Analysis of Properties of Fuzzy Graphs

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ABSTRACT

The purpose of this thesis is to investigate the properties of the combination of graph theory and fuzzy set theory called fuzzy graph. The crisp and fuzzy graphs are compared. The operations on fuzzy relations are performed. The max-min and max-product compositions of fuzzy relations are illustrated. α-cut of fuzzy relations is presented. The different types of fuzzy relations are investigated. The Cartesian product, union and join operations on fuzzy graph are studied. Such properties of fuzzy graph as fuzzy and partial fuzzy subgraphs, complement of fuzzy graph, degrees and total degrees of vertices of fuzzy graph, regular and totally regular fuzzy graphs, complete fuzzy graph, and fuzzy tree are analyzed.

Keywords: Fuzzy graph, Fuzzy compositions, Fuzzy relations, Cartesian product, Join and union operations, Fuzzy subgraph, Complement of fuzzy graph, Degree and total degree of a vertex, Regular and totally regular fuzzy graphs, Complete fuzzy graph, Fuzzy tree.

ÖZ

Bu tezin amacı graf teorisi ve bulanık küme teorisinin sentezi olan bulanık grafın özellilerini araştırmaktır. Keskin ve bulanık bağıntılar kıyaslanır. Bulanık bağıntılar üzerinde operasyonlar irdelenir. Bulanık bağıntıların max-min ve max-çarpım bileşimleri gösterilir. Bulanık bağıntının alfa-kesimi sunulur. Farklı türde bulanık bağıntılar araştırılır. Bulanık graflar üzerinde Kartezyen çarpımı, bileşim ve birleştirme operasyonları irdelenir. Bulanık grafların bazı özellikleri – bulanık ve kısmi bulanık alt graflar, bulanık küme tümlemesi, bulanık grafın dügümlerinin dereceleri ve toplam dereceleri, düzenli ve tamamen düzenli bulanık graflar, tam bulanık graf ve bulanık ağaç gibi kavramlar incelenir.

Anahtar Kelimeler: Bulanık graf, Bulanık bileşimler, Bulanık bağıntılar, Kartezyen çarpım, Bileşim ve birleştirme operasyonları, Bulanık alt graf, Bulanık graf tümlemesi, Düğümlerin dereceleri ve toplam dereceleri, Düzenli ve tamamen düzenli bulanık graflar, Tam bulanık graf, Bulanık ağaç.

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Chapter 1

INTRODUCTION

Graphs are graphical models of relations where objects are made of vertices and relations between them are denoted as edges. The graph $H = (X, E)$ contains two sets of X and E, where the elements of X are vertices while the elements of E are the edges which join endpoints (a set of related vertices) together. The connections depending on their structure could be unidirectional (operating in a single direction) or not, and in case all involved connections are defined as unidirectional, the graph is called a digraph. The graph can be considered as a mathematical structure to represent different relations. In other words, the relations defined on a finite set can be pictorially represented by graphs.

The following characteristics of a graph are known:

- Neighbor vertices are vertices that are adjacent to one another by having a common endpoint;

- The endpoint of edge w is vertex q if q is incident on w and vice versa;

- A proper edge is that which connects two individual vertices;

- Simple adjacency is a term used to describe an instance where only one edge exists between two vertices;

- At other times, an edge may join a single endpoint to itself; such edge is a self-loop or simply loop;

- The number of edges between a pair of vertices defines the edge-multiplicity between the pair of vertices.

An application of graph theory has received a major boost and diversification into new fields where the representation of objects and relationships is important. The graph theory has received great application in such fields as communication, computer science, biology etc.

As mentioned above, graphs are categorized into two classes: directed and undirected graphs.

A directed graph $\boldsymbol{H} = (\boldsymbol{X}, \boldsymbol{E})$ consists of $\boldsymbol{V}$ which is a set of vertices and a set of directed edges represented as $\boldsymbol{E}$ where every edge is a pair of vertices with an order.

An undirected graph by property is a graph the vertices of which are unordered pairs, having no direction. The edges in this instance cannot be differentiated by their related directional links as edge (p, q) is identical to edge (q, p), they are only related as sets.

From set theory, the fundamental concept describes an object to either be a part or member of a collection or not, there is an unambiguous definition as to what defines an element and as such, only discrete values are computed. Interestingly in fuzzy sets, the applicable theories of sets require partial membership and as such cannot fall within the scope of the classical set theory.

In a classical set theory, the function of X_Q in a set Q is described by:

$$X_Q(x) = \begin{cases} 1, if\ x \in Q \\ 0, if\ x \notin Q \end{cases}$$

which denotes distinctly the members and non-members of the set. But in the instance where an element is partially a member of the defined set, a membership grade is assigned to such member denoting the degree of membership the element possesses.

The fuzzy set is a category of objects with a series of membership grade, and this theory was proposed by Lotfi Zadeh. It is impossible to have membership of multiple sets in some situations but most natural phenomenon like temperature change defy this, and can be described more accurately using fuzzy notations as differences which cannot be accurately discretized.

The possibility and evaluation of partial membership using the degree of membership is the key element to fuzzy logic which is an approach of multiple-value logic where truth values are acceptable in the form of degrees of truth rather than a discrete Boolean logic.

If Q is a set of points, and an element of set Q is denoted by q, then $Q = \{q\}$.

A fuzzy set P in Q is described by a membership function $f_P(q)$ which assigns a real number in the range of $[0,1]$ to every selected point in Q and the value of $f_P(q)$ at selected point q is the membership grade of q in P. Generally, the closer the value of $f_P(q)$ to unity (1), the higher the membership grade assigned to q in P.

The crisp relation is a representation of absence or presence of relationship, connection or association between the elements of two or more sets. This representation can be extended to different degrees of association (also called strength of association) or relationship between various elements and this extension is called a membership grade.

As mentioned above, graphs provide an easy way of representing information regarding the relationship between multiple objects. In instances when the description or the relationship between objects seems to be vague, a fuzzy graph model is needed to describe the relation.

A fuzzy graph is a pair $G: (\sigma, \mu)$ with σ being a fuzzy subset of another set S while μ holds a fuzzy relation on σ. It is taken that set S is finite and nonempty while μ is symmetric and reflexive.

Fuzzy graphs have a wide range of disciplinary applications such as logic, topology, algebra, analysis, pattern recognition, information theory, operations research, neural networks, planning etc.

Because the human mind comprehends graphical data easily than complex numerical data, it is essential to put fuzzy graph into proper application to obtain reliable inferences. Amongst others, fuzzy graphs have the following advantages:

- They are vital in analyzing vague/fuzzy information which otherwise would be near impossible;

- They are essential in making reasonable decisions over fuzzy occurrences;

- They provide detailed information about events;

- They can be used to model information like human cognition and evaluation which are complex to model as they contain fuzzy information.

Chapter 2

REVIEW OF EXISTING LITERATURE ON FUZZY GRAPHS

The fuzzy graph notion was firstly introduced by Rosenfield in 1975 aiming to deal with relations containing uncertainty. It is to note that the fuzzy graph considers the fuzzy analogs of different structures of the crisp graph.

In this chapter the summary of literature review of some works on fuzzy graphs is presented.

An extension of fuzzy labeling to fuzzy tree results in a new tool called fuzzy labeling tree [1]. The fuzzy labeling trees as well as bipartite fuzzy labeling trees possessing the various properties are discussed. The proposed algorithm is used to find the maximum spanning, strong arcs and fuzzy bridges of any fuzzy labeling graph.

Some definitions and results for the calculation of the degree of any vertex and the distance between any two vertices of the fuzzy graph are given in [2]. The metric in fuzzy graph is considered.

If-Then rules with one of the basic types of inference called Modus Ponens are used in fuzzy graph [3]. The experimental results show the efficiency of the proposed method in comparison with a classical crisp version.

The regular fuzzy graphs, total degree and totally regular fuzzy graphs are presented in [4]. The examples are solved in order to describe the difference of regular fuzzy graphs and totally regular fuzzy graphs. The important conditions for making the regular and totally regular fuzzy graphs equivalent are presented. Several features of regular fuzzy graphs are examined for totally regular fuzzy graphs. The examples make the understanding of the concepts of regular and totally regular fuzzy graphs easier.

The modification of the definition of the complement of fuzzy graphs, and the properties of the self-complementary fuzzy graphs are investigated in [5]. The complement, union, join and composition operations are performed. It is shown that the complement of the union of two fuzzy graphs is the join of the complements of these graphs, and vice versa, the complement of join of two fuzzy graphs is the union of their complements.

The fuzzy graphs can be successfully applied to neural networks and clustering problems. One of the most important properties of fuzzy graph is its connectivity. In [6] the criterion for the connectivity of the fuzzy graph and its complement is defined. The complement of fuzzy cycles is examined with some examples. Each node of the graphs considered in examples has the membership value 1 unless another membership value is specified.

The properties of different kinds of fuzzy graph and fuzzy hypergraph as well as their structures are investigated in [7]. The summary of the work carried out for defining such properties as regular fuzzy graph, complementary fuzzy graph, bipolar fuzzy hyper graph, irregular and irregular bipolar fuzzy graphs, interval-valued fuzzy graph etc. is briefly described.

[8] presents the sum distance metric in fuzzy graph. The realization of sum distance matrix, the diameter and radius in fuzzy graph is performed by using the algorithms developed by authors. The first algorithm is used to find the adjacency matrix, the second algorithm intends to find the sum distance from this adjacency matrix, and the third algorithm is applied to define the eccentric nodes, diameter and radius from the distance matrix. The self-centered fuzzy graph is established, and the necessary conditions for the existence of such graph are provided.

The paper [9] is about the conditions under which the isolated graph exists. The author mentions that in order the fuzzy graph becomes the isolated fuzzy graph, it is necessary the condition meets the requirement that the complement of the fuzzy graph is a complete fuzzy graph.

[10] investigates both edge domination and total edge domination in a fuzzy graph. The numbers of edge domination and total edge domination are determined for some classes of fuzzy graph. The fuzzy edge cardinality and its relationship with another parameter are considered.

Some classes of intuitionistic fuzzy graph are put forward by the author in [11], and the concept of the edge domination number, total edge domination number, and independent edge number is introduced. The edge and total edge domination numbers with their related bounds are determined.

The paper [12] deals with the efficient analysis of inexact information. The fuzzy graph is used to study the fuzzy relation. The extension of fuzzy graph is done, and the presented fuzzy node fuzzy graph is transformed to crisp node fuzzy graph by applying T-norms. The relational structure of fuzzy information and some properties of fuzzy node fuzzy graph are considered. The fuzzy partial graph and new triangular forms are presented.

In [13] the concept of total strong (weak) domination in fuzzy graphs is proposed. The fuzzy domination number and strong (weak) fuzzy domination number are defined. The domination of strong (weak) and total strong (weak) vertices is investigated.

Some properties of the concept of strong intuitionistic fuzzy graphs are studied in [14]. The strong intuitionistic fuzzy graphs are characterized by the self-complementary and self-weak complementary properties. The intuitionistic fuzzy line graphs are focused on.

The paper [15] studies the degree, effective degree and neighborhood degree of a vertex in interval-valued fuzzy graphs, and discusses a strong interval-valued, a regular interval valued, a semi-regular and a semi-complete interval valued fuzzy graphs. Some examples are solved to better understand the above definitions.

In the paper [16] the degree and the total degree of an edge are obtained. Using these operations, the degree of an edge in the interval-valued fuzzy graph is determined. The importance of the interval-valued fuzzy graph in comparison with the classical and fuzzy versions of the graph is justified. The Cartesian product is performed to find the degree of an edge. The composition of two interval-valued fuzzy graphs is made. The properties of free and busy nodes in interval-valued fuzzy graph are considered.

The determination of the isomorphism of two fuzzy graphs is an important problem. The application of the algorithm for the fuzzy graph isomorphism is discussed in [17]. The performance of the proposed algorithm is estimated in terms of fuzzy matrix equality. It is defined that the fuzzy matrices used for the isomorphic fuzzy graphs should be unequal.

Being the subclass of the fuzzy graph, fuzzy planar graph have some properties [18]. The fuzzy dual graph and its close association with fuzzy planar graph are discussed. The provided examples illustrate the above graphs and fuzzy multi graphs. The properties of the isomorphism of the fuzzy planar graph are given.

The irregular bipolar fuzzy graphs and their properties are given in [19]. The highly and neighborly irregular bipolar fuzzy graphs are related with each other. Some conditions are provided to bring the regular fuzzy graph to the form of regular fuzzy bipolar graph.

In [20] presented method is based on fuzzy graph, and focuses on data security, in particular, on encrypting and decrypting images. The matrix of the pixels of the image generates the fuzzy graph, and then the encryption process occurs. The proposed method is differed from other methods by high security and speed. The experimental results show a high efficiency of the method tested on the images with different formats and sizes.

The coloring in fuzzy graphs has been important for several years. In [21] the concept of edge coloring is extended to a fuzzy graph to define k-fuzzy edge coloring. The applicability of Vising's theorem for the fuzzy edge coloring problem is proved. The conditions for edge chromatic number of both complete fuzzy graph and fuzzy trees are given.

The algorithm for obtaining the complement of the fuzzy graph and edge chromatic number for this graph is presented in [22].The edge chromatic number means that the number for coloring this fuzzy graph should be as minimal as possible. The coloring function based on α cut is defined for coloring the complement fuzzy graph. By using the proposed algorithm, all the edges of the complement fuzzy graph can be colored.

Coloring a fuzzy graph by using alpha cut method is discussed in [23]. This fuzzy graph is depicted by vertices and edges, and namely by fuzzy vertices and fuzzy edges with their membership values. The properties of alpha cut sets are described by fuzzy chromatic number. The properties of the union, join and complement operations are described. It is defined that the increment of chromatic number of fuzzy graph occurs due to the decrement of the value of alpha cut.

The edge coloring concept can be also effectively used for the fuzzy graph to overcome the possible conflicts in course time tabling problem involving such factors as instructors, classes and courses. In [24] developed algorithm is convenient for satisfying the certain constraints related to the course time table.

Chapter 3

FUZZY RELATIONS AND GRAPHS

3.1 Crisp Relation Versus Fuzzy Relation

In the classical or crisp relation, there is clarity on the existence of interconnectedness or relationship between the elements of multiple sets which is depicted by the existence (1) or inexistence (0) of a relationship. In crisp sets and relations, an element belongs clearly to a collection or not. There is an existence of a clear-cut definition of the properties of acceptable elements and as such, only discrete criteria are acceptable.

In technical and practical scenarios, discrete criteria are not always applicable to every situation, and as such, the fuzzy (non-distinct, ambiguous) nature of group belonging becomes crucial.

In fuzzy relations, less clarity exists in the discrete association between the elements of multiple sets, and as such, requires more complexity in computation. To identify the level of interconnectedness or association in a fuzzy relation, degree of membership is used. The fuzzy relation depicts the strength of connection between the elements of multiple sets.

A fuzzy graph functions to represent an indistinctly defined dependency. Fuzzy relations happen to be fuzzy subsets of $X \times Y$, a mapping from $X \rightarrow Y$.

The relation $R \subseteq S \times S$ on a set S can be identified to define a graph having vertex set S and edge set R. The graph has a pair (S, R), with S is the set while R is the relation on the set S. Also, any fuzzy relation $\tilde{R}$ related to a fuzzy subset $\tilde{A}$ belonging to set S is described to be defining a weighted or fuzzy graph, where edge weight is

$$(x, y) \in S \times S \,;\, R(x, y) \in [0,1].$$

Fuzzy relations due to their nature have gained natural applications in some fields of study. Some fields of application of fuzzy graphs are economics, medicine, sociology, psychology, image processing etc.

The Cartesian product is a cross product of the crisp sets X and Y, which is denoted by $X \times Y$, is the crisp set of every ordered pair whose first element per pair exists in X and the second exists in Y.

$$X \times Y = \{(x, y), x \in X, y \in Y\}$$

It is to note that if $X \neq Y$, then $X \times Y \neq Y \times X$.

The crisp relation from the set X to the set Y is represented below according to the Cartesian product $X \times Y$, where $X = \{x_1, x_2, x_3\}$, and $Y = \{y_1, y_2, y_3\}$:

R	y_1	y_2	y_3
x_1	1	1	1
x_2	1	1	1
x_3	1	1	1

The graphical representation of the crisp relation is given in Figure 1.

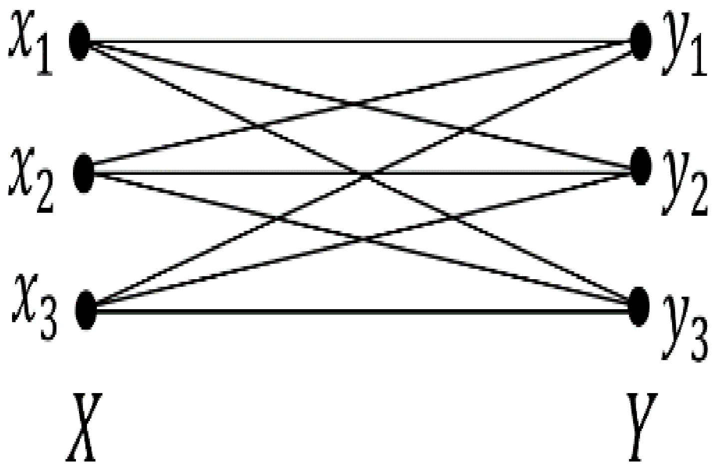

Figure 1: Graphical Representation of Crisp Relation

The fuzzy relations refer to fuzzy sets that are defined on universal sets of Cartesian products (Figure 1). The membership function of the relation R for the pair (x, y), $R(x, y) = 1$ shows that two elements in consideration x and y are fully related while

$R(x, y) = 0$ shows that they are fully unrelated. Values in between 0 and 1 denote a partial relationship. It is needful to know that in the Cartesian product there are distinct sets called the dimensions of the relation.

Assume that X and Y are fuzzy sets described as $X = 0.1/x_1 + 0.4/x_2 + 1/x_3$, and $Y = 0.8/y_1 + 0.3/y_2 + 0.2/y_3$. The Cartesian product of X and Y consists of pairs from X to Y with the minimum corresponding membership functions are given below:

$X \times Y = \{[(x_1, y_1)/\min(0.1,0.8)], [(x_2, y_1)/\min(0.4,0.8)], [(x_3, y_1)/min(1,0.8)],$

$[(x_1, y_2)/\min(0.1,0.3)], [(x_2, y_2)/\min(0.4,0.3)], [(x_3, y_2)/\min(1,0.3)],$

$[(x_1, y_3)/\min(0.1,0.2)], [(x_2, y_3)/\min(0.4,0.2)], [(x_3, y_3)/\min(1,0.2)]\} =$

$\{[(x_1, y_1)/0.1], [(x_2, y_1)/0.4], [(x_3, y_1)/0.8], [(x_1, y_2)/0.1], [(x_2, y_2)/$

$0.3], [(x_3, y_2)/0.3], [(x_1, y_3)/0.1], [(x_2, y_3)/0.2], [(x_3, y_3)/0.2)]\}.$

The fuzzy relation according to the Cartesian product of $X \times Y$ of above sets $X = \{x_1, x_2, x_3\}$, and $Y = \{y_1, y_2, y_3\}$ is given below, and graphically is represented in Figure 2:

R	y_1	y_2	y_3
x_1	0.1	0.1	0.1
x_2	0.4	0.3	0.2
x_3	0.8	0.3	0.2

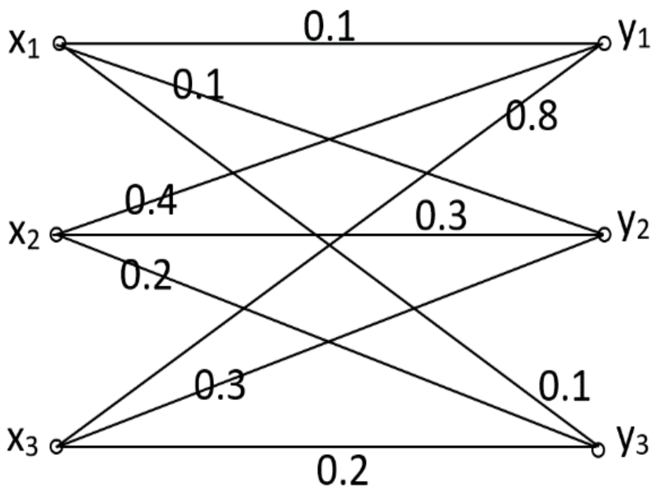

Figure 2: Graphical Representation of Fuzzy Relation

Let’s consider another example with the fuzzy relation. Assume the sets X and Y are given: $X = \{3,4,5\}$, $Y = \{3,4,5,6,7\}$, and the membership functions between the elements of the sets X and Y are defined as follows:

$$\mu_R(x,y) = \begin{cases} (y-x)/(y+x+2), & if\ y > x \\ 0, & if\ y \leq x \end{cases}$$

This can be further expressed as the following matrix representing the membership grades between the elements of the sets X and Y:

$$R= \begin{array}{c|ccccc} & 3 & 4 & 5 & 6 & 7 \\ \hline 3 & 0 & 0.11 & 0.2 & 0.27 & 0.33 \\ 4 & 0 & 0 & 0.09 & 0.17 & 0.23 \\ 5 & 0 & 0 & 0 & 0.08 & 0.14 \end{array}$$

3.2 Operations on Fuzzy Relations

3.2.1 Unary Operations

Unary operations, relating to a single item are of importance in fuzzy operation and relations. This identifies a mathematical computation performed on a single element or member of a set per time. Some of these operations are specific to fuzzy relations while others are generated as fuzzy extension of a crisp counterpart. The inverse or transpose of a fuzzy relation is a peculiar operation from the specific unary operations for fuzzy relations [25].

The following fuzzy relation $R^{-1} \subseteq Y \times X$ is the inverse of $R \subseteq Y|X \times Y$ if the following condition holds:

$$R^{-1}(y,x) = R(x,y)$$

for all pairs $(y,x) \in Y \times X$.

Also, noteworthy is it that for all $(y,x) \in Y \times X$,

$$R^{-1}(y,x) = R(x,y) \leftrightarrow \mu_{R^{-1}}(y,x) = \mu_R(x,y)$$

3.2.2 Binary Operations

Because fuzzy relations are known to be special cases of fuzzy sets, operations that are possible on fuzzy relations are equally possible on fuzzy sets as well. Examples of such operations are standard fuzzy union and standard fuzzy intersection [25].

Standard Fuzzy Union: In a situation when $A \subseteq X \times Y$ and $B \subseteq X \times Y$ fuzzy relations have some compatibility, the standard fuzzy union of A and B is a fuzzy relation

$$A \cup B \subseteq X \times Y$$

with degree of membership

$$\mu_{A \cup B}(x, y) = \vee\,(\mu_A(x, y),\ \mu_B(x, y))$$

for all pairs $(x, y) \in X \times Y$. Below the fuzzy relations and their defined standard fuzzy union are given.

R_1	y_1	y_2	y_3
x_1	0.1	0.4	0.2
x_2	0.5	0.3	0.1
x_3	0.6	0.2	0.1

$\cup$

R_2	y_1	y_2	y_3
x_1	0.2	0.3	0.1
x_2	0.4	0.5	0.2
x_3	0.5	0.1	0.3

=

$R_1 \cup R_2$	y_1	y_2	y_3
x_1	0.2	0.4	0.2
x_2	0.5	0.5	0.2
x_3	0.6	0.2	0.3

Standard Fuzzy Intersection: In a situation when $A \subseteq X \times Y$ and $B \subseteq X \times Y$ fuzzy relations have some compatibility, the standard fuzzy intersection of A and B is a fuzzy relation

$$A \cap B \subseteq X \times Y$$

with degree of membership

$$\mu_{A \cap B}(x, y) = \wedge\, (\mu_A(x, y),\ \mu_B(x, y))$$

for all pairs $(x, y) \in X \times Y$.

Below the fuzzy relations and their standard fuzzy intersection are given.

R_1	y_1	y_2	y_3
x_1	0.2	0.5	0.1
x_2	0.1	0.6	0.3
x_3	0.8	0.1	0.1

$\cap$

R_2	y_1	y_2	y_3
x_1	0.1	0.4	0.5
x_2	0.3	0.2	0.4
x_3	0.6	0.2	0.2

=

$R^1 \cap R^2$	y_1	y_2	y_3
x_1	0.1	0.4	0.1
x_2	0.1	0.2	0.3
x_3	0.6	0.1	0.1

3.3 Compositions of Fuzzy Relations

3.3.1 Fuzzy Max-min Composition and Fuzzy Max-product Composition

If $R_1(x, y), (x, y) \in X \times Y$ and $R_2(y, z), (y, z) \in Y \times Z$ are two fuzzy relations, the fuzzy max-min composition will be defined as follows [25]:

$$R_1 \circ R_2 = \left[(x, z), \underset{y}{\max}\{\min\{\mu_{R_1}(x, y), \mu_{R_2}(y, z)\}\}\right] x \in X, y \in Y, z \in Z$$

The fuzzy max-product composition is mathematically described as

$$R_1 \bullet R_2(x, z) = \left\{\left[(x, z), \underset{y}{max}\{\mu_{R_1}(x, y)\ .\ \mu_{R_2}(y, z)\}\right]\middle|\ x \in X, y \in Y, z \in Z\right\}$$

Let's apply fuzzy max-min composition and fuzzy max-product composition on fuzzy relations. Suppose the following fuzzy relations from the set X to the set Y and from the set Y to the set Z are given which are graphically depicted in Figure 3:

R_1	y_1	y_2	y_3
x_1	0.9	0.1	0.2
x_2	0.8	0.5	0.4
x_3	0.6	0.4	0.1

R_2	z_1	z_2	z_3
y_1	0.1	0.5	0.7
y_2	0.6	0.9	0.5
y_3	0.3	0.2	0.7

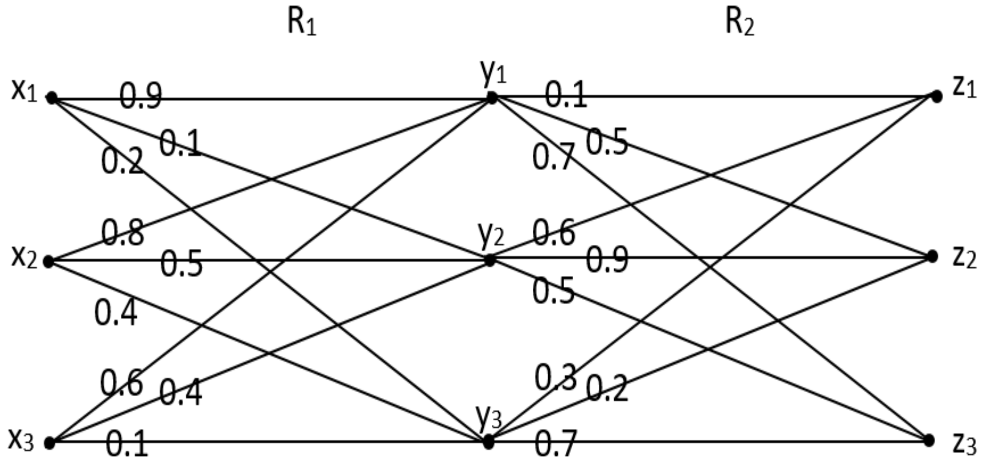

Figure 3: Graph of Fuzzy Relations

After using the fuzzy max-min composition we have the relation

R	z_1	z_2	z_3
x_1	0.2	0.5	0.7
x_2	0.5	0.5	0.7
x_3	0.4	0.5	0.6

After using the fuzzy max-product composition we have the following relation:

R	z_1	z_2	z_3
x_1	0.09	0.45	0.63
x_2	0.30	0.45	0.56
x_3	0.24	0.36	0.42

The new relations after applying the fuzzy max-min and fuzzy max-product compositions are graphically represented in Figure 4 and Figure 5, respectively.

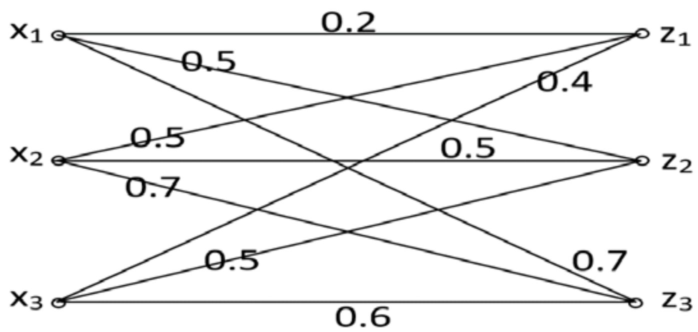

Figure 4: Graphical Representation of Relation after Applying Fuzzy Max-min Composition

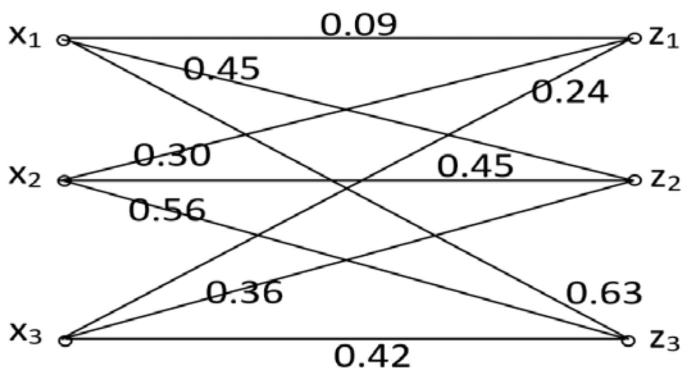

Figure 5: Graphical Representation of Relation after Applying Fuzzy Max-product Composition

3.4 α-cut of Fuzzy Relation

If $R \subseteq A \times B$, and $R_{\propto}$ is a α-cut relation, it follows directly that [25]

$$R_{\propto} = \{(x, y) | \mu_R(x, y) \geq \propto, \quad x \in A, y \in B\}$$

Below the fuzzy relation R and α-cut of fuzzy relation R with the coefficients of α=0.3, α=0.6, and α=0.9 are given.

	0.1	0.3	0.9	0.4
	0.2	0.0	1	0.7
$M_R =$	0.6	0.5	0.2	0.3
	0.4	0.3	0.1	0.4

	0	1	1	1
	0	0	1	1
$M_{R\ 0.3} =$	1	1	0	1
	1	1	0	1

	0	0	1	0
	0	0	1	1
$M_{R\ 0.6} =$	1	0	0	0
	0	0	0	0

$$M_{R\,0.9} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.5 Types of Fuzzy Relations

The following types of fuzzy relations exist to be defined on $A \times$ A.

- Fuzzy Reflexive Relation: R is defined to be fuzzy reflexive relation in $X \times X$ if $\mu_R(x, x) = 1$ for all x that exist in A ($\forall x \in A$).

The following is the fuzzy reflexive relation:

R	x_1	x_2	x_3	x_4
x_1	1	0.7	0.8	0.3
x_2	0.8	1	0.6	0.3
x_3	0.5	0.3	1	0.4
x_4	0.3	0.5	0.4	1

The graphical representation of above fuzzy reflexive relation is in Figure 6:

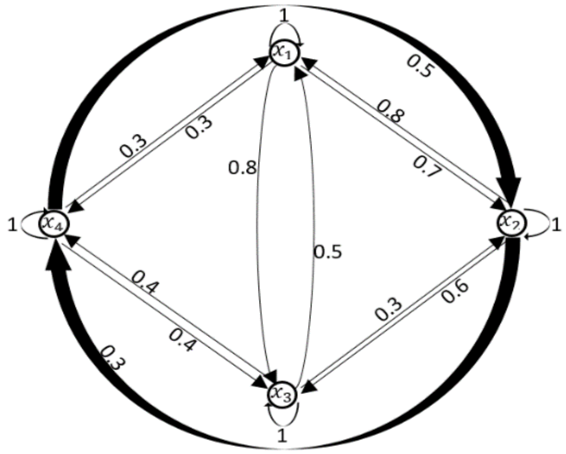

Figure 6: Graph of Fuzzy Reflexive Relation

- Fuzzy Symmetric Relation: R is fuzzy symmetric relation, if

$$R(x, y) = R(y, x), \text{ for } \forall (x, y) \in A \times A$$

The following is the fuzzy symmetric relation:

R	x_1	x_2	x_3	x_4
x_1	1	0.5	0.3	0.7
x_2	0.5	1	0.6	0.2
x_3	0.3	0.6	1	0.1
x_4	0.7	0.2	0.1	1

The graphical representation of above fuzzy symmetric relation is in Figure 7:

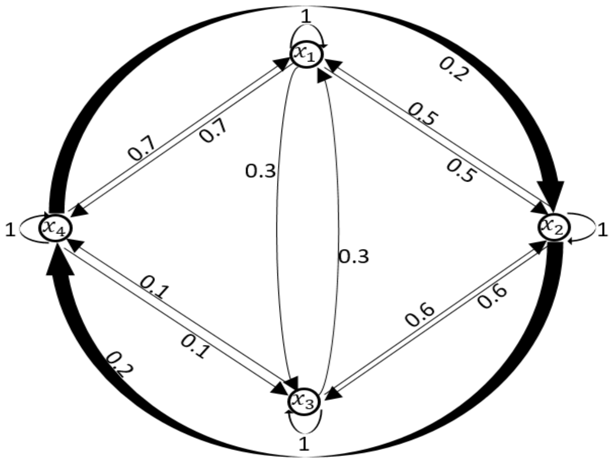

Figure 7: Graph of Fuzzy Symmetric Relation

- Fuzzy Anti-symmetric Relation: Conversely, any fuzzy relation is described to be anti-symmetric ($x \neq y$), if:

$$\mu_R(x, y) \neq \mu_R(y, x), \text{ for } \forall (x, y) \in A \times A$$

or

$$\mu_R(x, y) = \mu_R(y, x) = 0, \text{ for } \forall (x, y) \in A \times A$$

The following is fuzzy anti-symmetric relation:

R	x_1	x_2	x_3	x_4
x_1	0	0.2	0.6	0.3
x_2	0.1	0	0.3	0.7
x_3	0.3	0.5	0	0.5
x_4	0.2	0.8	0.4	0

The graphical representation of above fuzzy anti-symmetric relation is in Figure 8:

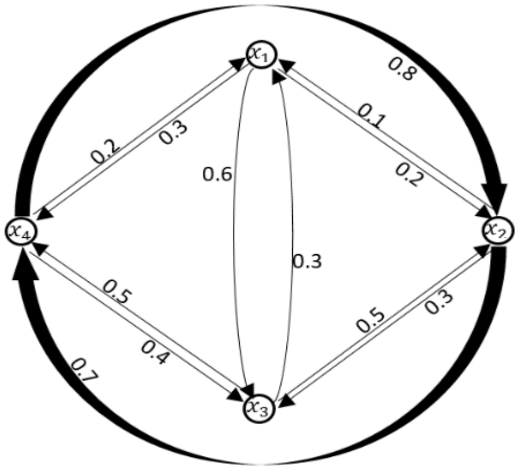

Figure 8: Graph of Fuzzy Anti-symmetric Relation

- Fuzzy Transitive Relation: R is fuzzy transitive relation if

$$\forall(x,y),(y,z),(x,z) \in A \times A$$

$$\mu_R(x,z) \geq \max[\min(\mu_R(x,y),\mu_R(y,z))] = \mu_{R\bullet R}(\mathrm{x,z})$$

Let R be the fuzzy relation given below:

R	x_1	x_2	x_3	x_4
x_1	0.2	1	0.4	0.6
x_2	0.1	0.7	0.3	0.5
x_3	0.1	1	0.4	0.5
x_4	0.1	0.4	0.3	0.4

Then, $R \circ R$ becomes

R	x_1	x_2	x_3	x_4
x_1	0.2	1	0.4	0.6
x_2	0.1	0.7	0.3	0.5
x_3	0.1	1	0.4	0.5
x_4	0.1	0.4	0.3	0.4

$\circ$

R	x_1	x_2	x_3	x_4
x_1	0.2	1	0.4	0.6
x_2	0.1	0.7	0.3	0.5
x_3	0.1	1	0.4	0.5
x_4	0.1	0.4	0.3	0.4

$$Max[min(0.2,0.2), min(1,0.1), min(0.4,0.1), min(0.6,0.1)]$$

$$= Max[0.2,0.1,0.1,0.1] = 0.2$$

$$Max[min(0.2,1), min(1,0.7), min(0.4,1), min(0.6,0.4)]$$

$$= Max[0.2,0.7,0.4,0.4] = 0.7$$

$$Max[min(0.2,0.4), min(1,0.3), min(0.4,0.4), min(0.6,0.3)]$$

$$= Max[0.2,0.3,0.4,0.3] = 0.4$$

$$Max[min(0.2,0.6), min(1,0.5), min(0.4,0.5), min(0.6,0.4)] =$$

$$Max[0.2,0.5,0.4,0.4] = 0.5$$

$$Max[min(0.1,0.2), min(0.7,0.1), min(0.3,0.1), min(0.5,0.1)]$$

$$= Max[0.1,0.1,0.1,0.1] = 0.1$$

$$Max[min(0.1,1), min(0.7,0.7), min(0.3,1), min(0.5,0.4)] =$$

$$Max[0.1,0.7,0.3,0.4] = 0.7$$

$$Max[min(0.1,0.4), min(0.7,0.3), min(0.3,0.4), min(0.5,0.3)]$$

$$= Max[0.1,0.3,0.3,0.3] = 0.3$$

$$Max[min(0.1,0.6), min(0.7,0.5), min(0.3,0.5), min(0.5,0.4)] =$$

$$Max[0.1,0.5,0.3,0.4] = 0.5$$

$$Max[min(0.1,0.2), min(1,0.1), min(0.4,0.1), min(0.5,0.1)] =$$

$$Max[0.1,0.1,0.1,0.1] = 0.1$$

$$Max[min(0.1,1), min(1,0.7), min(0.4,1), min(0.5,0.4)] =$$

$$Max[0.1,0.7,0.4,0.4] = 0.7$$

$$Max[min(0.1,0.4), min(1,0.3), min(0.4,0.4), min(0.5,0.3)]=$$

$$Max[0.1,0.3,0.4,0.3] = 0.4$$

$$Max[min(0.1,0.6), min(1,0.5), min(0.4,0.5), min(0.5,0.4)] =$$

$$Max[0.1,0.5,0.4,0.4] = 0.5$$

$$Max[min(0.1,0.2), min(0.4,0.1), min(0.3,0.1), min(0.4,0.1)] =$$

$$Max[0.1,0.1,0.1,0.1] = 0.1$$

$$Max[min(0.1,1), min(0.4,0.7), min(0.3,1), min(0.4,0.4)] =$$

$$Max[0.1,0.4,0.3,0.4] = 0.4$$

$$Max[min(0.1,0.4), min(0.4,0.3), min(0.3,0.4), min(0.4,0.3)] =$$

$$Max[0.1,0.3,0.3,0.3] = 0.3$$

$$Max[min(0.1,0.6), min(0.4,0.5), min(0.3,0.5), min(0.4,0.4)] =$$

$$Max[0.1,0.4,0.3,0.4] = 0.4$$

So the final form is

$R \circ R$	x_1	x_2	x_3	x_4
x_1	0.2	0.7	0.4	0.5
x_2	0.1	0.7	0.3	0.5
x_3	0.1	0.7	0.4	0.5
x_4	0.1	0.4	0.3	0.4

It can be observed that $\mu_{R \circ R}(x, y) \leq \mu_R(x, y)$ holds for all $x, y \in X$.

In Figure 9 the graph of fuzzy transitive relation R is depicted.

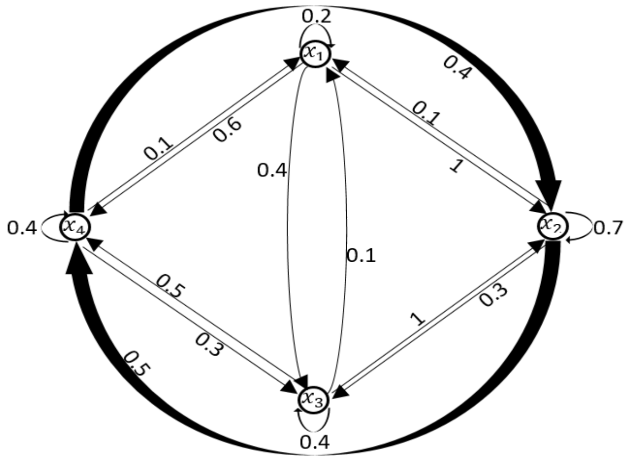

Figure 9: Graph of Fuzzy Transitive Relation

- Fuzzy Equivalence Relation: Any fuzzy relation R with reflexive, symmetric and transitive properties is a fuzzy equivalence relation.

The following is the fuzzy equivalence relation:

R	x_1	x_2	x_3	x_4
x_1	1	0.9	0.8	1
x_2	0.9	1	0.8	0.9
x_3	0.8	0.8	1	0.8
x_4	1	0.9	0.8	1

The graphical representation of above fuzzy equivalence relation is in Figure 10:

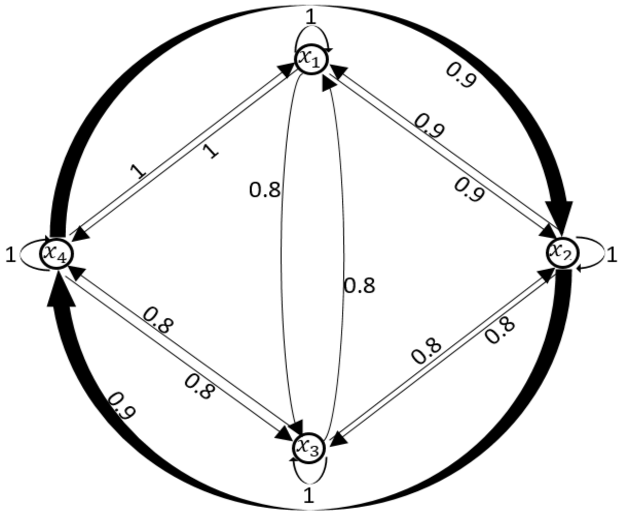

Figure 10: Graph of Fuzzy Equivalence Relation

- Fuzzy Partial Order Relation: Any fuzzy relation *R* with reflexive, anti-symmetric and transitive properties is a fuzzy partial order relation.

The following is the fuzzy partial order relation:

R	x_1	x_2	x_3	x_4
x_1	1	0.6	0.1	0
x_2	0.3	1	0.1	0
x_3	0	0	1	0
x_4	0.1	0.1	0.1	1

The graphical representation of above fuzzy partial order relation is in Figure 11:

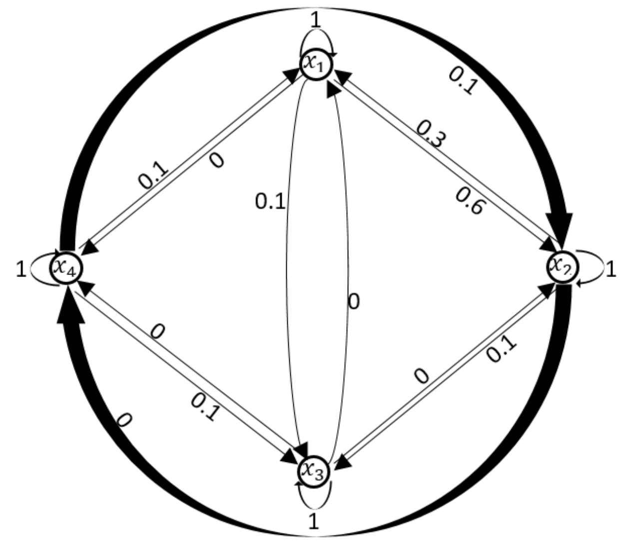

Figure 11: Graph of Fuzzy Partial Order Relation

Chapter 4

OPERATIONS ON FUZZY GRAPHS

4.1 Characteristics of Fuzzy Graphs

Various operations are permitted on fuzzy graphs. We try to describe a partial fuzzy subgraph of graph (xv, xx) by the partial fuzzy subgraph of graph (V, X) where X are edges. If we have $G = (V, X)$ as a graph, the partial fuzzy subgraph in G can be described as the ordered pair (μ, ρ) with μ being a fuzzy subset of V and ρ being a symmetric fuzzy relation on V. Barring a loss of generality, ρ also could have been evaluated as the fuzzy subset of X and as such, (μ, ρ) can also be evaluated to be a partial fuzzy subgraph of G.

We assume that (μ_i, ρ_i) is a partial fuzzy subgraph of $G_i = (V_i, X_i)$ where $0 < i \leq 2$.

The various operations of Cartesian product, union and join are defined on (μ_1, ρ_1) and (μ_2, ρ_2). We refer to the edge between two known vertices u and v as (uv) because the vertex of the graph from any Cartesian product is always an ordered pair, where $i = 1, 2,$ if graph G is obtained from G_1 and G_2 using any of the operations of join and union, we obtain the necessary and sufficient conditions for any arbitrary partial fuzzy subgraph of G to be obtained from partial fuzzy subgraphs of G_1 and G_2 through same operations.

4.2 Cartesian Product of Fuzzy Graphs

Consider the following Cartesian product

$G = G_1 \times G_2 = (V, X)$ of graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$.

Then $V = V_1 \times V_2$ and

$$X = \{(u, u_2)(u, v_2) |\ u \in V_1, u_2v_2 \in X_2\} \cup \{(u_1, w)(v_1, w) | w \in V_2, u_1u_2 \in X_1\}$$

μ_i is a the fuzzy subset of V_i and ρ_i is also a fuzzy subset of X_i, where $0 < i \leq 2$. The fuzzy subsets in $\mu_1 \times \mu_2$ of V and $\rho_1\rho_2$ of X is thus defined:

$\forall\ (u_1, u_2) \in V, (\mu_1 \times \mu_2)(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2)$;

$\forall u \in V_1, \forall u_2v_2 \in X_2, \rho_1\rho_2(\ (u, u_2,)(u, v_2) = \mu_1(u) \wedge \rho_2(u_2v_2)$.

4.3 Union and Join Operations of Fuzzy Graphs

Union Operation: If G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) are fuzzy graphs having G_{1*}: (V_1, E_1) just as G_{2*}: (V_2, E_2). G=$G_1 \cup G_2$: $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a fuzzy graph which describes the union of two fuzzy graphs G_1 and G_2, which is also described as

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u)\ if\ u \in V_1 - V_2 \\ \sigma_2(u)\ if\ u \in V_2 - V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(uv) = \begin{cases} \mu_1(u)\ if\ uv \in E_1 - E_2 \\ \mu_2(u)\ if\ uv \in E_2 - E_1 \end{cases}$$

Join Operation: For the join operation $G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs having arcs E' connecting nodes V_1 and V_2 (assuming that $V_1 \cap V_2 = \emptyset$). Hence, the join of two fuzzy graphs is a fuzzy graph [26].

Joining fuzzy graphs G_1 and $G_2(G_1 + G_2)$ produces another fuzzy graph G: $(\sigma_1 + \sigma_2$, $\mu_1 + \mu_2)$ which can be further defined by $(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u)$, where $u \in V_1 \cup V_2$ and as such

$$(\mu_1 + \mu_2)(uv) = \begin{cases} (\mu_1 \cup \mu_2)(uv) & if\ uv \in E_1 \cup E_2 \\ \sigma_1(u) \wedge \sigma_2(v) & if\ uv \in E' \end{cases}$$

4.4 Fuzzy Subgraph and Partial Fuzzy Subgraph

$Q: (S, U)$ is a partial fuzzy subgraph of $G: (\sigma, \mu)$ if $S(u) \leq \sigma(u) \forall\, u$ and $U(u, v) \leq \mu(u, v)\ \forall\, u$ and v, and as such, any partial fuzzy subgraph $(Q: (S, U))$ is described to be a subgraph of $G: (\sigma, \mu)$ if for each u in $S^*$and $U(u, v) = \mu(u, v)$ for each arc (u, v) in $U^*, s(u) = \sigma(u)$. The fuzzy subgraph $Q: (S, U)$ covers $G: (\sigma, \mu)$ if $S = \sigma$.

By definition, $H = (C, D, f)$ is a fuzzy subgraph of $F = (A, B, f)$ if $C \subseteq A$ and $D \subseteq B$.

In Figure 12 the fuzzy graph (a) and its possible fuzzy subgraph (b) are described.

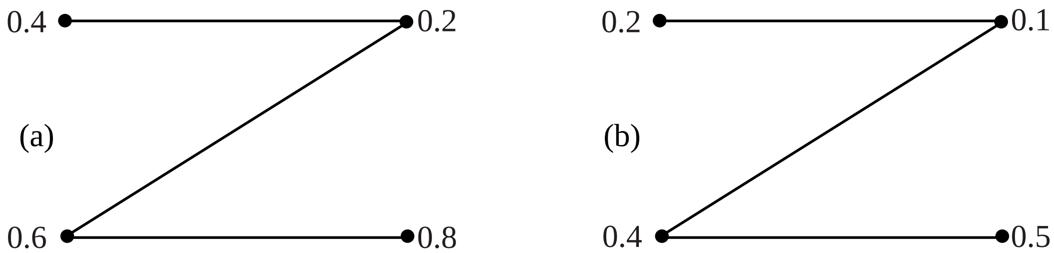

Figure 12: Fuzzy Graph (a) and Fuzzy Subgraph (b)

4.5 Complement of Fuzzy Graph

The complement of the fuzzy graph $G:(\sigma,\mu)$ is a fuzzy graph $\bar{G}:(\bar{\sigma},\bar{\mu})$, where $\bar{\sigma} \equiv \sigma$, and [27]

$$\bar{\mu}(u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v) \forall\, u, v \in V$$

Figures 13 and 14 show the fuzzy graph G_1 and its complement $\bar{G}_1$, and fuzzy graph G_2 and its complement $\bar{G}_2$, respectively.

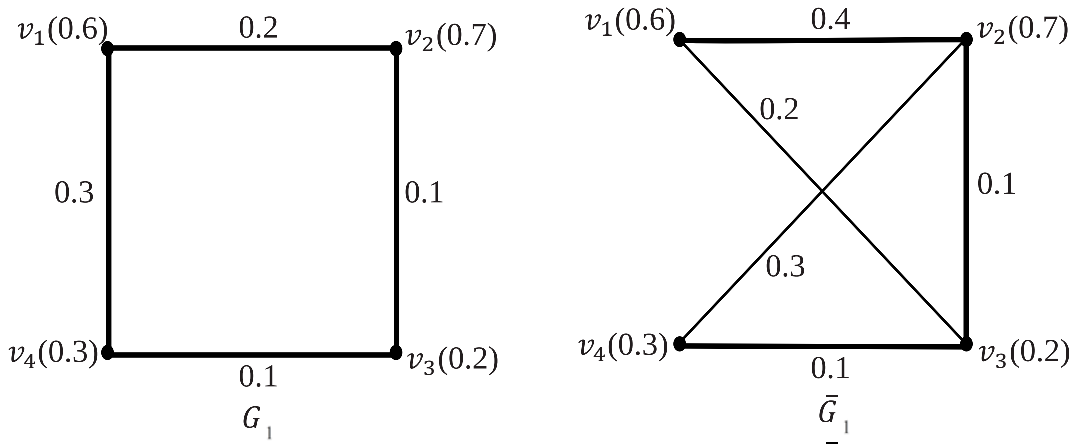

Figure 13: Fuzzy Graph G_1 and its Complement $\bar{G}_1$

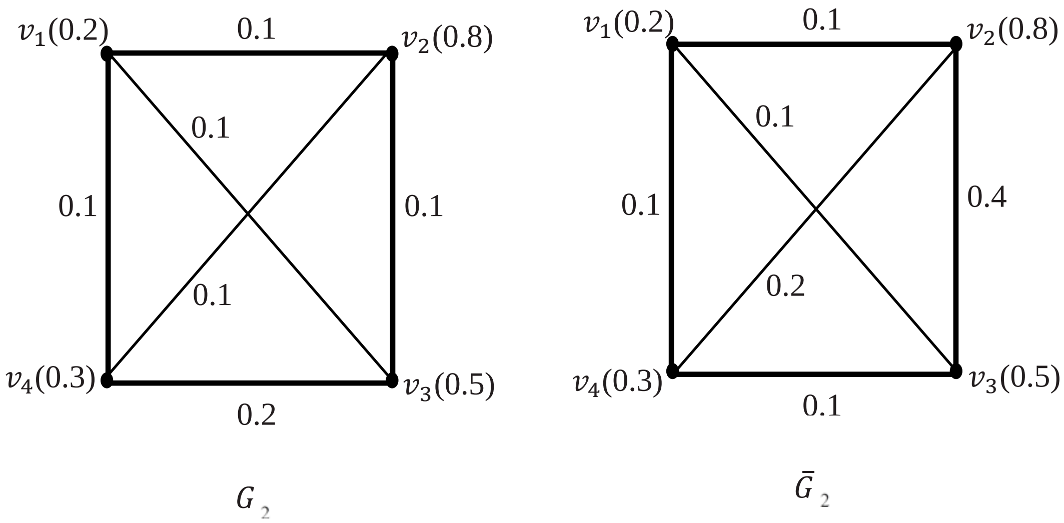

Figure 14: Fuzzy Graph G_2 and its Complement $\bar{G}_2$

4.6 Degrees and Total Degrees of Vertices of Fuzzy Graph

A vertex and an edge are said to be incident to each other when the vertex $\sigma(u_i)$ happens to be an end vertex of the edge $\mu(u_i, v_j)$ in a fuzzy graph $G: (\sigma, \mu)$. The degree of the aforementioned vertex $\sigma(u_i)$ is described to be the summation of the degree of membership of edges incident at the vertex $\sigma(u_i)$. The degree of a vertex is denoted by $d(\sigma(u_i))$.

The Degree of a Vertex: If $G: (\sigma, \mu)$ is a fuzzy graph on $G^*: (V, E)$. The degree of vertex u is $d_G(u) = \sum_{u \neq v} \mu(uv)$ [28].

The Total Degree of a Vertex: If $G: (\sigma, \mu)$ is a fuzzy graph on $G^*: (V, E)$. The total degree of vertex $u \in V$ is $td_G(u) = \sum_{u \neq v} \mu(uv) + \sigma(u)$ [28].

Regular Fuzzy Graph: Let $G: (\sigma, \mu)$ to be a fuzzy graph on $G^*: (V, E)$, and if the degree k is the same for all the vertices of G, then G is called a regular fuzzy graph [28].

Totally Regular Fuzzy Graph: Let $G: (\sigma, \mu)$ to be a fuzzy graph on $G^*: (V, E)$, and if the total degrees of the graph G are same, then G is called totally regular fuzzy graph [29].

The graph in Figure 15 is regular fuzzy graph since the degrees of all the vertices of the graph are same, but this graph is not totally regular fuzzy graph because the vertices have distinct total degrees.

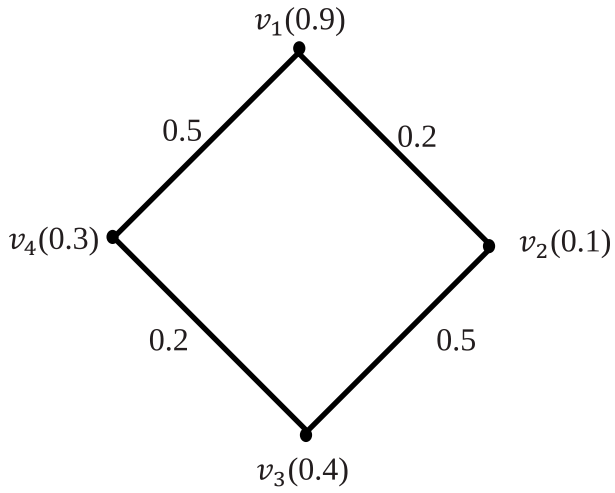

Figure 15: Regular Fuzzy Graph

The degrees and total degrees of the vertices of above regular fuzzy graph are calculated below:

$$\text{Degree:}v_1 = 0.5 + 0.2 = 0.7$$

$$\text{Degree:}v_2 = 0.5 + 0.2 = 0.7$$

$$\text{Degree:}v_3 = 0.5 + 0.2 = 0.7$$

$$\text{Degree:}v_4 = 0.5 + 0.2 = 0.7$$

$$\text{Total degree:}v_1 = 0.9 + 0.5 + 0.2 = 1.6$$

$$\text{Total degree:}v_2 = 0.1 + 0.5 + 0.2 = 0.8$$

$$\text{Total degree:}v_3 = 0.4 + 0.5 + 0.2 = 1.1$$

$$\text{Total degree:} v_4 = 0.3 + 0.5 + 0.2 = 1$$

The graph in Figure 16 is the totally regular fuzzy graph because all the vertices have the same total degrees.

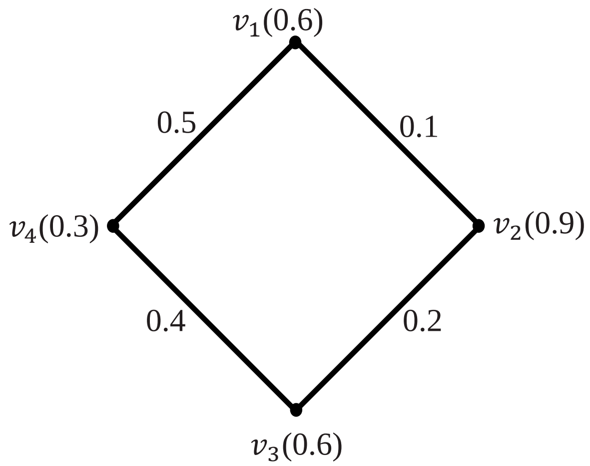

Figure 16: Totally Regular Fuzzy Graph

The total degrees of all the vertices of the graph in Figure 16 are same:

$$\text{Total degree:} v_1 = 0.6 + 0.5 + 0.1 = 1.2$$

$$\text{Total degree:} v_2 = 0.9 + 0.1 + 0.2 = 1.2$$

$$\text{Total degree:} v_3 = 0.6 + 0.4 + 0.2 = 1.2$$

$$\text{Total degree:} v_4 = 0.3 + 0.5 + 0.4 = 1.2$$

4.7 Complete Fuzzy Graph

The graph $G: (\mu, \rho)$ is complete fuzzy graph if the following condition holds:

$$\rho(u, v) = \mu(u) \wedge \mu(v) \forall u, v \in V$$

Figures 17, 18, and 19 represent complete fuzzy graphs with three vertices (K_3), four vertices (K_4), and five vertices (K_5), respectively.

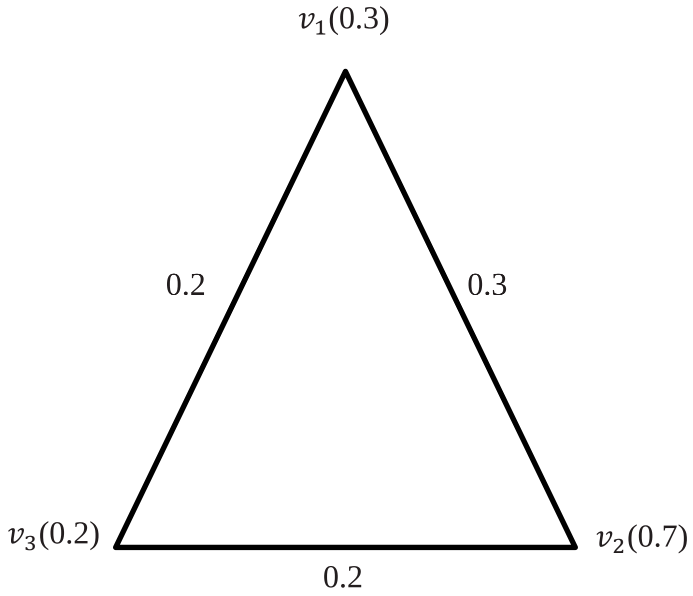

Figure 17: Complete Fuzzy Graph with Three Vertices K_3

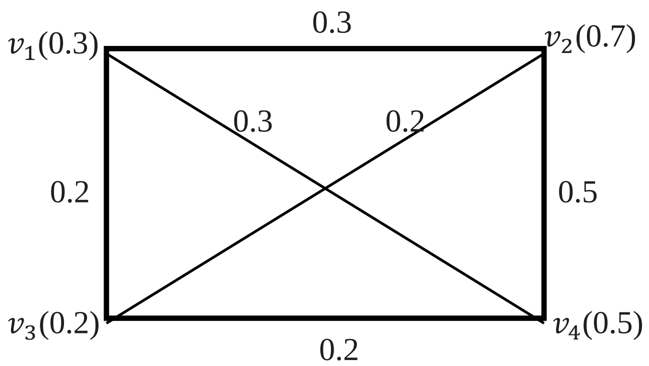

Figure 18: Complete Fuzzy Graph with Four Vertices K_4

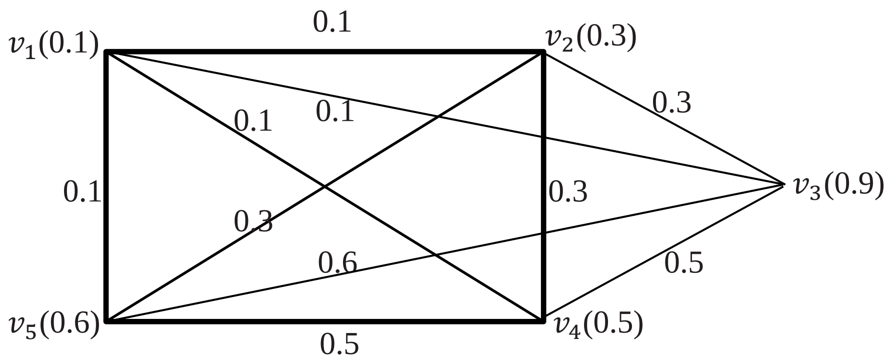

Figure 19: Complete Fuzzy Graph with Five Vertices K_5

4.8 Fuzzy Tree

Fuzzy tree is defined as a part of connected fuzzy graph $G: (\sigma, \mu)$ to be a fuzzy spanning subgraph $F: (\sigma, v)$. For all the edges (u, v) not in F, must be $\mu\ (u, v) < v^{\infty}(u, v)$[30].

In Figure 20 the connected fuzzy graph (a), and fuzzy tree (b) formed from this connected fuzzy graph are represented.

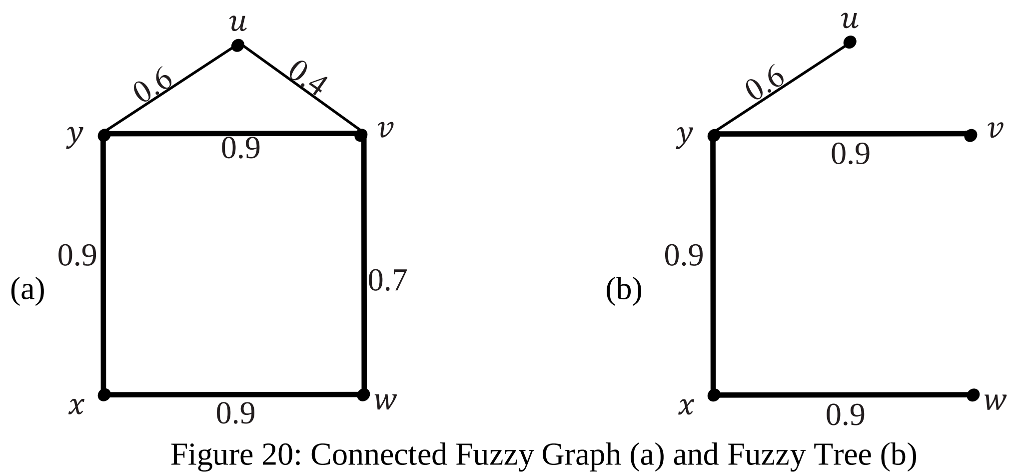

Figure 20: Connected Fuzzy Graph (a) and Fuzzy Tree (b)

The fuzzy tree in Figure 20 is formed by deletion of two edges from the connected fuzzy graph defined as follows:

$$\mu(u, v) = 0.4 < 0.6 = v^{\infty}(u, v)$$

$$\mu(v, w) = 0.7 < 0.9 = v^{\infty}(v, w)$$

Chapter 5

CONCLUSION

In this master thesis the various properties of fuzzy graphs are studied. The fuzzy relations and fuzzy graphs are described, and some of their major characteristics are identified. The comparison of crisp and fuzzy relations is done. The unary and binary operations on fuzzy relations are carried out. The compositions of fuzzy relations including the fuzzy max-min composition and fuzzy max-product composition are obtained. The fuzzy relations are also described by using α-cut concept.

Furthermore, the operations on fuzzy graphs and their properties considered. The Cartesian product, the union and join operations on fuzzy graphs are represented. The fuzzy subgraph, partial fuzzy subgraph and the complement of fuzzy graph are discussed. The degrees and total degrees of vertices of fuzzy graph are defined. The properties of regular and totally regular fuzzy graphs are studied. Finally, the complete fuzzy graph and the fuzzy tree are discussed.

REFERENCES

[1] Nagoorgani, A., & Subahashini, D. R. (2014). Fuzzy labeling tree. *International Journal of Pure and Applied Mathematics*, 90(2), 131-141.

[2] Vaishnaw, Y., & Sharma, S. (2012). Some analogues results on fuzzy graphs. Int. J. Math. Sci. Appl, 2, 535-539.

[3] Nirmala, G., & Prabavathi, S. (2014). Application of Fuzzy If-Then Rule in a Fuzzy Graph with Modus Ponens. *International Journal of Science and Research (IJSR), Volume 3, Issue 11*, pp. 837-841.

[4] Gani, A. N., & Radha, K. (2008). On regular fuzzy graphs. *Journal of Physical Sciences, Vol. 12*, pp. 33-40.

[5] Sunitha, M. S., & Vijaya Kumar, A. (2002). Complement of a Fuzzy Graph. *Indian J. pure appl. Math., 33(9),* pp. 1451-1464.

[6] Narayan, K. S. S., & Sunitha, M. S. (2012). Connectivity in a Fuzzy Graph and its Complement. *Gen. Math. Notes, Vol. 9, No. 1,* pp. 38-43.

[7] Sharma, A. K., Padamwar, B. V., & Dewangan, C. L. (2013). Trends in Fuzzy Graphs. *International Journal of Innovative Research in Science, Engineering and Technology, Vol.2, Issue 9,* pp. 4636-4640.

[8] Tom, M., & Sunitha, M. S. (2014). Sum Distance in Fuzzy Graphs. *Annals of Pure and Applied Mathematics, Vol. 7, No. 2,* pp. 73-89.

[9] Rahurikar, S., (2014). On Isolated Fuzzy Graph. *International Journal of Research in Engineering Technology and Management, Volume 2, Issue 6*, pp. 1-3.

[10] Ponnappan, C. Y., Ahamed, S. B., & Surulinathan, P. (2015). Edge Domination in Fuzzy Graphs - New Approach. *International Journal of IT, Engineering and Applied Sciences Research (IJIEASR), Volume 4, No. 1,* pp. 14-17.

[11] Velammal, S. (2012). Edge Domination in Intuitionistic Fuzzy Graphs. *International Journal of Computational Science and Mathematics. Volume 4, Number 2,* pp. 159-165.

[12] Chandrasekaran, E., & Sathyaseelan, N. (2012). Fuzzy Node Fuzzy Graph and its Cluster Analysis. *International Journal of Engineering Research and Applications (IJERA), Vol. 2, Issue 3,* pp.733-738.

[13] Jayalakshmi, P. J., Revathi, S., & Harinarayanan, C. V. R. (2014). Independent and Total Strong (Weak) Domination in Fuzzy Graphs. *International Journal of Computational Engineering Research, Vol.4, Issue 1,* pp. 1-4.

[14] Akram, M., & Davvaz, B. (2012). Strong intuitionistic fuzzy graphs. *Filomat 26:1,* pp. 177-196.

[15] Mohideen, B. A. (2015). Strong and regular interval-valued fuzzy graphs. *Journal of Fuzzy Set Valued Analysis 2015, No. 3,* pp. 215-223.

[16] Pal, M., Samanta, S., & Rashmanlou, H. (2015). Some Results on Interval-Valued Fuzzy Graphs. *International Journal of Computer Science and Electronics Engineering (IJCSEE), Volume 3, Issue 3,* pp. 205-211.

[17] Ganesamoorthy, K., Robinson, P. J., & Amirtharaj, E. C. H. (2009). Isomorphism of a Fuzzy Graph Using Fuzzy Matrix Equality. *International Journal of Recent Trends in Engineering, Vol. 1, No. 2,* pp. 276-278.

[18] Samanta, S., Pal, A., & Pal, M. (2014). New concepts of fuzzy planar graphs.*(IJARAI) International Journal of Advanced Research in Artificial Intelligence, Vol. 3, No. 1,* pp. 52-59.

[19] Samanta, S., & Pal, M. (2012). Irregular Bipolar Fuzzy Graphs. *International Journal of Applications of Fuzzy Sets, Vol. 2,* pp. 91-102.

[20] Qaid, G. R. S., Talbar, S. N., & AL–Kubati, A. A. M. (2014). Image Security by Using Fuzzy Graph. *International Journal of Engineering and Innovative Technology (IJEIT), Volume 3, Issue 12,* pp. 154-157.

[21] Govindarajan, R., & Lavanya, S. (2012). Fuzzy Edge Coloring of Fuzzy Graphs. *Fuzzy Sets, Rough Sets and Multivalued Operations and Applications, Vol. 4, No. 1,* pp. 1-5.

[22] Dey, A., Ghosh, D., & Pal, A. (2012). Edge Coloring of a Complement Fuzzy Graph. *International Journal of Modern Engineering Research (IJMER) Vol.2, Issue.4*, pp. 1929-1933.

[23] Savithiri, D., & Gayathrridevi, A. (2014). Application of Alpha Cut Coloring of a Fuzzy Graph. *International Journal of Computer Engineering and Technology (IJCET), Volume 5, Issue 8,* pp. 1-12.

[24] Poornima, B., & Ramaswamy, V. (2010). Application of Edge Coloring of a Fuzzy Graph. *International Journal of Computer Applications, Volume 6, No.2,* pp. 14-19.

[25] http://slidegur.com/doc/1462389/fuzzy-relation.

[26] Venugopalam, D., Kumari, N. M., & Kumar, M. V. (2013). Operations on fuzzy graphs. *South Asian Journal of Mathematics, Vol.3 (5),* pp. 333-338.

[27] Kumari, N. N. M., Venugopalam, D., & Kumar, M. V. (2013). Complement of graph and fuzzy graph. *International Journal of Mathematical Archive-4(7),* pp. 14-18.

[28] Radha, K., & Kumaravel, N. (2014). On Edge Regular Fuzzy Graphs. *Internaional Journal of Mathematical Archive-5(9),* pp. 100-112.

[29] Pathinathan, T., & Rosline, J. J. (2014). Characterization of Fuzzy Graphs into different categories using Arcs in Fuzzy Graphs. *Journal of Fuzzy Set Valued Analysis, Volume 2014,* 6 pages.

[30] Nagarajan, S., & Chandrasekharan, M. (2014). Characterization of Fuzzy Bridges and Fuzzy Cutnodes. *International Journal of Science and Research (IJSR), Volume 3, Issue 4,* pp. 143-146.