The Binary Mathematical Morphology on the Triangular Grid

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ABSTRACT

Mathematical morphology is a part of digital image processing which has strong mathematical foundation and also has several applications. In digital image processing images are understood on (usually, a finite segment of) a grid. Historically, the square grid is the most used one, theory on the square grid has been developed first and it is applied in the most cases. However, it is also known that other grids have some advantages over the square grid. There are two other regular tessellations of the plane, the hexagonal and the triangular grids. In this thesis, we considered mathematical morphology on the triangular grid. The two basic operations of mathematical morphology are the dilation and the erosion. The input image is changed by the help of structural elements with these operations. Since the triangular grid is not a point lattice we have needed to face to some difficulties when defining dilations and erosions, namely, the triangular grid is not closed under vector addition. We have proposed four possible solutions, namely the”strict”, the”weak”, the”strong” and the”independent” approaches. Definitions, examples and properties of the operations are investigated in each case.

Keywords: Mathematical morphology, dilation, erosion, non-traditional grids, triangular grid, adjunction relation, digital image processing, binary images

ÖZ

Matematiksel morfoloji, güçlü matematiksel temeli olan ve ayrıca çeşitli uygulamalara sahip dijital görüntü işlemenin bir parçasıdır. Dijital görüntü işlemelerinde, görüntüler bir ızgara (genellikle, sonlu bir segment) üzerinde anlaşılmaktadır. Tarihsel olarak, kare ızgara en çok kullanılanıdır, teori ilk olarak kare ızgara üzerinde geliştirilmiştir ve çoğu durumda bu yaklaşım uygulanır. Bununla birlikte, diğer ızgaraların kare ızgaraya göre bazı avantajları olduğu da bilinmektedir. Düzlemin diğer iki düzenli döşemesi, altıgen ve üçgen ızgaralardır. Bu tezde üçgen ızgara üzerinde matematiksel morfoloji ele alınmıştır. Matematiksel morfolojinin iki temel çalışması, genişleme ve erozyondur. Giriş görüntüsü bu işlemlerle yapısal elemanların yardımı ile değiştirilir. Üçgensel ızgara nokta kafes olmadığından, genişleme ve erozyonların tanımlanması sırasında bazı zorluklarla karşılaşmamız olasıdır; kısaca üçgen ızgara vektör toplamı altında kapalı değildir. Dört olası çözümü, yani “katı”, “zayıf”, “güçlü” ve “bağımsız” yaklaşımları önerdik. Her durumda operasyonların tanımları, örnekleri ve özellikleri incelenir.

Anahtar Kelimeler: Matematiksel morfoloji, genişleme, erozyon, geleneksel olmayan ızgaralar, üçgen ızgara, birleşim bağıntısı, dijital görüntü işleme, ikili görüntüler

DEDICATION

To My Family

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2D	Two-dimensional space
T	Triangular grid
Δ	Even triangular pixel
∇	Odd triangular pixel
Trixel	Triangle pixels
$\mathbb{R}^N$	N-dimensional continuous space
$\mathbb{Z}^N$	N-dimensional discrete space or digital space
$\mathbb{E}^N$	N-dimensional euclidean space
$\mathcal{R}$	Binary relation
$\leq$	Less than or equal
$\subseteq$	Inclusion operation
$\in$	Belong
sup or $\vee$	supremum
inf or $\wedge$	infimum
δ	Binary Dilation operator
A_i	An indexed family of sets
$\mathcal{E}$	Binary Erosion operator
$\forall$	for all
id	The identity map
$\mathcal{P}(E)$	The set of all subsets of E
$\emptyset$	The Empty set
A^c	A complement of set A

$\notin$	Not belong
$\breve{B}$	Refection of set B by 180°
$\cup$	Union operation
$\cap$	Itersection operation
$\oplus$	Binary dilation
$\ominus$	Binary Erosion
$\exists$	Exists
O	The original of the grid
hexels	The pixels of the hexagonal Grid
G^0	The set of even pixels
G^+	The set of odd pixels
$\oplus_\Delta$	Strict dilation
$-p$	Inverse of point/vector
$\neq$	Not equal operation
$\ominus_\Delta$	Strict erosion
$\oplus_w$	Weak dilation
$\ominus_w$	Weak erosion
$\Leftrightarrow$	If and only if
$\Rightarrow$	implies
$\nsubseteq$	Not subset nor equal to
$\oplus_s$	Strong dilation
$\ominus_s$	Strong erosion
B_e	Union of B_e^0 and B_e^+
B_e^0	The set of even vectors with the sum of its coordinate values equal to zero

B_e^+	The set of odd vectors with the sum of its coordinate values equal to one
B_o	Union of B_o^0 and B_o^-
B_o^0	The set of even vectors with the sum of its coordinate values equal to zero
A^0	A part of an image A that contains only the even points
A^+	A part of an image A that contains only the odd points
G^-	The set of all vectors with sum of their coordinate values equal to (-1)
$\oplus_i$	Independent dilation
$\ominus_i$	Independent erosion
$3D$	Three-dimensional space
B_o^-	The set of vectors with the sum of its coordinate values equal to (-1)

Chapter 1

INTRODUCTION

1.1 The Historical Context for Mathematical Morphology

In 1903, Minkowski found the first mathematical morphology operations (Minkowski addition and subtraction) in his study to identify integral measures of individual open sets. Later Hadwiger in 1950 redefined Minkowski operations, and he defined duality between two operations, which, later on, become known as dilation and erosion. Matheron investigated the duality between the opening and closing operations. Mathematical morphology has been formalized since the 1960's by Matheron (Matheron, 1975) and Serra (Jean Serra, 1983),at the Center de Morphologies, who found the geometry, and edge properties of ores (see also, (Ghosh & Deguchi, 2009; Gonzalez & Woods, 2006; Soille, 2013)). In the 1960's the hexagonal lattice was involved in this field by Golay (Golay, 1969). In that period, i.e., the late 1960's to 1970's, mathematical morphology was based on set theory, but later, was developed under the lattice concept. Mathematical morphology is recognized as a reliable tool for signal and image analysis. It is based on the analysis of the geometrical attributes of shapes or objects by using a structuring element (SE) that could provide benefits in the analysis of a specific objector shape (Najman & Talbot, 2013). Mathematical morphology has been extended to grayscale and color images. The first approach of grayscale image processing was developed based on local min/max operators and level sets (Meyer, 1978; Jean Serra, 1983) while its second approach is based on the umbra (the hypograph of a function) by

Sternberg (Sternberg, 1986). Serra and Matheron (Matheron, 1975; J Serra, 1988) enlarged the approach to a more general framework (lattice). The work of Heijmans and Ronse (H. J. A. M. Heijmans & Ronse, 1990) dealt with extending the idea of morphological invariance operation on the complete lattice; they replace translation invariance by a type of Abelian group of automorphisms. Maragos extended the idea of circular morphology by using Affine morphology (Maragos, 1990). Another study of mathematical morphology was based on graph morphology (H. Heijmans & Vincent, 1993; Vincent, 1989). Moreover, Roedrink (Roerdink, 2000) extended invariance notation under a general non-Abelian group. Serra gives a more general idea of a structuring function to define the essential transform of dilation and erosion with the non-variant property. Mathematical morphology has been widely used in medical fields such as magnetic resonance imaging (MRI) (Yang & Li, 2015). Moreover, Mathematical morphology has been applied in several areas. Such as radar imagery (Hou, Wu, & Ma, 2004), robot navigation (Ortiz, Puente, & Torres, n.d.), Intelligent transportation systems (Fomani & Shahbahrami, 2017; Hedberg, Dokladal, & Owall, 2009), remote sensing, robot vision, video processing, color processing, image sharpening, and restoration (Burger & Burge, 2010; Dougherty, 1992; H. J. A. M. Heijmans & Ronse, 1990; Shih, 2009; Wilson & Ritter, 2000).

In a 2D square grid, a binary image comprises a finite set of elements. The elements of the grid are called pixels, and they can be addressed with two coordinate values. Each pixel has the value zero or value one. The black color usually represents pixels with value one which forms the foreground while the white color represents value zero (background) pixels. The primary binary morphological operations, the dilation and the erosion have been defined on the square grid by using two images. One of them is the active (or current) image; it is the image of the object in which we are interested. The other one is called the structuring element, and its action (Shih, 2009) modifies/examines the active image. The structuring element has its size and shape and is used to redefine the shape of the original image.

Two of the fundamental transform operations (dilation and erosion) have a common characteristic property in that they are increasing operations. To obtain an opening or closing process, one needs to combine the last two operations: either erosion followed by dilation or vice versa, with the condition that both processes must satisfy an adjunction relation. The underlying intention behind the morphological opening and closing is to describe an operation that tends to recover as much of the initial shape of the image arrangements as possible, that have been either first eroded or dilated following a dilation or erosion operation (Soille, 2013). Matheron and Serra (Matheron, 1975; Jean Serra, 1983) examined this process, by defined opening and closing by using Minkowski subtraction rather than what is now termed" erosion" (Shih, 2009).

1.2 Motivation and Problem Statement

So far, binary digital image processing has been applied for different types of regular tessellation such as square and hexagonal grids (Klette & Rosenfeld, 2004; Soille, 2013). However, no one has tried to apply it in a triangular grid (denoted by *T*). Defining two binary operations (i.e., dilation and erosion) on the triangular grid is not straightforward, since these operations are, in general, translation-invariant by any vector of the square grid. In this study, we rely on the symmetric coordinate frame for triangular tiling (B Nagy, 2001; Benedek Nagy, 2015). This frame addresses every pixel by a coordinate triplet. In this way, all pixels of orientation Δ are addressed by triplets with zero-sum (they are called even pixels), while the pixels of orientation ∇ have triplets where the sum of the coordinate values is one (they are the odd pixels). We will raise some issues when we attempt to define dilation and erosion related to this symmetric frame.

- The points of the triangular grid described by this frame are not the points of a lattice (i.e., the even and odd pixels/points together does not comprise a discrete subgroup of the Euclidean space). That is, the grid points, or pixels are not closed under the addition operation. Further, the only points of the triangular grid that map the grid to itself are the even points (B. Nagy, 2009). On the other hand, the translating of an odd point by a similar point results in a vector outside the grid.
- Dilation and the erosion have been defined on the square or hexagonal grid by using two images. One of them is the active (or input) image; this is the image of the object in which we are interested. The other one is called a structuring element (Soille, 2013). On these type of grids, it is easy to exchange the roles of pixels and grid vectors. However, in the triangular grid, the situation is quite different.
- In the square grid, 180° rotation of the structuring element about the origin de-fines the reflection (Ghosh & Deguchi, 2009; Shih, 2009). However, this 180° rotation in the triangular grid is not a transformation of the triangular grid because it produces also vectors with sum (-1), and they are therefore outside the grid (B. Nagy, 2009). We have previously defined the reflection term by using the inverse sign of giving triples (Abdalla & Nagy, 2017).

In the following points, we show some advantages of the triangular grid and of using the symmetric coordinate system. These motivate us to define a binary dilation and erosion on that grid.

- The two integer coordinate values can address the pixels of the hexagonal grid (Luczak & Rosenfeld, 1976), but, there is a more efficient solution using three coordinate values and obtaining a symmetric description: addressing hexagons by zero-sum triplets (Her, 1995). Likewise, for the triangular tiling, i.e., the triangular grid is described by three coordinates (B Nagy, 2001; Benedek Nagy, 2004). Furthermore, hexagonal and triangular grids have more symmetry axes than the square grid.
- Rotations with a smaller angle (60°) can transform these non- traditional grids to themselves rather than the angle (90°) needed for a similar transformation on the square grid. On the hexagonal, and square grid, there are one and two types of usual neighbor relation among pixels, respectively. However, on the triangular grid, there are three types of usual neighborhood relations (Deutsch, 1972).
- Another reason to support the triangular grid is that the various types of neighbor relations give more freedom in applications. For example, approximating Euclidean disks with chamfer distances based on three weights can be done in a much better way on the triangular grid than on the square grid (Mir-Mohammad-Sadeghi & Nagy, 2017) (see also a binary application by using four approaches in Subsections 2.4.5 and 2.5.5.) Furthermore, in (Sarkar, Biswas, Dutt, Bhowmick, & Bhattacharya, 2017), triangular hulls of digital objects are computed efficiently on the triangular grid. Topology

related topics, e.g., cell complexes (Benedek Nagy, 2015; Wiederhold & Morales, 2008), and various thinning algorithms (Kardos & Palágyi, 2012, 2017; Wiederhold & Morales, 2008) have also been examined recently on the triangular grid. By chamfer distances, based on the usual three types of neighbor relations, much better results are gained than the best-known results on the square grid, even by using 5×5 neighborhood instead of the traditional 3×3 (Mir-Mohammad-Sadeghi & Nagy, 2017; Benedek Nagy, 2014).

We develop these ideas further by taking advantage of the above-mentioned points that motivate us to ask the following inquiries and then attempt to answer them.

- How we can define the most basic binary morphological operations (i.e., dilation and erosion) on the triangular grid? They are translation-invariant on the square lattice by any point of that grid.
- Does the suggested definition of dilation, or an erosion, form an adjunction relation?
- What type of structuring element should we use? Should it be a small subset of the grid as in the case of the square lattice? Is the use of an image and structuring element interchangeable or not?

This study aims to answer all the above open questions and proposes well-defined translation operations (i.e., binary dilation and erosion). By introducing four different approaches in (Chapter 2), in each approach, we provide definitions for both dilation and erosion. In the first approach, (the so-called" strict method") we apply a constraint on using the structuring element: it can contain only even pixels (zero-sum coordinates triplets). In this way, the result of strict dilation and erosion always belongs to the grid. In a second approach (the" weak method") we lift this constraint. In weak dilation and erosion, we allow the structuring elements that contain vectors with a non zero-sum to be used. However, we display only the pixels of the triangular grid (*tr*ixels) in the result. Therefore, the effect of some operations may not be *tr*ixels (or may contain points outside the triangular grid), and we do not consider them as part of the weak dilation and erosion results. The third approach (the" strong method") is more general than the previous ones, and we keep and work with points outside T, however; we display only the part inside T. To avoid losing some information (as the case of weak dilation and erosion) the strong approach seems to be the most applicable, since it inherits pleasant properties of the three-dimensional grid and uses any of the usual neighborhood structures of the triangular grid. Since the strong approach seems to be the most effective one, merely keeping more information could lead to memory consumption issue. In this matter, we need a new and efficient method that can remedy all the previous-mentioned problems. Thus a new method emerges that is the fourth approach (i.e., independent approach) Section 2.5, in which we define the set of structuring elements independently for the two types of pixels (of the image). Concerning the structuring element (SE): it contains two sets of vectors, one for working with the even pixels of the image, and one for the odd. The odd part may contain vectors with the property of having (-1) as the sum of their coordinates. In this method, the input images and structural elements are different types of entities. Notice that in all of above approaches, we deal with the case of a fixed structuring element (i.e., it does not change its shape or size during the task).

1.3 Preliminaries on Square Grid (Necessary Definitions)

We recall common definitions of binary morphology on traditional grids (using the terminology of (Aiello, Pratt-Hartmann, & van Benthem, 2007; Ghosh & Deguchi, 2009; Gonzalez & Woods, 2006; Najman & Talbot, 2013; Pitas & Venetsanopoulos, 1990; Shih, 2009; Soille, 2013). Let us start with point lattices. They are specific regularly-spaced arrays of points. Formally, a point lattice is a discrete subgroup of the Euclidean space $\mathbb{R}^N$ containing the origin, (i.e., a type of the subgroup that is closed under the operations of addition and inversion). Moreover, every point has a neighborhood in which the only lattice point is itself. Well-known examples are the grids $\mathbb{Z}^N$. The basis of vectors describes a point of lattices, and integer coordinates address their points. It is an essential property of the grids, $\mathbb{Z}^N$ that they are self - dual. That is, working with the N-dimensional (hyper) voxels (i.e., pixels if N = 2), their (neighborhood) structure is the same as the (neighborhood) structure of the grid points. Point lattices are usually simply called lattices. However, in lattice theory, this term has a different, and in a sense more general meaning as we recall in Definition 1.3.5. An image was delineated in Euclidean space as a set of corresponding vectors. Let $\mathbb{E}^N$ be N-dimensional Euclidean space (it could be discrete $\mathbb{Z}^N$ or continuous $\mathbb{R}^N$) where $\mathbb{E}^N$ is the set of all points $p = (x_1, .., x_N)$ in $\mathbb{E}^N$. Since we are interested in digitalimages, i.e., images in discrete space, we may consider in most cases that our space is $\mathbb{Z}^N$. A binary image A is a subset of the binary space $\mathbb{E}^N$, where the value of each pointp of $\mathbb{E}^N$ is either black or white: p is black if and only if $p \in A$, otherwise p is white. The following definitions are based on (Aiello et al., 2007; Birkhoff, 1940; Gierz et al., 1980; H. J. A. M. Heijmans & Ronse, 1990; Herrlich & Hušek, 1986).

Definition 1.3.1. Let $L \subseteq \mathbb{R}^N$ be non-empty set, a binary relation $\mathcal{R}$ on a set $\mathcal{L}$ is an order relation if it satisfies the following properties:

(a) For $x \in \mathcal{L}$ such that $x\mathcal{R}x$ (reflexivity)

(b) For $x, y \in \mathcal{L}$ such that $x\mathcal{R}y$ and $y\mathcal{R}x$ yield that $x = y$ (anti-symmetry)

(c) For $x, y, z \in \mathcal{L}$ such that $x\mathcal{R}y$ and $y\mathcal{R}z$ yield that $x\mathcal{R}z$ (transitivity)

Then $(\mathcal{L}, \mathcal{R})$ is called a partially ordered set or (poset).

Remark 1.3.1. We will use the symbol "$\leq$ " which read "less than or equal" for the order of between elements. Also, we use "$\subseteq$" which means inclusion between arbitrary set, however, if the relationship is clear as to what it is, then we use a proper symbol to express that relation.

Definition 1.3.2. Let $(\mathcal{L}, \leq)$ be poset and letting $\mathcal{S} \subseteq \mathcal{L}$ we have the following

(a) An upper bound for $\mathcal{S}$ is an element $a \in \mathcal{L}$ such that $s \leq a, \forall s \in \mathcal{S}$.

(b) An upper bound a of $\mathcal{S}$ is called the least upper bound or (supremum) if and only if for any other upper bound b of $\mathcal{S}$ we have $a \leq b$.

(c) A lower bound for $\mathcal{S}$ is an element $l \in \mathcal{L}$ such that $l \leq s, \forall s \in \mathcal{S}$.

(d) A lower bound d of $\mathcal{S}$ is call the greatest lower bound or (infimum) if and only if for any other lower bound c of $\mathcal{S}$ we have $c \leq d$.

Definition 1.3.3. A function $f: \mathcal{L} \longrightarrow \mathcal{M}$ between two posets $(\mathcal{L}, \leq)$, $(\mathcal{M}, \leq)$ is called (monotone) or order-preserving if, for any $x, y \in \mathcal{L}$, $x \leq y$ implies that $f(x) \leq f(y)$.

Definition 1.3.4. Let $(\mathcal{L}, \leq)$, $(\mathcal{M}, \leq)$ be two posets, let $f: \mathcal{L} \longrightarrow \mathcal{M}$ and $g: \mathcal{M} \longrightarrow \mathcal{L}$ we say that f, g form a Galois connection by assuming that they have the same order relation

(a) if f, g are both monotone, and

(b) $\forall x \in \mathcal{L}, y \in \mathcal{M}, f(x) \leq y \Leftrightarrow x \leq g(y)$.

We say that f is lower adjoint of g and g is upper adjoint of f.

Definition 1.3.5. A poset $(\mathcal{L}, \leq)$ is called a lattice when any non-empty finite subset S of $\mathcal{L}$ has an infimum, and a supremum. We use the notation $inf(S)$ and $sup(S)$ respectively if they exist.

Definition 1.3.6. A lattice $\mathcal{L}$ is said to be complete if an infimum and a supremum exist for any non-empty subset of $\mathcal{L}$.

Definition 1.3.7. Let $(\mathcal{L}, \preccurlyeq)$, $(\mathcal{M}, \grave{\preccurlyeq})$ be two complete lattices (equal or distinct).

(a) An operator $\delta: \mathcal{L} \rightarrow \mathcal{M}$ is an (abstract) dilation if it preserves the supremum, i.e., for every family of subsets X_i of $\mathcal{L}$: $\delta(\bigvee_i X_i) = \grave{\bigvee}_i \delta(X_i)$ where $\bigvee$ is the supremum in $\mathcal{L}$ related to $\preccurlyeq$ and $\grave{\bigvee}$ is the supremum for $\grave{\preccurlyeq}$ in $\mathcal{M}$.

(b) An operator $\mathcal{E}: \mathcal{M} \rightarrow \mathcal{L}$ is an (abstract) erosion if it preserves the infimum, i.e., for every family of subsets Y_i of $\mathcal{M}$: $\mathcal{E}(\grave{\bigwedge}_i Y_i) = \bigwedge_i \mathcal{E}(Y_i)$ where $\bigwedge$ is the infimum related to $\preccurlyeq$ in $\mathcal{L}$ and $\grave{\bigwedge}$ is the infimum for $\grave{\preccurlyeq}$ in $\mathcal{M}$.

(c) The above two operators form an adjunction $(\mathcal{E}, \delta)$ if $\forall X \in \mathcal{L}$ and $\forall Y \in \mathcal{M}$: $\delta(X) \grave{\preccurlyeq} Y$ if and only if $X \preccurlyeq \mathcal{E}(Y)$.

Definition 1.3.8. Let $\mathcal{L}, \mathcal{M}$ be two complete lattices, the set of all operators mapping from $\mathcal{L}$ into $\mathcal{M}$ under the same ordering relation, forms a complete lattice, where the order relation is defined by $\psi \le \rho \Leftrightarrow \psi(X) \le \rho(X), \forall X \in \mathcal{L}$. Now by letting $\mathcal{L} = \mathcal{M}$, the set of all operators from $\mathcal{L}$ to itself, i.e., $O(\mathcal{L})$ forms a complete lattice that inherits the properties of $\mathcal{L}$. The following hold:

(a) The composition of two operators $\psi, \rho \in O(\mathcal{L})$ is $\psi\rho\ \forall X \in \mathcal{L}$ is defined by $\psi\rho(X) = \psi(\rho(X))$. In the same manner $\psi^2 = \psi\psi$.

(b) The identity map (id) which maps every element onto itself is defined by $id : X \to X,\ id(X) = X,\ \forall X \in \mathcal{L}$.

(c) $\psi \in O(\mathcal{L})$ is extensive if $X \le \psi(X), \forall X \in \mathcal{L}$. i.e., $id \le \psi$.

(d) $\psi \in O(\mathcal{L})$ is anti-extensive if $\psi(X) \le X, \forall X \in \mathcal{L}$. i.e., $\psi \le id$.

The following definitions are based on the notation of lattice of sets on $\mathcal{P}(\mathbb{E})$ the set of all subsets of $\mathbb{E}$ with inclusion order ($\subseteq$). That is for any $A, B \in \mathcal{P}(\mathbb{E}), A \subseteq B$ if and only if $a \in A \Longrightarrow a \in B$. Where the supremum and infimum, are given by the union and intersection. This type of lattice is a complete lattice.

In binary image processing a grid, e.g., $\mathbb{E}^N$ is given and the set of its subsets, the set of images, plays the role of $\mathcal{L}$; the subset relation $\subseteq$ is the partial order, the infimum coincides with the intersection, and the supremum coincides with the union.

Definition 1.3.9. Let $A \subset \mathbb{E}^N$ be a binary image. If $A = \emptyset$, then A is empty (sometimes is also called null). Otherwise, a pixel of the image $a \in A$ is addressed by a vector (x, y).

Definition 1.3.10. Let $A \subset \mathbb{E}^2$ be a binary image, the complement of A is also a binary image, it is defined by $A^c = \{p: p \notin A\}$, i.e., it is obtained by interchanging the roles of black and white pixels.

Definition 1.3.11. Let $B \subset \mathbb{E}^2$, the reflection of an image B is denoted by $\breve{B}$ and it is defined by $\breve{B} = \{p: p = -b, \forall b \in B\}$. Note here the important fact that $\breve{B} \subset \mathbb{E}^2$ on the square grid. Moreover, it is denoting the symmetric set of B with respect to the origin or reflection for the set B about the origin (Ghosh & Deguchi, 2009).

Definition 1.3.12. The union of two binary images $A, B \subset \mathbb{E}^2$ is a binary image such that a pixel is black if it is black in A or in B, formally, $A \cup B = \{p: p \in A$ or $p \in B\}$.

Definition 1.3.13. The intersection of two binary images $A, B \subset \mathbb{E}^2$ is a binary image containing those pixels that are black both in A and in B, i.e., $A \cap B = \{p: p \in A, p \in B\}$.

Definition 1.3.14. Let $A \subset \mathbb{E}^N$, $b \in \mathbb{E}^N$, then the translation of A by b, denoted by $(A)_b$ is defined as $(A)_b = \{p \in \mathbb{E}^N: p = a + b,\ \exists a \in A\}$.

Definition 1.3.15. Let $A, B \subset \mathbb{E}^N$, the *binary dilation* of A by structuring element B is denoted by $A \oplus B = \{p \in \mathbb{E}^N : p = a + b, \exists a \in A, b \in B\}$. It can also be defined by

$$A \oplus B = \bigcup_{b \in B} A_b = \bigcup_{a \in A} B_a \qquad (0.2.1)$$

Note here the similar and interchangeable role of the image A and the structuring element B on the square grid.

Definition 1.3.16. Let $A, B \subset \mathbb{E}^N$, the *binary erosion* of A by structuring element B is denoted by $A \ominus B$. Formally, it is $A \ominus B = \{p \in \mathbb{E}^N: p + b \in A, \forall b \in B\}$, it can be written, equivalently, $A \ominus B = \bigcap_{b \in B} A_{-b}$, it can be defined as $A \ominus B = \{p \in \mathbb{E}^N: (B)_p \subseteq A\}$. Also, it is defined as

$$A \ominus B = \bigcap_{b \in \breve{B}} A_b \qquad (0.2.2)$$

Where, $\breve{B}$ is defined in Definition 1.3.11.

1.4 Properties of Dilation and Erosion on Square Grid

In this section, we list the basic properties of dilation and erosion. These properties are well known in the Euclidean space (Ghosh & Deguchi, 2009; Gonzalez & Woods, 2006; Jean Serra, 1983; Shih, 2009; Soille, 2013). Let $A, B, C, D \subset \mathbb{E}$ where $\mathbb{E}$ is the Euclidean space, assume that all of these four sets can play the role of the active (e.g., input) image and also the role of the structuring element. (Notes that this is a usual and valid assumption on lattices.) Let O be the origin of $\mathbb{E}$ and let $p, t \in \mathbb{E}$.

1.4.1 Dilation Properties

We should know the following facts: for any arbitrary subset A of $\mathbb{E}$, and for the empty set $\emptyset$, where $\mathbb{E}$ is the whole Euclidean space, $A \oplus \emptyset = \emptyset$, $\mathbb{E} \oplus A = A \oplus \mathbb{E} = \mathbb{E}$. Moreover, if $A \oplus B = \emptyset$, then at least one of A or B is the empty set $\emptyset$.

Property D1.

(a) $A \oplus \{O\} = A$.

(b) $A \oplus \{O\} = \{O\} \oplus A$. (Unit element)

Property D2.

(a) $A \oplus \{p\}=\{p\} \oplus A$.

(b) $A \oplus \{p\} = A_p$.

Property D3. If D contains the origin O, then $A \subseteq A \oplus D$.

Property D4. $A \oplus C = C \oplus A$. (Commutativity)

Property D5. $B \oplus (A \oplus C) = (B \oplus A) \oplus C$. (Associativity)

Property D6.

(a) $(A)_p \oplus C = (A \oplus C)_p$.

(b) $A \oplus (C)_p = (A \oplus C)_p$. (Translation invariance)

Property D7. $(A)_p \oplus (C)_{-p} = A \oplus C$.

Property D8. If $A \subseteq B$, then $A \oplus C \subseteq B \oplus C$. (Increasing property, monotonicity)

Property D9. $(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$.

Property D10. $(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$. (Distributivity over union of images)

Property D11. $A \oplus (C \cup D) = (A \oplus C) \cup (A \oplus D)$.

Property D12. $A \oplus (C \cap D) \subseteq (A \oplus C) \cap (A \oplus D)$.

1.4.2 Erosion Properties

We should also know that for an arbitrary subset A of $\mathbb{E}$, $A \ominus \emptyset = \mathbb{E}$, $\emptyset \ominus A = \emptyset$, $\mathbb{E} \ominus A = \mathbb{E}$. Let us see the other well-known facts involving erosion.

Property E1. If C contains the origin, then $A \ominus C \subseteq A$.

Property E2. $A \ominus C \neq C \ominus A$. (Not commutative)

Property E3. $(A)_t \ominus C = (A \ominus C)_t$. (Translation invariance)

Property E4. If $A \subseteq B$, then $A \ominus C \subseteq B \ominus C$. (Increasing property, monotonicity)

Property E5. $B \ominus (A \oplus C) = (B \ominus A) \ominus C$.

Property E6. $B \oplus (A \ominus C) \subseteq (B \oplus A) \ominus C$.

Property E7. $(A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C)$. (Distributivity over intersection)

Property E8. $(A \cup B) \ominus C \supseteq (A \ominus C) \cup (B \ominus C)$.

Property E9. $A \ominus (C \cup D) = (A \ominus C) \cap (A \ominus D)$.

Property E10. $A \ominus (C \cap D) \supseteq (A \ominus C) \cup (A \ominus D)$.

Property E11. $A \ominus (C)_t = (A \ominus C)_{-t}$.

1.4.3 Abstract Dilation and Erosion and Their Adjunction Relation

Up to now, we define dilation and erosion on a complete lattice of sets where its elements are subsets of a complete lattice $\mathcal{P}(\mathbb{E})$ (H. J. A. M. Heijmans & Ronse, 1990). Here we recall the abstract dilation and erosion.

(a) An operator $\delta: \mathcal{P}(\mathbb{E}) \longrightarrow \mathcal{P}(\mathbb{E})$, $\delta(A) = A \oplus B$ is dilation when for every family $A_i \in \mathcal{P}(\mathbb{E})$ and for any structuring element B we have $(\bigcup_i A_i) \oplus B = \bigcup_i (A_i \oplus B)$. This means that dilation preserves the union operation.

(b) An operator $\mathcal{E}: \mathcal{P}(\mathbb{E}) \longrightarrow \mathcal{P}(\mathbb{E}), \mathcal{E}(A) = A \ominus B$ is erosion when for every family $A_i \in \mathcal{P}(\mathbb{E})$ and for any structuring element B we have $(\bigcap_i A_i) \ominus B = \bigcap_i (A_i \ominus B)$. Hence, erosion preserves the intersection operation.

From (a) and (b) these two conditions are clearly mentioned in Property D10 of dilation and Property E7 of erosion. Also, these conditions are needed for the next adjunction relation on the complete lattice.

Now dilation and erosion are linked by adjunction relation as following (Ronse, 1990). Let $A, C, B \in \mathcal{P}(\mathbb{E})$, then

$$A \oplus B \subseteq C \Leftrightarrow A \subseteq C \ominus B. \qquad (0.3.1)$$

This comes directly from part (b) of the Definition 1.3.4. On point lattices, the dilation (Definition 1.3.15) and erosion (Definition 1.3.16) are also abstract dilation and erosion satisfying the adjunction relation.

1.5 Examples of Dilation and Erosion on Square Grid

Example 1.5.1. Let $A = \{(1,1),(2,1),(2,2)\}$ and $B = \{(-1,0),(0,0),(0,1)\}$ where $(0,0)$ is the origin of the square grid and also the origin of the structuring element. We have that dilation of an image A by the structuring element B is given by $A \oplus B = \{(0,1),(1,2),(2,1),(2,3),(2,2),(1,1)\}$. And $A \ominus B = \{(2,1)\}$.

(See Figure 1).

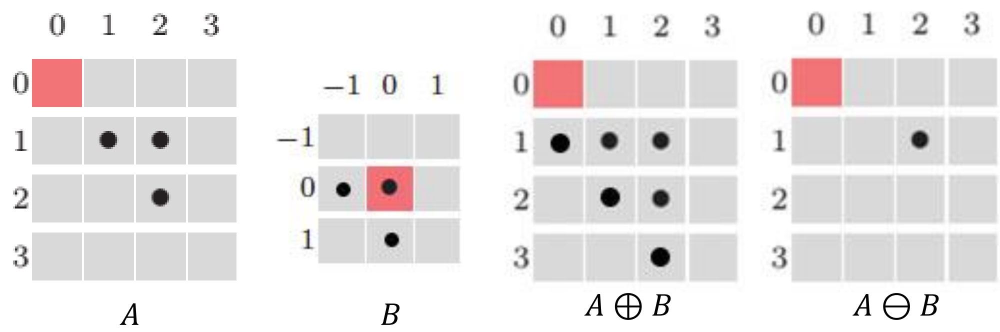


Figure 1: Dilation and erosion of an image A by the structuring element B, where the red square indicates the origin

Example 1.5.2. Let $A = \{(1,0),(2,0),(3,0),(1,1),(2,1),(1,2),(2,2),(1,3),(2,3),(3,3),(2,4),(3,4)\}$. And let $B = \{(1,0),(2,0),(1,1),(2,1)\}$, then the dilation and erosion of A by B is given in Figure 2.

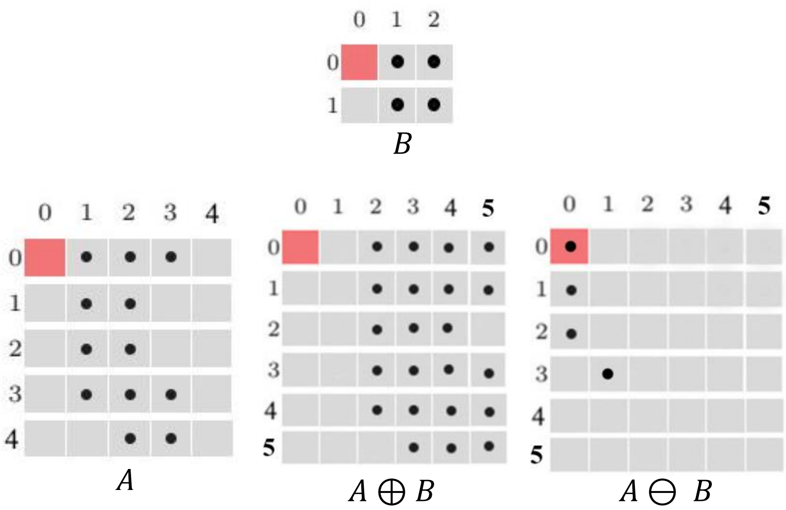


Figure 2: The result of dilation and erosion in Example 1.5.2

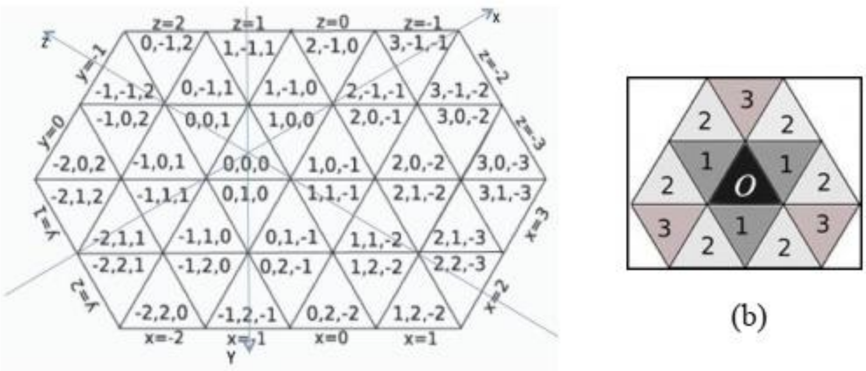
Chapter 2

DILATION AND EROSION ON THE TRIANGULAR TILING: FOUR PROPOSED SOLUTIONS

2.1 Preliminaries: Description of the Triangular Tiling

In digital geometry, integer coordinate values address the pixels (sometimes they are also referred as points). The square grid is often used for numerous applications since the well-known Cartesian coordinate system (that is a part of standard elementary mathematics) describes it. To use a different grid in image processing and/or in computer graphics, one requires a good, flexible coordinate system. Image processing on the hexagonal lattice is also examined due to some of its attractive properties, e.g., there is only one type of usual neighbor relation between pixels (they are also called hexels in this case), and the grid has better symmetric properties than the square grid. The pixels of the hexagonal grid can be addressed with two integers (Luczak & Rosenfeld, 1976), but there is a more elegant solution using three coordinate values and obtaining a symmetric description: addressing hexagons by zero-sum triplets (Her, 1995). Similarly, the triangular tiling, i.e., the triangular grid is described by three coordinates (B. Nagy, 2003, 2009; B Nagy, 2001; Benedek Nagy, 2004, 2015) as it is shown in Figure 3 (a). There are two orientations of the used triangle pixels (trixels), the sum of their coordinate values is zero and one, and they are called even and odd points (pixels), respectively. There are three types of normal neighborhood relations on the triangular grid (Deutsch, 1972).

The triangular tiling consists of *tri*angle p*ixels*, i.e., trixels. It is described by a symmetric coordinate system addressing every trixel by a coordinate triplet, see, e.g., (B. Nagy, 2003, 2009; B Nagy, 2001; Benedek Nagy, 2015) The origin, as a trixel, is addressed by (0,0,0). The coordinate axes are lines cutting this pixel to halves see Figure 3 (a). They are directed such that their angles are 120°. Every trixel has three closest neighbor pixels sharing one of the sides of the triangle. Notice that, although each pixel is a triangle, there are two different orientations of them: Δ, ∇. A trixel and its closest neighbors have opposite orientations. From a trixel having coordinates (x, y, z) with $x + y + z = 0$, its closest neighbor trixels can be reached by a step in the direction of one of the coordinate axes. Consequently, the respective coordinate value is increased by one: the three neighbors are addressed by $(x + 1, y, z)$, $(x, y + 1, z)$, and $(x, y, z + 1)$, respectively. For a trixel (x, y, z) with $x + y + z = 1$, its closest neighbor trixels can be reached by a step to the direction opposite to one of the coordinate axes, and thus, the three neighbors are addressed by $(x - 1, y, z)$, $(x, y - 1, z)$, and $(x, y, z - 1)$. In this way, all trixels of orientation Δ are addressed by triplets with zero-sum (they are called even pixels), while the trixels of orientation ∇ have triplets where the sum of the coordinate values is one (they are the odd pixels). There are three types of neighborhood relations (Deutsch, 1972): two triangles are 1-neighbours if they share a side, i.e., an edge of the grid. They are exactly the closest neighbors. Two triangles are strict 2-neighbours if they have a common 1-neighbor triangle. Two trixels are strict 3-neighbours if they share exactly one point on their boundaries (vertex of the grid) but they are not 2-neighbors. See also, Figure 3 (b), where these three types of neighbors are shown for an even trixel. Each trixel has three 1-neighbours, nine 2-neighbours (including 1-neighbours and six strict 2-negihbours) and twelve 3-neighbours (the nine 2-neighbours and three strict 3-neighbours). Actually, the coordinate triplets of two strict k-neighbor (k = 1,2,3) trixels mismatch exactly in k places, and the difference in each mismatch position is ±1. Triplets of two k-neighbor trixels could mismatch at most in k places and the difference in each mismatch position is ±1. As an example, consider the trixel having triplet (1,1,–1). As one can also observe in Figure 3 (a), its three 1-neighbors are addressed by the coordinate triplets (0,1,–1), (1,0,–1) and (1,1,–2). The strict 2-neighbors are (0,1,0), (1,0,0), (2,0,–1), (2,1, −2), (1,2,–2) and (0,2,–1). Further, (0,0,0), (2,0,−2) and (0,2,−2) are addressing the strict 3-neighbors of the pixel (1,1,–1).



(a) (b)

Figure 3: (a) Symmetric coordinate frame for triangular tiling. (b) Various neighborhood relations on the triangular grid of a trixel O

We will use the notation T for the triangular grid, we also use the notation G^0 for the set of even trixels of T ($G^0 \subset T$), and G^+ for the set of odd trixels of T ($G^+ \subset T$). We should notice that $G^0 \cap G^+ = \emptyset$ and $G^0 \cup G^+ = T$. Notice that the triangular grid can also be seen as a special subset of the cubic lattice, i.e., $T \subset \mathbb{Z}^3$, (B. Nagy, 2003; Benedek Nagy, 2004).

We may also call vectors the elements of $\mathbb{Z}^3$, and specially, even and odd vectors for those that are also elements of T.

2.1.1 Transformations on Triangular Grid

We recall an important observation from (B. Nagy, 2009) reflecting the fact that the triangular grid is not a point lattice.

Proposition 2.1.1.1. Translation with vector $v\ (x, y, z)$ maps the grid to itself if and only if $x + y + z = 0$, i.e., the coordinate sum of the vector equals to zero.

Proposition 2.1.1.2. Addition of two points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$ in T results $p_1(x_1, y_1, z_1) + p_2(x_2, y_2, z_2) = p(x_1 + x_2, y_1 + y_2, z_1 + z_2)$. The resulting point is in the triangular grid, i.e., it is a trixel, if

(a) $p_1, p_2 \in G^0$, then $p_1 + p_2 \in G^0$.

(b) $p_1 \in G^+, p_2 \in G^0$, then $p_1 + p_2 \in G^+$.

(c) $p_1 \in G^0, p_2 \in G^+$, then $p_1 + p_2 \in G^+$.

Proposition 2.1.1.3.

(a) Let $p_1 \in G^0$, then $-(p_1) \in G^0$.

(b) Let $p_2 \in G^+$, then $-(p_2) \notin T$, but if it is allowed to use it, $-\big(-(p_2)\big) = p_2$.

(c) If $p_1 \in G^0, p_3 \in G^0$, then $-(p_1) + p_3 = p_3 + (-(p_1)) = p \in G^0$.

(d) If $p_2 \in G^+, p_1 \in G^0$, then $-(p_2) + p_1 = p_1 + (-(p_2)) = -(p)$, with $p \in G^+$.

(e) If $p_1 \in G^0, p_2 \in G^+$, then $-(p_1) + p_2 = p_2 + (-(p_1)) = p$, where $p \in G^+$.

(f) If $p_2, p_4 \in G^+$, then $-(p_2) + p_4 = p_4 + (-(p_2)) = p$, where $p \in G^0$.

Remark 2.1.1.1. If $p_1 \in G^+, p_2 \in G^+$, then $p(x, y, z) = p_1(x_1 + y_1 + z_1) + p_2(x_2 + y_2 + z_2) \notin T$ since the sum of coordinate values of p is equal to 2 ($x + y + z = 2$). This operation can be allowed but keeping in mind that the resulted point is not in the grid.

Example 2.1.1.1. Let $p_1 = (-1, 1, 1)$, $p_2 = (0, 1, 0)$, then $p_1 + p_2 = (-1, 2, 1) \notin T$.

Remark 2.1.1.2. For any point (vector), $p(x, y, z) \in T$ we are using the notation $-p(-x, -y, -z)$ for its *inverse*, as a kind of 3-dimensional reflection. However, we should notice that it is not a rotation by 180 degrees, as the analogous transformation was on the square grid. Moreover, it is not even a transformation of the triangular grid, since it maps odd points to triplets with sum -1, and they are clearly not points in T. Even the inverse of an odd element is not in T, we could use it, e.g., in a strong dilation (see later) to have some effects.

Remark 2.1.1.3. We note here that in the triangular grid rotation having a center in the Origin (at the meeting point of the axes) can be used only if the degree is a multiplier of 120° (and 180° is not like that). There are also other types of mirroring and rotations on the triangular grid, e.g., with the center in the corner of a trixel (see (B. Nagy, 2009), for details).

The result of a translation of an odd point by another odd point is not in the grid. This type of translations does not map the grid into itself: this grid is not a point lattice, and hence, the extensions of the morphological operations to the triangular grid are not straightforward. In next sections, we recommend some possible definitions for dilation and erosion solving various ways the above problem.

2.2 Strict Dilation and Erosion on the Triangular Grid

Since the triangular grid is not a lattice, the types of points of the structuring element play importance. By the first and simplest solution, it is allowed to use only such transformations of the image which gives the resulted points inside the grid: the image points can be translated only by even points, a restriction on the structuring element is used: it must be a subset of G^0. This, so-called strict dilation and erosion are described in this section. The second option is when the translation is also allowed by odd points which may produce point(s) outside of the grid (see weak and strong dilations and erosions which allow that option also, in Sections 2.3 and 2.4, respectively).

Definition 2.2.1. (Strict Dilation)

Let $A \subset T, B \subset G^0$, then the *strict dilation* of A by set B is defined as $A \oplus_\Delta B = \{p \in T: p(x,y,z), x = x_1 + x_2,\ y = y_1 + y_2,\ z = z_1 + z_2,\ \exists\, p_1(x_1,y_1,z_1) \in A, p_2(x_2,y_2,z_2) \in B\}$. The notation refers for the fact that the structuring element must contain only even vectors.

We can write the above definition in the simplest formula as following:

$$A \oplus_\Delta B = \{p \in T: p = a + b, \exists a \in A, b \in B \subset G^0\} = \bigcup_{b \in B} A_b. \qquad (1.2.1)$$

Remark 2.2.1. Notice that in general $A \oplus_\Delta B \neq B \oplus_\Delta A$ unless, $A, B \subset G^0$ (see the Property D4Δ).

The idea of strict dilation is a restriction on the structuring element B: to force the resulted points to be trixels in T, the condition $B \subset G^0$ is applied.

Example 2.2.1. Let $A = \{(0,2,-1),(0,2,-2),(0,3,-2),(-1,2.-1),(-1,3,-2),$ $(-1,3,-1)\}$ be a binary image (its points have the value equal to one, their color is black). Now, let $B = \{(0,-1,1),(1,-1,0)\}$, then $A \oplus_\Delta B = \{(-1,1,0),(0,1,0),$ $(0,1,-1),(1,1,-1),(1,1,-2),(-1,2,0),(-1,2,-1),(0,2,-1),(0,2,-2),(1,2,-2)\}$ (See Figure 4:).

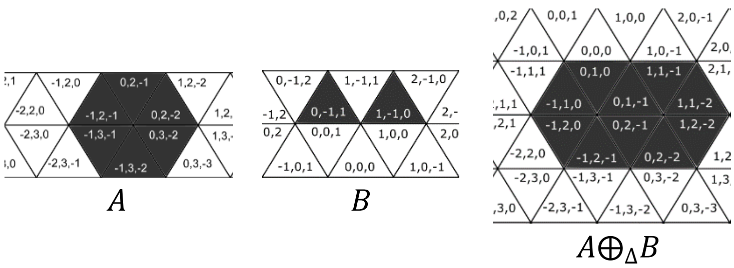


Figure 4: Strict dilation Example 2.2.1 with the even structuring element B

Definition 2.2.2. (**Strict Erosion**)

Let $A \subset T$ and $B \subset G^0$, then the *strict erosion* of the image A by structuring element B is defined by $A \ominus_\Delta B = \{p \in T: p + b \in A, \forall b \in B\}$. Also it can be defined by using the following equation.

$$A \ominus_\Delta B = \bigcap_{b \in B} A_{-b} = \{p \in T: B_p \subseteq A\}\,. \tag{1.2.2}$$

In strict erosion, it is guaranteed that each vector p obtained in this way is automatically belonging to the grid T, i.e. $p \in T$. To guarantee this fact a restriction is used for the structuring element, namely its inverse is also belonging to the grid.

Example 2.2.2. Let $B = \{(0,0,0),(0,1,-1)\}$, $A = \{(-1,0,1),(-1,1,1),(-2,1,1),$ $(-1,1,0)\}$. Then $A \ominus_\Delta B = \{(-1,0,1)\}$. In this example $A \ominus_\Delta B \subset A$ holds. (See Figure 5 and Subsection 2.2.2 for more details.)

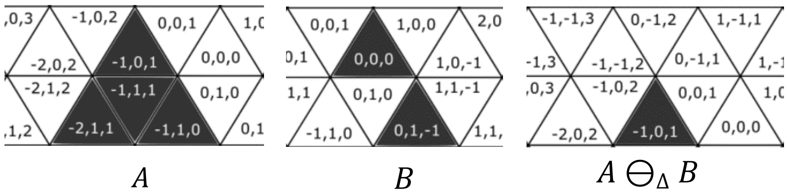


Figure 5: Example 2.2.2 shows the strict erosion that is a subset of an image A

2.2.1 Properties of Strict Dilation

In this subsection, we will present some properties of strict dilation, where we will prove the similar properties as it was used in Subsection 1.4.1. We should notice that since strict dilation is defined only if the second operand, the structuring element is a subset of G^0, in some of the original properties we need some restrictions to have all the appearing formulae defined. In the following descriptions we specify these restrictions, if any. In the following, we use $A, B \subset T$ as input images and $D, C \subset G^0$ as structuring elements, if further restrictions are not applied.

Property D1Δ.

(a) Let $O = (0,0,0) \in G^0$ be origin of triangular grid, then $A \oplus_\Delta \{O\} = A$.

(b) Moreover, if and only if $A \subset G^0$, then the right side of the equation $A \oplus_\Delta \{O\} = \{O\} \oplus_\Delta A$ is defined and the equivalence holds.

Proof. Part (a) comes directly from the Definition 2.2.1 and Eq. (1.2.1) of strict dilation. At part (b), by Definition 2.2.1, A can be used as structuring element only with the given condition, then the statement is straightforward by applying the definition on both sides of the equation. □

Property D2Δ. The formulae of the equations

(a) $A \oplus_{\Delta} \{p\} = \{p\} \oplus_{\Delta} A$

(b) $A \oplus_{\Delta} \{p\} = A_p$

Are defined if and only if $p \in G^0$ and at (a), $A \subset G^0$. When they are defined, they hold.

Proof. By Definition 2.2.1 it is easy to see that the given conditions are necessary for the formulae to be well defined. Now, let us start with equation (b). If p is an even point, then by applying Definition 2.2.1 (left side) and Definition 1.3.14 (for the right side) the same set of trixels is obtained. Equation (a) is verified by Definition 2.2.1. Notice that in this case $A \oplus_{\Delta} \{p\} \subset G^0$. □

Property D3Δ. Let $A \subset T$ be an input image and $D \subset G^0$ be a structuring element. If D contains the origin O, i.e., $O \in D$, then $A \subseteq A \oplus_{\Delta} D$.

Proof. By the Definition 2.2.1 and by Property D1Δ(a) we have: $A = A \oplus_{\Delta} \{O\}$. Let $p \in A$, then $p \in A \oplus_{\Delta} \{O\}$ and so, $p \in A \oplus_{\Delta} D$ which proves our statement. □

Property D4Δ. Let A be an input image and C be structuring element. The formulae of the equality $A \oplus_{\Delta} C = C \oplus_{\Delta} A$ are defined if and only if $A \subset G^0$. In this case the statement holds.

Proof. The condition $A \subset G^0$ is necessary for the right side to be defined. Now, since $A, C \subset G^0$, then by the Definition 2.2.1 we have $A \oplus_{\Delta} C = \{p \in T : p = a + c, \exists a \in A, c \in C\} = \{p \in T : p = c + a, \exists c \in C, a \in A\} = C \oplus_{\Delta} A$. Notice that in this case $A \oplus_{\Delta} C \subset G^0$ also holds. □

Property D5Δ. Let A, B be input images and C be a structuring element. The right side of the equality $B \oplus_\Delta (A \oplus_\Delta C) = (B \oplus_\Delta A) \oplus_\Delta C$ is defined if and only if $A \subset G^0$, and, then, it holds.

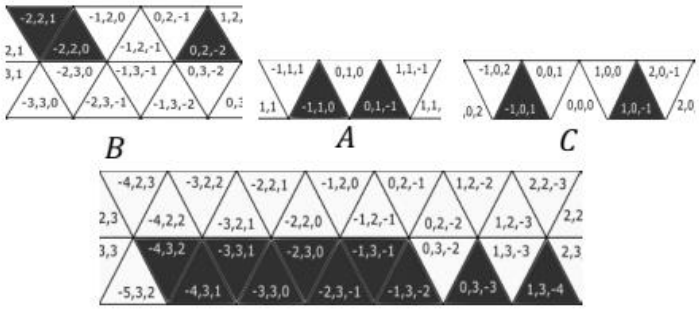
Proof. It is clear that, since A plays the role of a structuring element on the right side, the condition is necessary (even it has a role of an image of the left side). Now, to prove the equivalence of the two sides, let $p \in B \oplus_\Delta (A \oplus_\Delta C)$. Then, by Definition 2.2.1 $p = b + t$ for some $b \in B, t \in (A \oplus_\Delta C)$. Applying the Definition 2.2.1 for *t*: $t = a + c$ for some $a \in A, c \in C$, i.e., $p = b + (a + c) = (b + a) + c$ by the associativity of vector addition, and thus, $p \in (B \oplus_\Delta A) \oplus_\Delta C$ if and only if $p \in B \oplus_\Delta (A \oplus_\Delta C)$. □

Example 2.2.1.1. Let $B = \{(-2,2,1), (-2,2,0), (0,2,-2)\}$,$C = \{(-1,0,1)(1,0,-1)\}$, $A = \{(-1,1,0), (0,1,-1)\}$.Then $A \oplus_\Delta C = \{(-1,1,0), (1,1,-2)$,$(-2,1,1)$, $(0,1,-1)\}$, $B \oplus_\Delta (A \oplus_\Delta C) = \{(0,3,-3), (-1,3,-2), (-2,3,-1), (-1,3,-1)$, $(-3,3,1), (-4,3,1), (-3,3,0), (-2,3,0), (-4,3,2), (1,3,-4)\} = (B \oplus_\Delta A) \oplus_\Delta C$. (See Figure 6).

Property D6Δ. The strict dilation is translation invariant by translations with vectors with zero-sum. Formally: Let $p \in G^0$ be a vector, $A \subset T$ be an image and $C \subset G^0$ be a structuring element. Then

(a) $(A)_p \oplus_\Delta C = (A \oplus_\Delta C)_p$ and

(b) $A \oplus_\Delta (C)_p = (A \oplus_\Delta C)_p$.



$$B \oplus_{\Delta} (A \oplus_{\Delta} C) = (B \oplus_{\Delta} A) \oplus_{\Delta} C$$

Figure 6: Associative property of strict dilation

Proof. Let us prove the first statement (a). Let $t \in ((A)_p \oplus_{\Delta} C)$, then $\exists a \in A, c \in C$ such that $t = (a + p) + c$, since vector addition is commutative and associative $t = (a + c) + p$. On the other side, $(A \oplus_{\Delta} C)_p$ is well defined, since $C \subset G^0$ and p is a zero sum vector. Thus, $t \in (A \oplus_{\Delta} C)_p$. Since each step was if and only if, this yields that $(A)_p \oplus_{\Delta} C = (A \oplus_{\Delta} C)_p$. Similarly for (b) $A \oplus_{\Delta} (C)_p = (A \oplus_{\Delta} C)_p$. □

Property D7Δ. If $p \in G^0$, $A \subset T$ is an image and $C \subset G^0$ is a structuring element, then $(A)_p \oplus_{\Delta} (C)_{-p} = A \oplus_{\Delta} C$.

Proof. Let $t \in A \oplus_{\Delta} C$, it is if and only if $t \in (A \oplus_{\Delta} C)_{p+(-p)}$. However, by the previous Property (**D6**Δb) $(A \oplus_{\Delta} C)_{p+(-p)} = (A \oplus_{\Delta} (C)_{-p})_p$ and, thus, $t \in (A \oplus_{\Delta} (C)_{-p})_p$. Similarly, by Property (**D6**Δa) it is if and only if $t \in ((A)_p \oplus_{\Delta} (C)_{-p})$. □

Example 2.2.1.2. Let $A = \{(0,2,-2), (1,0,-1), (1,1,-2), (1,1,-1), (1,2,-2), (2,1,-3), (2,1,-2)\}$ and let $p = (-2,-1,3)$.Then, $(A)_p = \{(-2,1,1), (-1,-1,2), (-1,0,1), (-1,0,2), (-1,1,1), (0,0,0), (0,0,1)\}$, $(-p) = (2,1,-3)$. Let $C =$

$\{(-1,1,0), (0,1,-1)\}$, then, $(C)_{-p} = \{(1,2,-3), (2,2,-4)\}$. Further $A \oplus_\Delta C = \{(-1,3,-2), (0,1,-1), (0,2,-2), (0,2,-1), (0,3,-3), (0,3,-2), (1,1,-2), (1,2,-3), (1,2,-2), (1,3,-3), (2,2,-4), (2,2,-3)\} = (A)_p \oplus_\Delta (C)_{-p}$.

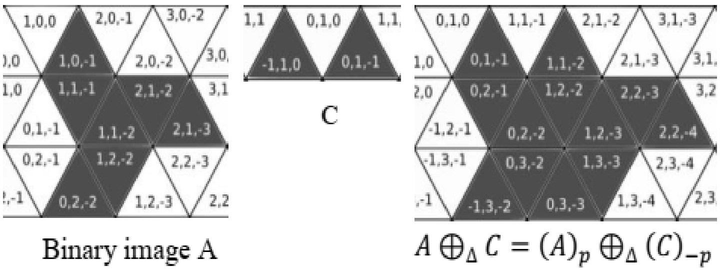


Figure 7: The sets of Example 2.2.1.2 illustrating Property D7Δ

Property D8Δ. The strict dilation has the increasing property: Let $A, B \subset T$ be input images and $C \subset G^0$ be a structuring element, If $A \subseteq B$, then $A \oplus_\Delta C \subseteq B \oplus_\Delta C$.

Proof. Let $p \in A \oplus_\Delta C$, then by Definition 2.2.1 $\exists a \in A, c \in C: p = a + c$ and since $A \subseteq B$, then, also $a \in B$, and thus, $p = a + c$ with $a \in B, c \in C$ gives that $p \in B \oplus_\Delta C$. □

Example 2.2.1.3. Let $A = \{(3,-3,0), (2,-3,1)\}$, $C = \{(0,-1,1), (1,-1,0)\}$, $B = \{(3,-4,1), (2,-3,1), (3,-3,0)\}$. Then, $A \oplus_\Delta C = \{(4,-4,0), (3,-4,1), (2,-4,2)\}$, $B \oplus_\Delta C = \{(4,-4,0), (2,-4,2), (4,-5,1), (3,-4,1), (3,-5,2)\}$. Thus $A \oplus_\Delta C \subseteq B \oplus_\Delta C$.

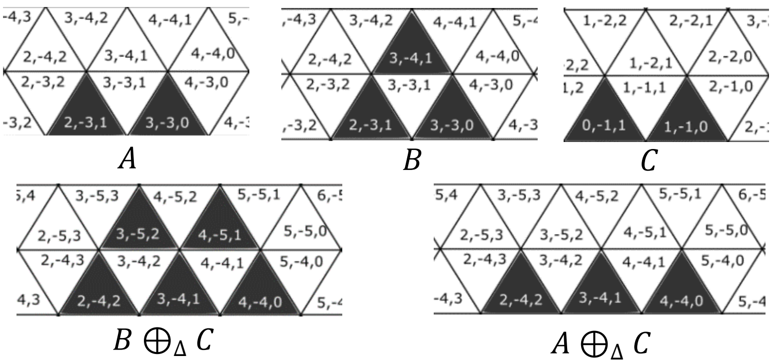


Figure 8: Illustration the (Property D8Δ) Example 2.2.1.3

Property D9Δ. If $A, B \subset T$ are input images and $C \subset G^0$ is a structuring element, then $(A \cap B) \oplus_\Delta C \subseteq (A \oplus_\Delta C) \cap (B \oplus_\Delta C)$.

Proof. Let $p \in (A \cap B) \oplus_\Delta C$, then, by Definition 2.2.1 $\exists\ t \in A \cap B, c \in C$ such that $p = t + c$. Since, $t \in A \cap B$ we have $p = t + c \in A \oplus_\Delta C$ and $p = t + c \in B \oplus_\Delta C$, in this way there is a $c \in C$ such that $p \in (A \oplus_\Delta C) \cap (B \oplus_\Delta C)$. Therefore, $(A \cap B) \oplus_\Delta C \subseteq (A \oplus_\Delta C) \cap (B \oplus_\Delta C)$. $\square$

Property D10Δ. The strict dilation is distributive over the union of images: Let $A, B \subset T$ be images and $C \subset G^0$ be a structuring element. Then, $(A \cup B) \oplus_\Delta C = (A \oplus_\Delta C) \cup (B \oplus_\Delta C)$.

Proof. Let $p \in (A \cup B) \oplus_\Delta C$. Then, by Definition 2.2.1 it is if and only if $\exists k \in A \cup B, c \in C$ such that $p = k + c$. However, this holds if and only if either $k \in A, c \in C$ and thus, $p = k + c \in A \oplus_\Delta C$ or $k \in B$ and $c \in C$: $p = k + c \in B \oplus_\Delta C$. But this is equivalent to $p \in (A \oplus_\Delta C) \cup (B \oplus_\Delta C)$. The statement is proven. $\square$

Property D11Δ. The strict dilation is distributive over union of structuring elements. Formally: If $A \subset T$ and $C, D \subset G^0$ are structuring elements, then $A \oplus_\Delta (C \cup D) = (A \oplus_\Delta C) \cup (A \oplus_\Delta D)$.

Proof. Let $p \in A \oplus_\Delta (C \cup D) \Leftrightarrow \exists a \in A, l \in (C \cup D)$ such that $p = a + l \Leftrightarrow p = a + l \in A \oplus_\Delta C$ or $p = a + l \in A \oplus_\Delta D \Leftrightarrow p \in (A \oplus_\Delta C) \cup (A \oplus_\Delta D)$. □

Property D12Δ. Let $A \subset T$ be an input image, $C, D \subset G^0$ be structuring elements. Then, $A \oplus_\Delta (C \cap D) \subseteq (A \oplus_\Delta C) \cap (A \oplus_\Delta D)$.

Proof. Let $p \in A \oplus_\Delta (C \cap D) \Rightarrow \exists a \in A, l \in (C \cap D)$ such that $p = a + l \Rightarrow p = a + l \in A \oplus_\Delta C$ and $p = a + l \in A \oplus_\Delta D \Rightarrow p \in (A \oplus_\Delta C) \cap (A \oplus_\Delta D)$. □

2.2.2 Properties of Strict Erosion

Property E1Δ. Let $A \subset T$ be an input binary image and $C \subset G^0$ be a structuring element. If $O \in C$ where, $O = (0,0,0)$, then $A \ominus_\Delta C \subseteq A$.

Proof. Let $p \in A \ominus_\Delta B$, then, by Definition 2.2.2 $p = p + c \in A, \forall c \in C$, but $(0,0,0) \in C$, therefore, $p \in A$. □

Property E2Δ. If $A \subset T$ and $C \subset G^0$, then, generally $A \ominus_\Delta C \neq C \ominus_\Delta A$.

Proof. By the Definition 2.2.2 the right hand side is defined only if $A \subset G^0$. However, the property does not necessarily hold in this case neither as the next example shows:

Example 2.2.2.1. Let $C = \{(-1,0,1), (0,0,0)\}$, $A = \{(-1,0,1), (-1,1,0), (0,-1,1), (0,0,0)\}$ then $A \ominus_\Delta C = \{(0,0,0)\}$, $C \ominus_\Delta A = \emptyset$. □

Property E3Δ. The strict erosion is translation invariant with grid vectors of the triangular gird. Formally, if $A \subset T$, $C \subset G^0$ and $t \in G^0$, then $(A)_t \ominus_\Delta C = (A \ominus_\Delta C)_t$.

Proof. Let $p \in (A)_t \ominus_\Delta C \Leftrightarrow p + c \in (A)_t, \forall\, c \in C \Leftrightarrow (p - t) + c \in A, \forall\, c \in C \Leftrightarrow (p - t) \in A \ominus_\Delta C \Leftrightarrow p \in (A \ominus_\Delta C)_t$. □

Property E4Δ. If $A \subseteq B \subset T$ are images and $C \subset G^0$ is a structuring element, then $A \ominus_\Delta C \subseteq B \ominus_\Delta C$.

Proof. Let $p \in A \ominus_\Delta C \Rightarrow p + c \in A, \forall c \in C$, since $A \subseteq B \Rightarrow p + c \in B, \forall\, c \in C \Rightarrow p \in B \ominus_\Delta C$, yield that $A \ominus_\Delta C \subseteq B \ominus_\Delta C$. □

Property E5Δ. If A, B are input images, C is structuring element such that $A \subset G^0$ also holds, then $(B \ominus_\Delta A) \ominus_\Delta C = B \ominus_\Delta (A \oplus_\Delta C)$.

Proof. For the left hand side to be defined it is obviously needed the condition $A \subset G^0$. Now, let $p \in (B \ominus_\Delta A) \ominus_\Delta C \Leftrightarrow p + c \in (B \ominus_\Delta A), \forall c \in C \Leftrightarrow p + c + a = p + (a + c) \in B, \forall\, a \in A, c \in C$. On the other side, using the notation $t = (a + c)$ it is equivalent to write $p + t \in B, \forall\, t \in (A \oplus_\Delta C) \Leftrightarrow p \in B \ominus_\Delta (A \oplus_\Delta C)$. This yields that $(B \ominus_\Delta A) \ominus_\Delta C = B \ominus_\Delta (A \oplus_\Delta C)$. □

Property E6Δ. If A, B are input images, C is structuring element, moreover $A \subset G^0$, then $B \oplus_\Delta (A \ominus_\Delta C) \subseteq (B \oplus_\Delta A) \ominus_\Delta C$.

Proof. It is clear that condition $A \subset G^0$ is necessary to have a well-defined right side. Now, let $p \in B \oplus_\Delta (A \ominus_\Delta C) \Rightarrow \exists b \in B, t \in (A \ominus_\Delta C)$ such that $p = b + t$. Since $t \in (A \ominus_\Delta C)$, we have $t + c \in A, \forall c \in C \Rightarrow (b + t + c) \in (B \oplus_\Delta A), \forall c \in C \Rightarrow p \in (B \oplus_\Delta A) \ominus_\Delta C$. □

Property E7Δ. The strict erosion is distributive over intersection of images. Formally, if $A, B \subset T$ and $C \subset G^0$, then $(A \cap B) \ominus_\Delta C = (A \ominus_\Delta C) \cap (B \ominus_\Delta C)$.

Proof. Let $p \in (A \cap B) \ominus_\Delta C \iff p + c \in (A \cap B), \forall c \in C$, now since $p + c \in (A \cap B) \iff p + c \in A$ and $p + c \in B, \forall c \in C \iff p \in (A \ominus_\Delta C)$ and $p \in (B \ominus_\Delta C) \iff p \in (A \ominus_\Delta C) \cap (B \ominus_\Delta C)$. □

Property E8Δ. If A, B are an input images and C is a structuring element, then $(A \cup B) \ominus_\Delta C \supseteq (A \ominus_\Delta C) \cup (B \ominus_\Delta C)$.

Proof. Let $p \in (A \ominus_\Delta C) \cup (B \ominus_\Delta C) \Rightarrow p \in (A \ominus_\Delta C)$ or $p \in (B \ominus_\Delta C) \Rightarrow p + c \in A, \forall c \in C$ or $p + c \in B, \forall c \in C \Rightarrow p + c \in (A \cup B), \forall c \in C \Rightarrow p \in (A \cup B) \ominus_\Delta C$. □

Property E9Δ. If A is an input image and C, D are structuring elements, then $A \ominus_\Delta (C \cup D) = (A \ominus_\Delta C) \cap (A \ominus_\Delta D)$.

Proof. Let $p \in A \ominus_\Delta (C \cup D) \iff p + q \in A, \forall q \in (C \cup D) \iff p + q \in A, \forall q \in C$ and $p + q \in A, \forall q \in D \iff p \in (A \ominus_\Delta C)$ and $p \in (A \ominus_\Delta D) \iff p \in (A \ominus_\Delta C) \cap (A \ominus_\Delta D)$. □

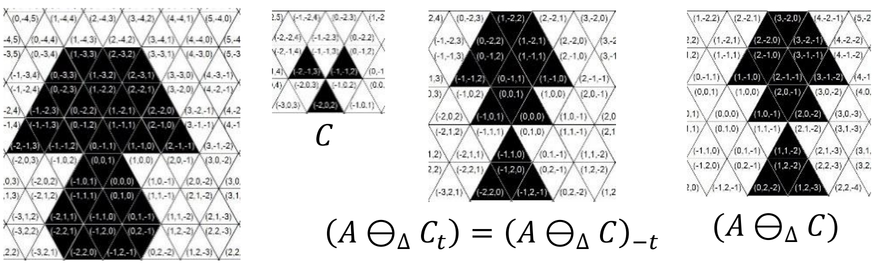
Property E10Δ. If A is an input image and C, D are structuring elements, then $A \ominus_\Delta (C \cap D) \supseteq (A \ominus_\Delta C) \cup (A \ominus_\Delta D)$.

Proof. Let $p \in (A \ominus_\Delta C) \cup (A \ominus_\Delta D)$, then $p \in (A \ominus_\Delta C)$ or $p \in (A \ominus_\Delta D)$. If $p \in (A \ominus_\Delta C)$, then $p + q \in A, \forall q \in C$; if $p \in (A \ominus_\Delta D)$, then $p + q \in A, \forall\, q \in D$. Thus, in both cases, $p + q \in A, \forall q \in (C \cap D)$ therefore, $p \in A \ominus_\Delta (C \cap D)$. □

Property E11Δ. If A is an input image, C is a structuring element and $t \in G^0$, then $A \ominus_\Delta (C)_t = (A \ominus_\Delta C)_{-t}$.

Proof. Let $p \in (A \ominus_\Delta C)_{-t} \Leftrightarrow p + t \in (A \ominus_\Delta C) \Leftrightarrow (p + t) + c \in A, \forall c \in C$. Since the vector addition is associative, it can be written as $p + (c + t) \in A, \forall c \in C$ and that holds if and only if $p \in A, \forall c \in (C)_t$, i.e., $p \in A \ominus_\Delta (C)_t$. □

Example 2.2.2.2. Let *A* be the image specified in Figure 9, and *C* be the structuring element given in the same figure, let $t = (2,0,-2)$. Then the obtained images $(A \ominus_\Delta C_t)$ and $(A \ominus_\Delta C)$ are also shown.



C

$(A \ominus_\Delta C_t) = (A \ominus_\Delta C)_{-t}$

$(A \ominus_\Delta C)$

Input binary image A

Figure 9: Shows the Property E11Δ

2.2.3 The Criteria and Adjunction between Strict Dilation and Erosion

(a) Strict dilation preserves the union operation this is very clear from the Property D10Δ.

(b) Strict erosion preserves the intersection operation it's already proven, (see Property E7Δ).

(c) There is adjunction between strict dilation and erosion see, the following theorem.

Theorem 2.2.3.1. Strict dilation and erosion form an adjunction relation by the following equation $A, C, B \subset T$, then $A \oplus_\Delta B \subseteq C \Leftrightarrow A \subseteq C \ominus_\Delta B$.

Proof. Let $A, C \subset T$ and let $B \subset G^0$ be structuring element we will show that $A \oplus_\Delta B \subseteq C$ if and only if $A \subseteq C \ominus_\Delta B$. Let assume that $A \oplus_\Delta B \subseteq C$ is hold and show that $A \subseteq C \ominus_\Delta B$. Let $a \in A$ and let $b \in B$, then $a + b \in A \oplus_\Delta B$. Since $A \oplus_\Delta B \subseteq C$ is hold, then $a + b \in C$ this true for any $b \in B$. By the Definition 2.2.2 yield that $a \in C \ominus_\Delta B$. For the other direction let $A \subseteq C \ominus_\Delta B$ hold and show that $A \oplus_\Delta B \subseteq C$. To do that let $p \in A \oplus_\Delta B$, then by Definition 2.2.1 there exist $a \in A$, $b \in B$ such that $p = a + b$, but since $A \subseteq C \ominus_\Delta B$ yield that $a \in C \ominus_\Delta B$, by the Definition 2.2.2 this mean $a + d \in C, \forall d \in B$ now for an element $b \in B$ yield that $p = a + b \in C$ furthermore $A \oplus_\Delta B \subseteq C$. □

2.3 Weak Dilation and Erosion on the Triangular Grid

In weak dilation and erosion it is allowed that structuring elements contain vectors with not zero-sum, however, in the result only pixels of the triangular grid are allowed. Also translations by any zero-sum or one-sum vectors are allowed. Basically, we work with trixels as with 3-dimensional vectors. By a translation of the triangular grid by an odd vector only the information of those pixels are kept in the result which are mapped to grid points. Thus, the result of some operations may not be trixels (or may contain some points outside of the triangular grid), and these points will not be considered as parts of the results of weak dilation and erosion. In this section, we give the formal definitions and properties.

Definition 2.3.1.(Weak Dilation)

Let $A, B \subset T$, then the *weak dilation* of A by the set B is defined by

$$A \oplus_w B = (A \oplus B) \cap T \tag{1.3.1}$$

where $A \oplus B = \bigcup_{p \in A} B_p = \bigcup_{p \in B} A_p$ as dilation of 3-dimensional vectors. In weak dilation we keep only those resulting vectors that belong to the grid T.

Example 2.3.1. Let $A = \{(2,-1,0),(1,-1,0),(1,0,0)\}$, $B = \{(-2,1,1),(-1,1,1),(-1,1,0)\}$. Then $A \oplus B = \{(0,1,1),(-1,1,1),(1,0,0),(0,0,1),(1,0,1),(-1,0,1),(0,0,0),(0,1,0)\}$. We have here points, e.g., $(0,1,1)$ such that the sum of its coordinate values equals to 2. These points are removed and do not appear in $A \oplus_w B$, thus the result is $A \oplus_w B = \{(-1,1,1),(1,0,0),(0,0,1),(-1,0,1),(0,0,0),(0,1,0)\}$.

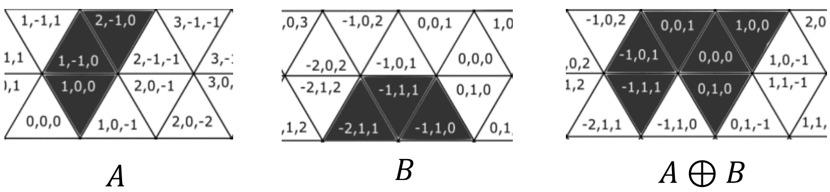


Figure 10: Example 2.3.1 show a weak dilation by resulting only points belong to T and deleting the other points that do not belong to T, i.e., (1,0,1) and (0,1,1)

Definition 2.3.2. (Weak Erosion)

Let $A, B \subset T$, then the *weak erosion* of the image A with structuring element B is defined by

$$A \ominus_w B = (A \ominus B) \cap T \qquad (1.3.2)$$

where $A \ominus B = \{p : p + b \in A, \forall b \in B\}$, i.e., the erosion defined on the cubic grid. We keep and display only points that belong to T and satisfy $p + b \in A, \forall\ b \in B$ and delete the other points of $A \ominus B$ that do not belong to T.

Remark 2.3.1. There could be vectors p not belonging to T, but satisfying $p + b \in A$. If the result of $A \ominus B$ contains vectors with the sum −1, then by weak erosion, these points are lost and there might be nothing to display (see Example 2.3.2).

Example 2.3.2. Let $A = \{(-3,1,2),(2,0,-2),(3,0,-3)\}$, $B = \{(1,0,0)\}$. Then, $A \ominus B = \{(-4,1,2),(1,0,-2),(2,0,-3)\}$ And $A \ominus_w B = \emptyset$.

Example 2.3.3. Let $A = \{(3,-1,-1),(2,-1,-1),(2,0,-1),(2,0,-2),(3,0,-2),(3,-1,-2)\}, B = \{(0,0,1)\}$. Then, $A \ominus B = \{(3\,,-1\,,-2),(2,-1,-2),(2\,,0\,,-2),(2,0\,,-3),(3\,,0\,,-3),(3,-1,-3)\}$ thus, $A \ominus_w B = \{(3\,,-1\,,-2),(2\,,0\,,-2),(3\,,0\,,-3)\}$. (The shaded elements of $A \ominus B$ have coordinate sum −1, they are excluded.)

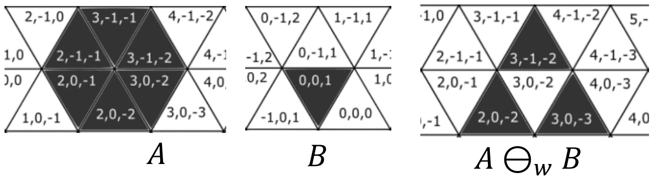


Figure 11: Example 2.3.3 is shown: a weak erosion

2.3.1 Properties of Weak Dilation

Notice that, since in weak dilation odd vectors are allowed in structuring element, the result of a weak dilation can be the empty set even none of the image and the structuring element is empty. This could happen e.g., if both of them contains only odd trixels/vectors. Now let us see which of the usual properties are satisfied by that type of definition. In the numbering of the properties the letter w is indicating weak operation.

Property D1w. Let $O = (0,0,0)$ be the origin of triangular grid, and A be input image where, $A \subset T$, then $A \oplus_w \{O\} = \{O\} \oplus_w A = A$.

Proof. From the Definition 2.3.1, Eq.(1.3.1), $A \oplus_w \{O\} = (A \oplus \{O\}) \cap T = (\{O\} \oplus A) \cap T = (A \cap T) = A$. □

Property D2w. If $p \in T$ and $A \subset T$ is an input image, then the following equalities hold:

(a) $A \oplus_w \{p\} = \{p\} \oplus_w A$,

(b) $A \oplus_w \{p\} = A_p$.

Proof. For the part (a) Let $t \in A \oplus_w \{p\} \Leftrightarrow t \in ((A \oplus \{p\}) \cap T) \Leftrightarrow$

$t \in ((\{p\} \oplus A) \cap T) \Leftrightarrow t \in \{p\} \oplus_w A$.

For the part (b) $A_p = (A \oplus p) \cap T = A \oplus_w \{p\}$. □

Remark 2.3.1.1. Actually, in case $p \in G^+$ and $A \subset G^+$, both sides of the previous two equations become the empty set $\emptyset$. This is a new property, previously, up to our knowledge, there were not defined any dilation that could give the empty set without having the empty set as an operand itself.

Property D3w. Let A be input binary image, D be a structuring element, i.e., $A, D \subset T$. If D contains the origin O, i.e., $O \in D$, then $A \subseteq A \oplus_w D$.

Proof. Let $p \in A \Rightarrow p \in (A \oplus \{O\}) \cap T \subseteq (A \oplus D) \cap T \Rightarrow p \in A \oplus_w D$ □

Property D4w. The weak dilation is commutative: Let A be an input binary image, C be a structuring element (i.e., $A, C \subset T$), then $A \oplus_w C = C \oplus_w A$.

Proof. Let $p \in A \oplus_w C \Leftrightarrow p \in (A \oplus C) \cap T$, from fact that dilation on the cubic grid is commutative, this is equivalent to $p \in (C \oplus A) \cap T \Leftrightarrow p \in C \oplus_w A$. □

Remark 2.3.1.2. Here, it should be noted that in case of $A, C \subset G^+$ the empty set is obtained.

Property D5w. The weak dilation is associative. Formally, let $A, B, C \subset T$, then $B \oplus_w (A \oplus_w C) = (B \oplus_w A) \oplus_w C$.

Proof. Let $p \in B \oplus_w (A \oplus_w C)$. By definition, Eq.(1.3.1), it is if and only if $p \in (B \oplus ((A \oplus C) \cap T) \cap T) \Leftrightarrow \exists a \in A, b \in B, c \in C$ such that $a + c \in T$ and $p = b + (a + c) \in T$ since vector addition is associative, $p = (b + a) + c \in T$, moreover since $a, b, c \in T$ this is possible only if $b + a \in T$, and thus, it is if and only if $p \in ((((B \oplus A) \cap T) \oplus C) \cap T) \Leftrightarrow p \in (B \oplus_w A) \oplus_w C)$. □

Remark 2.3.1.3. In the previous property, if at least two of A, B, C have only odd trixels, then both sides of the equation become $\emptyset$.

Property D6w. The weak dilation is translation invariant. Formally, let $A, C \subset T$, where A is an input image, C is a structuring element. If $p \in T$, then

(a) $(A)_p \oplus_w C = (A \oplus_w C)_p$.

(b) $A \oplus_w (C)_p = (A \oplus_w C)_p$.

Proof. (a) Let $t \in ((A)_p \oplus_w C) \Leftrightarrow t \in ((A)_p \oplus C) \cap T \Leftrightarrow \exists a \in A,\ c \in C$ such that $t = (a + p) + c \in T$. Since addition operation on vectors is associative, it is if and only if $t = (a + c) + p \Leftrightarrow t \in ((A \oplus C)_p) \cap T \Leftrightarrow t \in (A \oplus_w C)_p$.

The proof of part (b) is similar to the proof of part (a). □

Remark 2.3.1.4. In Property D6w, again, the empty set could be obtained if at least two of A, C and $\{p\}$ contain only odd elements.

Property D7w. Let A be an input image and C be a structuring element: $A, C \subset T$. If $p \in G^0$, then $A \oplus_w C = (A)_p \oplus_w (C)_{-p}$. (For $p \in G^+$ the vector $-p$ is not defined in this approach.)

Proof. Let $t \in (A \oplus_w C) \Leftrightarrow t \in (A \oplus_w C)_{p+(-p)}$. From Property D6w, we may write $t \in \left(A \oplus_w (C)_{-p}\right)_p \Leftrightarrow t \in ((A)_p \oplus_w (C)_{-p}) \Leftrightarrow t \in (A_p \oplus_w (C)_{-p})$. □

Again, we have that if $A, C \subset G^+$, then $A \oplus_w C$ is empty, and also the other side of the equality defines the empty set.

Property D8w. The weak dilation has the increasing property: Let $A, B, C \subset T$, where A, B are input images, C is a structuring element. If $A \subseteq B$, then $A \oplus_w C \subseteq B \oplus_w C$.

Proof. Let $p \in A \oplus_w C \Rightarrow p \in (A \oplus C) \cap T \Rightarrow p \in \left((A \oplus C) \cap T\right)$. Moreover, since $A \subseteq B$, $\left((A \oplus C) \cap T\right) \subseteq (B \oplus C) \cap T \Rightarrow p \in (B \oplus C) \cap T \Rightarrow$
$p \in (B \oplus_w C)$. □

In the previous property, with the condition $B, C \subset G^+$ both sides of the formula give the empty set.

Property D9w. Let A, B be input images and C be a structuring element ($A, B, C \subset T$). Then, $(A \cap B) \oplus_w C \subseteq (A \oplus_w C) \cap (B \oplus_w C)$.

Proof. Let $p \in (A \cap B) \oplus_w C \Rightarrow p \in \left((A \cap B) \oplus C\right) \cap T \Rightarrow \exists t \in A \cap B, c \in C$ such that $p = t + c \in T$. Now, since $t \in A \cap B \Longrightarrow t \in A$ and $t \in B$, $\Rightarrow p = t + c \in (A \oplus C) \cap T$ and $p \in (B \oplus C) \cap T$. Then, $p \in (A \oplus_w C) \cap (B \oplus_w C)$, and thus the statement $\Rightarrow (A \cap B) \oplus_w C \subseteq (A \oplus_w C) \cap (B \oplus_w C)$ is proven. □

Having the condition $A, B, C \subset G^{+}$ both sides of the formula result the empty set. The same observation applies also to the next property (D10w).

Property D10w. The weak dilation is distributive over union of images. Formally, let $A, B, C \subset T$. Then $(A \cup B) \oplus_w C = (A \oplus_w C) \cup (B \oplus_w C)$.

Proof. Let $p \in (A \cup B) \oplus_w C \Leftrightarrow p \in \big((A \cup B) \oplus C\big) \cap T \Leftrightarrow p \in T$ and $\exists t \in A \cup B,\ c \in C$ such that $p = t + c \in (A \oplus C)$ or $p = t + c \in (B \oplus C)$ that is if and only if $p \in (A \oplus_w C) \cup (B \oplus_w C)$. □

Property D11w. The weak dilation is distributive over a union of structural elements: Let A be an input images and C, D be structuring elements ($A, C, D \subset T$), then $A \oplus_w (C \cup D) = (A \oplus_w C) \cup (A \oplus_w D)$.

Proof. The statement can be proven in a similar manner to the previous property. □

Property D12w. Let A be an input image, C, D be structuring elements ($A, C, D \subset T$). Then $A \oplus_w (C \cap D) \subseteq (A \oplus_w C) \cap (A \oplus_w D)$.

Proof. Let $p \in A \oplus_w (C \cap D) \Rightarrow p \in (A \oplus (C \cap D)) \cap T \Rightarrow \exists a \in A, t \in (C \cap D)$ such that $p = a + t \in T \Rightarrow p \in (A \oplus C) \cap T$ and also $p \in (A \oplus D) \cap T$, thus p is in $(A \oplus_w C) \cap (A \oplus_w D)$, and therefore $A \oplus_w (C \cap D) \subseteq (A \oplus_w C) \cap (A \oplus_w D)$. □

In the last two properties, if $A, C, D \subset G^{+}$, then both sides of the formulae are the empty set.

2.3.2 Properties of Weak Erosion

We should notice that in weak erosion we keep and display only points p which belong to T and satisfy $p + b \in A,$ for $\forall\ b \in B$ and we delete the other points of $A \ominus B$ that do not belong to T. By definition, these latter points are removed automatically, but it is important to know about them, thus we will mention them in the following properties.

Property E1w. Let A be an input image and C be a structuring element ($A, C \subset T$) such that $O \in C$, where $O = (0,0,0)$, then $A \ominus_w C \subseteq A$.

Proof. From the Definition 2.3.2, Eq.(1.3.2), let $p \in A \ominus_w C \Rightarrow p \in (A \ominus C) \cap T \Rightarrow \exists\ p \in T$ such that $p + b \in A, \forall b \in C$. Now, since $O \in C$, we can write $p + O \in A$. This implies that $p + O = p \in A \Rightarrow A \ominus_w C \subseteq A$. □

Property E2w. The weak erosion is not commutative. Formally, let $A, C \subset T$ (A be an input image and C be a structuring element), then, generally, $A \ominus_w C \neq C \ominus_w A$.

Proof. It is obvious by the definition of weak erosion, one can find easily examples of A and C fulfilling the inequality. □

Although the weak erosion is generally not commutative, one may find special examples where $A \ominus_w C = C \ominus_w A$, e.g., in case of $C = A$.

The next property has also some interesting details using weak erosion.

Property E3w.

(a) The weak erosion is translation invariant for translations with even vectors. Formally: Let $A, C \subset T$ where A is an input image and C is a structuring element. If $t \in G^0$, then $(A)_t \ominus_w C = (A \ominus_w C)_t$.

(b) The weak erosion is generally not translation invariant for translations with odd vectors, i.e., if $t \in G^+$, then it may occur that $(A)_t \ominus_w C \neq (A \ominus_w C)_t$.

Proof. The proof also goes by cases according to the parity, i.e., the sum of coordinates of points. For part (a) Let $t \in G^0$ and let $p \in (A)_t \ominus_w C \Leftrightarrow p \in (A_t \ominus C) \cap T \Leftrightarrow p + c \in A_t, \forall\, c \in C$, then, either $p \in G^0$ and then, the above condition is satisfied if and only if $(p - t) + c \in A, \forall\, c \in C \Leftrightarrow (p - t) \in (A \ominus C) \cap T \Leftrightarrow p \in ((A \ominus C)_t \cap T) \Leftrightarrow p \in (A \ominus_w C)_t$; or $p \in G^+$ and $C \subset G^0$ (since $p + c \in A_t \subset G, \forall\, c \in C$ must hold), in this case $p + c \in A_t, \forall\, c \in C \Leftrightarrow p + c - t \in A, \forall\, c \in C$ that could be if and only if $p - t \in (A \ominus C)$, and also it is known that $p - t \in T$ and thus, $\Leftrightarrow p - t \in (A \ominus_w C) \Leftrightarrow p \in (A \ominus_w C)_t$. Case a, the statement with even *t* is has been proven. For the part (b) Let $t = (x, y, z) \in G^+$, then we show an example for which the inequality holds. Let A = {(0,0,0)} and C = {(*x*,*y*,*z*)}, then, obviously, $(A)_t = \{(x, y, z)\}$ and, thus, by definition, $(A)_t \ominus_w C$ = {(0,0,0)}. On the other side, $(A \ominus_w C) = \emptyset$, since $(A \ominus C) = \{(-x, -y, -z)\}$, but this point is not in *T*, and it also implies $(A \ominus_w C)_t = \emptyset$. Thus the inequality holds in this case. However, also in this case, for any trixel $p \in ((A)_t \ominus_w C) \Leftrightarrow p \in (A_t \ominus C) \cap T \Leftrightarrow p + c \in A_t, \forall\, c \in C$. Moreover, if $p \in G^+$, then the above formula is equivalent to $(p - t) + c \in A, \forall\, c \in C \Leftrightarrow (p - t) \in (A \ominus C) \cap T \Leftrightarrow p \in (A \ominus C)_t \cap T \Leftrightarrow p \in (A \ominus_w C)_t$; the transitivity holds. In the opposite case, when $p \in G^0$, this may not hold as we have already shown. (In this case, $p + c \in A_t, \forall\, c \in$

C, and then, since A_t can contain only odd trixels and $C \subset T$, the condition $C \subset G^+$ must hold. But the equality may not be fulfilled.) □

Property E4w. The weak erosion has the increasing property: Let $A, B, C \subset T$. If $A \subseteq B$, then $A \ominus_w C \subseteq B \ominus_w C$.

Proof. Let $p \in A \ominus_w C \Rightarrow p \in (A \ominus C) \cap T \Rightarrow p + c \in A, \forall c \in C$ but, since $A \subseteq B \Rightarrow p + c \in B$ also holds for $\forall c \in C \Rightarrow p \in (B \ominus C) \cap T \Rightarrow p \in (B \ominus_w C)$. □

The next two analyzed properties of erosion (combined by dilation) do, generally, not hold in the weak approach, as we prove them below.

Property E5w. Let $A, B, C \subset T$. Then, $(B \ominus_w A) \ominus_w C \neq B \ominus_w (A \oplus_w C)$ could hold.

Proof. We show an example such that the two sides of the formula results different sets. Let $A = \{(1,0,0)\}$, $B = \{(0,0,0)\}$, $C = \{(0,1,0)\}$, then one can easily compute that $(B \ominus_w A) = \emptyset$, and, therefore, $(B \ominus_w A) \ominus_w C = \emptyset$. On the other side, we have $(A \oplus_w C) = \emptyset$, and, thus, $B \ominus_w (A \oplus_w C) = T$. □

Property E6w. If $A, B, C \subset T$, then $B \oplus_w (A \ominus_w C) \supset (B \oplus_w A) \ominus_w C$ may hold.

Proof. The proof goes by an example. Let $A = \{(-1,0,2), (-2,0,2)\}$, $C = \{(0,0,0), (1,0,0)\}$, $B = \{(0,1,0)\}$. Then, on the left side, $(A \ominus_w C) = \{(-2,0,2)\}$ and $B \oplus_w (A \ominus_w C) = \{(-2,1,2)\}$. On the right side, $(B \oplus_w A) = \{(-2,1,2)\}$, and

$(B \oplus_w A) \ominus_w C = \emptyset$, thus we have $B \oplus_w (A \ominus_w C) \nsubseteq (B \oplus_w A) \ominus_w C$ proving that the original Property E6 does not hold in case of weak erosion and dilation. □

Property E7w. The weak erosion is distributive over the intersection of images: Let A, B be an input image and C be a structuring element, i.e., $A, B, C \subset T$. Then, $(A \cap B) \ominus_w C = (A \ominus_w C) \cap (B \ominus_w C)$.

Proof. Let $p \in (A \cap B) \ominus_w C \Leftrightarrow p \in ((A \cap B) \ominus C) \cap T \Leftrightarrow \forall c \in C$: $p + c \in (A \cap B) \cap T \Leftrightarrow \forall c \in C: p + c \in A$ and $p + c \in B \Leftrightarrow p \in (A \ominus C) \cap T$ and $p \in (B \ominus C) \cap T \Leftrightarrow p \in (A \ominus_w C) \cap (B \ominus_w C)$. □

Property E8w. Let $A, B, C \subset T$. Then, $(A \cup B) \ominus_w C \supseteq (A \ominus_w C) \cup (B \ominus_w C)$.

Proof. Let $t \in (A \ominus_w C) \cup (B \ominus_w C) \Rightarrow t \in \Big(\big((A \ominus C) \cap T\big) \cup \big((B \ominus C) \cap T\big)\Big)$. That is, $t \in (A \ominus C) \cap T$ or $t \in (B \ominus C) \cap T \Rightarrow t \in T, t + c \in A, \forall\, c \in C$ or $t \in T, t + c \in B, \forall\, c \in C$. Since $A, B \subset T$, we can write $t + c \in (A \cup B) \cap T,\ \forall\, c \in C \Rightarrow t \in ((A \cup B) \ominus C) \cap T \Rightarrow t \in (A \cup B) \ominus_w C$. □

Property E9w. Let A be an input image and $C, D \subset T$ be structuring elements. Then, $A \ominus_w (C \cup D) = (A \ominus_w C) \cap (A \ominus_w D)$.

Proof. Let $p \in A \ominus_w (C \cup D) \Leftrightarrow p \in \big(A \ominus (C \cup D)\big) \cap T \Leftrightarrow$ $p \in T, p + q \in A, \forall q \in (C \cup D) \Leftrightarrow p + q \in A, \forall\, q \in C$ and $\forall q \in D \Leftrightarrow p + q \in A, \forall\, q \in C$ and $p + q \in A, \forall\, q \in D \Leftrightarrow p \in (A \ominus C) \cap T$ and $p \in (A \ominus D) \cap T \Leftrightarrow$ $p \in (A \ominus_w C) \cap (A \ominus_w D)$. □

Property E10w. Let A be an input image, C, D be structuring elements. Then, $A \ominus_w (C \cap D) \supseteq (A \ominus_w C) \cup (A \ominus_w D)$.

Proof. Let $p \in (A \ominus_w C) \cup (A \ominus_w D)$, then $p \in (A \ominus_w C)$ or $p \in (A \ominus_w D)$. If $p \in (A \ominus_w C) \Rightarrow p \in (A \ominus C) \cap T \Rightarrow p + q \in A, \forall q \in C$. If $p \in (A \ominus_w D)$, then $p \in (A \ominus D) \cap T \Rightarrow p + q \in A, \forall q \in D$. Thus, in both cases, $p + q \in A, \forall q \in (C \cap D) \Rightarrow p \in A \ominus (C \cap D) \cap T \Rightarrow p \in A \ominus_w (C \cap D)$. □

Property E11w. Let A be an input image, C is structuring element, $(A, C \subset T)$ and let $t \in G^0$. Then, $A \ominus_w (C)_t = (A \ominus_w C)_{-t}$.

Proof. For translations by even vectors the proof is similar to the proof of part a. of Property E3w. It should be noted that the right side of the equation is not defined if *t* is an odd vector. □

2.3.3 About Adjunction Relation between the weak Dilation and Erosion

(a) Weak dilation preserves the union operation it's clear from the Property D10w.

(b) Weak erosion preserves the intersection operation it's already proven see Property E7w.

(c) There is no adjunction relation between weak dilation and erosion for an evident see the next example.

Example 2.3.3.1. Let $A = \{(-1,0,1), (0,-1,1), (0,-1,2), (0,0,1), (1,-1,1)\}$ and $B = \{(0,-1,2), (0,0,1), (1,-1,1)\}$, also let $C = A \oplus_w B$, now $C \ominus_w B = (A \oplus_w B) \ominus_w B = \{(-1,0,1), (0,-1,1)\}$, but $A \nsubseteq (A \oplus_w B) \ominus_w B = C \ominus_w B$ this observe that for $A \oplus_w B \subseteq C \nRightarrow A \subseteq C \ominus_w B$.

2.4 Strong Dilation and Erosion on the Triangular Grid

As we have seen, in the strict approach we had some restrictions on some sets and points. In the weak case, without this restriction some of the usual properties of the operations do not hold, since we may get some points outside of the grid, and we have lost them. In this section, we use an approach that is more general than the previous ones, we keep and work with points outside *T*, however, when a set should be displayed, only the part inside *T* is shown. Not to lose some of the information, as in the case of weak dilation and erosion, one can use the strong dilation and erosion.

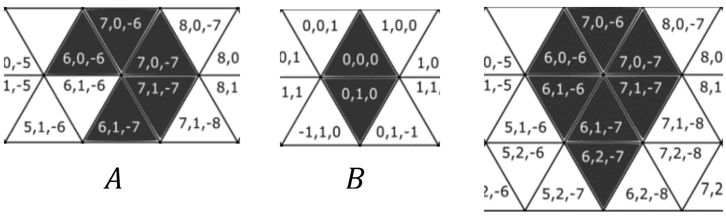
Definition 2.4.1. (**Strong Dilation)**

Let $A, B \subset \mathbb{Z}^3$, then the *strong dilation* of *A* by the set *B* is defined by

$$A \oplus_s B = A \oplus B \tag{1.4.1}$$

keeping also the resulted points outside of *T*, but displaying only points which belong to grid *T*: $p \in (A \oplus B) \cap T$.

Example 2.4.1. Let $B = \{(0,0,0)(0,1,0)\}$, $A = \{(6,0,-6),(7,0,-6),(7,0,-7),$ $(7,1,-7),(6,1,-7)\}$ Then, $A \oplus_s B = \{(6,0,-6),(7,0,-6),(7,0,-7),(7,1,-7),$ $(6,1,-7),(6,1,-6),(7,1,-6),(7,2,-7),(6,2,-7)\}$. (See Figure 12)



A *B* $A \oplus_s B$: plus keeping {(7,1, -6), (7, 2, -7)}

Figure 12: An example of strong dilation by displaying resulted points $p \in T$ and keep the other points $p \notin T$ in the result of $(A \oplus_s B)$

Definition 2.4.2. (**Strong Erosion)**

Let $A, B \subset \mathbb{Z}^3$ then, the *strong erosion* is defined by

$$A \ominus_s B = \{p \in \mathbb{Z}^3 \colon p + b \in A, \forall b \in B\} \qquad (1.4.2)$$

However, working on T one can display only the part $\{p \in T \colon p + b \in A, \forall b \in B\}$. The strong dilation and erosion can be written in a way that underlines which part is displayable and which part is not. For instance, the strong erosion is written as

$$A \ominus_s B = \{p \in T \colon p + b \in A, \forall b \in B\} \cup \{p \notin T \colon p + b \in A, \forall b \in B\} \qquad (1.4.3)$$

by having the pixels that can be displayed (points which belong to T and satisfy $p + b \in A$ for every $b \in B$) and keeping also all the other points which do not belong to T, but satisfy $p + b \in A$ for every $b \in B$.

Remark 2.4.1. Notice that in strong erosion, similarly to the strong dilation, it is not required that *A* and *B* are in *T*, instead all integer triplets are allowed, but only those triplets can be displayed that are element of *T*.

Example 2.4.2. In Example 2.3.3 we have seen that $A \ominus B = \{(3\,,-1\,,-2),$ $(2,-1,-2), (2\,,0\,,-2), (\,2, 0\,,-3), (3\,,0\,,-3), (3,-1,-3)\}$. Here we keep vectors outside of the grid, e.g., with sum of coordinate values equal to –1: (2, 0,-3), (3,-1,-3), (2,-1,-2), but we can display only those resulting points which are in the grid ($p \in T$).

2.4.1 Properties of Strong Dilation and Erosion

First we must see that, by their definition, the strong dilation and the strong erosion are entirely the same as the dilation and erosion on the cubic grid, therefore all the well-known properties listed in Section 1.4 are fulfilled by them, respectively. However, as we can display the result on the triangular grid, we may have some interesting phenomena in which seemingly the strong dilation and erosion differ from their original operations on the cubic grid. Especially, it may happen that, even we have trixels in each sets of the expression, maybe there is nothing to display in the result, although it is not empty. In this section we analyze these operations from this point of view. Letter s is used to refer to strong operations at the names of the properties. In the following we assume that the input images and the structuring elements contain some trixels (maybe also some points outside *T*): let, *A*,*B*,*C*,*D* be point sets of the cubic grid ($\mathbb{Z}^3$) such that $A \cap T \neq \emptyset$, $B \cap T \neq \emptyset$, $C \cap T \neq \emptyset$ and $D \cap T \neq \emptyset$.

In Property D2s, if $p \in G^+$ and $A \subset G^+$, then nothing to display in the expressions of the property (both sides).

In Property D4s, for instance in case of $A, C \subset G^+$ the results are completely outside of T, thus they do not have displayable parts.

Further, in Properties D5s, D8s, D9s, D10s in cases of $A, B, C \subset G^+$ the results do not contain any points of T, thus there is nothing to display. In Property D6s, with the condition $A, C \subset G^+$ and $p \in G^+$ the results are completely outside of T. In Property D7s, $A, C \subset G^+$ implies no displayable result. In Properties D11s and D12s, if each of $A, C, D \subset G^+$, then the results cannot be displayed on T.

Now, let us see the properties of the erosion with some examples. There is nothing to display from the result in Properties E3s and E11s, if, for instance, $A \subset G^0, C \subset G^+$ and $t \in G^0$. At Properties E4s, E5s, E6s, E7s and E8s, $A, B \subset G^0, C \subset G^+$ implies that there is no trixel in the result of the corresponding expressions, consequently, there is nothing to display. Finally, at Properties E9s and E10s in cases $A \subset G^0, C, D \subset G^+$, the resulted expressions do not have part that is displayable.

2.4.2 The Criteria and Adjunction between Strong Dilation and Erosion

Strong dilation and erosion satisfy the criteria and adjunction relation on $\mathbb{Z}^3$. Due to strong case inherit this property from cubic grid.

2.4.3 A Note on Usable Vectors Outside of the Grid

In case of strong dilation and erosion, we may use vectors both for the image and the structuring element which are outside of the grid, e.g., have coordinate sum -1.

Example 2.4.3.1. Let $A = \{(1,2,-2), (0,2,-2), (1,2,-3), (0,3,-2), (0,3,-3), (1,3,-3)\}$ and let $B = \{(0,0,-1), (0,-1,0)\}$, then $A \oplus_s B = \{(0,3,-3), (1,1,-2), (0,2,-2), (0,2,-3), (1,2,-4), (1,1,-3), (1,2,-3), (0,3,-4), (0,1,-2), (1,3,-4)\}$. The strong erosion $A \ominus_s B = \{(1,3,-2), (0,3,-2)\}$.

The result of the strong dilation and erosion may contain some vectors outside the grid, thus we may keep this information for using in some next operations, e.g., in a dilation of an odd point with this vector yield to an even point of the grid. Thus in strong operations it is allowed for both the image and the structuring element to contain vectors outside of the grid. However, only those vectors can be displayed as pixels that have sum 0 or 1. In the next subsection we show a special case of applications of vectors outside the grid *T*.

2.4.4 Traditional Neighborhood and Summary Tables

In this subsection, we briefly analyze how we can use the traditional neighborhood structure in morphological operations. It is well-known that in the most cases, at applications the neighborhood of the image is used for these operations, i.e., on the square grid, the structuring element contains the origin with its cityblock or chessboard neighbors. Based on the restricted definitions not allowing to generate or obtain any vectors outside of the triangular grid T, the structuring element can contain only even vectors. Observing the possible neighborhood relations, this implies that only strict 2-neighborhood can be applied for structural element, that is, the structural element $C_2 = \{(0,0,0),(1,-1,0),(-1,1,0),(1,0,-1),(-1,0,1),(0,1,-1),(0,-1,1)\}$ is used including the origin itself. We note that these vectors as neighbors correspond exactly the neighborhood in the hexagonal grid.

In the weak case, since we can use only vectors with coordinate sum zero or one (after each operation we lose the resulted vectors not having this property), we can use only strict 2-neighborhood again (set C_2). For odd points to define 1-neighbors or strict 3-neighbors one need vectors with coordinate sum -1. And, actually, this is the point why the strong case could be the most useful. For 1-neighborhood, as structuring element, one can use the set $C_1 = \{(0,0,0),(1,0,0),(-1,0,0),(0,1,0),(0,-1,0),(0,0,1), (0,0,-1)\}$. Also strict 2-neighborhood, C_2 is allowed, however we may use all the 2-neighbors including the 1-neighbors by the structuring element $C_1 \cup C_2$. The strong definitions also allow to use strict 3-neighborhood $C_3 = \{(0,0,0),(1,1,-1),(1,-1,1),(1,-1,-1),(-1,1,1),(-1,1,-1),(-1,-1,1)\}$; and 3-neighborhood $C_1 \cup C_2 \cup C_3$ as structuring element.

Tables 1 and 2 summarize which properties of the operations are inherited to the triangular grid using various definitions of the dilations and erosions, respectively. For the numbering we refer to Section 1.4 where these properties are listed. Let A and B be images, C and D be structuring elements based on various definitions of dilation and erosion on the triangular grid. Let p and t be three-dimensional vectors. In some cases, to have well defined expressions, conditions are applied. As we have seen, some of the properties hold only with some additional conditions.

Table 1: Summary of dilation properties according to their definitions

Properties	1D	2D	3D	4D	5D	6D	7D	8D	9D	10D	11D	12D
strict	a.√ b. Δ	a.Δ b.p^{Δ}	√	Δ	Δ	p^{Δ}	p^{Δ}	√	√	√	√	√
weak	√	+	√	+	+	+	p^{Δ},+	+	+	+	+	+
strong	√	√$^{+}$	√	√$^{+}$	√$^{+}$	√$^{+}$	√$^{+}$	√$^{+}$	√$^{+}$	√$^{+}$	√$^{+}$	√$^{+}$

- √ the property holds on the triangular grid without any additional conditions.
- Δ the strict dilation is defined only with the condition $A \subset G^0$. With this condition all the expressions are defined and the property holds.
- p^{Δ} the translation is defined with the condition that p is even pixel. With the condition $p \in G^0$ the property holds.
- + the property is fulfilled, but the result could be the empty set even none of the operands are the empty set.
- √$^{+}$ the property holds without any conditions, but maybe nothing to display even the operands contain trixels.

Table 2: Summary of erosion properties according to their definitions

Properties	1E	2E	3E	4E	5E	6E	7E	8E	9E	10E	11E
strict	√	Δ	t^{Δ}	√	Δ	Δ	√	√	√	√	t^{Δ}
weak	√	√	t^{Δ}	+	-	-	+	+	+	+	t^{Δ}
strong	√	√	√+	√+	√+	√+	√+	√+	√+	√+	√+

- √ the property holds on the triangular grid without any additional conditions.
- Δ the expressions are defined only with the condition $A \subset G^0$. With this condition the property holds.
- t^{Δ} if t is a vector with zero sum, then the property holds. (In the strict approach translations are defined only under this condition, in weak approach translations by –*t* are defined only under this condition.)
- + the property is fulfilled, but the result could be the empty set.
- - the property does not hold in general.
- √+ the property holds without any conditions, but maybe nothing to display although the operands contain trixels.

2.4.5 Application Using Different Type of Neighbors Structuring

In Figure 13 (a), an image of a bone implant is shown. This implant was used to insert in a leg of a rabbit. Similar picture was also used in (Benedek Nagy & Lukić, 2016) for a binary tomography problem. Here, strict dilation and erosion of this figure are obtained by the structural element C_2 on the triangular grid as they are shown in Figure 13 (b) and (c), respectively. Those parts of the strong dilations by the 1-, 2-and 3-neighborhood (i.e., by the sets $C_1, C_1 \cup C_2$ and $C_1 \cup C_2 \cup C_3$) that belong to the grid T are shown in Figure 13 (d), (f) and (h), respectively. Similarly, strong erosions obtained by the same sets of structuring elements are shown in Figure 13 (e), (g) and (i), respectively.

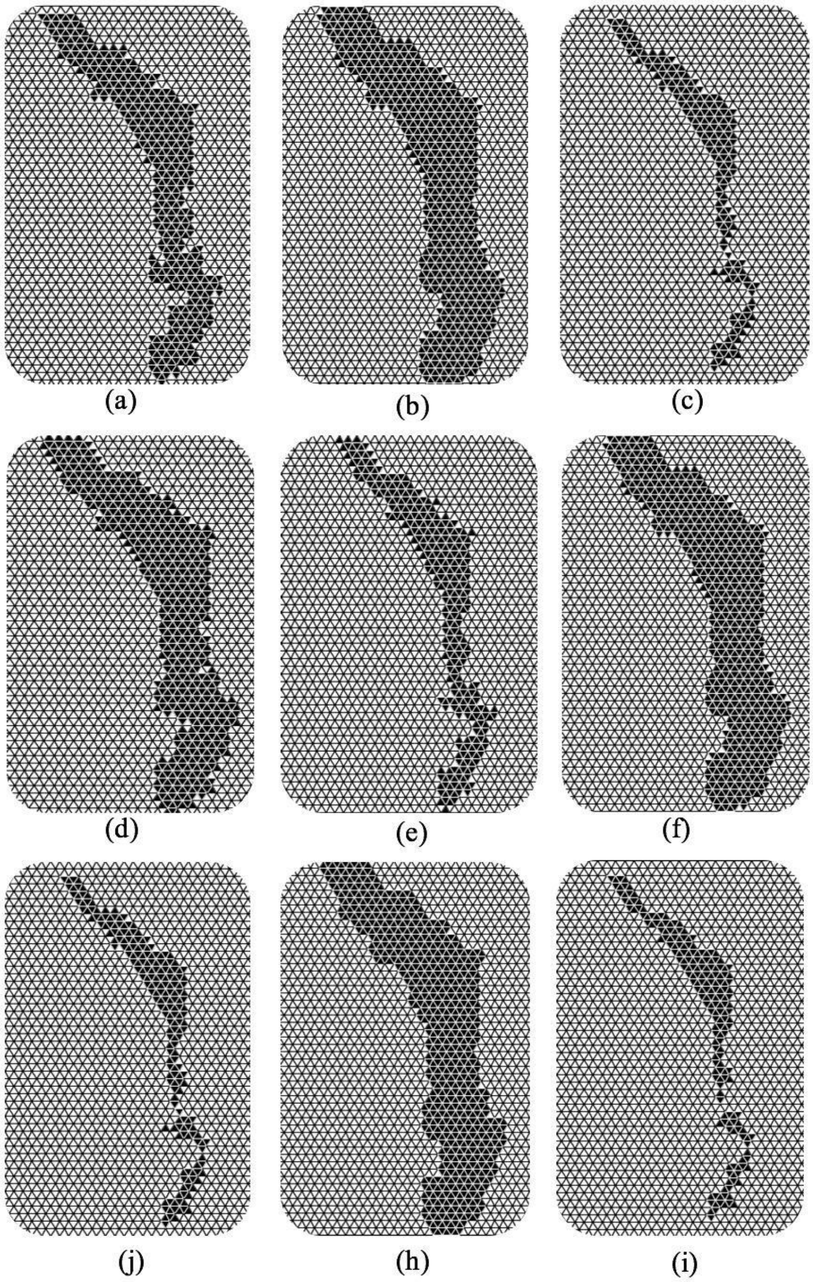


Figure 13: (a) resampled image of a bone implant inserted in a leg of a rabbit, (b)-strict dilation by structuring element C_2, (c) strict erosion by C_2. (d)-(i) the displayable result of strong dilations and erosions, by C_1 (d) and (e), by $C_1 \cup C_2$ (f) and (j), and by $C_1 \cup C_2 \cup C_3$ (h) and (i), respectively

2.5 The “Independent” Approach for Morphology

If we add the coordinate triplets of two odd pixels, the sum of the result is 2 and it is not located within the grid; the triangular grid is non-closed under the addition. The only translations that map the grid to itself are the ones with vectors with sum zero (even vectors, (B. Nagy, 2009)). We should notice that the translation issue exists in the traditional grid (i.e., square grid) in some particular location when translating an image by vector reach the edge of the grid windows or the boundary. In this case, we need somehow to define either translate operation or give a proper structuring element. In these cases, there are two offered solutions, one of them is called Spatially-Variant, and the other is Adaptive Morphology (Maragos, 2013). Similarly, in the sense of these problems, the triangular grid is not a point lattice, even and odd points/vectors have different behavior that’s not “a discrete subgroup of Euclidean space”. Therefore, we give previously three solutions for defining morphological operations on triangular grid (see (Abdalla & Nagy, 2017)). Moreover, now we introduce a novel method to solve the previous issue, the “independent dilation and erosion.” This method aims to define the operations such that they always result in images of the grid (see also (Abdalla & Nagy, 2018)). In this method, we separate the even and odd pixels of the image, and we work independently with these sets. We alter the concept of structuring element (SE), in this approach, it contains two sets of vectors, one for working with the even, and one for the odd pixels (also allowing vectors with sum –1 in the latter one). Figure 14 is illustrating the idea.

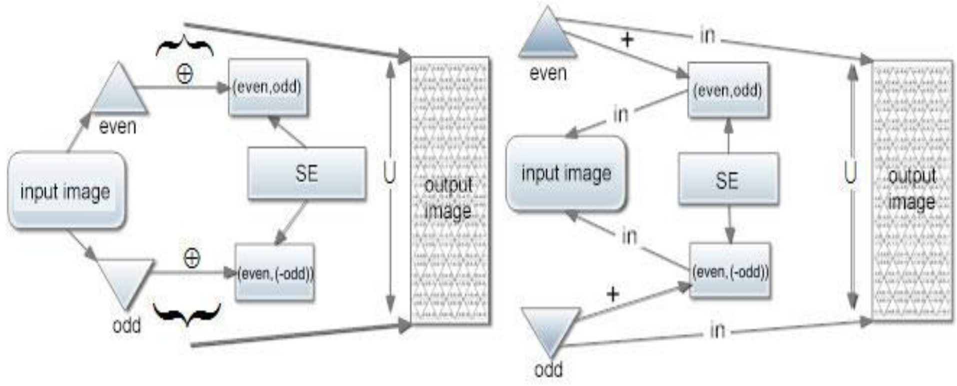


Figure 14: The idea of independent dilation (left) and erosion (right)

We deal with structuring element (SE) that contains two parts. The first part, it is dented by B_e, might contain even (B_e^0) and odd (B_e^+) vectors. The other, B_o, might contain even vectors and vectors with sum -1 (we also use the name –odd for them). Although B_e and B_o is given independently, they may or may not contain the same (even) vectors. We write $B = (B_e, B_o)$ where $B_e = B_e^0 \cup B_e^+$ and $B_o = B_o^0 \cup B_o^-$.

Definition 2.5.1. Let B, C be structuring elements where, $B = (B_e, B_o)$, $C = (C_e, C_o)$. We will define the union of B, C as new structuring element, $D = B \cup C$ where, $D = (D_e, D_o) = (B_e \cup C_e, B_o \cup C_o)$, where $D_e = B_e \cup C_e$ and $D_o = B_o \cup C_o$. We can also define the intersection of B, C written as $D = B \cap C$, where $D = (D_e, D_o) = (B_e \cap C_e, B_o \cap C_o)$, with $D_e = B_e \cap C_e$ and $D_o = B_o \cap C_o$.

Remark 2.5.1. Due to the fact that exactly the even vectors translate (map) the grid to itself (see (B. Nagy, 2009)) we can use these vectors for translations. Images are translated pixelwise (similarly to images on the square grid, see Definition 1.3.14), i.e., for an image A and a vector $t \in G^0$ the translated image is $(A)_t = \{p + t : p \in$

$A\}$. Further, we can also define translations by even vectors on the structuring elements.

Definition 2.5.2. Let $B = (B_e,\ B_o)$ be a structuring element and let $t \in G^0$ be an even vector. The translation of the SE B by t is $(B)_t = ((B_e)_t, (B_O)_t) = (\{b + t: b \in B_e\}, \{b + t: b \in B_o\})$.One can easily show by Definition 1.3.15 that the translation can also be expressed by the dilation, as follows:

Proposition 2.5. 1. Let $A \subset T$ be an input binary image, and $B = (B_e,\ B_o)$ be a structuring element and let $t \in G^0$ be an even vector. Then,

(a) $(A)_t = (A^0 \oplus \{t\}) \cup (A^+ \oplus \{t\})$, where $A = A^0 \cup A^+, A^0 \subset G^0, A^+ \subset G^+$.

(b) translation of SE B by vector t is defined by translate both parts of B by the same vector:$(B)_t = ((B_e)_t, (B_o)_t)$, moreover, $(B_e)_t = \left((B_e^0 \oplus \{t\}) \cup (B_e^+ \oplus \{t\})\right)$, $(B_o)_t = \left((B_o^0 \oplus \{t\}) \cup (B_o^- \oplus \{t\})\right)$.

Notice that for each even vector *t*, its inverse, – *t* is also an even vector, and, thus translations by – *t* are also defined.

We will start with the definition of independent dilation and let use the following notion, let *A* be an active binary image (input image) which consists of two parts: it's even pixels A^0 and it's odd pixels A^+.

Definition 2.5.3. (Independent Dilation)

Let $A \subset\ T$ be an input image. Let $A^0 \subseteq G^0, A^+ \subseteq G^+$ be two sets of points such that $A = A^0 \cup A^+$. Further, let B be a structural element (SE) including two sets: $B = (B_e, B_o)$, where $B_e = B_e^0 \cup B_e^+$ with $B_e^0 \subseteq G^0$, $B_e^+ \subseteq G^+$, and $B_o = B_o^0 \cup B_o^-$

with $B_o^0 \subseteq G^0$, $B_o^- \subseteq G^-$. Then the *independent dilation* of A by SE B is defined as follows

$$A \oplus_i B = (A^0 \oplus B_e) \cup (A^+ \oplus B_o). \qquad (1.4.4)$$

Remark 2.5.2. Notice that the dilation of 3D vectors is used on the right hand side (i.e., the concept of dilation on the cubic grid), and the index (i) denotes that the "independent" operation is used on the left side. We can write Eq.(1.4.4) In the following form:

$$A \oplus_i B = \Big(\big((A^0 \oplus B_e^0) \cup (A^+ \oplus B_o^-)\big) \cup \big((A^0 \oplus B_e^+) \cup (A^+ \oplus B_o^0)\big)\Big) \qquad (1.4.5)$$

One can easily check that $A \oplus_i B$ is an image of the triangular grid, i.e., $A \oplus_i B \subseteq T$. We can write Eq.(2.5.2) for the simplicity as the following:

$$A \oplus_i B = (V^0 \cup V^+) \qquad (1.4.6)$$

Where $V^0 = ((A^0 \oplus B_e^0) \cup (A^+ \oplus B_o^-))$, $V^+ = ((A^0 \oplus B_e^+) \cup (A^+ \oplus B_o^0))$.

Example 2.5.1. Let $A = \{(-1,3,-1), (-2,3,-1), (-1,3,-2), (-2,4,-1), (-2,4,-2)\}$, let $B_e = \{(-2,1,1), (-1,1,1)\}$, and $B_o = \{(-2,1,1), (1,-3,1)\}$. Then, $A^0 \oplus B_e = \{(-4,4,0), (-3,5,-1), (-3,4,0), (-4,5,-1), (-3,4,-1), (-2,4,-1)\}$, $A^+ \oplus B_o = \{(-3,4,0), (-1,1,0), (-4,5,0), (0,0,0)\}$, and thus, $A \oplus_i B = \{(-1,1,0), (-3,5,-1), (-4,4,0), (0,0,0), (-3,4,0), (-4,5,0), (-4,5,-1), (-3,4,-1), (-2,4,-1)\}$.

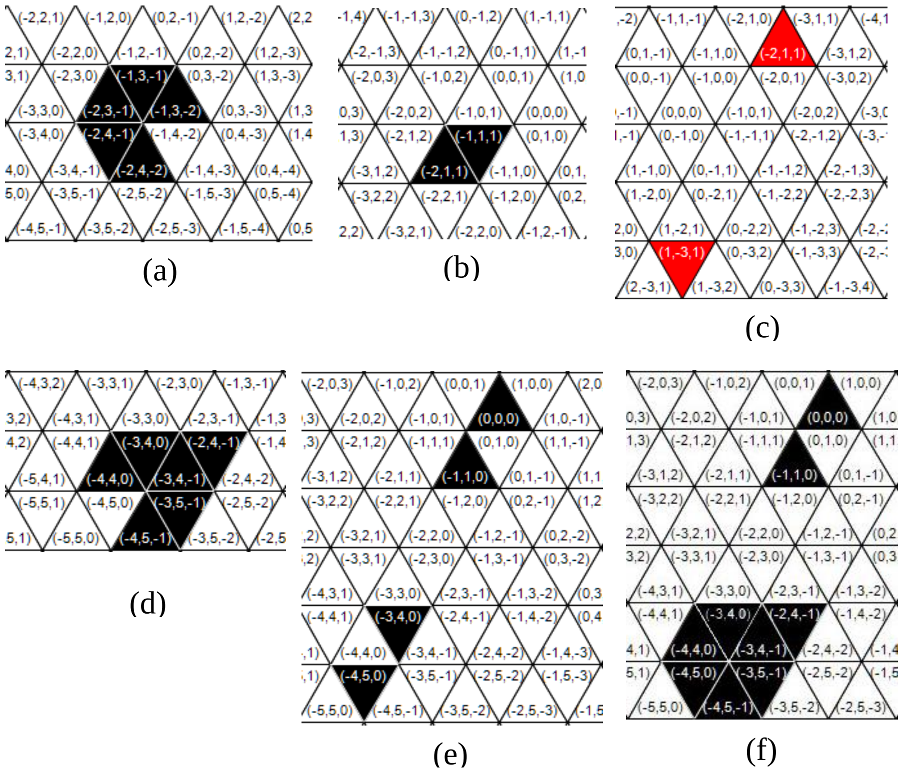


Figure 15: Example 2.5.1 illustrates the independent dilation. (a) input image and (b) the first part of SE(B_e), (c) the second part of SE(B_o), (d) the result of dilation $A^0 \oplus B_e$, (e) the result of $A^+ \oplus B_o$, (f) is the final result, the image $A \oplus_i B$.

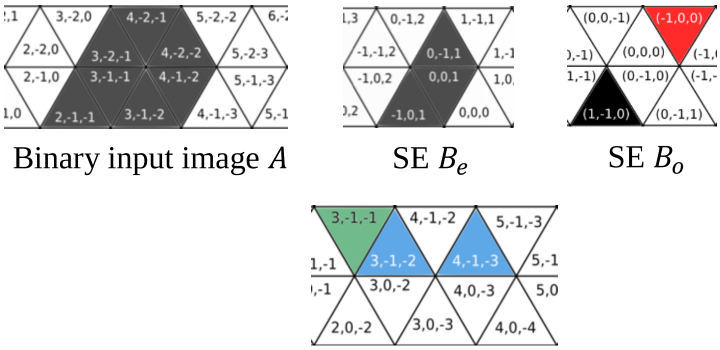
Remark 2.5.3. Notice that the red color in Figure 15 (c) shows how might a – odd, and – even vector be represented although – odd is not located in the triangular grid.

Definition 2.5.4. (**Independent Erosion)**

Let $A \subset T$ be the binary input image and B be an SE (structuring element) that contains two parts (B_e, B_o), where $B_e = B_e^0 \cup B_e^+$ with $B_e^0 \subseteq G^0$ and $B_e^+ \subseteq G^+$, $B_o = B_o^0 \cup B_o^-$ with $B_o^0 \subseteq G^0$ and $B_o^- \subseteq G^-$. Then, the *independent erosion* of A by B defined as follows:

$$A \ominus_i B = \{p \in G^0 : p + b \in A, \forall b \in B_e\} \cup \{t \in G^+ : t + d \in A, \forall d \in B_o\} \quad (1.4.7)$$

Example 2.5.2. Let $A = \{(2,-1,-1),(3,-1,-1),(3,-2,-1),(4,-2,-1),$ $(3,-1,-2),(4,-2,-2),(4,-1,-2)\}$, $B_e = \{(0,-1,1),(0,0,1),(-1,0,1)\}$ and $B_o = \{(1,-1,0),(-1,0,0)\}$.Then, $A \ominus_i B = \{(3,-1,-2),(4,-1,-3),(3,-1,-1)\}$.



Binary input image A SE B_e SE B_o

Result of $A \ominus_i B$

Figure 16: This figure illustrates the result of Example 2.5.2

Remark 2.5.4. The last figure represents the erosion of input image A by B where the green pixels indicate pixels which transformed by B_o inside A, blue pixels indicate pixels which transformed B_e inside of A. The red color in SE B_o indicates a vector not referring to a pixel, but having coordinate sum -1. (It is shown where its invers located in the triangular grid.)

Proposition 2.5. 2. We can rewrite the Eq.(1.4.7) in the following formula

$$A \ominus_i B = \Big(\big((A \ominus B_e) \cap G^0\big) \cup \big((A \ominus B_o) \cap G^+\big)\Big) \qquad (1.4.8)$$

Notice that the sign $\ominus$ on the right hand side indicates that we use the usual erosion on the 3D cubic grid.

Proof. If $p \in A \ominus_i B$, then by the Eq.(1.4.7) We have, either $p \in G^0$ such that $p + b \in A, \forall b \in B_e$ if and only if $p \in G^0$ and $p \in (A \ominus B_e)$, thus it is if and only if

$p \in \big((A \ominus B_e) \cap G^0\big)$. On the other hand, by Eq.(1.4.7), if $p \in G^+$, such that $p + d \in A, \forall d \in B_o$ then, it is if and only if $p \in G^+$ and $p \in (A \ominus B_o)$, yield that $p \in \big((A \ominus B_o) \cap G^+\big)$. □

Proposition 2.5. 3. We can rewrite the Eq.(1.4.7) as following :

$$A \ominus_i B = \Big(\big((A^0 \ominus B_e^0) \cap (A^+ \ominus B_e^+)\big) \cup \big((A^0 \ominus B_o^-) \cap (A^+ \ominus B_o^0)\big)\Big) \qquad (1.4.9)$$

Proof. Now we begin the proof, by showing that Eq.(1.4.8) is the same as Eq.(1.4.9). For left hand side in Eq.(1.4.8) if $p \in \big((A \ominus B_e) \cap G^0\big)$ then, p is even element and $p \in (A \ominus B_e^0) \cap (A \ominus B_e^+)$ this is mean that $p + k \in A, \forall k \in B_e^0$ and $p + d \in A, \forall d \in B_e^+$. Now since p is even, yield that $p + k \in A^0, \forall k \in B_e^0$ and $p + d \in A^+, \forall d \in B_e^+$ that is $p \in \big((A^0 \ominus B_e^0) \cap (A^+ \ominus B_e^+)\big)$. Similar proof for other case that, $p \in \big((A^0 \ominus B_o^-) \cap (A^+ \ominus B_o^0)\big)$ and p is odd element. □

2.5.1 Properties of Independent Dilation

In this section, we show the properties of our new dilation concept. The most important properties are inherited from the original dilation (defined on the cubic grid), but there are also significant differences. From now on A, L are input binary images and B, C, D are structuring elements. (We do not repeat this fact in the forthcoming theorems.)

Theorem 2.5.1.1. Let $O = (0,0,0\,)$ representing the origin of the triangular grid, if $O \in\ B_e^0$ and $O \in\ B_o^0$, then $A \subseteq\ A \oplus_i B$.

Proof. According to Eq.(1.4.4) we may write $A = (A^0 \oplus \{O\}) \cup (A^+ \oplus \{O\}) \subseteq (A^0 \oplus B_e) \cup (A^+ \oplus B_o) = A \oplus_i B$. □

However, the converse of Theorem 2.5.1.1 does not necessarily hold, i.e., it is not an "if and only if" theorem, as the next example shows.

Example 2.5.1.1. Let $A = \{(0,0,0),(0,0,1)\}$ be an image and B with $B_e = \{(0,0,1)\}$, $B_o = \{(0,0,-1)\}$ be an SE. Then, $A \oplus_i B = \{(0,0,1),(0,0,0)\}$, and thus, $A \subseteq A \oplus_i B$, but $O \notin B_e$ and $O \notin B_o$.

Theorem 2.5.1.2. The independent dilation is not commutative and not associative. That's $A \oplus_i B \neq B \oplus_i A$ and $(A \oplus_i B) \oplus_i C \neq A \oplus_i (B \oplus_i C)$.

Proof. Since input image and structural elements are different types of entities the operation is not defined if the first operand is not an image of the grid and/or the second operand is not a pair of vector sets. □

Theorem 2.5.1.3. If $t \in G^0$, then $(A)_t \oplus_i B = (A \oplus_i B)_t$, i.e., the independent dilation is translation invariant by even vectors.

Proof. Let $p \in (A)_t \oplus_i B \Leftrightarrow p \in \big((A^0 \oplus \{t\}) \cup (A^+ \oplus \{t\})\big) \oplus_i B$, Since $B = (B_e, B_o)$, $(A^0 \oplus \{t\}) \subset G^0$ and $(A^+ \oplus \{t\}) \subset G^+$, applying Eq.(1.4.4) one gets that $p \in \big((A^0 \oplus \{t\}) \oplus B_e\big) \cup \big((A^+ \oplus \{t\}) \oplus B_o\big)$. Now, since the dilation $\oplus$ (defined on the cubic grid) is commutative and associative, the previous formula can be written as $p \in \big((A^0 \oplus B_e) \oplus \{t\}\big) \cup \big((A^+ \oplus B_o) \oplus \{t\}\big)$. Further, it is equivalent to $p \in \big((A^0 \oplus B_e) \cup (A^+ \oplus B_o)\big) \oplus \{t\} \Leftrightarrow p \in \big((A^0 \oplus B_e) \cup (A^+ \oplus B_o)\big)_t$ that is, $p \in (A \oplus_i B)_t$. Since each step is if and only if step, the equation of the theorem is proven. □

Theorem 2.5.1.4. If $t \in G^0$ is an even vector, then $A \oplus_i B = (A)_t \oplus_i (B)_{-t}$.

Proof. By the Definition 2.5.3 and by the translation invariant property of the dilation on the cubic grid with respect to the structuring element, we can write the following:

$(A \oplus_i B) = \big((A^0 \oplus B_e) \cup (A^+ \oplus B_o)\big)_{t-t} = (A^0 \oplus (B_e)_{-t})_t \cup (A^+ \oplus (B_o)_{-t})_t$.

Applying the previous theorem for these formulae,

$(A^0 \oplus (B_e)_{-t})_t = (A_t^0 \oplus (B_e)_{-t}) = \big((A^0 \oplus (B_e)\big)$ and also $(A^+ \oplus (B_o)_{-t})_t = (A_t^+ \oplus (B_o)_{-t}) = \big((A^+ \oplus (B_o)\big)$.Hence,

$(A \oplus_i B) = \big((A_t^0 \oplus (B_e)_{-t}) \cup (A_t^+ \oplus (B_o)_{-t})\big) = (A)_t \oplus_i (B)_{-t}$. □

Theorem 2.5.1.5. If $A \subseteq L$, then $A \oplus_i C \subseteq L \oplus_i C$, i.e., the independent dilation has the increasing property.

Proof. If $A \subseteq L \Rightarrow A^0 \cup A^+ \subseteq L^0 \cup L^+$ and $A^0 \oplus C_e \subseteq L^0 \oplus C_e$, $A^+ \oplus C_o \subseteq L^+ \oplus C_o$, thus, $(A^0 \oplus C_e) \cup (A^+ \oplus C_o) \subseteq (L^0 \oplus C_e) \cup (L^+ \oplus C_o)$ and, therefore, $(A \oplus_i C) \subseteq (L \oplus_i C)$. □

Theorem 2.5.1.6. $(A \cap L) \oplus_i C \subseteq (A \oplus_i C) \cap (L \oplus_i C)$.

Proof. From the Eq.(1.4.4), $(A \cap L) \oplus_i C = ((A \cap L)^0 \oplus C_e) \cup ((A \cap L)^+ \oplus C_o)$, but we have $A^0 \subset A^0 \cup A^+, L^0 \subset L^0 \cup L^+$, and also, $A^+ \subset A^0 \cup A^+, L^+ \subset L^0 \cup L^+$. Moreover, the dilation on the 3D cubic grid has the propriety that $((A \cap L)^0 \oplus C_e) = \big((A^0 \cap L^0) \oplus C_e\big) \subseteq \big((A^0 \oplus C_e) \cap (L^0 \oplus C_e)\big)$, and similarly, $((A \cap L)^+ \oplus C_o) = \big((A^+ \cap L^+) \oplus C_o\big) \subseteq \big((A^+ \oplus C_o) \cap (L^+ \oplus C_o)\big)$. This means that

$$\left((A^0 \cap L^0) \oplus C_e\right) \cup \left((A^+ \cap L^+) \oplus C_o\right)$$

$$\subseteq \left((A^0 \oplus C_e) \cap (L^0 \oplus C_e)\right) \cup \left((A^+ \oplus C_o) \cap (L^+ \oplus C_o)\right)$$

$$= \left((A^0 \oplus C_e) \cup (A^+ \oplus C_o)\right) \cap \left((L^0 \oplus C_e) \cup (L^+ \oplus C_o)\right)$$

$$\cap \left((A^0 \oplus C_e) \cup (L^+ \oplus C_o)\right) \cap \left((L^0 \oplus C_e) \cup (A^+ \oplus C_o)\right)$$

$$\subseteq \left((A^0 \oplus C_e) \cup (A^+ \oplus C_o)\right) \cap \left((L^0 \oplus C_e) \cup (L^+ \oplus C_o)\right)$$

yielding the statement of the theorem. □

Theorem 2.5.1.7. $A \oplus_i (B \cap C) \subseteq (A \oplus_i B) \cap (A \oplus_i C)$.

Proof. By the Definition 2.5.3 and the intersection of SE we have $A \oplus_i (B \cap C) = \left(\left(A^0 \oplus (B_e \cap C_e)\right) \cup \left(A^+ \oplus (B_o \cap C_o)\right)\right)$. According to a similar property of 3D dilation we have, $\left(A^0 \oplus (B_e \cap C_e)\right) \subseteq \left((A^0 \oplus B_e) \cap (A^0 \oplus C_e)\right)$ and $\left(A^+ \oplus (B_o \cap C_o)\right) \subseteq \left((A^+ \oplus B_o) \cap (A^+ \oplus C_o)\right)$. By taking the union of the previous two inequalities one gets

$$\left(A^0 \oplus (B_e \cap C_e)\right) \cup \left(A^+ \oplus (B_o \cap C_o)\right)$$

$$\subseteq \left(\left((A^0 \oplus B_e) \cap (A^0 \oplus C_e)\right) \cup \left((A^+ \oplus B_o) \cap (A^+ \oplus C_o)\right)\right)$$

$$= \left(\left((A^0 \oplus B_e) \cup (A^+ \oplus B_o)\right) \cap \left((A^0 \oplus C_e) \cup (A^+ \oplus C_o)\right)\right)$$

$$\cap \left(\left((A^0 \oplus B_e) \cup (A^+ \oplus C_o)\right) \cap \left((A^0 \oplus C_e) \cup (A^+ \oplus B_o)\right)\right)$$

$$\subseteq \left(\left((A^0 \oplus B_e) \cup (A^+ \oplus B_o)\right) \cap \left((A^0 \oplus C_e) \cup (A^+ \oplus C_o)\right)\right)$$

$$= (A \oplus_i B) \cap (A \oplus_i C)$$

Hence the theorem. □

Theorem 2.5.1.8. $(A \cup L) \oplus_i C = (A \oplus_i C) \cup (L \oplus_i C)$, i.e., the independent dilation is distributive over the union of images.

Proof. From Eq.(1.4.4) We have $(A \cup L) \oplus_i C = ((A \cup L)^0 \oplus C_e) \cup ((A \cup L)^+ \oplus C_o)$. Moreover, $((A \cup L)^0 \oplus C_e) = (A^0 \oplus C_e) \cup (L^0 \oplus C_e)$ and similarly, $((A \cup L)^+ \oplus C_o) = (A^+ \oplus C_o) \cup (L^+ \oplus C_o)$. Thus,

$$(A \cup L) \oplus_i C =$$

$$\big((A^0 \oplus C_e) \cup (L^0 \oplus C_e)\big) \cup \big((A^+ \oplus C_o) \cup (L^+ \oplus C_o)\big) =$$

$$\big((A^0 \oplus C_e) \cup (A^+ \oplus C_o)\big) \cup \big((L^0 \oplus C_e) \cup (L^+ \oplus C_o)\big) =$$

$$(A \oplus_i C) \cup (L \oplus_i C)$$

The result has been proven. □

By the last theorem we have seen that the independent dilation is an abstract dilation: it is compatible with the union: commute with the supremum, i.e., preserves it.

2.5.2 Properties of Independent Erosion

Let us assume again that A, L are binary input images and B, C, D are structuring elements, where $B = (B_e, B_o)$, $C = (C_e, C_o)$, $D = (D_e, D_o)$.

Theorem 2.5.2.1. If $O \in B_e$ and $O \in B_o$ where $O = (0,0,0)$ is the origin of triangular grid, then $A \ominus_i B \subseteq A$.

Proof. If $O \in B_e$ and $O \in B_o$, then by Eq.(1.4.7), for an even $p \in (A \ominus_i B_e) \Rightarrow p + O = p \in A$ and also, for an odd $t \in (A \ominus_i B_o) \Rightarrow t + O = t \in A$. □

Example 2.5.2.1. Let input image and SE be as in the Figure 17 where $O \in B_e$ and $O \in B_o$.

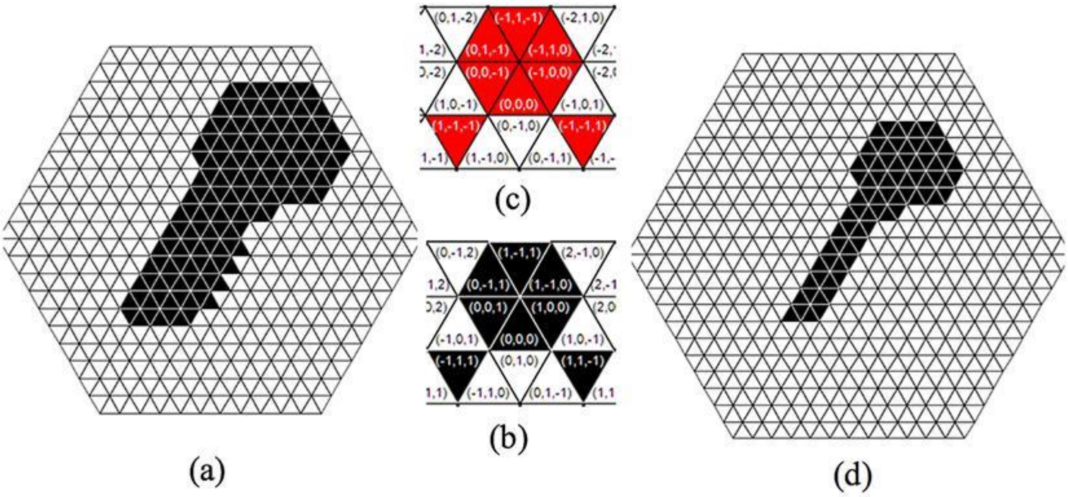


Figure 17: This example show that if the origin belong to B_e and B_o, then the result of independent erosion is subset of input image where: (a) input image, (b) B_e, (c) B_o, (d) is the result of independent erosion

Theorem 2.5.2.2. The independent erosion is not commutative and not associative.

Proof. Image and SE are different type of entities, thus the operation can be defined only in one order. Dilation and erosion on SE is not defined. □

Theorem 2.5.2.3. If $t \in G^0$ is an even vector, then $(A)_t \ominus_i B = (A \ominus_i B)_t$, i.e., the independent erosion is translation invariant under translations with even vectors.

Proof. By the definition of translation $p \in (A \ominus_i B)_t$ if and only if $(p - t) \in (A \ominus_i B)$. Then, applying Eq.(1.4.7), $(p - t) \in (A \ominus_i B)$, if and only if either $(p - t) + b \in A, \forall b \in B_e$ (if $(p - t) \in G^0$) or $(p - t) + d \in A, \forall d \in B_o$ (in case

$(p-t) \in G^+)$. Assume first that $(p-t)+b \in A, \forall b \in B_e$ with $(p-t) \in G^0 \Leftrightarrow (p+b)-t \in A, \forall b \in B_e$ and $p \in G^0 \Leftrightarrow (p+b) \in A_t, \forall b \in B_e$. Now, assume the other case, $(p-t)+d \in A, \forall d \in B_o$ with $(p-t) \in G^+ \Leftrightarrow (p+d)-t \in A, \forall d \in B_o$ and $p \in G^+ \Leftrightarrow (p+d) \in A_t, \forall d \in B_o$. These yield that $(p+b) \in A_t, \forall b \in B_e$ if $p \in G^0$ and $(p+d) \in A_t, \forall d \in B_o$ if $p \in G^+$. Thus,

$p \in (A_t \ominus_i B) = \{p \in G^0 : p+b \in A_t, \forall b \in B_e\} \cup \{p \in G^+ : p+d \in A_t, \forall d \in B_o\}$.□

Theorem 2.5.2.4. If $t \in G^0$, then $(A \ominus_i C_t) = (A \ominus_i C)_{-t}$.

Proof. Let $p \in (A \ominus_i C_t)$. By Eq. (1.4.7) it means that

$\{p \in G^0 : p+b \in A, \forall b \in (C_e)_t\}$ or $\{p \in G^+ : p+d \in A, \forall d \in (C_o)_t\}$. If $p \in G^0$ such that $p+b \in A, \forall b \in (C_e)_t$, this means by Eq. (1.4.8) that $p \in \left((A \ominus (C_e)_t) \cap G^0\right)$ but, $p+b \in A, \forall b \in (C_e)_t$ if and only if $(p-t)+b \in A, \forall b \in C_e$, $p \in G^0$. That is, if and only if, $(p-t) \in \left((A \ominus C_e) \cap G^0\right), (p-t) \in G^0$ which is equivalent to the condition $p \in ((A \ominus C_e)_{-t} \cap G^0)$. This yields that

$\left((A \ominus (C_e)_t) \cap G^0\right) = ((A \ominus C_e)_{-t} \cap G^0)$ for the even pixels. Now, if $p \in G^+$ such that $p+d \in A, \forall d \in (C_o)_t$, it is equivalent to $p \in \left((A \ominus (C_o)_t) \cap G^+\right)$. But, in this case $p+d \in A, \forall d \in (C_o)_t \Leftrightarrow (p-t)+d \in A, \forall d \in C_o, p \in G^+$

$\Leftrightarrow (p-t) \in \left((A \ominus C_o) \cap G^+\right), (p-t) \in G^+ \Leftrightarrow p \in ((A \ominus C_o)_{-t} \cap G^+)$. Thus, we have $\left((A \ominus (C_o)_t) \cap G^+\right) = ((A \ominus C_o)_{-t} \cap G^+)$ for odd pixels. Summarizing for the equation for the even and odd pixels, we have obtained

$$\left(\left((A \ominus (C_e)_t) \cap G^0\right) \cup \left((A \ominus (C_o)_t) \cap G^+\right)\right) =$$
$$\left(((A \ominus C_e)_{-t} \cap G^0) \cup ((A \ominus C_o)_{-t} \cap G^+)\right)$$

Thus, $(A \ominus_i C_t) = (A \ominus_i C)_{-t}$, the statement to be proven. □

Theorem 2.5.2.5. If $A \subseteq L$, then $(A \ominus_i B) \subseteq (L \ominus_i B)$, i.e., the independent erosion has the increasing (or monotonicity) property.

Proof. Let $A \subseteq L$ and let $p \in (A \ominus_i B)$. Then by Eq.(1.4.7), either $\{p \in G^0 : p + b \in A, \forall b \in B_e\}$ or $\{p \in G^+ : p + d \in A, \forall d \in B_o\}$. If $p \in G^0$ such that $p + b \in A, \forall b \in B_e$, then because of $A \subseteq L$, $p + b \in L, \forall b \in B_e$ also hold.

If $p \in G^+$ such that $p + d \in A, \forall d \in B_o$, then by $A \subseteq L$, $p + d \in L, \forall d \in B_o$ also holds. Now, uniting the two cases, we have that $p \in (L \ominus_i B) = \{p \in G^0 : p + b \in L, \forall b \in B_e\} \cup \{p \in G^+ : p + d \in L, \forall d \in B_o\}$. □

Theorem 2.5.2.6. $(A \cap L) \ominus_i C = (A \ominus_i C) \cap (L \ominus_i C)$, i.e., the independent erosion is distributive with respect to intersection.

Proof. Let $p \in (A \cap L) \ominus_i C$. From Eq.(1.4.7), either $\{p \in G^0 : p + b \in (A \cap L), \forall b \in C_e\}$ or $\{p \in G^+ : p + d \in (A \cap L), \forall d \in C_o\}$. In the first case, if $p \in G^0$ and $p + b \in (A \cap L), \forall b \in C_e$, then by Eq.(1.4.8), $p \in \left(\left((A \cap L) \ominus C_e\right) \cap G^0\right)$. Also, $p + b \in A, \forall b \in C_e$ and $p + b \in L, \forall b \in C_e$, and thus, in the case $p \in G^0$ it implies if and only if $\{p \in G^0 : p + b \in A, \forall b \in C_e\} \cap \{p \in G^0 : p + b \in L, \forall b \in C_e\}$ means that $p \in \left(\left((A \ominus C_e) \cap G^0\right) \cap \left((L \ominus C_e) \cap G^0\right)\right)$. This yields that

$$\left(\left((A \cap L) \ominus C_e\right) \cap G^0\right) = \left(\left((A \ominus C_e) \cap G^0\right) \cap \left((L \ominus C_e) \cap G^0\right)\right).$$

In the other case, if $p \in G^+$ such that $p + d \in (A \cap L), \forall d \in C_o$, it is equivalent to $p \in \left(\left((A \cap L) \ominus C_o\right) \cap G^+\right)$. Further, we also have $p + d \in A, \forall d \in C_o$ and $p + d \in L, \forall d \in C_o$ where $p \in G^+$. However, it is if and only if,

$\{p \in G^+ : p + d \in A, \forall d \in C_o\} \cap \{p \in G^+ : p + d \in L, \forall\, d \in C_o\}$. This yields that $p \in \Big(\big((A \ominus C_o) \cap G^+\big) \cap \big((L \ominus C_o) \cap G^+\big)\Big)$ and furthermore,

$\Big(\big((A \cap L) \ominus C_o\big) \cap G^+\Big) = \Big(\big((A \ominus C_o) \cap G^+\big) \cap \big((L \ominus C_o) \cap G^+\big)\Big)$.

Now by taking the union of the results obtained for both cases, we have that

$$\Big(\big((A \cap L) \ominus C_e\big) \cap G^0\Big) \cup \Big(\big((A \cap L) \ominus C_o\big) \cap G^+\Big) =$$

$\bigg(\Big(\big((A \ominus C_e) \cap G^0\big) \cup \big((A \ominus C_o) \cap G^+\big)\Big) \cap \Big(\big((L \ominus C_e) \cap G^0\big) \cup \big((L \ominus C_o) \cap G^+\big)\Big)\bigg) = (A \cap L) \ominus_i C = (A \ominus_i C) \cap (L \ominus_i C)$. □

As a corollary of the previous theorem we can state that the independent erosion is an abstract erosion.

Theorem 2.5.2.7. $A \ominus_i (B \cup C) = (A \ominus_i B) \cap (A \ominus_i C)$.

Proof. Based on Eq.(1.4.7), if $p \in A \ominus_i (B \cup C)$, then either $\{p \in G^0 : p + b \in A, \forall b \in (B_e \cup C_e)\}$ or $\{p \in G^+ : p + d \in A, \forall d \in (B_o \cup C_o)\}$.

In the first case, if $p \in G^0$ such that $p + b \in A, \forall b \in (B_e \cup C_e)$, it is if and only if $p \in \Big(\big((A \ominus (B_e \cup C_e)\big) \cap G^0\big)\Big)$ Also, this is if and only if $p + b \in A, \forall b \in B_e$ and $p + b \in A, \forall b \in C_e$, where $p \in G^0$. But this is equivalent to $p \in \Big(\big((A \ominus B_e) \cap G^0\big) \cap \big((A \ominus C_e) \cap G^0\big)\Big)$. This implies that

$\Big(\big((A \ominus (B_e \cup C_e)\big) \cap G^0\big)\Big) = \Big(\big((A \ominus B_e) \cap G^0\big) \cap \big((A \ominus C_e) \cap G^0\big)\Big)$ in one hand.

In the second case, if $p \in G^+$ such that $p + d \in A, \forall d \in (B_o \cup C_o)$, it is if and only if $p \in \Big(\big((A \ominus (B_o \cup C_o)\big) \cap G^+\big)\Big)$, furthermore $p + d \in A, \forall d \in B_o$ and $p + d \in$

$A, \forall d \in C_o$, where $p \in G^+$. However, it means exactly $p \in \Big(\big((A \ominus B_o) \cap G^+\big) \cap \big((A \ominus C_o) \cap G^+\big)\Big)$ yielding that

$\Big(\big((A \ominus (B_o \cup C_o)) \cap G^+\big)\Big) = \Big(\big((A \ominus B_o) \cap G^+\big) \cap \big((A \ominus C_o) \cap G^+\big)\Big)$ on the other hand. Now form these by taking their union we get

$\Big(\big((A \ominus (B_e \cup C_e)) \cap G^0\big) \cup \big((A \ominus (B_o \cup C_o)) \cap G^+\big)\Big) = \Big(\big((A \ominus B_e) \cap G^0\big) \cup \big((A \ominus B_o) \cap G^+\big)\Big) \cap \Big(\big((A \ominus C_e) \cap G^0\big) \cup \big((A \ominus C_o) \cap G^+\big)\Big)$, that is,

$A \ominus_i (B \cup C) = (A \ominus_i B) \cap (A \ominus_i C)$ what we wanted to proof. □

Theorem 2.5.2.8. $A \ominus_i (C \cap D) \supseteq (A \ominus_i C) \cup (A \ominus_i D)$.

Proof. Let $p \in (A \ominus_i C) \cup (A \ominus_i D)$ implying that either $p \in (A \ominus_i C)$ or $p \in (A \ominus_i D)$. If $p \in (A \ominus_i C)$, then by Eq.(1.4.7), we $\{p \in G^0 : p + b \in A, \forall b \in C_e\}$ or $\{p \in G^+ : p + d \in A, \forall d \in C_o\}$. Now if $p \in G^0$ such that $p + b \in A, \forall b \in C_e$, then $p \in \big((A \ominus C_e) \cap G^0\big)$ or if $p \in G^+$ such that $p + d \in A, \forall d \in C_o$, then $p \in \big((A \ominus C_o) \cap G^+\big)$, i.e., $p \in \big((A \ominus C_e) \cap G^0\big)$ or $p \in \big((A \ominus C_o) \cap G^+\big)$.

Let us assume the first case, $p \in G^0$ such that $p + b \in A, \forall b \in C_e$, then because of $(C_e \cap D_e) \subset C_e$, $p + b \in A, \forall b \in (C_e \cap D_e)$ also holds, yielding that if $p \in G^0$, then $p \in \big((A \ominus (C_e \cap D_e) \cap G^0)\big)$. In the other case, when $p \in G^+$ such that $p + d \in A, \forall d \in C_o$, since $(C_o \cap D_o) \subset C_o$, $p + d \in A, \forall d \in (C_o \cap D_o)$ also holds, that is $p \in \big((A \ominus (C_o \cap D_o) \cap G^+)\big)$, in case of $p \in G^+$.Uniting the two cases, and applying the Eq.(1.4.7), we obtained that p is in $A \ominus_i (C \cap D)$ also. The proof goes in a similar (symmetric) way when $p \in (A \ominus_i D)$. □

Theorem 2.5.2.9. $(A \cup L) \ominus_i C \supseteq (A \ominus_i C) \cup (L \ominus_i C)$.

Proof. The proof is similar to the proof of the previous theorem. □

Theorem 2.5.2.10. The following classical morphological properties do not work since some of the formulae are not defined in the independent approach.

1. $A \ominus_i (B \oplus_i C) = (A \ominus_i B) \ominus_i C$.
2. $A \oplus_i (C \ominus_i D) \subseteq (A \oplus_i C) \ominus_i D$.

2.5.3 Adjunction Relation on the Independent Case

First we may recall that the independent dilation and erosion preserve the union and intersection, respectively (Theorem 2.5.1.8, and Theorem 2.5.2.6). Second, to complete the picture, we will show that the independent dilation and erosion with the same SE form an adjunction:

Theorem 2.5.3.1. There is an adjunction relation between the independent dilation and erosion such that for any $A, L \subset T$, and (SE) B, $A \oplus_i B \subseteq L \Leftrightarrow A \subseteq L \ominus_i B$.

Proof. Let us start with direction ($\Rightarrow$): Let $a \in A$ arbitrarily chosen. Assume that $A \oplus_i B \subseteq L$. Then depending on the parity of a we have two cases. If $a \in A^0$ ($a \in A^+$, resp.), then for every $b \in B_e$ ($b \in B_o$ resp.), $p = a + b$ is in $A \oplus_i B$, and thus, in L. However, for any point q in G^0 (G^+ resp.), $q + b \in L$ for all $b \in B_e$ ($b \in B_o$ resp.) implies that q is in $L \ominus_i B$. But we have just proven this fact for an arbitrary point $a \in A$, thus $A \subseteq L \ominus_i B$ holds. Let us consider the other direction ($\Leftarrow$). Assume that $A \subseteq L \ominus_i B$. That means, by Eq.(1.4.7), that for any point $a \in A$, depending on its parity, $a + b \in L, \forall b \in B_e$ if $a \in G^0$ (or $a + b \in L, \forall b \in B_o$ if $a \in G^+$). However, in both cases, it also implies that every point of $A \oplus_i B$ that can be obtained by the help of a by adding any vector of the SE B is also in L, thus, $A \oplus_i B \subseteq L$. The proof is finished. □

2.5.4 Summary Tables for Independent Dilation and Erosion

In this section, we review the independent dilation and erosion and we compare them to the other aproaches based on the listed properties in Section 1.4.

Table 3: Summary of independent dilation properties

Property	1D	2D	3D	4D	5D	6D	7D	8D	9D	10D	11D	12D
condition	a.√ b.N_c	$a. N_c$ $b. t^0$	*	N_c	N_a	t^0	t^0	√	√	√	√	√

- √- holds without any condition.
- N_c - is not commutative or not in commutative way.
- *- holds with the condition that the origin point (0,0,0) must be included in both part of SEs.
- N_a - is not associative.
- t^0- hold with the condition that $t \in G^0$.

Table 4: Summary of independent erosion properties

Property	1E	2E	3E	4E	5E	6E	7E	8E	9E	10E	11E
condition	*	N_c	t^0	√	X	X	√	√	√	√	t^0

- √- holds without any condition.
- N_c - is not commutative or not in commutative way.

- - holds with the condition that the origin point (0,0,0) must be in both part of SEs.
- t^0- hold with the condition that $t \in G^0$.
- X - does not hold.

2.5.5 Applications of Using Various Type of Neighbors in Structuring Elements

Example 2.5.5.1. In this application we will use a binary image (tree) as it presented in Figure 18 and we will use different types of neighborhood of the grid namely,1-neighborhood one should use B_e^1 = {(0,0,0),(0,0,1),(0,1,0),(1,0,0)} and B_o^1 = {(0,0,0),(0,0,-1),(0,-1,0),(-1,0,0)}. For 2-neighbourhood we will use B_e^2 = B_e^1 ∪{(0,1,-1),(1,0,-1),(1,-1,0),(0,-1,1),(-1,0,1),(-1,1,0)} and B_o^2 = B_o^1 ∪{(0,1,-1),(1,0,-1),(1,-1,0), (0,-1,1),(-1,0,1),(-1,1,0)}. For 3-neighbrhood we will use B_e^3 = B_e^2 ∪{(1,1,-1),(1,-1,1),(-1,1,1)}, B_o^3 = B_o^2 ∪{(-1,-1,1),(-1,1,-1),(1,-1,-1)}.We will get a different type of result for both dilation and erosion. Observe that: in the Figure 18, we use $B_1 = B_e^1 \cup B_o^1$, $B_2 = B_e^2 \cup B_o^2$, $B_3 = B_e^3 \cup B_o^3$.

Example 2.5.5.2. In this example, we utilize the similar example of (a rabbit bone leg implant) that is shown in Subsection 2.4.5. Earlier, we used the same neighborhood for even and odd pixels, i.e., C_1,C_2 and C_3. The strong approach could use the traditional neighborhood in this way. Here we do not repeat the same examples, but we show applications which cannot be made with the previous methods by using different sets of structuring elements. Figure 19, we used $B = (B_e, B_o) = (B_e^1, B_o^3)$. In Figure 20 we use $B = (B_e, B_o)$ where $B_e = B_e^2$ and $B_o = \{(0,0,0),(0,0,-1),(-1,0,0),(0,-1,0),(1,-1,0),(0,-1,1)\}$. In Figure 21 we used $B = (B_e, B_o) = (B_e^3, B_o^1)$.

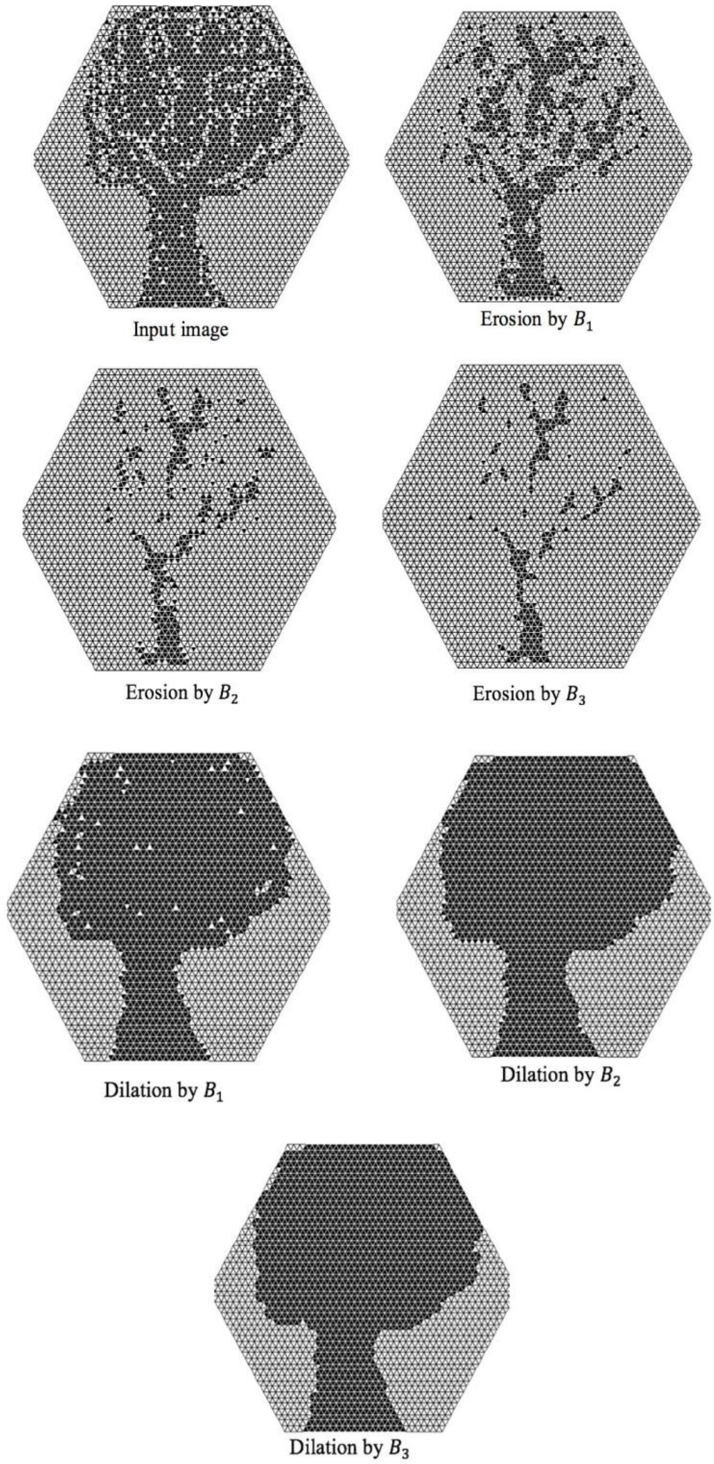
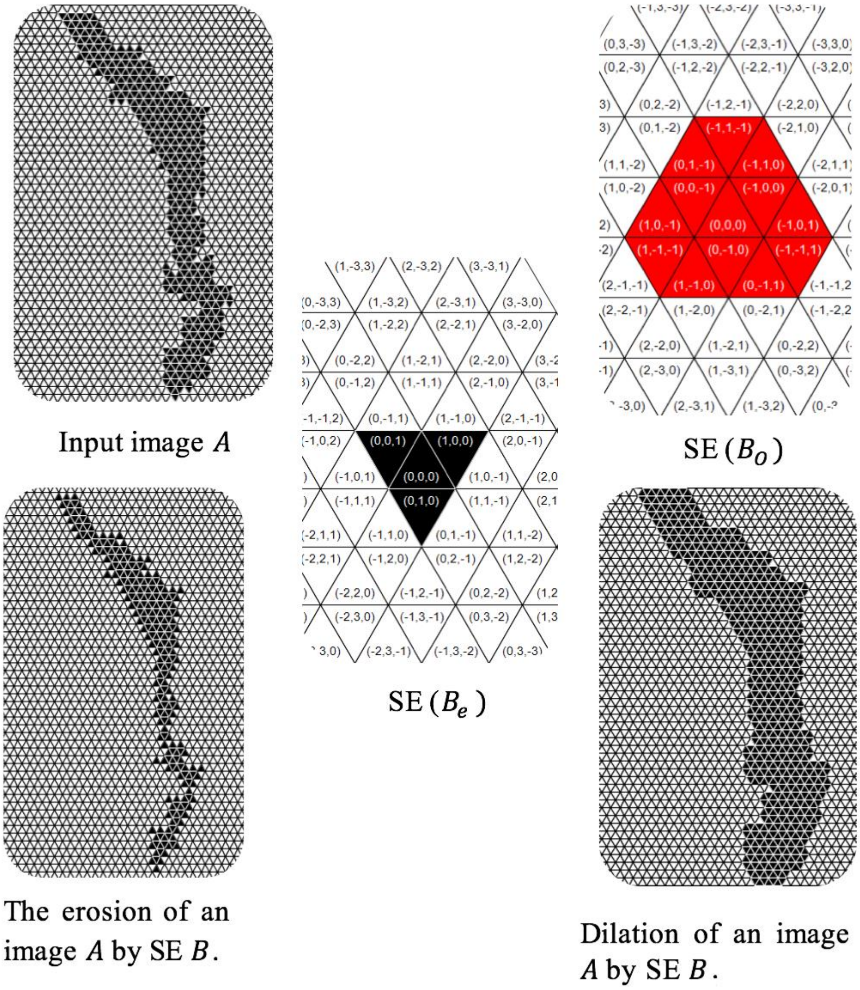


Figure 18: The results of different types of dilation and erosion by using different types of neighborhoods structuring



Input image A

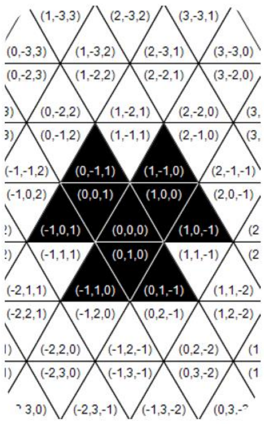
SE (B_e)

SE (B_O)

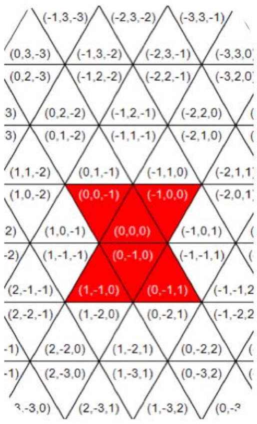
The erosion of an image A by SE B.

Dilation of an image A by SE B.

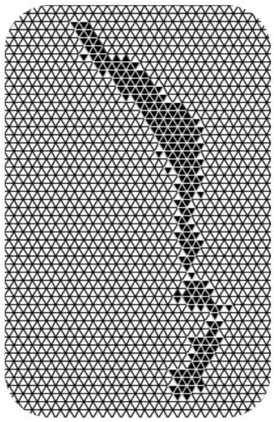
Figure 19: Erosion and dilation of a binary image (a rabbit bone leg implant) by SE applying 1-neighborhood for even pixels and 3-neighborhood for odd pixels



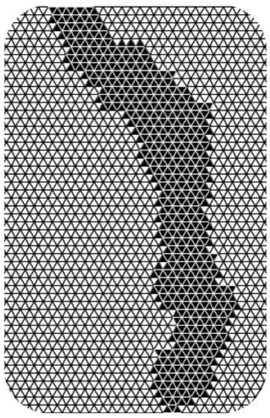
SE (B_e)



SE (B_o)

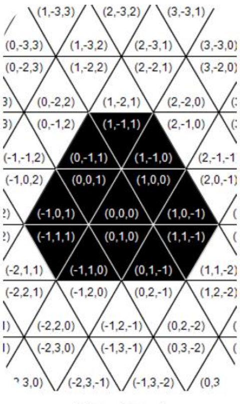


The erosion of an image A by SE B.

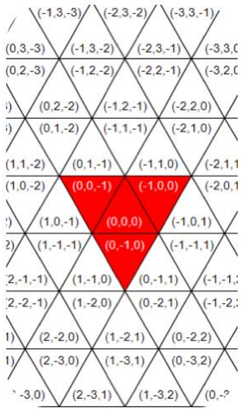


Dilation of an image A by SE B.

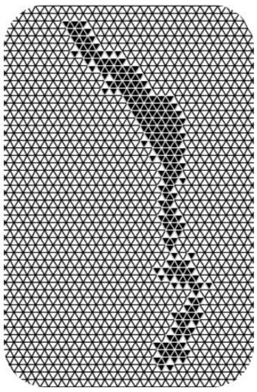
Figure 20: Erosion and dilation (bottom) of the image that shows above in Figure 19 with SE shown in the top



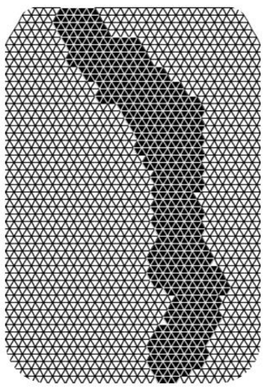
SE (B_e)



SE (B_o)



The erosion of an image A by SE B.



Dilation of an image A by SE B.

Figure 21: Erosion and dilation (bottom) of the image of Figure 19 with SE presented at in the top

Chapter 3

CONCLUSION

The binary image processing is applied in different areas. So far, the most applications are related to the square grid (lattice) where the role of an input image and structuring element can be interchanged. In this study, we used one of the regular tessellations, namely, the triangular grid. However, this grid is not a point lattice. For that reason, we need to define a suitable way to solve the translation issue that causing by using odd member/pixel on that grid when we attempt to establish a binary morphological operation dilation and erosion. One of these methods is a strict case where we use only even structuring element to translate an image with the structuring element, and that makes all of the strict properties are satisfied including the adjunction relation. While in the weak case, its allow to use a mixed structuring element (i.e., even and odd pixels), but we may lose information that is shifted outside the grid when we translate odd part of the image with similar one of the structuring element. However, by losing this information some of the weak dilation properties hold, but the result could be empty set even the operands are not empty. In the same context, some of the weak erosion properties may not hold such as Property E5W, E6W. Furthermore, the adjunction relation on the weak case may not hold. In the strong case, the triangular grid is seen, as a subspace of the cubic grid (B. Nagy, 2003; Benedek Nagy, 2004). Moreover, it is allowed to use vectors with the sum of the coordinate values 0, 1 or with the sum of their coordinate values equal to (-1) for both the image and the structuring element. After the translation process, we may keep information that cannot be displayed (somehow in a similar manner as topological coordinates can be used (Benedek Nagy, 2015)). Despite that, we have been able to give the necessary definitions for both dilation and erosion, but they still have a disadvantage in term of using only the even structuring element as in the strict case, losing information as in the weak case or keep more maybe useless information as in the case of the strong case. The last one could lead to memory consumption.To solve those problems we need a reliable method to handle the translation issue. For that, we provide a reasonable solution, i.e., the independent approach, where we treat the problem with the translations issue by using two of the structuring elements one work with the even and the other with odd part of an image. In this way, both of independent dilation and erosion are a well-defined and closed operation, and there are no outside results. We notice that the most of independent dilation properties are held with or without additional condition except for the cases of commutativity and associativity. Similarly, independent erosion is not commutative and not associative. Furthermore, some of the traditional erosion properties do not hold. Since the input image and structuring element are different types of entity (see the Theorem 2.5.2.10).We have proven that the independent approach forms an adjunction relation, in this way, they can be considered as a pair of abstract dilation and erosion.

REFERENCES

Abdalla, M., & Nagy, B. (2017). Concepts of Binary Morphological Operations Dilation and Erosion on the Triangular Grid. In *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)* (Vol. 10149 LNCS, pp. 89–104). Springer International Publishing AG 2017. https://doi.org/10.1007/978-3-319-54609-4_7

Abdalla, M., & Nagy, B. (2018). Dilation and Erosion on the Triangular Tessellation: An Independent Approach. *IEEE Access*, *6*, 23108–23119. https://doi.org/10.1109/ACCESS.2018.2827566

Aiello, M., Pratt-Hartmann, I., & van Benthem, J. (2007). *Handbook of Spatial Logics*. (M. Aiello, I. Pratt-Hartmann, & J. Van Benthem, Eds.) (Vol. 4). Dordrecht: Springer Netherlands. https://doi.org/10.1007/978-1-4020-5587-4

Birkhoff, G. (1940). *Lattice Theory*. American Mathematical Society.

Burger, W., & Burge, M. J. (2010). *Principles of Digital Image Processing: Fundamental Techniques*. Springer Science & Business Media.

Deutsch, E. S. (1972). Thinning algorithms on rectangular, hexagonal, and triangular arrays. *Communications of the ACM*, *15*(9), 827–837. https://doi.org/10.1145/361573.361583

Dougherty, E. (1992). *Mathematical Morphology in Image Processing*. CRC Press.

Fomani, B. A., & Shahbahrami, A. (2017). License plate detection using adaptive morphological closing and local adaptive thresholding. In *Pattern Recognition and Image Analysis (IPRIA), 2017 3rd International Conference on* (pp. 146–150).

Ghosh, P. K., & Deguchi, K. (2009). Mathematics Of Shape Description. *John Wiley Sons*.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W., & Scott, D. S. (1980). *A Compendium of Continuous Lattices*. Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-67678-9

Golay, M. J. E. (1969). Hexagonal Parallel Pattern Transformations. *IEEE Trans. Comput.*, *18*(8), 733–740. https://doi.org/10.1109/T-C.1969.222756

Gonzalez, R. C., & Woods, R. E. (2006). *Digital Image Processing (3rd Edition)*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc.

Hedberg, H., Dokladal, P., & Owall, V. (2009). Binary Morphology With Spatially Variant Structuring Elements: Algorithm and Architecture. *IEEE Transactions on Image Processing*, *18*(3), 562–572. https://doi.org/10.1109/TIP.2008.2010108

Heijmans, H. J. A. M., & Ronse, C. (1990). The algebraic basis of mathematical morphology I. Dilations and erosions. *Computer Vision, Graphics and Image Processing*, *50*(3), 245–295. https://doi.org/10.1016/0734-189X(90)90148-O

Heijmans, H., & Vincent, L. (1993). Graph morphology in image analysis.

Her, I. (1995). Geometric transformations on the hexagonal grid. *IEEE Transactions on Image Processing*, *4*(9), 1213–1222. https://doi.org/10.1109/83.413166

Herrlich, H., & Hušek, M. (1986). Galois connections BT - Mathematical Foundations of Programming Semantics: International Conference Manhattan, Kansas, April 11–12, 1985 Proceedings. In A. Melton (Ed.) (pp. 122–134). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/3-540-16816-8_28

Hou, S., Wu, S., & Ma, S. (2004). Doppler frequency extraction of radar echo signal based on time-frequency analysis and morphological operation. In *Signal Processing, 2004. Proceedings. ICSP'04. 2004 7th International Conference on* (Vol. 3, pp. 2017–2020).

Kardos, P., & Palágyi, K. (2012). On Topology Preservation for Triangular Thinning Algorithms. In R. P. Barneva, V. E. Brimkov, & J. K. Aggarwal (Eds.), *Combinatorial Image Analaysis: 15th International Workshop, IWCIA 2012, Austin, TX, USA, November 28-30, 2012. Proceedings* (pp. 128–142). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-34732-0_10

Kardos, P., & Palágyi, K. (2017). Unified Characterization of P-Simple Points in Triangular, Square, and Hexagonal Grids. In R. P. Barneva, V. E. Brimkov, & J. M. R. S. Tavares (Eds.), *Computational Modeling of Objects Presented in Images. Fundamentals, Methods, and Applications: 5th International Symposium, CompIMAGE 2016, Niagara Falls, NY, USA, September 21-23, 2016, Revised Selected Papers* (pp. 79–88). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-54609-4_6

Klette, R., & Rosenfeld, A. (2004). *Digital Geometry: Geometric Methods for Digital Picture Analysis*. Elsevier.

Luczak, E., & Rosenfeld, A. (1976). Distance on a Hexagonal Grid. *IEEE Transactions on Computers*, *C-25*(5), 532–533. https://doi.org/10.1109/TC.1976.1674642

Maragos, P. (1990). Affine morphology and affine signal models (Vol. 1350, pp. 31–43). Retrieved from http://dx.doi.org/10.1117/12.23574

Maragos, P. (2013). Representations for Morphological Image Operators and Analogies with Linear Operators. *Advances in Imaging and Electron Physics*, *177*, 45–187.

Matheron, G. (1975). Random sets and integral geometry. *Willey, New York*.

Meyer, F. (1978). Contrast Feature Extraction. *Quantitative Analysis of Microstrustures in Material Sciences, Biology and Medicine*. Retrieved from https://ci.nii.ac.jp/naid/10000149192/en/

Mir-Mohammad-Sadeghi, H., & Nagy, B. (2017). On the Chamfer Polygons on the Triangular Grid. In V. E. Brimkov & R. P. Barneva (Eds.), *Combinatorial Image Analysis: 18th International Workshop, IWCIA 2017, Plovdiv, Bulgaria, June 19-21, 2017, Proceedings* (pp. 53–65). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-59108-7_5

Nagy, B. (2001). Finding shortest path with neighbourhood sequences in triangular grids. In *ISPA 2001. Proceedings of the 2nd International Symposium on Image and Signal Processing and Analysis. In conjunction with 23rd International Conference on Information Technology Interfaces (IEEE Cat. No.01EX480)* (pp. 55–60). Univ. Zagreb. https://doi.org/10.1109/ISPA.2001.938603

Nagy, B. (2003). A family of triangular grids in digital geometry. In *3rd International Symposium on Image and Signal Processing and Analysis, 2003. ISPA 2003. Proceedings of the* (Vol. 1, pp. 101–106). IEEE. https://doi.org/10.1109/ISPA.2003.1296876

Nagy, B. (2004). Generalized triangular grids in digital geometry. *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, *20*(1), 63–78.

Nagy, B. (2009). Isometric transformations of the dual of the hexagonal lattice. In *Proceedings of 6th International Symposium on Image and Signal Processing and Analysis, 2009. ISPA 2009* (pp. 432–437).

https://doi.org/10.1109/ISPA.2009.5297709

Nagy, B. (2014). Weighted Distances on a Triangular Grid. In R. P. Barneva, V. E. Brimkov, & J. Šlapal (Eds.), *Combinatorial Image Analysis* (pp. 37–50). Springer International Publishing. https://doi.org/10.1007/978-3-319-07148-0_5

Nagy, B. (2015). Cellular topology and topological coordinate systems on the hexagonal and on the triangular grids. *Annals of Mathematics and Artificial Intelligence*, *75*(1–2), 117–134. https://doi.org/10.1007/s10472-014-9404-z

Nagy, B., & Lukić, T. (2016). Dense Projection Tomography on the Triangular Tiling. *Fundamenta Informaticae*, *145*(2), 125–141. https://doi.org/10.3233/FI-2016-1350

Najman, L., & Talbot, H. (2013). *Mathematical Morphology*. (L. Najman & H. Talbot, Eds.). Hoboken, NJ, USA: John Wiley & Sons, Inc. https://doi.org/10.1002/9781118600788

Ortiz, F., Puente, S., & Torres, F. (n.d.). Mathematical Morphology and Binary Geodesy for Robot Navigation Planning.

Pitas, I., & Venetsanopoulos, A. N. (1990). *Nonlinear Digital Filters* (Vol. 84). Boston, MA: Springer US. https://doi.org/10.1007/978-1-4757-6017-0

Roerdink, J. B. T. M. (2000). Group morphology. *Pattern Recognition*, *33*(6), 877–

895. https://doi.org/10.1016/S0031-3203(99)00152-1

Ronse, C. (1990). Why mathematical morphology needs complete lattices. *Signal Processing*, *21*(2), 129–154.

Sarkar, A., Biswas, A., Dutt, M., Bhowmick, P., & Bhattacharya, B. B. (2017). A linear-time algorithm to compute the triangular hull of a digital object. *Discrete Applied Mathematics*, *216*(Part 2), 408–423. https://doi.org/https://doi.org/10.1016/j.dam.2016.01.014

Serra, J. (1983). *Image Analysis and Mathematical Morphology*. Orlando, FL, USA: Academic Press, Inc.

Serra, J. (1988). Image Analysis and Math. Morphology Theoretical Advances, vol. 2. Academic Press.

Shih, F. Y. (2009). *Image Processing and Mathematical Morphology: Fundamentals and Applications*. CRC Press.

Soille, P. (2013). *Morphological Image Analysis: Principles and Applications*. Springer Science & Business Media.

Sternberg, S. R. (1986). Grayscale morphology. *Computer Vision, Graphics, and Image Processing*, *35*(3), 333–355. Retrieved from http://www.sciencedirect.com/science/article/pii/0734189X86900046

Vincent, L. (1989). Graphs and mathematical morphology. *Signal Processing*, *16*(4), 365–388. https://doi.org/https://doi.org/10.1016/0165-1684(89)90031-5

Wiederhold, P., & Morales, S. (2008). Thinning on Quadratic, Triangular, and Hexagonal Cell Complexes. In V. E. Brimkov, R. P. Barneva, & H. A. Hauptman (Eds.), *Combinatorial Image Analysis* (pp. 13–25). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-540-78275-9_2

Wilson, J. N., & Ritter, G. X. (2000). *Handbook of Computer Vision Algorithms in Image Algebra*. CRC Press.

Yang, S., & Li, J. (2015). The design of composite adaptive morphological filter and applications to Rician noise reduction in MR images. *International Journal of Imaging Systems and Technology*, *25*(1), 15–23. https://doi.org/10.1002/ima.22116