Quantum Integral Inequalities on Finite Intervals

Farhad Mustafa Taher

Submitted to the
Institute of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
June 2018
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Assoc. Prof. Dr. Ali Hakan Ulusoy
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Sonuç Zorlu Oğurlu
Supervisor

Examining Committee

1. Prof. Dr. Nazim Mahmudov

2. Prof. Dr. Sonuç Zorlu Oğurlu

3. Asst. Prof. Dr. Halil Gezer

ABSTRACT

The Integral Inequalities can be used for the study of qualitative and quantitative properties of integrals and they perform an important role in the theory of differential equations. The study of the fractional q-integral inequalities is also of great importance.

The purpose of this thesis is to study q-calculus analogs of some classical integral inequalities. In particular, some of the greatest significant integral inequalities of analysis are extended to Quantum calculus. We will work on the q-generalization of the Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Chebysev integral inequalities. The analysis is based on the notions of q-derivative and q-integral on finite intervals presented recently by the author in [9].

Keywords: Quantum Integral Inequalities; Hölder’s inequality, Hermite-Hadamard’s inequality, Ostrowski's Inequality, Grüss-Chebysev integral inequality.

ÖZ

İntegral eşitsizlikleri, integrallerin nitel ve nicel özelliklerinin incelenmesi için kullanılabilir ve diferansiyel denklemler teorisinde temel bir rol oynar. Kesirli q-integral eşitsizliklerinin incelenmesi de büyük önem taşımaktadır.

Bu çalışmanın amacı bazı klasik integral eşitsizliklerinin q-Kalkülüs analoglarını bulmaktır. Özellikle analizin en önemli integral eşitsizliklerinin bazılarının kuantum Kalkülüs'e genelleştirmelerini incelenecektir. Bunlar, Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss ve Grüss-Čebyšev integral eşitsizlikleri olacaktır. Yapılan çalışmalar ve analizler, son zamanlarda J. Tariboon ve S. Ntouyas v.s. araştırmacıların çalıştığı sınırlı aralıklarda q-türev ve q-integral kavramlarına dayanmaktadır.

Anahtar Kelimeler: Quantum İntegral eşitsizlikleri, Hölder eşitsizliği, Hermite-Hadamard eşitsizliği, Ostrovski eşitsizliği, Grüss-Chebysev eşitsizliği, Konvekslik

To My Family

ACKNOWLEDGMENT

Gratefulness is the biggest gifts given to someone. I would like to express a great thanking to my almighty Allah, who illuminate my life and blessed on me uncountable things. Also guide me whole my life. Really without him, I cannot do anything in this world. You are alone and we worship you we seek help from you. My Lord give us good in this world and good in the hereafter and guard us from the punishment of the fire. My Allah you deserve more than thankfulness.

Next, I should express thank to my beloved family who helped and encouraged me in whole my life. These people are my father, mother, siblings, uncles, cousins and nephews. Without them I couldn’t reach far distance. Thank you so much for everything they have did for me. I am really great thankful to them.

Furthermore, I should also express gratitude and recognition to my supervisor Prof. Dr. Sonuç Zorlu Oğurlu for given me a chance supervise me while she had a lot of students. Although she was busy, I believed for her extraordinary hardworking would give a chance to work with her. I appreciate her help, encouragement and support that were backbone for complete my work. I am so grateful to have her as my supervisor. I cannot find words to express deep thank to her.

However, I would like to thank to all teachers who put their faith in me and advised me to do well.

At last, I thank all classmates and friends. I will miss you all and a best thank to EMU University.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

1.1 Historical Background

Calculus is theory where there is a differential $h \to h'$ and integral $h \to R, h$ operation in such a way that$(R\ h)' = h$. This does not necessarily mean that the notion of derivative or integral are the classical notions.

A q-calculus is a version of calculus without limits. Derivatives are differences and anti-derivatives are sums. With a suitable generalization of integration of differential forms on curves and surfaces, the great notions of multi-variable calculus simplify same Stoke theorem. In fact, this expansion make a lot of proofs easier. There are many similarity in theory since 'curves' are on the same basis than one forms.

Moreover, q-Calculus were found by Euler in the 18th century. Also In 1910, Jackson [2] presented the correct from the definite q-integral. He was the first to grasp q-calculus in a methodical approach. In the 2^{nd} half of the 20th century, there was a substantial rise of movement in the part of the q-calculus due to requests of the q-calculus in fields of Mathematics and Physics.

In recent years, many attentions have been given to the study of the *q*-calculus, which has important and significant applications in quantum physics and in many branches of sciences.

Integral inequalities have been an important part in the theory of differential equations. The study of fractional q-integral inequalities is also have a great significance. Integral Inequalities have been worked broadly via number of scholars also in the classical analysis or in Quantum sense. See [2]-[7]

1.1.1 Development of Integral Inequality in Quantum Calculus

For a long time, studying, investigating and developing calculus had based on using limits. Later it had appeared in a calculus without limits called q −calculus. The quantum calculus started with F.H. Jackson in the beginning of last century as emerging area of mathematics, although it had been discovered already and vigorously studied by Euler.

Quantum calculus worked in from the beginning of this century represented as a link between mathematics and physics. Most of the scientific community, which benefit from quantum calculus, is physicists.

The area witnessed a great expansion, because of using foundations of hypergeometric series to the different subjects of combinatorics, quantum theory, number theory, statistical mechanics that are continuously discovered.

Integral inequalıtıes play an important role in the theory of differentıal equations and the study of q-integral inequalities is also of great importance. Integral inequalities have been studied by many authors in classical analysis as well as quantum analysis.

However, we have used an important theorem which is called Lagrangian Mean Value Theorem (LMVT) is one of the fundamental theorem.

The goal of the thesis is to study q-calculus analogs of several classical integral inequalities. We study the q-generalizations of the Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Chebysev integral inequalities.

The thesis is arranged in the following order: In Chapter 2, we give some definitions and helpful consequences that will help us to demonstrate our important results. In Chapter 3, we build up our important consequences.

Chapter 2

BASIC OF QUANTUM CALCULUS

2.1 Prefaces and Supplementary Results

For the reader, in this section we will provide an outline of the mathematical notions and definitions that will be used throughout the thesis.

Let us start with q-analogue of differentiation. Consider

$$\lim_{s\to s_0}\frac{h(s)-h(s_0)}{s-s_0}=\frac{dh}{ds}\ ,$$

which gives derivative of a function $h(s)$ at $s=s_0$.

If $s=qs_0$ where $0<q<1$ is a stable number and without limits, then we go into the world of Quantum calculus. The q-derivative of s^m is $[m]s^{m-1}$, and

$$[m]=\frac{q^m-1}{q-1}.$$

As q-analogue of m in the sense that m is a limit of $[m]$ as q tends to 1.

Now, we are going to give the definition of q-derivative of a function h.

Definition 2.1.1 The q-derivative has defined as following:

$$D_q h(s)=\frac{h(qs)-h(s)}{(q-1)s}. \tag{1}$$

If q tends to 1, then we get ordinary derivative.

We are moving towards q-antiderivative of a function.

Definition 2.1.2 The function $H(s)$ is a q-antiderivative of $h(s)$ if $D_q H(s) = h(s)$. denoted via

$$\int h(s) d_q s. \tag{2}$$

Next definition is Jackson integral ,

Definition 2.1.3

Jackson Integral of $h(s)$ is defined as

$$\int h(s) d_q s = (1-q) \sum_{m=0}^{\infty} q^m h(q^m s) \tag{3}$$

From the above definition it is obvious that

$$\int h(s) D_q k(s) d_q s$$

$$= (1-q)s \sum_{m=0}^{\infty} q^m h(q^m s) D_q k(q^m s)$$

$$= (1-q)s \sum_{m=1}^{\infty} q^m h(q^m s) \frac{k(q^m s) - k(q^{m+1} s)}{(1-q) q^m s}.$$

Definite q-integral is defined as follows:

Definition 2.1.4 [1] Assume that $0 < c < d$. The definite q-integral is defined as

$$\int_0^d h(s) d_q s = (1-q) d \sum_{m=0}^{\infty} q^m h(q^m d) \ , \tag{4}$$

provided the sum converges absolutely.

Overall, formula for definite integral is given as

$$\int_0^d h(s)kd_s s = \sum_{m=0}^{\infty} h(q^m\, d)\big(k(q^m d) - k(q^{m+1} d)\big).$$

Note that above definition of definite q-integral in a generic interval $[c, d]$ gives

$$\int_c^d h(s)d_q s = \int_0^d h(s)d_q s - \int_0^c h(s)d_q s.$$

Definition 2.1.5 Assume that $I := [c, d] \subset \mathbf{R}$ be an interval and $0 < q < 1$, $h : I \to \mathbf{R}$ is a continuous function. Then the q-derivative of a function $h: I \to \mathbf{R}$ at the point $s \in I$ on $[c, d]$ is expressed as follows,

and assume that $s \in I$. Then

$$_cD_q h(s) = \frac{h(s) - h(qs + (1-q)c)}{(1-q)(s-c)}, s \neq c,$$

$$_cD_q h(c) = \lim_{s \to c} {_cD_q h(s)}, \tag{2.1}$$

and named as q-derivative on I of function h at s.

We say that f is q-differentiable on I provided that $_cD_q h(s)$ exists for all $s \in I$. Recall that if $c = 0$ in (2.1), then $_cD_q h = D_q h$, where D_q is known q-derivative of the function $h(s)$ well-defined as

$$D_q h(s) = \frac{h(s) - h(qs)}{(1-q)s}. \tag{2.2}$$

Moreover, we must define the advanced q-derivative of function h on I.

Definition 2.1.6

Suppose $h : I \to \mathbf{R}$ is a continuous function having the 2nd order q-derivative on interval I, which is indicated as ${}_cD_q^2 h$, provided ${}_cD_q h$ is q-differential on I with ${}_cD_q^2 h = {}_cD_q\left({}_cD_q h\right) : I \to \mathbf{R}$. Likewise, it we can now state higher order q-derivative on I, ${}_cD_q^m : I_k \to \mathbf{R}$.

Convexity theory has played an important and fundamental role in the development of various areas of applied and pure sciences. This theory provides a unified, natural and general framework to study a wide ranges of classes of non-related problems.

Due to its importance, the concepts of convex sets and convex functions have been extended in different ways. An important generalization of convex functions is known as pre-invex functions, introduced in early 1980's which inspired many researchers to deal with some complicated problems.

We now recall the definition of a convex function.

Definition 2.1.7

The function h on $[c,d]$ is called

$$h\big((1-z)c + zd\big) \le (1-z)h(c) + zh(d),$$

the function h on $[c,d]$ satisfying is called a convex function

for all $z \in [0,1]$.

Let us now solve some examples.

Example 2.1.1 Suppose that $s \in [c,d]$ on $0 < q < 1$. Then, for $s \neq c$, we have

$$ {}_cD_q s^2 = \frac{s^2 - (qs + (1-q)c)^2}{(1-q)(s-c)} $$

$$ = \frac{(1+q)s^2 - 2qcs - (1-q)c^2}{s-c} $$

$$ = (1+q)s + (1-q)c. $$

For $s = c$, we have $\lim_{s \to c}\left({}_cD_q s^2\right) = 2c$.

Lemma 2.1.1 [9] Let $\alpha \in \mathbf{R}$, then we have

$$ {}_cD_q (s-c)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(s-c)^{\alpha-1}. $$

Let us now define q-integral on interval *I*.

Definition 2.1.8 Assume that $h : I \to \mathbf{R}$ is a continuous function, the q-Integral on *I* is expressed by

$$ \int_c^s h(z)\ {}_cd_q z = (1-q)(s-c) \sum_{m=0}^{\infty} q^m h(q^m s + (1-q^m)c) \ , \tag{2.4} $$

for $s \in I$.

Besides, if $a \in (c, s)$ then the define q-integral on *I* is defined via

$$ \int_a^s h(z) = \int_c^s h(z)\ {}_cd_q t - \int_c^a h(z)\ {}_cd_q z $$

$$ (1-q)(s-c) \sum_{m=0}^{\infty} q^m h(q^m s + (1-q^m)c) $$

$$ - (1-q)(a-c) \sum_{m=0}^{\infty} q^m h\,(q^m a + (1-q^m)c). $$

Note that if $c = 0$, (2.4) turns out to classical q-Integral of $h(s)$ which is defined as

$$\int_0^s h(z)\ {}_0d_qz = (1-q)s\sum_{m=0}^{\infty} q^m h(q^m s),$$

for $s \in [0,\infty)$. See [8].

Example 2.1.2 Assume that $h(s)=s$ for $s \in I$, then

$$\int_c^s h(z)\ {}_cd_qz = \int_c^s z\ {}_cd_qz$$

$$= (1-q)(s-c)\sum_{m=0}^{\infty} q^m(q^m s + (1-q^m)c)$$

$$= \frac{(s-c)(s+qc)}{1+q}.$$

Example 2.1.3 Assume that $a \in I$, then

$$\int_a^d (z-a)\ {}_cd_qz = \int_c^d (z-a)\ {}_cd_qz - \int_c^a (z-a)\ {}_cd_qz$$

$$= \left[\frac{(z-c)(z+qc)}{1+q} - az\right]_c^d - \left[\frac{(z-c)(z+qc)}{1+q} - az\right]_c^a$$

$$= \left[\frac{(d-c)(d+qc)}{1+q} - ad + ac\right] - \left[\frac{(a-c)(a+qc)}{1+q} - a^2 + ac\right]$$

$$= \frac{d^2 + qdc - cd - qc^2 - (1+q)ad + (1+q)ac}{1+q}$$

$$- \frac{a^2 - qac + ca + qc^2 + (1+q)a^2 - (1+q)ac}{1+q}$$

$$= \frac{d^2-(1+q)da+qa^2}{1+q} - \frac{c(1-q)(d-a)}{1+q}. \qquad (2.5)$$

Notice that if q tends to 1, then (2.5) reduced to be the classical integration which is

$$\int_a^d (z-a)dz = \frac{(d-a)^2}{2}.$$

Theorem 2.1.1 [9]

Assume that $h: I \to \mathbf{R}$ is a continuous function. Then the following hold:

1. ${}_cD_q \int_c^s h(z)\ {}_cd_q z = h(s)$;

2. $\int_a^s\ {}_cD_q h(z)\ {}_cd_q z = h(s) - h(a)$ for $a \in (c,s)$.

Theorem 2.1.2 [9]

Assume that $h, k : I \to \mathbf{R}$ are continuous functions and c, $\alpha \in \mathbf{R}$. For $s \in I$,

1. $\int_c^s [h(z) + k(z)]\ {}_cd_q z = \int_c^s h(z)\ {}_cd_q z + \int_c^s k(z)\ {}_cd_q z$;

2. $\int_c^s (ch)(z)\ {}_cd_q z = c\int_c^s h(z)\ {}_cd_q z$;

3. $\int_a^s h(z)\ {}_cD_q k(z)\ {}_cd_q z = (hk)|_a^s - \int_a^s k(qz + (1-q)c)\ {}_cD_q h(z)\ {}_cd_q z$ for $a \in (c,s)$.

For the other properties of q-derivative and q-integral on finite intervals, see [9].

Lemma 2.1.2

For $c \in \mathbf{R}$ \{-1}, then we have

$$\int_c^s (z-c)^\alpha\ {}_cd_q z = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(s-c)^{\alpha+1}. \tag{2.6}$$

Proof: Assume $h(s) = (s-c)^{\alpha+1}, s \in I$ and $\alpha \in \mathbf{R}$ \{-1}, then, by **Definition 2.1.5**, we get

$${}_cD_q h(s) = \frac{(s-c)^{\alpha+1} - (qs + (1-q)c - c)^{\alpha+1}}{(1-q)(s-c)}$$

$$= \frac{(s-c)^{\alpha+1} - q^{\alpha+1}(s-c)^{\alpha+1}}{(1-q)(s-c)}$$

$$= \left(\frac{1-q^{\alpha+1}}{1-q}\right)(s-c)^{\alpha}. \tag{2.7}$$

Using q-integral on I on (2.7), the result will be (2.6) as desired. ■

Example 2.1.4 Assume that $s \in [c,d]$ and $0 < q < 1$. Applying q-integral by parts and using **Lemmas 2.1.1** and **2.1.2**, we obtain the following.

$$\int_c^s z(z-c)\ {}_c d_q z = \frac{1}{1+q}\int_c^s z\ {}_c D_q (z-c)^2\ {}_c d_q z$$

$$= \frac{1}{1+q}\left[z(z-c)^2|_c^s - \int_c^s (qz+(1-q)c-c)^2\ {}_c d_q z\right]$$

$$= \frac{1}{1+q}\left[s(s-c)^2 - q^2\int_c^s (z-c)^2\ {}_c d_q z\right]$$

$$= \frac{1}{1+q}\left[s(s-c)^2 - \frac{q^2(s-c)^3}{1+q+q^2}\right]$$

$$= \frac{(s-a)^2}{1+q}\left[\frac{s(1+q)+q^2c}{1+q+q^2}\right].$$

Lagrangian Mean Value Theorem (LMVT)

Assume that $f: [c,d] \to \mathbb{R}$ is a continuous function and differentiable on the open interval(c,d). Then there exists some $a \in (c,d)$ such that

$$f'(a) = \frac{f(d)-f(c)}{d-c}.$$

Chapter3

QUANTUM INTEGRAL INEQUALITIES DEFINED ON FINITE INTERVALS

We give generalization of some integral inequalities in quantum sense which are further studied in quantum calculus.

3.1 The q-Hölder Integral Inequality

Let us consider the q-Hölder inequality on interval $I = [c, d]$.

Theorem 3.1.1

Assume that $s \in I, 0 < q < 1, p, \mathrm{r} > 1$ such that $\frac{1}{p} + \frac{1}{\mathrm{r}} = 1$. Then we get

$$\int_c^s |h(z)||k(z)| \ {}_c d_q z \leq \left(\int_c^s |h(z)|^p \ {}_c d_q z\right)^{\frac{1}{p}} \left(\int_c^s |k(z)|^{\mathrm{r}} \ {}_c d_q z\right)^{\frac{1}{\mathrm{r}}}. \qquad (3.1)$$

Proof: Here we use Definition 2.1.8 besides the discrete Hölder inequality to prove the q-Hölder Integral inequality on $I = [c, d]$.

$$\int_c^s |h(z)||k(z)| \ {}_c d_q z$$

$$= (1-q)(s-c) \sum_{m=0}^{\infty} q^m \, h|(q^m s + (1-q^m)c)||k(q^m s + (1-q^m)c)|$$

$$= (1-q)(s-c) \sum_{m=0}^{\infty} (|h(q^m s + (1-q^m)c)|(q^m)^{\frac{1}{p}})$$

$$\times (|k(q^m s + (1-q^m)c)|(q^m)^{\frac{1}{\mathrm{r}}})$$

$$\leq \left((1-q)(s-c) \sum_{m=0}^{\infty} |h(q^m s + (1-q^m)c)|^p q^m \right)^{\frac{1}{\mathrm{p}}}$$

$$\times \left((1-q)(s-c) \sum_{m=0}^{\infty} |k(q^m s + (1-q^m)c)|^{\mathrm{r}} q^m \right)^{\frac{1}{\mathrm{r}}}$$

$$= (\int_c^s |h(z)|^p \ {}_c d_q z)^{\frac{1}{\mathrm{p}}} (\int_c^s |k(z)|^{\mathrm{r}} \ {}_c d_q tz)^{\frac{1}{r}}.$$

Hence the theorem about q-Hölder inequality is obtained. ■

Remark 3.1.1 If $c = 0$, then the inequality (3.1) reduce to the classical q-Hölder integral in [[2], p.604].

3.2 The q-Hermite-Hadamard Integral Inequality

Now, we will give the important integral inequality defined on $[c, d]$, the q-Hermite-Hadamard integral inequality.

Theorem 3.1.2

Let the function $h: I \to \mathbf{R}$ be a convex continuous function on I and $0 < q < 1$. Then

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d h(z) \ {}_c d_q z \leq \frac{qh(c)+h(d)}{1+q} \tag{3.2}$$

holds.

Proof: Using **Definition 2.1.8**, and using the convex condition in quantum sense with respect to z on $[c, d]$, we obtain

$$h\big((1-z)c+zd\big) \leq (1-z)h(c)+zh(d) \quad , \tag{3.3}$$

for all $z \in [0,1]$.

$$\int_0^1 h\left((1-z)c+zd\right)\ {}_0d_qz \leq h(c)\int_0^1 (1-z)\ {}_0d_qz + h(d)\int_0^1 z\ {}_0d_qz. \tag{3.4}$$

It is clear from **Example 2.1.2**, that

$$\int_0^1 z\ {}_0d_qz = \frac{1}{1+q} \text{ and } \int_0^1 (1-z)\ {}_0d_qz = \frac{q}{1+q}.$$

Definition of q-Integration on I leads

$$\int_0^1 h(\,(1-z)c+zd)\ {}_0d_qz = (1-q)\sum_{m=0}^{\infty} q^m h\big((1-q^m)c+q^m d\big)$$

$$= \frac{(1-q)(d-c)}{(d-c)}\sum_{m=0}^{\infty} q^m h\big((1-q^m)c+q^m d\big)$$

$$= \frac{1}{d-c}\int_c^d h(z)\ {}_cd_qz,$$

which gives the second part of (3.2) by using (3.4).

To prove the first part of (3.2), we use the convexity property of h as follows:

$$\frac{1}{2}\left[h\big((1-z)c+zd\big)+h(zc+(1-z)d)\right]$$

$$\geq h\left(\frac{(1-z)c+zd+zc+(1-z)d}{2}\right)$$

$$= h\left(\frac{c+d}{2}\right).$$

Next, we take q-integral of the inequality above with respect to z on $[0,1]$ and change the variables to obtain

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{2}[\int_0^1 h\big((1-z)c+zd\big)\ {}_0d_qz + \int_0^1 h(zc+(1-z)d)\ {}_0d_qz]$$

$$= \frac{1}{d-c}\int_c^d h(z)\ \ {}_cd_qz\ .$$

We get the result.

Remark 3.1.2 If q tends to 1 then inequality (3.2) becomes the Hermite-Hadmard integral inequality

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d h(z)dz \leq \frac{h(c)+h(d)}{2}.$$

See [4, 10].

3.3 The q-Trapezoid Integral Inequality

The following inequality is the q-Trapezoid inequality on the interval $I = [c,d]$.

Here for the typical supremum, we use the notation $||\cdot||$ on $[c,d]$.

Theorem 3.1.3

Assume that $h: I \to \mathbf{R}$ is a q-differentiable function with ${}_cD_qh$ continuous on $[c,d]$ and $0 < q < 1$. Then,

$$\left|\int_c^d h(qz+(1-q)c)\ \ {}_cd_qz - (d-c)\left(\frac{h(d)+h(c)}{2}\right)\right| \leq \frac{(d-c)^2}{2(1+q)}\left\|\ {}_cD_qh\right\|. \tag{3.5}$$

Proof : Applying q-integral by parts on interval I leads to the following,

$$\int_c^d \left(z - \frac{c+d}{2}\right) \ {}_cD_q h(z) \ {}_cd_q z$$

$$= (d-c)\left(\frac{h(d)+h(c)}{2}\right) - \int_c^d h(qz+(1-q)c) \ {}_cd_q z \,. \tag{3.6}$$

Then by using the modulus's properties for (3.6), we get

$$\left|\int_c^d h(qz+(1-q)c) \ {}_cd_q z - (d-c)\left(\frac{h(d)+h(c)}{2}\right)\right|$$

$$\leq \int_c^d \left|z - \frac{c+d}{2}\right| | \ {}_cD_q h(z)| \ {}_cd_q z$$

$$\leq \| \ {}_cD_q z\| \int_c^d \left|z - \tfrac{c+d}{2}\right| \ {}_cd_q z. \tag{3.7}$$

Applying **Example 2.1.2** and **Example 2.1.3**, give us

$$\int_c^d \left|z - \frac{c+d}{2}\right| \ {}_cd_q z = \int_c^{\frac{c+d}{2}} \left(\frac{c+d}{2} - z\right) \ {}_cd_q z + \int_{\frac{c+d}{2}}^d \left(z - \frac{c+d}{2}\right) \ {}_cd_q z$$

$$= \left(\frac{c+d}{2}\right)\left(\frac{d-c}{2}\right) - \left(\frac{d-c}{2}\right)\left(\frac{(1+2q)c+d}{1+q}\right)$$

$$+ \frac{d^2 - d(1+q)\left(\frac{(c+d)}{2}\right) + q((c+d)/2)^2}{1+q} - c\left(\frac{1-q}{2}\right)\left(\frac{d-c}{1+q}\right)$$

$$= \frac{(d-c)^2}{2(1+q)}. \tag{3.8}$$

Joining (3.7) with (3.8), we get (3.5) as desired. ■

Remark 3.1.3 If q tends to 1 then inequality (3.5) reduces to the well-known Trapezoid inequality given as

$$\left|\int_c^d h(z)dz - (d-c)\left(\frac{h(d)+h(c)}{2}\right)\right| \le \frac{(d-c)^2}{4}\|h'\|.$$

See references therein [4, 10].

In the next thoerem, we consider the q-Trapezoid inequality with second order q-derivative on $[c, d]$.

Theorem 3.1.4

Assume that $h : I \to \mathbf{R}$ is a twice q-differentiable function where ${}_cD_q^2 h$ continuous on $[c, d]$ and $0 < q < 1$. We have

$$\left|\int_c^d h(q^2 z + (1-q^2)c)\ {}_cd_q z - \frac{(d-c)}{1+q}(qh(qd + (1-q)c + h(c))\right|$$

$$\le \frac{q^2(d-c)^3}{(1+q)^2(1+q+q^2)}\left\|\ {}_cD_q^2 h\right\|. \tag{3.9}$$

Proof

Appling q-Integration by parts on the interval I twice and having in mind **Example 2.1.1**, we get

$$\int_c^d (z-c)(d-z)\ {}_cD_q^2 h(z)\ \ {}_cd_q z$$

$$= -\int_c^d (qc + d - (1+q)z)\ \ {}_cD_q h(qz + (1-q)c)\ \ {}_cd_q z$$

$$= [-(qc + d - (1+q)z)h(qz + (1-q)c)]_c^d$$

$$+\int_c^d h(q^2z+(1-q^2)c)\ {}_cD_q(qc+d-(1+q)z)\ {}_cd_qz$$

$$=q(d-c)h(qd+(1-q)c+(d-c)h(a)$$

$$-(1+q)\int_c^d h(q^2z(1-q^2)c)\ {}_cd_qz\ .$$

Hence,

$$\left|\int_c^d h(q^2+(1-q^2)c)\ {}_cd_qz-\frac{(d-c)}{1+q}(qh(qd+(1-q)c)+h(c))\right|$$

$$\leq\frac{1}{1+q}\int_c^d(z-c)(d-z)\left|\ {}_cD_q^2h(z)\right|\ {}_cd_qz$$

$$\leq\frac{\|\ {}_cD_q^2h\|}{1+q}\int_c^d(z-c)(d-z)\ {}_cd_qz\,. \tag{3.10}$$

Using **Lemma 2.1.2** and **Example 2.1.4**,

$$\int_c^d(z-c)(d-z)\ {}_cd_az=d\int_c^d(z-c)\ \ {}_cd_qz-\int_c^d z(z-c)\ \ {}_cd_qz$$

$$=d\left[\frac{(z-c)(z+qc)}{1+q}\right]_c^d-\frac{1}{1+q}\int_c^d z\ \ {}_cD_q(z-c)^2\ {}_cd_qz$$

$$=d\left[\frac{(d-c)(d+qc)}{1+q}-cd+c^2\right]$$

$$-\frac{1}{1+q}\left[z(z-c)^2|_c^d-\int_c^d(qz+(1-q)c-c)^2\ {}_cd_qz\right.$$

$$=d\left[\frac{(d-c)(d+qc)-(1+q)cd+(1+q)c^2}{1+q}\right]$$

$$-\frac{1}{1+q}\left[d(d-c)^2 - q^2\int_c^d (z-c)^2\ {}_c d_q z\right]$$

$$= d\left[\frac{d^2 + qcd - cd - qc^2 - (1+q)cd + (1+q)c^2}{1+q}\right]$$

$$-\frac{1}{1+q}\left[d(d-c)^2 - \frac{q^2(d-c)^3}{1+q+q^2}\right]$$

$$= d\left[\frac{d^2 + (q-1-1-q)cd + (-q+1+q)c^2}{1+q}\right] - \frac{(d-c)^2}{1+q}\left[\frac{d(1+q)+q^2c}{1+q+q^2}\right]$$

$$= \frac{d(d-c)^2}{1+q} - \frac{(d-c)^2}{1+q}\left[\frac{d(1+q)+q^2c}{1+q+q^2}\right]$$

$$= \frac{(d-c)^2[d(1+q+q^2) - d(1+q) + q^2c]}{(1+q)(1+q+q^2)}$$

$$= \frac{(d-c)^2(d+dq+dq^2-d-dq-q^2c)}{(1+q)(1+q+q^2)}$$

$$= \frac{q^2(d-c)^3}{(1+q)(1+q+q^2)}. \tag{3.11}$$

Joining (3.10) with (3.11), we realize that the inequality (3.9) is true. ■

Remark 3.1.4

If q tends to 1, then Inequality (3.9) reduced to the Trapezoid inequality expressed in terms of second derivative that is,

$$\left|\int_c^d h(z)dz - \frac{(d-c)}{2}(h(d)+h(c))\right| \le \frac{(d-c)^3}{12}\|h''\|.$$

See [4, 10].

3.4 The q-Ostrowski Integral Inequality

We consider q-Ostrowski integral inequality on the interval I, as following,

Theorem 3.1.5 Assume that $h : I \to \mathbf{R}$ is a q-differentiable function and $\boldsymbol{{}_cD_q h}$ continuous on $[c, d]$, then

$$\left| h(s) - \frac{1}{d-c} \int_c^d h(z) \ {}_cd_q z \right|$$

$$\leq \| {}_cD_q h \| (d-c) \left[\frac{2q}{1+q} \left(\frac{s - \frac{(3q-1)c+(1+q)d}{4q}}{d-c} \right)^2 + \frac{(-q^2+6q-1)}{8q(1+q)} \right]. \qquad (3.12)$$

Proof: Using the Lagrangian Mean Value Theorem, for $\boldsymbol{s}, \boldsymbol{z} \in \boldsymbol{I}$, it follows that

$$\left| h(s) - \frac{1}{d-c} \int_c^d h(z) \ {}_cd_q z \right| = \left| \frac{1}{d-c} \int_c^d (h(s) - h(z)) \ {}_cd_q z \right|$$

$$\leq \frac{1}{d-c} \int_c^d |h(s) - h(z)| \ {}_cd_q z$$

$$\leq \frac{\| {}_cD_q h \|}{d-c} \int_c^d |s - z| \ {}_cd_q z$$

$$= \frac{\| {}_cD_q h \|}{d-c} \left[\int_c^s (s-z) \ {}_cd_q z + \int_s^d (z-s) \ {}_cd_q z \right]. \qquad (3.13)$$

Recalling **Example 2.1.2** and **Example 2.1.3** for $s, z \in I$, we get

$$\int_c^s (s-z) \ {}_cd_q z + \int_s^d (z-s) \ {}_cd_q z$$

$$= \left[\frac{qs^2 - 2qcs + qc^2}{1+q} \right] + \left[\frac{d^2 - (1+q)ds + qs^2}{1+q} - \frac{c(1-q)(d-s)}{1+q} \right]$$

$$= \frac{2q}{1+q} \left[s^2 - \left(\frac{(3q-1)c + (1+q)d}{2q} \right) s \right] + \frac{qc^2 + d^2 - (1-q)cd}{1+q}$$

$$= \frac{2q}{1+q}\left(s - \frac{(3q-1)c + (1+q)d}{4q}\right)^2 + \frac{qc^2 + d^2 - (1-q)cd}{1+q}$$

$$- \frac{(3q-1)^2c^2 + 2cd(3q-1)(1+q) + (1+q)^2d^2}{8q(1+q)}$$

$$= \frac{2q}{1+q}\left(s - \frac{(3q-1)c + (1+q)d}{4q}\right)^2$$

$$+ \frac{8q^2c^2 + 8qd^2 - 8q(1-q)cd - (3q-1)^2c^2 - 2cd(3q-1)(1+q) - (1+q)^2d^2}{8q(1+q)}$$

$$= \frac{2q}{1+q}\left(s - \frac{(3q-1)c + (1+q)d}{4q}\right)^2$$

$$+ \frac{(8q^2 - (3q-1)^2)c^2 - \big(4q(1-q) + (3q-1)\big)(1+q)2cd + (8q - (1-q)^2)d^2}{8q(1+q)}$$

$$= \frac{2q}{1+q}\left(s - \frac{(3q-1)c + (1+q)d}{4q}\right)^2$$

$$+ \frac{(8q^2 - 9q^2 + 6q - 1)c^2 - (4q - 4q^2 + 3q + 3q^2 - 1 - q)2cd + (8q - 1 - 2q - q^2)d^2}{8q(1+q)}$$

$$= \frac{2q}{1+q}\left(s - \frac{(3q-1)c + (1+q)d}{4q}\right)^2$$

$$+ \frac{(-q^2 + 6q - 1)c^2 - (-q^2 + 6q - 1)2cd + (-q^2 + 6q - 1)d^2}{8q(1+q)}$$

$$= \frac{2q}{1+q}\left(s - \frac{(3q-1)c + (1+q)d}{4q}\right)^2 + \frac{(-q^2 + 6q - 1)}{(1+q)8q}(d-c)^2. \tag{3.14}$$

We reach to the inequality (3.12). ■

Remark 3.1.5 If q leads to 1, then inequality (3.12) reduced to the classical Ostrowski integral inequality such that

$$\left|h(s)-\frac{1}{d-c}\int_c^d h(z)\,dz\right| \leq \left[\frac{1}{4}+\left(\frac{s-\frac{c+d}{2}}{d-c}\right)^2\right](d-c)\|h''\|.$$

See [4, 10].

3.5 The q-Korkine Identity

We are going to prove q-Korkine identity on interval I.

Lemma 3.1.1

Assume that $h, k : I \to \mathbf{R}$ are two continuous functions on I and $0 < q < 1$, then

$$\frac{1}{2}\int_{\boldsymbol{c}}^{\boldsymbol{d}}\int_{\boldsymbol{c}}^{\boldsymbol{d}}\big(h(s)-h(w)\big)\big(k(s)-k(w)\big)\ {}_c d_q s\ {}_c d_q w$$

$$= (d-c)\int_c^d h(s)k(s)\ {}_c d_q s - \left(\int_c^d h(s)\ {}_c d_q s\right)\left(\int_c^d k(s)\ {}_c d_q s\right). \quad (3.15)$$

Proof

Using Definition 2.1.8, we get

$$\int_c^d\int_c^d (h(s)-h(w))\,(k(s)-k(w))\ {}_c d_q s\ {}_c d_q w$$

$$= \int_c^d\int_c^d [h(s)k(s)-h(s)k(w)-h(w)k(s)+h(w)k(w)]\ {}_c d_q s\ {}_c d_q w$$

$$= (1-q)(d-c)\sum_{m=0}^{\infty} q^m h(q^m d+(1-q^m)c)\,k(q^m d+(1-q^m)c)(d-c)$$

$$-(1-q)^2(d-c)^2\left(\sum_{m=0}^{\infty} q^m h(q^m d+(1-q^m)c)\right)\left(\sum_{m=0}^{\infty} q^m k(q^m d+(1-q^m)c)\right)$$

$$-(1-q)^2(d-c)^2\left(\sum_{m=0}^{\infty} q^m k(q^m d+(1-q^m)c)\right)\left(\sum_{m=0}^{\infty} q^m h(q^m d+(1-q^m)c)\right)$$

$$+(1-q)(d-c)\sum_{m=0}^{\infty} q^m h(q^m d+(1-q^m)c)k(q^m d+(1-q^m)c)(d-c)$$

$$= 2(d-c)\int_c^d h(s)k(s)\ {}_c d_q s - 2\left(\int_c^d h(s)\ {}_c d_q s\right)\left(\int_c^d k(s)\ {}_c d_q s\right),$$

from which one can obtain the inequality (3.15). ■

3.6 The q-Cauchy-Bunyakovsky-Schwarz Integral Inequality

Now, let us prove the q-Cauchy-Bunyakovsky-Schwartz integral inequality for double integrals on $[c, d]$.

Lemma 3.1.2: Assume that $h, k : I \to \mathbf{R}$ are two continuous functions on I and $0 < q < 1$. Then

$$\left|\int_c^d \int_c^d h(s,w)k(s,w)\ {}_c d_q s\ {}_c d_q w\right|$$

$$\leq \left[\int_c^d \int_c^d h^2(s,w)\ {}_c d_q s\ {}_c d_q w\right]^{\frac{1}{2}} \left[\int_c^d \int_c^d k^2(s,w)\ {}_c d_q s\ {}_c d_q w\right]^{\frac{1}{2}}. \qquad (3.16)$$

Proof: Using **Definition 2.1.8** and doubling q-integral on I as

$$\int_c^d \int_c^d h(s,w)\ {}_c d_q s\ {}_c d_q w$$

$$= \int_c^d \left((1-q)(d-c)\sum_{m=0}^{\infty} q^m h(q^m d+(1-q^m)c, w)\right)\ {}_c d_q w$$

$$= (1-q)^2(d-c)^2 \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} q^{m+i} h(q^m d + (1-q^m)c, q^i b + \left(1-q^i\right)c),$$

using the discrete Cauchy-Schwarz inequality, we get

$$\left(\int_c^d \int_c^d h(s,w)g(s,w) \ {}_c d_q s \ {}_c d_q w\right)^2$$

$$= \left((1-q)^2(d-c)^2 \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} q^{m+i} h(q^m d + (1-q^m)c, q^i d + (1-q^i)c)\right.$$

$$\left. \times\, k(q^m d(1-q^m)c, q^i d + \left(1-q^i\right)c)\right)^2$$

$$\leq \left((1-q)^2(d-c)^2 \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} q^{m+i} h^2(q^m d + (1-q^m)c, q^i d + \left(1-q^i\right)c)\right)$$

$$\times \left((1-q)^2(d-c)^2 \sum_{m=0}^{\infty}\sum_{i=0}^{\infty} q^{m+i} k^2(q^m d + (1-q^m)c, q^i d + \left(1-q^i\right)c)\right)$$

$$= \left(\int_c^d \int_c^d h^2(s,w) \ {}_c d_q s \ {}_c d_q w\right)^{\frac{1}{2}} \left(\int_c^d \int_c^d k^2(s,w) \ {}_c d_q s \ {}_c d_q w\right)^{\frac{1}{2}}.$$

Here we get (3.16)

Remark 3.1.6 If q tends to 1, then **Lemmas 3.1.1** and **3.1.2** are reduced to the usual Korkine identity and Cauchy-Bunyakovsky-Schwarz Integral Inequality for dual integrals, respectively. See [4] and [10].

Now we express q-Chebysev function $T(h,k)$ on interval I by

$$T(h,k) = \frac{1}{d-c}\int_c^d h(s)k(s) \;\, _c d_q s$$

$$-\left(\frac{1}{d-c}\int_c^d h(s) \;\, _c d_q s\right)\left(\frac{1}{d-c}\int_c^d k(s) \;\, _c d_q s\right). \tag{3.17}$$

3.7 The q-Grüss Integral Inequality

Using **Lemmas 3.1.1** and **3.1.2** joined (3.17), we get q-Grüss integral inequality on interval $[c,d]$. The theorem is almost same with the classical Grüss integral inequality as in [4.10]. Thus, we omit it.

$$\left|\frac{1}{d-c}\int_c^d h(s)k(s) \;\, _c d_q s - \left(\frac{1}{d-c}\int_c^d h(s) \;\, _c d_q s\right)\left(\frac{1}{d-c}\int_c^d k(s) \;\, _c d_q s\right)\right|$$

$$\leq \frac{1}{4}\left(M_f - m_f\right)\left(M_g - m_g\right),$$

Where $m_f \leq f(s) \leq M_f$ and $m_g \leq g(s) \leq M_g$

Theorem 3.1.6

Assume that $h, k : I \to \mathbf{R}$ are two continuous functions on $[c,d]$ such that

$\emptyset \leq h(s) \leq \varphi, \gamma \leq k(s) \leq r$ for all $s \in [c,d], \emptyset, \varphi, \gamma, r \in \mathbf{R}$.

Then, the following inequality holds:

$$\left|\frac{1}{d-c}\int_c^d h(s)g(s) \;\, _c d_q s - \left(\frac{1}{d-c}\int_c^d h(s) \;\, _c d_q s\right)\left(\frac{1}{d-c}\int_c^d k(s) \;\, _c d_q s\right)\right|$$

$$\leq \tfrac{1}{4}(\varphi - \emptyset)(\mathrm{r} - \gamma) \tag{3.19}$$

3.8 The q-Grüss-Chebysev Integral Inequality

It is now the time to prove our next and last inequality the q-Grüss-Chebysev integral inequality on interval $[c, d]$.

Theorem 3.1.7 Assume that $h, k : I \to \mathbf{R}$ be L_1, L_2-Lipschitzian continuous functions on $[c, d]$ such that

$$|h(s) - k(w)| \leq L_1|s - w|, |k(s) - k(w)| \leq L_2|s - w|, \qquad (3.20)$$

for all $s, w \in [c, d]$. Then we get the inequality

$$\left|\frac{1}{d-c}\int_c^d h(s)k(s) \ {}_cd_qs - \left(\int_c^d h(s) \ {}_cd_qs\right)\left(\int_c^d k(s) \ {}_cd_qs\right)\right|$$

$$\leq \frac{qL_1L_2}{(1+q+q^2)(1+q)^2}(d-c)^2. \qquad (3.21)$$

Proof: Remember that q-Korkine identity on the interval I was

$$(d-c)\int_c^d h(s)k(s) \ {}_cd_qs - \left(\int_c^d h(s) \ {}_cd_qs\right)\left(\int_c^d k(s) \ {}_cd_qs\right)$$

$$= \frac{1}{2}\int_c^d \int_c^d (h(s) - h(w))(k(s) - k(w)) \ {}_cd_qs \ {}_cd_qw. \qquad (3.22)$$

Using (3.20), for all $s, w \in [c, d]$, we have

$$\left|\left(h(s) - h(w)(k(s) - k(w))\right)\right| \leq L_1L_2(s - w)^2 \qquad (3.23)$$

The double q-integral on (3.23) on $\boldsymbol{I} \times \boldsymbol{I}$ gives

$$\int_c^d \int_c^d |(h(s) - h(w))(k(s) - k(w))| \ {}_cd_qs \ {}_cd_qw$$

$$\leq L_1 L_2 \int_c^d \int_c^d (s-w)^2 \ {}_c d_q s \ {}_c d_q w$$

$$= L_1 L_2 \int_c^d \int_c^d (s^2 - 2sw + w^2) \ {}_c d_q s \ {}_c d_q w$$

$$= L_1 L_2 \left[2(d-c) \int_c^d s^2 \ {}_c d_q s - 2 \left(\int_c^d s \ {}_c d_q s \right)^2 \right] . \qquad (3.24)$$

Indeed,

$$\int_c^d s^2 \ {}_c d_q s = \int_c^d (s - c + c)^2 \ {}_c d_q s$$

$$= \int_c^d (s-c)^2 \ {}_c d_q s + 2c \int_c^d (s-c) \ {}_c d_q s + c^2 \int_c^d \ {}_c d_q s$$

$$= \frac{(d-c)^3}{1+q+q^2} + 2c \frac{(d-c)^2}{1+q} + c^2 (d-c)$$

$$= \frac{(d-c)((1+q)d^2 + 2q^2 cd + q(1+q^2)c^2)}{(1+q)(1+q+q^2)}. \qquad (3.25)$$

Recall that if q tends to 1, then (3.25) turns out to be the integral

$$\int_c^d s^2 ds = \frac{d^3 - c^3}{3}.$$

Then, it is obvious that,

$$(d-c) \int_c^d s^2 \ {}_c d_q s - \left(\int_c^d s \ {}_c d_q s \right)^2 = \frac{q(d-c)^4}{(1+q+q^2)(1+q)^2}. \qquad (3.26)$$

Therefore, from (3.24) and (3.26), we get

$$\int_c^d \int_c^d |(h(s)-h(w))(k(s)-k(w))| \ {}_c d_q s \ {}_c d_q w \leq \frac{2q(d-c)^4}{(1+q+q^2)(1+q)^2} L_1 L_2 \, .$$

By use of (3.22), we reach to (3.21). ■

Remark 3.1.8 Assume that q tends to 1, then inequality (3.21) reduced to classical Grüss-Chebysev integral inequality given as :

$$\left| \frac{1}{d-c} \int_c^d h(s)k(s)\, ds - \left(\frac{1}{d-c} \int_c^d h(s) ds \right) \left(\frac{1}{d-c} \int_c^d k(s) ds \right) \right|$$

$$\leq \frac{L_1 L_2}{12} (d-c)^2.$$

For more details, See [4,10].

Chapter 4

CONCLUSION

We discussed in this thesis some integral inequalities from perspective of quantum calculus. We studied q-analogues of some integral inequalities such as Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Chebysev. In our thesis we had focused on proving some theorems related to q-integral inequalities. Also some examples are given to illustrate the results.

REFERENCES

[1] Jackson F. H.: on a q-definite integrals, Quarterly J. pure Appl. Math. 41, 193-203, (1910).

[2] Anastassiou GA: Intelligent Mathematics: Computational Analysis. Springer, New York; 2011.

[3] Belarbi S, Dahmani Z: on some new fractional integral inequalities. JIPAM. J. Inequal. Pure Apple. Math. 2009. 10: Article ID 86

[4] Cerone P, Dragomir SS: Mathematical Inequalities. CRC press, New York; 2011.

[5] Dahmani Z: new inequalities in fractional integral. Int. J. nonlinear Sci. 2010, 9:493-497.

[6] Dragomir SS: some integral inequalities of Gruss type. Indian J. pure

[7] Ogunmez H, Ozkan UM: fractional quantum integral inequalities. J. inequal. Appl. 2011. 2011: Article ID 787939

[8] Kac V, Cheung p: quantum calculus. Springer, New York; 2002.

[9] Tariboon J, Ntouyas SK: Quantum calculus on finite intervals and applications to implusive difference equations. Adv. Differ. Equ. 2013. 2013: article ID 282

[10] Pachpatte BG: analytic inequalities. Atlantic press, paris; 2012.

[11] Florea A, Niculescu CP: A note on Ostrwski's inequality. J. Inequal. Appl. 2005, 2005(5): 459-468.

[12] Gauchman H: Integral inequalities in q-calculus. Comput. Math. Appl. 2004, 47: 281-300. 10. 1016/S0898-1221(04)90025-9.

[13] The inequalities for some types of q- integrals, computer and mathematics with applications 56 (2008) 2490-2498.