A Study on A New Class of q-Bernoulli, q-Euler and q-Genocchi Polynomials

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ABSTRACT

This thesis is aimed to study a new class of Bernoulli, Euler and Genocchi polynomials in the means of quantum forms. To achieve this aim, we introduce a new class of q-exponential function and using properties of this function; we reach to the interesting formulae. The q-analogue of some familiar relations as an addition theorem for these polynomials is found. Explicit relations between these classes of polynomials are given. In addition, the new differential equations related to these polynomials are studied. Moreover, improved q-exponential function creates a new class of q-Bernoulli numbers and like the ordinary case, all the odd coefficient becomes zero and leads us to the relation of these numbers and q-trigonometric functions. At the end we introduce a unification form of q-exponential function. In this way all the properties of these kinds of polynomials investigated in a general case. We also focus on two important properties of q-exponential function that lead us to the symmetric form of q- Euler, q-Bernoulli and q-Genocchi numbers. These properties and the conditions of them are studied.

Keywords: q-Exponential Function, q-Calculus, q-Polynomials, q-Bernoulli, q-Euler, q-Genocchi, q-Trigonometric Functions.

ÖZ

Bu tez kuantum formları uasıtası ile Bernoulli, Euler ve Gennochi polinomların yeni bir sınıfını incelemeyi amaçlamaktadır. Bu amaca ulaşmak için, q-üstel fonksiyonları ve bu fonksiyonların özellikleri kullanılarak yeni bir sınıf tanıtılmıştır.Bu polinomlar için bazı bilindik ilıskilerin q-uyarlamları ek teorem olarak bulunmuştur.

Bu tür polinom sınıfları arasındaki kapalı ilişkiler verilmiştir. Ayrıca bu polinomlarla ilişkili yeni diferansiyel denklemler çalışılmıştır. Geliştirilmiş q-üstel fonksiyonlarnin yeni bir q-Bernoulli sayısı oluşturduğu da gösterilmiştir. Buna göre bilinen q-1 durumunda oldügu gibi tüm tek katsayların sıfır olarak q-üstel fonksiyonları, için birleşme formu, elde edilmiştir. Böylelıkle, bu tür polinomların tüm özellikleri genelleştirilmiştir. Ayrıca, q-Euler, q-Bernoulli ve q-Gennochi sayılaının simetri formlarını elde etmemızi sağloyon, iki önemli q-üstel fonksiyon özelliğine de odaklanılmıştır.

Anahtar Kelimeler: q-Üstel Fonksiyonlar, q-Kalkülüs, q-Polinomlar, q-Bernoulli, q-Euler, q-Genocchi, q-Trigonometrik Fonksiyonla.

To the spirit of my father

To my compassionate unique mother

And

To my lovely Brother and sisters

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Chapter 1

INTRODUCTION

1.1 Classical Bernoulli Numbers

Two thousand years ago, Greek mathematician Pythagoras noted about triangle numbers that is $1+2+\cdots+n$. after that time, Archimedes proposed

$$1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1). \tag{1.1.1}$$

Later, Aryabhata, The Indian mathematician, found out

$$1^3+2^3+\cdots+n^3=\left(\frac{1}{2}n(n+1)\right)^2. \tag{1.1.2}$$

But Jacobi was the first who gave the vigorous proof in 1834. AL-Khwarizm, the Arabian mathematician found this summation's result for the higher power. He showed that

$$1^4+2^4+\cdots+n^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1). \tag{1.1.3}$$

The more generalized formula for $\sum_{k=1}^{n} k^r$, where r is any natural number is studied in the last few centuries. Among this investigation, the Bernoulli numbers are much significant. In this chapter, we present an elementary examination of the development of Bernoulli numbers throughout the time. We also aim to explore its properties and its application in other fields of mathematics. In addition, by introducing the generating function, we will take a look to the exponential function and its generalization to the quantum form [12].

Swiss mathematician Jacob Bernoulli (1654-1705), was the person who found the general formula for this kind of the summation. He found that [3]

$$S_n(r) = \sum_{k=1}^{n-1} k^r = \sum_{k=0}^{r} \frac{B_k}{k!} \frac{r!}{(r-k+1)!} n^{r-k+1}. \tag{1.1.4}$$

The B_k's numbers here are independent of r and named Bernoulli numbers. The first few Bernoulli's numbers are as following:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots$$

It is tempting to guess that $|B_n| \to 0$ as $n \to \infty$. However, if we consider some other numbers in the sequence,

$$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \ldots$$

We notice their values are general growing with alternative sign.

An equivalent definition of the Bernoulli's numbers is obtained by using the series expansion as a generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{1.1.5}$$

In fact, by writing the Taylor expansion for e^{kt} and adding them together, we have:

$$\sum_{m=0}^{\infty} S_m(n) \frac{t^m}{m!} = 1 + e^t + e^{2t} + \cdots + e^{(n-1)t} = \frac{1 - e^{nt}}{1 - e^t} = \frac{1 - e^{nt}}{t} \times \frac{t}{1 - e^t}$$

$$= \left(\sum_{k=1}^{\infty} n^k \frac{t^{k-1}}{k!} \right) \left(\sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m-k+1} \right) \frac{t^m}{m!} \frac{1}{m+1}.$$

This calculation, lead us to (1.1.4). This technique of proof is used temporary. In addition, this definition of Bernoulli's number connects them to the trigonometric function as following:

$$\frac{x}{e^x-1}+\frac{x}{2}=\frac{x}{2}\left(\frac{2}{e^x-1}+1\right)=\frac{x}{2}\frac{e^x+1}{e^x-1}=\frac{x}{2}Coth\left(\frac{x}{2}\right). \qquad (1.1.6)$$

It can be rewritten

$$\frac{x}{2}Coth\left(\frac{x}{2}\right)=\sum_{n=0}^{\infty}\frac{B_{2n}x^{2n}}{(2n)!}. \qquad (1.1.7)$$

If we substitute x with $2ix$, then it gives

$$xCot(x)=\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}(2x)^{2n}}{(2n)!} \qquad x\in[-\pi,\pi]. \qquad (1.1.8)$$

The following identities can be written

$$tanh(x)=\sum_{n=1}^{\infty}\frac{2(4^n-1)B_{2n}(2x)^{2n-1}}{(2n)!} \qquad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \qquad (1.1.9)$$

$$tan(x)=\sum_{n=1}^{\infty}(-1)^n\frac{2(1-4^n)B_{2n}(2x)^{2n-1}}{(2n)!} \qquad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right). \qquad (1.1.10)$$

Another equivalent definition of Bernoulli number, which is useful, is the recurrence formula for these numbers

$$\begin{cases}\displaystyle\sum_{j=0}^{n-1}\binom{n}{j}B_j=0, \quad n\geq 2.\\ \qquad\qquad B_1=1\end{cases} \qquad (1.1.11)$$

This definition can be found easily from the series definition, by writing the Taylor expansion for exponential function and using the Cauchy product. This formula can be used to evaluate the Bernoulli numbers inductively (see [5]).As we saw, the exponential function made a main role in these definitions, and by changing the alternative functions; we can reach to a new definitions. This idea was leading a lot of mathematician to find a new version of the Bernoulli numbers; we also did one of them.

The Bernoulli polynomials are defined in the means of Taylor expansion as following

$$\frac{ze^{zx}}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n. \tag{1.1.12}$$

For each nonnegative integer n, these $B_n(x)$, are the polynomials with respect to x. Taking derivative on both sides of (1.1.12) a derivative with respect to x, we get

$$\sum_{n=0}^{\infty}\frac{B'_n(x)}{n!}z^n=z\frac{ze^{zx}}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^{n+1}. \tag{1.1.13}$$

Equating coefficient of z^n, where $n\geq 1$, leads us to another important identity

$$B'_n(x)=nB_{n-1}(x). \tag{1.1.14}$$

The fact $B_0(x)=1$, can be yield by tending z to zero at (1.1.13). This and the above identity together, show that $B_n(x)$ is a polynomial in degree of n and it begins with the coefficient that is unity. If we use this identity and know what the constant terms are, then we could evaluate $B_n(x)$ (Bernoulli polynomials) one by one. It is clear that, by putting $x=0$, we reach to the Bernoulli's numbers. There are too many interesting properties for this polynomials and numbers, which are studied in the last few centuries. The application of them is going forward to the many branches of mathematics and physics, as combinatorics, theory of numbers, quantum information, etc. (see [20], [24]) We will list some of these properties without proof. The proofs can be found at [14].

$$B_m(x+1)-B_m(x)=mx^{m-1}, \tag{1.1.15}$$

$$B_m(1-x)=(-1)^mB_m(x), \tag{1.1.16}$$

$$B_m = \sum_{k=0}^{m} \frac{1}{k+1} \left(\sum_{r=0}^{k} (-1)^r \binom{k}{r} r^m \right), \tag{1.1.17}$$

$$m! = \sum_{k=0}^{m} (-1)^{k+m} \binom{m}{k} B_m(k), \tag{1.1.18}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \rightarrow 2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}. \tag{1.1.19}$$

Where $\zeta(s)$ is known as Riemann-zeta function at (1.1.19).

1.2 Quantum Calculus and q-Exponential Function

The fascinate world of quantum calculus has been started by the definition of derivative, where the limit has not been taken.

Consider the derivative expression without any limit, as $\frac{f(x)-f(x_0)}{x-x_0}$. The familiar definition of derivative $\frac{df}{dx}$ of a function $f(x)$ at $x = x_0$ can be yield by tending x to x_0. However, if we take $x = qx_0$ (where $q \neq 1$ and is a fixed number) and do not take a limit, we will arrive at the world of quantum calculus. The corresponding expression is a definition of the q-derivative of $f(x)$. The theory of quantum calculus can be traced back at a century ago to Euler and Gauss [7] [2]. Moreover, the significant contributions of Jackson made a big role [9]. Recently in these days, a lot of scientifics are working in this field to develop and apply the q-calculus in mathematical physics, especially concerning quantum mechanics [18] and special functions [10], many papers were mentioned the various models of elementary functions, including trigonometric functions and exponential by deforming formula of the functions in the means of quantum calculus [1]. For instance, many notions and results is discovered along the traditional lines of ordinary calculus, the q-derivative of x^n becomes $[n]_q x^{n-1}$, where $[n]_q = \frac{q^n-1}{q-1}$. This q-analogue of the number helps us to define the new version of familiar functions such an exponential one. We redefine these functions by their Taylor expansion and the different notation. In this case, two kind of q-exponential functions are defined as follows

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} = \prod_{j=0}^{\infty} \frac{1}{(1-(1-q)q^j x)} \quad 0 < |q| < 1, |x| < \frac{1}{|1-q|}, \tag{1.2.1}$$

$$E_q^x = \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}} x^j}{[j]_q!} = \prod_{j=0}^{\infty} \left(1+(1-q)q^j x\right) \quad 0 < |q| < 1, x \in \mathbb{C}. \tag{1.2.2}$$

These definitions are coming from q-Binomial theorem that will discuss in the next chapter.

Many mathematicians encourage defining the q-functions for the preliminary functions, such that, they coincide with the classic properties of them (1.2.1) and (1.2.2) identities are discovered by Euler. In general $e_q^x e_q^y \neq e_q^{x+y}$. But additive property of the q-exponentials holds true if x and y are not commutative i.e. $yx = qxy$. It is rarely happened, and make a lot of restrictions to use them. So, we used the improved exponential function as follows [6]

$$\varepsilon_q^x = e_q^{\frac{x}{2}} E_q^{\frac{x}{2}} = \sum_{j=0}^{\infty} \frac{x^j (1+q) \ldots (1+q^{j-1})}{2^{j-1} [j]_q!} = \prod_{j=0}^{\infty} \frac{(1+(1-q)q^j \frac{x}{2})}{(1-(1-q)q^j \frac{x}{2})} \tag{1.2.3}$$

For the remained, we assume that $0 < q < 1$. The improved q-exponential function is analytic in the disk $|z| < \frac{2}{1-q}$. The important property of this function is

$$\varepsilon_q^{-z} = \frac{1}{\varepsilon_q^z} \quad , \quad \left|\varepsilon_q^{ix}\right| = 1 \quad z \in \mathbb{C}, x \in \mathbb{R} \tag{1.2.4}$$

At the end, we will unify the q-exponential functions and investigate the properties of this general case. In addition q-analogue of some classical relations will be given.

Chapter 2

PRELIMINARY AND DEFINITIONS

2.1 Definitions and Notations

In this section we introduce some definitions and also some related theorem about q-calculus. Base of these information are [7], [14] and [24]. All of these definitions and notations can be found there.

Definition 2.1. Let us assume that $f(x)$ is an arbitrary function, then q-differential is defined by the following expression

$$d_q\big(f(x)\big) = f(qx) - f(x). \tag{2.1.1}$$

We can call the following expression by q-derivative of $f(x)$

$$D_q\big(f(x)\big) = \frac{d_q\big(f(x)\big)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x} \quad 0 \neq x \in \mathbb{C}, \qquad |q| \neq 1 \tag{2.1.2}$$

Note that $\lim_{q \to 1^-} D_q\big(f(x)\big) = \frac{df(x)}{dx}$, where $f(x)$ is a differentiable function. The q-analogue of product rule and quotient rule can be demonstrated by

$$D_q(f(x)g(x)) = f(qx)D_q\big(g(x)\big) + g(x)D_q\big(f(x)\big), \tag{2.1.3}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q\big(f(x)\big) - f(x)D_q\big(g(x)\big)}{g(x)g(qx)}. \tag{2.1.4}$$

By symmetry, we can interchange f and g, and obtain another form of these expressions as well. Let us introduce the q-number's notation

$$[n]_q = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + 1, \quad q \in \mathbb{C}\backslash\{1\}, n \in \mathbb{C}, q^n \neq 1. \tag{2.1.5}$$

In a natural way $[n]_q!$ can be defined by

$$[0]_q! = 1, \qquad [n]_q! = [n-1]_q!\ [n]_q. \tag{2.1.6}$$

Remark 2.2. It is clear that

$$\lim_{q\to 1}[n]_q = n, \quad \lim_{q\to 1}[n]_q! = n!\,. \tag{2.1.7}$$

Definition 2.3. For any complex number b, we can define q-shifted factorial inductively as following

$$(b;q)_0 = 1, \quad (b;q)_n = (b;q)_{n-1}\big(1-q^{n-1}b\big), \ \ n \in \mathbb{N}. \tag{2.1.8}$$

and in a case that $n \to \infty$, we have

$$(b;q)_\infty = \prod_{k=0}^{\infty}\big(1-q^k b\big), \quad |q| < 1, b \in \mathbb{C}. \tag{2.1.9}$$

Definition 2.4. The q-binomial coefficient can be defined as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, \quad k, n \in \mathbb{N}. \tag{2.1.10}$$

We can present q-binomial coefficient by q-shifted factorial

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} \tag{2.1.11}$$

Theorem 2.5. Suppose that f is a real function on a closed interval $[a,b], n$ is a positive integer, $f^{(n)}(x)$ exists for every $x \in (a,b)$ and $f^{(n-1)}$ is continuous on this interval. Let α, β be two distinct numbers of this interval, and define

$$P(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k, \tag{2.1.12}$$

Thus there exist a number y, such that $\alpha < y < \beta$ and

$$f(\beta) = P(\beta) + \frac{f^{(n)}(y)}{n!}(\beta-\alpha)^n. \tag{2.1.13}$$

Remark 2.6. The previous theorem is Taylor theorem [21], the general form of theorem is presented next [24]. By using the next theorem we lead to the definitions of classical q-exponential functions.

Theorem 2.7. Let b be an arbitrary number and D is defined as a linear operator on the space of polynomials and $\{P_0(x), P_1(x), P_2(x), ...\}$ be a sequence of polynomials satisfying three conditions:

1) $P_0(b) = 1$ and $P_n(b) = 0$ for any $n \geq 1$;
2) Degree of $P_n(x)$ is equal to n
3) For any $n \geq 1$, $D(1) = 0$ and $D\big(P_n(x)\big) = P_{n-1}(x)$.

Then, for any polynomial $f(x)$ of degree N, one has the following generalized Taylor formula:

$$f(x) = \sum_{n=0}^{N} (D^n f)(a) P_n(x). \tag{2.1.14}$$

Definition 2.8. The q-analogue of n-th power of $(x + a)$ is $(x + a)_q^n$ and defined by the following expression

$$(x + a)_q^n = \begin{cases} 1 & if\ n = 0 \\ (x + a)(x + qa) \dots (x + q^{n-1}a) & if\ n \geq 1 \end{cases}. \tag{2.1.15}$$

Corollary 2.9. Gauss's binomial formula can be presented by

$$(x + a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} a^j x^{n-j}. \tag{2.1.16}$$

Definition 2.10. Suppose that $0 < q < 1$. If for some $0 \leq \vartheta < 1$, value of $\left|f(x)x^{\vartheta}\right|$ is bounded on the interval $(0, B]$, then the following integral is defined by the series that converged to a function $F(x)$ on $(0, B]$ and is called Jackson integral

$$\int f(x) d_q x = (1 - q)x \sum_{j=0}^{\infty} q^j f(q^j x). \tag{2.1.17}$$

Definition 2.11. Classical q-exponential functions are defined by Euler [24]

$$e_q(x) = \sum_{m=0}^{\infty} \frac{x^m}{[m]_q!} = \prod_{m=0}^{\infty} \frac{1}{(1 - (1 - q)q^m x)}, \quad 0 < |q| < 1, |x| < \frac{1}{|1 - q|},$$

$$E_q(x) = \sum_{m=0}^{\infty} \frac{x^m q^{m(m-1)/2}}{[m]_q!} = \prod_{m=0}^{\infty} (1 + (1-q)q^m x), \;\; 0 < |q| < 1, x \in \mathbb{C}.$$

Proposition 2.12. Some properties of q-exponential functions are listed as follow

(a) $\left(e_q(x)\right)^{-1} = E_q(x), \quad e_{\frac{1}{q}}(x) = E_q(x),$

(b) $D_q E_q(x) = E_q(qx), \;\; D_q e_q(x) = e_q(x),$

(c) $e_q(x+y) = e_q(x).e_q(y) \quad if \quad yx = qxy$

Definition 2.13. These q-exponential functions that defined at 2.11 generate two pair of the q-trigonometric functions. We have

$$sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i}, \qquad Sin_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}$$

$$cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2}, \qquad Cos_q(x) = \frac{E_q(ix) + E_q(-ix)}{2}.$$

Proposition 2.14. We can easily derive some properties of standard q-trigonometric functions by taking into account the properties of q-exponential function

(a) $cos_q(x)Cos_q(x) + sin_q(x)Sin_q(x) = 1,$

(b) $Cos_q(x)sin_q(x) = cos_q(x)Sin_q(x),$

(c) $D_q\left(sin_q(x)\right) = cos_q(x), \;\; D_q\left(cos_q(x)\right) = -sin_q(x),$

(d) $D_q\left(Sin_q(x)\right) = Cos_q(qx), \;\; D_q\left(Cos_q(x)\right) = -Sin_q(qx).$

Remark 2.15. The corresponding tangents and cotangents coincide $tan_q(x) = Tan_q(x), \; cot_q(x) = Cot_q(x).$

2.2 Improved q-Exponential Function

Definition 2.16. Improved q-exponential function is defined as follows

$$\varepsilon_q(x) := e_q\left(\frac{x}{2}\right) E_q\left(\frac{x}{2}\right) = \prod_{m=0}^{\infty} \frac{1 + (1-q)q^m \frac{x}{2}}{1 - (1-q)q^m \frac{x}{2}}, \tag{2.2.1}$$

Theorem 2.17. The q-exponential function $\varepsilon_q(x)$ which is defined at (2.2.1) is analytic in the disk $|x| < R_q$, where R_q is as follows

$$R_q = \begin{cases} \dfrac{2}{1-q} & if \;\; 0 < q < 1 \\ \dfrac{2q}{1-q} & if \quad q > 1 \\ \infty & if \quad q = 1. \end{cases}$$

Moreover, we can write the following expansion for $\varepsilon_q(x)$ [6]

$$\varepsilon_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \frac{(-1;q)_n}{2^n}. \qquad (2.2.2)$$

Theorem 2.18. For $z \in \mathbb{C}$, $x \in \mathbb{R}$, improved q-exponential function $\varepsilon_q(z)$, has the following property

(a) $\left(\varepsilon_q(z) \right)^{-1} = \varepsilon_q(-z), \quad \left| \varepsilon_q(ix) \right| = 1,$

(b) $\varepsilon_q(z) = \varepsilon_{\frac{1}{q}}(z), \quad D_q\left(\varepsilon_q(z)\right) = \frac{\varepsilon_q(z) + \varepsilon_q(qz)}{2}.$

Remark 2.19. As we mentioned it before, in a general case $e_q(x+y) \neq e_q(x).e_q(y)$. One of the advantages of using improved q-exponential is the property (a) at the previous theorem. The form of improved q-exponential, motivates us to define the following

Definition 2.20. The q-addition and q-subtraction can be defined as follow

$$\left(x \oplus_q y\right)^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1;q)_k (-1;q)_{n-k}}{2^n} x^k y^{n-k}, \quad n = 0,1,2,..., \qquad (2.2.3)$$

$$\left(x \ominus_q y\right)^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1;q)_k (-1;q)_{n-k}}{2^n} x^k (-y)^{n-k}, \quad n = 0,1,2,.... \qquad (2.2.4)$$

The direct consequence of this definition is the following identity

$$\varepsilon_q(tx)\, \varepsilon_q(ty) = \sum_{n=0}^{\infty} \left(x \oplus_q y\right)^n \frac{t^n}{[n]_q!}. \qquad (2.2.5)$$

Remark 2.21. The properties that mentioned at (2.18) encourage some mathematicians to use this improved q-exponential in their works [11], [17]. Recently, we are working on other applications of improved q-exponential.

2.3 The New Class of q-Polynomials

In this section, we study a new class of q-polynomials including q-Bernoulli, q-Euler and q-Genocchi polynomials. First, we discuss about the classic definitions of them.

Definition 2.22. Classic Bernoulli, Euler and Genocchi polynomials can be defined by their generating functions as following. We named them $B_n(x), E_n(x)$ and $G_n(x)$ repectively.

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\qquad \frac{2e^{tx}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\qquad \frac{2te^{tx}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}.$$

If we put $x=0$, we have $B_n(0)=b_n$, $E_n(0)=e_n$, $G_n(0)=g_n$, which we call them Bernoulli numbers, Euler numbers and Genocchi numbers respectively. Also we can define them by the recurrence formula, which is coming from the Cauchy product of both sides of generating function. Main idea of introducing the new version of these polynomials is coming from these definitions. The q-Bernoulli numbers and polynomials have been studied by Carlitz, when he modified the form of recurrence formula. [4] Srivastava and Pint´er demonstrated a few theorems and they found the explicit relations that exist between the Euler polynomials and Bernoulli polynomials in [19]. Also they generalize some of these polynomials. Some properties of the q-analogues of Bernoulli polynomials and Euler polynomials and Genocchi polynomials are found by Kim et al. In [23], [13]. Some recurrence relations were given in these papers. In addition, the extension of Genocchi numbers in the means of quantum is studied by Cenkci et al. in a different manner at [15].

Definition 2.23. Assume that q is a complex number that $0 < |q| < 1$. Then we can define q-Bernoulli numbers $b_{n,q}$ and polynomials $B_{n,q}(x,y)$ as follows by using generating functions

$$\widehat{B_q} := \frac{t\, e_q\left(-\frac{t}{2}\right)}{e_q\left(\frac{t}{2}\right) - \, e_q\left(-\frac{t}{2}\right)} = \frac{t}{\varepsilon_q(t) - 1} = \sum_{n=0}^{\infty} b_{n,q} \frac{t^n}{[n]_q!} \; where \quad |t| < 2\pi, \tag{2.3.1}$$

$$\frac{t\, \varepsilon_q(tx)\, \varepsilon_q(ty)}{\varepsilon_q(t) - 1} = \sum_{n=0}^{\infty} B_{n,q}(x,y) \frac{t^n}{[n]_q!} \; where \quad |t| < 2\pi. \tag{2.3.2}$$

Definition 2.24. Assume that q is a complex number that $0 < |q| < 1$. Then we can define q-Euler numbers $e_{n,q}$ and polynomials $E_{n,q}(x,y)$ as follows by using generating functions

$$\widehat{E_q} := \frac{2\, e_q\left(-\frac{t}{2}\right)}{e_q\left(\frac{t}{2}\right) + \, e_q\left(-\frac{t}{2}\right)} = \frac{2}{\varepsilon_q(t) + 1} = \sum_{n=0}^{\infty} e_{n,q} \frac{t^n}{[n]_q!} \; where \quad |t| < \pi, \tag{2.3.2}$$

$$\frac{2\, \varepsilon_q(tx)\, \varepsilon_q(ty)}{\varepsilon_q(t) + 1} = \sum_{n=0}^{\infty} E_{n,q}(x,y) \frac{t^n}{[n]_q!} \; where \quad |t| < 2\pi. \tag{2.3.3}$$

Definition 2.25 Assume that q is a complex number that $0 < |q| < 1$. Then in a same way, we can define q-Genocchi numbers $g_{n,q}$ and polynomials $G_{n,q}(x,y)$ as follows by using generating functions

$$\widehat{G_q} := \frac{2t\, e_q\left(-\frac{t}{2}\right)}{e_q\left(\frac{t}{2}\right) + \, e_q\left(-\frac{t}{2}\right)} = \frac{2t}{\varepsilon_q(t) + 1} = \sum_{n=0}^{\infty} g_{n,q} \frac{t^n}{[n]_q!} \; where \quad |t| < \pi, \tag{2.3.4}$$

$$\frac{2t\, \varepsilon_q(tx)\, \varepsilon_q(ty)}{\varepsilon_q(t) + 1} = \sum_{n=0}^{\infty} G_{n,q}(x,y) \frac{t^n}{[n]_q!} \; where \quad |t| < \pi. \tag{2.3.5}$$

Definition 2.26. Like the previous definitions, we will assume that q is a complex number that $0 < |q| < 1$. The q-tangent numbers $\mathfrak{T}_{n,q}$ can be defined as following by using generating functions as following

$$tanh_q t = itan_q(it) = \frac{e_q(t) - e_q(-t)}{e_q(t) + e_q(-t)} = \frac{\varepsilon_q(2t) - 1}{\varepsilon_q(2t) + 1} = \sum_{n=1}^{\infty} \mathfrak{T}_{2n+1,q} \frac{(-1)^n t^{2n+1}}{[2n+1]_q!}.$$

Remark 2.27. The previous definitions are q-analogue of classic definitions of Bernoulli, Euler and Genocchi polynomials. By tending q to 1 from the left side, we derive to the classic form of these polynomials. That means

$$\lim_{q\to 1^-} B_{n,q}(x) = B_n(x), \quad \lim_{q\to 1^-} b_{n,q} = b_n, \tag{2.3.8}$$

$$\lim_{q\to 1^-} E_{n,q}(x) = E_n(x), \quad \lim_{q\to 1^-} e_{n,q} = e_n, \tag{2.3.6}$$

$$\lim_{q\to 1^-} G_{n,q}(x) = G_n(x), \quad \lim_{q\to 1^-} g_{n,q} = g_n. \tag{2.3.7}$$

In the next chapter, we will introduce some theorems and we reach to a few properties of these new q-analogues of Bernoulli, Euler and Genocchi polynomials.

Chapter 3

APPROACH TO THE NEW CLASS OF q-BERNOULLI, q-EULER AND q-GENNOCHI POLYNOMIALS

3.1 Relations to The q-Trigonometric Functions

This chapter is based on [17], we discovered some new results corresponding to the new definition of q-Bernoulli, q-Euler and q-Genocchi polynomials. The results are presented one by one as a lemma and propositions. Here, the details of proof and techniques are given.

Lemma 3.1. Following recurrence formula is satisfied by q-Bernoulli numbers $b_{n,q}$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1;q)_{n-k}}{2^{n-k}} b_{k,q} - b_{n,q} = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \tag{3.1.1}$$

Proof. The statement can be found by the simple multiplication on generating function of q-Bernoulli numbers (2.3.1). We have

$$\widehat{B_q}(t)\varepsilon_q(t) = t + \widehat{B_q}(t)$$

This implies that

$$\sum_{n=o}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} b_{k,q} \frac{t^n}{[n]_q!} = t + \sum_{n=o}^{\infty} b_{n,q} \frac{t^n}{[n]_q!}.$$

Comparing t^n-coefficient observe the expression.

Similar relations are hold for q-Euler and q-Genocchi numbers. If we do the same thing to their generating functions, we will find the following recurrence formulae

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} e_{k,q} + e_{n,q} = \begin{cases} 2, & n = 0, \\ 0, & n > 0, \end{cases} \tag{3.1.2}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} g_{k,q} + g_{n,q} = \begin{cases} 2, & n = 1, \\ 0, & n > 1. \end{cases} \tag{3.1.3}$$

These recurrence formulae help us to evaluate these numbers inductively. The first few q-Bernoulli, q-Euler, q-Genocchi numbers are given as following. The interesting thing over here is that, these values coincide with the classic values better than the previous one. Actually, the odd terms of q-Bernoulli numbers are zero as classic Bernoulli numbers and lead us to make a connection to a relation with trigonometric functions.

$$b_{0,q} = 1, \qquad b_{1,q} = -\frac{1}{2}, \qquad b_{2,q} = \frac{[3]_q[2]_q - 4}{4[3]_q}, \qquad b_{3,q} = 0$$

$$e_{0,q} = 1, \qquad e_{1,q} = -\frac{1}{2}, \qquad e_{2,q} = 0, \qquad e_{3,q} = \frac{q(1+q)}{8},$$

$$g_{0,q} = 0, \qquad g_{1,q} = 1, \qquad g_{2,q} = -\frac{q+1}{2}, \qquad g_{3,q} = 0.$$

Lemma 3.2. The odd coefficients of the q-Bernoulli numbers except the first one are zero, which means that $b_{n,q} = 0$ for $n = 2r+1, r \in \mathbb{N}$.

Proof. It is the direct consequence of the fact, that the function

$$f(t) = \sum_{n=0}^{\infty} b_{n,q} \frac{t^n}{[n]_q!} - b_{1,q} t = \frac{t}{\varepsilon_q(t) - 1} + \frac{t}{2} = \frac{t}{2}\left(\frac{\varepsilon_q(t) + 1}{\varepsilon_q(t) - 1}\right) \tag{3.1.4}$$

is even, and the coefficient of t^n in the McLaurin series of any arbitrary even function like $f(t)$, for all odd power n will be vanished. It's based on this fact that if f is an even function, then for any n we have $f^{(n)}(t) = (-1)^n f^{(n)}(-t)$ therefore for any odd n we lead to $f^{(n)}(0) = (-1)^n f^{(n)}(0)$. Since $\varepsilon_q(-t) = \left(\varepsilon_q(t)\right)^{-1}$, It could be happen.

Corollary 3.3. The following identity is true

$$\sum_{k=1}^{2n-2} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k}}{2^{2n-k}} b_{k,q} = -1 \quad n > 1 \tag{3.1.5}$$

Proof. Since $b_{2n-1,q} = 0$ for $n > 1$, and $b_{0,q} = 1$, simple substitution at recurrence formula lead us to this expression.

Next lemma shows q-trigonometric functions with q-Bernoulli and q-Genocchi's demonstration. We will expand $tanh_q(x)$ and $cot_q(x)$ in terms of q-Genocchi and q-Bernoulli numbers respectively

Lemma 3.4. The following identities are hold

$$\begin{aligned} tcot_q(x) &= 1 + \sum_{n=1}^{\infty} b_{n,q} \frac{(-4)^n t^{2n}}{[2n]_q!} \quad tanh_q(x) \\ &= -\sum_{n=1}^{\infty} g_{2n+2,q} \frac{(2t)^{2n+1}}{[2n+2]_q!} \end{aligned} \tag{3.1.6}$$

Proof. Substitute t by 2it at the generating function of q-Bernoulli number and remember that $\varepsilon_q(2it) = \, e_q(it)\, E_q(it) = \, e_q(it) \left(\, e_q(-it) \right)^{-1}$,

$$\begin{aligned} 1 - it + \sum_{n=2}^{\infty} b_{n,q} \frac{(2it)^n}{[n]_q!} = \sum_{n=o}^{\infty} b_{n,q} \frac{(2it)^n}{[n]_q!} &= \hat{B}(2it) = \frac{2it}{\varepsilon_q(2it) - 1} \\ &= \frac{te_q(-it)}{sin_q(t)} = \frac{t}{sin_q(t)} \left(cos_q(t) - isin_q(t) \right) \\ &= tcot_q(t) - it \end{aligned} \tag{3.1.7}$$

Since $tcot_q(t)$ is even and the odd coefficient of $b_{n,q}$ are zero, the first expression is true. For the next one, putting $z = 2it$ at (2.3.4) which is the generating function for q-Genocchi numbers and we reach to

$$2it + \sum_{n=2}^{\infty} g_{n,q} \frac{(2it)^n}{[n]_q!} = \sum_{n=o}^{\infty} g_{n,q} \frac{(2it)^n}{[n]_q!} = \hat{G}(2it) = \frac{4it}{\varepsilon_q(2it) + 1}$$
$$= \frac{te_q(-it)}{cos_q(t)} \tag{3.1.8}$$

In a same way, we reach to $tan_q(t) = \sum_{n=1}^{\infty} g_{n,q} \frac{(-1)^n (2t)^{2n-1}}{[2n]_q!}$. To find the expression, put $x = it$ instead of x at $tan_q(x)$. This and writing q-tangent numbers as the following, together lead us to the interesting identity, which is presented as follow

$$tanh_q t = -itan_q(it) = \frac{e_q(t) - e_q(-t)}{e_q(t) + e_q(-t)} = \frac{\varepsilon_q(2t) - 1}{\varepsilon_q(2t) + 1} = \sum_{n=1}^{\infty} \mathfrak{T}_{2n+1,q} \frac{(-1)^k t^{2n+1}}{[2n+1]_q!}$$

And at the end,

$$\mathfrak{T}_{2n+1,q} = g_{2n+2,q} \frac{(-1)^{k-1} 2^{2n+1}}{[2n+2]_q!}. \tag{3.1.9}$$

3.2 Addition and Difference Equations and Corollaries

In this section, by using the q-addition formula, we approach to the new formula for q-polynomials including q-Bernoulli, q-Euler and q-Genocchi polynomials. This is the q-analogue of classic expression for these polynomials. In addition, by taking the ordinary differentiation and q-derivative, we will present some new results as well. Next lemma presents the q-analogue of additional theorem.

Lemma 3.5. For any complex numbers x, y, the following statements are true

$$B_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_{k,q}\left(x \oplus_q y\right)^{n-k}, \quad B_{n,q}(x,y)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} B_{k,q}(x) y^{n-k}$$

$$E_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e_{k,q}\left(x \oplus_q y\right)^{n-k}, \quad E_{n,q}(x,y)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} E_{k,q}(x) y^{n-k}$$

$$G_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q g_{k,q}\left(x \oplus_q y\right)^{n-k}, \quad G_{n,q}(x,y)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} G_{k,q}(x) y^{n-k}$$

Proof. The proof is on a base of definition of q-addition, we will do it for q-Bernoulli polynomials and the remained will be as the same. It is the consequence of the following identity

$$\sum_{n=0}^{\infty} B_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{t}{\varepsilon_q(t) - 1} \varepsilon_q(tx) \varepsilon_q(ty)$$

$$= \sum_{n=0}^{\infty} b_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \left(x \oplus_q y\right)^{n-k} \frac{t^n}{[n]_q!} \qquad (3.2.1)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_{k,q}\left(x \oplus_q y\right)^{n-k} \frac{t^n}{[n]_q!}.$$

This is the direct consequence of the definition of q-addition. Actually, q-addition was defined such that to make q-improved exponential commutative, that means according to this definition, we can reach to $\varepsilon_q(tx)\varepsilon_q(ty) = \sum_{n=0}^{\infty} \left(x \oplus_q y\right)^{n-k} \frac{t^n}{[n]_q!}$, because the simple calculation for the expansion of $\varepsilon_q(tx)$ and $\varepsilon_q(ty)$ respectively shows

$$\varepsilon_q(tx)\varepsilon_q(ty) = \left(\sum_{n=0}^{\infty} \frac{(-1,q)_n}{2^n} \frac{(tx)^n}{[n]_q!}\right)\left(\sum_{m=0}^{\infty} \frac{(-1,q)_m}{2^m} \frac{(ty)^m}{[m]_q!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_k(-1,q)_{n-k}}{2^n} \frac{x^k y^{n-k} t^n}{[n]_q!} \tag{3.2.2}$$

$$= \sum_{n=0}^{\infty} \left(x \oplus_q y\right)^{n-k} \frac{t^n}{[n]_q!}.$$

Corollary 3.6. For any complex number , we have the following statements

$$B_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} b_{k,q} x^{n-k}, \quad B_{n,q}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} B_{k,q}(x)$$

$$E_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} e_{k,q} x^{n-k}, \quad E_{n,q}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} E_{k,q}(x)$$

$$G_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} g_{k,q} x^{n-k}, \quad G_{n,q}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} G_{k,q}(x)$$

Proof. It's easy to substitute y by the suitable values to reach the statements, first put $y = 0$ then put $y = 1$, at the previous lemma complete the proof.

Actually, these formulae are the q-analogue of the classic forms, which are

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x), \;\; E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x), \; G_n(x+1)$$

$$= \sum_{k=0}^{n} \binom{n}{k} G_k(x)$$

Corollary 3.7. q-derivative of q-Bernoulli polynomial is as following

$$D_q\left(B_{n,q}(x)\right) = [n]_q \frac{B_{n-1,q}(x) + B_{n-1,q}(xq)}{2} \tag{3.2.3}$$

Proof. According to the previous corollary, if we know the value of q-Bernoulli numbers, then we can present q-Bernoulli polynomials in terms of them. Therefore, by taking q-derivatives from the summation we lead to

$$D_q\left(B_{n,q}(x)\right) = D_q\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} b_{k,q} x^{n-k}\right)$$
$$= \sum_{k=0}^{n-1} \frac{[n]_q!}{[k]_q!\,[n-k-1]_q!} \frac{(-1,q)_{n-k}}{2^{n-k}} b_{k,q} x^{n-k-1}$$

If we change the order of the summation, we have

$$D_q\left(B_{n,q}(x)\right) = \frac{[n]_q}{2} \sum_{k=0}^{n-1} \frac{[n-1]_q!}{[k]_q!\,[n-k-1]_q!} \frac{(-1,q)_{n-k-1}}{2^{n-k-1}} (1+q^{n-k-1}) b_{k,q} x^{n-k-1}$$

$$= \frac{[n]_q}{2}\left(\sum_{k=0}^{n-1} \frac{[n-1]_q!}{[k]_q!\,[n-k-1]_q!} \frac{(-1,q)_{n-k-1}}{2^{n-k-1}} b_{k,q} x^{n-k-1}\right.$$

$$\left. + \sum_{k=0}^{n-1} \frac{[n-1]_q!}{[k]_q!\,[n-k-1]_q!} \frac{(-1,q)_{n-k-1}}{2^{n-k-1}} b_{k,q} (qx)^{n-k-1}\right)$$

$$= [n]_q \frac{B_{n-1,q}(x) + B_{n-1,q}(xq)}{2}$$

In a same way, we can demonstrate the q-derivative of $E_{n,q}(x)$ and $G_{n,q}(x)$ by the following identities

$$D_q\left(E_{n,q}(x)\right) = \frac{[n]_q}{2}\left(E_{n-1,q}(x) + E_{n-1,q}(xq)\right),$$
$$D_q\left(G_{n,q}(x)\right) = \frac{[n]_q}{2}\left(G_{n-1,q}(x) + G_{n-1,q}(xq)\right). \tag{3.2.4}$$

Next lemma shows another property of these polynomials, which we named them difference equations.

Lemma 3.8. The following identities are true

$$B_{n,q}(x,1) - B_{n,q}(x) = \frac{(-1,q)_{n-1}}{2^{n-1}} [n]_q x^{n-1} \quad n \geq 1, \tag{3.2.5}$$

$$E_{n,q}(x,1) - E_{n,q}(x) = 2\frac{(-1,q)_{n}}{2^{n}} [n]_q x^{n} \quad n \geq 0, \tag{3.2.6}$$

$$G_{n,q}(x,1) - G_{n,q}(x) = 2\frac{(-1,q)_{n-1}}{2^{n-1}} [n]_q x^{n-1} \quad n \geq 1. \tag{3.2.7}$$

Proof. Since proofs of all statements are similar, we prove it only for q-Bernoulli difference equation. This can be found from the identity

$$\frac{t\varepsilon_q(t)}{\varepsilon_q(t)-1}\varepsilon_q(tx) = t\varepsilon_q(tx) + \frac{t}{\varepsilon_q(t)-1}\varepsilon_q(tx) \tag{3.2.8}$$

It follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} B_{k,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\frac{(-1,q)_n}{2^n}x^n\frac{t^{n+1}}{[n]_q!} + \sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{[n]_q!}$$

Equating the coefficient of t^n completes the proof.

The following familiar expansions will be demonstrated to the means of q-calculus in the following corollary

$$x^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x), \tag{3.2.9}$$

$$x^n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}E_k(x) + E_n(x)\right), \tag{3.2.10}$$

$$x^n = \frac{1}{2(n+1)}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}G_k(x) + G_{n+1}(x)\right). \tag{3.2.11}$$

Corollary 3.9. The following identities hold true

$$x^n = \frac{2^n}{[n]_q(-1,q)_n}\sum_{k=0}^{n}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{(-1,q)_{n+1-k}}{2^{n+1-k}}B_{k,q}(x), \tag{3.2.12}$$

$$x^n = \frac{2^{n-1}}{(-1,q)_n}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}}E_{k,q}(x) + E_{n,q}(x)\right), \tag{3.2.13}$$

$$x^n = \frac{2^{n-1}}{[n+1]_q(-1,q)_n}\left(\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \frac{(-1,q)_{n+1-k}}{2^{n+1-k}} G_{k,q}(x) \right. \tag{3.2.14}$$
$$\left. + G_{n+1,q}(x)\right)$$

Proof. Evaluate x^n at the difference equation (3.2.5) and use corollary 3.6 then the last terms at the summation is vanished and equation is yield.

Lemma 3.10. The following identities hold true

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} B_{k,q}(x,y) - B_{n,q}(x,y) = [n]_q\left(x\oplus_q y\right)^{n-1}, \tag{3.2.15}$$

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} E_{k,q}(x,y) + E_{n,q}(x,y) = 2\left(x\oplus_q y\right)^{n}, \tag{3.2.16}$$

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} G_{k,q}(x,y) - G_{n,q}(x,y) = 2[n]_q\left(x\oplus_q y\right)^{n-1}. \tag{3.2.17}$$

Proof. The same technique that we used at (3.8), leads us to these expressions. In fact, we will use the following identity

$$\frac{t\varepsilon_q(t)}{\varepsilon_q(t)-1}\varepsilon_q(tx)\varepsilon_q(ty) = t\varepsilon_q(tx)\varepsilon_q(ty) + \frac{t}{\varepsilon_q(t)-1}\varepsilon_q(tx)\varepsilon_q(ty), \tag{3.2.18}$$

It follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{2^{n-k}} B_{k,q}(x,y)\frac{t^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty}\frac{(-1,q)_n(-1,q)_{n-k}}{2^n} x^n y^{n-k}\frac{t^{n+1}}{[n]_q!} + \sum_{n=0}^{\infty} B_{n,q}(x,y)\frac{t^n}{[n]_q!},$$

By equating the coefficient of t^n, we complete the proof.

3.3 Differential Equations Related to q-Bernoulli Polynomials

The classical Cayley transformation $z \to Cay(z,a) := \frac{1+az}{1-az}$, is a good reason to motivate us to interpret the new formula for $\varepsilon_q(qt)$. This result leads us to a new generation of formulae, that we called them q-differential equations for q-Bernoulli polynomials. Similar results can be done for another q-function. We will start it by the next proposition.

Proposition 3.11. Assume that $n \geq 1$ is a positive integer, then we have

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_{k,q} b_{n-k,q}\, q^k$$

$$= -q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_{k,q} e_{n-k,q}\, [k-1]_q - \frac{q}{2} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q b_{k,q} e_{n-k-1,q}\, [n]_q$$

Proof. By knowing the definition of the improved q-exponential as a production of some terms, we can write

$$\varepsilon_q(t) = \prod_{k=0}^{\infty} \frac{1 + q^k(1-q)\frac{t}{2}}{1 - q^k(1-q)\frac{t}{2}} \to\ \varepsilon_q(qt) = \frac{1 + (1-q)\frac{t}{2}}{1 - (1-q)\frac{t}{2}}\ \varepsilon_q(t)$$
$$= Cay\left(-\frac{t}{2}, 1-q\right)\ \varepsilon_q(t) \tag{3.3.1}$$

Now use this expression at generating function of q-Bernoulli numbers (2.3.1) and multiplies it by itself,

$$\widehat{B_q}(qt)\widehat{B_q}(t) = \left(\widehat{B_q}(qt) - q\widehat{B_q}(t)\left(1 + (1-q)\frac{t}{2}\right)\right)\frac{1}{1-q}\frac{2}{\varepsilon_q(t)+1} \tag{3.3.2}$$

The last terms of the above expression can be demonstrated by q-Euler numbers, this expression lead us to the identity that we assumed at the proposition.

Proposition 3.12. For all $n \geq 1$, we have

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q b_{k,q} b_{2n-k,q}\, q^k$$

$$= -q \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k}}{2^{2n-k}} b_{k,q}\, [k-1]_q (-1)^k$$

$$+ \frac{q(1-q)}{2} \sum_{k=0}^{2n-1} \begin{bmatrix} 2n-1 \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k-1}}{2^{2n-k-1}} b_{k,q} [k-1]_q (-1)^k,$$

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q b_{k,q} b_{2n-k+1,q}\, q^k$$

$$= q \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k+1}}{2^{2n-k+1}} b_{k,q}\, [k-1]_q (-1)^k$$

$$- \frac{q(1-q)}{2} \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \frac{(-1,q)_{2n-k}}{2^{2n-k}} b_{k,q} [k-1]_q (-1)^k.$$

Proof. By knowing quotient rule for q-derivative, take q-derivative from the generating function (2.3.1), also using (3.3.1) and the fact that $D_q\Big(\varepsilon_q(t)\Big) = \frac{1}{2}\Big(\varepsilon_q(qt) + \varepsilon_q(t)\Big)$ together, we reach

$$\widehat{B_q}(qt)\widehat{B_q}(t) = \frac{2 + (1-q)t}{2\, \varepsilon_q(t)(q-1)} \Big(q\widehat{B_q}(t) - \widehat{B_q}(qt)\Big) \tag{3.3.3}$$

Expanding the above expression, and equating t^n-coefficient, leads us to the proposition. Also we can do the same thing for q-Genocchi and q-Euler generators to find similar identities.

Proposition 3.13. The following differential equations hold true

$$\frac{\partial}{\partial t}\widehat{B_q}(t) = \widehat{B_q}(t)\left(\frac{1}{t} - \frac{(1-q)\, \varepsilon_q(t)}{\varepsilon_q(t) - 1}\left(\sum_{k=0}^{\infty} \frac{4q^k}{4 - (1-q)^2 q^{2k}}\right)\right) \tag{3.3.4}$$

$$\frac{\partial}{\partial q}\widehat{B_q}(t) = -\widehat{B_q}^{\,2}(t)\,\varepsilon_q(t)\left(\sum_{k=0}^{\infty}\frac{4t(kq^{k-1}-(k+1)q^k)}{4-(1-q)^2q^{2k}}\right) \tag{3.3.5}$$

Proof. First and second identity can be reached by taking the normal derivatives respect to t and q respectively. We used product rule temporarily, and demonstrate it as a summation of these expressions. (3.3.6) is the combination of the (3.3.4) and (3.3.5).

$$\begin{aligned}&\frac{\partial}{\partial t}\widehat{B_q}(t) - \frac{\partial}{\partial q}\widehat{B_q}(t)\\ &= \frac{\widehat{B_q}(t)}{t} + \frac{\widehat{B_q}^{\,2}(t)\,\varepsilon_q(t)}{t}\left(\sum_{k=0}^{\infty}\frac{4t(kq^{k-1}-(k+1)q^k)-q^k(1-q)}{4-(1-q)^2q^{2k}}\right)\end{aligned} \tag{3.3.6}$$

3.4 Explicit Relationship Between q-Bernoulli and q-Euler Polynomials

We will study a few numbers of explicit relationships that exist between q-analogues of two new classes of Euler and Bernoulli polynomials in this section. For this reason, we will investigate some q-analogues of known results, and some new formulae and their special cases will be obtained in the following. We demonstrate some q-extensions of the formulae that are given before at [16].

Theorem 3.14. For any positive integer n, we have the following relationships

$$\begin{aligned}B_{n,q}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q m^{k-n}\Bigg(&B_{k,q}(x)\\ &+\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q \frac{(-1,q)_{k-j}}{2^{k-j}}\frac{B_{j,q}(x)}{m^{k-j}}\Bigg)E_{n-k,q}(my)\end{aligned} \tag{3.4.1}$$

$$= \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}\left(B_{k,q}(x) + B_{k,q}\left(x, \frac{1}{m}\right)\right)E_{n-k,q}(my).$$

Proof. To reach (3.4.1), let us start with the following identity

$$\frac{t}{\varepsilon_q(t)-1}\varepsilon_q(tx)\varepsilon_q(ty)$$

$$= \frac{t}{\varepsilon_q(t)-1}\varepsilon_q(tx).\frac{\varepsilon_q\left(\frac{t}{m}\right)+1}{2}.\frac{2}{\varepsilon_q\left(\frac{t}{m}\right)+1}.\varepsilon_q\left(\frac{t}{m}my\right) \tag{3.4.2}$$

Now expanding the above expression leads us to

$$\sum_{n=0}^{\infty} B_{n,q}(x,y)\frac{t^n}{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} E_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{(-1,q)_n}{2^n m^n}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!}$$

$$+ \frac{1}{2}\sum_{n=0}^{\infty} E_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!} =: I_1 + I_2$$

We assumed the first part of addition as I_1 and the second part as I_2. Indeed, for the second part we have

$$I_2 = \frac{1}{2}\sum_{n=0}^{\infty} E_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}B_{k,q}(x)E_{n-k,q}(my)\frac{t^n}{[n]_q!}.$$

On the other hand, the first part or I_1, can be rewritten as

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty} E_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q\frac{(-1,q)_{n-j}}{2^{n-j}}B_{j,q}(x)\frac{t^n}{m^{n-j}[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q E_{n-k,q}(my)\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q\frac{(-1,q)_{k-j}}{2^{k-j}}\frac{B_{j,q}(x)}{m^{n-k}m^{k-j}}\frac{t^n}{[n]_q!}.$$

Now if we combine the results at I_1 and I_2 ,we lead to the expression that is equal to $\sum_{n=0}^{\infty} B_{n,q}(x,y)\frac{t^n}{[n]_q!}$. That means

$$\sum_{n=0}^{\infty} B_{n,q}(x,y)\frac{t^n}{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}\Bigg(B_{k,q}(x) \tag{3.4.3}$$

$$+ \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{2^{k-j}}\frac{B_{j,q}(x)}{m^{k-j}}\Bigg)E_{n-k,q}(my)\frac{t^n}{[n]_q!}.$$

It remains to equating t^n-coefficient to complete the proof.

Corollary 3.15. For any nonnegative integer n, we have the following statement

$$B_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(B_{k,q}(x) + \frac{(-1,q)_{k-1}}{2^{k-1}}[k]_q x^{k-1}\right)E_{n-k,q}(y). \tag{3.4.4}$$

Proof. This is the special case of previous theorem, where $m = 1$. It can be assumed as a q-analogue of Cheon's main result [22]

Theorem 3.16. For any positive integer n, the following relationship between q-analogue of Bernoulli polynomials and Euler polynomials is true

$$E_{n,q}(x,y) = \frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n+1-k}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \Bigg(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{2^{k-j}}\frac{E_{j,q}(x)}{m^{k-j}} \tag{3.4.5}$$

$$- E_{k,q}(y)\Bigg)B_{n+1-k,q}(mx)$$

Proof. Like the previous theorem, we start by the similar identity. In fact, we can reach to the proof by using the following identity

$$\frac{2}{\varepsilon_q(t)+1}\varepsilon_q(tx)\varepsilon_q(ty)=\frac{2}{\varepsilon_q(t)+1}\varepsilon_q(ty).\frac{\varepsilon_q\left(\frac{t}{m}\right)-1}{t}.\frac{t}{\varepsilon_q\left(\frac{t}{m}\right)-1}.\varepsilon_q\left(\frac{t}{m}mx\right)$$

According to the definition of q-Euler polynomials, by substituting and expanding the terms we have

$$\sum_{n=0}^{\infty}E_{n,q}(x,y)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}E_{n,q}(y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(-1,q)_n}{m^n2^n}\frac{t^{n-1}}{[n]_q!}\sum_{n=0}^{\infty}B_{n,q}(mx)\frac{t^n}{m^n[n]_q!}$$

$$-\sum_{n=0}^{\infty}E_{n,q}(y)\frac{t^{n-1}}{[n]_q!}\sum_{n=0}^{\infty}B_{n,q}(mx)\frac{t^n}{m^n[n]_q!}=:I_1-I_2$$

Indeed,

$$I_2=\frac{1}{t}\sum_{n=0}^{\infty}E_{n,q}(y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}B_{n,q}(mx)\frac{t^n}{m^n[n]_q!}$$

$$=\frac{1}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1}{m^{n-k}}E_{k,q}(y)B_{n-k,q}(mx)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_q\frac{1}{m^{n+1-k}}E_{k,q}(y)B_{n+1-k,q}(mx)\frac{t^n}{[n]_q!}.$$

In addition for I_1, we have

$$I_1=\frac{1}{t}\sum_{n=0}^{\infty}B_{n,q}(mx)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}E_{k,q}(y)\frac{t^n}{m^{n-k}[n]_q!}$$

$$=\frac{1}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{n-k,q}(mx)\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q\frac{(-1,q)_{k-j}}{2^{k-j}}\frac{E_{j,q}(y)}{m^{n-k}m^{k-j}}\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\sum_{j=0}^{n+1}\begin{bmatrix}n+1\\j\end{bmatrix}_q B_{n+1-j,q}(mx)\sum_{k=0}^{j}\begin{bmatrix}j\\k\end{bmatrix}_q\frac{(-1,q)_{j-k}}{2^{j-k}}\frac{E_{k,q}(y)}{m^{n+1-k}}\frac{t^n}{[n]_q!}.$$

Combining I_1 and I_2, and equating t^n- coefficient together complete the proof.

Theorem 3.17. For any nonnegative integer n, the relationship between q-analogue of Bernoulli polynomials and Genocchi polynomials can be described as following

$$G_{n,q}(x,y) = \frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n-k}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q\frac{(-1,q)_{k-j}}{2^{k-j}}\frac{G_{j,q}(x)}{m^{k-j}} - G_{k,q}(x)\right)B_{n+1-k,q}(my) \tag{3.4.6}$$

$$B_{n,q}(x,y) = \frac{1}{2[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n-k}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q\frac{(-1,q)_{k-j}}{2^{k-j}}\frac{B_{j,q}(x)}{m^{k-j}} + B_{k,q}(x)\right)G_{n+1-k,q}(my) \tag{3.4.7}$$

Proof. At this theorem, we use the same technique to find a relationship between q-analogue of Genocchi and Bernoulli polynomials, which is completely new.

Proof is straightforward. Like the previous one, first assume the following identity,

$$\frac{2t}{\varepsilon_q(t)+1}\varepsilon_q(tx)\varepsilon_q(ty)$$

$$= \frac{2t}{\varepsilon_q(t)+1}\varepsilon_q(tx).\left(\varepsilon_q\left(\frac{t}{m}\right)-1\right)\frac{m}{t}.\frac{\frac{t}{m}}{\varepsilon_q\left(\frac{t}{m}\right)-1}.\varepsilon_q\left(\frac{t}{m}my\right)$$

Now, substitute the generating function (2.3.4). A simple calculation shows that

$$\sum_{n=0}^{\infty} G_{n,q}(x,y)\frac{t^n}{[n]_q!}$$

$$= \frac{m}{t}\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(-1,q)_n}{m^n 2^n}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} B_{n,q}(my)\frac{t^n}{m^n[n]_q!}$$

$$-\frac{m}{t}\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} B_{n,q}(my)\frac{t^n}{m^n[n]_q!}$$

By using the Cauchy product of two series, it can be rewritten

$$\sum_{n=0}^{\infty} G_{n,q}(x,y)\frac{t^n}{[n]_q!}$$

$$=\frac{m}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1,q)_{n-k}}{m^{n-k}2^{n-k}} G_{k,q}(y) - G_{n,q}(x)\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} B_{n,q}(my)\frac{t^n}{m^n[n]_q!}$$

$$=\frac{m}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{m^{n-k}}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}} G_{j,q}(y) - G_{k,q}(x)\right) B_{n-k,q}(my)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n-k}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}} G_{j,q}(y) - G_{k,q}(x)\right) B_{n+1-k,q}(my)\frac{t^n}{[n]_q!}.$$

Now equating t^n- coefficient to find the first identity. The second one can be proved in a same way.

We will finish the chapter by focusing on the symmetric properties of the given polynomials. In fact, part (b) at theorem (2.17) and definition of q-polynomials by the generating function (2.3.1) leads us to the following proposition

Proposition 3.18. q-Bernoulli, q-Euler and q-Gennoci numbers has the following property

$$q^{-\binom{n}{2}} b_{n,q} = b_{n,q^{-1}}\,,\ \ q^{-\binom{n}{2}} e_{n,q} = e_{n,q^{-1}}\ and\ \ q^{-\binom{n}{2}} g_{n,q} = g_{n,q^{-1}} \qquad (3.4.8)$$

Proof. The proof is based on the fact that $[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$. Since $\varepsilon_q(z) = \varepsilon_{\frac{1}{q}}(z)$, we have

$$\frac{t}{\varepsilon_q(t)-1} = \sum_{n=0}^{\infty} b_{n,q}\frac{t^n}{[n]_q!} = \frac{t}{\varepsilon_{q^{-1}}(t)-1} = \sum_{n=0}^{\infty} b_{n,q^{-1}}\frac{t^n}{[n]_{q^{-1}}!} \qquad (3.4.9)$$

Equating t^n-coefficient, leads us to (3.4.1).

Remark 3.19. The previous proposition, gives us a tool to evaluate the corresponding values of $b_{n,q^{-1}}$ by knowing $b_{n,q}$.

Chapter 4

UNIFICATION OF q-EXPONENTIAL FUNCTION AND RELATED POLYNOMIALS

4.1 Preliminary Results

We will define the new class of q-exponential function in this section. In fact, we will add a parameter to the old definition. In this way, we reach to the unification of q-exponential function and by changing this parameter; we lead to the different kind of the q-exponential functions that defined before. Moreover, this parameter helps us to lead to a group of new q-exponential functions as well. We will study the important properties of q-exponential function, by taking some restrictions on this parameter.

Definition 4.1. Let $\alpha(q,n)$ be a function of q and n, such that $\alpha(q,n) \to 1$, where q tends to one from the left side. We define new general q-exponential function as following

$$\varepsilon_{q,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \alpha(q,n) \tag{4.1.1}$$

In the special case where $\alpha(q,n) = 1$, $\alpha(q,n) = q^{\binom{n}{2}}$ and $\alpha(q,n) = \frac{(-1,q)_n}{2^n}$ we reach to $e_q(z)$, $E_q(z)$ and $\varepsilon_q(z)$ respectively.

At the next lemma, we will discuss about the conditions that make $\varepsilon_{q,\alpha}(z)$ convergent. There are some restrictions, which have to be considered. Since $\varepsilon_{q,\alpha}(z)$

is the q-analogue of exponential function, $\alpha(q,n)$ approaches to 1, where q tends one from the left side.

Lemma 4.2. If $\lim_{n\to\infty}\left|\frac{\alpha(q,n+1)}{[n+1]_q!\alpha(q,n)}\right|$ does exist and is equal to l ,then $\varepsilon_{q,\alpha}(z)$ as a q-exponential function is analytic in the region $|z| < l^{-1}$.

Proof. Radius of convergence can be obtained by computing the following limit

$$\lim_{n\to\infty}\left|\frac{z^{n+1}\alpha(q,n+1)}{[n+1]_q!}\right|\left|\frac{[n]_q!}{z^n\alpha(q,n)}\right| = \lim_{n\to\infty}\left|\frac{\alpha(q,n+1)}{[n+1]_q!\,\alpha(q,n)}\right|\,|z| \qquad (4.1.2)$$

Then, for $q \neq 1$ we can use d'Alembert's test and we find the radius of convergence.

Example 4.3. Let $\alpha(q,n) = 1$, $\alpha(q,n) = q^{\binom{n}{2}}$ and $\alpha(q,n) = \frac{(-1,q)_n}{2^n}$, then we reach to, $e_q(z), E_q(z)$ and improved q-exponential function $\varepsilon_q(z)$ respectively. Then the radius of convergence becomes $\frac{1}{|1-q|}$, infinity and $\frac{2}{|1-q|}$ respectively where $0 < |q| < 1$.

Now, with this q-exponential function, we define the new class of q-Bernoulli numbers and polynomials. Next definition denotes a general class of these new q-numbers and polynomials.

Definition 4.4. Assume that q is a complex number such that $0 < |q| < 1$. Then we can define q-analogue of the following functions in the meaning of generating function including Bernoulli numbers $b_{n,q,\alpha}$ and polynomials $B_{n,q,\alpha}(x,y)$ and Euler numbers $e_{n,q,\alpha}$ and polynomials $E_{n,q,\alpha}(x,y)$ and the Genocchi numbers $g_{n,q,\alpha}$ and polynomials $G_{n,q,\alpha}(x,y)$ in two variables x, y respectively

$$\widehat{B_q} \coloneqq \frac{t}{\varepsilon_{q,\alpha}(t)-1} = \sum_{n=0}^{\infty} b_{n,q,\alpha}\frac{t^n}{[n]_q!}, \quad |t| < 2\pi, \qquad (4.1.3)$$

$$\frac{t}{\varepsilon_{q,\alpha}(t)-1}\ \varepsilon_{q,\alpha}(tx)\ \varepsilon_{q,\alpha}(ty)=\sum_{n=0}^{\infty}B_{n,q,\alpha}(x,y)\frac{t^n}{[n]_q!},\quad |t|<2\pi, \tag{4.1.4}$$

$$\widehat{E_q}:=\frac{2}{\varepsilon_{q,\alpha}(t)+1}=\sum_{n=0}^{\infty}e_{n,q,\alpha}\frac{t^n}{[n]_q!},\quad |t|<\pi, \tag{4.1.5}$$

$$\frac{2}{\varepsilon_{q,\alpha}(t)+1}\ \varepsilon_{q,\alpha}(tx)\ \varepsilon_{q,\alpha}(ty)=\sum_{n=0}^{\infty}E_{n,q,\alpha}(x,y)\frac{t^n}{[n]_q!},\quad |t|<2\pi, \tag{4.1.6}$$

$$\widehat{G_q}:=\frac{2t}{\varepsilon_{q,\alpha}(t)+1}=\sum_{n=0}^{\infty}g_{n,q,\alpha}\frac{t^n}{[n]_q!},\quad |t|<\pi, \tag{4.1.7}$$

$$\frac{2t}{\varepsilon_{q,\alpha}(t)+1}\ \varepsilon_{q,\alpha}(tx)\ \varepsilon_{q,\alpha}(ty)=\sum_{n=0}^{\infty}G_{n,q,\alpha}(x,y)\frac{t^n}{[n]_q!},\quad |t|<\pi. \tag{4.1.8}$$

If the convergence conditions are hold for q-exponential function, it is obvious that by tending q to 1 from the left side, we lead to the classic definition of these polynomials. We mention that $\alpha(q,n)$ is respect to q and n. In addition by tending q to 1^{-}, $\varepsilon_{q,\alpha}(z)$ approach to the ordinary exponential function. That means

$$b_{n,q,\alpha}=B_{n,q,\alpha}(0),\quad \lim_{q\to1^{-}}B_{n,q,\alpha}(x,y)=B_n(x,y),\quad \lim_{q\to1^{-}}b_{n,q,\alpha}=b_n, \tag{4.1.9}$$

$$e_{n,q,\alpha}=E_{n,q,\alpha}(0),\quad \lim_{q\to1^{-}}E_{n,q,\alpha}(x,y)=E_n(x,y),\quad \lim_{q\to1^{-}}e_{n,q,\alpha}=e_n, \tag{4.1.10}$$

$$g_{n,q,\alpha}=G_{n,q,\alpha}(0),\quad \lim_{q\to1^{-}}G_{n,q,\alpha}(x,y)=G_n(x,y),\quad \lim_{q\to1^{-}}g_{n,q,\alpha}=g_n. \tag{4.1.11}$$

Our purpose in this chapter is presenting a few results and relations for the newly defined q-Bernoulli and q-Euler polynomials. In the next section we will discuss about some restriction for $\alpha(q,n)$, such that the familar results discovered. we will focus on two main properties of q-exponential function, first in which situation $\varepsilon_{q,\alpha}(z)=\varepsilon_{q^{-1},\alpha}(z)$, second we investigate the conditions for $\alpha(q,n)$ such that $\varepsilon_{q,\alpha}(-z)=\left(\varepsilon_{q,\alpha}(z)\right)^{-1}$. A lot of classical results are found by these two properties.The form of new type of q-exponential function, motivate us to define a new q-addition and q-subtraction like a Daehee formula as follow

$$\left(x \oplus_q y\right)^n := \sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, n-k)\alpha(q, k)x^k y^{n-k}, \;\; n = 0,1,2, \ldots \qquad (4.1.12)$$

$$\left(x \ominus_q y\right)^n := \sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, n-k)\alpha(q, k)x^k (-y)^{n-k}, \;\; n = 0,1,2, \ldots \qquad (4.1.13)$$

4.2 New Exponential Function and Its Properties

We shall provide some conditions on $\alpha(q, n)$ to reach two main properties that discussed before. First we try to find out, in which situation $\varepsilon_{q,\alpha}(z) = \varepsilon_{q^{-1},\alpha}(z)$. Following lemma is related to this property.

Lemma 4.5. The new q-exponential function $\varepsilon_{q,\alpha}(z)$, satisfy $\varepsilon_{q,\alpha}(z) = \varepsilon_{q^{-1},\alpha}(z)$, if and only if $q^{\binom{n}{2}}\alpha(q^{-1}, n) = \alpha(q, n)$.

Proof. The proof is based on the fact that $[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$, therefore

$$\varepsilon_{q^{-1},\alpha(q^{-1})}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!}\alpha(q^{-1}, n) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}\alpha(q, n) = \varepsilon_{q,\alpha}(z) \qquad (4.2.1)$$

Corollary 4.6. If $\alpha(q, n)$ is in a form of polynomial that means $\alpha(q, n) = \sum_{i=0}^{m} a_i q^i$, to satisfy $\varepsilon_{q,\alpha}(z) = \varepsilon_{q^{-1},\alpha}(z)$, we have

$$\begin{gathered} deg\big(\alpha(q, n)\big) = m = \binom{n}{2} - j \leq \binom{n}{2}, \\ a_{j+k} = a_{m+k} \;\; and \; k = 0,1, \ldots, m-j \end{gathered} \qquad (4.2.2)$$

Where j is the leading index, such that $a_j \neq 0$ and for $0 \leq k < j, a_k = 0$.

Proof. First, we want to mention that $\sum_{i=0}^{m} a_i = 1$, becuase $\alpha(q, n)$ approches to 1, where q tends one from the left side. In addition as we assumed $\alpha(q, n) = \sum_{i=0}^{m} a_i\, q^i$, by substituting q^{-1} instead of q we have

$$q^{\binom{n}{2}}\alpha(q^{-1},n) = q^{\binom{n}{2}-m}\sum_{i=0}^{m} a_i\, q^i = \alpha(q,n) = \sum_{i=0}^{m} a_i\, q^i \tag{4.2.3}$$

Now equate the coefficient of q^k , to reach the statement.

Example 4.7. Simplest example of the previous corollary will be happened when $(q,n) = q^{\frac{\binom{n}{2}}{2}}$. This case leads us to the following exponential function

$$\varepsilon_{q,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} q^{\frac{\binom{n}{2}}{2}} \ \ \& \ \ \varepsilon_{q^{-1},\alpha(q^{-1})}(z) = \varepsilon_{q,\alpha}(z) \tag{4.2.4}$$

Another example will be occurred if $\alpha(q,n) = \frac{(-1,q)_n}{2^n} = \frac{(1+q)(1+q^2)\dots(1+q^n)}{2^{n-1}}$. Now use q-binomial formula (2.8) to reach $\alpha(q,n) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}_q q^{\frac{i(i-1)}{2}}$. As we expect, where q tends 1 from the left side, $\alpha(q,n)$ approach to 1. This presentation is not in a form of previous corollary, however $q^{\binom{n}{2}}\alpha(q^{-1},n) = \alpha(q,n)$. This parameter leads us to the improved q-exponential function as following

$$\varepsilon_q(z) = \varepsilon_{q,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}\frac{(-1,q)_n}{2^n} \ \ \& \ \ \varepsilon_{q^{-1}}(z) = \varepsilon_q(z) \tag{4.2.5}$$

Some properties of q-Bernoulli polynomials that are corresponding to this improved q-exponential function were studied at previous chapter.

Remark 4.8. It's obvious that if we substitute q to q^{-1}, in any kind of q-exponential function and derive to another q-analogue of exponential function, the parameter $\alpha(q,n)$ will change to $\beta(q,n)$, and $q^{\binom{n}{2}}\alpha(q^{-1},n) = \beta(q,n)$. The famous case is standard q-exponential function

$$e_{q^{-1}}(z) = \varepsilon_{q^{-1},\alpha(q^{-1})}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}\, q^{\binom{n}{2}} = E_q(z) \tag{4.2.6}$$

$$q^{\binom{n}{2}}\alpha(q^{-1},n) = q^{\binom{n}{2}} = \beta(q,n) \tag{4.2.7}$$

Proposition 4.9. The general q-exponential function $\varepsilon_{q,\alpha}(z)$ satisfy $\varepsilon_{q,\alpha}(-z) = \left(\varepsilon_{q,\alpha}(z)\right)^{-1}$, if and only if

$$2\sum_{k=0}^{p-1}\binom{n}{k}_q(-1)^k\alpha(q,n-k)\alpha(q,k) = \binom{n}{p}_q(-1)^{p+1}\alpha^2(q,p)\&$$
$$\alpha(q,0) = \pm 1 \ \ where\ n = 2p\ and\ p = 1,2,\ldots \tag{4.2.8}$$

Proof. This condition can be rewritten as $\left(1 \ominus_q 1\right)^n = 0$ for any $n \in \mathbb{N}$. Since $\varepsilon_{q,\alpha}(-z)\varepsilon_{q,\alpha}(z) = 1$ has to be hold, we write the expansion for this equation, then

$$\varepsilon_{q,\alpha}(-z)\varepsilon_{q,\alpha}(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k\alpha(q,n-k)\alpha(q,k)\right) = 1 \tag{4.2.9}$$

Let call the expression on a bracket as $\beta_{k,q}$. If n is an odd number, then

$$\begin{aligned}\beta_{n-k,q} &= \binom{n-k}{k}_q(-1)^{n-k}\alpha(q,n-k)\alpha(q,k)\\ &= \binom{n}{k}_q(-1)^k\alpha(q,n-k)\alpha(q,k)\\ &= -\beta_{k,q} \qquad where\ k = 0,1,\ldots,n\end{aligned} \tag{4.2.10}$$

Therefore, for n as an odd number, we have the trivial equation. Since $\binom{n-k}{k}_q = \binom{n}{k}_q$ The same discussion for even n and equating z^n-coefficient together lead us to the proof.

Remark 4.10. The previous proposition can be rewritten as a system of nonlinear equations. For instance, let write $\alpha(q,k)$ as α_k, then the following system shows a condition for α_k. We mention that $\alpha_k = 1$ where $q \to 1^-$ and $\alpha_0 = \pm 1$.

$$\begin{cases} 2\alpha_2\alpha_1 - \binom{2}{1}_q \alpha_0\alpha_0 = 0 \\ 2\alpha_4\alpha_1 - 2\binom{4}{1}_q \alpha_3\alpha_2 + \binom{4}{2}_q \alpha_2\alpha_2 = 0 \\ 2\alpha_6\alpha_1 - 2\binom{6}{1}_q \alpha_5\alpha_2 + 2\binom{6}{2}_q \alpha_4\alpha_3 - 2\binom{6}{3}_q \alpha_3\alpha_3 = 0 \\ \vdots \\ 2\alpha_n\alpha_1 - 2\binom{n}{1}_q \alpha_{n-1}\alpha_2 + 2\binom{n-2}{2}_q \alpha_{n-2}\alpha_3 - \cdots + (-1)^{n/2}\binom{n}{n/2}_q \alpha_{n/2}\alpha_{n/2} = 0 \end{cases}$$

For even n, we have $n/2$ equations and n unknown variables. In this case we can find α_k respect to $n/2$ parameters by the recurence formula. For example, some few terms can be found as follow

$$\begin{cases} \alpha_0 = \pm 1 \\ \alpha_2 = \dfrac{1+q}{2}\dfrac{1}{\alpha_1} \\ \alpha_4 = \dfrac{[4]_q}{2{\alpha_1}^2}\left([2]_q\alpha_3 - \dfrac{[3]_q!}{4\alpha_1}\right) \\ \alpha_6 = \binom{6}{1}_q + \binom{6}{3}_q - \dfrac{1}{2}\left(\binom{6}{2}_q\left(\dfrac{1+q}{2}\dfrac{1}{\alpha_1}\right)\left(\dfrac{[4]_q}{2{\alpha_1}^2}\left([2]_q\alpha_3 - \dfrac{[3]_q!}{4\alpha_1}\right)\right)\right) \end{cases}$$

The familiar solution of this system is $\alpha(q,k) = \frac{(-1,q)_k}{2^k}$. This $\alpha(q,k)$ leads us to the improved exponential function. On the other hand, we can assume that all α_k for odd k are 1. Then by solving the system for these parameters, we reach to another exponential function that satisfies $\varepsilon_{q,\alpha}(-z) = \left(\varepsilon_{q,\alpha}(z)\right)^{-1}$.

In the next lemma, we will discuss about the q-derivative of this q-exponential function. Here, we assume the special case, which covers well known q-exponential functions.

Lemma 4.11. If $\frac{\alpha(q,n+1)}{\alpha(q,n)}$ can be demonstrated as a polynomial of q, that means $\frac{\alpha(q,n+1)}{\alpha(q,n)} = \sum_{k=0}^{m} a_k\, q^k$, then $D_q\left(\varepsilon_{q,\alpha}(z)\right) = \sum_{k=0}^{m} a_k\ \varepsilon_{q,\alpha}\left(zq^{\frac{k}{n}}\right)$.

Proof. We can prove it by using the following identity

$$D_q\left(\varepsilon_{q,\alpha}(z)\right)=\sum_{n=1}^{\infty}\frac{z^{n-1}}{[n-1]_q!}\alpha(q,n)=\sum_{n=0}^{\infty}\frac{z^{n}}{[n]_q!}\alpha(q,n)\left(\sum_{k=0}^{m}a_k\,q^k\right)$$
$$=\sum_{k=0}^{m}a_k\sum_{n=0}^{\infty}\frac{\left(zq^{\frac{k}{n}}\right)^n}{[n]_q!}\alpha(q,n)=\sum_{k=0}^{m}a_k\ \varepsilon_{q,\alpha}\left(zq^{\frac{k}{n}}\right). \tag{4.2.11}$$

Example 4.12. For $\alpha(q,n)=1,\ \alpha(q,n)=q^{\binom{n}{2}}$ and $\alpha(q,n)=\frac{(-1,q)_n}{2^n}$, the ratio of $\frac{\alpha(q,n+1)}{\alpha(q,n)}$ becomes $1, q^{\mathrm{n}}$ and $((1+q^{\mathrm{n}})/2)$ respectively. Therefore the following derivatives hold true

$$D_q\left(e_q(z)\right)=e_q(z)\ \&\ D_q\left(E_q(z)\right)=E_q(zq)\ \&\ D_q\left(\varepsilon_q(z)\right)=\frac{\varepsilon_q(z)+\ \varepsilon_q(zq)}{2}$$

4.3 Related q-Bernoulli Polynomial

In this section, we will study the related q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials. The discussion of properties of general q-exponential at the previous section, give us the proper tools to reach to the general properties of these polynomials related to $\alpha(q,n)$.

Lemma 4.13. The condition $\varepsilon_{q,\alpha}(-z)=\left(\varepsilon_{q,\alpha}(z)\right)^{-1}$ and $\alpha(q,1)=1$ together provides that the odd coefficient of related q-Bernoulli numbers except the first one becomes zero. That means $b_{n,q,\alpha}=0$ where $n=2r+1, r\in\mathbb{N}$.

Proof. The proof is similar to (3.2) and based on the fact that the following function is even. I mention that the condition $\alpha(q,1)=1$ and (4.2.8) together imply that $b_{1,q,\alpha}=-\frac{1}{2}$.

$$f(t)=\sum_{n=0}^{\infty}b_{n,q,\alpha}\frac{t^n}{[n]_q!}-b_{1,q,\alpha}t=\frac{t}{\varepsilon_{q,\alpha}(t)-1}+\frac{t}{2}=\frac{t}{2}\left(\frac{\varepsilon_{q,\alpha}(t)+1}{\varepsilon_{q,\alpha}(t)-1}\right) \tag{4.3.1}$$

Lemma 4.14. If $\alpha(q,n)$ as a parameter of $\varepsilon_{q,\alpha}(z)$ satisfy $\frac{\alpha(q,n+1)}{\alpha(q,n)} = \sum_{k=0}^{m} a_k\, q^k$, then we have

$$D_q\left(B_{n,q,\alpha}(x)\right) = [n]_q \sum_{k=0}^{m} a_k\, B_{n-1,q,\alpha}\left(xq^{\frac{k}{n}}\right) \tag{4.3.2}$$

Proof. Use lemma 4.11 similar corollary 3.7, we reach to the relation. Moreover in a same way for q-Euler and q-Genocchi polynomials we have

$$D_q\left(E_{n,q,\alpha}(x)\right) = [n]_q \sum_{k=0}^{m} a_k\, E_{n-1,q,\alpha}\left(xq^{\frac{k}{n}}\right) \tag{4.3.3}$$

$$D_q\left(G_{n,q,\alpha}(x)\right) = [n]_q \sum_{k=0}^{m} a_k\, G_{n-1,q,\alpha}\left(xq^{\frac{k}{n}}\right). \tag{4.3.4}$$

4.4 Unification of q-Numbers

In this section, we study some properties of related q-numbers including q-Bernoulli, q-Euler and q-Gennochi numbers. For this reason we investigate these numbers that is generated by the unified q-exponential numbers. We reach to the general case of these numbers. In addition, any new definition of these q-numbers can be demonstrate in this form and we can study the general case of them by applying that two properties of q-exponential function which is discussed in the previous sections.

As I mention it before, all the lemma and propositions at the previous chapter can be interpreted in a new way. The proofs and techniques are as the same. Only some format of the polynomials will be changed. For example the symmetric proposition can be rewritten as following

Proposition 4.15. If the symmetric condition is hold, that means $\varepsilon_{q,\alpha}(z) = \varepsilon_{q^{-1},\alpha}(z)$, then q-Bernoulli, q-Euler and q-Gennoci numbers has the following property

$$q^{-\binom{n}{2}} b_{n,q,\alpha} = b_{n,q^{-1},\alpha}\ ,\ \ q^{-\binom{n}{2}} e_{n,q,\alpha} = e_{n,q^{-1},\alpha}\ and\ \ q^{-\binom{n}{2}} g_{n,q,\alpha} = g_{n,q^{-1},\alpha} \quad (4.4.1)$$

Since the proof is similar, it is not given. The symmetric conditions for $\varepsilon_{q,\alpha}(z)$ can be hold from 4.5 to 4.8.

Another interesting property of these numbers can be found in the next lemma. We can demonstrate all q-Bernoulli numbers by knowing their values where q belongs to the interval of [0, 1]. When q is a real number, we can assume this interval as a space of the range of the probability random variable.

Lemma 4.16. If the symmetric condition is hold, that means $\varepsilon_{q,\alpha}(z) = \varepsilon_{q^{-1},\alpha}(z)$, then related q-Bernoulli, q-Euler and q-Gennochi numbers can be found if we have the value of them for $0 < |q| < 1$.

Proof. If $0 < |q| < 1$, then by the assumption of lemma, the value of $b_{n,q,\alpha},\ e_{n,q,\alpha}$ and $g_{n,q,\alpha}$ are known. If q is outside of this region, q^{-1} will be inside of this region and by using the previous proposition we can compute the value of these numbers.

In the middle of the last century, q-analogues of the Bernoulli numbers were introduced by Carlitz [4]. The recurrence relation that he used for these numbers was as following

$$\sum_{k=0}^{m} \binom{m}{k} B_k q^{k+1} - B_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1. \end{cases} \quad (4.4.2)$$

In the next proposition we will give the recurrence formulae for the related q-Bernoulli numbers and another q-numbers as well. These presentations are based on the generating function of these numbers by using the unification of q-exponential function.

Proposition 4.17. Related q-Bernoulli numbers, q-Euler numbers and q-Gennochi numbers to the new q-exponential function can be evaluated by the following recurrence formulae

$$\sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, n-k) b_{k,q,\alpha} - b_{n,q,\alpha} = \begin{cases} 1, & n = 1 \\ 0, & n > 1, \end{cases} \tag{4.4.3}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, n-k) e_{k,q,\alpha} + e_{n,q,\alpha} = \begin{cases} 2, & n = 0, \\ 0, & n > 0, \end{cases} \tag{4.4.4}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, n-k) g_{k,q,\alpha} + g_{n,q,\alpha} = \begin{cases} 2, & n = 1, \\ 0, & n > 1. \end{cases} \tag{4.4.5}$$

Proof. Related q-numbers are defined by the means of the generating function at (4.1.3), (4.1.5) and (4.1.7) respectively. If we multiply each side by the corresponding terms and using Cauchy product of the series, we reach to these recurrence relations.

Remark 4.18. The previous proposition can be used to evaluate these numbers one by one. In this case, by assuming that $\alpha_0 = \alpha(q, 0) = 1$, few numbers of related q-Bernoulli, q-Euler and q-Gennochi numbers can be obtained as following

$$b_{0,q,\alpha} = \frac{1}{\alpha_1}, \qquad b_{1,q,\alpha} = -\frac{\alpha_2}{(\alpha_1)^2[2]_q}, \qquad b_{2,q,\alpha} = \frac{1}{[3]_q \alpha_1}\left(-\frac{\alpha_3}{\alpha_1} + \frac{(\alpha_2)^2[3]_q}{(\alpha_1)^2[2]_q}\right),$$

$$e_{0,q,\alpha} = 1, \qquad e_{1,q,\alpha} = -\frac{\alpha_1}{2}, \quad e_{2,q,\alpha} = \frac{1}{2}\left(\frac{(\alpha_2)^2[2]_q}{2} - \alpha_2\right),$$

$$g_{0,q,\alpha} = 0, \qquad g_{1,q,\alpha} = 1, \qquad g_{2,q,\alpha} = -\frac{q+1}{2}\alpha_1.$$

If we continue to evaluate the sequence of q-Bernoulli numbers, we can see that the lemma 4.13 hold true. That means the odd coefficients of the general q-Bernoulli numbers are zero, in a condition that the related q-exponential function has the symmetric property. In addition, by knowing the recurrence relation of the q-

Bernoulli numbers we can reach to the corresponding q-exponential function. For instance, the q-Bernoulli numbers that were introduced by Carlitz can be traced in a same way. By comparing the terms we can see that

$$\alpha(q, n-k) = \frac{[n-k]_q!}{(n-k)!}\frac{[k]_q!}{k!}\frac{n!}{[n]_q!}q^{k+1} \tag{4.4.6}$$

Now, we can also find the corresponding q-exponential function such that this relation is hold. This form of unification leads us to a lot of new q-exponential function with interesting properties.

This unification of q-exponential function gives us a tool to redefine the generating functions according to this q-exponential function. In addition we can change the form of generating function to lead a general class of these polynomials as well.

For instance, we can assume the following definition.

Definition 4.19. For the polynomial of a degree m $P_m(t)$, we can define the sequence of numbers $\{\gamma_n\}$ by the means of generating function as follows

$$\frac{P_m(t)}{\varepsilon_{q,\alpha}(t) \pm 1} = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{[n]_q!}. \tag{4.4.7}$$

In the special case, when the degree of $P_m(t)$ is one and simply is equal to t, 2 and 2t we can reach to the q-Bernoulli, q-Euler and q-Gennochi numbers respectively. In a case that the degree of $P_m(t)$ is higher than one, these numbers can be demonstrated as a combination of these numbers.

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