\underline{\mu}^2),\ \alpha(\bar{\mu}^2)\right)$	$\left(\frac{-2}{\left(1-\underline{\mu}^2\right)^{1/2}},\ 0\right)$
$\left(\beta(\underline{\mu}^2),\ \alpha(\bar{\mu}^2)\right)$	$(0,\ 2]$
$\left(\beta(\underline{\mu}^2),\ \alpha(\bar{\mu}^2)\right)$	$\left[2,\ \frac{2}{\left(1-\underline{\mu}^2\right)^{1/2}}\right)$

Proof: [1] page 154. Let us first notice that the matrix A satisfies the requirement of the Theorem 4. So the eigenvalues λ of the matrix $L_{r,w}$ holds the property (2.32) with μ being the eigenvalue of matrix $L_{0,1}$. The equation (2.32) can be written as

$$\lambda^2 - \left[2(1-w) + rw\mu^2\right]\lambda + (w-1)^2 + (r-w)w\mu^2 = 0. \tag{2.34}$$

$M_{r,w}$ method converges if and only if $\rho(L_{r,w}) < 1$. Therefore form (2.34)

$$\left|(w-1)^2 + (r-w)w\mu^2\right| < 1, \tag{2.35}$$

$$\left|2(1-w) + rw\mu^2\right| < 1 + (w-1)^2 + (r-w)w\mu^2. \tag{2.36}$$

From equation (2.35),

$(w-1)^2 + (r-w)w\mu^2 > -1,$

$\Rightarrow w^2 - 2w + 1 + rw\mu^2 - w^2\mu^2 > -1$

$\Rightarrow rw\mu^2 > -2 - w^2 + 2w + w^2\mu^2$

$$rw\mu^2 > -2 + 2w - (1-\mu^2)w^2. \tag{2.37}$$

Moreover $(w-1)^2 + (r-w)w\mu^2 < 1,$

$\Rightarrow w^2 - 2w + 1 + rw\mu^2 - w^2\mu^2 < 1$

$\Rightarrow rw\mu^2 < -w^2 + 2w + w^2\mu^2$

$$rw\mu^2 < -(1-\mu^2)w^2 + 2w. \tag{2.38}$$

From equation (2.36) we obtain

$\Rightarrow 2(1-w) + rw\mu^2 > -1 - (w-1)^2 - (r-w)w\mu^2$

$$\Rightarrow 2rw\mu^2 > -4+4w - w^2 + w^2\mu^2$$

$$\Rightarrow 2rw\mu^2 > -4+4w - (1-\mu^2)w^2$$

$$rw\mu^2 > -2+2w - \frac{1}{2}(1-\mu^2)w^2. \tag{2.39}$$

Also

$$\Rightarrow 2(1-w) + rw\mu^2 < 1+(w-1)^2 + (r-w)w\mu^2$$

$\Rightarrow 0 < (1-\mu^2)w^2$ dividing both side by w^2 we get

$$\mu^2 < 1. \tag{2.40}$$

The inequality (2.40) provides one of the necessary and sufficient conditions for the $M_{r,\,w}$ method to be convergent, that is $\bar{\mu} < 1$. The inequalities (2.37), (2.38), (2.39) can be combined as

$$-\frac{1}{2}\left(1-\mu^2\right)w^2 + 2w - 2 < rw\mu^2 < -\left(1-\mu^2\right)w^2 + 2w. \tag{2.41}$$

From which results the relation $\left(1-\mu^2\right)w^2 < 4$. Obviously this gives the next inequality

$$-\frac{2}{\left(1-\underline{\mu}^2\right)^{1/2}} < \underset{\neq 0}{w} < \frac{2}{\left(1-\underline{\mu}^2\right)^{1/2}}. \tag{2. 42}$$

At this point, all the possible values of the overrelaxation parameter w are determined using the above. From inequality (2.41), let us now find the corresponding values of r based on the analysis of the following two cases.

Case 1: $\underline{\mu} \neq 0.$ if $w > 0,$ then equation (2.41) is written as

$$\alpha(z) \equiv \frac{1}{wz}\left(\frac{1}{2}w^2 z - \frac{1}{2}w^2 + 2w - 2\right) < r < \frac{1}{z}(wz - w + 2) \equiv \beta(z), \quad (2.43)$$

with $z = \mu^2$. It is clear that the parameter r in (2.43) is bounded by

$$\max_z \alpha(z) < r < \min_z \beta(z). \quad (2.44)$$

If the $w < 0$, then (2.41) is now equivalent to $\beta(z) < r < \alpha(z)$; for this case inequalities are fulfilled for r satisfying

$$\max_z \beta(z) < r < \min_z \alpha(z). \quad (2.45)$$

The sing of partial derivatives of $\beta(z)$ and $\alpha(z)$ with respect to the variable z, are given by Table 2.4 to present the behavior of these functions, depending on z.

$\beta(z) \equiv \frac{1}{z}(wz - w + 2)$, which can be written as $\beta(z) = w - wz^{-1} + 2z^{-1}$ taking the derivative of $\beta(z)$ we get

$\beta'(z) = wz^{-2} - 2z^{-2} = z^{-2}[w - 2]$, with $z^2 = \frac{1}{\mu^4}$. Thus $\beta'(z) = z^{-2}(w - 2)$

substituting $z^{-2} = \mu^4$ we get $\beta'(\mu) = \mu^4(w - 2) = \mu^4(\varphi(w))$ Similarly

$\alpha(z) = \frac{1}{wz}\left(\frac{1}{2}w^2 z - \frac{1}{2}w^2 + 2w - 2\right)$, differentiating with respect to z we get

$\alpha'(z) = \left(\frac{1}{2z^2}w - \frac{2}{z^2} + \frac{2}{wz^2}\right)$ and substituting $z^{-2} = \mu^4$ we get

$\alpha'(\mu) = \mu^4\left(\frac{1}{2}w - 2 + \frac{2}{w}\right) = \mu^4(\Omega(w))$.

Figure 2.1 represents the function $\varphi(w) = (w - 2)$, and Figure 2.2 displays the function $\Omega(w) = \frac{1}{2}w - 2 + \frac{2}{w}$.

Table 2.4 represents the sign of $\alpha'(\mu)$ and $\beta'(\mu)$ on the intervals I_w obviously the sign of $\beta'(\mu)$ is given by the sign of $\varphi(w) = (w-2)$ and the sign of $\alpha'(\mu)$ is given by the sign of $\Omega(w) = \frac{1}{2}w - 2 + \frac{2}{w}$,

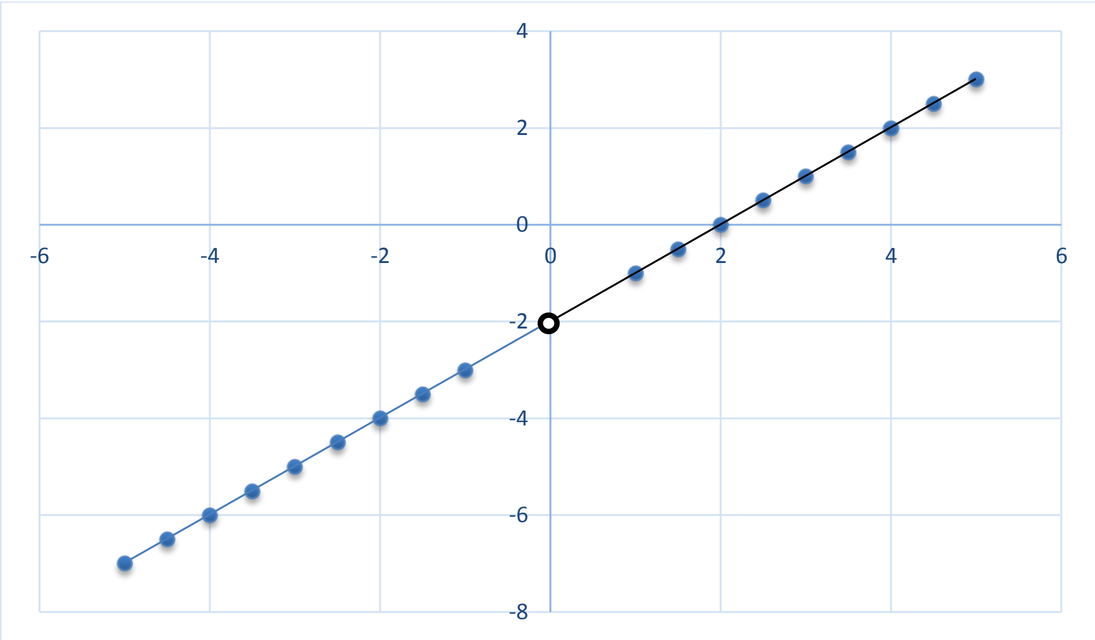

Figure 2.1: Graph of $w \neq 0$

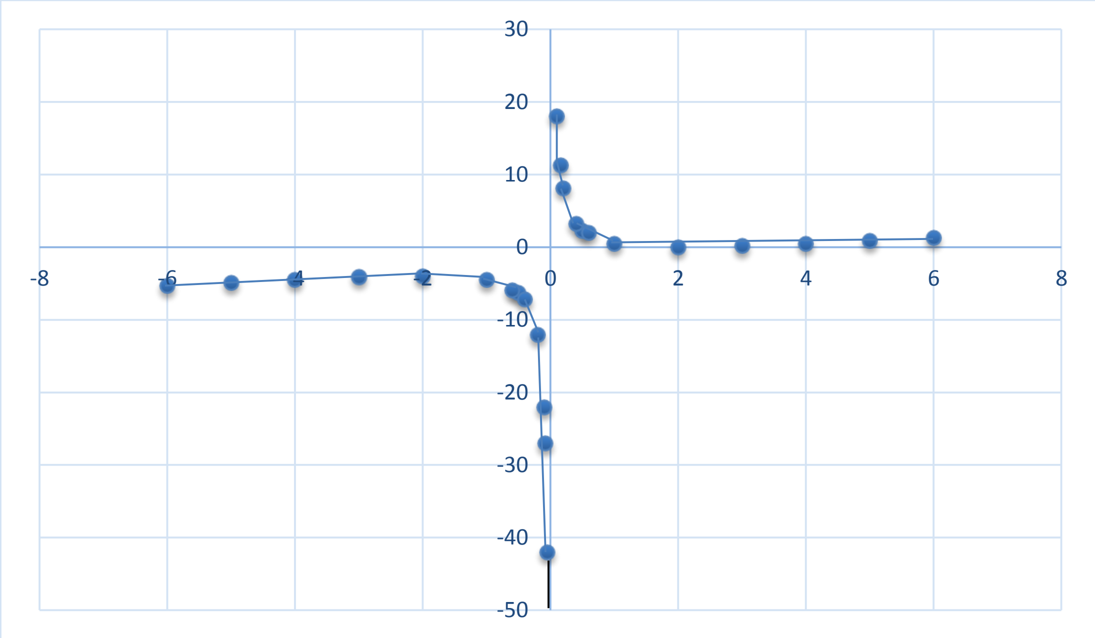

Figure 2.2: Graph of $\Omega(w)=\frac{1}{2}w-2+\frac{2}{w},\ w\neq 0$

Table 2.4**:** Sign of $\alpha'(\mu)$ and $\beta'(\mu)$ on I_w

I_w	$\alpha'(\mu)=\mu^4(\Omega(w))$.	$\beta'(\mu)=\mu^4(\varphi(w))$
$(\frac{-2}{(1-\underline{\mu}^2)^{1/2}},\ 0)$	negative	negative
$(0,\ 2]$	Positive	negative
$\left[2,\ \frac{2}{(1-\underline{\mu}^2)^{1/2}}\right)$	Positive	Positive

Based on the equations (2.44) and (2.45) and also on the Table 2.4, one can build easily a table which shows the values range of I_r for the parameter r, and this is given in Table 2.5.

Table 2.5: Values range of I_r for the intervals I_w

I_w	I_r
$\left(\frac{-2}{\left(1-\underline{\mu}^2\right)^{1/2}},\ 0\right)$	$\left(\beta\left(\underline{\mu}^2\right),\ \alpha\left(\bar{\mu}^2\right)\right)$
$(0,\ 2]$	$\left(\alpha\left(\bar{\mu}^2\right),\ \beta\left(\bar{\mu}^2\right)\right)$
$\left[2,\ \frac{2}{\left(1-\underline{\mu}^2\right)^{1/2}}\right)$	$\left(\alpha\left(\bar{\mu}^2\right),\ \beta\left(\underline{\mu}^2\right)\right)$

It is clear that the first case and the third case exist provided that the following $\beta\left(\underline{\mu}^2\right)<\alpha\left(\bar{\mu}^2\right)$ and $\alpha\left(\bar{\mu}^2\right)<\beta\left(\underline{\mu}^2\right)$ hold respectively.

Case 2: $\underline{\mu}=0.$ Since the inequality (2.41) must be satisfied for $\mu=0$ and $\mu\neq 0$, two sub cases have to be distinguished. If the minimum value $\underline{\mu}=0,$ then the relationships (2.41) leads to $0<w<2,$ whereas if $\mu\neq 0,$ then the analysis given in the study of Case 1 is still valid as well as the values demonstrated in Table 2.5. For this case w and r are given from the intervals respectively $I_w \equiv (0,2)$ and $I_r \equiv\left(\alpha\left(\bar{\mu}^2\right),\beta\left(\bar{\mu}^2\right)\right)$ respectively, because they must satisfy (2.41).

Theorem 6: [1], page 155. Let A have nonvanishing diagonal elements and be a consistently ordered matrix, and if $L_{0,\,1}$ has real eigenvalues $\mu_i \mid i=1(1)n$ such that $0<\mu=\underline{\mu}=\min_i |\mu_i|=\bar{\mu}=\max_i |\mu_i|<1$, then for

$$(r,\ w) = \left(\frac{2(1+(1-\mu^2)^{1/2})}{\mu^2}, -\frac{1}{(1-\mu^2)^{1/2}} \right) \text{ or } \left(\frac{2}{1+(1-\mu^2)^{1/2}}, \frac{1}{(1-\mu^2)^{1/2}} \right),$$

$\rho(L_{r,\,w}) = 0.$

Proof: [1], page 155. Based on Lemma 1, μ will be an eigenvalue of $L_{0,\,1}$. Because μ^2 has unique fixed value it is easy to determine for $r \neq 0$ so that (2.27) has two roots expressed as:

$$r_1 = \frac{2(1+(1-\mu^2)^{1/2})}{\mu^2}\ ,\ r_2 = \frac{2}{1+(1-\mu^2)^{1/2}} \tag{2.46}$$

and a double root v with the value

$$v = \frac{2(1-r)+r^2\mu^2}{2}. \tag{2.47}$$

Because v is a double root we can determine w from (2.10) so that $\lambda = 0.$ For this we must have

$$w = \frac{r}{(1-v)}. \tag{2.48}$$

Thus from (2.46), (2.47) and (2.48) we finally obtain

$$w_1 = \frac{-1}{(1-\mu^2)^{1/2}}\ ,\ w_2 = \frac{1}{(1-\mu^2)^{1/2}}.$$

$\rho(L_{r,\,w}) = 0$ for the calculated values $(r_1,\ w_1)$ and $(r_2,\ w_2)$.

2.4 Realization of Accelerated Overrelaxation Method (AOR) for the Solution of Laplace's Equation with Dirichlet Boundary Conditions on a Rectangle

Let $R=\{(x,y):0<x<a,0<y<b\}$ be an open rectangle γ^j, $j=1.2.3.4$ be the sides of this rectangle which the vertices are included. Let the numbering be in counterclockwise direction starting from the side $y=0$. The Dirichlet Laplace's equation on a rectangle is

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ on R} \tag{2.49}$$

$$u=\varphi_m \ on \ \ \gamma^m, \ m=1,2,3,4 \tag{2.50}$$

2.4.1 The Discrete Laplace Problem

Based on [4], let us draw two systems of parallel lines on the plane:

$$x=x_0+ih=x_i. \tag{2.51}$$

Consider the node (i,k) of the net, and take the four nodes closest to it which are $(i+1,\ k),(i,\ k+1),(i-1,\ k),(i,\ k-1)$ as shown in the figure below

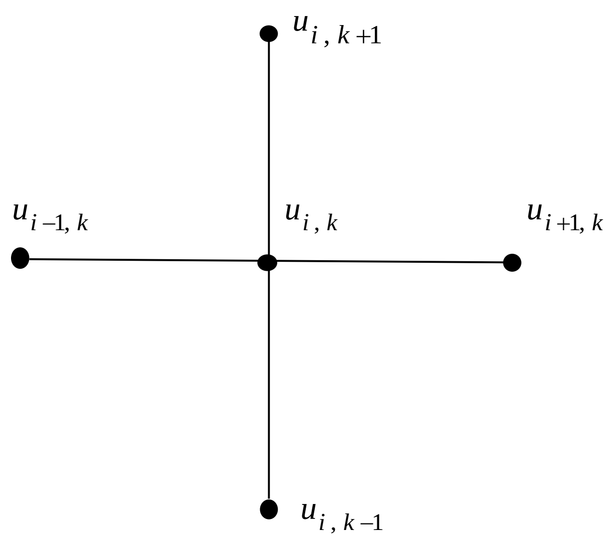

Figure 2.3: 5-point Stencil.

We aim to find an approximate expression for Δu at the node $(i,\ k)$. From Taylor's formula the expressions for the neighboring points of u_{ik} are as follows:

$$u_{i+1,\,k} - u_{i,\,k} = hu_x + \frac{h^2}{2!}u_{x^2} + \frac{h^3}{3!}u_{x^3} + \frac{h^4}{4!}u_{x^4} + \ldots$$

$$u_{i-1,\,k} - u_{i,\,k} = -hu_x + \frac{h^2}{2!}u_{x^2} - \frac{h^3}{3!}u_{x^3} + \frac{h^4}{4!}u_{x^4} + \ldots$$

$$u_{i,\,k+1} - u_{i,\,k} = hu_y + \frac{h^2}{2!}u_{y^2} + \frac{h^3}{3!}u_{y^3} + \frac{h^4}{4!}u_{y^4} + \ldots$$

$$u_{i,\,k-1} - u_{i,\,k} = -hu_y + \frac{h^2}{2!}u_{y^2} - \frac{h^3}{3!}u_{y^3} + \frac{h^4}{4!}u_{y^4} + \ldots \qquad (2.52)$$

we look for Δu as linear combination of the differences in (2.52). The expression is obtained for Δu depending on the derivatives by adding the equations in (2.52) term by term.

$$u_{i,\,k} = u_{i+1,\,k} + u_{i.\,k+1} + u_{i-1,\,k} - 4u_{i,\,k} =$$
$$2\left[\frac{h^2}{2!}\left(u_{x^2} + u_{y^2}\right) + \frac{h^4}{4!}\left(u_{x^4} + u_{y^4}\right) + \frac{h^6}{6!}\left(u_{x^6} + u_{y^6}\right) + \ldots\right] \qquad (2.53)$$

Which yields

$$\frac{1}{h^2}u_{i,\,k} = \Delta u + E_{i,\,k} \qquad (2.54)$$

where

$$E_{i,\,k} = \frac{2h^2}{4!}\left(u_{x^4} + u_{y^4}\right) + \frac{2h^4}{6!}\left(u_{x^6} + u_{y^6}\right) + \ldots \qquad (2.55)$$

is the remainder term. Taking the values of derivatives up to fourth orders, and evaluating the fourth order derivative at the mean points $E_{i,\,k}$ becomes an expression of the form

$$E_{i,k} = \frac{4h^2}{4!} c M_4 \tag{2.56}$$

where,

$$M_4 = \max\left\{\left|\frac{\partial^4 u}{\partial x^4}\right|, \left|\frac{\partial^4 u}{\partial y^4}\right|\right\}. \tag{2.57}$$

For the Laplace's equation (2.50) and ignoring the remainder term $E_{i,k}$ in (2.54) we get

$$\frac{1}{h^2} u_{i,k} = 0. \tag{2.58}$$

Assign a square mesh R_h, with $h = \frac{a}{n_1} = \frac{b}{n_2}$, $n_1 \geq 2$, $n_2 \geq 2$ are integers, obtained with the lines in (2.51) as $x = 0 + ih$, $y = 0 + kh$, $i = 0,1,...,n_1$, $k = 0,1,...,n_2$. Denoting the set of grids on γ^k by γ_{h^k}, $k = 1,2,3,4$ and $\gamma_h = U_{k=1}^4 \gamma_{h^k}$, $\overline{R_h} = R_h \text{ U } \gamma_h$ we obtain the following difference problem for (2.49), (2.50).

$$u_h = Bu_h \quad on \quad R_h \tag{2.59}$$

$$u_h = \varphi_{h^m} \quad on \quad \gamma_{h^m}, \quad m = 1,2,3,4. \tag{2.60}$$

φ_{h^m} is the trace of φ_m on γ_{h^m} and

$$Bu(x, y) = (u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h)). \tag{2.61}$$

Consider the difference problem given in equations (2.59), (2.60) for grid values on the boundary γ_{h^m}, $m = 1,2,3,4$, $u_{i,k}$ is known for the boundary data (2.60),

$$u_{i,0} = \varphi_{h1}(ih,\ 0) \text{ for } i = 0,1,\dots,n_1$$

$$u_{n_1,k} = \varphi_{h2}(n_1 h,\ kh) \text{ for } k = 0,1,\dots,n_2 \tag{2.62}$$

$$u_{i,n_2} = \varphi_{h3}(ih,\ n_2 h) \text{ for } i = 0,1,\dots,n_1$$

$$u_{0,k} = \varphi_{h4}(0,\ kh) \text{ for } k = 0,1,\dots,n_2.$$

The number of unknown $u_{i,k}$ is $(n_1 - 1) \times (n_2 - 1)$ which is the number of inner grid points. The algebraic system of equations is obtained by using lexicographical ordering [5] for the inner points and by eliminating the boundary values (2.62) which appears in (2.59), we form the commonly used matrix form $Ax = b$, where A is of order $(n_1 - 1)(n_2 - 1)$.

The coefficient matrix A obtained for the difference problem (2.59), (2.60) using Lexicographical ordering has the following structure as given in Figure 2.4

$$A = \begin{bmatrix} D & -I & & & & & \\ -I & D & -I & & & & \\ & -I & D & -I & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & \ddots & & \ddots & \\ & & & & & D & -I \\ & & & & & -I & D \\ & & & & & & \end{bmatrix} \qquad D = \begin{bmatrix} 4 & -1 & & & & & \\ -1 & 4 & -1 & & & & \\ & -1 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & \ddots & \ddots & \\ & & & & & 4 & -1 \\ & & & & & -1 & 4 \\ & & & & & & \end{bmatrix}$$

Figure 2.4: Structure of the Coefficient Matrix Using 5-point Scheme and Lexicographical Ordering

Chapter 3

NUMERICAL APPLICATIONS AND RESULT

3.1 Introduction

A test problem is chosen and its solution is obtained via numerical simulation by implementing the (AOR) method to solve the algebraic linear systems obtained.

3.2 Description of the Problem

Let us consider a rectangle R defined as follows

$R = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, consider the problem:

$\Delta u = 0$ on R

$u = 2x - x^3$ on $\gamma^1 : \{0 \le x \le 1,\ y = 0\}$

$u = 1 + 3y^2$ on $\gamma^2 : \{0 \le y \le 1,\ x = 1\}$

$u = 5x - x^3$ on $\gamma^3 : \{0 \le x \le 1,\ y = 1\}$

$u = 0$ on $\gamma^4 = \{0 \le y \le 1,\ x = 0\}$.

The exact solution of this problem is $u(x, y) = 2x - x^3 + 3xy^2$. This model problem, is represented in Figure 3.1, with mesh step $h = \frac{1}{4}$.

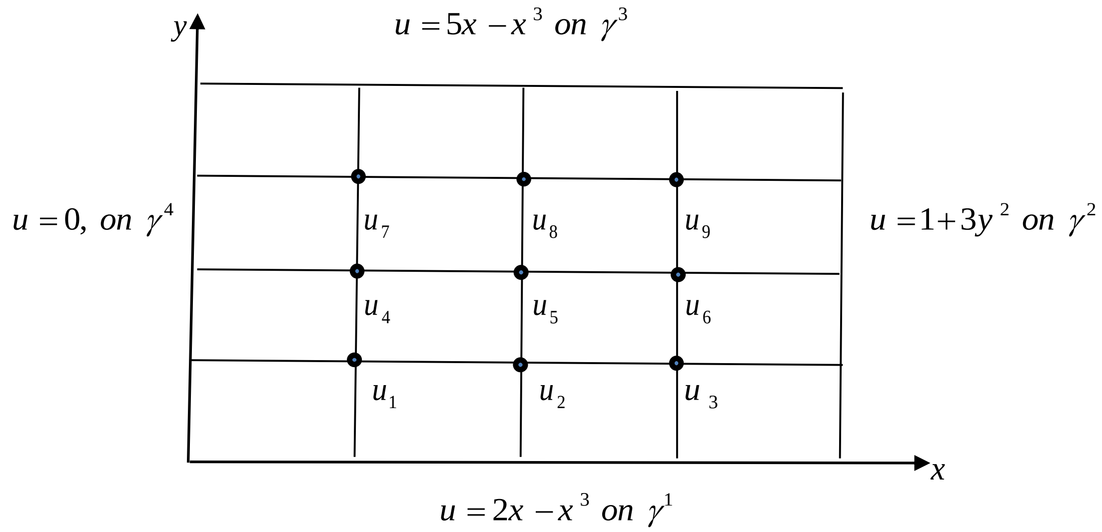

Figure 3.1: The Model Problem and Representation of Inner Grids for $h = \frac{1}{4}$

To control the iterations in (2.8) we used $\|r^{(m)}\|_\infty \le \varepsilon$ where $r^{(m)} = A - bx^{(m)}$ and ε is the preassigned accuracy. All the calculations are carried in Matlab. Table 3.1, Table 3.2, Table 3.3 and Table 3.4 represent the maximum errors and iteration numbers for various mesh size h when $w = 0.6,$ $r = 0.5$ are fixed and $\varepsilon = 10^{-3},$ $\varepsilon = 10^{-4},$ $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-6}$ respectively. Analyzing these results we see that for each value of the mesh step h given as $h = \frac{1}{4},$ $h = \frac{1}{8}$ and $h = \frac{1}{16}$ the iteration number increases when ε decreases.

Table 3.2, Table 3.5 and Table 3.6 represent the maximum errors and iteration numbers for various values of h and for fixed value of $w = 0.6,$ $\varepsilon = 10^{-4}$ with respect to different values of r as $r = 0.5,$ $r = 0.3$ and $r = 0.9$ respectively. One can view firm these Tables that when $r = 0.9,$ for the fixed value of $w = 0.6$ and $\varepsilon = 10^{-4},$ the iteration numbers are fewer, for each value of step size $h = \frac{1}{4},$ $h = \frac{1}{8}$ and $h = \frac{1}{16}.$

Table 3.3, Table 3.7, Table 3.8 and Table 3.9 display the maximum errors and iteration numbers for various mesh sizes h when $r=0.5$ and $\varepsilon=10^{-4}$ are fixed and w is changing as $w=0.6$, $w=0.3$, $w=0.9$ and $w=1.2$ respectively. Analyzing these tables we conclude that the number of iterations are fewer in Table 3.10 when $r=0.5$, $w=1.2$ for $\varepsilon=10^{-4}$, for $h=\frac{1}{4}$, $h=\frac{1}{8}$ and $h=\frac{1}{16}$.

Table 3.1: Maximum errors and iteration numbers for the test problem when $w=0.6$, $r=0.5$ and $\varepsilon=10^{-3}$, for $h=\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\|r_i\|_\infty$	$\|u-u_h\|_\infty$	Iterations
$\frac{1}{4}$	7.719953428551030E-4	5.789965071413272E-4	33
$\frac{1}{8}$	8.854114214058573E-4	7.747349937301251E-4	106
$\frac{1}{16}$	9.773674105089114E-4	8.086305468561761E-4	339

Table 3.2: Maximum errors and iteration numbers for the test problem when w=0.6, $r=0.5$ and, $\varepsilon=10^{-4}$, for $h=\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\|r_i\|_\infty$	$\|u-u_h\|_\infty$	iterations
$\frac{1}{4}$	8.661972382340011E-5	6.496479286755008E-5	42
$\frac{1}{8}$	9.685021519501014E-5	8.473938295633587E-5	142
$\frac{1}{16}$	9.284305543472954E-5	9.219661438568394E-5	478

Table 3.3: Maximum errors and iteration numbers obtained for the choice of $w = 0.6$, $r = 0.5$ and, $\varepsilon = 10^{-5}$, when $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\lVert r_i \rVert_\infty$	$\lVert u - u_h \rVert_\infty$	iterations
$\frac{1}{4}$	9.7190114395725978E-6	7.3892607967944833E-6	51
$\frac{1}{8}$	7.790860572676195E-6	6.817003001091671E-6	184
$\frac{1}{16}$	9.7190114395725978E-6	9.277454353084913E-6	626

Table 3.4: Results obtained for the choice of $w = 0.6$, $r = 0.5$ and, $\varepsilon = 10^{-6}$, when $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\lVert r_i \rVert_\infty$	$\lVert u - u_h \rVert_\infty$	iterations
$\frac{1}{4}$	8.552102821468566E-7	6.414077116101424E-7	61
$\frac{1}{8}$	9.636935358603438E-7	8.432318438778008E-7	218
$\frac{1}{16}$	9.354819796580927E-7	8.77014355929619E-7	777

Table 3.5: Results obtained for the choice of $w = 0.6$, $r = 0.3$ and $\varepsilon = 10^{-4}$, when $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\lVert r_i \rVert_\infty$	$\lVert u - u_h \rVert_\infty$	iterations
$\frac{1}{4}$	9.17405713736219E-5	6.880542853021643E-5	46
$\frac{1}{8}$	8.936046951735221E-5	7.819041082768319E-5	161
$\frac{1}{16}$	9.730621214387725E-5	9.122457388488492E-5	535

Table 3.6: Maximum errors and iteration numbers when $w = 0.6$, $r = 0.9$ and, $\varepsilon = 10^{-4}$ when $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\|r_i\|_\infty$	$\|u - u_h\|_\infty$	iterations
$\frac{1}{4}$	7.431212691799693E-5	5.647721645767767E-5	34
$\frac{1}{8}$	8.952204016410281E-5	7.833178514358996E-5	108
$\frac{1}{16}$	9.908821543369584E-5	1.471871658512147E-5	458

Table 3.7: Maximum errors and iteration numbers when $w = 0.3$, $r = 0.5$ and $\varepsilon = 10^{-4}$ when $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\|r_i\|_\infty$	$\|u - u_h\|_\infty$	iterations
$\frac{1}{4}$	9.687479335678772E-5	6.815699561759079E-5	89
$\frac{1}{8}$	4.370856289259706E-5	3.824499253102243E-5	195
$\frac{1}{16}$	9.848360080422225E-5	9.232837575395836E-5	888

Table 3.8: Maximum errors and iteration numbers when $w = 0.9$, $r = 0.5$ and $\varepsilon = 10^{-4}$ when $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\|r_i\|_\infty$	$\|u - u_h\|_\infty$	iterations
$\frac{1}{4}$	9.02344145394806E-5	6.7675810990461050E-5	26
$\frac{1}{8}$	5.646451917806772E-5	4.9406454208080925E-5	108
$\frac{1}{16}$	7.883796281715760E-5	7.3910590108525964E-5	351

Table 3.9: Maximum errors and iteration numbers obtained for the choice of $w = 1.2$, $r = 0.5$, and $\varepsilon = 10^{-4}$ where $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

h	$\|r_i\|_\infty$	$\|u - u_h\|_\infty$	iterations
$\frac{1}{4}$	7.381525327199157E-5	5.536143995399367E-5	26
$\frac{1}{8}$	9.378812138116643E-5	8.20646062088526E-5	70
$\frac{1}{16}$	9.718071447006871E-5	9.110691981568941E-5	238

Figure 3.2 presents the maximum error $\|u - u_h\|_\infty$ with respect to the first 20 iterations, when $h = \frac{1}{16}$, $r = 0.5$ and $\varepsilon = 10^{-4}$ for various values of w. It can be viewed that $M_{0.5,\,1.4}$ does not converge to the exact solution for $h = \frac{1}{8}$ and $h = \frac{1}{8}$. This happens because $\rho\left(L_{0.5,\,1.4}\right) > 1$ for the chosen step sizes $h = \frac{1}{8}, \frac{1}{16}$ as presented in Table 3.10. In table 3.11 we present the $\rho\left(L_{0.5,\,1.2}\right)$ for the chosen step size $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.

Table 3.10: Spectral radius $\rho\left(L_{0.5,\,1.4}\right)$

h	$\rho\left(L_{0.5,\,1.4}\right)$
$\frac{1}{4}$	0.94654348123091
$\frac{1}{8}$	1.063405009247968
$\frac{1}{16}$	1.0855302002759997

Table 3.11: Spectral radius $\rho\left(L_{0.5,\ 1.2}\right)$

h	$\rho\left(L_{0.5,\ 1.2}\right)$
$\frac{1}{4}$	0.668465843842649
$\frac{1}{8}$	0.880764899425652
$\frac{1}{16}$	0.969000123805952

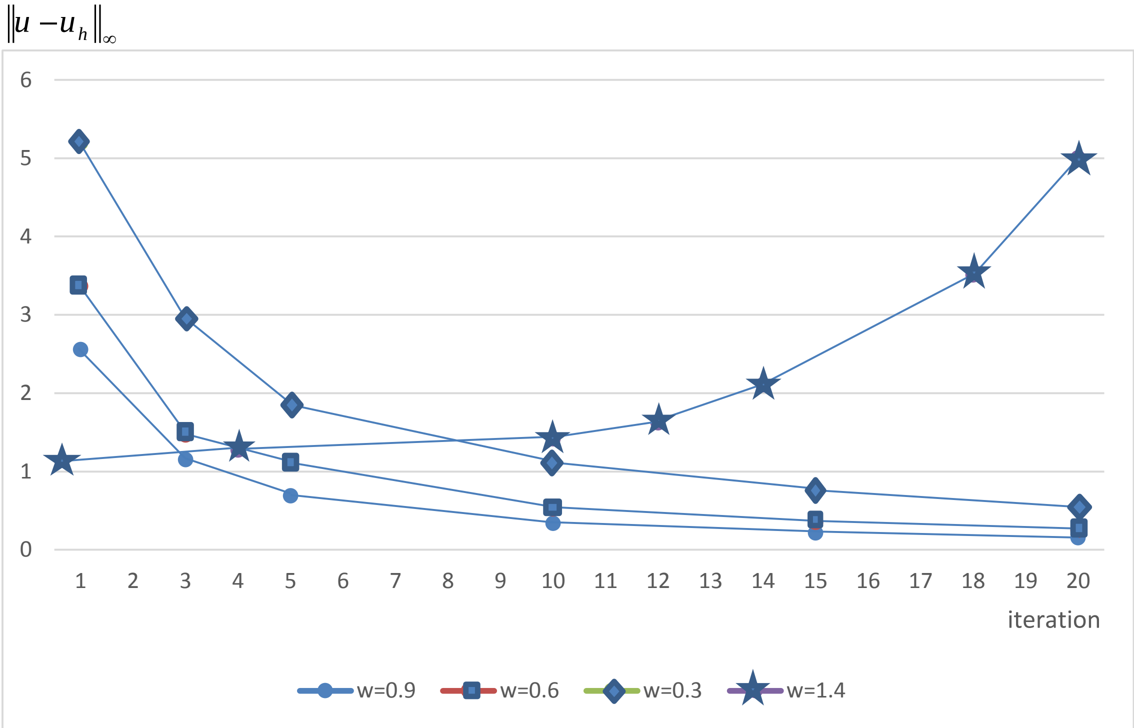

Figure 3.2: Maximum error $\|u-u_h\|_\infty$ with respect to iteration numbers when $r=0.5,\ h=\frac{1}{16}$ and $\varepsilon=10^{-4}$

Chapter 4

CONCLUSION

In this thesis we analyed the solution of Discrete Laplace equation on a rectangle with Dirichlet boundary conditions by (AOR) method. The matrix obtained from the difference problem using 5-point scheme is symmetric and diagonally dominent matrix which is consistenly ordered. A is also on L-matrix we choosed a test problem and solved this problem by (AOR) method using different values of the two parameters r and w and for mesh step $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. The numerical results obtained show that for the mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ when $w = 1.2$, $r = 0.5$ the iteration numbers are fewer than the iteration number for other choices of w and r which considered, for $\varepsilon = 10^{-4}$. Also when $w = 1.4$ and $r = 0.5$ the method showed divergence form the solution for $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ because $\rho\left(L_{0.5,\,1.4}\right) > 1$ for these values of h, as presented in Table 3.10.

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