Stability of Spherically Symmetric Timelike Thin-shells in General Relativity

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ABSTRACT

In this thesis we study spherically symmetric timelike thin-shells in 3+1-dimensional bulk spacetime. We first introduce the cut and paste formalism which is used to make thin-shells in general relativity and then we investigate the stability of such thin-shells. Basically, in 3+1-dimensional bulk spacetime a timelike thin-shell is a 2+1-dimensional hyperplane whose normal 4-vector is a spacelike vector at any point on the hyperplane. A thin-shell connects two different parts of the bulk, therefore it has to satisfy some conditions which are called the Israel-junction conditions. In accordance with these conditions, the first fundamental form of the thin-shell must be continuous while its second fundamental form is not and it requires energy-momentum tensor on the thin-shell.

Keywords: General relativity; Thin-shell; Timelike hypersurface; Stability; Linear perturbation;

ÖZ

Bu tezde 3+1 boyutlu geniş uzay zamanda küresel simetrik zaman-tipli ince kabukları incelemekteyiz. Önce ince kabukların genel görelikteki kesip-yapıştırma özelliklerini gösterip bu ince kabukların kararlılıklarını inceliyoruz. Temel olarak, 3+1 geniş uzay zamanda bir zaman-tipli ince kabuk, 2+1 boyutlu bir hiperdüzlemdir ve bunun 4-vektörü hiperdüzlemde her hangi bir noktadaki uzay-tipli vektördür. Ince bir kabuk, uzay-zaman iki farklı bölümünü birbirine bağlar ve bundan dolayı İsrael-sınır koşulları olarak bilinen bazı koşulları sağlaması gerekir. Bu koşullara göre, ince kabuğun ilk temel formu devamlı olmalıyken ikinci temel formu değildir ve ince kabuk üzerinde enerji-momentum tensörüne ihtiyaç duyar.

Anahtar Kelimeler: Genel görelilik, İnce kabuk, Zaman-tipli hiperdüzlem, Kararlılık, Doğrusal düzensizlik.

DEDICATION

This thesis dedicated to:

- my dear parents, thanks for everything you did for me.
- my lovely wife, Mahabad; who always encourages me in my study and life.
- my son, Zhanyar; you are the pleasure of my life.

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INRODUCTION

Time-like thin-shell in spherically symmetric static spacetime is one of the most interesting cosmological object which can be constructed in general relativity. Such model of cosmological objects have been used to analyze some astrophysical phenomenon such as gravitational collapse and supernovae. The seminal work of Israel in 1966 [11, 12] provided a concrete formalism for constructing the time-like shells, in general, by gluing two different manifolds at the location of the thin-shell. As it was shown in [11], although the metric tensor of the shell which is induced by the bulk spacetime presented in both sides of the shell must be continuous the extrinsic curvature across the shell is not continuous and therefore matter has to be presented on the shell. This formalism has been employed to study shells in general relativity by many authors, for instance, a good review paper has been worked out by Kijowski et al in Ref. [13]. In 1990 an exact solution for a static shell which surrounds a black hole was found by Frauendiener et al [7] and its stability was also studied in Refs. [2] and [21]. In the formalism introduced by Israel there are some conditions which are called Israel junction conditions. These conditions provide a systematic method of finding the energy momentum tensor presented on the shell. In [19] a computer program was prepared to apply the Junction conditions on the thin shells in general relativity using computer algebra. Models of stars and circumstellar shells in general relativity was studied in [22]. In [10, 16] the stability of spherically symmetric thin-shells was studied while the gravitational collapse of thin shells was considered in [3] and [4]. Thin-shells in Gauss-Bonnet theory of gravity has been studied in [9] while the rotating thin-shells has been introduced in Ref. [8]. Stability of charged thin shells was studied by Eiroa and Simeone in [6] while its collapse in isotropic coordinates was investigated in [1]. In [17] charge screening by thin-shells in a 2+1–dimensional regular black hole has been studied while the thermodynamics, entropy, and stability of thin shells in 2+1 flat spacetimes have been given in [14] and [15]. Recently in [20] the stability of thin-shell interfaces inside compact stars has been studied by Pereira et al; which is very interesting as they consider a compact star with the core and the crust with different energy momentum tensor and consequently with different metric tensor. Screening of the Reissner-Nordström charge black hole by a thin-shell of dust matter has also been introduced recently in [18]. Finally one of the last work published in this context is about thin shells joining local cosmic string geometries [5].

Our aim in this thesis is to establish a self-contained, clear and complete formalism on the construction of timelike thin-shells in spherically symmetric bulk. The details of the calculation which are not usually mentioned or given in the papers are discussed here and the results are given all in closed analytical and generic form which can be used for any further study in future. In addition we study the stability analysis of such a thin shell and again in a closed and generic form the results are presented. Finally we apply our formalism for two specific examples including a thin- shell connecting two different spacetimes consisting of cloud of strings and a vacuum connected to a Schwarzschild spacetime.

Chapter 2

SPHERICALLY SYMMETRIC TIMELIKE THIN-SHELLS: THE FORMALISM

In 3+1–dimensional spherically symmetric bulk spacetime a 2+1–dimensional thin-shell divides the spacetime into two parts which we shall call them inside the shell and outside the shell. The line elements of the spacetime in different sides must be different otherwise the thin-shell becomes a trivial invisible object. For our future convenient we label the spacetime inside the shell as 1 and outside the shell as 2. Hence the line element of each side may be written as

$$ds_a^2 = -f_a(r_a)dt_a^2 + \frac{dr_a^2}{f_a(r_a)} + r_a^2(d\theta_a^2 + sin^2\theta_a\, d\varphi_a^2), \tag{2.1}$$

in which $a = 1,2$ for inside and outside respectively. Let's add that in general the coordinates i.e., t_a, r_a, θ_a and φ_a need not be the same. In general a thin-shell is a constraint relation on the coordinates of the bulk spacetime but in our study the thin-shell is defined by

$$F := r_a - R(\tau) = 0, \tag{2.2}$$

in which τ is the proper time measured by an observer on the shell such that on both sides we define

$$-f_a(r_a)dt_a^2 + \frac{dr_a^2}{f_a(r_a)} = -d\tau^2, \tag{2.3}$$

or equivalently

$$-f_a(R)\left(\frac{dt_a}{d\tau}\right)^2 + \frac{1}{f_a(R)}\left(\frac{dR}{d\tau}\right)^2 = -1. \qquad (2.4)$$

Considering (2.3) we find the induced metric on the shell for each side given by

$$ds^2_{a(ts)} = -d\tau^2 + R^2(\tau)(d\theta_a^2 + sin^2\theta_a\, d\varphi_a^2). \qquad (2.5)$$

As one of the Israel junction condition, $ds^2_{a(ts)}$ from one side to other side of the thin-shell must be continuous. Hence, the coordinates on the shell i.e., τ , θ_a and φ_a have to be identical on both sides which we shall remove the sub a. This results in a unique induced metric on the shell which is applicable in both sides which is given by

$$ds^2_{(ts)} = -d\tau^2 + R^2(\tau)(\, d\theta^2 + sin^2\theta\, d\varphi^2). \qquad (2.6)$$

Before we move on let us note that although the proper time in different sides of the shell is the same, the coordinate time t_a are different and they are found from (2.4) i.e.,

$$\dot{t}_a^2 = \frac{f_a(R)+\dot{R}^2}{f_a^2(R)}, \qquad (2.7)$$

in which a dot stands for the derivative with respect to the proper time τ. Let's also add that t_1 is measured by an observer inside the shell while t_2 is measured by an observer outside the shell and being different is due to the different line element they use and is very acceptable.

For future use we set our coordinate systems of the bulk i.e., $ds_a^2 = g^{(a)}_{\mu\nu}dx^{(a)\mu}dx^{(a)\nu}$ and the shell i.e., $ds^2_{(ts)} = h_{ij}\,d\xi^i d\xi^j$ as follow: for the bulk spacetime $x^{(a)\mu} = t_a, r_a, \theta_a, \varphi_a$ and $\xi^i = \tau, \theta, \varphi$ for the thin shell. The second fundamental form or extrinsic curvature tensor of the shell in each side can be found as

$$K_{ij}^{(a)} = -n_\gamma^{(a)} \left(\frac{\partial^2 x^{(a)\gamma}}{\partial \xi^i \, \partial \xi^j} + \Gamma_{\alpha\beta}^{(a)\gamma} \frac{\partial x^{(a)\alpha}}{\partial \xi^i} \frac{\partial x^{(a)\beta}}{\partial \xi^j} \right), \tag{2.8}$$

in which $n_\gamma^{(a)}$ is the four normal spacelike vector on each side of the thin-shell pointing outward and given by

$$n_\gamma^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}} \frac{\partial F}{\partial x^{(a)\gamma}} , \tag{2.9}$$

where

$$\Delta^{(a)} = g^{(a)\mu\nu} \frac{\partial F}{\partial x^{(a)\mu}} \frac{\partial F}{\partial x^{(a)\nu}} . \tag{2.10}$$

We note that as $n_\gamma^{(a)}$ is spacelike it satisfies

$$n_\gamma^{(a)} n^{(a)\gamma} = 1, \tag{2.11}$$

and the thin-shell defined by F in (2.2) where $n_\gamma^{(a)}$ is its four normal, is a timelike hypersurface. In order to calculate the second fundamental forms of the thin-shell on each side we have to first obtain the four normal $n_\gamma^{(a)}$. This can be done by using the definition of the thin-shell in (2.2). Hence, we find

$$n_t^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}} \frac{\partial \left(r_a - R(\tau) \right)}{\partial t_a}, \tag{2.12}$$

$$n_r^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}} \frac{\partial \left(r_a - R(\tau) \right)}{\partial r_a}, \tag{2.13}$$

$$n_{\theta}^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}}\frac{\partial\left(r_a - R(\tau)\right)}{\partial\theta_a}, \tag{2.14}$$

and

$$n_{\varphi}^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}}\frac{\partial\left(r_a - R(\tau)\right)}{\partial\varphi_a}. \tag{2.15}$$

We first recall that in the bulk the coordinates are independent i.e., $\frac{\partial x^{(a)\alpha}}{\partial x^{(a)\beta}} = \delta_{\beta}^{\alpha}$.

Therefore. the Eq.s (2.12)-(2.15) yield

$$n_t^{(a)} = -\frac{1}{\sqrt{\Delta^{(a)}}}\frac{\partial R(\tau)}{\partial t_a}, \tag{2.16}$$

$$n_r^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}}, \tag{2.17}$$

$$n_{\theta}^{(a)} = 0 \tag{2.18}$$

and

$$n_{\varphi}^{(a)} = 0. \tag{2.19}$$

Also using the chain rule one finds

$$\frac{\partial}{\partial t_a} = \frac{1}{\frac{\partial t_a}{\partial\tau}}\frac{\partial}{\partial\tau}\ , \tag{2.20}$$

where

$$\frac{\partial t_a}{\partial\tau} = \dot{t}_a = \frac{\sqrt{f_a(R) + \dot{R}^2}}{f_a(R)}. \tag{2.21}$$

Finally

$$n_{\gamma}^{(a)} = \frac{1}{\sqrt{\Delta^{(a)}}}\left(-\frac{\dot{R}(\tau)}{\dot{t}_a}, 1,0,0\right), \tag{2.22}$$

where

$$\Delta^{(a)} = -\frac{1}{f_a(R)}\left(-\frac{\dot{R}(\tau)}{\dot{t}_a}\right)^2 + f_a(R). \tag{2.23}$$

One can simplify the latter equation as

$$\Delta^{(a)} = \frac{f_a^2(R)}{f_a(R)+\dot{R}^2}, \tag{2.24}$$

which is clearly equal to the inverse of $\dot{t}_a^2$ i.e.,

$$\Delta^{a} = \frac{1}{\dot{t}_a^2}. \tag{2.25}$$

Considering the closed from of $\Delta^{(a)}$ in the four normal (2.22) we find

$$n_{\gamma}^{(a)} = \left(-\dot{R}(\tau)\, , \dot{t}_a\, , 0\, ,0\, \right). \tag{2.26}$$

Next, we apply $n_{\gamma}^{(a)}$ in the definition of the extrinsic curvature given by (2.8) to find the nonzero components of the second fundamental form tensor. Before that we need to find the components of the Christoffel symbol which is defined as

$$\Gamma_{\alpha\beta}^{(a)\gamma} = \frac{1}{2} g^{(a)\gamma\lambda}\left(g_{\lambda\beta,\alpha}^{(a)} + g_{\alpha\lambda,\beta}^{(a)} - g_{\alpha\beta,\lambda}^{(a)}\right). \tag{2.27}$$

The closed form of the nonzero components of the Christoffel symbol are found to be

$$\Gamma_{tt}^{(a)r} = \frac{1}{2} f_a(r_a) f_a^{'}(r_a), \tag{2.28}$$

$$\Gamma^{(a)t}_{tr} = \Gamma^{(a)t}_{rt} = \frac{f'_a(r_a)}{2f_a(r_a)}, \tag{2.29}$$

$$\Gamma^{(a)r}_{rr} = -\frac{f'_a(r_a)}{2f_a(r_a)}, \tag{2.30}$$

$$\Gamma^{(a)\theta}_{r\theta} = \Gamma^{(a)\theta}_{\theta r} = \Gamma^{(a)\varphi}_{r\varphi} = \Gamma^{(a)\varphi}_{\varphi r} = \frac{1}{r_a}, \tag{2.31}$$

$$\Gamma^{(a)r}_{\theta\theta} = -r_a f_a\ (r_a), \tag{2.32}$$

$$\Gamma^{(a)r}_{\varphi\varphi} = -r_a f_a\ (r_a)\ sin^2\theta_a, \tag{2.33}$$

$$\Gamma^{(a)\theta}_{\varphi\varphi} = -sin\theta_a cos\theta_a, \tag{2.34}$$

and

$$\Gamma^{(a)\varphi}_{\theta\varphi} = \Gamma^{(a)\varphi}_{\varphi\theta} = \frac{cos\theta_a}{sin\theta_a}, \tag{2.35}$$

in which a prime stands for the derivative with respect to r_a. Hence, we get

$$K^{(a)}_{\tau\tau} = -n^{(a)}_t\left(\frac{\partial^2 t_a}{\partial\tau^2} + 2\Gamma^{(a)t}_{tr}\frac{\partial t_a}{\partial\tau}\frac{\partial r_a}{\partial\tau}\right) - n^{(a)}_r\left(\frac{\partial^2 r_a}{\partial\tau^2} + \Gamma^{(a)r}_{tt}\frac{\partial t_a}{\partial\tau}\frac{\partial t_a}{\partial\tau} + \Gamma^{(a)r}_{rr}\frac{\partial r_a}{\partial\tau}\frac{\partial r_a}{\partial\tau}\right), \tag{2.36}$$

$$K^{(a)}_{\theta\theta} = -n^{(a)}_r\left(\Gamma^{(a)r}_{\theta\theta}\right), \tag{2.37}$$

and

$$K^{(a)}_{\varphi\varphi} = -n^{(a)}_r\left(\Gamma^{(a)r}_{\varphi\varphi}\right). \tag{2.38}$$

To obtain the explicit form of the nonzero components of the extrinsic curvature tensor we need to find $\ddot{t}_a = \frac{\partial^2 t_a}{\partial \tau^2}$ This can be done by using (2.7) which yields

$$2\ddot{t}_a \dot{t}_a = \frac{f_a^{'}(R)\dot{R}+2\ddot{R}\dot{R}}{f_a^2(R)} - \frac{2\left[f_a(R)+\dot{R}^2\right]f_a^{'}(R)\dot{R}}{f_a^3(R)}, \tag{2.39}$$

therefore

$$\ddot{t}_a = \frac{\dot{R}}{2\sqrt{f_a(R)+\dot{R}^2}} \frac{2\ddot{R}f_a(R)-f_a(R)f_a^{'}(R)-2\dot{R}^2 f_a^{'}(R)}{f_a^2(R)}. \tag{2.40}$$

The nonzero components of the extrinsic curvature are found to be

$$K_{\tau\tau}^{(a)} = -\frac{2\ddot{R}(\tau)+f_a^{'}(R)}{2\sqrt{f_a(R)+\dot{R}^2}}, \tag{2.41}$$

$$K_{\theta\theta}^{(a)} = R(\tau)\sqrt{f_a(R)+\dot{R}^2}, \tag{2.42}$$

and

$$K_{\varphi\varphi}^{(a)} = R(\tau)\sqrt{f_a(R)+\dot{R}^2} sin^2\theta. \tag{2.43}$$

It is observed that unlike the first fundamental form, the second fundamental form is not continuous in general. However, if $f_a(R)$ and $f_a^{'}(R)$ are the same in both sides of the shell then K_{ij} is continuous as well as h_{ij} . Although K_{ij} is not continuous but still it satisfies the other Israel junction condition which implies

$$\left[K_i^j\right] - \delta_i^j [K] = -8\pi G S_i^j, \tag{2.44}$$

in which $\left[K_i^j\right] = K_i^{(2)j} - K_i^{(1)j}$, $[K] = tra\left[K_i^j\right] = \left[K_i^i\right]$,

and

$$S_i^j = diag(-\sigma, p, p), \tag{2.45}$$

is the energy- momentum of the thin-shell. Herein, σ is the energy density and p is the lateral pressure. We note that, as we have considered the bulk to be spherically symmetric, the pressures in θ and φ directions are identical and the energy momentum tensor is of a perfect fluid type. By applying (2.44) we need to find the mixed tensor $K_i^{(a)j}$ which is defined as

$$K_i^{(a)j} = h^{(a)jk} K_{ik}^{(a)}, \tag{2.46}$$

in which

$$h^{(a)jk} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{R^2(\tau)} & 0 \\ 0 & 0 & \frac{1}{R^2(\tau) sin^2 \theta} \end{pmatrix}. \tag{2.47}$$

The explicit calculation reveals

$$K_i^{(a)j} = \begin{pmatrix} \frac{2\ddot{R}(\tau)+f_a'(R)}{2\sqrt{f_a(R)+\dot{R}^2}} & 0 & 0 \\ 0 & \frac{\sqrt{f_a(R)+\dot{R}^2}}{R(\tau)} & 0 \\ 0 & 0 & \frac{\sqrt{f_a(R)+\dot{R}^2}}{R(\tau)} \end{pmatrix}, \tag{2.48}$$

and consequently the total curvature is found to be

$$K^a = traK_i^{(a)j} = K_i^{(a)i} = \frac{2\ddot{R}(\tau)+f_a'(R)}{2\sqrt{f_a(R)+\dot{R}^2}} + \frac{2\sqrt{f_a(R)+\dot{R}^2}}{R(\tau)}. \tag{2.49}$$

The effective extrinsic curvature tensor defined by $[K_i^j] = K_i^{(2)j} - K_i^{(1)j}$, can be written as

$$[K_i^j] = \begin{pmatrix} \frac{2\ddot{R}(\tau)+f_2'(R)}{2\sqrt{f_2(R)+\dot{R}^2}} - \frac{2\ddot{R}(\tau)+f_1'(R)}{2\sqrt{f_1(R)+\dot{R}^2}} & 0 & 0 \\ 0 & \frac{\sqrt{f_2(R)+\dot{R}^2}}{R(\tau)} - \frac{\sqrt{f_1(R)+\dot{R}^2}}{R(\tau)} & 0 \\ 0 & 0 & \frac{\sqrt{f_2(R)+\dot{R}^2}}{R(\tau)} - \frac{\sqrt{f_1(R)+\dot{R}^2}}{R(\tau)} \end{pmatrix}, \tag{2.50}$$

and the effective total curvature becomes

$$[K] = \frac{2\ddot{R}(\tau)+f_2'(R)}{2\sqrt{f_2(R)+\dot{R}^2}} - \frac{2\ddot{R}(\tau)+f_1'(R)}{2\sqrt{f_1(R)+\dot{R}^2}} + \frac{2\sqrt{f_2(R)+\dot{R}^2}}{R(\tau)} - \frac{2\sqrt{f_1(R)+\dot{R}^2}}{R(\tau)}. \tag{2.51}$$

Finally we find the explicit form of the energy momentum tensor presented on the shell given by

$$S_i^j = \begin{pmatrix} -\sigma & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \tag{2.52}$$

in which

$$\sigma = -\frac{1}{4\pi G}\left(\frac{\sqrt{f_2(R)+\dot{R}^2} - \sqrt{f_1(R)+\dot{R}^2}}{R(\tau)}\right), \tag{2.53}$$

and

$$p = \frac{1}{8\pi G}\left(\frac{2\ddot{R}(\tau)+f_2'(R)}{2\sqrt{f_2(R)+\dot{R}^2}} - \frac{2\ddot{R}(\tau)+f_1'(R)}{2\sqrt{f_1(R)+\dot{R}^2}} + \frac{\sqrt{f_2(R)+\dot{R}^2} - \sqrt{f_1(R)+\dot{R}^2}}{R(\tau)}\right). \tag{2.54}$$

To conclude this chapter we would like to add that using the thin-shell formalism and the Israel junction conditions we established a dynamic spherical symmetric timelike thin-shell whose energy momentum tensor has to be of the form we found in latter equations i.e., (2.53) and (2.54). In addition, we comment that in the case of static thin-shell one has to set $R(\tau) = R_0$ which is a constant. Consequently the form of the energy momentum tensor takes its static form given by

$$S_i^j = \begin{pmatrix} -\sigma_0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{pmatrix}, \tag{2.55}$$

in which

$$\sigma_0 = -\frac{1}{4\pi G}\left(\frac{\sqrt{f_2(R_0)} - \sqrt{f_1(R_0)}}{R_0}\right), \tag{2.56}$$

and

$$p_0 = \frac{1}{8\pi G}\left(\frac{f_2'(R_0)}{2\sqrt{f_2(R_0)}} - \frac{f_1'(R_0)}{2\sqrt{f_1(R_0)}} + \frac{\sqrt{f_2(R_0)} - \sqrt{f_1(R_0)}}{R_0}\right). \tag{2.57}$$

In the coming Chapter we shall use the results found in this Chapter to investigate the stability of such a dynamic thin-shell wormhole. This, furthermore, needs a relation between the energy density on the shell and the lateral pressure which in general is called an equation of state and is expressed as $p = \psi(\sigma)$ in which ψ is in general a function of σ.

Chapter 3

STABILITY ANALYSIS OF THE SPHERICALLY SYMMETRIC THIN-SHELL

As we have already constructed the timelike dynamic thin-shells in spherically symmetric bulk spacetime in the previous Chapter, a natural question a rises; are these thin-shells stable against an external perturbation?. To answer this question one must find a way to analyze the motion of the thin-shell after such kind of perturbation. In addition we must have a clear definition for a stable motion and accordingly we may state the stability or instability of such thin-shells. These will be our aim to be investigated in this Chapter.

3.1 General Formalism

Let's assume that our constructed thin-shell is at equilibrium at $R = R_0$ which means $\dot{R} = \ddot{R} = 0$ and therefore the energy density and the pressures are given by Eqs. (2.56) and (2.57). Any radial perturbation causes the radius of the shell to be changed in a dynamical sense. In other words R after the radial perturbation is a function of proper time τ and consequently the energy density and the lateral pressure are found to be as of Eqs. (2.53) and (2.54). Before we go further let's add that σ and p given in these equations are related via a differential equation. To find that we start from

$$\sigma = -\frac{1}{4\pi G}\left(\frac{\sqrt{f_2(R)+\dot{R}^2}-\sqrt{f_1(R)+\dot{R}^2}}{R(\tau)}\right), \tag{3.1}$$

σ whose derivative with respect to τ is given by

$$\dot{\sigma} = -\frac{1}{4\pi G}\left(-\dot{R}\frac{\left(\sqrt{f_2(R)+\dot{R}^2}-\sqrt{f_1(R)+\dot{R}^2}\right)}{R(\tau)^2}+\frac{\dot{R}}{R(\tau)}\left(\frac{f_2^{'}(R)+2\ddot{R}}{2\sqrt{f_2(R)+\dot{R}^2}}-\frac{f_1^{'}(R)+2\ddot{R}}{2\sqrt{f_1(R)+\dot{R}^2}}\right)\right). \tag{3.2}$$

In terms of σ we may write it as

$$-\frac{R(\tau)}{\dot{R}}\dot{\sigma} = \sigma + \frac{1}{4\pi G}\left(\frac{f_2^{'}(R)+2\ddot{R}}{2\sqrt{f_2(R)+\dot{R}^2}}-\frac{f_1^{'}(R)+2\ddot{R}}{2\sqrt{f_1(R)+\dot{R}^2}}\right). \tag{3.3}$$

Next we look at

$$p = \frac{1}{8\pi G}\left(\frac{2\ddot{R}(\tau)+f_2^{'}(R)}{2\sqrt{f_2(R)+\dot{R}^2}}-\frac{2\ddot{R}(\tau)+f_1^{'}(R)}{2\sqrt{f_1(R)+\dot{R}^2}}+\frac{\sqrt{f_2(R)+\dot{R}^2}-\sqrt{f_1(R)+\dot{R}^2}}{R(\tau)}\right), \tag{3.4}$$

which surprisingly can be written as

$$-2p = \sigma - \frac{1}{4\pi G}\left(\frac{2\ddot{R}(\tau)+f_2^{'}(R)}{2\sqrt{f_2(R)+\dot{R}^2}}-\frac{2\ddot{R}(\tau)+f_1^{'}(R)}{2\sqrt{f_1(R)+\dot{R}^2}}\right). \tag{3.5}$$

Finally adding (3.3) and (3.5) we get

$$-\frac{R(\tau)}{\dot{R}}\dot{\sigma} - 2p = 2\sigma, \tag{3.6}$$

which, more conveniently , can be written as

$$\frac{\dot{\sigma}}{\dot{R}} + \frac{2}{R}(p+\sigma) = 0, \tag{3.7}$$

or after applying the chain rule it becomes

$$\frac{d\sigma}{dR} + \frac{2}{R}(p+\sigma) = 0. \tag{3.8}$$

This equation is the dynamical relation which connects p and σ after the perturbation. Furthermore, any kind of fluid presented on the shell has to satisfy an equation of state (EoS) which is nothing but a relation between p and σ. This relation is traditionally expressed as

$$p = p(\sigma), \tag{3.9}$$

but in our study we use a more general EoS given by

$$p = \psi(R, \sigma). \tag{3.10}$$

A substitution into (3.8) yields

$$\frac{d\sigma(R)}{dR} + \frac{2}{R}\big(\psi(R, \sigma(R)) + \sigma(R)\big) = 0, \tag{3.11}$$

which is a principal equation that connects σ to R after the perturbation. In addition to this, from the explicit form of σ in Eq. (3.1) we also find

$$\dot{R}^2 + V\big(R, \sigma(R)\big) = 0, \tag{3.12}$$

in which

$$V\big(R, \sigma(R)\big) = \frac{f_1(R)+f_2(R)}{2} - \frac{\big(f_1(R)-f_2(R)\big)^2}{\big(8\pi GR\sigma(R)\big)^2} - \big(2\pi GR\sigma(R)\big)^2. \tag{3.13}$$

This is a one dimensional equation of motion for the radius of the thin-shell after the perturbation. This equation together with Eq. (3.11) give a clear picture of the motion of the thin-shell after the perturbation. To be more precise, the solution of Eq. (3.11) is used in (3.12) and the general motion of the radius of the thin-shell, in principle is found by solving Eq. (3.12). The nature of the motion after the perturbation depends on the form of the function $\psi(R, \sigma)$ given by EoS and the metric functions $f_1(R)$and $f_2(R)$.

3.1.1 A Linearized Equation of Motion

The general one-dimensional equation of motion (3.12) is highly non-linear. In general we do not expect an exact closed form solution for the radius of the thin-shell after the perturbation. A linearized version of this equation helps us to know the general behavior of the motion of the thin-shell after the perturbation without going through the complete solution. As we have stated the thin-shell is in equilibrium at $R = R_0$ we expand $V\big(R, \sigma(R)\big)$ in Eq. (3.13) about $R = R_0$ and we keep it to the first order. This is called a linearized radial perturbation. Let's expand $V\big(R, \sigma(R)\big)$ about $R = R_0$, which is given by

$$V\big(R, \sigma(R)\big) = V\big(R_0, \sigma(R_0)\big) + \frac{dV}{dR}\bigg|_{R=R_0} (R - R_0) + \frac{1}{2}\frac{d^2V}{dR^2}\bigg|_{R=R_0} (R - R_0)^2) + O((R - R_0)^3). \tag{3.14}$$

Since, $R = R_0$ is the equilibrium point, it is very clear that $V\big(R_0, \sigma(R_0)\big)$ and $\frac{dV}{dR}\big|_{R=R_0}$ both are zero, the first because $\dot{R}_0^2 = 0$ and second because $R = R_0$ is the equilibrium in the sense that the force is zero there. Introducing $x = R - R_0$ up to the first non-zero term we get

$$\dot{x}^2 + \omega^2 x^2 \simeq 0, \tag{3.15}$$

in which

$$\omega^2 = \frac{1}{2}\frac{d^2V}{dR^2}\bigg|_{R=R_0}. \tag{3.16}$$

A derivative with respect to τ implies

$$\ddot{x} + \omega^2 x \simeq 0, \tag{3.17}$$

which clearly with $\omega^2 > 0$ represents an oscillation about $x = 0$. This, however, means the radius of the thin-shell moves on an oscillation about the equilibrium radius $R = R_0$. This is what we means by a stable state. In other words if

$$\frac{1}{2}\frac{d^2V}{dR^2}\Big|_{R=R_0} > 0, \tag{3.18}$$

the thin-shell oscillates and remains stable. Unlike $\omega^2 > 0$, if $\omega^2 < 0$ then the motion of the radius of the thin-shell grows exponentially which indicates that it does not come back to its equilibrium point; an indication of unstable thin-shell.

To proceed with the formalism we need to calculate $\frac{d^2V}{dR^2}\Big|_{R=R_0}$ and as $V = V(R,\sigma)$ we shall need σ' and σ''. From Eq.(3.11) we have already found

$$\sigma' = -\frac{2}{R}(\psi(R,\sigma) + \sigma), \tag{3.19}$$

which upon applying a differentiation with respect to R yields

$$\sigma'' = \frac{2}{R^2}(\psi(R,\sigma) + \sigma) - \frac{2}{R}\left(\frac{\partial\psi(R,\sigma)}{\partial R} + \frac{\partial\psi(R,\sigma)}{\partial\sigma}\sigma' + \sigma'\right), \tag{3.20}$$

where a prime stands for the derivative with respect to R. Using the explicit form of σ' in the latter equation it implies

$$\sigma'' = \frac{2}{R^2}(\psi(R,\sigma) + \sigma) - \frac{2}{R}\left[\frac{\partial\psi(R,\sigma)}{\partial R} + \left(\frac{\partial\psi(R,\sigma)}{\partial\sigma} + 1\right)\left(-\frac{2}{R}(\psi(R,\sigma) + \sigma)\right)\right], \tag{3.21}$$

or in a simplified form

$$\sigma'' = \frac{2}{R^2}(\psi + \sigma)\left[2\frac{\partial\psi}{\partial\sigma} + 3\right] - \frac{2}{R}\frac{\partial\psi}{\partial R}. \tag{3.22}$$

Now, we are ready to find $V''(R)$ at $R = R_0$ by applying σ' and σ'' whenever we need them. The first derivative of the potential with respect to R is given by

$$V' = \frac{dV}{dR} = \frac{f_1'+f_2'}{2} - \frac{2(f_1'-f_2')(f_1-f_2)}{(8\pi GR\sigma)^2} - \frac{16\pi G(2\psi+\sigma)(f_1-f_2)^2}{(8\pi GR\sigma)^3} + 8\pi^2G^2(2\psi+\sigma)R\sigma, \quad (3.23)$$

in which we have used $R\sigma' = -2(\psi+\sigma)$. The second derivative of the potential is found to be

$$V'' = \frac{dV'}{dR} = \frac{f_1''+f_2''}{2} - \frac{2(f_1''-f_2'')(f_1-f_2)+2(f_1'-f_2')^2}{(8\pi GR\sigma)^2} + \frac{2\left(8\pi G(\sigma+R\sigma')\right)}{(8\pi GR\sigma)^3} -$$
$$\frac{16\pi G\left(2\left(\frac{\partial\psi}{\partial R}+\frac{\partial\psi}{\partial\sigma}\sigma'\right)+\sigma'\right)(f_1-f_2)^2+16\pi G(2\psi+\sigma)2(f_1-f_2)(f_1'-f_2')}{(8\pi GR\sigma)^3} +$$
$$\frac{16\pi G(2\psi+\sigma)(f_1-f_2)^2 3\left(8\pi G(\sigma+R\sigma')\right)}{(8\pi GR\sigma)^4} + 8\pi^2G^2\left(2\left(\frac{\partial\psi}{\partial R}+\frac{\partial\psi}{\partial\sigma}\sigma'\right)+\sigma'\right)R\sigma +$$
$$8\pi^2G^2(2\psi+\sigma)(\sigma+R\sigma'). \quad (3.24)$$

Substituting σ' finally one finds

$$V'' = \frac{dV'}{dR} = \frac{f_1''+f_2''}{2} - \frac{2(f_1''-f_2'')(f_1-f_2)+2(f_1'-f_2')^2}{(8\pi G)^2(R\sigma)^2} - \frac{2(2\psi+\sigma)}{(8\pi G)^2(R\sigma)^3} -$$
$$\frac{\left(\frac{\partial\psi}{\partial R}-\frac{1}{R}\left(2\frac{\partial\psi}{\partial\sigma}+1\right)(\psi+\sigma)\right)(f_1-f_2)^2+(2\psi+\sigma)(f_1-f_2)(f_1'-f_2')}{(4\pi G)^2(R\sigma)^3} - \frac{6(2\psi+\sigma)^2(f_1-f_2)^2}{(8\pi G)^2(R\sigma)^4} +$$
$$16\pi^2G^2\left(\frac{\partial\psi}{\partial R}-\frac{1}{R}\left(2\frac{\partial\psi}{\partial\sigma}+1\right)(\psi+\sigma)\right)R\sigma - 8\pi^2G^2(2\psi+\sigma)^2\,. \quad (3.25)$$

At the equilibrium point both V and V' vanish and we are left with nonzero V'' given by

$$V_0'' = -\frac{16\pi GF_0H_0}{F_0-H_0}\psi_{,R} + \frac{2\left[H_0\left(2F_0^2-f_{10}'R_0\right)-F_0\left(2H_0^2-f_{20}'R_0\right)\right]}{(F-H)R_0^2}\psi_{,\sigma} +$$
$$\frac{\left[4F_0^4-2R_0\left(f_{10}'+R_0f_{10}''\right)F_0^2+R_0^2f_{10}'^2\right]H_0^3-\left[4H_0^4-2R_0\left(f_{20}'+R_0f_{20}''\right)H_0^2+R_0^2f_{20}'^2\right]F_0^3}{2(F_0-H_0)F_0^2H_0^2R_0^2}, \quad (3.26)$$

in which $F_0 = \sqrt{f_{10}}$, $H_0 = \sqrt{f_{20}}$, $\psi_{,R} = \frac{\partial \psi}{\partial R}\Big|_{R=R_0}$ and $\psi_{,\sigma} = \frac{\partial \psi}{\partial \sigma}\Big|_{R=R_0}$. Then our next step will be to check the sign of $V''(R_0)$ for a specific Eos and the bulk metrics.

Chapter 4

APPLICATIONS

What we have found in the previous two chapters are completely generic which can be applied to any spherically symmetric bulks and any EoS. To show some applications of the formalism, in this chapter, we study two specific examples including a thin-shell of cloud of strings and a Schwarzschild thin-shell.

4.1 Thin-shell Connecting Two Spacetimes of Cloud of Strings

In our first application of the formalism let's consider $f_1 = \kappa_1$ and $f_2 = \kappa_2$ in which κ_1 and κ_2 are two positive not equal constants with $\kappa_1 > \kappa_2$. The energy momentum tensor at the equilibrium is given by

$$S_i^j = \frac{\sqrt{\kappa_1}-\sqrt{\kappa_2}}{4\pi G R_0} diag\left(1, -\frac{1}{2}, -\frac{1}{2}\right), \tag{4.1}$$

which means

$$\sigma_0 = \frac{\sqrt{\kappa_1}-\sqrt{\kappa_2}}{4\pi G R_0}, \tag{4.2}$$

and

$$p_0 = -\frac{1}{2}\sigma_0. \tag{4.3}$$

This energy momentum tensor satisfies the weak energy condition which states that $\sigma_0 \geq 0$, and $\sigma_0 + p_0 \geq 0$ and therefore is physical. To proceed with the stability analysis we must choose an EoS. For the first example we set

$$\frac{d\psi}{dR} = \omega_{1,} \tag{4.4}$$

and

$$\frac{d\psi}{d\sigma} = \omega_2, \tag{4.5}$$

in which both ω_1 and ω_2 are constants. The general form of $V''(R_0)$ given in Eq. (3.25) yields

$$V''(R_0) = -\frac{2\sqrt{\kappa_1\kappa_2}\left[8\pi G\,\omega_1\,R_0^2-(2\omega_2+1)(\sqrt{\kappa_1}-\sqrt{\kappa_2})\right]}{R_0^2(\sqrt{\kappa_1}-\sqrt{\kappa_2})}, \tag{4.6}$$

In order to have $V''(R_0) > 0$ one must impose

$$8\pi G\omega_1\,R_0^2 - (2\omega_2+1)(\sqrt{\kappa_1}-\sqrt{\kappa_2}) < 0\,, \tag{4.7}$$

which in turn implies

$$\omega_1 < \frac{(\sqrt{\kappa_1}-\sqrt{\kappa_2})}{8\pi G\,R_0^2}(2\omega_2+1). \tag{4.8}$$

The case where $\omega_1 = 0$ implies a linear perfect fluid which is stable for $\omega_2 > -\frac{1}{2}$ and unstable for $\omega_2 < -\frac{1}{2}$. For the case where $\omega_1 \neq 0$ the situation becomes more complicated. In Fig. 1 we plot

$$\omega_1 = \frac{(\sqrt{\kappa_1}-\sqrt{\kappa_2})}{8\pi G\,R_0^2}(2\omega_2+1), \tag{4.9}$$

for $\frac{(\sqrt{\kappa_1}-\sqrt{\kappa_2})}{8\pi G\,R_0^2} = 0.1, 0.2, 0.3$ and $0,4$. As it is imposed from the condition (4.9) the values of ω_1 and ω_2 under the lines for each specific choice of $\frac{(\sqrt{\kappa_1}-\sqrt{\kappa_2})}{8\pi G\,R_0^2}$ implies the region of stability while the opposite side (above the lines) stands for the values of ω_1 and ω_1 result in an unstable thin-shell.

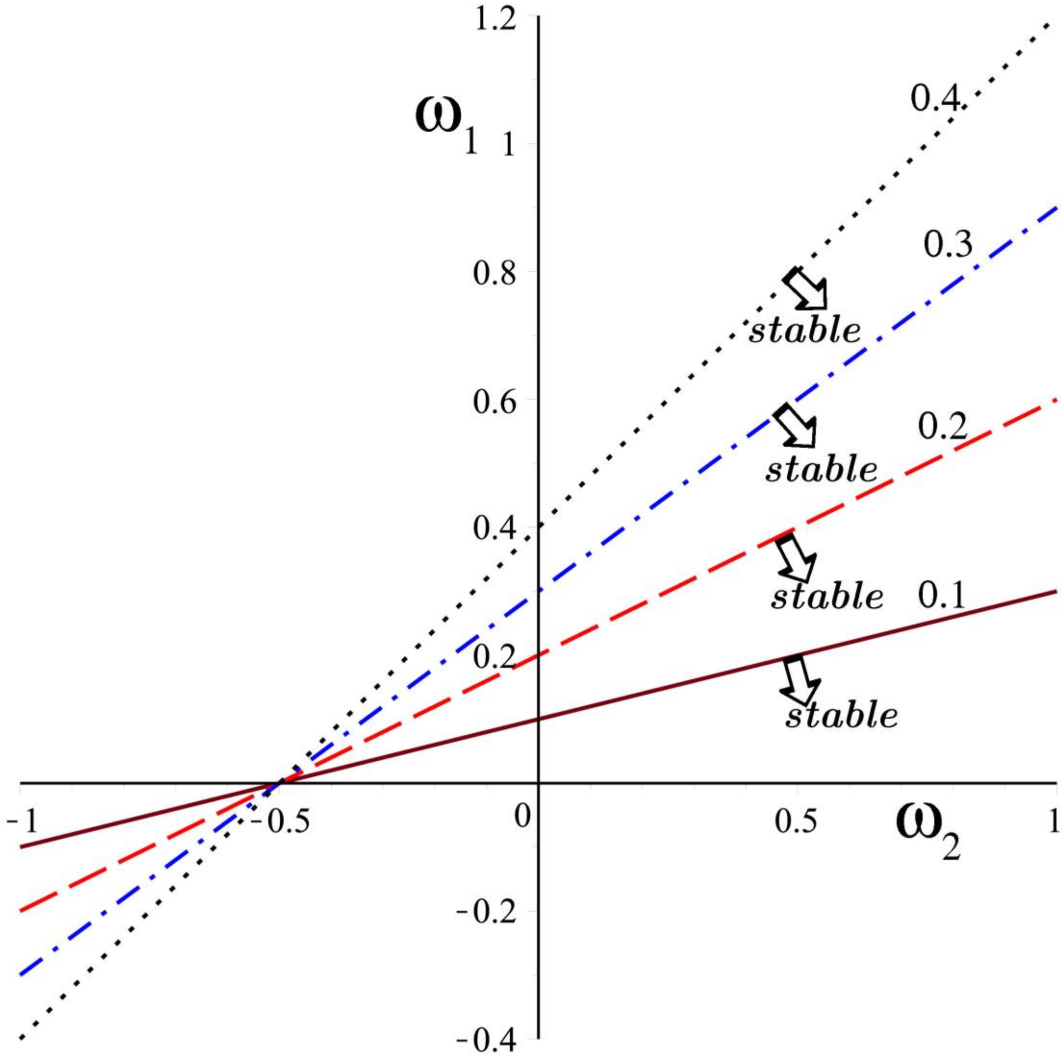

Figure 4.1: A plot of ω_1 with respect to ω_2 for various values of $\frac{(\sqrt{\kappa_1}-\sqrt{\kappa_2})}{8\pi G R_0^2} =$ 0.1,0.2,0.3 and 0.4 . The arrows show the region of stability while the opposite side is the unstable zone for each case.

4.2 Thin-shell Connecting Vacuum to Schwarzschild

In our second explicit application we consider the inner spacetime to be flat with $f_1 = 1$ and the outer spacetime to be the Schwarzschild with $f_2(r_2) = 1 - \frac{2m}{r_2}$. The closed forms of σ_0 and p_0 are found to be

$$\sigma_0 = \frac{1-\sqrt{1-\frac{2m}{R_0}}}{4\pi G R_0}, \tag{4.10}$$

and

$$p_0 = -\frac{m-R_0-R_0\sqrt{1-\frac{2m}{R_0}}}{8\pi G R_0^2\sqrt{1-\frac{2m}{R_0}}}. \tag{4.11}$$

These clearly satisfy the weak energy conditions i.e., $\sigma_0 \geq 0$ and $\sigma_0 + p_0 \geq 0$ provided $R_0 > 2m$. Furthermore, one finds

$$V_0'' = \frac{16\pi G\Omega}{\Omega-1}\psi_{,R} + \frac{2\left(2\psi_{,\sigma}+1\right)\Omega}{R_0^2}. \tag{4.12}$$

in which $\Omega = \sqrt{1-\frac{2m}{R_0}} > 0$, $\psi_{,R} = \frac{\partial\psi}{\partial R}\Big|_{R=R_0}$ and $\psi_{,\sigma} = \frac{\partial\psi}{\partial\sigma}\Big|_{R=R_0}$. In this case also we set $\frac{\partial\psi}{\partial R} = \omega_1$ and $\frac{\partial\psi}{\partial\sigma} = \omega_2$ which implies

$$V_0'' = \frac{16\pi G\Omega}{\Omega-1}\omega_1 + \frac{2(2\omega_2+1)\Omega}{R_0^2}. \tag{4.13}$$

For the case $\omega_1 = 0$, which corresponds to a linear perfect fluid one finds $V_0'' \geq 0$ equivalent to $2\omega_2 + 1 \geq 0$ or equivalently $\omega_2 \geq -\frac{1}{2}$. For the case $\omega_1 \neq 0$ we have to work out the regions in the plane of ω_1 and ω_2 such that $V_0'' \geq 0$. To find the region where $V_0'' \geq 0$, we find ω_2 in terms of ω_1 such that $V_0'' = 0$. This yields

$$\omega_2 = -\frac{1}{2} + \chi\omega_1 , \tag{4.14}$$

in which

$$\chi = \frac{4\pi G R_0^2}{1-\sqrt{1-\frac{2m}{R_0}}}. \qquad (4.15)$$

Depending on the value of m and $R_0 > 2m$, one finds

$$4\pi G R_0^2 < \chi < \infty. \qquad (4.16)$$

In Fig. 2 we plot ω_2 versus ω_1 for various values for $\chi = 0.1, 0.2, 0.3$ and 0.4. Also the stability regions for each case is shown by an indicator.

Before we finish this section we would like to find the explicit form of the energy density σ after the perturbation. In both examples we have worked out in this chapter we assumed $\frac{\partial \psi}{\partial R} = \omega_1$ and $\frac{\partial \psi}{\partial \sigma} = \omega_2$ in which ω_1 and ω_2 are two constants. Integration with respect to R and σ results in

$$\psi = \omega_1 R + \omega_2 \sigma + C_0\ , \qquad (4.17)$$

in which C_0 is an integration constant. As $p = \psi$ should give the equilibrium pressure at $R = R_0$ we can find the value of C_0 as

$$C_0 = p_0 - \omega_1 R_0 - \omega_2 \sigma_0 \qquad (4.18)$$

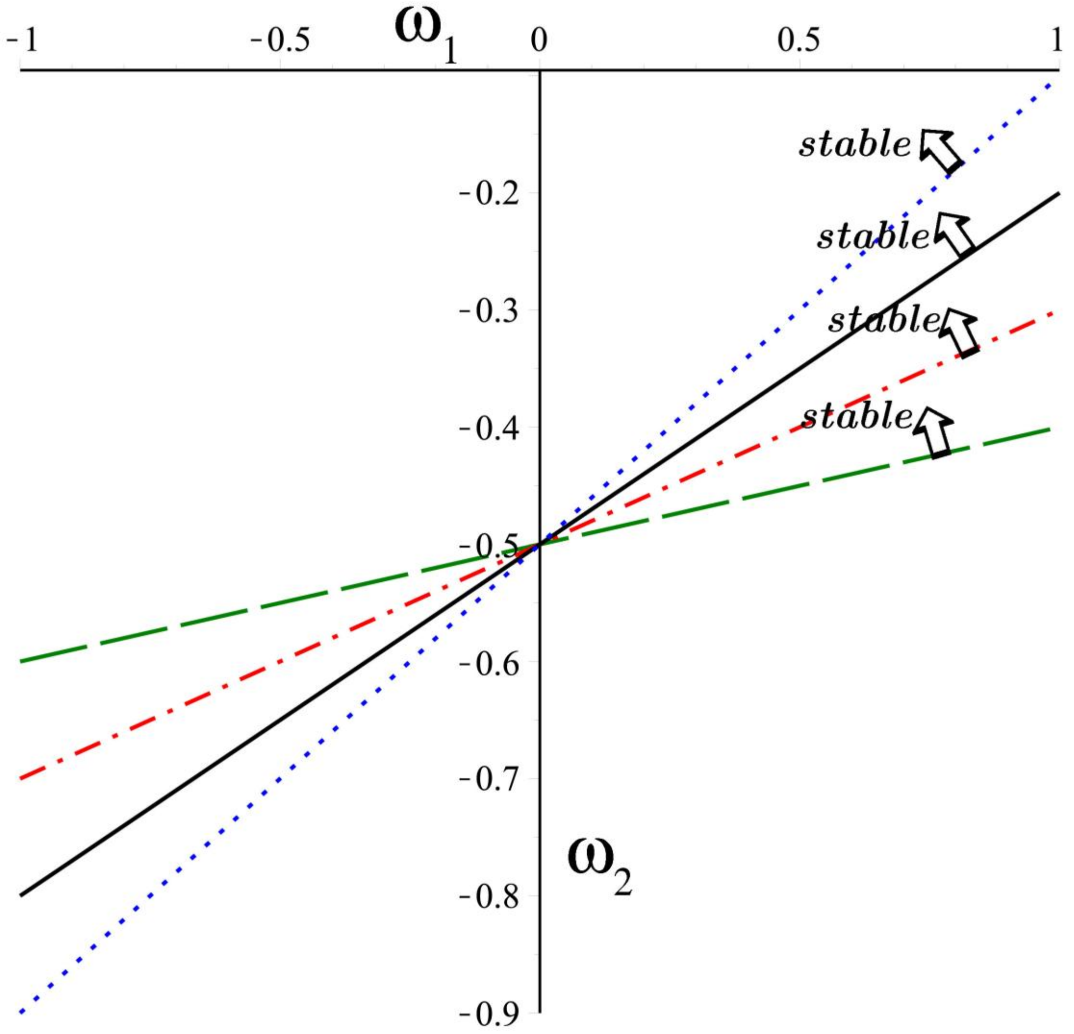

Figure 4.2: A plot of $\omega_2$versus ω_1 for various values of $\chi = 0.1, 0.2, 0.3$ and 0.4.The arrows show the region of stability while the opposite side is the unstable zone for each case.

And therefore the dynamic pressure becomes

$$p = \omega_1(R - R_0) + \omega_2(\sigma - \sigma_0) + p_{0.} \qquad (4.19)$$

This EoS together with Eq. (3.8) gives the differential equation

$$\frac{d\sigma}{dR} + \frac{2}{R}(\omega_1(R - R_0) + \omega_2(\sigma - \sigma_0) + p_0 + \sigma) = 0, \qquad (4.20)$$

which must be satisfied by σ. The solution of this equation is given by

$$\sigma(R) = \frac{\omega_2\sigma_0 - p_0 + \omega_1 R_0}{1+\omega_2} - \frac{2\omega_1 R}{3+2\omega_2} + \frac{C_1}{R^{2(\omega_2+1)}}, \qquad (4.21)$$

in which C_1 is an integration constant. Imposing $\sigma(R_0) = \sigma_0$ yields

$$C_1 = R_0^{2(\omega_2+1)} \left(\frac{\sigma_0+p_0}{1+\omega_2} - \frac{\omega_1 R_0}{3+5\omega_2+2\omega_2^2}\right). \tag{4.22}$$

Finally the closed form of the energy density is found to be

$$\sigma(R) = \frac{\omega_2\sigma_0 - p_0 + \omega_1 R_0}{1+\omega_2} - \frac{2\omega_1 R}{3+2\omega_2} + \left(\frac{R_0}{R}\right)^{2(\omega_2+1)} \left(\frac{\sigma_0+p_0}{1+\omega_2} - \frac{\omega_1 R_0}{3+5\omega_2+2\omega_2^2}\right). \tag{4.23}$$

We note that at $R = R_0$, $\sigma(R)$ reduces to σ_0 and it is a function of R as well as ω_1 and ω_2. The case $\omega_1 = 0$ admits

$$\sigma(R) = \frac{\omega_2\sigma_0 - p_0}{1+\omega_2} + \left(\frac{R_0}{R}\right)^{2(\omega_2+1)} \left(\frac{\sigma_0+p_0}{1+\omega_2}\right), \tag{4.24}$$

while when $\omega_2 = 0$ we find

$$\sigma(R) = -p_0 + \omega_1 R_0 - \frac{2\omega_1 R}{3} + \left(\frac{R_0}{R}\right)^2 \left(\sigma_0 + p_0 - \frac{\omega_1 R_0}{3}\right). \tag{4.25}$$

In the case both ω_1 and ω_2 are set to zero the energy density becomes

$$\sigma(R) = -p_0 + \left(\frac{R_0}{R}\right)^2 (\sigma_0 + p_0), \tag{4.26}$$

while

$$p = p_0 , \tag{4.27}$$

even after the perturbation. According to Fig. 2, this is one of the cases which the thin-shell is stable with

$$V_0'' = \frac{2\Omega}{R_0^2} , \tag{4.28}$$

which is clearly positive.

Chapter 5

CONCLUSION

In electromagnetism it is well-known that when crossing from one region to another the normal component of the electric field suffers a discontinuity if there is a source of charge as surface layer in between. In contrast to the discontinuity of the electric field vector the electric potential is a continuous function at the interface. In Einstein's general relativity we have similar situation: the metric tensor (the first fundamental form) is continuous whereas the extrinsic tensor is discontinuous if the two region are different . The discontinuity conditions were studied first by Israel.

We have studied the formalism known as the "Israel junction formalism" to construct timelike thin-shells in spherically symmetric spacetimes. Our 2+1-dimensional dynamical thin-shell is supported by an energy-momentum tensor which is linked to the discontinuity of the second fundamental form of the thin-shell hyperplane in 3+1-dimensional bulk. We analyzed very deeply the stability of the thin-shell against a radial perturbation and by a linearized approximation we found a general condition to be satisfied in order for having a stable spherically symmetric thin-shell. We applied our results to two explicit examples with certain EoS on the shell numerically as well as analytically to provide the stability regions.

Finally, we would like to state that the problem of thin-shells is not restricted only by spherical symmetry. Similar analysis can be carried out for cylindrical and planar symmetric geometries, for instance. These all are among our further projects to be considered seriously.

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