Fractional Differential Equations with Fractional Boundary Conditions

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ABSTRACT

This work is dedicated to investigate the existence and uniqueness of solutions for nonlinear fractional differential equations with boundary conditions involving the Caputo fractional derivative in a Banach space. After introducing some basic preliminaries and the important concepts of fractional calculus, we considered two models of boundary value problems of Caputo fractional derivative. The first one is nonlinear fractional differential equation with nonlocal four-point fractional boundary conditions. The second equation is nonlinear impulsive boundary value problem of multi-orders fractional supplemented with nonlocal four-point fractional boundary conditions. The existence and uniqueness of solution are obtained via Banach's fixed point theorem and Schauder's fixed point theorem for the two models. In addition, both results are provided by the illustrative examples to support them.

Keywords: Fractional integrals and derivatives, Fractional differential equations, Existence, Uniqueness, Fixed point theorems, Impulse, Multi-orders.

ÖZ

Bu çalışma Caputo kesirli türevi içeren sınır koşulları ile doğrusal olmayan fraksiyonel diferansiyel denklemlerin çözümleri varlığını ve tekliğini araştırmaktadır. Bazı temel tanımlar ve Kesirli analizin önemli kavramları tanıttıktan sonra Caputo kesirli türevi yardımıyla sınır değer problemleri için iki model verilecektir. İlki yerel olmayan dört nokta kesirli sınır koşulları ile doğrusal olmayan kesirli diferansiyel denklemdir. İkinci denklem kesirli yerel olmayan dört nokta kesirli sınır koşulları ile desteklenmiş çoklu siparişlerin doğrusal olmayan dürtüsel sınır değer problemidir. Çözümün varlığı ve tekliği iki model için Banach'sabit nokta teoremi ve Schauder'sabit nokta teoremi ile elde edilir. Buna ek olarak, her iki sonuç icin de açıklayıcı örnekler verilmektedir.

Anahtar kelimeler: Kesirli integraller ve türevler, Kesirli diferansiyel denklemler, Varlık, Teklik, Sabit nokta teoremleri, Dürtü.

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LIST OF SYMBOLS

Sets

$\mathbb{N}$	The set of natural numbers, $\mathbb{N} \coloneqq \{1,2,3,\ldots\}$
$\mathbb{N}_0$	The set of counting numbers,$\mathbb{N}_0 \coloneqq \{0,1,2,3,\ldots\}$
$\mathbb{R}$	The set Real numbers
$\mathbb{R}^+$	The set of positive real numbers, $\mathbb{R}^+ \coloneqq \{x \in \mathbb{R} : x > 0\}$
$\mathbb{C}$	The set of complex numbers, $\mathbb{C} \coloneqq \{x + iy \mid x, y \in \mathbb{R}, i \coloneqq \sqrt{-1}\}$
$AC^n[a,b]$	Set of functions with absolutely continuous derivative of order of $n-1$
$C[a,b]$	Set of continuous functions
$C^k[a,b]$	Set of continuous functions with kth derivative
$L_p[a,b]$	Lebesegue space
$PC^1([a,b],\mathbb{R})$	The space of all piecewise continuous function from $[a,b]$ into $\mathbb{R}$ which have left continuous derivative on $[a,b]$

Functions

$E_\alpha(z)$	Mittage-Leffler function in one parameter, α
$E_{\alpha,\beta}(Z)$	Mittage-Leffler function in two parameters,α,β
$\Gamma(z)$	Euler's continuous gamma function
$B(p,q)$	Beta function in two parameters, p,q
$\Vert f\Vert_\infty$	$sup_{a\le x\le b}\lvert f(x)\rvert$, Chebyshev norm $\Vert.\Vert_\infty$
$[\alpha]$	Greatest integer function
$\binom{\alpha}{k}$	The generalized binomial coefficient, $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}$
$T_j[f,a]$	Taylor polynomial of degree j for the function f centered at the point a

Operators

D^n	Classical differential operator, $n \in \mathbb{N}$
I^n	Cauchy n-fold integral operator, $n \in \mathbb{N}$
D_a^α	Riemann-Liouville fractional differentioal operator, $\alpha \in \mathbb{R}$
${}^cD_a^\alpha$	Caputo fractional differentional operator, $\alpha \in \mathbb{R}$
${}^{GL}D_a^\alpha$	Gr$\ddot{u}$nwald-Letnikov fractional differential operator, $\alpha \in \mathbb{R}^+$
I_a^α	Riemann-Liouville fractional integral operator, α
I	Identity operator

LIST OF ABBREVIATIONS

RHS	Right hand side
LHS	Left hand side
FDEs	Fractional differential equations
BVP	Boundary value problem
BCs	Boundary conditions

Chapter 1

INTRODUCTION

In this Chapter we want to provide a concise history of fractional calculus. The theory of fractional calculus emanated from the origin of classical calculus itself. Historically, classical calculus was developed by Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century and the latter (he) first brought out the conception of a symbolic method, more precisely his notation,

$$\frac{d^n y}{dx^n} = D^{\,n} y$$

for the n^{th}derivative of function $y(x)$, where n is a non-negative integer.

In [1], L'Hospital had written a letter to Leibniz in 1695 and asked about the likelihood of n beıng a fraction " What does $(\frac{d^n f(x)}{dX^{\,n}})$ mean if n=$\frac{1}{2}$? ". Leibniz ascertains that "It will lead a paradox". But predictably "from this apparent paradox, some day it would lead to useful consequences" [1]. In view of the increasing interest in the development of fractional calculus by means of many mathematicians, it can be extended to the n^{th} derivative of $D^n y$ to any number, where n may be rational, irrational or complex number.

Many other mathematicians such as Euler, Laplace, and Fourier have investigated fractional calculus in order to answer L'hospital's question. Each of them had unique notations and methodology and also proposed many divergent concepts of non-integer order integral or derivative. The first discussion of a derivative of fractional order in calculus was written by Lacroix in 1819 [2]. Lacroix expressed the precise formula for the n^{th} derivative which is defined by

$$D^n x^m = \frac{m!}{(m-n)!} x^{m-n}, \text{ where } n(\leq m) \text{ is integer,} \tag{1.1}$$

and he replaced the discrete factorial function with Euler's continuous Gamma function and obtained the following formula

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{1.2}$$

where α and β are fractional numbers.

In particular, he computed

$$D^{\frac{1}{2}} x = \frac{\Gamma(2)}{\Gamma(3/2)} x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}} \ . \tag{1.3}$$

The first application of fractional calculus was made by Niels Henrik Abel in [3] at the beginning of the nineteenth century. He used mathematical tools to solve an integral equation which arise from the tautochrone problem. This problem sımply deals with the determination of curve on the (x, y) plane through the origin in vertical plane such that the required time for a particle with a total mass (m) will be released at a time which is absolutely independent of the origin.

In this situation the physical law states that “the potential energy lost during the descent of the particle is equal to the kinetic energy the particle gains”:

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = mg(y_{0-y}), \tag{1.4}$$

where (m) is defined as the mass of the particle, s is the distance of the particle from origin along the curve and g implies acceleration due to gravity. The formula above can be solved by separating the variables which yields

$$\frac{-ds}{\sqrt{\mathrm{y}_{0-\mathrm{y}}}} = \sqrt{2g}dt$$

and integration from when time $t = 0$ to $t = T$

$$\sqrt{2g}T = \int_0^{y_0} (y_0 - y)^{-\frac{1}{2}} ds. \tag{1.5}$$

Assuming that the time a particle needs to reach the lowest point of the curve is constant. So the left hand side must be a constant, say k. If we denoted the path length s as a function of height $s = F(y)$, then, $\frac{ds}{dy} \equiv F'(y)$.

By changing the variables y_0 with x and y with t and putting $F' = f$ the tautochrone integral equation becomes

$$k = \int_0^x (x - t)^{-\frac{1}{2}} f(t)dt, \tag{1.6}$$

where f is the function to be determined.

After multiplying both sides of the integral equation with $\frac{1}{\Gamma(\frac{1}{2})}$, Abel got on the right hand side a fractional integral of order $\frac{1}{2}$

$$\frac{k}{\Gamma(\frac{1}{2})} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\mathrm{x}} (\mathrm{x} - \mathrm{t})^{-\frac{1}{2}} f(t)dt = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} f(x). \tag{1.7}$$

Or , equivalently,

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \frac{k}{\Gamma(\frac{1}{2})} = \frac{d^{1/2}}{dx^{1/2}} \frac{d^{-1/2}}{dx^{-1/2}} f(x) = \frac{d^0}{dx^0} f(x) = f(x). \tag{1.8}$$

So , we have the tautochrone solution given as follows

$$f(x) = \frac{1}{\Gamma(\frac{1}{2})} \frac{\mathrm{d}^{1/2}}{\mathrm{dx}^{1/2}} K = \frac{\mathrm{K}}{\pi\sqrt{\mathrm{x}}} , \tag{1.9}$$

where the Abel problem has a solution which ıs subjected to the condition that derivative constant k is not zero always.

Here, It is necessary to note that Abel not only give a solution to the tautochrone problem, but also gave the solution for more general integral equation

$$f(x) = \int_0^x \frac{f(t)}{(x-t)^\alpha} dt, x > a, 0 < \alpha < 1. \tag{1.10}$$

After Abel application of fractional operators to a problem in physics, the first series of papers were stated by Liouville (see e.g. [1-3]). Liouville extended the known integer order derivatives $D^n e^{ax} = a^n e^{ax}$ to a derivative of arbitrary order α (formally replacing n∈N with α∈ℂ) as follows:

$$D^\alpha e^{ax} = a^\alpha e^{ax}. \tag{1.11}$$

Liouville developed two definitions for fractional derivatives. The first definition of a derivative of arbitrary order α for certain class of functions involved an infinite series. Here the series must be convergent for some α. Based on the Gamma function, Loiuville formulated the second definition as follows:

$$\Gamma(\beta) x^{-\beta} = \int_0^\infty t^{\beta-1} e^{-xt} dt , \quad \beta > 0. \tag{1.12}$$

$$D^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta} , \ \beta > 0. \tag{1.13}$$

This definition is useful only for rational functions.

Another scholar who had contributed to the fractional calculus is Riemann [1]. Riemann developed the definition for fractional integral of order α of a given function $f(x)$. The most important definition which is known as Riemann-Liouville fractional integral and formulated as follows:

$$ {}_{\mathrm{c}}\mathrm{D}_{\alpha}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{c}}^{\mathrm{x}} (\mathrm{x}-\mathrm{t})^{\alpha-1} f(t)dt \ , Re(\alpha) > 0. \tag{1.14} $$

When c=0, expression (1.14) is the definition of Riemann integral, and when c=−∞, expression (1.14) represents the Liouville definition. In this regard, it can be shown that

$$ {}_{c}D_{x}^{\alpha} f(x) = {}_{c}D_{x}^{n-\beta} f(x) = {}_{c}D_{x}^{n} \, {}_{c}D_{x}^{-\beta} f(x) $$
$$ = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\beta)} \int_{c}^{x} (x-t)^{\beta-1} f(t)dt\right), \tag{1.15} $$

holds, which is known today as the Riemann-Liouville fractional derivative, where n=$[Re(\alpha)] + 1$ and 0< $\beta = n - \alpha < 1$.

On the other hand, Grünwald and Letnikov [4] generated the concept of fractional derivative which is the limit of a sum given by

$$ {}^{GL}D_{d+}^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(x-kh) \, , \alpha > 0, \tag{1.16} $$

where $\binom{\alpha}{k}$ is the generalized binomial coefficient . At this point in time, it is enough for mentioning the historical development of fractional calculus.

In the twentieth century, the generalization of fractional calculus has been subjected of several approaches. That is why there are various definitions that are proved equivalent, and their use is encouraged by researchers in different scientific fields. Although a great number of results of fractional calculus were presented in this century but the most interesting one was introduced by M.Caputo in [5] and was used extensively. Caputo defined a fractional derivative by

$$ {}^{c}D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1}\left(\frac{d}{ds}\right)^n f(s)ds, \qquad (1.17)$$

where f is a function with an (n−1) absolutely continuous derivative and n=[α]+1 . Nowadays, expression (1.17) named Caputo fractional derivative. This derivative (1.17) is strongly connected with Riemann-Liouville fractional derivative and is frequently used in fractional differential equations with initial conditions $x^{(k)}(0)=b_k, k=0,\ 1,\dots,n-1$.

Fractional calculus has grown and come to light in the late twentieth century. In 1974, the commencing conference related with the application and theory of fractional calculus was successfully showcased in the New Haven [6]. A number of books on fractional calculus have appeared in the same year. Finally in 2004 the huge conference on fractional differentiation and its application was held in Bordeaux.

From its birth (simple question from L'Hospital to Leibniz) to its today's wide use in numerous scientific areas fractional calculus has come a long way. Although it's as old as integer calculus, it has still proved good applicability on models describing complex real life problems.

After a review of the historical development of the fractional calculus this work will give a brief investigation to its main goal and form a cornerstone in the application that arise in engineering and other sciences. It is fractional differential equation which has played an important role in mathematical modeling of different specialization such as physics, bio-chemistry, economics, and engineering etc. We will be interested in the boundary conditions of fractional differential equation which involves Caputo derivative.

Recently, problems with boundary value for non-linear FDEs draw many researchers attention. For instance Ahmad, B. et al [7], investigated non-linear FDEs with fractional separated boundary conditions. Also in [8] , Ahmad, B. and Sivasundaram, S. studied the existence of solutions for impulsive integral boundary condition of non-linear fractional differential condition. By following this technique, I do consider two types of non-linear FDEs which are not the same with boundary value problems. The first one is concerned with FDEs with four points non-local fractional boundary condition; the second is associated with non-linear impulsive fractional differential equation with four points non-local boundary condition. In each of these we will obtain the existence solutions by means of fixed point theorems. Both results will be illustrated by examples.

The remaining structure of this work is arranged as follows: In Chapter 2, we presented briefly the essential facts and theorems from mathematical analysis and functional analysis such as functional spaces, special functions, normed space and fixed point theorems which are prerequisite for the upcoming chapters. In Chapter 3, we provide a solid foundation in fractional calculus which includes definitions. In Chapter 4, existence results are developed and implemented for fractional boundary value problems of non-linear FDEs and some useful existence and uniqueness theorems for boundary value problems for impulsive FDEs are given. Furthermore, the examples are given to explain the results.

Chapter 2

PRELIMINARIES

This Chapter is all about presentation of some principles, theorems and understandings that support what is to come in the upcoming chapters. It introduces a fruitful feedback from classical analysis which aim at refreshing and building a bridge between the fields of applied and pure mathematics and to explain the ideas concerned with generalization of fractional environment. Since some of the stated theorems are well known and one can refer to the books [9-10], Erdēlyi et al.[11], therefore, the proofs are omitted.

2.1 Basic Ideas from Functional Analysis

For the fractional calculus and its related FDEs, we need some classical methodology and conceptual framework from functional analysis and classical calculus. Namely, we require the normed space, metric space, and classical functions spaces to formulate some results in fractional calculus.

Definiton 2.1.1 A linear Vector space V on the field R or C consist of a set V with two different binary operations, which are the vector addition(+) defined on V×V to V and the scalar multiplication $(\cdot)$ which is defined on $\mathbb{R}\times V$ to V such that the preceding properties hold,

1. $\forall u, v \in V, u + v = v + u$ (Commutivity)
2. $\forall u, v, w \in V, (u + v) + w = u + (v + w)$ (Associativity)

3. $. \forall u \in V, \exists! 0 \in V \ s.t. 0 + u = u + 0 = u$

4. $\forall u \in V, \exists! (-u) \in V \ s.t. (-u) + u = u + (-u) = 0$

5. $\forall u \in V, 1. u = u$

6. $\forall a, b \in \mathbb{R} \ and \ \forall u \ \in V, (ab)u = a(bu)$

7. $\forall a \in \mathbb{R} \ and \ \forall u, v \ \in V, \ a(u + v) = au + bv$

8. $\forall a, b \in \mathbb{R} \ and \ \forall u \ \in V, (a + b)u = au + bu$

Definition 2.1.2 Let X be a vector space over ℝ. A function ||. || : X → ℝ is called a norm on X if it is satisfying the three properties below for every $u, v \in X$ and $\forall a \in \mathbb{R}$

1. $\|u\| \geq 0, and \ \|v\| = 0 \ \ if \ and \ only \ if \ \ u = 0$

2. $\|au\| = |a|. \|u\|$

3. $\|u + v\| \leq \|u\| + \|v\| \ (Triangle \ Inequality).$

A normed linear space (X,||·||) is linear vector space X equipped with a norm ||·||.

In what follows, a normed linear space (X, ||·||) will be written for abbreviation by X.

Definition 2.1.3 Let X≠ Φ be a set. A function $d: X \times \ X \ \rightarrow \mathbb{R}$ is defined as a metric (or rather a distance function) if the below axioms are satisfied for $\forall x, y, z \in X$.

(i) $d(x, y) \ \geq 0$

(ii) $d(x, y) = 0 \ if \ and \ only \ if \ x = 0$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq \ d(x, y) + d(y, z).$ (Triangle Inequality).

The set *X* together with the function *d* is called a metric space and denoted by (*X*,*d*).

Remark 2.1.4: If ||·|| is a norm on a vector space V , then the function $V \times V \to \mathbb{R}^+$ given by $d(x_1, x_2) := \|x_1 - x_2\|$ is called a metric on V. that is a normed vector space is automatically a metric space, by characterizing the metric in terms of the norm in the usual way. Moreover, a metric space may have no algebraic (vector) structure that is to say, it may not be a vector space; so the idea of a metric space is a generalized form of the concept of a normed vector space.

Definition 2.1.5 a. Let (X,||·||) be a normed space. If every Cauchy sequence in X is also convergent in X, then we say X is a complete normed space or a Banach space.

Definition 2.1.5 b. A metric space (X, d) can be called a complete metric space or a Banach space provided every Cauchy sequence converge.

Definition 2.1.6 Assuming, $k \in \mathbb{N}$ and $p \geq 1$. We mention the following definition.

L_p[a, b]:={ f:[a, b] → ℝ, f is measurable on [a,b] and $\int_b^a |f(x)|^p dx < \infty$},

$L_\infty[a.b] \coloneqq \{ f: [a, b] \to \mathbb{R}; f\ is\ measurable\ and\ essentially\ bounded\ on\ [a, b]\}$,

$C^k[a, b] \coloneqq \{f: [a, b] \to \mathbb{R}; f$ has a continuous kth derivative },

For $1 \leq p \leq \infty$, L_p[a,b] is the usual Lebesgue space.

Another function space is formulated here.

Definition 2.1.7 A function *f(x)* is called absolutely continuous on a compact interval [a,b] , if for any $\varepsilon > 0$, there exist a $\delta > 0$ so that for every finite set of pairwise non intersecting subintervals $[a_k, b_k] \subset [a, b]$, k=1,2,…,n such that $\sum_{k=1}^{n}(b_k - a_k) < \delta$ implies $\sum_{k=1}^{n} | f(b_k) - f(a_k)| < \varepsilon$. The space of these functions is denoted by *AC*.

Similar way for the characterization of this space is by the following definition.

Definition 2.1.8 The set of functions which have an absolutely continuous (n-1)st derivative are denoted by AC^n or $AC^n[a,b]$, i.e. the functions f at which there is (almost everywhere) a function $g \in L_1$[a , b] such that

$$f^{(n-1)}(x) = f^{(n-1)}(a) + \int_a^x g(t)dt. \tag{2.1}$$

In this case g is said to be the (generalized) nth derivative of f and we can write g=$f^{(n)}$

Theorem 2.1.9 (Taylor expansion) For $m \in N$, assume that $f \epsilon AC^m[a,b]$.Then, for every x, y ∈ [a,b], we have

$$f(x) = \sum_{k=0}^{m-1} \frac{(x-y)^k}{k!} D^k f(y) + J_y^m D^m f(x). \tag{2.2}$$

Definition 2.1.10 Let *f*(*x*) ϵC^{n} [a, b] and $x_0 \epsilon [a \, , b]$.The polynomial

$$T_n[f; x_0](x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} D^k f(x_0), \tag{2.3}$$

is called Taylor polynomial of degree *n* for *f* with centered at x_0.

In the sequel we shall have to deal with convolution integral operators

$$h * \varphi = (h*\varphi)\ (x) = \int_{-\infty}^{\infty} h(x-t)\varphi(t)dt, \tag{2.4}$$

where h and φ belong to a certain function space. Therefore, It is obvious that

$$h * \varphi \ = \varphi * h.$$

The boundedness theorem in L_p in the following theorem which is called the **Young's Theorem.**

Theorem 2.1.11 $If\ h(t)\ \in\ L_1(\ \mathbb{R}), \varphi(t)\ \epsilon\ L_p(\mathbb{R}),$ then

$(\ h * \varphi\ )(x)\ \epsilon\ L_p(\mathbb{R}), 1 \leq\ p\ \leq \infty$, the inequality$\|h * \varphi\|_p \leq \|h\|_1 \|\varphi\|_p$ holds. (2.5)

Also we shall need to interchange the order of integration with the following theorem.

Theorem 2.1.12 (Fubini's theorem) Let $[a, b]$ and $[c, d]$ be two intervals, and assume f is integrable function on $[a, b]\ \times\ [c, d]$. If g(y) $= \int_a^b f(x, y)dx$ exist for each fixed y $\in$ [c,d], then g is integrable on [c,d] and $\int_{[a,b]\times[c,d]} f(x, y)dxdy = \int_c^d \left(\int_a^b f(x, y)dx\right)dy$. Moreover, if $h(x) = \int_c^d f(x, y)dy$ exist for each fixed x∈[a,b], then $\int_a^b \left(\int_c^d f(x, y)\right)dx = \int_c^d \left(\int_a^b f(x, y)dx\right)dy = \int_{[a,b]\times[c,d]} f(x, y)dxdy.$ Furthermore, the following relation is special case of Fubini's Theorem namely

$$\int_a^b dx \int_a^x f(x, y)d\ =\ \int_a^b dy \int_y^b f(x, y)dx. \qquad (2.6)$$

It is supposed to be one of those integrals exist. This relation is called the **Dirichlet formula.**

2.2 Some Special Functions

The generalization of classical calculus to fractional calculus is connected with generalization of some functions, which are called special functions. Such functions are Gamma function which is a generalized form of the factorial function and the Mittag-Leffler which is a generalization of the exponential function and its takes a cogent position in the theory of ordinary FDEs.

Definition 2.2.1 The Euler's Gamma function Γ(z) is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.7}$$

Theorem 2.2.2 Euler's Gamma function satisfies the below properties

1. For Re(z)>0 , the first part of the Definition 2.2.1 is equivalent to

$$\Gamma(z) = \int_0^1 \left(ln\left(\frac{1}{t}\right)\right)^{z-1} dt.$$

2. For $z \epsilon \mathbb{C} \backslash \{0,-1,-2,-3,\ldots\}$, $\Gamma(z+1)=z\Gamma(z)$.

3. For $n \in \mathbb{N}, \Gamma(n) = (n-1)!$.

4. Euler's Gamma function is analytic for all $z \epsilon \mathbb{C} \backslash \{0,-1,-2,-3,\ldots.\}$.

5. Euler's Gamma function is never zero.

6. $\Gamma(z)=\lim_{n\to\infty} \frac{n!\ n^z}{z(z+1)(z+2)\ldots.(z+n-1)(z+n)}$. (2.8)

7. (Reflection Theorem). For all non-integer $z \epsilon \mathbb{C}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Directly connected to Euler's Gamma function is the definition of generalized binomial coefficients.

Definition 2.2.3 The binomial coefficients are defined for $\alpha \in \mathbb{R}$ and for $k \in \mathbb{N}_0 \coloneqq \{0,1,2,3,\ldots\}$ by

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots..(\alpha-k+1)}{k!} \tag{2.9}$$

Another important special function which is related to Euler' s Gamma function is the Beta function as defined by the following.

Definition 2.2.4 The Beta function $B(p,q)$ in two variables p, q is defined by

$$B(p\,,q)=\int_{0}^{1} t^{p-1}(1-t)^{q-1}dt\ \ ,\ \ Re\,p>0\,,Re\,q>0. \tag{2.10}$$

Gamma and Beta functions are connected with themselves through the following expression

$$B(p\,,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(\mathrm{p+q})}\quad . \tag{2.11}$$

It then follows that

$$B(p\,,q)=B(q\,,p). \tag{2.12}$$

Next, we will define the Mittag-Leffler function which again is strongly connected with Gamma function and plays basic role in theory of fractional calculus. Furthermore, information can be found in a number of books on special function such as 13, 14 and 15].

Definition 2.2.5 For z $\epsilon\mathbb{C}$ the Mittag-Leffler function $\mathrm{E}_{\alpha}(\mathrm{z})$ is defined by

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}\ ,\qquad \alpha\ >0\ , \tag{2.13}$$

and the generalized (a two-parameter) Mittag-Leffler function $E_{\alpha,\beta}(z)$ has of the form

$$E_{\alpha,\beta}(\mathrm{z})=\sum_{\mathrm{n}=0}^{\infty}\frac{\mathrm{z}^{\mathrm{k}}}{\Gamma(\alpha\mathrm{k}+\beta)},\qquad \alpha\,,\beta>0. \tag{2.14}$$

Theorem 2.2.6 The Mittag-Leffler function meets the following properties

1. The *k*th derivatives of one parameter and the two-parameter of Mittag-Leffer function are given ,respectively, by

$$E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!z^j}{j!\Gamma(\alpha j+\alpha k+1)} , \tag{2.15}$$

$$E_{\alpha,\beta}^{(k)}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!z^j}{j!\Gamma(\alpha j+\alpha k+\beta)}. \tag{2.16}$$

2. For $|z|<1$, the general form of Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t}t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}z)\,dt = \frac{1}{1-z} , \qquad |z| < 1$$

3. The Laplace transform of the function $t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})$ is given by

$$\int_0^{\infty} e^{-st} z^{\beta-1}E_{\alpha,\beta}(\lambda z^{\alpha})dt = \frac{s^{\alpha-\beta}}{s^{\alpha}-z} \quad , Re(s) > |z|^{\frac{1}{\alpha}} \tag{2.17}$$

4. The Laplace transform of the Mittag-leffeler function $E_{\alpha}(\lambda z^{\alpha})$ is determined by

$$\frac{s^{\alpha-1}}{s^{\alpha}-\lambda} \tag{2.18}$$

5. For the particular values of α and β, the Mittag-Leffer function is given by

(a) $E_1 = e^z$ (b) $E_2(z^2) = \cosh(z)$

(c) $E_2(-z^2) = \cos(z)$ (d) $E_{2,2}(z^2) = \frac{\sinh(z)}{z}$.

2.3 Some Fixed Point Theorems

For some proofs of solutions to existence and uniqueness for the theory of FDEs, we need two fixed point theorems. They are Banach's fixed point theorem and Schauder's fixed point theorem. A proof of these theorems may be found ,e.g. in [16] and [17]. In order to state Banach's fixed point theorem we introduce the following concept.

Definition 2.3.1 Contraction Mapping Assume that $(X\,, d)$ is a metric space $F: X \to X$ is said to be a contraction mapping on X if $\exists\ 0 \leq \alpha < 1$ such that

$$d(F(x), F(y)) \leq \alpha\, d(x, y), \forall x, y \in X.$$

Theorem 2.3.2 (Banach's Fixed Point Theorem) Assume that (U, d) is a nonempty complete metric space and let the mapping $T: U \to U$ be a contraction, that is

$$d(Tu, Tv) \leq\ \alpha\, d(u, v)\ \ \forall\, u, v\ \in\ U, \text{ and } 0 \leq \alpha < 1,$$

then T possesses a unique fixed point u^*.That is $Tu^* = u^*$.

Also, we will use slightly different result that gives the existence without uniqueness of a fixed point in this thesis. But before mentioning this theorem, we will give the following concepts.

Theorem 2.3.3 Let *X*, *Y* be normed spaces. An operator $T: M \subset\ X \longrightarrow Y$ is called compact operator or completely continuous if

I. *T* is continuous.

II. *T* maps bounded sets $U \subset M$ into relatively compact sets.

Definition 2.3.4 Let (E, d) be a metric space and $F \subseteq E$ .The set F is called relatively compact in *E* if the closure of *F* is a compact subset of *E*.

Theorem 2.3.5 (Schauder's Fixed Point Theorem) Suppose Q is a nonempty, bounded, convex, closed, subset of a Banach space *X*, and Let *T:Q⟶Q* is a compact operator .Then *T* has at least one fixed point .

Another useful result from Analysis is very important for theory of FDEs in the following theorem.

Theorem 2.3.6(Arzelà-Ascoli's Theorem). Assume that F is a subset of $C[a,b]$endowed with the Chebyshev norm. Then F is relatively compact in $C[a, b]$ if and only if F is equi-continuous (i.e. for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $f \in F$ and for each $x_1, x_2 \in [a, b]$ whenever $|x_1 - x_2| < \delta$ implies$|f(x_1) - f(x_2)| < \varepsilon$) and uniformly bounded (i.e. $\exists$ a constant $K > 0$ so that $\|f\|_\infty < K \; \forall f \in F$).

Chapter 3

FRACTIONAL CALCULUS

In chapter 1 a brief historical stages of fractional calculus has been stated and the powerful connected with the development of classical calculus was established. As seen in the brief historical outline, more than one approach to transfer integer order operations to the non-integer case was developed. Anyway, the structure of this chapter is devoted to study some of these approaches for the fractional integration and differentiation and can be found in various books [21, 22, 23]. We start with the most common one, the Riemann-Liouville operators for fractional differentiation and integration.

3.1 Riemann-Liouville Integrals

Definition 3.1.1 Let $\alpha \epsilon \mathbb{R}^{+}$. The operator I_a^α , defined on $L_1[a,b]$ by

$$(I_a^\alpha f)(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z-s)^{\alpha-1} f(s) ds \tag{3.1}$$

for $a \leq z \leq b$ is said to be the Riemann-Liouville fractional integral operator of order α. For $\alpha = 0$, we put $I_a^0 := I$, the identity operator.

It is worth mentioning that some books define the left-sided and right-sided Riemann-Liouville fractional integral as follows

Definition 3.1.2 (see [22]). Let $\alpha \epsilon \mathbb{R}^{+}$ and $f(x) \epsilon L_1[a,b]$. The left-sided and right-sided Riemann-Liouville integrals of order α are defined respectively by

$$(I_{a+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \text{ ,} x > a \tag{3.2}$$

$$(I_{b-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \,, x < b, \tag{3.3}$$

respectively. It is clear that the Definition 3.1.1 coincide with the first part of the Definition 3.1.2**.** So we adapt the the Definition 3.1.1 and drop the sign +.

Lemma 3.1.3(see [22]). Let $f(x) \epsilon C[a,b]$, then

$$I_{a+}^{\alpha} I_{a+}^{\beta} f \equiv I_a^{\alpha+\beta}, \;\; I_{b-}^{\alpha} I_{b-}^{\beta} f \equiv I_{b-}^{\alpha+\beta} \tag{3.4}$$

where $\alpha > 0, \beta > 0$.

Proof. Suppose that $f\ (x) \epsilon$ C[a,b] , then

$$I_{a+}^{\alpha} I_{a+}^{\beta} f = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^z (z-s)^{\alpha-1} \int_a^s (s-\tau)^{\beta-1} f(\tau) d\tau ds$$

By Fubini's Theorem 2.12 it is possible to change the order of integration and we have

$$I_{a+}^{\alpha} I_{a+}^{\beta} f = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_\tau^x (x-t)^{\alpha-1} (t-\tau)^{\beta-1} f(\tau) dt\, d\tau$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) \int_\tau^x (x-t)^{\alpha-1} (t-\tau)^{\beta-1} dt d\tau \,.$$

The substitution $t = \tau + s(x-t)$ produces

$$I_{a+}^{\alpha} I_{a+}^{\beta} f = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) \int_0^1 [(x-\tau)(1-s)]^{\alpha-1} [s(x-\tau)]^{\beta-1} (x-\tau) ds d\tau$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) (x-\tau)^{\alpha+\beta-1} \int_0^1 s^{\beta-1} (1-s)^{\alpha-1} ds d\tau.$$

The term $\int_0^1 \mathrm{s}^{\beta-1} (1-\mathrm{s})^{\alpha-1} \mathrm{ds}$ is the definition of Beta function (Definition 2.2.11) and $B(\beta, \alpha) = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)}$. Therefore,

$$I_a^{\alpha} I_{a+}^{\beta} f(\mathrm{x}) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-t)^{\alpha+\beta-1} f(\tau)\, d\tau = I_a^{\alpha+\beta} f(x).$$

In similar way we can prove the right-sided of Riemann-Louivelle fractional integral.

Remark 3.1.4 The equations in (3.4) are called semigroup property of the fractional integration.

In the next subject we investigate the exchangeability of limit operation and fractional integration in the following theorem.

Theorem 3.1.5 Let $\alpha > 0$ .Suppose that $(f_k)_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $[a,b]$. Then we can interchange between the limit process and integral operators, i.e.

$$(I_a^\alpha \, lim_{k\to\infty} f_k)(x) = (lim_{k\to\infty} I_a^\alpha f_k)(x).$$

Proof: Let the limit of the sequence $(f_k)_{k=1}^{\infty}$ be represented by f .Since the uniform limit of all sequence of continuous functions is also continuous, so f is continuous. Then we find

$$|I_a^\alpha f_k(x) - I_a^\alpha f(x)| \le \frac{1}{\Gamma(\alpha)} \int_a^x |f_k(t) - f(t)| \, (\mathrm{x}-\mathrm{t})^{\alpha-1} \mathrm{dt}$$

$$\le \frac{1}{\Gamma(\alpha+1)} \|f_k - f\|_\infty (b-a)^\alpha.$$

The term $\|f_k - f\|_\infty$ converges uniformly to f as $k \to \infty \; \forall \; x \epsilon [a,b]$. ■

We will give two examples on the fractional integration.

Example 3.1.6 Consider the power function

$$f\,(z) = (z-w)^c \quad \text{for some } c > -1 \text{ and } \alpha > 0 \text{ .Then}$$

$$I_a^\alpha f(z) = \frac{\Gamma(c+1)}{\Gamma(\alpha+c+1)} (z-w)^{\alpha+c}. \tag{3.5}$$

If $\alpha \in \mathbb{N}$ we obtain a familiar result in classical calculus. For the fractional case, we have,

$I_a^{\alpha} f(\text{z}) = \frac{1}{\Gamma(\alpha)} \int_w^z (t-w)^c (z-t)^{\alpha-1} dt$ by substituting $t = w + s(z-w)$.

We obtain

$$I_a^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} (z-w)^{\alpha+c} \int_0^1 s^c (1-s)^{\alpha-1} ds$$

$$= \frac{\Gamma(c+1)}{\Gamma(\alpha+c+1)} (z-w)^{\alpha+c}.$$

Example 3.1.7 Assume $f\ (x) = exp(\lambda x)$ for some > 0 , then

$$I_a^{\alpha} f(x) = x^{\alpha} E_{1,\alpha+1}(\lambda x), \tag{3.6}$$

where $E_{1,\alpha+1}(\lambda\text{x})$ is the Mittag-Leffler function of two parameters.

In the case $\alpha \in \mathbb{N}$, we clearly have $I_0^{\alpha} f(x) = \lambda^{-\alpha} exp(\lambda x)$.

In the case $\alpha \notin \mathbb{N}$ then by utilizing from the expansion of exponential function of the power series, Theorem 3.1.5 and Example 3.1.6 we have

$$I_0^{\alpha}(fx) = I_0^{\alpha}\left[\sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!}\right]$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} I_0^{\alpha}(x)^k$$

$$= x^{\alpha} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{\Gamma(k+\alpha+1)} = x^{\alpha} E_{1,\alpha+1}\ (\lambda x)$$

Corollary 3.1.8 Assume that f is analytic function in $(d-h, d+h\,)$ for some h>0, and let $\alpha > 0$ .Then

$$I_d^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x-d)^{k+\alpha}}{\Gamma(k+1+\alpha)} D^k f(x), \text{ for } d \le x < d + \frac{h}{2} \text{ and}$$

$I_d^{\alpha} f(x) = \sum_{k=0}^{\infty} \frac{(x-d)^{k+\alpha}}{\Gamma(k+1+\alpha)} D^k f(d)$, for $d \le x \le d+h$. In particular, $I_d^{\alpha} f$ is analytic in$(d, d+h)$.

Proof. Because of the analyticity of f, it can be written by a power series round x. And since $x \in \left[d, d+\frac{h}{2}\right)$,the power series is convergent in the whole interval of integration. By Theorem 3.1.5, it is allowing to exchange summation and integration. Then by using the formula (3.5) in Example 3.1.6, we get the first result. The second result can be achieved in a similar way by representing f into the power series round a not x .The analyticity of $I_a^\alpha f$comes from the second statement.

3.2 Riemann-Liouville Derivatives

Associated with the fractional integration, it is natural to define the fractional derivative and investigate its properties. So we have the following definition.

Definition 3.2.1 Let $\alpha \epsilon \mathbb{R}^+$ and $= [\alpha] + 1$, where $[\alpha]$ the integer part of α. The operator $\mathrm{D}_\mathrm{a}^\alpha$, defined by

$$D_a^\alpha f(x) := D^n I_a^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t) dt \tag{3.7}$$

for $a \leq x \leq b$, is said to be the Riemann-Liouville operator of order α .

Remark 3.2.2 If $\alpha \in \mathbb{N}$, say $\alpha = m$ then $D_a^\alpha f = D^m f$. This means that the operator D_a^α coincide with the usual operator D^m.

Again, as the same of fractional integrals definitions, the left-sided and right-sided fractional derivatives may be defined as follow

Definition 3.2.3 The left-sided $D_{a+}^\alpha f$ and right-sided $D_{b-}^\alpha f$ Riemann-Liouville derivatives $D_{a+}^\alpha f$and $D_{b-}^\alpha f$ of order $\alpha \epsilon \mathbb{R}^+$ are defined by

$$(D_a^\alpha f)(x) := \left(\frac{d}{dx}\right)^n I_a^{n-\alpha} f(x) = \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \tag{3.8}$$

where $n = [\alpha] + 1,\ x > a$ and

$$(D_{b-}^\alpha f)(x) := \left(-\frac{d}{dx}\right)^n I_{b-}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (-\frac{d}{dx})^n \int_x^b (t-x)^{n-\alpha-1} f(t)dt, \tag{3.9}$$

where $n = [\alpha] + 1,\ x < b$.

We see that the Definition 3.2.1 match the first part of the Definition 3.2.2, so we drop the sign +.

Lemma 3.2.4 Let $\alpha \epsilon \mathbb{R}^+$ and let $n \in \mathbb{N}$ so that $n \geq \alpha$.Then

$$D_a^\alpha = D^n I_a^{n-\alpha}.$$

Proof: The assumption on n yields $n \geq m = [\alpha] + 1$. Thus,

$$D^n I_a^{n-\alpha} = D^m D^{n-m} I_a^{n-m} I_a^{m-\alpha} = D^m I_a^{m-\alpha} = D_a^\alpha.$$

According to the semigroup property of fractional integral (3.4) and the fact that the integer derivative is left inverse to the integer integration.

The following Lemma provides a simple condition which is sufficient for the existence of $D_a^\alpha f$.

Lemma 3.2.5 Let $f \in AC[a,b]$ and $0<\alpha<1$ .Then $D_a^\alpha f$ exists almost everywhere in [a,b] .Furthermore $D_a^\alpha f \in L_p[a,b]$ for $1 \leq p < \frac{1}{\alpha}$ and

$$D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(x-a)^\alpha} + \int_a^x f'(t)(x-t)^{-\alpha} dt \right).$$

Proof: since $f \in AC[a,b]$ by assumption we employ the Riemann-Liouville differential operator .This yields

$$D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t) dt$$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x[(f(a)+\int_a^t f'(u)du](x-t)^{-\alpha}dt$$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}[f(a)\int_a^x\frac{dt}{(x-t)^{\alpha}}+\int_a^x\int_a^t f'(u)(x-t)^{-\alpha}dudt)]$$

$$=\frac{1}{\Gamma(1-\alpha)}[\frac{f(a)}{(x-a)^{\alpha}}+\frac{d}{dx}\int_a^x\int_a^t f'(u)(x-t)^{-\alpha}dudt].$$

Then we apply Fubini's theorem to alternate the integration order .This yields

$$D_a^{\alpha}f(x)=\frac{1}{\Gamma(1-\alpha)}[\frac{f(a)}{(x-a)^{\alpha}}+\frac{d}{dx}\int_a^x f'(u)\frac{(x-u)^{1-\alpha}}{1-\alpha}du]$$

$$=\frac{1}{\Gamma(1-\alpha)}[\frac{f(a)}{(x-a)^{\alpha}}+\int_a^x f'(t)(x-t)^{-\alpha}dt].$$

This is obtained from the rules on the derivatives of parameter integrals thus we get the required result.

It remains to prove that $D_a^{\alpha}f\in L_p[a,b]$ for $1\le p<\frac{1}{\alpha}$. To do this we will use the following Minkowsky inequality

$$\|f+g\|_{L_p}\le\|f\|_{L_p}+\|g\|_{L_p},$$

where $\|\varphi\|_{L_p}(\Omega)=\left\{\int_{\Omega}|\varphi(x)|^p dx\right\}^{\frac{1}{p}}$ and $\Omega=[a,b]$, $-\infty\le a<b\le\infty$.

So, we get

$$\|D_a^{\alpha}f\|_{L_p}\le\frac{1}{\Gamma(1-\alpha)}(|f(a)|\|(x-a)^{-\alpha}\|_{L_p}+\left\|\int_a^x f'(t)(x-t)^{-\alpha}\right\|_{L_p})$$

The first term belongs to L_p and the second term we apply Young's Theorem (Theorem 2.11) since $f\in AC[a,b]$ implies $f'\in L_1[a,b]$ and $(x-t)^{-\alpha}\in L_p[a,b]$ because $1\le p<\frac{1}{\alpha}$. ■

Example 3.2.6 Let $f(x) = (x-a)^c$ with some $c > -1$ and α>0. Then according to Example 3.1.6, we have

$$D_a^\alpha f(x) = D^n I_a^{n-\alpha} f(x) = \frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} D^n (x-a)^{n-\alpha+c} ,$$

where $n = [\alpha] + 1$.

In the case$(-\alpha + c) \in \mathbb{N}$, the RHS is the nth derivative of a classical polynomial of degree $(n - \alpha + c) \in \{0,1,2, \dots, n-1\}$ and thus yields the following result

$$D_a^\alpha [(t-a)^{\alpha-n}](x) = 0 \quad \text{for } n \epsilon \{1,2 \dots, [\alpha].$$

In the case $(-\alpha + c) \notin \mathbb{N}$ we find

$$D_a^\alpha [(t-a)^c](x) = \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} .$$

From example above we see that the Riemann-Liouville derivative of a constant is not zero that differs from the integer calculus.

Having presented both of definition, Riemann-Liouville integral and differential operator, we can now investigate the interaction between each other. One of the most important results is concerned with the inverse property of both operators.

Theorem 3.2.7 Let $\alpha \geq 0$ and for each$f \in L_1[a,b]$.

Then we have

$$D_a^\alpha I_a^\alpha f = f$$

almost everywhere .Moreover, if there is a function $g \in L_1[a,b]$ such that $f = I_a^\alpha g$, then

$$I_a^\alpha D_a^\alpha f = f \qquad \text{almost everywhere.}$$

We see from the first statement of the theorem above that the Riemann-Liouville operator is actually left inverse to the Riemann-Liouville integral operator .while in the second statement reads that the Riemann-Liouville operator is the right inverse to Riemann-Liouville differential operator under the constraint $f = I_a^{\alpha} g$, which is similarly for the integer case.

If f does not verify this condition then we obtain a different characterization for $I_a^{\alpha} D_a^{\alpha} f$ which is given in the following theorem.

Theorem 3.2.8 (see [24]). Let $\alpha > 0$ and $n = [\alpha] + 1$. Suppose that f is such that $I_a^{n-\alpha} f \in AC^n[a, b]$. Then we have

$$I_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{z \to a+} D^{n-k-1} I_a^{n-\alpha} f(z) .$$

In particular, for $0 < \alpha < 1$ we have

$$I_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{z \to a+} I_a^{1-\alpha} f(z) .$$

Unfortunately, the Riemann-Liouville derivatives have determined drawbacks when atempting to model complex real life proplems relating with FDEs. Therefore, we study the most important modification for the idea of a fractional derivative.

3.3 Caputo Operator

Definition 3.3.1 Let $\alpha \in \mathbb{R}^{+}$and $n = [\alpha] + 1$. The operator ${}^{C}D_a^{\alpha}$ defined by

$${}^{c}D_a^{\alpha} f(x) \coloneqq I_a^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t) dt \qquad (3.10)$$

for a$\leq x \leq b$, when $D^n f(x) \in L_1[a, b]$ is called the Caputo differential operator of order α.

We begin the analysis of this operator with a simple example.

Example 3.3.2 Let $\alpha \geq 0$, $n = [\alpha] + 1$ and $f(x) = (x-a)^c$ with some $c \geq 0$.Then

$$ {}^cD_a^\alpha = \begin{cases} 0 & if\ c \in \{0,1,2,\dots,n-1\} \\ \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)}(x-a)^{c-\alpha} & if\ c \in\ and\ c \geq n \\ & or\ c \notin \mathbb{N}\ and\ c > n-1. \end{cases} $$

A first connection result between Riemann-Liouville derivative and Caputo derivative as follows

Theorem 3.3.3 (see [21]). Let $\alpha \geq 0$ and $n = [\alpha] + 1$.Furthermore, let's assume $f \in AC^n[a,b]$. It follows that

$$ {}^cD_d^\alpha f = D_d^\alpha[f - T_{n-1}[f;d]], $$

where $T_{n-1}[f;d]$ stands for the Taylor polynomial with $n-1$ degrees with the function f,with a center d .

Remark 3.3.4 We see for $\alpha \in \mathbb{N}$ that $\alpha = n$, then

$$ {}^cD_d^\alpha f = D_d^\alpha\big[f - T_{n-1}[f;d]\big] = D^n - D^n(T_{n-1}[f;d]) = D^n\ . $$

Since $T_{n-1}[f;d]$ is a polynomial with $n-1$ degrees that is vanished by with the operator D^n, so in this case the Caputo derivative gives a conventional nth derivative of the function $f(t)$.

We will also mention in this regard an important thing that is in the Caputo setting the initial conditions associated with FDEs coincide with those in integer case.

Another way to state the correlation between the Riemann-Liouvelle operator and the Caputo operator is formulated by the following lemma.

Lemma 3.3.5 Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Suppose that f is such that both ${}^{c}D_{a}^{\alpha} f$ and $D_{a}^{\alpha} f$ exist .Then

$${}^{c}D_{a}^{\alpha} f(x) = D_{a}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{D^{k} f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}$$

Proof: By using Theorem 3.3.3 and and Example 3.2.6 we have

$${}^{c}D_{a}^{\alpha} f(x) = D_{a}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{D^{k} f(a)}{k!} D_{a}^{\alpha} [(t-a)^{k}](x)$$

$$= D_{a}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{D^{k} f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} \quad . \blacksquare$$

Lemma 3.3.6 Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Suppose that f is such that both ${}^{c}D_{a}^{\alpha} f$ and $D_{a}^{\alpha} f$ exist. Furthermore, let $D^{k} f(a) = 0$, $k = 0,1,2, \dots, n-1$. Then,

$${}^{c}D_{a}^{\alpha} f = D_{a}^{\alpha} f \ .$$

This lemma plays an essential role of differential equations of fractional order .It states, when the initial conditions are homogeneous then the differential equations corresponding to Riemann-Liuovile derivative agree with those equations corresponding to Caputo derivative.

On the other hand, in comparison with Example 3.2.6 for $f(x) = 1$ and $\alpha > 0$, $\alpha \notin \mathbb{N}$ we deduces that it cannot be replaced ${}^{c}D_{a}^{\alpha}$ by D_{a}^{α} here. This difference is confirmed by the following lemma.

Lemma 3.3.7 Let $\alpha > 0$,$\alpha \notin \mathbb{N}$ and $n = [\alpha] + 1$. Furthermore if $f \in C^{n}[a,b]$. Then, ${}^{c}D_{a}^{\alpha} f \in C[a,b]$ and ${}^{c}D_{a}^{\alpha} f(a) = 0$.

Proof: We will use the Definition 3.3.1 and Theorem 3.3.3 .Since ${}^{c}D_{a}^{\alpha} f = I_{a}^{n-\alpha} D^{n} f$ and $D^{n} f$ is continuous by assumption, then according to classical theory of integrals involving parameter we deduce that $I_a^{n-\alpha} f \epsilon C[a.b]$, hence ${}^cD_a^\alpha f \in C[a,b]$. Moreover, since ${}^cD_a^\alpha f \coloneqq D_a^\alpha(f - T_{n-1}[f;a])$, we have

$${}^cD_a^\alpha f(a) = D_a^\alpha[f(a) - T_{n-1}[f;a](a)] = D_a^\alpha[f(a) - f(a)] = 0 \ . \blacksquare$$

Chapter 4

EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL BOUNDARY CONDITIONS

The area of fractional differential equations (FDEs) has been widely researched in the last two decades and many features of this field of calculus have been examined. One of the causes for famousness of the fractional calculus is the nonlocal property of the fractional order operators which takes into consideration the hereditary characteristics of various materials and processes. Actually, FDEs are found to be more appropriate model than analogous of integer differential equations. More precisely, it has furnished a fine tool for the nature of many phenomena in different real word problems. Also, boundary value problems (BVP) of fractional order containing the collection boundary conditions have been studied by a number of researchers. It has chiefly due to the natural circumstance of FDEs in many fields of engineering and sciences. For more details and examples one can see the books [22-28].The recent development of this topic can be obtained in a series of papers (for example [30-38]. In this chapter, we investigate two models of BVP involving the Caputo fractional derivative. The first one is the BVP of nonlinear fractional differential equation with nonlocal four-point fractional boundary conditions. The second it deals with nonlinear impulsive BVP of multi-orders fractional with nonlocal four-point fractional boundary conditions. Moreover, we obtain some existence results for both these issues by means of Banach’s fixed point theorem and

Schauder's fixed point theorem. In fact, the existence of the solutions is the major results of my thesis.

4.1 Existence of Solutions for Nonlinear Fractional Differential Equations Subject to Nonlocal Four-point Fractional Boundary Conditions

As we have seen above, the BVP of fractional order play a vital role in mathematical modeling of systems and processes in applied sciences such as physical processes, chemistry, biology, chemical, engineering, economics, and so on. Therefore, it has encouraged the researchers to investigate the existence of solution of these PVB by using some fixed point theorems.

Recently, new existence results for nonlinear fractional differential equations with three-point integral boundary conditions are obtained in [39], existence of solution for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal fractional differential equations are discussed in [40] and the existence of solutions for nonlinear factional differential equations with ant-periodic type fractional boundary conditions are investigated in [41].

Stimulated mentioned works above, we consider the following nonlinear FDEs subject to nonlocal four-point fractional boundary conditions (FBCs).

$$\begin{aligned} &{}^{c}D^{\alpha} x(t) = f(t, x(t)), 1 < \alpha \leq 2, \ \ t \,\epsilon\, J = [0, T], T > 0, \\ &x(0) + \mu_0\, x(T) = \sigma_0\, x(\eta_0), \qquad 0 < \eta_0 < T \qquad\qquad (4.1) \\ &{}^{c}D^{\alpha} x(0) + \mu_1\, {c_D}^{p}\, x(T) = \sigma_1 x(\eta_1), \ \ 0 < \eta_1 < T, 0 < P < 1, \end{aligned}$$

where ${}^{c}D^{\alpha}$ represents the Caputo fractional derivative of order α and $\mu_0,\ \mu_1, \sigma_0\ , \sigma_1$ are real constants and $f\colon [0,T]\text{x}\ \mathbb{R}\ \to \mathbb{R}$ is a continuous function. Here, $(\mathbb{R},\|\,.\,\|)$ is a Banach space and $C = C([0,T],\mathbb{R})$ denotes the banach space of all continuous functions from $[0,\mathrm{T}] \to \mathbb{R}$ with sup-norm $\|x\| = \sup_{t\in[0,T]}|\mathrm{x(t)}|$.

Before proof of the new results, we will draw down the auxiliary lemmas.

Lemma 4.1.1(see [29]). For $\alpha > 0$, the general solution of the fractional differential equation ${}^{c}D^{\alpha}_{0^+}x(t) = 0$ is given by

$$\mathrm{x(t)} = c_0 + c_1 t + c_2\, t^2 + \cdots + c_{n-1}t^{n-1}, \tag{4.2}$$

where $c_i\ \in \mathbb{R}, \mathrm{i} = 0,1,\ldots,\mathrm{n}-1,\ (n = [\alpha]+1)$

In view of Lemma 4.1.1, it follows that

$$I^{\alpha}_{0^+}\ {}^{c}D^{\alpha}_{0^+}\mathrm{x(t)} = \mathrm{x(t)} + c_0\mathrm{t} + c_1\mathrm{t} + c_2\, t^2 + \cdots + c_{n-1}t^{n-1}. \tag{4.3}$$

The following lemma will play an important role in the forthcoming analysis.

Lemma 4.1.2 For any $f(t)\ \epsilon\ C\ ([0,T],\mathbb{R})$, the unique solution of the boundary value problem

$$ {}^{c}D^{\alpha}x(t) = f(t) \qquad t\epsilon\ [0,\mathrm{T}],\ 0 < \alpha \le 2 \tag{4.4}$$

$$x(0) + \mu_0 x(T) = \sigma_0\, x(\eta_0), \qquad 0 < \eta_0 < T$$

$${}^{c}D^{\alpha}x(0) + \mu_1\ {}^{c}D^{p}_{0^+}x(T) = \sigma_1 x(\eta_1)\ , 0 < \eta_1 < T,$$

with $0 < P < 1$ and $\mu_0, \mu_1, \sigma_0, \sigma_1 \in \mathbb{R}$

is given by

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds + \omega_0(t)I^{\alpha}f(\eta_0) + \omega_1(t)I^{\alpha}f(\eta_1) + \omega_2(t)I^{\alpha}f(T)$$

$$+\omega_3(t)I^{\alpha-p}f(T), \tag{4.5}$$

where

$$\rho = (1+\mu_0-\sigma_1)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p}-\sigma_1\eta_1\right)+\sigma_1(\mu_0 T-\sigma_0\eta_0) \neq 0, \tag{4.6}$$

$$\omega_0(t) = \frac{\sigma_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p}-\sigma_1\eta_1\right)+\frac{\sigma_0\sigma_1}{\rho}t, \tag{4.7}$$

$$\omega_1(t) = -\frac{\sigma_0}{\rho}(\mu_0 T-\sigma_0\eta_0)+\frac{\sigma_1(1+\mu_0-\sigma_1)}{\rho}t, \tag{4.8}$$

$$\omega_2(t) = -\frac{\mu_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p}-\sigma_1\eta_1\right)-\frac{\mu_0\sigma_1}{\rho}t, \tag{4.9}$$

$$\omega_3(t) = -\frac{\mu_1}{\rho}(\mu_0 T-\sigma_0\eta_0)-\frac{\mu_1(1+\mu_0-\sigma_1)}{\rho}t. \tag{4.10}$$

Proof: Observe that the general solution of FDE (4.4) is given by

$$x(t) = I^{\alpha}f(t)-c_0-c_1 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds-c_0-c_1 t \tag{4.11}$$

Using the fact

$${}^{c}D^{p}c = 0 \ (c \ is \ constant), \qquad {}^{c}D^{p}t = \frac{T^{1-p}}{\Gamma(2-\rho)}, \qquad {}^{c}D^{p}I^{\alpha}f(t) = I^{\alpha-p}f(t),$$

$${}^{c}D^{p}x(t) = \frac{1}{\Gamma(\alpha-\rho)}\int_0^t (t-s)^{\alpha-p-1}f(s)ds-\frac{c_1}{\Gamma(2-\rho)}t^{1-p}.$$

Applying boundary conditions, we find that

$$-c_0+\mu_0\left[\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s)ds-c_0-c_1 T\right] =$$

$$\sigma_0\left[\frac{1}{\Gamma(\alpha)}\int_0^{\eta_0} (\eta_0-s)^{\alpha-1}f(s)ds-c_0-c_1\eta_0\right]$$

$$\mu_1\left[\frac{1}{\Gamma(\alpha-p)}\int_0^T (T-s)^{\alpha-p-1}f(s)ds-\frac{c_1}{\Gamma(2-p)}T^{1-p}\right] =$$

$$\sigma_1\left[\frac{1}{\Gamma(\alpha)}\int_0^{\eta_1} (\eta_1-s)^{\alpha-1}f(s)ds-c_0-c_1\eta_1\right].$$

By solving these two equations and arranging we get

$$-c_0(1+\mu_0-\sigma_0)-c_1(\mu_0 T-\sigma_0\eta_0) = \sigma_0 I^{\alpha}f(\eta_0)-\mu_0 I^{\alpha}f(T)$$

$$c_0\sigma_1 - c_1\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right) = \sigma_1 I^\alpha f(\eta_1) - \mu_1 I^{\alpha-p} f(T)$$

$$-c_0\sigma_1(1+\mu_0-\sigma_0) - c_1\sigma_1(\mu_0 T - \sigma_0\eta_0) = \sigma_1\sigma_0 I^\alpha f(\eta_0) - \mu_0\sigma_1 I^\alpha f(T),$$

$$c_0\sigma_1(1+\mu_0-\sigma_0) - c_1(1+\mu_0-\sigma_0)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right) =$$

$$\sigma_1(1+\mu_0-\sigma_0)I^\alpha f(\eta_1) - \mu_1(1+\mu_0-\sigma_0)I^{\alpha-p} f(T).$$

$$-c_1\left[\sigma_1(\mu_0 T - \sigma_0\eta_0) + (1+\mu_0-\sigma_0)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right)\right] = \sigma_1\sigma_0 I^\alpha f(\eta_0) -$$

$$\mu_0\sigma_1 I^\alpha f(T) + \sigma_1(1+\mu_0-\sigma_0)I^\alpha f(\eta_1) - \mu_1(1+\mu_0-\sigma_0)I^{\alpha-p} f(T),$$

Set $\rho = \sigma_1(\mu_0 T - \sigma_0\eta_0) + (1+\mu_0-\sigma_0)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right)$.

$$-c_1 = \frac{\sigma_1\sigma_0}{\rho}I^\alpha f(\eta_0) - \frac{\mu_0\sigma_1}{\rho}I^\alpha f(T) + \frac{\sigma_1(1+\mu_0-\sigma_0)}{\rho}I^\alpha f(\eta_1) - \frac{\mu_1(1+\mu_0-\sigma_0)}{\rho}I^{\alpha-p} f(T).$$

$$-c_0(1+\mu_0-\sigma_0)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right) - c_1(\mu_0 T - \sigma_0\eta_0)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p}\sigma_1\eta_1\right) =$$

$$\sigma_0\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right)I^\alpha f(\eta_0) - \mu_0\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right)I^\alpha f(T),$$

$$-c_0\sigma_1(\mu_0 T - \sigma_0\eta_0) - c_1(\mu_0 T - \sigma_0\eta_0)\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right) =$$

$$-\sigma_1(\mu_0 T - \sigma_0\eta_0)I^\alpha f(\eta_1) + \mu_1(\mu_0 T - \sigma_0\eta_0)I^{\alpha-p} f(T).$$

$$-c_0 = \frac{\sigma_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right)I^\alpha f(\eta_0) - \frac{\mu_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right)I^\alpha f(T) -$$

$$\frac{-\sigma_1}{\rho}(\mu_0 T - \sigma_0\eta_0)I^\alpha f(\eta_1) + \frac{\mu_1}{\rho}(\mu_0 T - \sigma_0\eta_0)I^{\alpha-p} f(T).$$

Replacing $-c_0$ and $-c_1$ in (2.13), we get the solution (2.7) where

$$\omega_0(t) = \frac{\sigma_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-\rho)}T^{1-p} - \sigma_1\eta_1\right) + \frac{\sigma_0\sigma_1}{\rho}t,$$

$$\omega_1(t) = -\frac{\sigma_1}{\rho}(\mu_0 T - \sigma_0\eta_0) + \frac{\sigma_1(1+\mu_0-\sigma_0)}{\rho}t,$$

$$\omega_2(t) = -\frac{\mu_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-p)}T^{1-p} - \sigma_1\eta_1\right) - \frac{\mu_0\sigma_1}{\rho}t,$$

$$\omega_3(t) = +\frac{\mu_1}{\rho}(\mu_0 T - \sigma_0\eta_0) - \frac{\mu_1(1+\mu_0-\sigma_0)}{\rho}t.$$

Due to Lemma(4.1.2), Let an operator $F: C \to C$ be defined by

$$(Fx)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds + \omega_0(t) I^\alpha f(\eta_0) + \omega_1(t) I^\alpha f(\eta_1) +$$

$$\omega_2(t) I^\alpha f(T) + \omega_3(t) I^{\alpha-p} f(T). \tag{4.12}$$

Now for proving the main theorems, we put the following for the computational convenience:

$$\left|\frac{\sigma_0}{p}\left(\frac{\mu_1}{\Gamma(2-\rho)} T^{1-p} - \sigma_1 \eta_1\right)\right| + \left|\frac{\sigma_0 \sigma_1}{\rho}\right| T = Z_0, \tag{4.13}$$

$$\left|-\frac{\sigma_0}{\rho}(\mu_0 T - \sigma_0 \eta_0)\right| + \left|\frac{\sigma_1(1+\mu_0-\sigma_0)}{\rho}\right| T = Z_1, \tag{4.14}$$

$$\left|-\frac{\mu_0}{\rho}\left(\frac{\mu_1}{\Gamma(2-p)} T^{1-p} - \sigma_1 \eta_1\right)\right| + \left|\frac{\mu_0 \sigma_1}{\rho}\right| T = Z_2, \tag{4.15}$$

$$\left|\frac{\mu_1}{\rho}(\mu_0 T - \sigma_0 \eta_0)\right| + \left|\frac{\mu_1(1+\mu_0-\sigma_0)}{\rho}\right| T = Z_3. \tag{4.16}$$

Let us set

$$\Omega = \frac{1}{\Gamma(\alpha+1)}[T^\alpha + Z_0 \eta_0^\alpha + Z_1 \eta_1^\alpha + Z_2 T^\alpha] + \frac{Z_3 T^{\alpha-p}}{\Gamma(\alpha-p+1)}. \tag{4.17}$$

Theorem 4.1.3 Assume $f: [0, T] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function and satisfies Lipschitiz condition (that is)

$|f(t, x) - f(t, y)| \leq L|x - y|$, $\forall\, t \in [0, T]$, $L > 0$, $x, y \in \mathbb{R}$, where L is Lipschitiz constant, with $L\Omega < 1$, where Ω is given by (4.17). Then the boundary value problem (4.1) has a unique solution.

Proof: setting $sup_{t\in[0,T]}|f(t, 0)| = M$ and choosing $r \geq \frac{M\Omega}{1-L\Omega}$, we show that $FB_r \subset B_r$ where $B_r = \{x \in C: \|x\| \leq r\}$. For $x \in B_r$ we have

$$
\begin{aligned}
|F(x)(t)| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}\big|f\big(v,x(v)\big)\big|dv \\
&+\frac{Z_0}{\Gamma(\alpha)}\int_0^{\eta_0}(\eta_0-v)^{\alpha-1}\big|f\big(v,x(v)\big)\big|dv \\
&+\frac{Z_1}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-v)^{\alpha-1}\big|f\big(v,x(v)\big)\big|dv \\
&+\frac{Z_2}{\Gamma(\alpha)}\int_0^{T}(T-v)^{\alpha-1}\big|f\big(v,x(v)\big)\big|dv \\
&+\ \frac{Z_3}{\Gamma(\alpha-p)}\ \int_0^{T}(T-v)^{\alpha-p}\big|f\big(v,x(v)\big)\big|dv \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}\Big(\big|f\big(v,x(v)\big)-f(v,0)\big|+|f(v,0)|\Big)dv \\
&+\frac{Z_0}{\Gamma(\alpha)}\int_0^{\eta_0}(\eta_0-s)^{\alpha-1}\Big(\big|f\big(s,x(s)\big)-f(s,0)\big|+|f(s,0)|\Big)ds \\
&+\frac{Z_1}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-v)^{\alpha-1}\Big(\big|f\big(v,x(v)\big)-f(v,0)\big|+|f(v,0)|\Big)dv \\
&+\frac{Z_2}{\Gamma(\alpha)}\int_0^{T}(T-v)^{\alpha-1}\Big(\big|f\big(v,x(v)\big)-f(v,0)\big|+|f(v,0)|\Big)dv \\
&+\frac{Z_3}{\Gamma(\alpha-\rho)}\int_0^{T}(T-v)^{\alpha-p}\Big(\big|f\big(v,x(v)\big)-f(v,0)\big|+|f(v,0)|\Big)dv \\
&\le (Lr+M)\left[\frac{1}{\Gamma(\alpha+1)}\left([T^\alpha+Z_0\eta_0^\alpha+Z_1\eta_1^\alpha+Z_2\eta_2^\alpha]\right)\right. \\
&\left.+\frac{Z_3T^{\alpha-p}}{\Gamma(\alpha-\rho+1)}\right] \le (Lr+M)\Omega \le r\,.
\end{aligned}
$$

Now for any $x, y \in C$ and for each $t \in [t, 0]$, we obtain

$$
\begin{aligned}
&|(Fx)(t) - Fy(t)| \\
&\qquad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} \left| f\big(v, x(v)\big) - f\big(v, y(v)\big) \right| dv \\
&\qquad + \frac{Z_0}{\Gamma(\alpha)} \int_0^{\eta_0} (\eta_0 - v)^{\alpha-1} \left| f\big(v, x(v)\big) - f\big(v, y(v)\big) \right| dv \\
&\qquad + \frac{Z_1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1 - v)^{\alpha-1} \left| f\big(v, x(v)\big) - f\big(v, y(v)\big) \right| dv \\
&\qquad + \frac{Z_2}{\Gamma(\alpha)} \int_0^{T} (T - v)^{\alpha-1} \left| f\big(v, x(v)\big) - f\big(v, y(v)\big) \right| dv \\
&\qquad + \frac{Z_3}{\Gamma(\alpha-\rho)} \int_0^{T} (T - v)^{\alpha-p} \left| f\big(v, x(v)\big) - f\big(v, y(v)\big) \right| dv \\
&\le L|x-y| \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} dv + \frac{Z_0}{\Gamma(\alpha)} \int_0^{\eta_0} (\eta_0 - v)^{\alpha-1} dv \right. \\
&+ \frac{Z_1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1 - v)^{\alpha-1} dv + \frac{Z_2}{\Gamma(\alpha)} \int_0^{T} (T - v)^{\alpha-1} dv \\
&\left. + \frac{Z_3}{\Gamma(\alpha-p)} \int_0^{T} (T - v)^{\alpha-p} dv \right] \\
&\le L|x-y| \left[\frac{1}{\Gamma(\alpha+1)} \left([T^\alpha + Z_0 \eta_0^\alpha + Z_1 \eta_1^\alpha + Z_2 \eta_2^\alpha] \right) \right. \\
&\left. + \frac{Z_3 T^{\alpha-p}}{\Gamma(\alpha-\rho+1)} \right] = L\Omega|x-y|,
\end{aligned}
$$

where Ω is given by (3.5). We note that Ω dependent only on the parameters in the problem (4.1). Then by assumption of theorem $L\Omega < 1$, therefore F is a contraction. Thus by Banach fixed point theorem we conclude that F possesses a unique fixed point which is a unique solution of boundary value problem (4.1) on [0,T].■

The second existence result is based on Schauder's fixed point theorem.

Theorem (4.1.4): Assume that $f: [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exist $v \in C([0, T], \mathbb{R}^+)$ such that $|f(t, x)| \le v(t)$ for all $(t, x) \in [0, T] \times \mathbb{R}$ with $\|v\| = \max_{t\in[0,T]}|v(t)|$. Then the BVP (4.1) possesses at least one solution on $[0, T]$.

Proof: let us fix

$$\bar{r} \ge \frac{\|v\|}{\Gamma(\alpha+1)}([T^\alpha + Z_0\eta_0^\alpha + Z_1\eta_1^\alpha + Z_2T^\alpha]) + \frac{\|v\|Z_3T^{\alpha-p}}{\Gamma(\alpha-\rho+1)}, \tag{4.18}$$

or $\bar{r} \ge \|v\|\Omega$ for a positive constant $\bar{r}$ and Ω is given by the relation (4.17).Now consider $B_{\bar{\bar{r}}} = \{x \in C([0, T], \mathbb{R}): \|x\| \le \bar{r}\}$ it is easy to know that $B_{\bar{r}}$ is a nonempty, closed, bounded and convex subset of $C([0, T]), \mathbb{R})$. Now we define an operator on $B_{\bar{\bar{r}}}$ as:

$$(\Phi x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds + \omega_0(t)I^\alpha f(\eta_0) + \omega_1(t)I^\alpha f(\eta_1) + \omega_2(t)I^\alpha f(T) + \omega_3(t)I^{\alpha-p} f(T) \tag{4.19}$$

We show that $\Phi: B_{\bar{r}} \to B_{\bar{r}}$. Let $x \in B_{\bar{r}}$, then we have

$$\begin{aligned}\|(\Phi x)(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, x(s)\right)\right|ds + |\omega_0(t)|\|I^\alpha f(\eta_0)\| \\ &\quad + |\omega_1(t)|\|I^\alpha f(\eta_1)\| + |\omega_2(t)|\|I^\alpha f(T)\| + |\omega_3(t)|\|I^{\alpha-p} f(T)\| \\ &\le \|v\|\left\{\frac{1}{\Gamma(\alpha+1)}([T^\alpha + Z_0\eta_0^\alpha + Z_1\eta_1^\alpha + Z_2T^\alpha]) + \frac{Z_3T^{\alpha-p}}{\Gamma(\alpha-\rho+1)}\right\} \\ &\le \bar{r}, \end{aligned} \tag{4.20}$$

which means that $\Phi B_{\bar{r}} \subset B_{\bar{r}}$.

Continuity of f means that the operator Φ is continuous on$B_{\bar{r}}$ and Φ is uniformly bounded on $B_{\bar{\bar{r}}}$ since

$$\|\Phi \mathrm{x}\| \le \|v\|\Omega. \tag{4.21}$$

By assumption of theorem, we define $sup_{(t,x)\in[0,T]\times B_{\bar{r}}} \|f(t,x)\| = f_{max}$. Now showing that Φ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. For arbitrary $s_1, \mathrm{s}_2 \in [0,\mathrm{T}]$ with $s_1 < \mathrm{s}_2$ and $x \in B_{\bar{\bar{r}}}$, where $B_{\bar{\bar{r}}}$ is bounded set of $C \in ([0,T],\mathbb{R})$. Then we have

$$\|(\Phi x)(s_2) - (\Phi x)(s_1)\|$$

$$= \left\| \int_{s_1}^{s_2} \frac{(s_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big) ds \right.$$

$$+ \int_0^{s_1} \frac{(s_2 - s)^{\alpha-1} - (s_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big) ds$$

$$+ \frac{\sigma_0\sigma_1}{\rho}(s_2 - s_1) I^\alpha f(\eta_0) + \frac{\sigma_1(1 + \mu_0 - \sigma_0)}{\rho}(s_2 - s_1) I^\alpha f(\eta_1)$$

$$\left. - \frac{\mu_0\sigma_1}{\rho}(s_2 - s_1) I^\alpha f(T) - \frac{\mu_1(1 + \mu_0 - \sigma_1)}{\rho}(s_2 - s_1) I^{\alpha-p} f(T) \right\|$$

$$\le f_{max} \left\{ \frac{1}{\Gamma(\alpha+1)} \left[2(|s_2 - s_1|)^\alpha + |s_2^\alpha - s_1^\alpha| + \left|\frac{\sigma_0\sigma_1}{\rho}\right| (s_2 - s_1)\eta_0^\alpha \right.\right.$$

$$\left. + \left|\frac{\sigma_1(1 + \mu_0 - \sigma_0)}{\rho}\right| (s_2 - s_1)\eta_1^\alpha + \left|\frac{\mu_0\sigma_1}{\rho}\right| (s_2 - s_1) T^\alpha \right]$$

$$\left. + \left|\frac{\mu_1(1 + \mu_0 - \sigma_0)}{\rho}\right| \frac{(s_2 - s_1) T^{\alpha-p}}{\Gamma(\alpha - \rho + 1)} \right\}. \tag{4.22}$$

As $s_2 \to s_1$, the RHS of the inequality above tends to zero independently of $x \in B_{\bar{r}}$. Thus Φx is equicontinuous on interval $[0,T]$. Hence, by Arezola-Ascoli's theorem, the set $\{\Phi x; x \in B_{\bar{\bar{r}}}\}$ is a relatively compact subset of $C([0,T],\mathbb{R})$. Thus $\Phi: B_{\bar{r}} \to B_{\bar{r}}$ is compact operator. So by Schauder's fixed point theorem, we can say Φ has a fixed point on $B_{\bar{r}}$ which is a solution of BVP (4.1) on [0,T].

Example 4.1.5 Consider the following nonlinear four-point fractional boundary value problem:

$$^{c}D^{\frac{3}{2}}\,x(t) = \frac{1}{12(t+4)^3}\tanh(x)\,, t \in [0,4]$$

$$x(0) + \tfrac{1}{2}x(4) = 3x(1)$$

$$^{c}D^{\frac{1}{2}}x(0) + \tfrac{1}{3}\ {}^{c}D^{\frac{1}{2}}\,x(4) = x(\tfrac{3}{2})\,. \tag{4.23}$$

Here, $\alpha = \frac{3}{2}\,, \mu_0 = \frac{1}{2}\,, T = 4\,, \sigma_0 = 3\,, \eta_0 = 1\,, p = \frac{1}{2}\ ,\ \mu_1 = \frac{1}{3}\,,\ \sigma_1 = 1\,,\ \eta_1 = \frac{3}{2}$

$f(t,x) = \frac{1}{12(t+4)^3}\tanh x(t)$, with the given data, it is found that

$|f(t,x) - f(t,y)|$

$$\leq \left|\frac{1}{12(t+4)^3}\right| |\tanh(x) - \tanh(y)|\ \leq \frac{1}{768}|\tanh(x) - \tanh(y)|$$

$$\leq \frac{1}{768}|x-y|.$$

Here in the last step we use mean value theorem for $\tanh x(t)$. (Since $\tanh' x(t) = \left(\frac{2}{e^x+e^{-x}}\right)^2 < 1$).

So $L = \frac{1}{768}$. To find Ω, we calculate $\rho, Z_0, Z_1, Z_2\ and\ Z_3$.

$$\rho = \left(1 + \frac{1}{2} - 3\right)\left(\frac{4}{3\sqrt{\pi}} - (1)\left(\frac{3}{2}\right)\right) + (1)\left(\frac{1}{2}(4) - 3(1)\right) = 1.6216,$$

$$z_0 = \left|\frac{3}{1.6216}\left(\frac{4}{3\sqrt{\pi}} - (1)\left(\frac{3}{2}\right)\right)\right| + \frac{(3)(1)(4)}{1.6216} = 6.0167,$$

$$z_1 = \left|\frac{-1}{2(1.6216)}\left(\frac{1}{2}(4) - 3(1)\right)\right| + \left|\frac{(1)\left(1 + \frac{1}{2} - 3\right)}{1.6216}\right|.4 = 3.8511,$$

$$z_2 = \left|\frac{-1}{2(1.6.216)}\left(\frac{4}{3\sqrt{\pi}} - (1)\left(\frac{3}{2}\right)\right)\right| + \frac{\frac{1}{2}(1)(4)}{1.6216} = 1.0027,$$

$$z_3 = \left|\frac{1}{3(1.6216)}\left(\frac{1}{2}(4) - (3)(1)\right)\right| + \left|\frac{\left(\frac{1}{3}\right)\left(1 + \frac{1}{2} - 3\right)}{1.6216}\right|.4 = 1.4389,$$

$$\Omega = \frac{4}{3\sqrt{\pi}}[8 + (6.0167)(1) + (3.8511)(1.5)^{1.5} + (1.0027)(8)] + (1.4389)(4)$$

$$= 26.2301 .$$

Thus $L\Omega = \frac{1}{768} 26.2301 = 0.03415 < 1.$

Therefore all the assumptions of Theorem 4.1.3 are fulfilled. Hence, by the finalized form of Theorem 4.1.3, the problem (4.23) has a unique solution on [0,4].

Example 4.1.6 we still consider the same boundary value problem in Example 4.1.5 but $f(x) = \frac{e^{-x(t)^4}}{5\sqrt{1+t}} \ln(3 + sinx(t))$, t∈ $[0,4]$ .i.e.

$$^cD^{\frac{3}{2}}\, x(t) = \frac{e^{-x(t)^4}}{5\sqrt{1+t}} \ln(3 + sinx(t)),\, t \in [0,4]$$

$$x(0) + \frac{1}{2}x(4) = 3x(1),$$

$$^cD^{\frac{1}{2}}x(0) = \tfrac{1}{3}\, {}^cD^{\frac{1}{2}}\, x(4) = x(\tfrac{3}{2}) \,. \tag{4.24}$$

Obviously, $|f(t, x(t))| \leq \frac{\ln(4)}{5\sqrt{1+t}} = v(t)$ with $\|v(t)\| = \frac{\ln(4)}{5}$.

Therefore, the condition of Theorem 4.1.4 holds. Hence by applying the conclusion of Theorem 4.1.4, we get the BVP (4.24) has at least one solution on [0,4].

Here, we get the positive constant $\bar{\bar{r}}$ which is assigned in a proof of Theorem 4.1.4 from the formula $\Omega\|v\| \leq \bar{\bar{r}}$. So, $\bar{\bar{r}} \geq \frac{\ln(4)}{5}\times$ (26.2301) =7.2725.

4.2 Four-point Impulsive Multi-Orders Fractional Boundary Value Problems

Impulsive differential equations have extensively been studied in the past two decades. Impulsive differential equations are used to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in harvesting, earthquakes, diseases, and so forth. Recently, fractional impulsive differential equations have attracted the attention of many researchers. For the general theory and application of such equations we refer the interested reader to see the monographs of Bainov and Simeonov [48], Lakshmikantham et al.[49] and Benchohra et al.[50] and the references therein.

In [52], Kosmatov considered the following two impulsive problems:

$$^{C}D^{\alpha}u(t) = f\big(t,u(t)\big), 1 < \alpha < 2, t \in [0,1]\backslash\{t_1, t_2, \dots, t_p\},$$

$$^{C}D^{\gamma}u(t_k^{+}) - D^{\gamma}u(t_k^{-}) = I_k\big(u(t_k^{-})\big), t_k \in (0,1), k = 1, \dots, p,$$

$$u(0) = u_0, u'(0) = u_0, 0 < \gamma < 1,$$

and

$$^{L}D^{\alpha}u(t) = f\big(t,u(t)\big), 0 < \alpha < 1, t \in [0,1]\backslash\{t_1, t_2, \dots, t_p\},$$

$$^{L}D^{\gamma}u(t_k^{+}) - {}^{L}D^{\gamma}u(t_k^{-}) = I_k\big(u(t_k^{-})\big), t_k \in (0,1), k = 1, \dots, p,$$

$$I^{1-\alpha}u(0) = u_0, \qquad 0 < \gamma < \alpha < 1.$$

In [53], Feckan et al. studied the impulsive problem of the following form:

$$^{C}D^{\alpha}u(t) = f\big(t,u(t)\big), 0 < \alpha < 1, t \in [0,1]\backslash\{t_1, t_2, \dots, t_p\},$$

$$u(t_k^{+}) - u(t_k^{-}) = I_k\big(u(t_k^{-})\big), t_k \in (0,1), k = 1, \dots, p,$$

$$u(0) = u_0, \qquad 0 < \gamma < \alpha < 1.$$

Wang et al. [54] obtained some existence and uniqueness results for the following impulsive multipoint fractional integral boundary value problem involving multi-orders fractional derivatives and deviating

$$^{c}D_{t_k}^{\alpha_k}u(t) = f\left(t, u(t), u\big(\theta(t)\big)\right), 1 < \alpha_k < 2, t \in [0,T]\backslash\{t_1, t_2, \dots, t_p\},$$

$$\Delta u(t_k) = I_k\big(u(t_k^-)\big), \;\; \Delta u'(t_k) = J_k\big(u(t_k^-)\big), \;\; t_k \in (0,T), k = 1, \dots, p$$

$$u(0) = \textstyle\sum_{k=0}^{p} \lambda_k I_{t_k}^{\beta_k} u(\eta_k), \quad t_k < \eta_k < t_{k+1},$$

$$u'(0) = 0.$$

Yukunthorn et.al. [55] Studied the similar problem for multi-order Caputo-Hadamard fractional differential equations with nonlinear integral boundary conditions.

Motivated by the above works, in this section, we study the existence of solutions for nonlocal four-point boundary value problems of nonlinear impulsive equations of fractional order

$$^{c}D_{t_k}^{\alpha_k}u(t) = f\big(t, u(t), u'(t)\big), 1 + \beta \le \alpha \le 2, t \in [0,T]\backslash\{t_1, t_2, \dots, t_p\},$$

$$\Delta u(t_k) = I_k\big(u(t_k^-)\big), \Delta u'(t_k) = J_k\big(u'(t_k^-)\big), t_k \in (0,T), k = 1, \dots, p,$$

$$\alpha_1 u(0) + \mu_1 \, {}^{c}D_{0+}^{\beta} u(0) = \sigma_1 u(\eta_1), 0 < \eta_1 < t_1 < T,$$

$$\alpha_2 u(T) + \mu_2 \, {}^{c}D_{t_p}^{\beta} u(T) = \sigma_2 u(\eta_2), 0 < t_p < \eta_2 < T, 0 < \beta < 1, \tag{4.25}$$

where ${}^{c}D_{t}^{\alpha_k}, k = 1, \dots, p$ is the Caputo derivative, $f : |0,T| \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function $I_k, J_k; \mathbb{R} \to \mathbb{R}$, $\Delta u\ (t_k) =\ u\ (t_k^+) - u\ (t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u\ (t_k^+)$ and $u\ (t_k^-)$ represent the right hand limit and left hand limit of function $u\ (t)$ at $t = t_k$; and sequence $\{t_k\}$ satisfy that $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$.

The main difficulty of this problem is that the corresponding integral equation is very complex because of the impulse effects. By applying Banach's fixed point theorem and Schauder's fixed point theorem, some existence results are obtained.

The material in this section is basic in some sense. So, in order to prove the results we present in the following some useful preliminaries and notations.

Let $[0,T]^- = [0,T]\backslash\{t_1, t_2, \dots, t_p\}$ and

$PC([0,T],\mathbb{R}) = \{x: [0,T] \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+), x(t_k^-)$ exist and $x(t_k^-) = x(t_k),\ k = 1, \dots, p\}$ and

$PC^1([0,T],\mathbb{R}) = \{x \in PC([0,T],\mathbb{R});\ x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+), x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k), k = 1, \dots, p\}$,

where $PC([0,T],\mathbb{R})$ and $PC^1([0,T],\mathbb{R})$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{|x(t)|; t \in [0,T]\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Definition 4.2.1 Let $X = PC^1([0,T],\mathbb{R}) \cap C^2([0,T]^-,\mathbb{R})$. A function $x \in X$ whose Caputo derivative of order $\alpha_k, k = 1, \dots, p$ exists on $[0,T]^-$ is called a solution of problem (4.25) if it satisfies (4.25).

Throughout this section we will use the following notations.

$$\rho = \sigma_1\eta_1(1-\sigma_2) + \left(T + \mu_2 \frac{T^{1-\beta}}{\Gamma(2-\beta)} - \sigma_2\eta_2\right)(1-\sigma_1),$$

$$A_0 = \frac{\sigma_1}{1-\sigma_1} - \frac{\sigma_1}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \qquad B_0 = \frac{\sigma_1}{\rho},$$

$$A_p = \frac{(1-\sigma_1)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \qquad B_p = \frac{1-\sigma_1}{\rho}.$$

$$F_k(y,u,u')(t) = \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t} (t-s)^{\alpha_k - 1} y(s)ds$$

$$+\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_j-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-1}y(s)ds+\sum_{j=1}^{k}I_j\left(u(t_j^-)\right)$$

$$+\sum_{j=1}^{k}(t-t_j)\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t-s)^{\alpha_{j-1}-2}y(s)ds$$

$$+\sum_{j=1}^{k}(t-t_j)J_j\left(u(t_j^-)\right),$$

$$G_k(y,u,u')(t)=\frac{1}{\Gamma(\alpha_k-\beta)}\int_{t_k}^{t}(t-s)^{\alpha_k-\beta-1}y(s)ds$$

$$+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}y(s)ds$$

$$+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}J_j\left(u'(t_j^-)\right).$$

$$F'_k(y,u,u')(t)=\frac{1}{\Gamma(\alpha_k-1)}\int_{t_k}^{t}(t-s)^{\alpha_k-2}y(s)ds$$

$$+\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}y(s)ds+\sum_{j=1}^{k}J_j\left(u'(t_j^-)\right).$$

The following lemma will take a major role to define the solutions of the problem (4.25).

Lemma 4.2.2 Let $y\in C\,[0,1]$. A function $u\in PC^1[0,T]$ is a solution of the boundary value problem

$$D_{t_k}^{\alpha_k}u(t)=y(t),1<\alpha_k\le 2,t\in[0,T]\backslash\{t_1,t_2,\dots,t_p\},$$

$$\Delta u(t_k)=I_k\left(u(t_k^-)\right),\Delta u'(t_k)=J_k\left(u'(t_k^-)\right),t_k\in(0,T),k=1,\dots,p,$$

$$u(0)+\mu_1D^{\beta}u(0)=\sigma_1u(\eta_1),0<\eta_1<t_1<T,$$

$$\alpha_2u(T)+\mu D_{t_p}^{\beta}u(T)=\sigma_2u(\eta_2),0<t_p<\eta_2<T,0<\beta<1, \tag{4.26}$$

If and only if

$$u(t) = F_k(y,u,u')(t) - \frac{\sigma_1}{1-\sigma_1} F_0(y,u)(\eta_1)$$

$$-\frac{\sigma_1}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right) F_0(y,u,u')(\eta_1)$$

$$+\frac{\sigma_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right) F_p(y,u,u')(\eta_2)$$

$$-\frac{(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right) F_p(y,u,u')(\mathrm{T})$$

$$-\frac{\mu_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right) G_p(y,u,u')(\mathrm{T}). \tag{4.27}$$

Proof. Suppose that u is a solution of (4.26). For $0 \le \mathrm{t} \le t_1$, we have

$$u(t) = I_{0^+}^{\alpha_0} y(t) - c_1 - c_2 t$$

$$= \frac{1}{\Gamma(\alpha_0)}\int_0^t (t-s)^{\alpha_0-1} y(s)ds - c_1 - c_2 t,\ \ c_1, c_2 \in \mathbb{R} \tag{4.28}$$

Then differentiating (4.28), we get

$$D_{0+}^{\beta} u(t) = \frac{1}{\Gamma(\alpha_0-\beta)}\int_0^t (t-s)^{\alpha_0-\beta-1} y(s)ds - c_2\frac{t^{1-\beta}}{\Gamma(2-\beta)},$$

$$u'(t) = \frac{1}{\Gamma(\alpha_0-1)}\int_0^t (t-s)^{\alpha_0-2} y(s)ds - c_2.$$

If $t_1 < t \le t_2$, then for some d_1, $d_2 \in R$ we have

$$u(t) = \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-s)^{\alpha_1-1} y(s)ds - d_1 - d_2(t-t_1),$$

$$u'(t) = \frac{1}{\Gamma(\alpha_1-1)}\int_{t_1}^t (t-s)^{\alpha_1-2} y(s)ds - d_2,$$

$$D_{t_1^+}^{\beta} u(t) = \frac{1}{\Gamma(\alpha_1-\beta)}\int_{t_1}^t (t-s)^{\alpha_1-\beta-1} y(s)ds - d_2\frac{t^{1-\beta}}{\Gamma(2-\beta)}.$$

Thus,

$$u(t_1^-) = \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} (t_1-s)^{\alpha_0-1} y(s)ds - c_1 - c_2 t_1,$$

$$u(t_1^+) = -d_1$$

$$u'(t_1^-) = \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s) ds - c_2,$$

$$u'(t_1^+) = -d_{2.}$$

In view of

$$u(t_1^+) - u(t_1^-) = I_1\big(u(t_1^-)\big), \qquad u'(t_1^+) - u'(t_1^-) = J_1\big(u'(t_1^-)\big),$$

we find that

$$-d_1 = \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s) ds + I_1\big(u(t_1^-)\big) - c_1 - c_2 t_1,$$

$$-d_2 = \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s) ds + J_1\big(u'(t_1^-)\big) - c_2.$$

Hence we obtain for $t_1 < \text{t} \leq t_2$

$$u(t) = \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t - s)^{\alpha_1 - 1} y(s) ds$$

$$+\frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s) ds + I_1\big(u(t_1^-)\big)$$

$$+(t - t_1) \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s) ds + (t - t_1) J_1\big(u'(t_1^-)\big)$$

$$-c_1 - c_2 t_1, \qquad t_1 < t \leq t_2.$$

In a similar way, for $k = 1, 2, \ldots, p$ we can obtain

$$u(t) = \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t - s)^{\alpha_k - 1} y(s) ds$$

$$+ \sum_{j=1}^{k} \frac{1}{\Gamma\big(\alpha_{j-1}\big)} \int_{t_{j-1}}^{t_j} \big(t_j - s\big)^{\alpha_{j-1} - 1} y(s) ds + \sum_{j=1}^{k} I_j \left(u\big(t_j^-\big) \right)$$

$$+ \sum_{j=1}^{k} \big(t - t_j\big) \frac{1}{\Gamma\big(\alpha_{j-1} - 1\big)} \int_{t_{j-1}}^{t_j} \big(t_j - s\big)^{\alpha_{j-1} - 2} y(s) ds$$

$$+\sum_{j=1}^{k}(t-t_j)J_j\left(u'(t_j^-)\right)-c_1-c_2t, \quad t_k<t\le t_{k+1.} \tag{4.29}$$

Moreover,

$$ {}^cD_{t_k}^{\beta}u(t)=\frac{1}{\Gamma(\alpha_k-\beta)}\int_{t_k}^{t}(t-s)^{\alpha_k-\beta-1}y(s)ds$$

$$+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}y(s)ds$$

$$+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}J_j\left(u'(t_j^-)\right)-c_2\frac{t^{1-\beta}}{\Gamma(2-\beta)}.$$

Now applying the boundary conditions

$$u(0)+\mu_1D_{0^+}^{\beta}u(0)=\sigma_1u(\eta_1),0<\eta_1<\mathrm{t}_1<T,$$

$$u(T)+\mu_2D_{t_p}^{\beta}u(T)=\sigma_2u(\eta_2),0<\mathrm{t}_p<\eta_2<T,0<\beta<1,$$

we get

$$-c_1=\sigma_1F_0(y,u,u')(\eta_1)-\sigma_1c_1-c_2\sigma_1\eta_1,$$

$$F_p(y,u,u')(T)-c_1-c_2T+\mu_2G_p(y,u)(T)-\mu_2c_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}$$

$$=\sigma_2F_p(y,u,u')(\eta_2)-c_1\sigma_2-c_2\sigma_2\eta_2.$$

By solving this system for c_1 and c_2 we find that

$$-c_1(1-\sigma_1)+c_2\sigma_1\eta_1=\sigma_1F_0(y,u,u')(\eta_1),$$

$$-c_1-c_2\left(T+\mu_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}-\sigma_2\eta_2\right)=\sigma_2F_p(y,u,u')(\eta_2)-F_p(y,u)(T)-$$

$$\mu_2G_p(y,u,u')(T)-c_1(1-\sigma_1)+c_2\sigma_1\eta_1=\sigma_1F_0(y,u,u')(\eta_1),$$

$$-c_1(1-\sigma_1)-c_2\left(T+\mu_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}-\sigma_2\eta_2\right)(1-\sigma_1)$$

$$=\sigma_2(1-\sigma_1)F_p(y,u,u')(\eta_2)-(1-\sigma_1)F_p(y,u,u')(T)$$

$$-\mu_2(1-\sigma_1)G_p(y,u,u')(T)\,.$$

$$c_2\left(\sigma_1\eta_1+\left(T+\mu_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}-\sigma_2\eta_2\right)(1-\sigma_1)\right)=$$

$$\sigma_1F_0(y,u,u')(\eta_1)-\sigma_2(1-\sigma_1)F_0(y,u,u')(\eta_2)+(1-\sigma_1)F_p(y,u,u')(T)$$

$$+\mu_2(1-)G_p(y,u,u')(T).$$

$$-c_2=-\frac{\sigma_1}{\rho}F_0(y,u,u')(\eta_1)+\frac{\sigma_2(1-\sigma_1)}{\rho}F_p(y,u,u')(\eta_2)$$

$$-\frac{(1-\sigma_1)}{\rho}F_p(y,u,u')(T)\quad-\frac{\mu_2(1-\sigma_1)}{\rho}G_p(y,u,u')(T).$$

$$-c_1=\frac{\sigma_1}{1-\sigma_1}F_0(y,u,u')(\eta_1)-c_2\frac{\sigma_1\eta_1}{1-\sigma_1}$$

$$=\frac{\sigma_1}{1-\sigma_1}F_0(y,u,u')(\eta_1)-\frac{\sigma_1\eta_1}{1-\sigma_1}\frac{\sigma_1}{\rho}F_0(y,u,u')(\eta_1)$$

$$+\frac{\sigma_2(1-\sigma_1)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}F_p(y,u,u')(\eta_2)$$

$$-\frac{(1-\sigma_1)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}F_p(y,u,u')(T)-\frac{\mu_2(1-\sigma_1)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}G_p(y,u,u')(T).$$

After inserting these values of c_1 and c_2 into (4.29) we get

$$u(t)=F_k(y,u,u')(t)+\tfrac{\sigma_1(1-\sigma_2)}{1-\sigma_1}F_0(y,u,u')(\eta_1)-c_2\left(\tfrac{\sigma_1\eta_1}{1-\sigma_1}+t\right)$$

$$=F_k(y,u.u')(t)+\frac{\sigma_1(1-\sigma_2)}{1-\sigma_1}F_0(y,u,u')(\eta_1)$$

$$-\frac{\sigma_1(1-\sigma_2)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_0(y,u,u')(\eta_1)$$

$$+\frac{\sigma_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u,u')(\eta_2)$$

$$-\frac{(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u,u')(T)$$

$$-\frac{\mu_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)G_p(y,u.u')(T),$$

where $\rho=\sigma_1\eta_1(1-\sigma_2)+\left(T+\mu_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}-\sigma_2\eta_2\right)(1-\sigma_1)$.

Conversely, assume that u is a solution of the impulsive fractional integral equation (4.27). Then by a direct computation, it follows that the solution given by (4.27) satisfies (4.26).This complete the proof. ■

Now, before investigating the existence and uniqueness of the solutions of BVP (4.25) we put the following assumption. In this sequel we assume that

(A_1) $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous function such that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|$$
$$\leq l_f(|x_1-x_2|+|y_1-y_2|),\ l_f>0,$$
$$0\leq t\leq T\ ,x_1,x_2,y_1,y_2\in\mathbb{R}$$

(A_2) $I_k,J_k:\mathbb{R}\to\mathbb{R}$ are continuous and satisfy

$$|I_k(x)-I_k(y)|\leq l_1(x-y),$$
$$|J_k(x)-J_k(y)|\leq l_2(x-y),$$
$$l_1>0,l_2>0,0\leq t\leq T\ ,x,y\in\mathbb{R}\ ,$$

and for convenience, we will give some notations:

$$T^*=max\{T^{\alpha_k}:0\leq k\leq p\},\qquad \Gamma^*=min\{\Gamma(\alpha_k):0\leq k\leq p\}$$

$$\Delta_1=\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\ ,\ \Delta_2=\sum_{j=1}^{p}\frac{(T-t_j)(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})},$$

$$\Delta_3 = \frac{T^{1-\beta}}{\Gamma(2-\beta)} \sum_{J=1}^{P} \frac{(t_j - t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}, \quad \Delta_4 = \sum_{j=1}^{p} \frac{(t_j - t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})},$$

$$\Lambda_F := l_f \frac{T^*}{\Gamma^*} + l_f \Delta_1 + l_f \Delta_2 + p l_1 + l_2 p T,$$

$$\Lambda_G := l_f \frac{T^*}{\Gamma^*} + l_f \Delta_3 + l_2 p \frac{T^{1-\beta}}{\Gamma(2-\beta)},$$

$$\Lambda_{F'} := l_f \frac{T^*}{\Gamma^*} + l_f \Delta_4 + l_2 p .$$

Lemma 4.2.3 $F_k(f,u,u')$ and $G_k(f,u,u')$ are Lipschitzian operators

$$|F_k(f,u,u') - F_k(f,v,v')| \le \Lambda_F \|u-v\|_{PC^1}, \qquad \Lambda_F > 0,$$

$$|G_k(f,u,u') - G_k(f,v,v')| \le \Lambda_G \|u-v\|_{PC^1}, \ \Lambda_G > 0, \ \ u,v \in PC^1([0,T],\mathbb{R}).$$

Proof: For $u,v \in PC^1([0,T],\mathbb{R})$, we have

$$|F_k(f,u,u') - F_k(f,v,v')| \le$$

$$\frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \left| f\big(s,u(s),u'(s)\big) - f\big(s,v(s),v'(s)\big) \right| ds$$

$$+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} \big(t_j - s\big)^{\alpha_{j-1}-1} \left| f\big(s,u(s),u'(s)\big) - f(s,v(s),v'(s)) \right| ds$$

$$+ \sum_{j=1}^{k} \left| I_j(u\big(t_j^-\big) - I_j(v\big(t_j^-\big) \right|$$

$$+ \sum_{j=1}^{k} \frac{\big(t-t_j\big)}{\Gamma\big(\alpha_{j-1}-1\big)} \times \int_{t_{j-1}}^{t_j} \big(t_j - s\big)^{\alpha_{j-1}-2} \left| f\big(s,u(s),u'(s)\big) - f(s,v(s),v'(s)) \right| ds$$

$$+ \sum_{j=1}^{k} \big(t-t_j\big) \left| J_j\left(u'\big(t_j^-\big)\right) - J_j\left(v'\big(t_j^-\big)\right) \right|$$

$$\le l_f \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k} \left(|u(s) - v(s)| + |u'(s) - v'(s)| \right) ds$$

$$+l_f \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} \left(|u(s) - v(s)| + |u'(s) - v'(s)|\right) ds$$

$$+l_1 \sum_{j=1}^{k} \left|u(t_j^-) - v(t_j^-)\right| + l_f \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1} - 1)} (t - t_j)$$

$$\times \int_{t_{j-1}}^{t_j} \left(t_j - s\right)^{\alpha_{j-1}-2} \left(|u(s) - v(s)| + |u'(s) - v'(s)|\right) ds$$

$$+l_2 \sum_{j=1}^{k} (t - t_j) \left|u'\left(t_j^-\right) - v'\left(t_j^-\right)\right|$$

$$\leq \Lambda_F \|u - v\|_{PC^1}.$$

Similarly,

$$|G_k(f, u, u')(t) - G_k(f, v, v')(t)|$$

$$\leq \frac{1}{\Gamma(\alpha_k - \beta)} \int_{t_k}^{t} (t - s)^{\alpha_k - \beta - 1} \left|f\left(s, u(s), u'(s)\right) - f\left(s, v(s), v'(s)\right)\right| ds$$

$$+\frac{T^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1} - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} \left|f\left(s, u(s), u'(s)\right)\right.$$

$$\left. - f\left(s, v(s), v'(s)\right)\right| ds$$

$$+\frac{t^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{k} \left|J_j\left(u'\left(t_j^-\right)\right) - J_j\left(v'\left(t_j^-\right)\right)\right|$$

$$\leq \left(l_f \frac{(T - t_k)^{\alpha_k}}{\Gamma(\alpha_k - \beta + 1)}\right.$$

$$\left. + l_f \frac{T^{1-\beta}}{\Gamma(2-\beta)} \sum_{J=1}^{k} \frac{\left(t_j - t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)} + \frac{T^{1-\beta}}{\Gamma(2-\beta)} l_2 \right) \|u - v\|_{PC^1}$$

$$\leq \Lambda_G \|u - v\|_{PC^1}.$$

Also, we have

$$|F_k'(f,u,u')(t) - F_k'(f,v,v')(t)| \le \Lambda_{F'}\|u-v\|_{PC^1}. \blacksquare$$

In view of Lemma 4.2.2 we define an operator $\Theta: X \to X$ by

$$(\Theta u)(t) = F_k(f,u,u')(t) + (A_0 - B_0 t)F_0(f,u,u')(\eta_1)$$

$$+\sigma_2\left(A_p + B_p t\right)F_p(f,u,u')(\eta_2) - \left(A_p + B_p t\right)F_p(f,u,u')(T)$$

$$-\mu_2\left(A_P + B_p t\right)G_p(f,u,u'),$$

where

$$A_0 = \frac{\sigma_1}{1-\sigma_1} - \frac{\sigma_1}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \qquad B_0 = \frac{\sigma_1}{\rho},$$

$$A_p = \frac{(1-\sigma_1)}{\rho}, \qquad B_p = \frac{1-\sigma_1}{\rho}.$$

Let

$$\Lambda_\Theta := \max\{\Lambda_F, \Lambda_G, \Lambda_{F'}\}.$$

Theorem 4.2.4 Suppose that the assumption (A_1),(A_2) are satisfied. If

$$\Lambda := \Lambda_\Theta max\left\{\begin{matrix}(1+|A_0|+|B_0|T+(|\sigma_2|+|\mu_2|+1)\left(|A_p|+|B_p|T\right)) \\ ,(1+|B_0|+(|\sigma_2|+|\mu_2|+1)|B_p|) \\ ,\end{matrix}\right\} < 1,$$

then the BVP (4.25) has a unique solution on $[0,T]$.

Proof. Let $u,v \in PC^1([0,T],\mathbb{R})$.For $u,v \in (t_k,t_{k+1}]$, $k=1,\dots,p$, we have

$$|(\Theta u)(t) - (\Theta v)(t)| \le |F_k(f,u,u')(t) - F_k(f,v,v')(t)|$$

$$+|A_0 - B_0 t||F_0(f,u,u')(\eta_1) - F_0(f,v,v')(\eta_1)|$$

$$+|\sigma_2|\left|A_p + B_p t\right|\left|F_P(f,u,u')(\eta_2) - F_p(f,v,v')(\eta_1)\right|$$

$$+\left|A_P + B_p t\right|\left|F_p(f,u,u')(T) - F_p(f,v,v')(T)\right|$$

$$+|\mu_2|\left|A_P + B_p t\right|\left|G_p(f,u,u')(T) - G_p(f,v,v')(T)\right|$$

$$\leq \Lambda_\Theta(1+|A_0|+|B_0|T+(|\sigma_2|+|\mu_2|+1)(|A_p|+|B_p|T))\|u-v\|_{PC^1}.$$

Similarly, for $t \in (t_k, t_{k+1}]$ we have

$$|(\theta u)'(t)-(\theta v)'(t)| \leq |F_k'(f,u,u')(t)-F_k'(f,v,v')(t)|$$
$$+|B_0||F_0(f.u.u')(\eta_1)-F_0(f.v.v')(\eta_1)|$$
$$+|\sigma_2||B_p||F_p(f,u,u')(\eta_2)-F_p(f.v.v')(\eta_2)|$$
$$+|B_p||F_p(f,u,u')(T)-F_p(f,v.v')(T)|$$
$$+|\mu_2||B_p||G_p(f,u,u')(T)-G_p(f,v,v')(T)|$$
$$\leq \Lambda_\Theta(1+|B_0|+(|\sigma_2|+|\mu_2|+1)|B_p|)\|u-v\|_{PC^1}.$$

It follows that

$$\|\Theta u-\Theta v\|_{PC^1} \leq \Lambda\|u-v\|_{PC^1}.$$

Since $\Lambda < 1, \Theta$ is a contraction. According to the Banach fixed point theorem Θ has a unique fixed point that is the problem (4.25) has a unique solution. ■

Again, to study the existence of solutions of BVP (4.25), we precede it with the following conditions:

(A_3) $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous function and there exists $h \in C([0,T],\mathbb{R}^+)$ such that

$$|f(t,u,v)| \leq h(t)+b_1|u|^{\varrho_1}+b_2|v|^{\varrho_2}, (t,u,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}, 0<\varrho_1,\varrho_2<1.$$

(A_4) $I_k, J_k:\mathbb{R}\to\mathbb{R}$ are continuous functions and there are $L_2>0, L_3>0$ such that

$$|I_k(x)| \leq L_2, \quad |J_k(x)| \leq L_3, \quad x \in \mathbb{R}.$$

For convenience, we will give some notations:

$$C_1 := \left(1+|A_0|+|B_0|T+(|\sigma_2|+1)\left(|A_p|+|B_p|T\right)\right)$$

$$\times\left(\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)(\|h\|)+pL_2+PTL_3\right)$$

$$+|\mu_2|\left(|A_p|+|B_p|T\right)\left(\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)(\|h\|)+\frac{T^{1-\beta}}{\Gamma(2-\beta)}pL_3\right)+\left(\frac{T^*}{\Gamma^*}+\Delta_4\right)(\|h\|),$$

$$C_2 \coloneqq \left(1+|A_0|+|B_0|T+(|\sigma_2|+1)(|A_p|+|B_p|T)\right)\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)$$

$$+|\mu_2|\left(|A_p|+|B_p|T\right)\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)+\frac{T^*}{\Gamma^*}+\Delta_4.$$

Lemma4.2.5

If

$$R \geq \max\left\{3C_1,(3b_1C_2)^{\frac{1}{1-\varrho_1}},(3b_2C_2)^{\frac{1}{1-\varrho_2}}\right\},$$

then Θ maps $B(0,R)$ into itself, where $B(0,R) \coloneqq \{u \in PC^1([0,T],\mathbb{R}): \|u\|_{PC^1} \leq R\}$.

Proof. Assume that

$$R \geq \max\left\{3C_1,(3b_1C_2)^{\frac{1}{1-\varrho_1}},(3b_2C_2)^{\frac{1}{1-\varrho_2}}\right\}.$$

Then for$t \in (t_k,t_{k+1}]$, $k=0,\dots,p$, we have

$$|F_k(f,u,u')(t)|$$

$$\leq \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\left|f\left(s,u(s),u'(s)\right)\right|ds$$

$$+\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-1}|f(s,u(s),u'(s))|ds+\sum_{j=1}^{k}\left|I_j\left(u(t_j^-)\right)\right|$$

$$+\sum_{j=1}^{k}\frac{(t-t_j)}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}|f(s,u(s),u'(s)|ds$$

$$+\sum_{j=1}^{k}(t-t_j)\left|J_j\left(v'(t_j^-)\right)\right|,$$

$$|F_k(f,u,u')(t)|$$

$$\leq \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\,(h(s)+b_1|u(s)|^{\varrho_1}\ +b_2|u'(s)|^{\varrho_2})ds$$

$$+\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-1}\,(h(s)+b_1|u(s)|^{\varrho_1}+b_2|u'(s)|^{\varrho_2}ds$$

$$+\sum_{j=1}^{k}\left|I_j\left(u(t_j^-)\right)\right|$$

$$+\sum_{j=1}^{k}\frac{(t-t_j)}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}\,(h(s)+b_1|u(s)|^{\varrho_1}+b_2|u'(s)|^{\varrho_2})ds$$

$$+\sum_{j=1}^{k}(t-t_j)J_j\left(u(t_j^-)\right)$$

$$\leq \frac{T^{\alpha_k}}{\Gamma(\alpha_{j-1}-1)}(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})$$

$$+\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}(\,\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+pL_2$$

$$+\sum_{j=1}^{p}\frac{(t=t_j)(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2}+pTL_3,$$

$$\leq\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+pl_2+pTL_3,$$

$$|G_k(f,u,u')(t)|$$

$$\leq\frac{T^{\alpha_K-\beta}}{\Gamma(\alpha_K-\beta+1)}(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})$$

$$+\frac{T^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+\frac{T^{1-\beta}}{\Gamma(2-\beta)}pL_3$$

$$\leq\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+\frac{T^{1-\beta}}{\Gamma(2-\beta)}pl_3+pL_3$$

$$\leq\left(\frac{T^*}{\Gamma^*}+\Delta_4\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+pl_3.$$

It follows that

$|(\Theta u)(t)|$

$$\leq\left(1+|A_0|+|B_0|T+(|\sigma_2|+1)\left(\left|A_p\right|+\left|B_p\right|T\right)\right)$$

$$\times\left(\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)(\|h\|+b_1\|h\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+pL_2+pTL_3\right)$$

$$+|\mu_2|(\left|A_p\right|+\left|B_p\right|T)\left(\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2}+\frac{T^{1-\beta}}{\Gamma(2-\beta)}pL_3\right)$$

$$\leq C_1+C_2b_1\|u\|^{\varrho_1}+C_2b_2\|u'\|^{\varrho_2},$$

and

$|(\Theta u)'(t)|$

$$\leq\left(\frac{T^{\alpha_k-1}}{\Gamma(\alpha_k)}+\sum_{j=1}^{k}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})+L_3p$$

$$+\left(|B_0|+|\sigma_2|\left|B_p\right|+\left|B_p\right|\right)\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})$$

$$+pL_2+pTL_3+|\mu_2|\left|B_p\right|\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)(\|h\|+b_1\|u\|^{\varrho_1}+b_2\|u'\|^{\varrho_2})$$

$$+\frac{T^{1-\beta}}{\Gamma(2-\beta)}L_3$$

$$\leq C_1+C_2b_1\|u\|^{\varrho_1}+C_2b_2\|u'\|^{\varrho_2}.$$

Thus,

$$\|(\Theta u)\|_{PC^1} \leq C_1 + C_2 b_1 R^{\varrho_1} + C_2 b_2 R^{\varrho_2} \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R. \blacksquare$$

Theorem 4.2.6 Assume that the conditions (A_3) and (A_4) are satisfied. Then the problem (4.25) has at least one solution.

Proof. Firstly, we prove that $\Theta: PC^1([0,T],\mathbb{R}) \to PC^1([0,T],\mathbb{R})$ is completely continuous operator. It is clear, the continuity of functions f, I_k and J_k implies the continuity of the operator Θ.

Let $\Omega \subset PC^1([0,T],\mathbb{R})$ be bounded. Then there exist positive constants L_1,L_2 and L_3 such that

$$|F_k(f,u,u')| \leq L_1, \quad |I_k(u)| \leq L_2, \quad |J_k(u)| \leq L_3,$$

for all $u \in \Omega$. Thus, for any $u \in \Omega$, we have

$$|F_k(f,u,u')(t)| \leq L_1 \left(\frac{T^*}{\Gamma^*} + \Delta_1 + \Delta_2\right) + pL_2 + L_3 pT.$$

Similarly,

$$|G_k(f,u,u')(t)| \leq L_1 \left(\frac{T^*}{\Gamma^*} + \Delta_3\right) + \frac{T^{1-\beta}}{\Gamma(2-\beta)} PL_3.$$

It follows that

$$|(\Theta u)(t)| \leq \Lambda_\Theta^1, \text{ where } \Lambda_\Theta^1 \text{ is constant.}$$

In a like manner,

$$|F_k'(f,u,u')(t)| \leq L_1 \left(\frac{T^*}{\Gamma^*} + \Delta_4\right) + L_3 P.$$

It follows that

$|(\Theta u)'(t)|$

$$\leq L_1 \left(\frac{T^*}{\Gamma^*} + \Delta_4\right) + L_3 P + (|B_0| + |\sigma_2||B_p| + |B_p|)\Lambda_F$$

$$+|\mu_2||B_p|\Lambda_G \coloneqq \Lambda_\Theta^2 \text{ , where } \Lambda_\Theta^2 \text{ is a constant.}$$

Thus,

$$\|(\Theta u)\|_{PC'} \leq \Lambda_\Theta^1 + \Lambda_\Theta^2 = \psi, \text{ where } \psi \text{ is constant.}$$

On the other hand, for $\tau_1, \tau_2 \in (t_k, t_{k+1}]$ with $\tau_1 < \tau_2$ and we have

$$|(\Theta u)(\tau_2) - (\Theta u)(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |(\Theta u)'(s)| ds \leq \Lambda_\Theta^2 (\tau_2 - \tau_1).$$

Similarly,

$$|(\Theta u)'(\tau_2) - (\Theta u)'(\tau_1)| \leq \Pi_\Theta (\tau_2 - \tau_1),$$

where Π_Θ is constant. This implies that Θu is equicontinuous on all $t \in (t_k, t_{k+1}], k = 0,1 \dots, p$. Consequently, Arzela-Ascoli theorem ensures us that the operator Θ is a completely continuous operator and by Lemma 4.2.5 $\Theta: B(0,R) \to B(0,R)$. Hence, we conclude that $\Theta: B(0,R) \to B(0,R)$ is completely continuous. It follows that from Schauder's fixed point theorem that Θ has at a least one fixed point. That is the problem (4.2.5) has at least one solution on$[0,T]$.■

Example 4.2.7 For p$= 1, t_1 = \frac{1}{4}, T = 1, \beta = \frac{1}{2}, \mu_1 = 2, \sigma_1 = \frac{1}{2}, \mu_2 = 3, \sigma_2 = \frac{1}{10}, \eta_1 = \frac{1}{5}, \eta_2 = \frac{2}{3}, \alpha_0 = \frac{3}{2}, \alpha_k = \frac{3}{2}$, we consider the following impulsive multi-orders fractional differential equation:

$$\begin{aligned}
&{}^cD_{t_k}^{\alpha_k} u(t) = \frac{1}{100}\cos u(t) + \frac{|u'(t)|}{|u'(t)|+100} + t, \quad 0 \leq t \leq 1, t \neq \frac{1}{4},\\
&\Delta u\left(\frac{1}{4}\right) = \frac{\left|u\left(\frac{1}{4}\right)\right|}{\left|u\left(\frac{1}{4}\right)\right| + 50}, \quad \Delta u'\left(\frac{1}{4}\right) = \frac{\left|u'\left(\frac{1}{4}\right)\right|}{\left|u'\left(\frac{1}{4}\right)\right| + 70},\\
&u(0) + 2\, {}^cD_{0+}^{\frac{1}{2}} u(0) = \frac{1}{2} u\left(\frac{1}{5}\right),\\
&u(1) + 3\, {}^cD_{0+}^{\frac{1}{2}} u(1) = \frac{1}{10} u\left(\frac{2}{3}\right).
\end{aligned} \tag{4.30}$$

It is clear that

$$|f(t,x,x_1) - f(t,y,y_1)|$$

$$\leq 0.02(|x-y| + |x_1 - y_2|), \quad 0 \leq t \leq 1\,, x, y, x_1, y_2 \in \mathbb{R}\,.$$

One can easily calculate that

$$\Lambda = 0.2178 < 1.$$

Therefore, all assumptions of Theorem 4.2.4 hold. Thus, the impulsive multi-orders fractional BVP (4.30) has a unique solution on [0,1].

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