Existence of Solutions of Fractional Differential Equations

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ABSTRACT

This thesis is aimed to study the existence of mild solutions of a class of fractional differential equations with different boundary conditions. By using fixed point theorems, the existence results about mild solutions are expected to obtain.

A strong motivation for studing fractional differential equations comes from the fact, that is essential in various fields of science, engineering and economics.

Keywords: Differential equations, integral boundary conditions, irregular boundary conditions, p-laplacian operator.

ÖZ

Bu tez farklı sınırlı koşullar altında verilen kesirli diferansiyel denklemlerin çözümleri üzerinde çalışmayı amaçlamaktadır. Çözümlerin analitik sonuçları, sabit nokta teoremleri ve uygulamaları kullanılarak bulunmuştur.

Kesirli diferansiyel denklem çalışmalarındaki etkin motivasyon, bu konunun bilim, mühendislik ve ekonomide gerekliliğinden ileri gelmektedir.

Anahtar Kelimeler: Diferansiyel denklemler, integral sınır koşulları, düzensiz sınır koşulları, p-laplasyan operatörü.

To My Beloved Family

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LIST OF ABBREVIATIONS

FDE	Fractional Differential Equation
ODE	Ordinary Differential Equation (only ordinary derivative)
PDE	Partial Differential Equation (with partial derivative)
BVP	Boundary Value Problem

Chapter 1

INTRODUCTION

Mathematics is one of the oldest science in history. In ancient times, it is defined as the science of numbers and figures. Now, by the improvements, its size can not be explained with few sentences. For some philosophers, mathematics is the production of human mind to dominate and explain nature by using special symbols and figures. When we examine mathematics' history and related works, two main groups are seen. The first group is "Ancient Greek Mathematicians"; Thales (624-547 BC), Pythagor (569-500 BC, known as Phythagoras), Zero (495-435 BC), Archimedes (287-212 BC), Apollonius (260 BC?-200?)... The second group is "Western World Mathematicians"; Johann Müller (1436-1476), Cardano (1501-1596), Descartes (1596-1650), Fermat (1601-1665), Pascal (1623-1662), Isaac Newton (1642-1727), Lebniz (1647-1716), Euler (1707-1783), Lagrange (1776-1813), Gauss (1777-1855), Cauchy (1789-1857), Riemann (1826-1866)... Their works indicated the information of basic systems and theorems. The mentioned first group, Ancient Greek Mathematicians, lived between 8^{th} century BC and 2^{nd} century AD, also Western World Mathematicians lived between 16^{th} and 20^{th} century. Moreover between 7^{th} and 16^{th} century, islamic world improved Greek mathematicians' works. Not only the new systems and new concepts, but also the new theorems and the proofs are found. The basis of modern mathematics is formed by these developments. The first written book on algebra by Harezmi, described the basic knowledge of trigonometry by el-Battani, tangent and cotangent by

Ebu'l Vefa; improved Bionomial formulas by Ömer Hayyam are some examples from islamic world mathematicians and their studies.

The first studies on DEs were started on the second half of the 17^{th} century by the British mathematician Newton (1642-1727) and the German mathematician Leibnitz (1641-1716). In 18^{th} century, these works are improved by Bernoulli brothers, Euler, Lagrange, Monge and, in 19^{th} century, Chrystal, Cauchy, Jacobl, Darboux, Picard are studied on the related works. With the help of these mathematicians, the current high-leveled version of the DEs is formed.

This research is basically purposed to study existence of solution of $\alpha-$order three point BVP with integral conditions. To do this, in the first section of Chapter 2, the frequently used preliminaries and definitions are given. Moreover, in second section of Chapter 2, the collected works about DEs are included.

The properties and usage fields and also the related works of the fractional three point BVP with integral conditions are given in Chapter 3.

In Chapter 4, the existence and uniqueness of the solution for given $\alpha-$ordered non-linear FDE is shown. In Chapter 4; the existence, uniqueness and existence results are shown by Green functions and the related fixed point theorems, respectively.

In Chapter 5, the related works of the FDEs with p-laplacian operator and irregular, integral conditions are given. By Green function and the related fixed point theorems, the existence and uniqueness of the solution for given FDEs with p-laplacian operator and irregular, integral boundary conditions is given in.

At last, in Chapter 6, conclusions and some examples are given and illustration of the results are shown.

Chapter 2

PRELIMINARIES AND DEFINITIONS

The proof of the existence of the solution of certain types of DEs under some conditions is named as existence theory which is found by French mathematician A.L. Cauchy between 1820 and 1830. In addition, the existence theory is studied and developed by other mathematicians. The following are some kind of DE types which are developed by famous mathematicians: The British mathematician Newton started his researches on 1665, and on 1671, he defined three types of DEs which are first, second, and third degree differential equations. The German mathematician, Leibnitz, studied DEs between 1684 and 1686, and on 1690, he developed new solution methods with Bernoulli brothers. Another German mathematician, Euler, studied on degrating the equation degrees. He found the algebraic solution of Abel's theory which is important for eliptic functions.

Let us recall some basic definitions see [49], [64], [66].

Definition 2.0.1 *The Riemann Liouville fractional integral of order* α *for a function* $f:[0,\infty)\to\mathbb{R}$ *which is provided the integral exists, and defined as*

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad \alpha>0.$$

Definition 2.0.2 *For a function* $f:[0,\infty)\to\mathbb{R}$ *the Caputo derivative fractional order* α *is defined where* $[\alpha]$ *denotes the integral part of the real number* α*. and given as;*

$$\mathfrak{D}_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds; \;\; n-1 < \alpha < n, \; n = [\alpha]+1.$$

Lemma 2.0.3 *Let $\alpha > 0$. The differential equation $\mathfrak{D}_{0^+}^{\alpha} f(t) = 0$ has solutions*

$$f(t) = k_0 + k_1 t + k_2 t^2 + ... + k_{n-1} t^{n-1}.$$

Also $$I_{0^+}^{\alpha} \mathfrak{D}_{0^+}^{\alpha} f(t) = f(t) + k_0 + k_1 t + k_2 t^2 + ... + k_{n-1} t^{n-1}.$$

Here $k_i \in \mathbb{R}$ and $i = 1,2,3,...,n-1$, $n = [\alpha]+1$.

Definition 2.0.4 *Let $(X.d)$ be a metric space. A mapping $T : X \to X$ is contraction mapping if there exists a nonnegative constant k which is $0 \leq k < 1$, for each $x,y \in X$. such that*

$$d(T(x), T(y)) \leq kd(x,y).$$

Theorem 2.0.5 *(Nonlinear alternative) Let X be a Banach space, let B be a closed, convex subset of X, let W be an open subset of B and $0 \in W$. Suppose that $F : \overline{W} \to B$ is a continuous and compact map. Then either (a) F has a fixed point in $\overline{W}$, or (b) there exist an $x \in \partial W$ (the boundary of W) and $\lambda \in (0,1)$ with $x = \lambda F(x)$.*

Theorem 2.0.6 *(The Banach Contraction Principle) If $T : X \to X$ is contraction mapping on complete metric space $(X.d)$, then there exists one solution $x \in X$ such that $T(x) = x$ fixed point of T. The Banach fixed point theorem is also called the contraction mapping theorem.*

Theorem 2.0.7 *(Arzela-Ascoli Theorem) Let R be a region in $\mathbb{C}$, and let F be a uniformly bounded, equi-continuous family complex-valued functions on R . Then every sequence $\{f_n\}$ in F has convergent subsequence, the convergence being uniform on compact subsets.*

Theorem 2.0.8 *(Schaefer fixed point theorem) Let X be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact and convex set. Then given any continuous mapping $f : K \to K$ there exists $x \in K$ such that $f(x) = x$.*

Theorem 2.0.9 *(Krasnoselskii's fixed point theorem) Let M be a closed convex non-empty subset of a Banach space $(X, \|.\|)$. Suppose that A and B map M into X such that, if the given conditions hold then there exists $y \in M$ with $y = Ay + By$.*

(i) *if $x, y \in M$, then $Ax + By \in M$,*

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Theorem 2.0.10 *(Leray-Schauder fixed point theorem) If D is a non-empty, convex, bounded and closed subset of Banach space B and $T : D \to D$ a compact and continuous map, then T has a fixed point in D.*

Remark 2.0.11 *The Caputo fractional derivative of order $n-1 < \alpha < n$ for t^{γ}, is given as*

$$\mathfrak{D}_{0^+}^{\alpha} t^{\gamma} = \begin{cases} \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \gamma \in \mathbb{N} \text{ and } \gamma \geq n \text{ or } \gamma \notin \mathbb{N} \text{ and } \gamma > n-1, \\ 0, & \gamma \in \{0, 1, ..., n-1\}. \end{cases} \tag{2.0.1}$$

2.1 Introduction

Today, algebraic geometry, algebraic techniques and DEs are used for modeling robots and computer games. Also the DEs and numerical analysis techniques are available for modeling aircraft, satelitte production, measuring the change of dynamical systems. For this reason, DEs are attracted the attention of many researchers. There are many works about the DEs with boundary conditions. Some of them are given in the following.

The multiplicity and the existence of non-negative solutions for non-linear FDEs with boundary conditions are discussed in [24]. Here, $0 < t < 1$, and for real α, $1 < \alpha \leq 2$. They used standard Riemann-Liouville differentiation for function f. Some existence and multiplicity results are found on cone by fixed point theorems

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0,$$

$$u(0) = 0,$$

$$u(1) = 0.$$

In [83], the multiplicity and the existence of non-negative solutions for non-linear FDEs with boundary conditions are studied. Here, $0 < t < 1$, and for real α, $1 < \alpha \leq 2$. The Caputo's fractional derivative ${}^{c}D_{0+}^{\alpha}$ is used with continuous f. Some existence and multiplicity results are found on cone by fixed point theorems.

$$D_{0+}^{\alpha} u(t) = f(t, u(t)),$$

$$u(0) + u'(0) = 0,$$

$$u(1) + u'(1) = 0.$$

In [50], α order Riemann-Liouville differential operator and continuous functions f and a are used. It is given as follows. Here, $0 < t < 1$, and for real α, $1 < \alpha \leq 2$. The sufficient conditions for the existence of at least one and at least three non-negative solutions to the non-linear fractional boundary value problem are given.

$$D_{0+}^{\alpha} u + a(t) f(u) = 0,$$

$$u(0) = 0,$$

$$u'(1) = 0.$$

The existence and uniqueness of boundary value problem for FDEs is discussed in [2], where $t \in [0,T]$, $2 < \alpha \leq 3$, with continuous f, with real constants y_0, y_0^{*}, y_0^{*} and Caputo fractional derivative ${}^{c}D_{0+}^{\alpha}$ is used. Three results are found by Banach fixed point, Schaefer's fixed point and Leray-Schauder fixed point theorem.

$$ {}^{c}D_{0+}^{\alpha} y(t) = f(t,y), $$

$$ y(0) = y_0, $$

$$ y'(0) = y_0^{*}, $$

$$ y''(T) = y_0^{*}. $$

The existence of solutions of the equation is studied by S. Zhang in [84] which is given as follows, and here $\delta \in (1,2)$, $\alpha, \beta \neq 0$ and $t \in [0,1]$. Here ${}^{c}D_{0+}^{\alpha}$ is Caputo fractional derivative and g is continuous function. The Schauder fixed point theorem is used.

$$ {}^{c}D_{0+}^{\delta} u(x) = g(x, u(x)), $$

$$ u(0) = \alpha, $$

$$ u(0) = \beta. $$

In [36], the existence of solutions of the FDE with boundary conditions is studied where $\delta \in (1,2)$, $\alpha, \beta \neq 0$ and $t \in [0,1]$. The Bohnenblust–Karlin fixed point theorem is studied.

Bashir Ahmad, [9], is obtained irregular boundary value problem by using Banach fixed point theorem. The problem is in the following, where ${}^{c}D^{q}$ is Caputo derivative, f continuous function, $t \in [0,1]$, $1 < q \leq 2$, $\theta = 0, 1$ and b is not zero.

$$
\begin{aligned}
{}^{c}D^{q}x(t) &= f(t,x(t)),\\
x'(0)+(-1)^{\theta}x'(\pi)+bx'(\pi) &= 0,\\
x(0)+(-1)^{\theta+1}x(\pi) &= 0.
\end{aligned}
$$

A coupled system of non-linear fractional differential equations with three point boundary conditions are discussed in [8] by Shauder fixed point theorem. In this study, Riemann Liouville fractional derivative and continuous functions f and g are used. For $t \in (0,1)$, $1 < \alpha, \beta < 2$, p, q, γ are non-negative, $0 < \eta < 1$, $\alpha - q \succeq 1$, $\beta - p \succeq 1$, $\gamma\eta^{\alpha-1} < 1$, $\gamma\eta^{\beta-1} < 1$.

By using some fixed point theorems, the existence and multiplicity results of positive solutions of the following non-linear DEs are obtained in [54], which is given, where D_{0+}^{α} is denoted the standard Riemann Liouville fractional order derivative.

$$
\begin{aligned}
D_{0+}^{\alpha}u(t)+f(t,u(t)) &= 0, 0<t<1 \text{ and } 1<\alpha<2,\\
u(0) &= 0,\\
D_{0+}^{\beta}u(1) &= aD_{0+}^{\beta}u(\xi).
\end{aligned}
$$

In [68], a boundary value problem for a coupled differential system of fractional order is studied. Riemann–Liouville differential operator is taken. The existence results of the solution is found by Schauder's fixed point theorem. It is given as follows, where $0 < t < 1$, $1 < \alpha, \beta < 2$,with non-negative $\mu, \nu, \alpha - \nu \succeq 1$, $\beta - \mu \succeq 1$ and f, g are continuous functions. It is given as:

$$D_{0+}^{\alpha} u(t) = f\left(t, v(t), D_{0+}^{\alpha} v(t)\right),$$

$$D_{0+}^{\beta} v(t) = g(t, u(t), D^{q} u(t)),$$

$$u(t) = u(1) = 0,$$

$$v(0) = v(1) = 0.$$

Chapter 3

DEs WITH BOUNDARY CONDITIONS

3.1 Introduction

The FDEs are used in different fields of science. The dynamical systems, modern physics, chemistery, biology and genetics are some of them. Especially for dynamical systems and modern physics, fractional differentials are used to construct self-replication machines and re-construction of lost pieces of digital data from space stations to world and are used to produce small volumed and/or large surfaced antenna. In chemical engineering and biology, while calculating bounded and/or unbounded equations with boundary conditions, the common techniques are used to explain blood flow models, arrangement of blood vessels, cellular systems.

In [55], the existence and uniqueness results of solutions for FDEs with integral boundary conditions are discussed which is given as follows:

$$D_{0+}^{\alpha} x(t) + f\left(t, x(t), x'(t)\right) = 0,$$

$$x(0) = \int_0^1 g_0(s, x(s))\, ds,$$

$$x(1) = \int_0^1 g_1(s, x(s))\, ds,$$

$$x^{(k)}(0) = 0,$$

$$k = 2, 3, \ldots [\alpha] - 1.$$

Here $t \in (0,1)$, D_{0+}^{α} is the Caputo fractional derivative $1 < \alpha \in \mathbb{R}$ and the new results on the existence and uniqueness are discussed by Banach fixed point principle.

The non-linear FDE of an arbitrary order with four-point non-local integral boundary conditions is discussed by the Banach fixed point theorem in [12]. Here $0 < t < 1$, $m-1 < q \leq m$, $0 < \xi, \eta < 1$. The Caputo fractional differentiation is used with order q where $\alpha, \beta \in \mathbb{R}$, is given in the following:

$$
\begin{aligned}
{}^{c}D_{0+}^{q}x(t) &= f(t,x(t)), \\
x(0) &= \alpha \int_{0}^{\xi} x(s)\,ds, \\
x'(0) &= 0, \\
x''(0) &= 0, \ldots, x^{(m-2)}(0) = 0, \\
x(1) &= \beta \int_{0}^{\xi} x(s)\,ds,
\end{aligned}
$$

In [62], the impulsive FDEs with two point and integral boundary conditions are studied. Here A and B are given $n \times n$ matrices where $\det(A+B) \neq 0$. The fixed point theorems are used. It is shown in the following:

$$
\begin{aligned}
{}^{c}D_{0+}^{\alpha}x(t) &= f(t,x(t)),\ t \in J' \\
x\left(t_j^{+}\right) - x\left(t_j\right) &= I_j\left(x\left(t_j\right)\right),\ j = 1,2,\ldots,p, \\
Ax(0) + Bx(T) &= \int_{0}^{T} g(s,x(s))\,ds.
\end{aligned}
$$

By the standard fixed point theorems and Leray-Schauder degree theory, some new existence results are found for BVP of non-linear FDEs of order $q \in (1,2]$ with three point integral boundary condition in [13] is given as, follows, where ${}^{c}D_{0+}^{q}$ is the Caputo derivative of order q, $0 < t < 1$, $0 < \eta < 1$ and $1 < q \leq 2$ with continuous function f.

$$
\begin{aligned}
{}^{c}D_{0+}^{q} x(t) &= f(t, x(t)), \\
x(0) &= 0, \\
x(1) &= \alpha \int_{0}^{\eta} x(s)\, ds,
\end{aligned}
$$

The multiple non-negative solutions for the FDE with integral boundary conditions are studied in [46]. They obtained some new results on the existence of at least three non-negative solutions by the Leggett-Williams fixed point theorem. It is given in the following form, where $k = 2, 3, ..., [\alpha]$.

The existence of positive solutions for a class of non-linear BVP of FDEs with integral boundary conditions is discussed in [29]. That is given as follows. Here ${}^{c}D_{0+}^{\alpha}$ is Caputo fractional derivation, $2 < \alpha < 3$, $0 < \lambda < 2$ and continuous function f.

$$
\begin{aligned}
{}^{c}D_{0+}^{\alpha} u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1 \\
u(0) = u''(0) &= 0, \\
u(1) &= \lambda \int_{0}^{1} u(s)\, ds,
\end{aligned}
$$

The existence, non-existence and multiplicity of positive solutions for a class of higher order non-linear fractional differential equations with integral boundary conditions are discussed in [39] by Krasnoselskii's fixed-point theorem in cones.

The existence of positive solutions for the following nonlinear FDEs with integral boundary conditions is obtained in [86] by Green functions and the fixed point theorems, which is given as follows, where D_{0+}^{α} is the Riemann-Liouville derivative,

$3 < \alpha \leq 4$, $0 < \eta \leq 1$. It is given as follows:

$$D_{0+}^{\alpha} u(t) + h(t) f(t, u(t)) = 0, \ \ 0 < t < 1,$$

$$u(0) = 0,$$

$$u^{'}(0) = 0,$$

$$u''(0) = 0,$$

$$u(1) = \lambda \int_{0}^{\eta} u(s)\, ds.$$

In [76], the authors are studied about the eigenvalue problem of the following nonlinear fractional DEs with integral boundary conditions which is shown, where $0 < t < 1$, $n < \alpha \leq n+1$, $n \geq 2$, $0 < \xi < 2$ and D_{0+}^{α} is the Caputo derivative. They studied by the Green's function and Guo-Krasnoselskii's fixed point theorem, which is;

$$D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \ \ 0 < t < 1,$$

$$u(0) = u''(0) = u\prime''(0) = ... = u^{(n)}(0) = 0,$$

$$u(1) = \xi \int_{0}^{1} u(s)\, ds.$$

Chapter 4

FDE WITH BOUNDARY CONDITIONS

In this chapter, the existence (and uniqueness) of solution for nonlinear FDEs of order $\alpha \in (2,3]$ is obtained when the nonlinearity of f depends on the fractional derivatives of the unknown function:

$$\mathfrak{D}^{\alpha}_{0^+}(t) = f\left(t, u(t), \mathfrak{D}^{\beta_1}_{0^+}u(t), \mathfrak{D}^{\beta_2}_{0^+}u(t)\right); \ \ 0 \le t \le T; \ 2 < \alpha \le 3. \tag{4.0.1}$$

The three point and integral boundary conditions:

$$\begin{cases} a_0 u(0) + b_0 u(T) = \lambda_0 \int\limits_0^T g_0(s, u(s))ds, \\ a_1 \mathfrak{D}^{\beta_1}_{0^+}u(\eta) + b_1 \mathfrak{D}^{\beta_1}_{0^+}u(T) = \lambda_1 \int\limits_0^T g_1(s, u(s))ds, \quad 0 < \beta_1 \le 1, \ \ 0 < \eta < T, \\ a_2 \mathfrak{D}^{\beta_2}_{0^+}u(\eta) + b_2 \mathfrak{D}^{\beta_2}_{0^+}u(T) = \lambda_2 \int\limits_0^T g_2(s, u(s))ds, \quad 1 < \beta_2 \le 2, \end{cases} \tag{4.0.2}$$

where $\mathfrak{D}^{\alpha}_{0^+}$ denotes the Caputo fractional derivative of order α, and f, g_i are continuous functions and $a_i, b_i, \lambda_i \in \mathbb{R}$ for $i = 0, 1, 2$.

Lemma 4.0.1 *For each $f, g_0, g_1, g_2 \in C([0,T];\mathbb{R})$, the unique solution of the fractional boundary value problem:*

$$\mathfrak{D}_{0^+}^{\alpha}u(t) = f(t);\;\; 0 \le t \le T,\; 2 < \alpha \le 3, \tag{4.0.3}$$

$$\begin{cases} a_0 u(0) + b_0 u(T) = \lambda_0 \int\limits_0^T g_0(s)ds, \\ a_1 \mathfrak{D}_{0^+}^{\beta_1} u(\eta) + b_1 \mathfrak{D}_{0^+}^{\beta_1} u(T) = \lambda_1 \int\limits_0^T g_1(s)ds, \quad 0 < \eta < T,\; 0 < \beta_1 \le 1, \\ a_2 \mathfrak{D}_{0^+}^{\beta_2} u(\eta) + b_2 \mathfrak{D}_{0^+}^{\beta_2} u(T) = \lambda_2 \int\limits_0^T g_2(s)ds. \quad 1 < \beta_2 \le 2. \end{cases}$$

(4.0.4)

$$u(t) = \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \sum_{i=0}^{2} \omega_i(t)\, b_i \int\limits_0^T \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s)ds$$
.
$$+ \sum_{i=1}^{2} \omega_i(t)\, a_i \int\limits_0^{\eta} \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s)ds - \sum_{i=0}^{2} \omega_i(t)\, \lambda_i \int\limits_0^T g_i(s)ds.$$

Proof. For $2 < \alpha \le 3$, the general solution of the equation $\mathfrak{D}_{0^+}^{\alpha}u(t) = f(t)$ is found by lemma 3, that can be given as follows:

$$u(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s)ds - k_0 - k_1 t - k_2 t^2. \tag{4.0.5}$$

Here $k_0, k_1, k_2 \in \mathbb{R}$ are arbitrary constants. By the formula of $\mathfrak{D}_{0^+}^{\alpha} t^{\gamma}$, the β_1 and β_2 order derivatives are given as:

$$\mathfrak{D}_{0^+}^{\beta_1} u(t) = I^{\alpha-\beta_1} f(t) - k_1 \frac{t^{1-\beta_1}}{\Gamma(2-\beta_1)} - 2k_2 \frac{t^{2-\beta_1}}{\Gamma(3-\beta_1)},$$
$$\mathfrak{D}_{0^+}^{\beta_2} u(t) = I^{\alpha-\beta_2} f(t) - 2k_2 \frac{t^{2-\beta_2}}{\Gamma(3-\beta_2)}.$$

The given algebraic system of equations for k_0, k_1, k_2 are found.

$$
\begin{aligned}
&-(a_0+b_0)k_0-b_0Tk_1-b_0T^2k_2\\
&=\lambda_0\int_0^T g_0(s)ds-b_0I_{0^+}^{\alpha}f(T),\\
&-\frac{a_1\eta^{1-\beta_1}+b_1T^{1-\beta_1}}{\Gamma(2-\beta_1)}k_1-2\frac{a_1\eta^{2-\beta_1}+b_1T^{2-\beta_1}}{\Gamma(3-\beta_1)}k_2\\
&=\lambda_1\int_0^T g_1(s)ds-a_1I_{0^+}^{\alpha-\beta_1}f(\eta)-b_1I_{0^+}^{\alpha-\beta_1}f(T),\\
&-2\frac{a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}}{\Gamma(3-\beta_2)}k_2\\
&=\lambda_2\int_0^T g_2(s)ds-a_2I_{0^+}^{\alpha-\beta_2}f(\eta)-b_2I_{0^+}^{\alpha-\beta_2}f(T).
\end{aligned}
$$

The following boundary conditions are used;

$$
\begin{aligned}
a_0u(0)+b_0u(T)&=\lambda_0\int_0^T g_0(s)ds,\\
a_1\mathfrak{D}_{0^+}^{\beta_1}u(\eta)+b_1\mathfrak{D}_{0^+}^{\beta_1}u(T)&=\lambda_1\int_0^T g_1(s)ds,\quad 0<\eta<T,\ \ 0<\beta_1\leq 1,\\
a_2\mathfrak{D}_{0^+}^{\beta_2}u(\eta)+b_2\mathfrak{D}_{0^+}^{\beta_2}u(T)&=\lambda_2\int_0^T g_2(s)ds,\quad 1<\beta_2\leq 2.
\end{aligned}
$$

Also for convenience, we set

$$
a_0+b_0\neq 0,\ \ a_1\eta^{1-\beta_1}+b_1T^{1-\beta_1}\neq 0,\ \ a_i\eta^{2-\beta_i}+b_iT^{2-\beta_i}\neq 0,
$$

and

$$
\begin{aligned}
\mu^{\beta_1}&:=\frac{\Gamma(3-\beta_1)}{2\left(a_1\eta^{2-\beta_1}+b_1T^{2-\beta_1}\right)},\ \mu^{\beta_2}:=\frac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}\right)},\\
\nu^{\beta_1}&:=\frac{\Gamma(2-\beta_1)}{a_1\eta^{1-\beta_1}+b_1T^{1-\beta_1}},\\
\omega_0&:=-\frac{1}{a_0+b_0},\quad \omega_1(t):=\nu^{\beta_1}\left(\frac{b_0}{a_0+b_0}T-t\right),\\
\omega_2(t)&:=\frac{b_0T^2}{a_0+b_0}\mu^{\beta_2}-\frac{b_0T}{a_0+b_0}\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}+\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}t-\mu^{\beta_2}t^2.
\end{aligned}
$$

Moreover, we assume $\beta_0=0$. Now, by using the first condition: $a_0u(0)+b_0u(T)=$

$\lambda_0 \int\limits_0^T g_0(s)ds$, we get

$$a_0\left[-k_0\right] + b_0\left[\int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds - k_0 - k_1 T - k_2 T^2\right]$$
$$= \lambda_0 \int\limits_0^T g_0(s)ds,$$
$$-k_0 a_0 + b_0 \int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds - b_0 k_0 - b_0 k_1 T - b_0 k_2 T^2$$
$$= \lambda_0 \int\limits_0^T g_0(s)ds,$$
$$k_0\left(-a_0 - b_0\right) + b_0 \int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds - b_0 k_1 T - b_0 k_2 T^2$$
$$= \lambda_0 \int\limits_0^T g_0(s)ds.$$

Then
$$k_0\left(-a_0 - b_0\right) - b_0 k_1 T - b_0 k_2 T^2$$
$$= \lambda_0 \int\limits_0^T g_0(s)ds - b_0 \int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$
$$-\left(a_0 + b_0\right) k_0 - b_0 T k_1 - b_0 T^2 k_2$$
$$= \lambda_0 \int\limits_0^T g_0(s)ds - b_0 \int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$
$$-\left(a_0 + b_0\right) k_0 - b_0 T k_1 - b_0 T^2 k_2$$
$$= \lambda_0 \int\limits_0^T g_0(s)ds - b_0 I_{0^+}^{\alpha} f(T).$$

By using the second boundary condition, we have

$$a_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}f(s)ds-a_1k_1\frac{\eta^{1-\beta_1}}{\Gamma(2-\beta_1)}-2k_2a_1\frac{\eta^{2-\beta_1}}{\Gamma(3-\beta_1)}$$
$$+b_1\int_0^{T}\frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}f(s)ds-k_1b_1\frac{T^{1-\beta_1}}{\Gamma(2-\beta_1)}-2k_2b_1\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}$$
$$=\lambda_1\int_0^{T}g_1(s)ds.$$

It is
$$k_1\left[-a_1\frac{\eta^{1-\beta_1}}{\Gamma(2-\beta_1)}-b_1\frac{T^{1-\beta_1}}{\Gamma(2-\beta_1)}\right]$$
$$+k_2\left[\frac{-2a_1\eta^{2-\beta_1}}{\Gamma(3-\beta_1)}-\frac{2b_1T^{2-\beta_1}}{\Gamma(3-\beta_1)}\right]$$
$$+a_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}f(s)ds+b_1\int_0^{T}\frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}f(s)ds$$
$$=\lambda_1\int_0^{T}g_1(s)ds.$$

Then
$$-k_1\left[\frac{a_1\eta^{1-\beta_1}+b_1T^{1-\beta_1}}{\Gamma(2-\beta_1)}\right]-2k_2\left[\frac{a_1\eta^{2-\beta_1}+2b_1T^{2-\beta_1}}{\Gamma(3-\beta_1)}\right]$$
$$+a_1\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}f(s)ds+b_1\int_0^{T}\frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}f(s)ds$$
$$=\lambda_1\int_0^{T}g_1(s)ds.$$

Thus

$$
\begin{aligned}
&- k_1 \left[\frac{a_1 \eta^{1-\beta_1} + b_1 T^{1-\beta_1}}{\Gamma(2-\beta_1)} \right] \\
&- 2k_2 \left[\frac{a_1 \eta^{2-\beta_1} + 2b_1 T^{2-\beta_1}}{\Gamma(3-\beta_1)} \right] \\
&= \lambda_1 \int_0^T g_1(s)ds - a_1 \int_0^\eta \frac{(\eta - s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s)ds \\
&- b_1 \int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s)ds.
\end{aligned}
$$

It also can be written as,

$$
\begin{aligned}
&- k_1 \left[\frac{a_1 \eta^{1-\beta_1} + b_1 T^{1-\beta_1}}{\Gamma(2-\beta_1)} \right] \\
&- 2k_2 \left[\frac{a_1 \eta^{2-\beta_1} + 2b_1 T^{2-\beta_1}}{\Gamma(3-\beta_1)} \right] \\
&= \lambda_1 \int_0^T g_1(s)ds - a_1 I_{0^+}^{\alpha-\beta_1} f(\eta) - b_1 I_{0^+}^{\alpha-\beta_1} f(T).
\end{aligned}
$$

At last, by using $a_2 \mathfrak{D}_{0^+}^{\beta_2} u(\eta) + b_2 \mathfrak{D}_{0^+}^{\beta_2} u(T) = \lambda_2 \int_0^T g_2(s, u(s))ds$, for $1 < \beta_2 \leq 2$, we get,

$$
\begin{aligned}
&a_2 \left[\int_0^\eta \frac{(\eta - s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s)ds - 2k_2 \frac{\eta^{2-\beta_2}}{\Gamma(3-\beta_2)} \right] \\
&+ b_2 \left[\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s)ds - 2k_2 \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} \right] \\
&= \lambda_2 \int_0^T g_2(s)ds.
\end{aligned}
$$

Then,

$$a_2\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds-2k_2a_2\frac{\eta^{2-\beta_2}}{\Gamma(3-\beta_2)}$$
$$+b_2\int_0^{T}\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds-2k_2b_2\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}$$
$$=\lambda_2\int_0^{T}g_2(s)ds.$$

That is
$$-2k_2a_2\frac{\eta^{2-\beta_2}}{\Gamma(3-\beta_2)}-2k_2b_2\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}$$
$$+a_2\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds+b_2\int_0^{T}\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds$$
$$=\lambda_2\int_0^{T}g_2(s)ds.$$

It is
$$-2k_2\left[\frac{a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}}{\Gamma(3-\beta_2)}\right]+a_2\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds$$
$$+b_2\int_0^{T}\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds$$
$$=\lambda_2\int_0^{T}g_2(s)ds.$$

Now,
$$-2k_2\left[\frac{a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}}{\Gamma(3-\beta_2)}\right]$$
$$=\lambda_2\int_0^{T}g_2(s)ds-a_2\int_0^{\eta}\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds$$
$$-b_2\int_0^{T}\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}f(s)ds.$$

That can be written as,

$$- 2k_2 \left[\frac{a_2 \eta^{2-\beta_2} + b_2 T^{2-\beta_2}}{\Gamma(3-\beta_2)} \right]$$
$$= \lambda_2 \int_0^T g_2(s) ds - a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) - b_2 I_{0^+}^{\alpha-\beta_2} f(T).$$

Thus

$$\begin{aligned} k_2 &= \frac{\Gamma(3-\beta_2)}{-2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} \lambda_2 \int_0^T g_2(s) ds \\ &- \frac{\Gamma(3-\beta_2)}{-2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) \\ &- \frac{\Gamma(3-\beta_2)}{-2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} b_2 I_{0^+}^{\alpha-\beta_2} f(T), \\ &= -\frac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} \lambda_2 \int_0^T g_2(s) ds \\ &+ \frac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) \\ &+ \frac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} b_2 I_{0^+}^{\alpha-\beta_2} f(T). \end{aligned}$$

Since $\mu^{\beta_2} := \dfrac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)}$, we found k_2 as:

$$k_2 = -\mu^{\beta_2} \lambda_2 \int_0^T g_2(s) ds + \mu^{\beta_2} a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) + \mu^{\beta_2} b_2 I_{0^+}^{\alpha-\beta_2} f(T).$$

With the help of the algebraic equation of k_2, k_0 and k_1 are given as follows:

$$- \frac{a_1 \eta^{1-\beta_1} + b_1 T^{1-\beta_1}}{\Gamma(2-\beta_1)} k_1 - 2 \frac{a_1 \eta^{2-\beta_1} + b_1 T^{2-\beta_1}}{\Gamma(3-\beta_1)} k_2$$
$$= \lambda_1 \int_0^T g_1(s) ds - a_1 I_{0^+}^{\alpha-\beta_1} f(\eta) - b_1 I_{0^+}^{\alpha-\beta_1} f(T).$$

We have the following:

$$-\left(\frac{a_1\eta^{1-\beta_1}+b_1T^{1-\beta_1}}{\Gamma(2-\beta_1)}\right)k_1$$

$$=2\frac{a_1\eta^{2-\beta_1}+b_1T^{2-\beta_1}}{\Gamma(3-\beta_1)}k_2+\lambda_1\int_0^T g_1(s)ds$$

$$-a_1I_{0^+}^{\alpha-\beta_1}f(\eta)-b_1I_{0^+}^{\alpha-\beta_1}f(T).$$

Thus,
$$k_1=\frac{-\nu^{\beta_1}}{\mu^{\beta_1}}k_2-\nu^{\beta_1}\lambda_1\int_0^T g_1(s)ds+\nu^{\beta_1}a_1I_{0^+}^{\alpha-\beta_1}f(\eta)$$

$$+\nu^{\beta_1}b_1I_{0^+}^{\alpha-\beta_1}f(T).$$

That is
$$k_1=\frac{-\nu^{\beta_1}}{\mu^{\beta_1}}\left[-\mu^{\beta_2}\lambda_2\int_0^T g_2(s)ds+\mu^{\beta_2}a_2I_{0^+}^{\alpha-\beta_2}f(\eta)\right.$$

$$\left.+\mu^{\beta_2}b_2I_{0^+}^{\alpha-\beta_2}f(T)\right]$$

$$-\nu^{\beta_1}\lambda_1\int_0^T g_1(s)ds+\nu^{\beta_1}a_1I_{0^+}^{\alpha-\beta_1}f(\eta)$$

$$+\nu^{\beta_1}b_1I_{0^+}^{\alpha-\beta_1}f(T).$$

Therefore
$$k_1=\frac{\nu^{\beta_1}}{\mu^{\beta_1}}\mu^{\beta_2}\lambda_2\int_0^T g_2(s)ds-\frac{\nu^{\beta_1}}{\mu^{\beta_1}}\mu^{\beta_2}a_2I_{0^+}^{\alpha-\beta_2}f(\eta)$$

$$-\frac{\nu^{\beta_1}}{\mu^{\beta_1}}\mu^{\beta_2}b_2I_{0^+}^{\alpha-\beta_2}f(T)$$

$$-\nu^{\beta_1}\lambda_1\int_0^T g_1(s)ds+\nu^{\beta_1}a_1I_{0^+}^{\alpha-\beta_1}f(\eta)$$

$$+\nu^{\beta_1}b_1I_{0^+}^{\alpha-\beta_1}f(T).$$

k_1 can also be written as:

$$k_1 = \nu^{\beta_1} b_1 I_{0^+}^{\alpha-\beta_1} f(T) + \nu^{\beta_1} a_1 I_{0^+}^{\alpha-\beta_1} f(\eta) - \nu^{\beta_1} \lambda_1 \int_0^T g_1(s) ds - \frac{\nu^{\beta_1}}{\mu^{\beta_1}} \mu^{\beta_2} b_2 I_{0^+}^{\alpha-\beta_2} f(T) - \frac{\nu^{\beta_1}}{\mu^{\beta_1}} \mu^{\beta_2} a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) + \frac{\nu^{\beta_1}}{\mu^{\beta_1}} \mu^{\beta_2} \lambda_2 \int_0^T g_2(s) ds.$$

It is

$$k_0 (-a_0 - b_0) - b_0 T \left[\nu^{\beta_1} b_1 I_{0^+}^{\alpha-\beta_1} f(T) + \nu^{\beta_1} a_1 I_{0^+}^{\alpha-\beta_1} f(\eta) - \nu^{\beta_1} \lambda_1 \int_0^T g_1(s) ds - \frac{\nu^{\beta_1}}{\mu^{\beta_1}} \mu^{\beta_2} b_2 I_{0^+}^{\alpha-\beta_2} f(T) - \frac{\nu^{\beta_1}}{\mu^{\beta_1}} \mu^{\beta_2} a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) + \frac{\nu^{\beta_1}}{\mu^{\beta_1}} \mu^{\beta_2} \lambda_2 \int_0^T g_2(s) ds \right]$$
$$- b_0 T^2 \left[-\frac{\Gamma(3-\beta_2)}{2\left(a_2 \eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} \lambda_2 \int_0^T g_2(s) ds + \frac{\Gamma(3-\beta_2)}{2\left(a_2 \eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} a_2 I_{0^+}^{\alpha-\beta_2} f(\eta) + \frac{\Gamma(3-\beta_2)}{2\left(a_2 \eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)} b_2 I_{0^+}^{\alpha-\beta_2} f(T) \right]$$
$$= \lambda_0 \int_0^T g_0(s) ds - b_0 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds.$$

Then

$$
\begin{aligned}
&k_0\left(-a_0-b_0\right)-b_0T\nu^{\beta_1}b_1I_{0^+}^{\alpha-\beta_1}f(T)\\
&-b_0T\nu^{\beta_1}a_1I_{0^+}^{\alpha-\beta_1}f(\eta)+b_0T\nu^{\beta_1}\lambda_1\int_0^T g_1(s)ds\\
&+b_0T\frac{\nu^{\beta_1}}{\mu^{\beta_1}}\mu^{\beta_2}b_2I_{0^+}^{\alpha-\beta_2}f(T)+b_0T\frac{\nu^{\beta_1}}{\mu^{\beta_1}}\mu^{\beta_2}a_2I_{0^+}^{\alpha-\beta_2}f(\eta)\\
&-b_0T\frac{\nu^{\beta_1}}{\mu^{\beta_1}}\mu^{\beta_2}\lambda_2\int_0^T g_2(s)ds\\
&+\frac{b_0T^2\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}\right)}\lambda_2\int_0^T g_2(s)ds\\
&-b_0T^2\frac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}\right)}a_2I_{0^+}^{\alpha-\beta_2}f(\eta)\\
&-b_0T^2\frac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2}+b_2T^{2-\beta_2}\right)}b_2I_{0^+}^{\alpha-\beta_2}f(T)\\
&=\lambda_0\int_0^T g_0(s)ds-b_0\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds.
\end{aligned}
$$

In epitome, the following algebraic expressions are used;

$$
k_2=b_2\mu^{\beta_2}I_{0^+}^{\alpha-\beta_2}f(T)+a_2\mu^{\beta_2}I_{0^+}^{\alpha-\beta_2}f(\eta)-\lambda_2\mu^{\beta_2}\int_0^T g_2(s)ds,
$$

$$
\begin{aligned}
k_1&=b_1\nu^{\beta_1}I_{0^+}^{\alpha-\beta_1}f(T)+a_1\nu^{\beta_1}I_{0^+}^{\alpha-\beta_1}f(\eta)\\
&\quad-\lambda_1\nu^{\beta_1}\int_0^T g_1(s)ds-b_2\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}I_{0^+}^{\alpha-\beta_2}f(T)\\
&\quad-a_2\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}I_{0^+}^{\alpha-\beta_2}f(\eta)+\lambda_2\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}\int_0^T g_2(s)ds.
\end{aligned}
$$

.

We get

$$
\begin{aligned}
k_0 = {} & \frac{b_0}{a_0+b_0} I_{0^+}^{\alpha} f(T) - \frac{\lambda_0}{a_0+b_0} \int_0^T g_0(s)ds \\
& - \frac{b_0 b_1 \nu^{\beta_1} T}{a_0+b_0} I_{0^+}^{\alpha-\beta_1} f(T) - \frac{b_0 a_1 \nu^{\beta_1} T}{a_0+b_0} I_{0^+}^{\alpha-\beta_1} f(\eta) \\
& + \frac{b_0 \lambda_1 \nu^{\beta_1} T}{a_0+b_0} \int_0^T g_1(s)ds + \frac{b_0 b_2 \nu^{\beta_1} T}{a_0+b_0} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(T) \\
& + \frac{b_0 a_2 \nu^{\beta_1} T}{a_0+b_0} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(\eta) - \frac{b_0 \lambda_2 \nu^{\beta_1} T}{a_0+b_0} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} \int_0^T g_2(s)ds \\
& - \frac{b_0 b_2 \mu^{\beta_2} T^2}{a_0+b_0} I_{0^+}^{\alpha-\beta_2} f(T) - \frac{b_0 a_2 \mu^{\beta_2} T^2}{a_0+b_0} I_{0^+}^{\alpha-\beta_2} f(\eta) \\
& + \frac{b_0 \lambda_2 \mu^{\beta_2} T^2}{a_0+b_0} \int_0^T g_2(s)ds.
\end{aligned}
$$

Inserting k_0, k_1 and k_2 into the expression, we get the desired representation for the solution of the BVP, which is given in the following:

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds \\
&- \Bigg[\frac{b_0}{a_0+b_0} I_{0^+}^{\alpha} f(T) - \frac{\lambda_0}{a_0+b_0}\int_0^T g_0(s)ds \\
&- \frac{b_0 b_1 \nu^{\beta_1} T}{a_0+b_0} I_{0^+}^{\alpha-\beta_1} f(T) - \frac{b_0 a_1 \nu^{\beta_1} T}{a_0+b_0} I_{0^+}^{\alpha-\beta_1} f(\eta) \\
&+ \frac{b_0 \lambda_1 \nu^{\beta_1} T}{a_0+b_0}\int_0^T g_1(s)ds + \frac{b_0 b_2 \nu^{\beta_1} T}{a_0+b_0}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(T) \\
&+ \frac{b_0 a_2 \nu^{\beta_1} T}{a_0+b_0}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(\eta) - \frac{b_0 \lambda_2 \nu^{\beta_1} T}{a_0+b_0}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}\int_0^T g_2(s)ds \\
&- \frac{b_0 b_2 \mu^{\beta_2} T^2}{a_0+b_0} I_{0^+}^{\alpha-\beta_2} f(T) - \frac{b_0 a_2 \mu^{\beta_2} T^2}{a_0+b_0} I_{0^+}^{\alpha-\beta_2} f(\eta) \\
&+ \frac{b_0 \lambda_2 \mu^{\beta_2} T^2}{a_0+b_0}\int_0^T g_2(s)ds\Bigg] \\
&- t\Bigg[b_1 \nu^{\beta_1} I_{0^+}^{\alpha-\beta_1} f(T) + a_1 \nu^{\beta_1} I_{0^+}^{\alpha-\beta_1} f(\eta) - \lambda_1 \nu^{\beta_1}\int_0^T g_1(s)ds \\
&- b_2 \nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(T) - a_2 \nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(\eta) \\
&+ \lambda_2 \nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}\int_0^T g_2(s)ds\Bigg] \\
&- \Bigg[b_2 \mu^{\beta_2} I_{0^+}^{\alpha-\beta_2} f(T) + a_2 \mu^{\beta_2} I_{0^+}^{\alpha-\beta_2} f(\eta) - \lambda_2 \mu^{\beta_2}\int_0^T g_2(s)ds\Bigg] t^2.
\end{aligned}
$$

That is

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} f(s)ds - \frac{b_0}{a_0+b_0} I_{0^+}^{\alpha} f(T) \\
&+ \frac{\lambda_0}{a_0+b_0}\int\limits_0^T g_0(s)ds + \frac{b_0 b_1 \nu^{\beta_1} T}{a_0+b_0} I_{0^+}^{\alpha-\beta_1} f(T) \\
&+ \frac{b_0 a_1 \nu^{\beta_1} T}{a_0+b_0} I_{0^+}^{\alpha-\beta_1} f(\eta) - \frac{b_0 \lambda_1 \nu^{\beta_1} T}{a_0+b_0}\int\limits_0^T g_1(s)ds \\
&- \frac{b_0 b_2 \nu^{\beta_1} T}{a_0+b_0}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(T) - \frac{b_0 a_2 \nu^{\beta_1} T}{a_0+b_0}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(\eta) \\
&+ \frac{b_0 \lambda_2 \nu^{\beta_1} T}{a_0+b_0}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}\int\limits_0^T g_2(s)ds + \frac{b_0 b_2 \mu^{\beta_2} T^2}{a_0+b_0} I_{0^+}^{\alpha-\beta_2} f(T) \\
&+ \frac{b_0 a_2 \mu^{\beta_2} T^2}{a_0+b_0} I_{0^+}^{\alpha-\beta_2} f(\eta) - \frac{b_0 \lambda_2 \mu^{\beta_2} T^2}{a_0+b_0}\int\limits_0^T g_2(s)ds \\
&+ -t b_1 \nu^{\beta_1} I_{0^+}^{\alpha-\beta_1} f(T) - t a_1 \nu^{\beta_1} I_{0^+}^{\alpha-\beta_1} f(\eta) \\
&+ t\lambda_1 \nu^{\beta_1}\int\limits_0^T g_1(s)ds + t b_2 \nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(T) \\
&+ t a_2 \nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}} I_{0^+}^{\alpha-\beta_2} f(\eta) - t\lambda_2 \nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}\int\limits_0^T g_2(s)ds \\
&- t^2 b_2 \mu^{\beta_2} I_{0^+}^{\alpha-\beta_2} f(T) - t^2 a_2 \mu^{\beta_2} I_{0^+}^{\alpha-\beta_2} f(\eta) \\
&+ t^2 \lambda_2 \mu^{\beta_2}\int\limits_0^T g_2(s)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} f(s)ds - \frac{b_0}{a_0+b_0} I_{0^+}^{\alpha} f(T) \\
& + \nu^{\beta_1} b_1 \left(\frac{b_0 T}{a_0+b_0} - t\right) I_{0^+}^{\alpha-\beta_1} f(T) \\
& + \left(\frac{b_0}{a_0+b_0} T^2 \mu^{\beta_2} b_2 - \frac{b_0 T}{a_0+b_0} \nu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} b_2 \right. \\
& \left. + \nu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} t b_2 - \mu^{\beta_2} t^2 b_2 \right) I_{0^+}^{\alpha-\beta_2} f(T) \\
& - \frac{a_0}{a_0+b_0} I_{0^+}^{\alpha} f(\eta) + \nu^{\beta_1} a_1 \left(\frac{b_0 T}{a_0+b_0} - t\right) I_{0^+}^{\alpha-\beta_1} f(\eta) \\
& + \left(\frac{b_0}{a_0+b_0} T^2 \mu^{\beta_2} a_2 - \frac{b_0 T}{a_0+b_0} \nu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} a_2 \right. \\
& \left. + \nu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} t a_2 - \mu^{\beta_2} t^2 a_2 \right) I_{0^+}^{\alpha-\beta_2} f(\eta) \\
& + \frac{b_0}{a_0+b_0} \lambda_0 \int\limits_0^T g_0(s)ds \\
& - \nu^{\beta_1} \lambda_1 \left(\frac{b_0 T}{a_0+b_0} - t\right) \int\limits_0^T g_1(s)ds \\
& + \left(\frac{-b_0}{a_0+b_0} T^2 \mu^{\beta_2} a_2 + \frac{b_0 T}{a_0+b_0} \nu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} a_2 \right. \\
& \left. - \nu^{\beta_1} \frac{\mu^{\beta_2}}{\mu^{\beta_1}} t a_2 + \mu^{\beta_2} t^2 a_2 \right) \int\limits_0^T g_2(s)ds.
\end{aligned}
$$

We found

$$
\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds \\
& + \omega_0(t)\, b_0 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
& + \omega_1(t)\, b_1 \int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s)ds \\
& + \omega_2(t)\, b_2 \int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s)ds \\
& + \omega_0(t)\, a_0 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
& + \omega_1(t)\, a_1 \int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s)ds \\
& + \omega_2(t)\, a_2 \int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s)ds \\
& - \omega_0(t)\, \lambda_0 \int_0^T g_0(s)ds \\
& - \omega_1(t)\, \lambda_1 \int_0^T g_1(s)ds - \omega_2(t)\, \lambda_2 \int_0^T g_2(s)ds.
\end{aligned}
$$

Thus it can be written as,

$$
\begin{aligned}
u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
& + \sum_{i=0}^{2} \omega_i(t)\, b_i \int_0^T \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s)ds \\
& + \sum_{i=1}^{2} \omega_i(t)\, a_i \int_0^\eta \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s)ds \\
& - \sum_{i=0}^{2} \omega_i(t)\, \lambda_i \int_0^T g_i(s)ds.
\end{aligned}
$$

Here $\beta_0 = 0$, $\mu^{\beta_1} := \dfrac{\Gamma(3-\beta_1)}{2\left(a_1\eta^{2-\beta_1} + b_1 T^{2-\beta_1}\right)}$, $\mu^{\beta_2} := \dfrac{\Gamma(3-\beta_2)}{2\left(a_2\eta^{2-\beta_2} + b_2 T^{2-\beta_2}\right)}$ and $\nu^{\beta_1} :=$

$$\frac{\Gamma(2-\beta_1)}{a_1\eta^{1-\beta_1}+b_1T^{1-\beta_1}}. \blacksquare$$

Remark 4.0.2 *The Green function of the BVP, is defined by*

$$G(t;s)=\begin{cases}\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+G_0(t;s), & 0\le s\le t\le T,\\[2ex] G_0(t;s), & 0\le t\le s\le T.\end{cases}$$

Here
$$\begin{aligned}G_0(t;s)&=\sum_{i=0}^{2}\omega_i(t)\,b_i\frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\\&+\sum_{i=1}^{2}\omega_i(t)\,a_i\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\chi_{(0,\eta)}(s),\end{aligned}$$

$$\chi_{(a,b)}(s):=\begin{cases}1, & s\in(a,b),\\ 0, & s\notin(a,b).\end{cases}$$

Remark 4.0.3 *For $\alpha=3$, $\beta_1=1$, $\beta_2=2$ and $\eta=0$, the BVP (4.0.1)-(4.0.2) can be written as follows:*

$$u'''(t)=f\left(t,u(t),u'(t),u''(t)\right),\ \ 0\le t\le T,$$

$$\begin{cases}a_0u(0)+b_0u(T)=\lambda_0\displaystyle\int_0^T g_0(s,u(s))ds,\\ a_1u'(0)+b_1u'(T)=\lambda_1\displaystyle\int_0^T g_1(s,u(s))ds,\\ a_2u''(0)+b_2u''(T)=\lambda_2\displaystyle\int_0^T g_2(s,u(s))ds.\end{cases}$$

In this case, the Green function can be written as follows:

$$G(t;s)=\begin{cases}\dfrac{(t-s)^{2}}{\Gamma(\alpha)}+G_0(t;s), & 0\le s\le t\le T,\\[2ex] G_0(t;s), & 0\le t\le s\le T.\end{cases}$$

Here

$$
\begin{aligned}
G_0(t;s) &= \frac{b_0}{a_0+b_0}\frac{(T-s)^2}{\Gamma(\alpha)} \\
&+\left(-\frac{b_0}{a_0+b_0}\frac{b_1}{a_1+b_1}T+\frac{b_1}{a_1+b_1}t\right)\frac{T-s}{\Gamma(\alpha-1)} \\
&+\left(\frac{b_0}{a_0+b_0}\frac{b_1}{a_1+b_1}\frac{b_2}{a_2+b_2}T-\frac{b_0}{a_0+b_0}\frac{b_2}{2\,(a_2+b_2)}T^2\right. \\
&\left.-\frac{2b_1}{a_1+b_1}\frac{b_2}{2\,(a_2+b_2)}t+\frac{b_2}{2\,(a_2+b_2)}t^2\right)\frac{1}{\Gamma(\alpha-2)}.
\end{aligned}
$$

Moreover, the following case is investigated in [22]:

$$a_0=1, b_0=0, a_1=0, b_1=1, a_2=1, b_2=0.$$

In this case,

$$
G(t;s)=\begin{cases} \dfrac{(t-s)^2}{\Gamma(\alpha)}+\dfrac{t(T-s)}{\Gamma(\alpha-1)}, & 0\le s\le t\le T, \\ \dfrac{t(T-s)}{\Gamma(\alpha-1)}, & 0\le t\le s\le T. \end{cases}
$$

4.1 Existence And Uniqueness Results For $\alpha \in (2,3]$ Order FBVP

In this section, while proving the existence and uniqueness result for the fractional BVP, the Banach fixed-point theorem is used. The following space is used

$$C_\beta\left([0,T];\mathbb{R}\right) := \left\{ v \in C\left([0,T];\mathbb{R}\right) : \mathfrak{D}^{\beta_1}_{0^+}v,\ \mathfrak{D}^{\beta_2}_{0^+}v \in C\left([0,T];\mathbb{R}\right) \right\}.$$

That is equipped with the norm

$$\|v\|_\beta := \|v\|_C + \left\|\mathfrak{D}^{\beta_1}_{0^+}v\right\|_C + \left\|\mathfrak{D}^{\beta_2}_{0^+}v\right\|_C.$$

Here $\|\cdot\|_C$ is the sup norm in $C\left([0,T];\mathbb{R}\right)$.

The following notations, formulae and estimations will be used throughout the thesis.

$$\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t) = -\frac{\nu^{\beta_1}t^{1-\beta_1}}{\Gamma(2-\beta_1)}, \quad \mathfrak{D}_{0^+}^{\beta_2}\omega_1(t) = 0,$$
$$\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t) = \frac{\nu^{\beta_1}\mu^{\beta_2}t^{1-\beta_1}}{\mu^{\beta_1}\Gamma(2-\beta_1)} - 2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}, \quad \mathfrak{D}_{0^+}^{\beta_2}\omega_2(t) = -2\frac{\mu^{\beta_2}t^{2-\beta_2}}{\Gamma(3-\beta_2)}.$$

We have $|\omega_0| = \dfrac{1}{|a_0+b_0|} =: \rho_0, \quad |\omega_1(t)| \le \left|\nu^{\beta_1}\right|(|\omega_0||b_0|+1)T := \rho_1,$

$$|\omega_2(t)| \le \frac{|b_0|\left|\mu^{\beta_2}\right|}{|a_0+b_0|}T^2 + \frac{|b_0|\left|\nu^{\beta_1}\right|}{|a_0+b_0|}\frac{\left|\mu^{\beta_2}\right|}{\left|\mu^{\beta_1}\right|}T + \frac{\left|\nu^{\beta_1}\right|\left|\mu^{\beta_2}\right|}{\left|\mu^{\beta_1}\right|}T + \left|\mu^{\beta_2}\right|T^2 := \rho_2.$$
$$\widetilde{\rho}_0 = 0, \quad \left|\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t)\right| \le \frac{\left|\nu^{\beta_1}\right|T^{1-\beta_1}}{\Gamma(2-\beta_1)} := \widetilde{\rho}_1,$$
$$\left|\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t)\right| \le \frac{\left|\mu^{\beta_2}\right|\left|\nu^{\beta_1}\right|T^{1-\beta_1}}{\left|\mu^{\beta_1}\right|\Gamma(2-\beta_1)} + 2\frac{\left|\mu^{\beta_2}\right|T^{2-\beta_1}}{\Gamma(3-\beta_1)} := \widetilde{\rho}_2,$$
$$\widehat{\rho}_0 = \widehat{\rho}_1 = 0, \quad \left|\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)\right| \le 2\frac{\left|\mu^{\beta_2}\right|T^{2-\beta_2}}{\Gamma(3-\beta_2)} := \widehat{\rho}_2.$$

Also
$$\Delta_0 := \frac{T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} + \sum_{i=0}^{2}\rho_i\left(|b_i|\frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)} + |a_i|\frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\right)\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau}.$$

And
$$\Delta_1 = \frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha-\beta_1)}\left(\frac{1-\tau}{\alpha-\beta_1-\tau}\right)^{1-\tau}\|l_f\|_{1/\tau} + \sum_{i=1}^{2}\widetilde{\rho}_i\|l_f\|_{1/\tau}\left(|b_i|\frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)} + |a_i|\frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\right)\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau}.$$

And
$$\Delta_2 = \frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}\|l_f\|_{1/\tau} + \widehat{\rho}_2\|l_f\|_{1/\tau}\left(|b_2|\frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)} + |a_2|\frac{\eta^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\right)\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}.$$

Theorem 4.1.1 *Assume that*

(H_1) The function $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is jointly continuous.

(H_2) There exists a function $l_f\in L^{\frac{1}{\tau}}([0,T];\mathbb{R}^+)$ with $\tau\in(0,\min(1,\alpha-\beta_2))$ for each $(t,u_1,u_2,u_3),(t,v_1,v_2,v_3)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, such that

$$|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)|\le l_f(t)(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|).$$

(H_3) The function $g_i:[0,T]\times\mathbb{R}\to\mathbb{R}$ is jointly continuous and there exists $l_{g_i}\in L^1([0,T],\mathbb{R}^+)$ for each $(t,u),(t,v)\in[0,T]\times\mathbb{R}$, such that

$$|g_i(t,u)-g_i(t,v)|\le l_{g_i}(t)|u-v|,\ i=0,1,2.$$

If there exists

$$(\Delta_0+\Delta_1+\Delta_2)\left\|l_f\right\|_{1/\tau}+\sum_{i=0}^{2}\rho_i|\lambda_i|\left\|l_{g_i}\right\|_1+\sum_{i=1}^{2}\widetilde{\rho}_i|\lambda_i|\left\|l_{g_i}\right\|_1+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1<1. \tag{4.1.1}$$

Then the boundary value problem has a unique solution on $[0,T]$.

Proof. We used the operator $\mathfrak{F}$ to transform the BVP into a fixed point problem.The operator $\mathfrak{F}:C_\beta([0,T];\mathbb{R})\to C_\beta([0,T];\mathbb{R})$ which is defined by

$$
\begin{aligned}
&(\mathfrak{F}u(t))\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\frac{b_0}{a_0+b_0}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\nu^{\beta_1}b_1\left(\frac{b_0T}{a_0+b_0}-t\right)\int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\Bigg(\frac{b_0}{a_0+b_0}T^2\mu^{\beta_2}b_2-\frac{b_0T}{a_0+b_0}\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}b_2\\
&+\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}tb_2-\mu^{\beta_2}t^2b_2\Bigg)\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\frac{a_0}{a_0+b_0}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\nu^{\beta_1}a_1\left(\frac{b_0T}{a_0+b_0}-t\right)\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\Bigg(\frac{b_0}{a_0+b_0}T^2\mu^{\beta_2}a_2-\frac{b_0T}{a_0+b_0}\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}a_2\\
&+\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}ta_2-\mu^{\beta_2}t^2a_2\Bigg)\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\frac{b_0}{a_0+b_0}\lambda_0\int_0^T g_0(s)ds-\nu^{\beta_1}\lambda_1\left(\frac{b_0T}{a_0+b_0}-t\right)\int_0^T g_1(s)ds\\
&+\Bigg(\frac{-b_0}{a_0+b_0}T^2\mu^{\beta_2}a_2+\frac{b_0T}{a_0+b_0}\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}a_2\\
&-\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}ta_2+\mu^{\beta_2}t^2a_2\Bigg)\int_0^T g_2(s)ds.
\end{aligned}
$$

We have

$$
\begin{aligned}
&(\mathfrak{F}u)(t)\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\frac{b_0}{a_0+b_0}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\nu^{\beta_1}b_1\left(\frac{b_0T}{a_0+b_0}-t\right)\int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\left(\frac{b_0T^2\mu^{\beta_2}b_2}{a_0+b_0}-\frac{b_0T\nu^{\beta_1}\mu^{\beta_2}b_2}{(a_0+b_0)\mu^{\beta_1}}\right.\\
&\left.+\frac{\nu^{\beta_1}\mu^{\beta_2}tb_2}{\mu^{\beta_1}}-\mu^{\beta_2}t^2b_2\right)\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\frac{a_0}{a_0+b_0}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\nu^{\beta_1}a_1\left(\frac{b_0T}{a_0+b_0}-t\right)\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\left(\frac{b_0}{a_0+b_0}T^2\mu^{\beta_2}a_2-\frac{b_0T}{a_0+b_0}\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}a_2\right.\\
&\left.+\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}ta_2-\mu^{\beta_2}t^2a_2\right)\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\frac{b_0}{a_0+b_0}\lambda_0\int_0^T g_0(s,u(s))ds-\nu^{\beta_1}\lambda_1\left(\frac{b_0T}{a_0+b_0}-t\right)\int_0^T g_1(s,u(s))ds\\
&+\left(\frac{-b_0}{a_0+b_0}T^2\mu^{\beta_2}a_2+\frac{b_0T}{a_0+b_0}\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}a_2\right.\\
&\left.-\nu^{\beta_1}\frac{\mu^{\beta_2}}{\mu^{\beta_1}}ta_2+\mu^{\beta_2}t^2a_2\right)\int_0^T g_2(s,u(s))ds.
\end{aligned}
$$

That is

$$(\mathfrak{F}u)(t)$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\omega_0(t)\,b_0\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\omega_1(t)\,b_1\int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\omega_2(t)\,b_2\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\omega_0(t)\,a_0\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\omega_1(t)\,a_1\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\omega_2(t)\,a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$-\omega_0(t)\,\lambda_0\int_0^T g_0(s,u(s))ds - \omega_1(t)\,\lambda_1\int_0^T g_1(s,u(s))ds$$
$$-\omega_2(t)\,\lambda_2\int_0^T g_2(s)ds.(s,u(s))ds.$$

Thus, the expression $(\mathfrak{F}u)(t)$ can be written as,

$$(\mathfrak{F}u)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\sum_{i=0}^{2}\omega_i(t)\,b_i\int_0^T \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\sum_{i=1}^{2}\omega_i(t)\,a_i\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds \tag{4.1.2}$$
$$-\sum_{i=0}^{2}\omega_i(t)\,\lambda_i\int_0^T g_i(s,u(s))ds. \tag{4.1.3}$$

We take β_1^{-th} and β_2^{-th} fractional derivative, we have

$$
\begin{aligned}
&\mathfrak{D}_{0^+}^{\beta_1}(\mathfrak{F}u)(t)\\
&=\int_0^t \frac{(t-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\frac{\nu^{\beta_1}t^{1-\beta_1}}{\Gamma(2-\beta_1)} b_1\int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\left(\frac{\nu^{\beta_1}\mu^{\beta_2}t^{1-\beta_1}}{\mu^{\beta_1}\Gamma(2-\beta_1)}-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}\right) b_2\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\frac{\nu^{\beta_1}t^{1-\beta_1}}{\Gamma(2-\beta_1)} a_1\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\left(\frac{\nu^{\beta_1}\mu^{\beta_2}t^{1-\beta_1}}{\mu^{\beta_1}\Gamma(2-\beta_1)}-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}\right) a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&-\left(-\frac{\nu^{\beta_1}t^{1-\beta_1}}{\Gamma(2-\beta_1)}\right)\lambda_1\int_0^T g_1(s,u(s))\,ds\\
&-\left(\frac{\nu^{\beta_1}\mu^{\beta_2}t^{1-\beta_1}}{\mu^{\beta_1}\Gamma(2-\beta_1)}-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}\right)\lambda_2\int_0^T g_2(s,u(s))\,ds.
\end{aligned}
$$

Thus
$$
\begin{aligned}
&\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t)=-\frac{\nu^{\beta_1}t^{1-\beta_1}}{\Gamma(2-\beta_1)},\\
&\mathfrak{D}_{0^+}^{\beta_2}\omega_1(t)=0,\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t)=\frac{\nu^{\beta_1}\mu^{\beta_2}t^{1-\beta_1}}{\mu^{\beta_1}\Gamma(2-\beta_1)}-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)},\\
&\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)=-2\frac{\mu^{\beta_2}t^{2-\beta_2}}{\Gamma(3-\beta_2)}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&\mathfrak{D}_{0^+}^{\beta_1}(\mathfrak{F}u)(t)\\
&=\int_0^t \frac{(t-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t)\,b_1\int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t)\,b_2\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t)\,a_1\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t)\,a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds\\
&+\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t)\,\lambda_1\int_0^T g_1(s,u(s))ds\\
&+\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t)\,\lambda_2\int_0^T g_2(s,u(s))ds.
\end{aligned}
$$

It is $\mathfrak{D}_{0^+}^{\beta_1}(\mathfrak{F}u)(t)$

$$
=\int_0^t \frac{(t-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds \tag{4.1.4}
$$

$$
+\sum_{i=1}^{2}\mathfrak{D}_{0^+}^{\beta_1}\omega_i(t)\,b_i\int_0^T \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds
$$

$$
+\sum_{i=1}^{2}\mathfrak{D}_{0^+}^{\beta_1}\omega_i(t)\,a_i\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds \tag{4.1.5}
$$

$$
-\sum_{i=1}^{2}\mathfrak{D}_{0^+}^{\beta_1}\omega_i(t)\,\lambda_i\int_0^T g_i(s,u(s))ds. \tag{4.1.6}
$$

The β_2^{th} derivative is in the following:

$$\mathfrak{D}_{0^+}^{\beta_2}(\mathfrak{F}u)(t)$$
$$=\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}b_2\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)}\lambda_2\int_0^T g_2(s,u(s))ds.$$

Since $\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)=\dfrac{-2\mu^{\beta_2}t^{2-\beta_2}}{\Gamma(3-\beta_2)}$ is given, $\mathfrak{D}_{0^+}^{\beta_2}(\mathfrak{F}u)(t)$ can be written as follows:

$$\mathfrak{D}_{0^+}^{\beta_2}(\mathfrak{F}u)(t)$$
$$=\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds \tag{4.1.7}$$
$$+\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)\,b_2\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds$$
$$+\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)\,a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))ds \tag{4.1.8}$$
$$-\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)\,\lambda_2\int_0^T g_2(s,u(s))ds.$$

Since the functions f, g_0, g_1 and g_2 are jointly continuous and, the expressions $(\mathfrak{F}u)(t)$, $\mathfrak{D}_{0^+}^{\beta_1}(\mathfrak{F}u)(t)$ and $\mathfrak{D}_{0^+}^{\beta_2}(\mathfrak{F}u)(t)$ are well defined. The Banach fixed point theorem is used to show existence and uniqueness of the solution of BVP (4.0.3)-(4.0.4) Thus we need to show that $\mathfrak{F}$ is contraction. Indeed,

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t)-(\mathfrak{F}v)(t)\right| \\
&\leq \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) - f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s)) \right| ds \\
&+|\omega_0(t)||b_0| \int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \right. \\
&\left. -f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s)) \right| ds \\
&+|\omega_1(t)||b_1| \int\limits_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} \left| f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \right. \\
&\left. -f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s)) \right| ds \\
&+|\omega_2(t)||b_2| \int\limits_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} \left| f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \right. \\
&\left. -f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s)) \right| ds \\
&+|\omega_1(t)||a_1| \int\limits_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} \left| f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \right. \\
&\left. -f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s)) \right| ds \\
&+|\omega_2(t)||a_2| \int\limits_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)} \left| f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \right. \\
&\left. -f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s)) \right| ds \\
&+|\omega_0(t)||\lambda_0| \int\limits_0^T |g_0(s,u(s))-g_0(s,v(s))| ds \\
&+|\omega_1(t)||\lambda_1| \int\limits_0^T |g_1(s,u(s))-g_1(s,v(s))| ds \\
&+|\omega_2(t)||\lambda_2| \int\limits_0^T |g_2(s,u(s))-g_2(s,v(s))| ds.
\end{aligned}
$$

Thus, it can be written as follows:

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t)-(\mathfrak{F}v)(t)\right| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds \\
&+\sum_{i=0}^{2}|\omega_i(t)||b_i|\int_0^T \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds \\
&+\sum_{i=1}^{2}|\omega_i(t)||a_i|\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds \\
&+\sum_{i=0}^{2}|\omega_i(t)||\lambda_i|\int_0^T |g_i(s,u(s))-g_i(s,v(s))|ds \\
&\leq \left\|l_f\right\|_{1/\tau}\frac{T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}\|u-v\|_\beta \\
&+\Bigg[\left\|l_f\right\|_{1/\tau}\sum_{i=0}^{2}\rho_i\left(|b_i|\frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}+|a_i|\frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\right)\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \qquad (4.1.9)\\
&+\sum_{i=0}^{2}\rho_i|\lambda_i|\left\|l_{g_i}\right\|_1\Bigg]\|u-v\|_\beta \\
&=\left[\Delta_0\left\|l_f\right\|_{1/\tau}+\sum_{i=0}^{2}\rho_i|\lambda_i|\left\|l_{g_i}\right\|_1\right]\|u-v\|_\beta . \qquad (4.1.10)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\mathfrak{D}_{0^+}^{\beta_1}\omega_1(t)=-\frac{\nu^{\beta_1}t^{1-\beta_1}}{\Gamma(2-\beta_1)}, \\
&\mathfrak{D}_{0^+}^{\beta_2}\omega_1(t)=0,\mathfrak{D}_{0^+}^{\beta_1}\omega_2(t)=\frac{\nu^{\beta_1}\mu^{\beta_2}t^{1-\beta_1}}{\mu^{\beta_1}\Gamma(2-\beta_1)}-2\frac{\mu^{\beta_2}t^{2-\beta_1}}{\Gamma(3-\beta_1)},
\end{aligned}
$$

$$
\mathfrak{D}_{0^+}^{\beta_2}\omega_2(t)=-2\frac{\mu^{\beta_2}t^{2-\beta_2}}{\Gamma(3-\beta_2)}.
$$

and

We have

$$
\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u)(t)-\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}v)(t)\right| \qquad (4.1.11)\\
&\leq \int_0^t \frac{(t-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds\\
&+\left|\mathfrak{D}_{0+}^{\beta_1}\omega_1(t)\right||b_1|\int_0^T \frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds\\
&+\left|\mathfrak{D}_{0+}^{\beta_1}\omega_2(t)\right||b_2|\int_0^T \frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds\\
&+\left|\mathfrak{D}_{0+}^{\beta_1}\omega_1(t)\right||a_1|\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds\\
&+\left|\mathfrak{D}_{0+}^{\beta_1}\omega_2(t)\right||a_2|\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\\
&-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds\\
&+\left|\mathfrak{D}_{0+}^{\beta_1}\omega_1(t)\right||\lambda_1|\int_0^T |g_1(s,u(s))-g_1(s,v(s))|ds\\
&+\left|\mathfrak{D}_{0+}^{\beta_1}\omega_2(t)\right||\lambda_2|\int_0^T |g_2(s,u(s))-g_2(s,v(s))|ds.
\end{aligned}
$$

It is given as

$$
\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u)(t)-\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}v)(t)\right| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}\left|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\right. \\
&\left.-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\right|ds \\
&+\sum_{i=1}^{2}\left|\mathfrak{D}_{0+}^{\beta_1}\omega_i(t)\right||b_i|\int_0^T \frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\left|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\right. \\
&\left.-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\right|ds \\
&+\sum_{i=1}^{2}\left|\mathfrak{D}_{0+}^{\beta_1}\omega_i(t)\right||a_i|\int_0^\eta \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\left|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))\right. \\
&\left.-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\right|ds \\
&+\sum_{i=1}^{2}\left|\mathfrak{D}_{0+}^{\beta_1}\omega_i(t)\right||\lambda_i|\int_0^T |g_i(s,u(s))-g_i(s,v(s))|\,ds.
\end{aligned}
$$

By
$$
\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u)(t)-\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}v)(t)\right| \\
&\leq \frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha-\beta_1)}\left(\frac{1-\tau}{\alpha-\beta_1-\tau}\right)^{1-\tau}\left\|l_f\right\|_{1/\tau}\|u-v\|_\beta \\
&+\sum_{i=1}^{2}\widetilde{\rho}_i\left[\left\|l_f\right\|_{1/\tau}\left(|b_i|\frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}+|a_i|\frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\right)\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau}\right. \\
&\left.+|\lambda_i|\left\|l_{g_i}\right\|_1\right]\|u-v\|_\beta .
\end{aligned}
$$

That is
$$
\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u)(t)-\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}v)(t)\right| \\
&\leq\left(\Delta_1\left\|l_f\right\|_{1/\tau}+\widetilde{\rho}_1|\lambda_1|\left\|l_{g_1}\right\|_1+\widetilde{\rho}_2|\lambda_2|\left\|l_{g_1}\right\|_1\right)\|u-v\|_\beta \\
&\leq\left(\Delta_1\left\|l_f\right\|_{1/\tau}+\sum_{i=1}^{2}\widetilde{\rho}_i|\lambda_i|\left\|l_{g_i}\right\|_1\right)\|u-v\|_\beta .
\end{aligned}
$$

Similarly for β_2^{th} derivative:

$$\left|\mathfrak{D}_{0+}^{\beta_2}\left(\mathfrak{F}u\right)(t)-\mathfrak{D}_{0+}^{\beta_2}\left(\mathfrak{F}v\right)(t)\right|$$
$$=\int_0^t\frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))$$
$$-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds$$
$$+2\frac{\left|\mu^{\beta_2}\right|T^{2-\beta_2}}{\Gamma(3-\beta_2)}|b_2|\int_0^T\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))$$
$$-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds$$
$$+2\frac{\left|\mu^{\beta_2}\right|T^{2-\beta_2}}{\Gamma(3-\beta_2)}|a_2|\int_0^\eta\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))$$
$$-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds$$
$$+2\frac{\left|\mu^{\beta_2}\right|T^{2-\beta_2}}{\Gamma(3-\beta_2)}|\lambda_2|\int_0^T|g_2(s,u(s))-g_2(s,v(s))|ds.$$

Then, $$\left|\mathfrak{D}_{0+}^{\beta_2}\left(\mathfrak{F}u\right)(t)-\mathfrak{D}_{0+}^{\beta_2}\left(\mathfrak{F}v\right)(t)\right|$$
$$\leq\int_0^t\frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s)) \tag{4.1.12}$$
$$-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds$$
$$+\left|\mathfrak{D}_{0+}^{\beta_2}\omega_2(t)\right||b_2|\int_0^T\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))$$
$$-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds$$
$$+\left|\mathfrak{D}_{0+}^{\beta_2}\omega_2(t)\right||a_2|\int_0^\eta\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}\Big|f(s,u(s),\mathfrak{D}_{0^+}^{\beta_1}u(s),\mathfrak{D}_{0^+}^{\beta_2}u(s))$$
$$-f(s,v(s),\mathfrak{D}_{0^+}^{\beta_1}v(s),\mathfrak{D}_{0^+}^{\beta_2}v(s))\Big|ds$$
$$+\left|\mathfrak{D}_{0+}^{\beta_2}\omega_2(t)\right||\lambda_2|\int_0^T|g_2(s,u(s))-g_2(s,v(s))|ds.$$

$$
\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}u)(t)-\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}v)(t)\right| \\
&\leq \frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}\left\|l_f\right\|_{1/\tau}\|u-v\|_\beta \\
&+\left[\widehat{\rho}_2\left(|b_2|\frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}+|a_2|\frac{\eta^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\right)\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}\left\|l_f\right\|_{1/\tau}+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1\right]\|u-v\|_\beta \\
&=\left[\Delta_2\left\|l_f\right\|_{1/\tau}+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1\right]\|u-v\|_\beta .
\end{aligned}
$$

Here, in estimations (4.1.10), Hölder's inequality is used,

$$
\begin{aligned}
&\int_0^t l_f(s)(t-s)^{\alpha-m-1}ds \\
&\leq\left(\int_0^t\left(l_f(s)\right)^{\frac{1}{\tau}}ds\right)^{\tau}\left(\int_0^t\left((t-s)^{\alpha-m-1}\right)^{\frac{1}{1-\tau}}ds\right)^{1-\tau} \\
&=\left\|l_f\right\|_{L^{1/\tau}}\left(\frac{1-\tau}{\alpha-m-\tau}\right)^{1-\tau}t^{\alpha-m-\tau},\quad \text{if } 0<\tau<\min(1,\alpha-m).
\end{aligned}
$$

From (4.1.10), it follows that

$$
\begin{aligned}
&\|(\mathfrak{F}u)-(\mathfrak{F}v)\|_\beta \\
&\leq\Big[(\Delta_0+\Delta_1+\Delta_2)\left\|l_f\right\|_{1/\tau}+\rho_0|\lambda_0|\left\|l_{g_0}\right\|_1+\rho_1|\lambda_1|\left\|l_{g_1}\right\|_1+\rho_2|\lambda_2|\left\|l_{g_2}\right\|_1 \\
&\widetilde{\rho}_1|\lambda_1|\left\|l_{g_1}\right\|_1+\widetilde{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1\Big]\|u-v\|_\beta .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|(\mathfrak{F}u)-(\mathfrak{F}v)\|_\beta \\
&\leq\left[(\Delta_0+\Delta_1+\Delta_2)\left\|l_f\right\|_{1/\tau}+\sum_{i=0}^{2}\rho_i|\lambda_i|\left\|l_{g_i}\right\|_1\right. \\
&\left.+\sum_{i=1}^{2}\widetilde{\rho}_i|\lambda_i|\left\|l_{g_i}\right\|_1+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1\right]\|u-v\|_\beta .
\end{aligned}
$$

Thus $\mathfrak{F}$ is a contraction mapping. Then $\mathfrak{F}$ has a fixed point which is the solution of the problem (4.0.1)-(4.0.2) by the Banach fixed point theorem by

$$(\Delta_0+\Delta_1+\Delta_2)\left\|l_f\right\|_{1/\tau}+\sum_{i=0}^{2}\rho_i|\lambda_i|\left\|l_{g_i}\right\|_1+\sum_{i=1}^{2}\widetilde{\rho}_i|\lambda_i|\left\|l_{g_i}\right\|_1+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1<1.$$

■

Remark 4.1.2 *In the assumptions (H_2) if l_f is a positive constant then the condition (4.1.1) can be replaced by*

$$\begin{aligned}&\frac{l_fT^{\alpha}}{\Gamma(\alpha+1)}+l_f\sum_{i=0}^{2}\rho_i\left(|b_i|\frac{T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}+|a_i|\frac{\eta^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\\&+\frac{l_fT^{\alpha-\beta_1}}{\Gamma(\alpha-\beta_1+1)}+l_f\sum_{i=1}^{2}\widetilde{\rho}_i\left(|b_i|\frac{T^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}+|a_i|\frac{\eta^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\\&+\frac{l_fT^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2+1)}+l_f\widehat{\rho}_2\left(|b_2|\frac{T^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2+1)}+|a_2|\frac{\eta^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2+1)}\right)\\&+\sum_{i=0}^{2}\rho_i|\lambda_i|\left\|l_{g_i}\right\|_1+\sum_{i=1}^{2}\widetilde{\rho}_i|\lambda_i|\left\|l_{g_i}\right\|_1+\widehat{\rho}_2|\lambda_2|\left\|l_{g_2}\right\|_1<1.\end{aligned}$$

4.2 Existence Results For Fractional Three Point BVP

We start with the existence of solutions for BVP (4.0.1)-(4.0.2),

Theorem 4.2.1 *Assume that*

(H_4) The functions $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $g_i:[0,T]\times\mathbb{R}\to\mathbb{R}$ ($i=0,1,2$) are jointly continuous.

(H_5) There exists non-decreasing functions $\varphi:[0,\infty)\to[0,\infty)$, $\psi_i:[0,\infty)\to[0,\infty)$ and functions $l_f\in L^{\frac{1}{\tau}}([0,T],\mathbb{R}^+)$, $l_{g_i}\in L^1([0,T],\mathbb{R}^+)$ with $\tau\in(0,\min(1,\alpha-\beta_2))$ such that $i=0,1,2$ for all $t\in[0,T]$ and $u,v,w\in\mathbb{R}$;

$$|f(t,u,v,w)|\le l_f(t)\,\varphi(|u|+|v|+|w|),$$

$$|g_i(t,u)|\le l_{g_i}(t)\,\psi_i(|u|)$$

(H_6) There exists a constant $K>0$ such that

$$\frac{K}{\varphi(K)\left\|l_f\right\|_{1/\tau}(\Delta_0+\Delta_1+\Delta_2)+\sum_{i=0}^{2}(\rho_i+\widetilde{\rho}_i+\widehat{\rho}_i)|\lambda_i|\,\psi_i(K)\left\|l_{g_i}\right\|_1}>1.$$

Then BVP (4.0.1)-(4.0.2) has at least one solution on $[0,T]$.

Proof. Let $B_r := \left\{u \in C_\beta\left([0,T];\mathbb{R}\right) : \|u\|_\beta \leq r\right\}$.

Step 1: The operator $\mathfrak{F} : C_\beta\left([0,T];\mathbb{R}\right) \to C_\beta\left([0,T];\mathbb{R}\right)$ is defined by (4.1.2) maps B_r into bounded set.For all $u \in B_r$, we get

$$\begin{aligned}
&\left|\left(\mathfrak{F}u\right)(t)\right| \\
&\leq \frac{\varphi(r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|l_f(s)\right|ds \\
&+\varphi(r)\left|\omega_0\right|\left|b_0\right|\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|l_f(s)\right|ds \\
&+\varphi(r)\left|\omega_1(t)\right|\left|b_1\right|\frac{1}{\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1-1}\left|l_f(s)\right|ds \\
&+\varphi(r)\left|\omega_2(t)\right|\left|b_2\right|\frac{1}{\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2-1}\left|l_f(s)\right|ds.
\end{aligned}$$

That can be

$$\begin{aligned}
&\left|\left(\mathfrak{F}u\right)(t)\right| \\
&\leq \frac{\varphi(r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|l_f(s)\right|ds \\
&+\varphi(r)\rho_0\left|b_0\right|\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|l_f(s)\right|ds \\
&+\varphi(r)\rho_1\left|b_1\right|\frac{1}{\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1-1}\left|l_f(s)\right|ds \\
&+\varphi(r)\rho_2\left|b_2\right|\frac{1}{\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2-1}\left|l_f(s)\right|ds.
\end{aligned}$$

Then,

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t)\right| \\
&\leq \frac{\varphi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|l_f(s)\right| ds \\
&+ \varphi(r) \sum_{i=0}^{2} \rho_i \left|b_i\right| \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^T (T-s)^{\alpha-\beta_i-1} \left|l_f(s)\right| ds \\
&+ \varphi(r) \sum_{i=1}^{2} \rho_i \left|a_i\right| \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^\eta (\eta-s)^{\alpha-\beta_i-1} \left|l_f(s)\right| ds \\
&+ \sum_{i=0}^{2} \rho_i \left|\lambda_i\right| \psi_i(r) \int_0^T \left|l_{g_i}(s)\right| ds.
\end{aligned}
$$

By using Hölder's inequality, we get

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t)\right| \\
&\leq \varphi(r) \left\|l_f\right\|_{1/\tau} \frac{T^{\alpha-\tau}}{\Gamma(\alpha)} \left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \\
&+ \varphi(r) \left\|l_f\right\|_{1/\tau} \sum_{i=0}^{2} \rho_i \left|b_i\right| \frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)} \left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \\
&+ \varphi(r) \left\|l_f\right\|_{1/\tau} \sum_{i=1}^{2} \rho_i \left|a_i\right| \frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)} \left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \\
&+ \sum_{i=0}^{2} \rho_i \left|\lambda_i\right| \psi_i(r) \left\|l_{g_i}\right\|_1 .
\end{aligned}
$$

That is

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t)\right| \\
&\leq \varphi(r) \left\|l_f\right\|_{1/\tau} \left(\frac{T^{\alpha-\tau}}{\Gamma(\alpha)} \left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \right. \\
&+ \sum_{i=0}^{2} \rho_i \left|b_i\right| \frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)} \left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \\
&\left. + \sum_{i=1}^{2} \rho_i \left|a_i\right| \frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)} \left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \right) \\
&+ \sum_{i=0}^{2} \rho_i \left|\lambda_i\right| \psi_i(r) \left\|l_{g_i}\right\|_1 \\
&= \varphi(r) \left\|l_f\right\|_{1/\tau} \Delta_0 + \sum_{i=0}^{2} \rho_i \left|\lambda_i\right| \psi_i(r) \left\|l_{g_i}\right\|_1 .
\end{aligned}
$$

Also for all $u \in B_r$, β_1^{th} derivative can be written as:

$$\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u)(t)\right| \\
&\leq \varphi(r)\left\|l_f\right\|_{1/\tau} \frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha-\beta_1)}\left(\frac{1-\tau}{\alpha-\beta_1-\tau}\right)^{1-\tau} \\
&+\varphi(r)\left\|l_f\right\|_{1/\tau} \sum_{i=1}^{2} \widetilde{\rho}_i\left|b_i\right| \frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \\
&+\varphi(r)\left\|l_f\right\|_{1/\tau} \sum_{i=1}^{2} \widetilde{\rho}_i\left|a_i\right| \frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \\
&+\sum_{i=1}^{2} \widetilde{\rho}_i\left|\lambda_i\right| \psi_i(r)\left\|l_{g_i}\right\|_1 .
\end{aligned}$$

That is

$$\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u)(t)\right| \\
&\leq \varphi(r)\left\|l_f\right\|_{1/\tau}\left(\frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha-\beta_1)}\left(\frac{1-\tau}{\alpha-\beta_1-\tau}\right)^{1-\tau}\right. \\
&+\sum_{i=1}^{2} \widetilde{\rho}_i\left|b_i\right| \frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau} \\
&\left.+\sum_{i=1}^{2} \widetilde{\rho}_i\left|a_i\right| \frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau}\right) \\
&+\sum_{i=1}^{2} \widetilde{\rho}_i\left|\lambda_i\right| \psi_i(r)\left\|l_{g_i}\right\|_1 \\
&=\varphi(r)\left\|l_f\right\|_{1/\tau} \Delta_1+\sum_{i=1}^{2} \widetilde{\rho}_i\left|\lambda_i\right| \psi_i(r)\left\|l_{g_i}\right\|_1 .
\end{aligned}$$

Similarly β_2^{th} derivative is given as follows:

$$\begin{aligned}
&\left|\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}u)(t)\right| \\
&\leq \varphi(r)\left\|l_f\right\|_{1/\tau} \frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau} \\
&+\varphi(r)\left\|l_f\right\|_{1/\tau} \widehat{\rho}_2 \frac{T^{\alpha-\beta_2-\tau}\left|b_2\right|}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau} \\
&+\varphi(r)\left\|l_f\right\|_{1/\tau} \widehat{\rho}_2 \frac{\eta^{\alpha-\beta_2-\tau}\left|b_2\right|}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau} \\
&+\widehat{\rho}_2\left|\lambda_2\right| \psi_2(r)\left\|l_{g_2}\right\|_1 .
\end{aligned}$$

That is
$$\left|\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}u)(t)\right|$$
$$\leq \varphi(r)\left\|l_f\right\|_{1/\tau}\left(\frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}\right.$$
$$+\widehat{\rho}_2\frac{T^{\alpha-\beta_2-\tau}|b_2|}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}$$
$$\left.+\widehat{\rho}_2\frac{\eta^{\alpha-\beta_2-\tau}|b_2|}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}\right)$$
$$+\widehat{\rho}_2|\lambda_2|\psi_2(r)\left\|l_{g_2}\right\|_1$$
$$=\varphi(r)\left\|l_f\right\|_{1/\tau}\Delta_2+\widehat{\rho}_2|\lambda_2|\psi_2(r)\left\|l_{g_2}\right\|_1.$$

Thus, we have
$$\|(\mathfrak{F}u)\|_\beta$$
$$\leq \varphi(r)\left\|l_f\right\|_{1/\tau}(\Delta_0+\Delta_1+\Delta_2)$$
$$+(\rho_0+\widetilde{\rho}_0+\widehat{\rho}_0)|\lambda_0|\psi_0(r)\left\|l_{g_0}\right\|_1$$
$$+(\rho_1+\widetilde{\rho}_1+\widehat{\rho}_1)|\lambda_1|\psi_1(r)\left\|l_{g_1}\right\|_1$$
$$+(\rho_2+\widetilde{\rho}_2+\widehat{\rho}_2)|\lambda_2|\psi_2(r)\left\|l_{g_2}\right\|_1.$$

Then,
$$\|(\mathfrak{F}u)\|_\beta$$
$$\leq \varphi(r)\left\|l_f\right\|_{1/\tau}(\Delta_0+\Delta_1+\Delta_2)+\sum_{i=0}^{2}(\rho_i+\widetilde{\rho}_i+\widehat{\rho}_i)|\lambda_i|\psi_i(r)\left\|l_{g_i}\right\|_1$$
$$=\varphi(r)\left\|l_f\right\|_{1/\tau}\left(\sum_{i=0}^{2}\Delta_i\right)+\sum_{i=0}^{2}(\rho_i+\widetilde{\rho}_i+\widehat{\rho}_i)|\lambda_i|\psi_i(r)\left\|l_{g_i}\right\|_1.$$

Here,
$$\Delta_0:=\frac{T^{\alpha-\tau}}{\Gamma(\alpha)}\left(\frac{1-\tau}{\alpha-\tau}\right)^{1-\tau}$$
$$+\sum_{i=0}^{2}\rho_i\left(|b_i|\frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}+|a_i|\frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\right)\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau}.$$

And $\Delta_1 := \frac{T^{\alpha-\beta_1-\tau}}{\Gamma(\alpha-\beta_1)}\left(\frac{1-\tau}{\alpha-\beta_1-\tau}\right)^{1-\tau}\left\|l_f\right\|_{1/\tau}$
$$+\sum_{i=1}^{2}\widetilde{\rho}_i\left\|l_f\right\|_{1/\tau}\left(|b_i|\frac{T^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}+|a_i|\frac{\eta^{\alpha-\beta_i-\tau}}{\Gamma(\alpha-\beta_i)}\right)\left(\frac{1-\tau}{\alpha-\beta_i-\tau}\right)^{1-\tau}.$$

Also $\Delta_2 := \frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}\left\|l_f\right\|_{1/\tau}$
$$+\widehat{\rho}_2\left\|l_f\right\|_{1/\tau}\left(|b_2|\frac{T^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}+|a_2|\frac{\eta^{\alpha-\beta_2-\tau}}{\Gamma(\alpha-\beta_2)}\right)\left(\frac{1-\tau}{\alpha-\beta_2-\tau}\right)^{1-\tau}.$$

Step 2: The given $\{\mathfrak{F}u : u\in B_r\}$, $\left\{\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u) : u\in B_r\right\}$, $\left\{\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}u) : u\in B_r\right\}$ families are equicontinuous. With the help of the continuity of $\omega_i(t)$ and assumption (H_5), we get

$$
\begin{aligned}
&|(\mathfrak{F}u)(t_2)-(\mathfrak{F}u)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)}\varphi(r)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}l_f(s)\,ds \\
&+\frac{1}{\Gamma(\alpha)}\varphi(r)\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)l_f(s)\,ds \\
&+\varphi(r)|\omega_0(t_2)-\omega_0(t_1)||b_0|\int_0^T\frac{(T-s)^{\alpha-\beta_0-1}}{\Gamma(\alpha-\beta_0)}l_f(s)\,ds \\
&+\varphi(r)|\omega_1(t_2)-\omega_1(t_1)||b_1|\int_0^T\frac{(T-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}l_f(s)\,ds \\
&+\varphi(r)|\omega_2(t_2)-\omega_2(t_1)||b_2|\int_0^T\frac{(T-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}l_f(s)\,ds \\
&+\varphi(r)|\omega_1(t_2)-\omega_1(t_1)||a_1|\int_0^\eta\frac{(\eta-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)}l_f(s)\,ds \\
&+\varphi(r)|\omega_2(t_2)-\omega_2(t_1)||a_2|\int_0^\eta\frac{(\eta-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha-\beta_2)}l_f(s)\,ds \\
&+|\omega_0(t_2)-\omega_0(t_1)||\lambda_0|\,\psi_0(r)\left\|l_{g_0}\right\|_1 \\
&+|\omega_1(t_2)-\omega_1(t_1)||\lambda_1|\,\psi_1(r)\left\|l_{g_1}\right\|_1 \\
&+|\omega_2(t_2)-\omega_2(t_1)||\lambda_2|\,\psi_2(r)\left\|l_{g_2}\right\|_1 \\
&\to 0 \quad \text{as} \quad t_2\to t_1.
\end{aligned}
$$

Then it can be written as,

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t_2)-(\mathfrak{F}u)(t_1)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\varphi(r)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}l_f(s)\,ds \\
&+\frac{1}{\Gamma(\alpha)}\varphi(r)\int_{0}^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)l_f(s)\,ds \\
&+\varphi(r)\sum_{i=0}^{2}\left|\omega_i(t_2)-\omega_i(t_1)\right|\left|b_i\right|\int_{0}^{T}\frac{(T-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}l_f(s)\,ds \\
&+\varphi(r)\sum_{i=1}^{2}\left|\omega_i(t_2)-\omega_i(t_1)\right|\left|a_i\right|\int_{0}^{\eta}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}l_f(s)\,ds \\
&+\sum_{i=0}^{2}\left|\omega_i(t_2)-\omega_i(t_1)\right|\left|\lambda_i\right|\psi_i(r)\left\|l_{g_i}\right\|_1 \\
&\to 0 \quad \text{as} \quad t_2\to t_1.
\end{aligned}
$$

Therefore, the family$\{\mathfrak{F}u : u\in B_r\}$ is equi-continuous. In similar way, the families $\left\{\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u) : u\in B_r\right\}$ and $\left\{\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}u) : u\in B_r\right\}$ are equicontinuous.

By the Arzela–Ascoli theorem, in $C([0,T];\mathbb{R})$, the family sets $\{\mathfrak{F}u : u\in B_r\}$, and $\left\{\mathfrak{D}_{0+}^{\beta_1}(\mathfrak{F}u) : u\in B_r\right\}$ and $\left\{\mathfrak{D}_{0+}^{\beta_2}(\mathfrak{F}u) : u\in B_r\right\}$ are relatively compact. Thus, $\mathfrak{F}(B_r)$ is a relatively compact subset of $C_\beta([0,T];\mathbb{R})$. Then, $\mathfrak{F}$ operator is compact.

Step 3: $\mathfrak{F}$ has a fixed point in $W=\left\{u\in C_\beta([0,T];\mathbb{R}) : \|u\|_\beta < K\right\}$. Lets get $u=\lambda(\mathfrak{F}u)$ for $0<\lambda<1$. Then

$$
\begin{aligned}
&\|u\|_{\beta}\\
&=\|\lambda(\mathfrak{F}u)\|_{\beta}\leq\varphi\left(\|u\|_{\beta}\right)\left\|l_{f}\right\|_{1/\tau}(\Delta_{0}+\Delta_{1}+\Delta_{2})\\
&+(\rho_{0}+\widetilde{\rho}_{0}+\widehat{\rho}_{0})\,|\lambda_{0}|\,\psi_{0}\left(\|u\|_{\beta}\right)\left\|l_{g_{0}}\right\|_{1}\\
&+(\rho_{1}+\widetilde{\rho}_{1}+\widehat{\rho}_{1})\,|\lambda_{1}|\,\psi_{1}\left(\|u\|_{\beta}\right)\left\|l_{g_{1}}\right\|_{1}\\
&+(\rho_{2}+\widetilde{\rho}_{2}+\widehat{\rho}_{2})\,|\lambda_{2}|\,\psi_{2}\left(\|u\|_{\beta}\right)\left\|l_{g_{2}}\right\|_{1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|u\|_{\beta}\\
&=\|\lambda(\mathfrak{F}u)\|_{\beta}\leq\varphi\left(\|u\|_{\beta}\right)\left\|l_{f}\right\|_{1/\tau}(\Delta_{0}+\Delta_{1}+\Delta_{2})\\
&+\sum_{i=0}^{2}(\rho_{i}+\widetilde{\rho}_{i}+\widehat{\rho}_{i})\,|\lambda_{i}|\,\psi_{i}\left(\|u\|_{\beta}\right)\left\|l_{g_{i}}\right\|_{1}.
\end{aligned}
$$

In other words,

$$
\frac{\|u\|_{\beta}}{\varphi\left(\|u\|_{\beta}\right)\left\|l_{f}\right\|_{1/\tau}(\Delta_{0}+\Delta_{1}+\Delta_{2})+\sum_{i=0}^{2}(\rho_{i}+\widetilde{\rho}_{i}+\widehat{\rho}_{i})\,|\lambda_{i}|\,\psi_{i}\left(\|u\|_{\beta}\right)\left\|l_{g_{i}}\right\|_{1}}\leq 1.
$$

As we know, there exists $K>0$ such that $K>\|u\|_{\beta}$ and also we have

$$
\frac{K}{\varphi(K)\left\|l_{f}\right\|_{1/\tau}\left(\sum_{i=0}^{2}\Delta_{i}\right)+\sum_{i=0}^{2}(\rho_{i}+\widetilde{\rho}_{i}+\widehat{\rho}_{i})\,|\lambda_{i}|\,\psi_{i}(K)\left\|l_{g_{i}}\right\|_{1}}>1.
$$

Then for each $u\in\partial W$, we get $u\neq\lambda(\mathfrak{F}u)$. The operator $\mathfrak{F}:\overline{W}\rightarrow C_{\beta}([0,T];\mathbb{R})$ is known to be continuous and compact, from Theorem 4.2.2, in other words, $\mathfrak{F}$ has fixed point in $\overline{W}$. ■

Chapter 5

DE WITH p-LAPLACIAN OPERATOR

5.1 Introduction

The p-laplacian operator is used in mechanics, dynamical systems and related fields of mathematical modeling. For some recent development on these topic, see [6], [15], [26], [30], [33], [37], [48], [56], [72], [59], [60], [63], [85] and references therein.

However, there are just few studies about FDEs with irregular boundary conditions and p-laplacian operator. More detailly, one can see [58], [90].

In [59], the following non-linear fractional impulsive differential equation is studied which is given as follows, where $0 < \alpha,\ \beta \leq 1,\ 1 < \alpha + \beta \leq 2,\ \phi_p$ is p-Laplacian operator, $f \in C([0,1] \times R, R)\,, u_0, u_1 \in R, k = 1,2,...,m, b_k \in R, I_k \in C(R,R)\,, J \in [0,1]\,,$ $0 = t_0 < t_1 < ... < t_m < t_{m+1} = 1,\ J' = J \backslash \{t_1,...,t_m\}\,,\ \Delta u(t_k) = u\left(t_k^+\right) - u\left(t_k^-\right)$. The given $u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ are right and left limits, respectively. The authors Liu, Lu, Szanto are applied some standard fixed point theorems to find new results of existence and uniqueness of the problem.

Liu and Zhi, in [90], studied the existence of positive solutions for non-local BVP of fractional DEs with p-Laplacian operator which is given in the following:

$$\left(\varphi_p\left(D_{0+}^{\alpha}u(t)\right)\right)'' = f(t,u(t),D_{0+}^{\beta}u(t)),\ t\in(0,1)$$
$$u(0) = u''(0) = 0,\ u_1(0) = \int_0^1 g(s)u(s)ds,$$
$$\left(\varphi_p\left(D_{0+}^{\alpha}u(0)\right)\right)' = \lambda_1\left(\varphi_p\left(D_{0+}^{\alpha}u(\xi_1)\right)\right)',$$
$$\varphi_p\left(D_{0+}^{\alpha}u(1)\right) = \lambda_2\left(\varphi_p\left(D_{0+}^{\alpha}u(\xi_2)\right)\right).$$

They used fixed point theorem in a cone to find the multiple solution of the BVP.

5.2 Fractional Differential Equation With p-Laplacian Operator

In this section, we focus on the existence of solutions of FDE with p-Laplacian operator, irregular and integral boundary conditions,

$$D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}u(t)\right) = f(t,u(t),D_{0+}^{\gamma}u(t)). \tag{5.2.1}$$

With

$$u'(0)+(-1)^{\theta}u'(1)+bu(1) = \int_0^1 g(s,u(s))ds, \tag{5.2.2}$$
$$u(0)+(-1)^{\theta+1}u(1) = \int_0^1 h(s,u(s))ds,$$
$$D_{0+}^{\alpha}u(0) = 0,$$
$$D_{0+}^{\alpha}u(1) = -\lambda D_{0+}^{\alpha}u(\eta).$$

Here $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the Caputo fractional derivatives with $1<\alpha\leq 2$, $1<\beta\leq 2$, $2<\alpha+\beta\leq 4$ and λ is non-negative parameter. The functions f, g, h are continuous. By Green's functions and the fixed point theorems, the existence and uniqueness results of the solutions are stated and proved.

Lemma 5.2.1 *Let $f,g,h\in C(0,1)$, and $1<\alpha\leq 2$. The following fractional boundary value problem:*

$$D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}u(t)\right) = f(t), \tag{5.2.3}$$

For

$$\begin{cases} u^{'}(0) + (-1)^{\theta} u^{'}(1) + bu(1) = \int\limits_0^1 g(s)ds, \\ u(0) + (-1)^{\theta+1} u(1) = \int\limits_0^1 h(s)ds \end{cases} \tag{5.2.4}$$

$$D_{0+}^{\alpha}u(0) = 0, \tag{5.2.5}$$

Then

$$D_{0+}^{\alpha}u(1) = -\lambda D_{0+}^{\alpha}u(\eta).$$

We get

$$(\mathscr{T}u)(t) = \int\limits_0^t G(t,s)\phi_q\left(\int\limits_0^1 H(t,\tau)f(\tau)d\tau\right)ds + \varepsilon_1 + \varepsilon_2 t$$

$$\varepsilon_1 = \frac{-((-1)^{\theta} - 1 - b)}{/b} + \frac{((-1)^{\theta})}{/b}]$$

$$\varepsilon_2 = -1 - \frac{((-1)^{\theta} - 1)}{((-1)^{\theta} + 1 + b)}.$$

Proof. By applying β^{th} integral to both sides of (5.2.3), we get

$$D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}u(t)\right) = f(t),$$

$$\phi_p\left(D_{0+}^{\alpha}u(t)\right) = I_{0+}^{\beta}f(t) - b_1 - b_2 t.$$

Written as,

$$\phi_p\left(D_{0+}^{\alpha}u(t)\right) = \int\limits_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f(s)ds - b_1 - b_2 t, \quad b_1, b_2 \in R.$$

That is

$$\begin{aligned} D_{0+}^{\alpha}u(t) &= \phi_q\left(I_{0+}^{\beta}f(t) - b_1 - b_2 t\right) \\ &= \phi_q\left(\int\limits_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f(s)ds - b_1 - b_2 t\right). \end{aligned}$$

We used the boundary conditions $D_{0+}^{\alpha}u(0) = 0$, and $D_{0+}^{\alpha}u(1) = -\lambda D_{0+}^{\alpha}u(\eta)$. Then we get

$$\phi_q(-b_1) = 0 \Longrightarrow b_1 = 0.$$

Then
$$D_{0+}^{\alpha}u(1) = \phi_q\left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds - b_2\right)$$
$$= -\lambda\phi_q\left(\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds - b_2\eta\right).$$

That means,
$$\phi_q\left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds - b_2\right)$$
$$= \phi_q\left(-\lambda^{\frac{1}{q-1}}\left(\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds - b_2\eta\right)\right)$$
$$= \phi_q\left(-\lambda^{p-1}\left(\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds - b_2\eta\right)\right)$$
$$= \phi_q\left(-\lambda^{p-1}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds + \lambda^{p-1}b_2\eta\right).$$

Since ϕ_p is one-to-one,
$$I_{0+}^{\beta}f(1) - b_2 = -\lambda^{p-1}\left(I_{0+}^{\beta}f(\eta) - b_2\eta\right)$$
$$= -\lambda^{p-1}I_{0+}^{\beta}f(\eta) + \lambda^{p-1}b_2\eta.$$

That is
$$\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds - b_2 = -\lambda^{p-1}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds + \lambda^{p-1}b_2\eta.$$
$$I_{0+}^{\beta}f(1) + \lambda^{p-1}I_{0+}^{\beta}f(\eta) = \left(1+\lambda^{p-1}\eta\right)b_2.$$
.
$$\left(1+\lambda^{p-1}\eta\right)b_2 = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds + \lambda^{p-1}\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds.$$

.

Therefore

$$b_2 = \frac{1}{(1+\lambda^{p-1}\eta)} I_{0+}^{\beta} f(1) + \frac{\lambda^{p-1}}{(1+\lambda^{p-1}\eta)} I_{0+}^{\beta} f(\eta)$$

$$= \frac{1}{(1+\lambda^{p-1}\eta)} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds$$

$$+ \frac{\lambda^{p-1}}{(1+\lambda^{p-1}\eta)} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds.$$

Since $\phi_p\left(D_{0+}^{\alpha} u(t)\right) = I_{0+}^{\beta} f(t) - b_1 - b_2 t$, we have

$$\phi_p\left(D_{0+}^{\alpha} u(t)\right) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,) ds$$

$$- \frac{t}{(1+\lambda^{p-1}\eta)} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds$$

$$- \frac{t\lambda^{p-1}}{(1+\lambda^{p-1}\eta)} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds$$

$$= \int_0^1 H(t,s) f(s) ds.$$

We get $$D_{0+}^{\alpha} u(t) = \phi_q\left(\int_0^1 H(t,s) f(s) ds\right).$$

Also $$u(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(t,s) f(s) ds\right) d\tau - c_1 - c_2 t. \qquad (5.2.6)$$

Its derivative can be written as follows:

$$u^{'}(t) = \int_0^t \frac{(t-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q\left(\int_0^1 H(t,s) f(s) ds\right) d\tau - c_2. \qquad (5.2.7)$$

To find c_1 and c_2, the boundary conditions are used. By $u^{'}(0) + (-1)^{\theta} u^{'}(1) + bu(1) = \int_0^1 g(s) ds$, we get

$$-c_2+(-1)^{\theta}\left[\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau-c_2\right]$$
$$+b\left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau-c_1-c_2\right]$$
$$=\int_0^1 g(s)ds.$$

It is

$$c_2\left(1+(-1)^{\theta}+b\right)$$
$$=(-1)^{\theta}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+b\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$-bc_1-\int_0^1 g(s)ds.$$

Then

$$c_2=\frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{b}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$-\frac{b}{\left(1+(-1)^{\theta}+b\right)}c_1-\frac{1}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 g(s)ds. \qquad (5.2.8)$$

By $u(0)+(-1)^{\theta+1}u(1)=\int_0^1 h(s)ds$, we have

$$-c_1+(-1)^{\theta+1}\left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau-c_1-c_2\right]$$
$$=\int_0^1 h(s)ds,$$
$$-c_1\left(1+(-1)^{\theta+1}\right)-(-1)^{\theta+1}c_2$$
$$+(-1)^{\theta+1}\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$=\int_0^1 h(s)ds.$$

It is

$$c_2=-c_1\frac{\left(1+(-1)^{\theta+1}\right)}{(-1)^{\theta+1}}+\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau \tag{5.2.9}$$
$$-\frac{1}{(-1)^{\theta+1}}\int_0^1 h(s)ds. \tag{5.2.10}$$

If we combine (5.2.8) and (5.2.9),

$$c_2=\left[-c_1\frac{\left(1+(-1)^{\theta+1}\right)}{(-1)^{\theta+1}}+\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\right.$$
$$\left.-\frac{1}{(-1)^{\theta+1}}\int_0^1 h(s)ds\right]$$
$$=\left[\frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\right.$$
$$+\frac{b}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$\left.-\frac{b}{\left(1+(-1)^{\theta}+b\right)}c_1-\frac{1}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 g(s)ds\right].$$

That is

$$-c_1\frac{\left(1+(-1)^{\theta+1}\right)}{(-1)^{\theta+1}}+c_1\frac{b}{\left(1+(-1)^{\theta}+b\right)}$$
$$=\left(\frac{b}{\left(1+(-1)^{\theta}+b\right)}-1\right)\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{1}{(-1)^{\theta+1}}\int_0^1 h(s)ds-\frac{1}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 g(s)ds.$$

Then

$$c_1\left(-\frac{\left(1+(-1)^{\theta+1}\right)}{(-1)^{\theta+1}}+\frac{b}{\left(1+(-1)^{\theta}+b\right)}\right)$$
$$=\left(\frac{b-1-(-1)^{\theta}-b}{1+(-1)^{\theta}+b}\right)\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{1}{(-1)^{\theta+1}}\int_0^1 h(s)ds-\frac{1}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 g(s)ds.$$

It can also be written as

$$c_1\left(\frac{b}{1+(-1)^{\theta}+b(-1)^{\theta}}\right)$$
$$=\left(\frac{-1-(-1)^{\theta}}{1+(-1)^{\theta}+b}\right)\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau$$
$$+\frac{1}{(-1)^{\theta+1}}\int_0^1 h(s)ds-\frac{1}{\left(1+(-1)^{\theta}+b\right)}\int_0^1 g(s)ds.$$

Then, we get

$$c_1 = \left(\frac{1+(-1)^{\theta}+b(-1)^{\theta}}{b} \right)$$

$$\left[\left(\frac{-1-(-1)^{\theta}}{1+(-1)^{\theta}+b} \right) \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau \right.$$

$$+ \frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau$$

$$\left. + \frac{1}{(-1)^{\theta+1}} \int_0^1 h(s)ds - \frac{1}{\left(1+(-1)^{\theta}+b\right)} \int_0^1 g(s)ds \right].$$

Therefore, c_1 is written as:

$$c_1 = \frac{1}{b} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau$$

$$- \frac{\left(1+(-1)^{\theta}\right)}{b} \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau$$

$$+ \frac{\left((-1)^{\theta+1}-1-b\right)}{b} \int_0^1 h(s)ds - \frac{(-1)^{\theta}}{b} \int_0^1 g(s)ds.$$

From (5.2.8),

$$c_2 = \frac{(-1)^{\theta}}{\left(1+(-1)^{\theta}+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau$$

$$+ \frac{b}{\left(1+(-1)^{\theta}+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau$$

$$- \frac{b}{\left(1+(-1)^{\theta}+b\right)} \left[\frac{1}{b} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau \right.$$

$$- \frac{\left(1+(-1)^{\theta}\right)}{b} \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s)f(s)ds \right) d\tau$$

$$\left. + \frac{\left((-1)^{\theta+1}-1-b\right)}{b} \int_0^1 h(s)ds - \frac{(-1)^{\theta}}{b} \int_0^1 g(s)ds \right]$$

$$- \frac{1}{\left(1+(-1)^{\theta}+b\right)} \int_0^1 g(s)ds.$$

That is

$$c_2 = \frac{(-1)^\theta}{\left(1+(-1)^\theta+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s) f(s) ds \right) d\tau$$
$$+ \frac{b}{\left(1+(-1)^\theta+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s) f(s) ds \right) d\tau$$
$$- \frac{1}{\left(1+(-1)^\theta+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s) f(s) ds \right) d\tau$$
$$+ \left(\frac{1+(-1)^\theta}{1+(-1)^\theta+b} \right) \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s) f(s) ds \right) d\tau$$
$$+ \int_0^1 h(s) ds + \frac{(-1)^\theta}{\left(1+(-1)^\theta+b\right)} \int_0^1 g(s) ds$$
$$- \frac{1}{\left(1+(-1)^\theta+b\right)} \int_0^1 g(s) ds.$$

It can be written as

$$c_2 = \frac{\left((-1)^\theta - 1\right)}{\left(1+(-1)^\theta+b\right)} \int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(\tau,s) f(s) ds \right) d\tau$$
$$+ \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau,s) f(s) ds \right) d\tau$$
$$+ \int_0^1 h(s) ds + \frac{\left((-1)^\theta - 1\right)}{\left(1+(-1)^\theta+b\right)} \int_0^1 g(s) ds.$$

We know $u(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(t,s) f(s) ds \right) d\tau - c_1 - c_2 t$, by inserting c_1 and c_2 into $u(t)$, we get

$$
\begin{aligned}
&u(t)\\
&=\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\\
&+\left(-\frac{1}{b}-\frac{\left((-1)^{\theta}-1\right)t}{\left(1+(-1)^{\theta}+b\right)}\right)\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\\
&+\left(\frac{\left(1+(-1)^{\theta}\right)}{b}-t\right)\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\\
&+\left(\frac{-\left((-1)^{\theta+1}-1-b\right)}{b}-t\right)\int_0^1 h(s)ds\\
&+\left(\frac{(-1)^{\theta}}{b}-\frac{\left((-1)^{\theta}-1\right)t}{\left(1+(-1)^{\theta}+b\right)}\right)\int_0^1 g(s)ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
u(t)=&\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\\
&+\left(-\frac{1}{b(-1)^{\theta}}\right)\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\\
&+\left(\frac{1+(-1)^{\theta}-b}{b}\right)\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,s)f(s)ds\right)d\tau\\
&+\left(\frac{(-1)^{\theta}+1}{b}\right)\int_0^1 h(s)ds\\
&+\left(\frac{1}{b}\right)\int_0^1 g(s)ds.
\end{aligned}
$$

Here $t\in[0,1]$. ■

Lemma 5.2.2 *The Green functions $G(t,s)$ and $H(t,s)$ are continuous on $[0,1]\times[0,1]$ and $H(t,s)$ satisies the following properties:*

1. $H(t,s)\le 0;$ *for* $t,s\in[0,1]$,

2. $H(t,s)\ge H(s,s);$ *for* $t,s\in[0,1]$,

3. The Green's function $H(t,s)$ satisfies

$$0 \leq \int_0^1 |H(t,s)|\, ds \leq \frac{\left(1+\lambda^{p-1}\right)}{(1+\lambda^{p-1}\eta)\,\Gamma(\beta+1)},$$

Proof. The proofs of properties (1)-(2) are given in [72]. Thus we will prove property 3 for any $t,s \in [0,1]$. The Green's function $H(t,s)$ is not positive, then,

$$\begin{aligned}
0 &\leq \int_0^1 |H(t,s)|\, ds \\
&= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds + \frac{t}{(1+\lambda^{p-1}\eta)} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&+ \frac{t\lambda^{p-1}}{(1+\lambda^{p-1}\eta)} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&\leq \int_0^1 |H(s,s)|\, ds \\
&\leq \frac{s}{(1+\lambda^{p-1}\eta)} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} ds + \frac{s\lambda^{p-1}}{(1+\lambda^{p-1}\eta)} \int_0^\eta \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} ds \\
&\leq \frac{\left(1+\lambda^{p-1}\right)}{(1+\lambda^{p-1}\eta)} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\
&\leq \frac{\left(1+\lambda^{p-1}\right)}{(1+\lambda^{p-1}\eta)\,\Gamma(\beta+1)}.
\end{aligned}$$

5.3 Existence And Uniqueness Results With $p-$Laplacian

■

The Banach fixed point theorem is used to state and prove the existence and uniqueness results of fractional BVP (5.2.1)-(5.2.2). We study on C_γ space:

$$C_\gamma([0,1],R) = \left\{u \in C([0,1],R),\ D_{0+}^{\gamma} u \in C([0,1],R)\right\},$$

is given in the form

$$\|u\|_\gamma = \|u\|_c + \left\|D_{0+}^{\gamma} u\right\|_c.$$

Here $\|.\|_c$ is the sup norm in $C([0,1],R)$. We use

$$\Delta_1 = \frac{1}{\Gamma(\alpha+1)}\left(\frac{2+|b|+\alpha}{|b|}\right),$$
$$\Delta_2 = \frac{2}{(2+|b|)\Gamma(2-\gamma)\Gamma(\alpha-\gamma)} + \frac{(1+\Gamma(2-\gamma))}{\Gamma(2-\gamma)\Gamma(\alpha-\gamma+1)},$$
$$\Delta_{h_1} = \frac{2}{|b|},$$
$$\Delta_{h_2} = \frac{1}{\Gamma(2-\gamma)},$$
$$\Delta_{g_1} = \frac{1}{|b|},$$
$$\Delta_{g_2} = \frac{2}{\Gamma(2-\gamma)(2+|b|)}.$$

By using the following conditions, we state and prove our first result.$(A1)$ *The function $f:[0,1]\times R\times R \to R$ is jointly continuous.*

$(A2)$ *There exists a function $l_f \in L^{\frac{1}{\tau}}([0,1],R^+)$ such that*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le l_f(t)(|u_1-v_1| + |u_2-v_2|).$$

For all $(t,u_1,u_2),(t,v_1,v_2) \in [0,1]\times R\times R$.

$(A3)$ *The functions g and h are jointly continuous and there exists l_g,$l_h \in L^1([0,1],R^+)$*

$$|g(t,u) - g(t,v)| \le l_g(t)|u-v|$$

and

$$|h(t,u) - h(t,v)| \le l_h(t)|u-v|.$$

For each $(t,u),(t,v) \in [0,1]\times R$.

Also we defined an operator T_0, which is $T_0 : C[0,1] \to C[0,1]$, it is given as follows:

$$T_0 x(t) = \phi_q\left(\int_0^1 H(t,s) f(s,x(s),D_{0+}^{\gamma}x(s))ds\right).$$

Moreover, the following properties of p-laplacian operator $\phi_p(u) = |u|^{p-2}u$, $p>1$ and $\phi_p^{-1} = \phi_q$, $\frac{1}{p}+\frac{1}{q}=1$ are used.

If $1 < p < 2$, $uv > 0$, $|u|, |v| \geq r > 0$, then

$$\left|\phi_p(u) - \phi_p(v)\right| \leq (p-1)r^{p-2}|u-v|.$$

If $p > 2, |u|,\ |v| \leq R$, then

$$\left|\phi_p(u) - \phi_p(v)\right| \leq (p-1)R^{p-2}|u-v|.$$

The two Green functions $G(t,s)$ and $H(t,s)$ are defined.

$$G(t,s) = \begin{cases} \dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\left(1+(-1)^{\theta+1}-b\right)}{b}\dfrac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ +\dfrac{-1}{b(-1)^{\theta}}\dfrac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} & ,t \geq \tau \\ \dfrac{\left(1+(-1)^{\theta+1}-b\right)}{b}\dfrac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ * & ,t \leq \tau. \\ +\dfrac{-1}{b(-1)^{\theta}}\dfrac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \end{cases}$$

Also

$$H(t,s) = \begin{cases} \dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} - \dfrac{t(1-s)^{\beta-1}}{(1+\lambda^{p-1}\eta)\Gamma(\beta)} & ,0 \leq s \leq t \leq 1; \eta \leq s \\ \dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} - \dfrac{t(1-s)^{\beta-1}}{(1+\lambda^{p-1}\eta)\Gamma(\beta)} - \dfrac{t\lambda^{p-1}(\eta-s)^{\beta-1}}{(1+\lambda^{p-1}\eta)\Gamma(\beta)} & ,0 \leq s \leq t \leq 1; \eta \geq s \\ -\dfrac{t(1-s)^{\beta-1}}{(1+\lambda^{p-1}\eta)\Gamma(\beta)} & ,0 \leq t \leq s \leq 1; \eta \leq s \\ -\dfrac{t(1-s)^{\beta-1}}{(1+\lambda^{p-1}\eta)\Gamma(\beta)} - \dfrac{t\lambda^{p-1}(\eta-s)^{\beta-1}}{(1+\lambda^{p-1}\eta)\Gamma(\beta)} & ,0 \leq t \leq s \leq 1; \eta \geq s. \end{cases}.$$

Lemma 5.3.1 *[63], Assume* $(A1)$*-*$(A3)$ *hold and* $q > 2$*. There exists a constant* $l_{T_0} > 0$ *such that*

$$|\mathscr{T}_0 u(t) - \mathscr{T}_0 v(t)| \leq l_{\mathscr{T}_0}|u-v|.$$

For all $u, v \in B_r$*, we have*

$$l_{\mathscr{T}_0} = (q-1)L^{q-2}\left\|l_f\right\|_{\infty}\int_0^1 |H(s,s)|\,ds \leq (q-1)L^{q-2}\left\|l_f\right\|_{\infty}\frac{\left(1+\lambda^{p-1}\right)}{(1+\lambda^{p-1}\eta)\Gamma(\beta+1)}.$$

Theorem 5.3.2 *Assume* $(A1)$-$(A3)$ *holds. If*

$$\left\{l_{\mathscr{T}_0}(\Delta_1+\Delta_2)+(\Delta_{g_1}+\Delta_{g_2})\left\|l_g\right\|_1+(\Delta_{h_1}+\Delta_{h_2})\left\|l_h\right\|_1\right\}<1. \tag{5.3.1}$$

Then BVP (5.2.1)-(5.2.2) has a unique solution on $[0,1]$.

Proof. Lets define the operator $\mathscr{T}:C_\gamma([0,1],R)\to C_\gamma([0,1],R)$ to transform problem (5.2.1)-(5.2.2) into the fixed point,

$$\begin{aligned}
&(\mathscr{T}u)(t)\\
&=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathscr{T}_0(f(s,u(s),D^\gamma_{0+}u(s))ds\\
&-\frac{1}{b(-1)^\theta}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\mathscr{T}_0(f(s,u(s),D^\gamma_{0+}u(s))ds\\
&+\frac{\left(1+(-1)^\theta-b\right)}{b}\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\mathscr{T}_0(f(s,u(s),D^\gamma_{0+}u(s))ds\\
&+\frac{\left(1+(-1)^\theta\right)}{b}\int_0^1 h(s,u(s))ds+\frac{1}{b}\int_0^1 g(s,u(s))ds.
\end{aligned} \tag{5.3.2}$$

We take the γ-th fractional derivative, and we get

$$\begin{aligned}
&D^\gamma_{0+}(\mathscr{T}u)(t)\\
&=\int_0^t\frac{(t-\tau)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathscr{T}_0(f(s,u(s),D^\gamma_{0+}u(s))ds\\
&-\frac{\left((-1)^\theta-1\right)}{\left(1+(-1)^\theta+b\right)}\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1\frac{(1-\tau)^{\alpha-\gamma-2}}{\Gamma(\alpha-\gamma-1)}\mathscr{T}_0(f(s,u(s),D^\gamma_{0+}u(s))ds\\
&-\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1\frac{(1-\tau)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathscr{T}_0(f(s,u(s),D^\gamma_{0+}u(s))ds\\
&-\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1 h(s,u(s))ds\\
&-\frac{\left((-1)^\theta-1\right)}{\left(1+(-1)^\theta+b\right)}\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1 g(s,u(s))ds.
\end{aligned} \tag{5.3.3}$$

We have $t \in [0,1]$. Since f, g, h are continuous, the expression (5.3.2) and (5.3.3) are well defined. The fixed point of the operator $\mathscr{T}$ is the solution of the BVP. The Banach fixed point theorem is used to show existence and uniqueness of the solution, then we showed $\mathscr{T}$ is contraction and get

$$\begin{aligned}
&|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \\
&\le \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} l_{\mathscr{T}_0} \|u-v\|_\gamma d\tau + \frac{1}{|b|}\int_0^1 \frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} l_{\mathscr{T}_0} \|u-v\|_\gamma d\tau \\
&+ \frac{(2+|b|)}{|b|}\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} l_{\mathscr{T}_0} \|u-v\|_\gamma d\tau + \frac{2}{|b|}\int_0^1 l_h(s)\left(|u(s)-v(s)|\right) ds \\
&+ \frac{1}{|b|}\int_0^1 l_g(s)\left(|u(s)-v(s)|\right) ds,
\end{aligned}$$

We have

$$\begin{aligned}
&|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \\
&\le \left\{ l_{\mathscr{T}_0}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|b|\Gamma(\alpha)} + \frac{2+|b|}{|b|\Gamma(\alpha+1)}\right)\right. && (5.3.4)\\
&+ \left.\frac{2}{|b|}\|l_h\|_1 + \frac{1}{|b|}\left\|l_g\right\|_1\right\}\|u-v\|_\gamma && (5.3.5)\\
&= \left\{ l_{\mathscr{T}_0}\left(\frac{2+|b|}{|b|\Gamma(\alpha+1)} + \frac{1}{|b|\Gamma(\alpha)}\right)\right. \\
&+ \left.\frac{2}{|b|}\|l_h\|_1 + \frac{1}{|b|}\left\|l_g\right\|_1\right\}\|u-v\|_\gamma && (5.3.6)\\
&= \left\{ l_{\mathscr{T}_0}\left(\frac{1}{\Gamma(\alpha+1)}\left(\frac{2+|b|+\alpha}{|b|}\right)\right)\right. \\
&+ + \left.\frac{2}{|b|}\|l_h\|_1 + \frac{1}{|b|}\left\|l_g\right\|_1\right\}\|u-v\|_\gamma && (5.3.7)\\
&\le \left\{ l_{\mathscr{T}_0}\Delta_1 + \|l_h\|_1 \Delta_{h_1} + \left\|l_g\right\|_1 \Delta_{g_1}\right\}\|u-v\|_\gamma. && (5.3.8)
\end{aligned}$$

Similarly, for γ^{th} derivative,

$$
\begin{aligned}
&\left|D_{0+}^{\gamma}(\mathscr{T}u)(t)-D_{0+}^{\gamma}(\mathscr{T}v)(t)\right| \\
&\leq \int_0^t \frac{(t-\tau)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} l_{\mathscr{T}_0}\|u-v\|_{\gamma} d\tau \\
&+\frac{2}{(2+|b|)\Gamma(2-\gamma)} \int_0^1 \frac{(1-\tau)^{\alpha-\gamma-2}}{\Gamma(\alpha-\gamma-1)} l_{\mathscr{T}_0}\|u-v\|_{\gamma} d\tau \\
&+\frac{1}{\Gamma(2-\gamma)} \int_0^1 \frac{(1-\tau)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} l_{\mathscr{T}_0}\|u-v\|_{\gamma} d\tau \\
&+\frac{1}{\Gamma(2-\gamma)} \int_0^1 l_h(s)(|u(s)-v(s)|) ds \\
&+\frac{2}{(2+|b|)\Gamma(2-\gamma)} \int_0^1 l_g(s)(|u(s)-v(s)|) ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\left|D_{0+}^{\gamma}(\mathscr{T}u)(t)-D_{0+}^{\gamma}(\mathscr{T}v)(t)\right| \\
&\leq \left\{ l_{\mathscr{T}_0}\left(\frac{1}{\Gamma(\alpha-\gamma+1)}+\frac{2}{(2+|b|)\Gamma(2-\gamma)\Gamma(\alpha-\gamma)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha-\gamma+1)}\right)\right. \\
&\left.+\frac{1}{\Gamma(2-\gamma)}\|l_h\|_1+\frac{2}{\Gamma(2-\gamma)(2+|b|)}\|l_g\|_1\right\}\|u-v\|_{\gamma} \\
&=\left\{ l_{\mathscr{T}_0}\left(\frac{2}{(2+|b|)\Gamma(2-\gamma)\Gamma(\alpha-\gamma)}+\frac{(1+\Gamma(2-\gamma))}{\Gamma(2-\gamma)\Gamma(\alpha-\gamma+1)}\right)\right. \\
&\left.+\frac{1}{\Gamma(2-\gamma)}\|l_h\|_1+\frac{2}{\Gamma(2-\gamma)(2+|b|)}\|l_g\|_1\right\}\|u-v\|_{\gamma} \\
&\leq \left\{ l_{\mathscr{T}_0}\Delta_2+\Delta_{h_2}\|l_h\|_1+\Delta_{g_2}\|l_g\|_1\right\}\|u-v\|_{\gamma}. \qquad (5.3.9)
\end{aligned}
$$

By (5.3.8)-(5.3.9), we found

$$
\begin{aligned}
&\|Tu-Tv\|_{\gamma} \\
&\leq \left\{ l_{\mathscr{T}_0}(\Delta_1+\Delta_2)+(\Delta_{g_1}+\Delta_{g_2})\|l_g\|_1+(\Delta_{h_1}+\Delta_{h_2})\|l_h\|_1\right\}\|u-v\|_{\gamma} \\
&=\left\{\left(\sum_{i=1}^{2}\Delta_i\right) l_{\mathscr{T}_0}+\left(\sum_{i=1}^{2}\Delta_{g_i}\right)\|l_g\|_1+\left(\sum_{i=1}^{2}\Delta_{h_i}\right)\|l_h\|_1\right\}\|u-v\|_{\gamma}.
\end{aligned}
$$

Thus $\mathscr{T}$ is a contraction mapping and by the Banach fixed point theorem, $\mathscr{T}$ has a fixed point which is the solution of the BVP. ■

5.4 Existence Results For FDE With $p-$Laplacian Operator

Theorem 5.4.1 *Assume,*

$(A4)$ *There exist increasing functions* $\varphi : [0,\infty)\times[0,\infty)\to[0,\infty)$ *and* $\psi_i : [0,\infty)\to[0,\infty)$, $i = 1,2$ *and the functions* $l_f \in L^{\frac{1}{\tau}}([0,1],R^+)$ *and* $l_g, l_h \in L^1([0,1],R^+)$ *such that*

$$|f(t,u,v)| \leq l_f(t)\varphi(|u|+|v|),$$

$$|g(t,u)| \leq l_h(t)\psi_1(|u|),$$

$$|h(t,u)| \leq l_g(t)\psi_2(|u|).$$

For all $t\in[0,1]$ *and* $u,v\in R$. $(A5)$ *There exists a constant* $\mathscr{N}>0$ *such that*

$$\left[\frac{\mathscr{N}}{\varphi(\|u\|_\gamma)l_{\mathscr{T}_0}\left(\sum\limits_{i=1}^{2}\Delta_i\right)+\left(\sum\limits_{i=1}^{2}\Delta_{h_i}\right)\psi_1(\|u\|_\gamma)\|l_h\|_1+\left(\sum\limits_{i=1}^{2}\Delta_{g_i}\right)\psi_2(\|u\|_\gamma)\left\|l_g\right\|_1}\right] > 1. \tag{5.4.1}$$

Thus the boundary value problem (5.2.1)-(5.2.2) *has at least one solution on* $[0,1]$.

Proof. Let $B_r = \left\{u\in C_\gamma([0,1],R) : \|u\|_\gamma \leq r\right\}$. Step 1: Let the operator $\mathscr{T} : C_\gamma([0,1],R)\to C_\gamma([0,1],R)$ is given in (5.3.2) and (5.3.3) which defines B_r into the bounded set. For each $u\in B_r$, we have

$$
\begin{aligned}
&|(\mathscr{T}u)(t)| \\
&\le \frac{\varphi(r)}{\Gamma(\alpha)} l_{\mathscr{T}_0} \int_0^t (t-\tau)^{\alpha-1} d\tau \\
&+ \frac{1}{|b|} \frac{\varphi(r)}{\Gamma(\alpha-1)} l_{\mathscr{T}_0} \int_0^1 (1-\tau)^{\alpha-2} d\tau \\
&+ \frac{(2+|b|)}{|b|} \frac{\varphi(r)}{\Gamma(\alpha)} l_{\mathscr{T}_0} \int_0^1 (1-\tau)^{\alpha-1} d\tau \\
&+ \frac{2}{|b|} \psi_1(r) \int_0^1 |l_h(s)|\, ds \\
&+ \frac{1}{|b|} \psi_2(r) \int_0^1 \left|l_g(s)\right| ds.
\end{aligned}
$$

In a similar manner

$$
\begin{aligned}
&\left|D_{0+}^{\gamma}(\mathscr{T}u)(t)\right| \\
&\le \frac{\varphi(r)}{\Gamma(\alpha-\gamma)} l_{\mathscr{T}_0} \int_0^t (t-\tau)^{\alpha-\gamma-1} d\tau \\
&+ \frac{2\varphi(r)}{(2+|b|)\,\Gamma(2-\gamma)\,\Gamma(\alpha-\gamma-1)} l_{\mathscr{T}_0} \int_0^1 (1-\tau)^{\alpha-\gamma-2} d\tau \\
&+ \frac{\varphi(r)}{\Gamma(\alpha-\gamma)\Gamma(2-\gamma)} l_{\mathscr{T}_0} \int_0^1 (1-\tau)^{\alpha-\gamma-1} d\tau \\
&+ \frac{1}{\Gamma(2-\gamma)} \psi_1(r) \int_0^1 |l_h(s)|\, ds \\
&+ \frac{2}{(2+|b|)\,\Gamma(2-\gamma)} \psi_2(r) \int_0^1 \left|l_g(s)\right| ds.
\end{aligned}
$$

By the Hölder Inequality, we get

$$
\begin{aligned}
&|(\mathscr{T}u)(t)| \\
&\le \frac{(2+2|b|)}{|b|\Gamma(\alpha+1)}\varphi(r)l_{\mathscr{T}_0} + \frac{\varphi(r)}{|b|\Gamma(\alpha)}l_{\mathscr{T}_0} \\
&+ \frac{2}{|b|}\psi_1(r)\|l_h\|_1 + \frac{1}{|b|}\psi_2(r)\left\|l_g\right\|_1 \\
&= \left[\frac{(2+2|b|)}{|b|\Gamma(\alpha+1)} + \frac{1}{|b|\Gamma(\alpha)}\right]\varphi(r)l_{\mathscr{T}_0} \\
&+ \frac{2}{|b|}\psi_1(r)\|l_h\|_1 + \frac{1}{|b|}\psi_2(r)\left\|l_g\right\|_1 \\
&\le \varphi(r)l_{\mathscr{T}_0}\Delta_1 + \Delta_{g_1}\psi_1(r)\left\|l_g\right\|_1 + \Delta_{h_1}\psi_2(r)\|l_h\|_1 .
\end{aligned}
$$

Also we have

$$
\begin{aligned}
&\left|D_{0+}^{\gamma}(\mathscr{T}u)(t)\right| \\
&\le \frac{\varphi(r)}{\Gamma(\alpha-\gamma+1)}l_{\mathscr{T}_0} + \frac{2\varphi(r)}{(2+|b|)\Gamma(2-\gamma)\Gamma(\alpha-\gamma)}l_{\mathscr{T}_0} \\
&+ \frac{\varphi(r)}{\Gamma(\alpha-\gamma+1)\Gamma(2-\gamma)}l_{\mathscr{T}_0} \\
&+ \frac{1}{\Gamma(2-\gamma)}\psi_1(r)\|l_h\|_1 + \frac{2}{(2+|b|)\Gamma(2-\gamma)}\psi_2(r)\left\|l_g\right\|_1 \\
&= \left[\frac{2}{(2+|b|)\Gamma(2-\gamma)\Gamma(\alpha-\gamma)} + \frac{(\Gamma(2-\gamma)+1)}{\Gamma(2-\gamma)\Gamma(\alpha-\gamma+1)}\right]\varphi(r)l_{\mathscr{T}_0} \\
&+ \frac{1}{\Gamma(2-\gamma)}\psi_1(r)\|l_h\|_1 + \frac{2}{(2+|b|)\Gamma(2-\gamma)}\psi_2(r)\left\|l_g\right\|_1 \\
&\le \varphi(r)l_{\mathscr{T}_0}\Delta_2 + \Delta_{h_2}\psi_1(r)\|l_h\|_1 + \Delta_{g_2}\psi_2(r)\left\|l_g\right\|_1 .
\end{aligned}
$$

Thus the following expression is found.

$$
\begin{aligned}
\|(\mathscr{T}u)\|_{\gamma} &\le \varphi(r)l_{\mathscr{T}_0}(\Delta_1+\Delta_2) + (\Delta_{g_1}+\Delta_{g_2})\psi_1(r)\left\|l_g\right\|_1 \\
&+ (\Delta_{h_1}+\Delta_{h_2})\psi_2(r)\|l_h\|_1 \\
&= \varphi(r)l_{\mathscr{T}_0}\left(\sum_{i=1}^{2}\Delta_i\right) + \psi_1(r)\|l_h\|_1\left(\sum_{i=1}^{2}\Delta_{h_i}\right) \\
&+ \psi_2(r)\left\|l_g\right\|_1\left(\sum_{i=1}^{2}\Delta_{g_i}\right).
\end{aligned}
$$

Step 2: The families $\{(\mathscr{T}u) : u \in B_r\}$ and $\left\{D_{0+}^{\gamma}(\mathscr{T}u) : u \in B_r\right\}$ are equicontinuous.

For $t_1 < t_2$, we have

$$\begin{aligned}
&|(\mathscr{T}u)(t_2) - (\mathscr{T}u)(t_1)| \\
&\leq \frac{\varphi(r)l_{\mathscr{T}_0}}{\Gamma(\alpha)}\left[\int\limits_0^{t_1}\left((t_1-\tau)^{\alpha-1} + (t_2-\tau)^{\alpha-1}\right)d\tau + \int\limits_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\right] \\
&+ \frac{|t_2-t_1|\,\varphi(r)l_{\mathscr{T}_0}}{\Gamma(\alpha)}\int\limits_0^{t_1}(1-\tau)^{\alpha-1}d\tau \\
&+ \frac{2\,|t_2-t_1|\,\varphi(r)}{|b|\,\Gamma(\alpha-1)}l_{\mathscr{T}_0}\int\limits_0^{t_1}(1-\tau)^{\alpha-2}d\tau \\
&+ \frac{2\,|t_2-t_1|}{|b|}\psi_1(r)\int\limits_0^1\left|l_g(s)\right|ds \\
&+ 2\,|t_2-t_1|\,\psi_2(r)\int\limits_0^1|l_h(s)|\,ds \\
&\to 0
\end{aligned}$$

as $t_2 \to t_1$.

Similarly, we have

$$
\begin{aligned}
&\left|D^{\gamma}_{0+}(\mathscr{T}u)(t_2)-D^{\gamma}_{0+}(\mathscr{T}u)(t_1)\right| \\
&\leq \frac{\varphi(r)l_{\mathscr{T}_0}}{\Gamma(\alpha-\gamma)}\left[\int_0^{t_1}\left((t_1-\tau)^{\alpha-\gamma-1}+(t_2-\tau)^{\alpha-\gamma-1}\right)d\tau+\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-\gamma-1}d\tau\right] \\
&+\frac{\varphi(r)l_{\mathscr{T}_0}\left|t_2^{1-\gamma}-t_1^{1-\gamma}\right|}{\Gamma(\alpha-\gamma)\Gamma(2-\gamma)}\int_0^1(1-\tau)^{\alpha-\gamma-1}d\tau \\
&+\frac{\varphi(r)l_{\mathscr{T}_0}2\left|t_2^{1-\gamma}-t_1^{1-\gamma}\right|}{|b|\,\Gamma(\alpha-\gamma-1)\Gamma(2-\gamma)}\int_0^1(1-\tau)^{\alpha-\gamma-2}d\tau \\
&+\frac{2\left|t_2^{1-\gamma}-t_1^{1-\gamma}\right|}{|b|\,\Gamma(2-\gamma)}\psi_1(r)\int_0^1\left|l_g(s)\right|ds \\
&+\frac{2\left|t_2^{1-\gamma}-t_1^{1-\gamma}\right|}{\Gamma(2-\gamma)}\psi_2(r)\int_0^1|l_h(s)|\,ds\to 0
\end{aligned}
$$

as $t_2\to t_1$.

By Arzela-Ascoli theorem, the families $\{(\mathscr{T}u):u\in B_r\}$ and $\left\{D^{\gamma}_{0+}(\mathscr{T}u):u\in B_r\right\}$ are equicontinuous and relatively compact in $C([0,1],R)$. Therefore $\mathscr{T}(B_r)$ is a relatively compact subset of $C_\gamma([0,1],R)$ and the operator $\mathscr{T}$ is compact.

Step 3: Let $u=\xi\,(Tu)$ and for $0<\xi<1$. For all $t\in[0,1]$, we define then operator $\bar{K}=\left\{u\in C_\gamma([0,1],R),\|u\|_\gamma<\mathscr{N}\right\}$ and then, we have

$$
\begin{aligned}
&\|u\|_\gamma \\
&\leq \varphi(\|u\|_\gamma)l_{\mathscr{T}_0}(\Delta_1+\Delta_2)+\psi_1(\|u\|_\gamma)\|l_h\|_1(\Delta_{h_1}+\Delta_{h_2}) \\
&+\psi_2(\|u\|_\gamma)\left\|l_g\right\|_1(\Delta_{g_1}+\Delta_{g_2}) \\
&\leq \varphi(\|u\|_\gamma)l_{\mathscr{T}_0}\left(\sum_{i=1}^{2}\Delta_i\right)+\psi_1(\|u\|_\gamma)\|l_h\|_1\left(\sum_{i=1}^{2}\Delta_{h_i}\right) \\
&+\psi_2(\|u\|_\gamma)\left\|l_g\right\|_1\left(\sum_{i=1}^{2}\Delta_{g_i}\right).
\end{aligned}
$$

That means

$$\frac{\|u\|_\gamma}{\varphi(\|u\|_\gamma) l_{\mathscr{T}_0}\left(\sum_{i=1}^{2}\Delta_i\right) + \psi_1(\|u\|_\gamma)\,\|l_h\|_1\left(\sum_{i=1}^{2}\Delta_{h_i}\right) + \psi_2(\|u\|_\gamma)\,\left\|l_g\right\|_1\left(\sum_{i=1}^{2}\Delta_{g_i}\right)} \leq 1.$$

For a positive $\mathscr{N}$ and $\|u\|_\gamma < \mathscr{N}$, the operator $\mathscr{T}$ which is defined in $\overline{\mathscr{K}}$ into $C_\gamma([0,1],R)$, is continuous and compact. Therefore $\mathscr{T}$ has a fixed point in $\overline{\mathscr{K}}$. ■

Chapter 6

CONCLUSION AND DISCUSSION

Firstly, to illustrate the results, the following examples are given.

6.1 Examples

Example 1. Consider the following BVP of FDEs:

$$\begin{cases} \mathfrak{D}_{0^+}^{5/2}u(t) = \frac{1}{11}\left(\dfrac{|u(t)|}{1+|u(t)|} + \dfrac{\left|\mathfrak{D}_{0^+}^{1/2}u(t)\right|}{1+\left|\mathfrak{D}_{0^+}^{1/2}u(t)\right|} + \tan^{-1}\left(\mathfrak{D}_{0^+}^{3/2}u(t)\right)\right), \quad 0 \le t \le 1, \\ u(0)+u(1) = \displaystyle\int_0^1 \frac{u(s)}{(1+s)^2}ds, \\ \mathfrak{D}_{0^+}^{1/2}u\left(\frac{1}{10}\right) + \mathfrak{D}_{0^+}^{1/2}u(1) = \dfrac{1}{2}\displaystyle\int_0^1 \left(\frac{e^s u(s)}{1+2e^s} + \frac{1}{2}\right)ds, \\ \mathfrak{D}_{0^+}^{3/2}u\left(\frac{1}{10}\right) + \mathfrak{D}_{0^+}^{3/2}u(1) = \dfrac{1}{3}\displaystyle\int_0^1 \left(\frac{u(s)}{1+e^s} + \frac{3}{4}\right)ds. \end{cases} \tag{6.1.1}$$

Here,

$$\alpha = 5/2,\ \beta_1 = 1/2,\ \beta_2 = 3/2,\ T = 1,\ \tau = \frac{1}{10},\ \ a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = 1,$$

$$\eta = \frac{1}{10},\ \ \lambda_0 = 1,\ \ \lambda_1 = \frac{1}{2},\ \ \lambda_2 = \frac{1}{3},\ l_{g_0} = l_{g_1} = l_{g_2} = 1,$$

The functions are defined as:

$$f(t,u,v,w) := \frac{1}{11}\left(\frac{u}{1+u} + \frac{v}{1+v} + \tan^{-1}(w)\right),\ \ l_f(t) = \frac{1}{11},$$

$$g_0(t,u) := \frac{u}{(1+t)^2},\quad g_1(t,u) := \frac{e^t u}{1+2e^t} + \frac{1}{2},$$

$$g_2(t,u) := \frac{u}{1+e^t} + \frac{3}{4}.$$

Since $1.77 < \Gamma(\frac{1}{2}) < 1.78,\ 0.88 < \Gamma(\frac{3}{2}) < 0.89,\ 1.32 < \Gamma(\frac{5}{2}) < 1.33$ and $3.32 < \Gamma(\frac{7}{2}) <$

3.33, with simple calculations, we show that

$$\Delta_0 = 2.34,\ \ \Delta_1 = 0.19,\ \ \Delta_2 = 0.15,$$

$$\rho_0 = 0.5,\ \ \ \rho_1 = 1.01,\ \ \ \rho_2 = 1.2,$$

$$\tilde{\rho}_0 = 0,\ \ \tilde{\rho}_1 = 0.76,\ \ \ \tilde{\rho}_2 = 0.9,$$

$$\hat{\rho}_0 = \hat{\rho}_1 = 0,\ \ \hat{\rho}_2 = 0.51.$$

Furthermore, we get the following

$$(\Delta_0 + \Delta_1 + \Delta_2)\left\|l_f\right\|_{1/\tau} + \sum_{i=0}^{2}\rho_i\left|\lambda_i\right|\left\|l_{g_i}\right\|_1 + \sum_{i=1}^{2}\widetilde{\rho}_i\left|\lambda_i\right|\left\|l_{g_i}\right\|_1 + \widehat{\rho}_2\left|\lambda_2\right|\left\|l_{g_2}\right\|_1$$
$$< 2.7\frac{1}{11} + 0.75 < 1.$$

Thus, all the assumptions of Theorem 4.1.1 are satisfied. Hence, the problem (6.1.1) has a unique solution on $[0,1]$.

Example 2. Consider the following BVP of FDEs:

$$\begin{cases} \mathfrak{D}_{0^+}^{5/2}u(t) = \dfrac{|u(t)|^3}{9(|u(t)|^3+3)} + \dfrac{\left|\sin\mathfrak{D}_{0^+}^{1/2}u(t)\right|}{9(\left|\sin\mathfrak{D}_{0^+}^{1/2}u(t)\right|+1)} + \dfrac{1}{12},\ \ t\in[0,1], \\ u(0)+u(1) = \displaystyle\int_0^1 \frac{u(s)}{3(1+s)^2}ds, \\ \mathfrak{D}_{0^+}^{1/2}u\left(\tfrac{1}{10}\right) + \mathfrak{D}_{0^+}^{1/2}u(1) = \dfrac{1}{2}\displaystyle\int_0^1 \frac{e^s u(s)}{3(1+e^s)^2}ds, \\ \mathfrak{D}_{0^+}^{3/2}u\left(\tfrac{1}{10}\right) + \mathfrak{D}_{0^+}^{3/2}u(1) = \dfrac{1}{3}\displaystyle\int_0^1 \frac{u(s)}{3(1+e^s)^2}ds, \end{cases}, \qquad (6.1.2)$$

where f is given by

$$f(t,u,v,w) = \frac{|u|^3}{10(|u|^3+3)} + \frac{|\sin v|}{9(|\sin v|+1)} + \frac{1}{12}.$$

$$\text{and}\qquad |f(t,u,v,w)| \le \frac{|u|^3}{9(|u|^3+3)} + \frac{|\sin v|}{9(|\sin v|+1)} + \frac{1}{12} \le \frac{11}{36}, \qquad u,v,w \in \mathbb{R}.$$

Thus we get the following by simple calculations,

$$|f(t,u,v,w)| \le \frac{11}{36} = l_f(t)\varphi\left(|u|+|v|+|w|\right), \quad \text{with} \quad l_f(t) = \frac{1}{3}, \varphi(t) = \frac{11}{12}.$$

Thus we get the following values,

$$\alpha = 5/2,\ \beta_1 = 1/2,\ \beta_2 = 3/2,\ T = 1,\ \tau = \frac{1}{10},\ a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = 1,$$
$$\eta = \frac{1}{10},\ \lambda_0 = 1,\ \lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{1}{3},\ l_{g_0} = l_{g_1} = l_{g_2} = \frac{1}{3}.$$

We get the following by shortenings,

$$\Delta_0 = 2.34,\ \Delta_1 = 0.19,\ \Delta_2 = 0.15,$$

$$\rho_0 = 0.5,\quad \rho_1 = 1.01,\quad \rho_2 = 1.2,$$

$$\tilde{\rho}_0 = 0,\ \tilde{\rho}_1 = 0.76,\quad \tilde{\rho}_2 = 0.9,$$

$$\hat{\rho}_0 = \hat{\rho}_1 = 0,\ \hat{\rho}_2 = 0.51.$$

Also, we have the functions

$$g_0(t,u) := \frac{u}{3(1+t)^2},\ \ g_1(t,u) := \frac{e^t u}{3(1+e^t)^2},\ \ g_2(t,u) := \frac{u}{3(1+e^t)^2},\ \ \psi_i(u) = u,\ \ i = 0,1,2.$$

By the given condition,

$$\frac{K}{\varphi(K)\left\|l_f\right\|_{1/\tau}(\Delta_0+\Delta_1+\Delta_2) + \sum_{i=0}^{2}(\rho_i+\widetilde{\rho}_i+\widehat{\rho}_i)\left|\lambda_i\right|\psi_i(K)\left\|l_{g_i}\right\|_1} > 1.$$

We found $\qquad\qquad K > 9.8.$

Similarly, all the conditions of Theorem 4.2.1 are satisfied. So, there exists at least one solution of problem (6.1.2) on $[0,1]$.

6.2 Discussion

In this thesis, the existence solutions of FDEs of unknown functions were discussed. At first, $\alpha \in (2,3]$ ordered FDEs with three point fractional boundary and integral conditions were obtained.

By fixed point theorems and , in second part of the thesis, the existence of solutions of FDEs with p-laplacian operator, irregular and integral boundary conditions were discussed. While stating and proving, the fixed point theorems and the Green functions were used.

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