On q-Analogue and (p,q)–Analogue of Gamma and Beta Functions

Areen Saber Al-Khateeb

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Prof. Dr. Mustafa Tümer
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in mathematics

Prof. Dr. Nazım Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of mathematics.

Prof. Dr. Nazım Mahmudov
Supervisor

Examining Committee

1. Prof. Dr. Nazım Mahmudov
2. Prof. Dr. Sonuç Zorlu Oğurlu
3. Asst. Prof. Dr. Mustafa Kara

ABSTRACT

In this study we discuss the integral representation of the q-analogue of two special functions, q-GF (gamma function) and q-BF (beta function). This discussion gives a very attractive q-constant.Also, we get the proof of the famous Jacobi triple product which contains the identity of Jacobi .After that we obtain a new proof for Ramanujan's equation. Furthermore, we introduce a new generalization of gamma function and beta function that are: the (p, q)-GF and the (p, q)-BF. Finally, we obtain an equivalent definitions for (p, q)-analogue for GF and BF.

Keywords: q-Gamma Function, q-Beta Function, (p, q)-Gamma Function, (p, q)-Beta Function.

ÖZ

Bu çalışmada, iki özel fonksiyon, q-gama fonksiyonu ve q-beta fonksiyonunun integral formu tartışılmaktadır. Ayrıca, (p, q) -gamma fonksiyonun ve (p, q) -beta fonksiyonun yeni bir gama fonksiyonu ve beta fonksiyonu genellemesi elde edilmiştir. Son olarak, gama ve beta fonksiyonları için (p, q) -analog için eşdeğer tanımlar verilmiştir.

Anahtar kelimeler: q-gama fonksiyonu, q-beta fonksiyonunun, (p, q) -gamma fonksiyonun, (p, q) -beta fonksiyonun.

To my husband, Son, daughter, and parents

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Chapter 1

INTRODUCTION

Thomae introduced the q–analogue for Euler's GF, $\Gamma_q(n)$, by the following formula

$$\Gamma_q(n) = \frac{(1-q)_q^{n-1}}{(1-q)^{n-1}}, \qquad 0 < q < 1, \quad t > 0 \,, \tag{1.1}$$

also, Jackson gave such a representation for $\Gamma_q(n)$

Here and in the next parts of the study we consider the following equations

$$(s+t)_q^n = (s+t)(s+qt)\ldots(s+q^{n-1}t) = \prod_{k=0}^{n-1}(s+q^k t), \text{ if } n \in \mathbb{Z}_+. \tag{1.2}$$

$$(1+s)_q^\infty = \prod_{m=0}^{\infty}(1+q^m s). \tag{1.3}$$

$$(1+s)_q^n = \frac{(1+s)_q^\infty}{(1+q^n s)_q^\infty}, \quad \text{if } n \in \mathbb{C}. \tag{1.4}$$

Note that, the infinite product (1.3) is convergent under our assumptions on q. Also, the formulas (1.2) and (1.4) are consistent.

The authors sometimes avoided using the q–integral representation of the q-GF despite what has been written about q-GF and its applications is more extensive.

Actually, it is not totally right to use q–integral representation as a rule. Here, we introduce the first correct integral representation of $\Gamma_q(n)$ is

$$\Gamma_q(n)=\int_0^{1/1-q} x^{n-1}\mathrm{E}_q^{-qx} d_q x. \tag{1.5}$$

Now, we want to introduce the two types of q–analogue of the EF (Exponential Function) are

$$\mathrm{E}_q^x=\sum_{j=0}^{\infty} q^{j(j-1)/2}\frac{x^j}{[j]!}=(1+(1-q)x)_q^{\infty}. \tag{1.6}$$

$$e_q^x=\sum_{j=0}^{\infty}\frac{x^j}{[j]!}=\frac{1}{(1-(1-q)x)_q^{\infty}}. \tag{1.7}$$

The q–integral is given by the following notation

$$\int_0^b f(x)d_q x=(1-q)\sum_{n=0}^{\infty} bq^n f(bq^n), \tag{1.8}$$

The q–BF was defined by Jackson and Thomae as

$$\mathrm{B}_q(n,t)=\frac{\Gamma_q(n)\Gamma_q(t)}{\Gamma_q(n+t)}. \tag{1.9}$$

and the q–integral representation of BF which is given by Euler's is

$$\mathrm{B}_q(n,t)=\int_0^1 x^{n-1}(1-qx)_q^{t-1} d_q x, \; if \;\; n,t>0 \tag{1.10}$$

but Jackson gave a different q–analogue of integral representation for BF which defined by

$$\mathrm{B}(n,t)=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{n+t}}dx. \tag{1.11}$$

Next, we present a q–integral representation of $\Gamma_q(n)$ depend on the q-EF e_q^x, and present a q–integral representation of $\mathrm{B}_q(n,t)$ for (1.11). Futhermore, the two representations are depend on the following important function

$$\mathrm{K}(z,n) = \frac{z^n}{1+z}\left(1+\frac{1}{z}\right)_q^n (1+z)_q^{1-n}, \tag{1.12}$$

where this is a q–constant function in z, it means that

$$\mathrm{K}(qz,n) = \mathrm{K}(z,n).$$

Moreover, for any n is an integer and it must independents on z, and it’s equivalent to $q^{n(n-1)/2}$. On the other hand, if $n \in (0,1)$ this function does depend on z, while for these n one has

$$\lim_{q\to 0} \mathrm{K}(z;n) = z^n + z^{n-1}.$$

Now we define our integral representations as

$$\Gamma_q(n) = \mathrm{K}(A;n)\int_0^{\infty/A(1-q)} x^{n-1} e_q^{-x} d_q x. \tag{1.13}$$

and

$$\mathrm{B}_q(n,t) = \mathrm{K}(A;n)\int_0^{\infty/A} x^{n-1} \frac{1}{(1+x)_q^{n+t}} d_q x. \tag{1.14}$$

The improper integral is described by

$$\int_0^{\infty/A} g(z) d_q z = (1-q)\sum_{k\in\mathbb{Z}} q^k \frac{1}{A} g\left(\frac{q^k}{A}\right), \tag{1.15}$$

The integrals in the two formulas depend on A when $\mathrm{K}(A,n)$ dependents on A. Jackson replaced "$\mathrm{K}(A,n)$" by "$q^{n(n-1)/2}$" in his formula which is acceptable only for an integer n.

Remark: The integral representation of $\Gamma_q(n)$ can be written also by using the improper integral (when $\mathrm{E}_q^{\frac{-q^t}{1-q}} = 0$ for $t \le 0$) as

$$\Gamma_q(n) = \int_0^{\infty/(1-q)} x^{n-1} \mathrm{E}_q^{-qx} d_q x, \tag{1.16}$$

In Chapter 3, we obtain the integral representation of q–BF by using formula(1.14) which is obviously symmetric in n and t .Also, we will get a q-analogue of TI (translation invariance) for some improper integrals. Also, we want to prove formula (1.13) which is corresponding to a family of triple product identities

$$\left(1-\frac{q}{z}\right)_q^{\infty} (1-z)_q^{\infty} (1-q)_q^{\infty} = \sum_{k=-\infty}^{\infty} q^{k(k-1)/2} (-1)^k z^k. \tag{1.17}$$

and formula (1.14) is corresponding to Ramanujan's identity

$$\sum_{k=-\infty}^{\infty} \frac{(1-a)_q^k}{(1-b)_q^k} x^k = \frac{(1-q)_q^{\infty} \left(1-\frac{b}{a}\right)_q^{\infty} (1-ax)_q^{\infty} (1-q/ax)_q^{\infty}}{(1-b)_q^{\infty} (1-q/a)_q^{\infty} (1-x)_q^{\infty} (1-b/ax)_q^{\infty}}. \tag{1.18}$$

Also, we want to show that the following identity is corresponding to the symmetric integral representation of the q –BF

$$\sum_{k=-\infty}^{\infty} \frac{(1-a)_q^k (1-q/a)_q^{-k}}{(1-b)_q^k (1-c)_q^{-k}} = \frac{(1-a)_q^{\infty} (1-q/a)_q^{\infty} (1-q)_q^{\infty} (1-bc/q)_q^{\infty}}{(1-b)_q^{\infty} (1-c)_q^{\infty} (1-b/a)_q^{\infty} (1-ac/q)_q^{\infty}}, \tag{1.19}$$

Finally, we will discuss the(p,q)-analogue of GF and BF and obtain the relation between (p,q)-GF and (p,q)-BF.

Chapter 2

THE q-GAMMA FUNCTION AND THE q-BETA FUNCTION

2.1 Definitions and preliminary results

In this study, we suppose $0 < q < 1$,where q is a fixed number.

Definition 1: The q–derivative of a function f is

$$\left(D_q f\right)(z) = \frac{f(qz) - f(z)}{(q-1)z},$$

Definition 2: The definite integral of Jackson for the function f is

$$\int_0^c f(z)\,d_q z = (1-q)c\sum_{k\geq 0} q^k f\left(cq^k\right),$$

Definition 3: The q–analogue of the product rule is

$$D_q\left(g(z)h(z)\right) = h(z)D_q g(z) + g(qz)D_q h(z),$$

Definition 4: The q-IBP (integration by parts) rule is given by

$$\int_0^b g(z)D_q f(z)\,d_q z = f(b)g(b) - f(0)g(0) - \int_0^b f(qz)D_q g(z)\,d_q z.$$

Definition 5: The Jackson integral in a basic interval $[a,b]$ is given by

$$\int_b^c f(z)d_q z = \int_0^c f(z)d_q z - \int_0^b f(z)d_q z,$$

also, we define the improper integrals as:

$$\int_0^{\infty/A} g(z)d_q z = (1-q)\sum_{k\in\mathbb{Z}} q^k \frac{1}{A} g\left(\frac{q^k}{A}\right), \tag{2.1.1}$$

Remark: Notice that, the series in equation (2.1.1) in order to be convergent; f must be satisfy the following conditions

$|f(z)| < Dz^{\beta}, \forall z \in [0, \varepsilon),$ for some $D > 0, \beta > -1, \varepsilon > 0.$

and

$|f(z)| < Cz^{\alpha}, \forall z \in [N, \infty),$ for some $C > 0, \alpha < -1, N > 0.$

Generally, if the two conditions are achieved, then the value of the summation in (2.1.1) is dependent on the value of the "constant A". To make the integral independent of the q–antiderivative of the function f should be has limits for ($z \to 0$ and $z \to +\infty$).

The following formulas are the reciprocation relations for one of them

$$\int_0^{\infty/A} f(x)d_q x = \int_0^{\infty.A} \frac{1}{x^2} f\left(\frac{1}{x}\right)d_q x, \tag{2.1.2}$$

If $u(x) = \alpha x^{\beta}$, then

$$\int_{u(a)}^{u(b)} f(u)d_q u = \int_a^b f(u(x))D_{q^{1/\beta}} u(x)\, d_{q^{1/\beta}} x,$$

Definition 6: For any positive integer number, the q – analogue of numbers is given by

$$[t]=\frac{1-q^{t}}{1-q}=1+q+...+q^{t-1},$$

Generally, we will refer to $[n]=\frac{1-q^{n}}{1-q}$ until for a non-integer n.

From the definitions (1.2), (1.3) and (1.4) of $(s+t)_q^n$ we get the following lemma

Lemma 2.1.1 Let $n, t, s \in \mathbb{Z}_+$ and $a,b,A,B\in\mathbb{R}$.

1. $D_q z^{s}=[s]z^{s-1}.$
2. $D_q\left(Ax+b\right)_q^{t}=[t]A\left(Ax+b\right)_q^{t-1}$
3. $D_q\left(a+Bx\right)_q^{t}=[t]B\left(a+Bqx\right)_q^{t-1}$
4. $D_q\left(1+Bx\right)_q^{n}=[n]B\left(1+Bqx\right)_q^{n-1}$
5. $D_q\frac{Ax^{t}}{\left(1+Bx\right)_q^{n}}=[t]\frac{Ax^{t-1}}{\left(1+Bx\right)_q^{n+1}}-B\left([n]-[t]\right)\frac{Ax^{t}}{\left(1+Bx\right)_q^{n+1}}$
6. $D_q\frac{\left(1+Ax\right)_q^{k}}{\left(1+Bx\right)_q^{j}}=[k]A\frac{\left(1+Aqx\right)_q^{k-1}}{\left(1+Bqx\right)_q^{j}}-B[j]\frac{\left(1+Ax\right)_q^{k}}{\left(1+Bx\right)_q^{j+1}}$
7. $\left(1+x\right)_q^{k+j}=\left(1+x\right)_q^{k}\left(1+q^{k}x\right)_q^{j}$
8. $\left(1+x\right)_q^{-s}=\frac{1}{\left(1+q^{-s}x\right)_q^{s}}$
9. $\left(1+q^{t}x\right)_q^{n}=\frac{\left(1+x\right)_q^{n+t}}{\left(1+x\right)_q^{t}}=\frac{\left(1+q^{n}x\right)_q^{t}}{\left(1+x\right)_q^{t}}\left(1+x\right)_q^{n},$

10. $\left(1+q^{-n}x\right)_q^s = \dfrac{\left(x+q\right)_q^n}{\left(q^s x+q\right)_q^n}\left(1+x\right)_q^s$

Proof (1):

$$D_q z^s = \frac{f\left(qz\right)-f\left(z\right)}{\left(q-1\right)z} = \frac{q^s z^s - z^s}{\left(q-1\right)z},$$

$$= \frac{z^s\left(q^s-1\right)}{z\left(q-1\right)} = z^{s-1}\frac{\left(q^s-1\right)}{\left(q-1\right)},$$

$$= \left[s\right]_q z^{s-1}.$$

Proof (5):

$$D_q \frac{Ax^t}{\left(1+Bx\right)_q^n} = \frac{\left(1+Bqx\right)_q^n D_q\left(Ax^t\right)-\left(Aq^t x^t\right)D_q\left(1+Bx\right)_q^n}{\left(1+Bx\right)_q^n\left(1+Bqx\right)_q^n},$$

$$= \frac{\left[t\right]Ax^{t-1}\left(1+Bqx\right)_q^n}{\left(1+Bx\right)_q^n\left(1+Bqx\right)_q^n} - \frac{\left[n\right]BAq^t x^t\left(1+Bqx\right)_q^{n-1}}{\left(1+Bx\right)_q^n\left(1+Bqx\right)_q^n},$$

$$= \frac{\left[t\right]Ax^{t-1}}{\left(1+Bx\right)_q^n} - \frac{\left[n\right]BAq^t x^t\left(1+Bqx\right)_q^{n-1}}{\left(1+Bx\right)_q^n\left(1+Bqx\right)_q^{n-1}\left(1+Bq^n x\right)},$$

$$= \left[t\right]\frac{\left(1+Bq^n x\right)Ax^{t-1}}{\left(1+Bq^n x\right)\left(1+Bx\right)_q^n} - \left[n\right]\frac{BAq^t x^t}{\left(1+Bx\right)_q^{n+1}},$$

$$= \frac{\left[t\right]Ax^{t-1}\left(1+Bq^n x\right)-\left[n\right]BAq^t x^t}{\left(1+Bx\right)_q^{n+1}},$$

$$= \frac{Ax^{t-1}}{\left(1+Bx\right)_q^{n+1}}\left[t\right]-\left(\left[t\right]q^n-\left[n\right]q^t\right)\frac{ABx^t}{\left(1+Bx\right)_q^{n+1}},$$

$$=\frac{1}{(1+Bx)_q^{n+1}}Ax^{t-1}[t]-BAx^t([n]-[t])\frac{1}{(1+Bx)_q^{n+1}}.$$

Where: $[t]q^n-[n]q^t=\frac{q^t-1}{q-1}q^n-\frac{q^n-1}{q-1}q^t,$

$$=\frac{q^{n+t}-q^n-q^{n+t}+q^t}{q-1}=\frac{(q^t-1)-(q^n-1)}{q-1}=[t]-[n],$$

Proof (6):

$$D_q(1+Ax)_q^k\frac{1}{(1+Bx)_q^j}=\frac{(1+Bx)_q^j D_q(1+Ax)_q^k-(1+Ax)_q^k D_q(1+Bx)_q^j}{(1+Bqx)_q^j(1+Bx)_q^j},$$

$$=\frac{A[k](1+Aqx)_q^{k-1}(1+Bx)_q^j-B[j](1+Bqx)_q^{j-1}(1+Ax)_q^k}{(1+Bx)_q^j(1+Bqx)_q^j},$$

$$=\frac{[k]A(1+Aqx)_q^{k-1}}{(1+Bqx)_q^j}-\frac{[j]B(1+Bqx)_q^{j-1}(1+Ax)_q^k}{(1+Bx)_q^j(1+Bqx)_q^{j-1+1}},$$

$$=\frac{[k]A(1+Aqx)_q^{k-1}}{(1+Bqx)_q^j}-\frac{[j]B(1+Ax)_q^k}{(1+Bx)_q^{j+1}}.$$

Proof (7):

$$(1+x)_q^{k+j}=(1+x)(1+qx)\ldots(1+q^{k-1}x)(1+q^kx)(1+q^{k+1}x)\ldots(1+q^{k+j-1}x),$$

$$=(1+x)_q^k(1+q^kx)(1+qq^kx)\ldots(1+q^{j-1}q^kx),$$

$$=(1+x)_q^k(1+q^kx)_q^j,$$

We defined the q–analogues of the EF in chapter (1) .We proved the equivalence between the infinite product expansion and the SE (Series Expansion) of e_q^x and E_q^x

(where the two expansions converge in the domain) after we take the limit in the Heine's and Gauss q–binomial formulas for $n \to \infty$.

Lemma 2.1.2 The q–EF properties are given by

(1) (i) $D_q e_q^x = e_q^x$,

Proof:

$$D_q e_q^x = \sum_{m=0}^{\infty} D_q x^m \frac{1}{[m]_q!} = \sum_{m=1}^{\infty} [m]_q \, x^{m-1} \frac{1}{[m]_q [m-1]_q!},$$

$$= \sum_{m=1}^{\infty} x^{m-1} \frac{1}{[m-1]_q!} = \sum_{m=0}^{\infty} x^m \frac{1}{[m]_q!} = e_q^x.$$

(ii) $D_q \mathrm{E}_q^x = \mathrm{E}_q^{qx}$.

Proof:

$$D_q \mathrm{E}_q^x = \sum_{m=0}^{\infty} q^{m(m-1)/2} D_q x^m \frac{1}{[m]_q!} = \sum_{m=1}^{\infty} q^{m(m-1)/2} x^{m-1} \frac{1}{[m-1]_q!},$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} x^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{q^n x^n}{[n]_q!},$$

$$= \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{(qx)^n}{[n]_q!} = \mathrm{E}_q^{qx}.$$

(2) $e_q^x \mathrm{E}_q^{-x} = \mathrm{E}_q^x e_q^{-x} = 1$,

Proof:

We know: $e_q^x = \dfrac{1}{\left(1-(1-q)x\right)_q^{\infty}}$,

and: $\mathrm{E}_q^{-x} = \left(1-(1-q)x\right)_q^{\infty}$, Then from these we get: $e_q^x \mathrm{E}_q^{-x} = 1.$

2.2 The Definitions of *q*–Gamma Function and *q*–Beta Function

Definition 1: For $n, s > 0$, the GF and BF which introduced by Euler are defined in the following formulas

$$\Gamma(n) = \int_0^\infty z^{n-1} e^{-z} dz. \tag{2.2.1}$$

$$\mathrm{B}(n,s) = \int_0^1 z^{n-1} (1-z)^{s-1} dz, \tag{2.2.2}$$

$$\mathrm{B}(n,s) = \int_0^\infty \frac{z^{n-1}}{(1+z)^{n+s}} dz. \tag{2.2.3}$$

Note that (2.2.3) comes from (2.2.2) after changing the variable $z = 1/(1+y)$. $\mathrm{B}(n,s)$ is symmetric in n and s we can see that clearly from the equation (2.2.2).

The most important properties of these functions are

$$\Gamma(s+1) = s\Gamma(s),\ \Gamma(1) = 1 \tag{2.2.4}$$

$$\mathrm{B}(s,n) = \Gamma(s)\Gamma(n)/\Gamma(s+n). \tag{2.2.5}$$

We are concerned in the q–analogue of these functions in this study. They are clarified in the next definitions.

Definition 2 : (i) For $s > 0,$, the q–GF is given by the following equation

$$\Gamma_q(s) = \int_0^{1/(1-q)} z^{n-1} \mathrm{E}_q^{-qx} d_q z, \tag{2.2.6}$$

(ii) The q–BF for $(n, s > 0)$is given by

$$\mathrm{B}_q(n,s) = \int_0^1 z^{n-1} (1-qz)_q^{s-1} d_q z, \tag{2.2.7}$$

where $\mathrm{B}_q(n,s)$ is the q-analogue of BF and $\Gamma_q(s)$ is the q–analogues of the GF, if $q \to 1$,then they reduce to $\mathrm{B}(n,s)$ and $\Gamma(s)$ in that order and they satisfy the properties analogues to the equations (2.2.4) and (2.2.5). This is explained in the next theorem.

Theorem 2.2.1: (a) The relation between q-analogue for GF and BF are stated in the following two formulas

$$\Gamma_q(s) = \frac{\mathrm{B}_q(s,\infty)}{(1-q)^n}, \tag{2.2.8}$$

$$\mathrm{B}_q(n,s) = \frac{\Gamma_q(s)\Gamma_q(n)}{\Gamma_q(s+n)}, \tag{2.2.9}$$

(b) $\Gamma_q(s)$ can be expressed also as

$$\Gamma_q(s) = (1-q)_q^{s-1}\frac{1}{(1-q)^{s-1}}. \tag{2.2.10}$$

specifically one has

$$\Gamma_q(1+s) = [s]_q \Gamma_q(s), \forall s > 0, \Gamma_q(1) = 1.$$

By using the definition of the q-GF we want to prove this property as

$$\Gamma_q(1+s) = \int_0^{1/(1-q)} x^s \mathrm{E}_q^{-qx} d_q x,$$

$$= -\int_0^{1/1-q} x^s d_q \mathrm{E}_q^{-x} = -x^s \mathrm{E}_q^{-x} \Big|_{x=0}^{x=1/1-q} + [s]\int_0^{1/1-q} x^{s-1}\mathrm{E}_q^{-qx} d_q x,$$

Since, $\mathrm{E}_q^0 = 1, \mathrm{E}_q^{-1/1-q} = 0$, we have

$$\Gamma_q(s+1) = [s]\int_0^{1/1-q} x^{s-1}\mathrm{E}_q^{-qx} d_q x = [s]\Gamma_q(s),$$

and

$$\Gamma_q(1)=\int_0^{\frac{1}{1-q}} \mathrm{E}_q(-qx)\,d_q x=\mathrm{E}_q(0)-\mathrm{E}_q\left(\frac{-1}{1-q}\right)=1.$$

Proof: a-(2.2.8)

$$\mathrm{B}_q(n,\infty)=\int_0^1 x^{n-1}(1-qx)_q^{\infty}\,d_q x$$

Now by formula

$$\mathrm{E}_q^x=(1+(1-q)x)_q^{\infty}$$

We have:

$$\mathrm{B}_q(n,\infty)=\int_0^1 x^{n-1}\mathrm{E}_q^{\frac{-qx}{(1-q)}}d_q x$$

Let $x=(1-q)y$ then $d_q x=(1-q)d_q y$

$$\mathrm{B}_q(n,\infty)=(1-q)^{n-1}\int_0^{1/1-q}(y)^{n-1}\mathrm{E}_q^{-qy}(1-q)\,d_q y,$$

$$=(1-q)^n\int_0^{1/1-q} y^{n-1}\mathrm{E}_q^{-qy}d_q y,$$

$$=(1-q)^n\,\Gamma_q(n).$$

We derive two recurrence relations for $\mathrm{B}_q(n,t)$ from q-IBP and using lemma 2.1.1, we have

For any $n,t>0$:

$$\text{(i)}\,\mathrm{B}_q(n+1,t)=\mathrm{B}_q(n,t+1)\frac{[n]}{[t]},$$

Proof:

$$\mathrm{B}_q(n+1,t)=\int_0^1 x^n(1-qx)_q^{t-1}\,d_q x,$$

Let $f(x)=x^n$, $g(x)=(1-x)_q^t$, $d_q g(x)=-[t](1-qx)_q^{t-1}$, then apply q-IBP

$$-[t]\int_0^1 x^n (1-qx)_q^{t-1}\, d_q x = -[n]\int_0^1 (1-qx)_q^{t} x^{n-1} d_q x$$

$$\int_0^1 x^n (1-qx)_q^{t-1}\, d_q x = \mathrm{B}_q(n,t+1)\frac{[n]}{[t]},$$

$$\text{(ii)}\ \mathrm{B}_q(n,t+1) = -q^t \mathrm{B}_q(n+1,t) + \mathrm{B}_q(n,t),$$

Proof:

$$\mathrm{B}_q(n,t+1) = \int_0^1 x^{n-1}(1-qx)_q^{t}\, d_q x,$$

$$= \int_0^1 x^{n-1}(1-qx)_q^{t-1+1}\, d_q x = \int_0^1 x^{n-1}(1-qx)_q^{t-1}\left(1-q^t x\right) d_q x,$$

$$= \int_0^1 x^{n-1}(1-qx)_q^{t-1}\, d_q x - q^t \int_0^1 x^{n}(1-qx)_q^{t-1}\, d_q x,$$

$$= -q^t \mathrm{B}_q(n+1,t) + \mathrm{B}_q(n,t),$$

Now from (i) and (ii) we obtain

$$\mathrm{B}_q(n,t+1) = \mathrm{B}_q(n,t)\frac{[t]}{[t+n]}.$$

Proof:

$$\mathrm{B}_q(n,t+1) = -q^t \mathrm{B}_q(n+1,t) + \mathrm{B}_q(n,t),$$

$$= -q^t \frac{[n]}{[t]}\mathrm{B}_q(n,t+1) + \mathrm{B}_q(n,t),$$

$$\mathrm{B}_q(n,t+1) + \frac{[n]}{[t]} q^t \mathrm{B}_q(n,t+1) = \mathrm{B}_q(n,t),$$

$$\mathrm{B}_q(n,t+1)\left(1 + \frac{[n]}{[t]} q^t\right) = \mathrm{B}_q(n,t),$$

$$\mathrm{B}_q(n,t+1)=\mathrm{B}_q(n,t)\frac{[t]}{[t]+q^t[n]},$$

$$=\mathrm{B}_q(n,t)\frac{\left(\frac{q^t-1}{q-1}\right)}{\left(\frac{q^t-1}{q-1}+\frac{q^nq^t-q^t}{q-1}\right)},$$

$$=\mathrm{B}_q(n,t)\frac{\left(\frac{q^t-1}{q-1}\right)}{\left(\frac{q^{t+n}-1}{q-1}\right)}=\mathrm{B}_q(n,t)\frac{[t]}{[t+n]}.$$

Since $\mathrm{B}_q(n,1)=\int_0^1 x^{n-1}d_qx=\frac{1}{[n]},$

and

$$\mathrm{B}_q(n,2)=\frac{[1]}{[1+n]}\mathrm{B}_q(n,1)=\frac{[1]}{[n]}\frac{[1]}{[1+n]},$$

We get

For $n>0$ any positive integer t

$$\mathrm{B}_q(n,t)=\frac{[1][2]...[t-1]}{[n][n+1]...[n+t-1]}=\frac{\left(\frac{q-1}{q-1}\right)\left(\frac{qq-1}{q-1}\right)...\left(\frac{qq^{t-1}-1}{q-1}\right)}{\left(\frac{q^n-1}{q-1}\right)\left(\frac{q^nq-1}{q-1}\right)...\left(\frac{q^nq^{t-1}-1}{q-1}\right)}$$

$$=\frac{(1-q)_q^{t-1}/(1-q)^{t-1}}{(1-q^n)_q^t/(1-q)^t}=(1-q)\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}, \tag{2.2.11}$$

where $(1-q)_q^{n+t-1}=(1-q)_q^{n-1}(1-q^{n-1}q)_q^t$ and $(1-q^n)_q^t=\frac{(1-q)_q^{n+t-1}}{(1-q)_q^{n-1}},$

When we take the limit for $t\to\infty$ in the expression (2.2.11) we obtain

$$\mathrm{B}_q(n,\infty)=(1-q)_q^{n-1}(1-q).$$

We want to prove (b-(2.2.10))

$$\Gamma_q(s)=\frac{B_q(s,\infty)}{(1-q)^s}=\frac{(1-q)(1-q)_q^{s-1}}{(1-q)^s}=\frac{(1-q)_q^{s-1}}{(1-q)^{s-1}},$$

We are left to prove $B_q(n,t)=\frac{\Gamma_q(n)\Gamma_q(t)}{\Gamma_q(n+t)}$, (is true for any positive integer value t)

$$B_q(n,t)=(1-q)\frac{(1-q)_q^{n-1}(1-q)_q^{t-1}}{(1-q)_q^{n+t-1}}\cdot\frac{(1-q)^{n-1}}{(1-q)^{n-1}}\cdot\frac{(1-q)^{t-1}}{(1-q)^{t-1}},$$

$$=(1-q)_q^{t-1}\frac{1}{(1-q)^{t-1}}(1-q)_q^{n-1}\frac{1}{(1-q)^{n-1}}(1-q)^{n+t-1}\frac{1}{(1-q)_q^{n+t-1}},$$

$$=\frac{1}{\Gamma_q(t+n)}\Gamma_q(t)\Gamma_q(n).$$

The left hand side of (2.2.9) can be written like

$$B_q(n,t)=\int_0^1 x^{n-1}(1-qx)_q^{t-1}d_qx,$$

$$=\int_0^1 x^{n-1}\frac{(1-qx)_q^{\infty}}{(1-q^tx)_q^{\infty}}d_qx,$$

Now, by apply q-integral formula we get

$$\int_0^1(1-qx)_q^{\infty}\frac{1}{(1-q^tx)_q^{\infty}}x^{n-1}d_qx=(1-q)\sum_{k=0}^{\infty}(1-qq^k)_q^{\infty}q^{nk}\frac{1}{(1-q^tq^k)_q^{\infty}},$$

$$=(1-q)\sum_{k=0}^{\infty}(1-q^{k+1})_q^{\infty}b^k\frac{1}{(1-cq^k)_q^{\infty}},$$ (where $b=q^n, c=q^t$)

and the right hand side of (2.2.9) as

$$\frac{\Gamma_q(n)\Gamma_q(t)}{\Gamma_q(n+t)}=\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}(1-q)=\frac{(1-q)_q^{\infty}(1-q)_q^{\infty}\left(1-qq^{n+t-1}\right)_q^{\infty}}{\left(1-qq^{t-1}\right)_q^{\infty}\left(1-qq^{n-1}\right)_q^{\infty}(1-q)_q^{\infty}}(1-q),$$

$$=\frac{1}{\left(1-q^t\right)_q^{\infty}\left(1-q^n\right)_q^{\infty}}(1-q)(1-q)_q^{\infty}\left(1-q^{n+t}\right)_q^{\infty}, \text{Let } (d=q^n, c=q^t)$$

$$=\frac{1}{(1-c)_q^{\infty}(1-d)_q^{\infty}}(1-cd)_q^{\infty}(1-q)(1-q)_q^{\infty}.$$

It can be considered the two equations as FPS (formal power series) in q with CRF (coefficients rational functions) in c and d.□

2.3 The analogous definition for q–Gamma and q-Beta Functions

We obtained the definition of $\Gamma_q(s)$ in the former section from equation (2.2.1) by replacing the integral with the function e^{-x} with its q–analogue E_q^{-qx} and Jackson integral.

Now, we will explain a new function in the following way where $(A>0)$

$$\gamma_q^{(A)}(n)=\int_0^{\infty/A(1-q)} x^{n-1}e_q^{-x}d_qx. \tag{2.3.1}$$

After we take the q–analogue of the integral equation (2.2.2) we obtained the function $\mathrm{B}_q(n,t)$ We will to explain the q–analogue for the integral term in equation (2.2.3). Hence we define it as

$$\beta_q^{(A)}(n,t)=\int_0^{\infty/A}\frac{x^{n-1}}{(1+x)_q^{n+t}}d_qx. \tag{2.3.2}$$

In this part, we want to show that the functions $\gamma_q^{(A)}(n)$ and $\beta_q^{(A)}(n,t)$ are related to q-analogue of GF and BF in that order .We will to use the reasoning in the results of theorem (2.2.1) to $\gamma_q^{(A)}(n)$ and $\beta_q^{(A)}(n,t)$.Firstly, in the definition of $\beta_q^{(A)}(n,t)$ take the limit $t \to \infty$ and use the infinite product expansion of e_q^x ,then replace x by $y(1-q)$.We obtain

$$\beta_q^{(A)}(n,\infty) = \int_0^{\infty/A} \frac{x^{n-1}}{(1+x)_q^\infty} d_q x = \int_0^{\infty/A} x^{n-1} e_q^{-x/1-q} d_q x,$$

(where $\frac{1}{(1+x)_q^\infty} = e_q^{-x/1-q}$ is the infinite product expansion of e_q^x)

Now change the variable $x = (1-q)y,\ d_q x = (1-q) d_q y$ in the right- hand side we get:

$$\beta_q^{(A)}(n,\infty) = (1-q)\int_0^{\infty/A(1-q)} e_q^{-y} \left((1-q)y\right)^{n-1} d_q y = (1-q)\int_0^{\infty/A(1-q)} e_q^{-y} y^{n-1} (1-q)^{n-1} d_q y,$$

$$= (1-q)^n \int_0^{\infty/A(1-q)} e_q^{-y} y^{n-1} d_q y,$$

$$= (1-q)^n \gamma_q^{(A)}(n).$$

we therefore proved

$$\gamma_q^{(A)}(n) = \frac{1}{(1-q)^n} \beta_q^{(A)}(n,\infty), \qquad (2.3.3)$$

Here we want to obtain the recursive relations for $\gamma_q^{(A)}(n)$ and $\beta_q^{(A)}(n,t)$

$\gamma_q^{(A)}(n+1) = \int_0^{\infty/A(1-q)} x^n e_q^{-x} d_q x.$ (Now apply q-IBP)

Let $f(x) = x^n, d_q g(x) = e_q^{-x} d_q x$ then $g(x) = -e_q^{-x}$

$\int_0^{\infty/A(1-q)} x^n e_q^{-x} d_q x = -[n]\int_0^{\infty/A(1-q)} -e_q^{-qx} x^{n-1} d_q x,$ (let $y = qx, d_q y = q d_q x$)

$$= [n]\int_0^{\infty/A(1-q)} \left(\frac{y}{q}\right)^{n-1} e_q^{-y} \frac{d_q y}{q}$$

$$= \frac{[n]}{q^n} \int_0^{\infty/A(1-q)} y^{n-1} e_q^{-y} d_q y,$$

$$= [n] q^{-n} \gamma_q^{(A)}(n).$$

we get: $\tilde{\gamma}_q^{(A)}(n+1) = q^{-n}[n]\tilde{\gamma}_q^{(A)}(n).$

We used here the fact that $x^n e_q^{-x}$ resort to zero as $x \to 0$ and $x \to +\infty$. Since $\gamma_q^{(A)}(1) = 1$, then we can the following way (for all positive integer t and $A > 0$)

$$q^{t(t-1)/2} \gamma_q^{(A)}(t) = [t-1]! = \Gamma_q(t). \tag{2.3.4}$$

Proof:

Since $\tilde{\gamma}_q^{(A)}(1) = 1.$

then by $\tilde{\gamma}_q^{(A)}(n+1) = [n] q^{-n} \tilde{\gamma}_q^{(A)}(n).$ we get

$$\gamma_q^{(A)}(2) = \frac{1}{q}[1]\gamma_q^{(A)}(1)$$

$$\gamma_q^{(A)}(3) = \frac{1}{q}\frac{1}{q^2}[1][2]\gamma_q^{(A)}(2)$$

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$$\gamma_q^{(A)}(n) = \frac{1}{q^{n(n-1)/2}}[1][2]...[n-1] = \frac{1}{q^{n(n-1)/2}}[n-1]!$$

$$q^{n(n-1)/2} \gamma_q^{(A)}(n) = [n-1]! = \Gamma_q(n).$$

Now consider the function $\beta_q^{(A)}(n,t)$ then we get for $n, t > 0$

$$\beta_q^{(A)}(n+1,t)=\beta_q^{(A)}(n,t)\,q^{-n}\frac{[n]}{[n+t]}.$$

Proof:

$$\beta_q^{(A)}(n+1,t)=\int_0^{\infty/A}\frac{x^n}{(1+x)_q^{n+t+1}}d_qx.$$

$$=\frac{1}{[n+t]}q^{-n}\int_0^{\infty/A}\frac{(qx)^n[n+t]}{(1+x)_q^{n+t+1}}d_qx.$$

$$=-\frac{1}{[n+t]}q^{-n}\int_0^{\infty/A}(qx)^n D_q\frac{1}{(1+x)_q^{n+t}}d_qx.\ \text{ (because } D_q\frac{1}{(1+x)_q^{n+t}}=-\frac{[n+t]}{(1+x)_q^{n+t+1}}\text{)}$$

Now apply q-IBP to find this integral

$$=\frac{1}{[n+t]}q^{-n}\int_0^{\infty/A}\frac{1}{(1+x)_q^{n+t}}D_qx^nd_qx=\frac{1}{[n+t]}q^{-n}\int_0^{\infty/A}[n]\frac{x^{n-1}}{(1+x)_q^{n+t}}d_qx,$$

$$=\beta_q^{(A)}(n,t)q^{-n}\frac{[n]}{[n+t]}. \qquad (2.3.5)$$

For $n=1$ we have

$$\beta_q^{(A)}(1,t)=\int_0^{\infty/A}\frac{1}{(1+x)_q^{t+1}}d_qx=\frac{1}{[t]}. \qquad (2.3.6)$$

Now, for $t>0, n\in\mathbb{Z}_+$ we get

$$q^{n(n-1)/2}\beta_q^{(A)}(n,t)=(1-q)\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{t+n-1}}=\mathrm{B}_q(n,t).$$

Proof:

For $n=1,\beta_q^{(A)}(1,t)=\dfrac{1}{[t]}.$

$$\beta_q^{(A)}(2,t)=q^{-1}\frac{1}{[t]}\frac{1}{[t+1]}.$$

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$$\beta_q^{(A)}(n,t)=\frac{1}{q^{1+2+...+(n-1)}}\frac{1}{[t]}\frac{1}{[t+1]}\frac{2}{[t+2]}\cdots\frac{[n-1]}{[t+n-1]},$$

$$=\frac{1}{q^{n(n-1)/2}}\frac{[n-1]!}{[t+n-1]!}.$$

$$q^{n(n-1)/2}\beta_q^{(A)}(n,t)=\frac{\Gamma_q(n)}{\Gamma_q(t+n)}.$$

$$=\frac{1}{(1-q)^{n-1}(1-q)_q^{t+n-1}}(1-q)_q^{n-1}(1-q)^{t+n-1}.$$

$$=\frac{1}{(1-q)_q^{t+n-1}}(1-q)_q^{n-1}(1-q)_q^{t-1}(1-q)=\beta_q^{(A)}(n,t). \qquad (2.3.7)$$

We also have

$$\beta_q^{(A)}(n,t+1)=\beta_q^{(A)}(n,t)q^t\frac{[t]}{[n+t]}.$$

Proof:

$$\beta_q^{(A)}(n,t+1)=\int_0^{\infty/A}x^{n-1}\frac{1}{(1+x)_q^{n+t+1}}d_qx,$$

$$=\frac{q^t}{[n+t]}\int_0^{\infty/A}\frac{1}{(qx)^t}[n+t]x^{n+t-1}\frac{1}{(1+x)_q^{n+t+1}}d_qx,$$

$$=\frac{q^t}{[n+t]}\int_0^{\infty/A}\frac{1}{(qx)^t}D_qx^{n+t}\frac{1}{(1+x)_q^{n+t}}d_qx,$$

(where $D_q\frac{x^{n+t}}{(1+x)_q^{n+t}}=[n+t]\frac{x^{n+t-1}}{(1+x)_q^{n+t+1}}$. by lemma 2.1.1 (5))

Now from q-IBP we have

$$=-\frac{1}{[n+t]}q^t\int_0^{\infty/A}x^{n+t}\frac{1}{(1+x)_q^{n+t}}D_q\frac{1}{x^t}d_qx,$$

$$=\frac{[t]}{[n+t]}q^t\int_0^{\infty/A}x^{n+t}\frac{1}{(1+x)_q^{n+t}}x^{-t-1}d_qx,$$

$$=\frac{[t]}{[n+t]}q^t\int_0^{\infty/A}x^{n-1}\frac{1}{(1+x)_q^{n+t}}d_qx,$$

$$=\beta_q^{(A)}(n,t)q^t\frac{[t]}{[n+t]}. \tag{2.3.8}$$

We need to compute $\tilde{\beta}_q^{(A)}(n,1)$

$$\beta_q^{(A)}(n,1)=\int_0^{\infty/A}x^{n-1}\frac{1}{(1+x)_q^{n+1}}d_qx,$$

$$=\frac{1}{[n]}\int_0^{\infty/A}[n]x^{n-1}\frac{1}{(1+x)_q^{n+1}}d_qx,$$

$$=\frac{1}{[n]}\int_0^{\infty/A}D_qx^n\frac{1}{(1+x)_q^{n}}d_qx, \text{(by lemma 2.1.1 (5))} \tag{2.3.9}$$

We have to be careful when we use the FTQC to calculate the right–hand side of (2.3.9), and the limit of the function $F(x)=\frac{x^n}{(1+x)_q^n}$ does not exist when $x\to+\infty$ .In contrast, by definition of Jackson integral and q-derivative, we get

$$\int_0^{\infty/A}D_qF(x)d_qx=\lim_{N\to\infty}F\left(\frac{1}{Aq^N}\right)-\lim_{N\to\infty}F\left(\frac{q^N}{A}\right).$$

We take the limit along the sequence of integer numbers N. Then we obtain from (2.3.9)

$$\beta_q^{(A)}(s,1) = \frac{1}{[s]}\left(\lim_{N\to\infty}\left(Aq^N\right)^s\left(1+\frac{1}{Aq^N}\right)_q^s\right)^{-1}. \tag{2.3.10}$$

Let $\mathrm{K}(A,s)$ is the limit in (2.3.10), and using Lemma 2.1.1 (10) to get

$$\mathrm{K}(A;s) = A^s \lim_{N\to\infty} q^{Ns}\left(1+\frac{q^{-N}}{A}\right)_q^s,$$

$$= A^s\left(1+\frac{1}{A}\right)_q^s \lim_{N\to\infty} q^{Ns}\left(\frac{1}{A}+q\right)_q^N \frac{1}{\left(\frac{q^s}{A}+q\right)_q^N}, \text{ (by using lemma 2.1.1(10))}$$

$$= A^s\left(1+\frac{1}{A}\right)_q^s \lim_{N\to\infty} q^{Ns}\frac{\left(1+qA\right)_q^N}{\left(1+q^{1-s}A\right)_q^N},$$

$$= A^s\left(1+\frac{1}{A}\right)_q^s \frac{\left(1+qA\right)_q^\infty}{\left(1+q^{1-s}A\right)_q^\infty} = A^s\left(1+\frac{1}{A}\right)_q^s \left(1+qA\right)_q^{-s},$$

$$= \frac{1}{(1+A)}A^s\left(1+\frac{1}{A}\right)_q^s\left(1+A\right)_q^{1-s}. \; where \; \left(1+A\right)_q^{1-s} = \left(1+A\right)\left(1+qA\right)_q^{-s}.$$

Now from (2.3.8) and (2.3.9) we can conclude: For all $n>0$ and positive integer t then

$$\mathrm{K}(A;n)\beta_q^{(A)}(n,t) = \frac{\left(1-q\right)_q^{t-1}\left(1-q\right)_q^{n-1}}{\left(1-q\right)_q^{t+n-1}}\left(1-q\right) = \mathrm{B}_q(n,t). \tag{2.3.11}$$

We mention some properties in the next lemma for the function

$$\mathrm{K}(M;n)=\frac{1}{(1+M)}M^{n}\left(1+\frac{1}{M}\right)_{q}^{n}(1+M)_{q}^{1-n}.$$

Lemma 2.3.1 (a) In the limit $q\to 1$ and 0 we have

$$\lim_{q\to 1}\mathrm{K}(M;n)=1,\qquad \forall M,n\in\mathbb{R}.$$

$$\lim_{q\to 1}\mathrm{K}(M;n)=M^{n}+M^{n-1},\qquad \forall n\in(0,1),\ M\in\mathbb{R}.$$

especially when $\mathrm{K}(M,n)$"is not constant in *M*".

(b) $\mathrm{K}(M,n)$ satisfies the following recurrence relation (as a function of n)

$$\mathrm{K}(M;n+1)=q^{n}\mathrm{K}(M;n).$$

Since clearly $\mathrm{K}(M;0)=\mathrm{K}(M;1)=1.$ in particular we have for any positive integer t

$$\mathrm{K}(M;t)=q^{t(t-1)/2}.\text{(by comparing 2.3.7 and 2.3.11)}$$

(c)Viewed as a function of $M,\mathrm{K}(M;n)$ is a "q-constant", that is

$$D_{q}\mathrm{K}(M;n)=0,\qquad \forall n,M\in\mathbb{R}.$$

In other expression $\mathrm{K}(q^{t}M;n)=\mathrm{K}(M;n)$ for all integer t.

Proof (a):

It's obviously, the limit of $\mathrm{K}(M,n)$for $q\to 1$ is equal 1. In the limit $q\to 0$ we have, for any $\beta>0$

$$(1+M)_{q}^{\beta}=\frac{(1+M)_{q}^{\infty}}{(1+qM)_{q}^{\infty}}=(1+M)\frac{(1+qM)_{q}^{\infty}}{(1+q^{\beta}M)_{q}^{\beta}}\ \to\ (1+M).$$

For $n\in(0,1)$ we have

$$\lim_{q\to 0}\mathrm{K}(M;n)=\lim_{q\to 0}\frac{1}{(1+M)}M^{n}\left(1+\frac{1}{M}\right)_q^{n}(1+M)_q^{1-n}.$$

$$=M^{n}\left(1+\frac{1}{M}\right).$$

Proof (b):

$$\mathrm{K}(M;n+1)=\frac{1}{(1+M)}M^{n+1}\left(1+\frac{1}{M}\right)_q^{n+1}(1+M)_q^{-n},$$

$$=\frac{1}{(1+M)}M^{n+1}\left(1+\frac{1}{M}\right)_q^{n}\left(1+\frac{q^{n}}{M}\right)\frac{(1+M)_q^{1-n}}{\left(1+q^{-n}M\right)},\quad \text{(by Lemma 2.1.1 (7))}$$

$$=M\left(1+\frac{q^{n}}{M}\right)\frac{1}{\left(1+q^{-n}M\right)}\frac{1}{(1+M)}M^{n}\left(1+\frac{1}{M}\right)_q^{n}(1+M)_q^{1-n},$$

$$=M\left(1+\frac{q^{n}}{M}\right)\frac{1}{\left(1+q^{-n}M\right)}\mathrm{K}(M;n),$$

$$=\left(\frac{M}{1+q^{-n}M}\right)\left(\frac{1+q^{-n}M}{q^{-n}M}\right)\mathrm{K}(M;n),$$

$$=q^{n}\mathrm{K}(M;n).$$

Proof (c):

We want to prove $\mathrm{K}(qM;n)=\mathrm{K}(M;n)$

$$\mathrm{K}(qM;n)=(qM)^{n}(1+qM)_q^{1-n}\frac{1}{(1+qM)}\left(1+\frac{1}{qM}\right)_q^{n},$$

Now we can use lemma 2.1.1 (9 and 10) to find: $\left(1+\frac{1}{qM}\right)_q^{n}$ and $(1+qM)_q^{1-n}$

$$\left(1+\frac{1}{qM}\right)_q^n = \frac{1}{\left(\frac{q^n}{M}+q\right)}\left(\frac{1}{M}+q\right)\left(1+\frac{1}{M}\right)_q^n,$$

$$=\frac{1}{\left(1+\frac{q^n}{qM}\right)}\left(1+\frac{1}{qM}\right)\left(1+\frac{1}{M}\right)_q^n.$$

$$\left(1+qM\right)_q^{1-n} = \frac{1}{\left(1+M\right)}\left(1+q^{1-n}M\right)\left(1+M\right)_q^{1-n}.$$

$$\mathrm{K}\left(qM;n\right) = \left(qM\right)^n \frac{1}{\left(1+qM\right)}\frac{1}{\left(\frac{q^n}{M}+q\right)}\frac{\left(1+q^{1-n}M\right)}{\left(1+M\right)}\left(1+M\right)_q^{1-n}\left(\frac{1}{M}+q\right)\left(1+\frac{1}{M}\right)_q^n,$$

$$=\frac{\left(\frac{1}{M}+q\right)\left(1+q^{1-n}M\right)}{\left(1+qM\right)\left(\frac{q^n}{M}+q\right)}q^n\left(1+M\right)_q^{1-n}M^n\left(1+\frac{1}{M}\right)_q^n\frac{1}{\left(1+M\right)}.$$

$$=\frac{\left(1+\frac{1}{qM}\right)\left(1+q^{1-n}M\right)}{\left(1+qM\right)\left(1+\frac{1}{q^{1-n}M}\right)}q^n\mathrm{K}\left(M;n\right).$$

but $\dfrac{q^n\left(1+\frac{1}{qM}\right)\left(1+q^{1-n}M\right)}{\left(1+qM\right)\left(1+\frac{1}{q^{1-n}M}\right)}=1,$ so $\mathrm{K}\left(qM;n\right)=\mathrm{K}\left(M;n\right).$

We conclude from (2.3.7), (2.3.11) and Lemma 2.3.1 the functions $\mathrm{K}\left(M;n\right)\beta_q^{(A)}\left(n,t\right)$ and $\mathrm{B}_q\left(n,t\right)$ "coincide" for all $M>0$ when either n or t is a non negative integer. Now we will to show that they are indeed "coincide" for any $n,t>0$.

Theorem 2.3.1. We have for all $A,n,t>0$

$$\text{(a)}\,\mathrm{K}(A;n)\gamma_q^{(A)}(n)=\Gamma_q(n), \tag{2.3.12}$$

$$\text{(b)}\,\mathrm{K}(A;n)\beta_q^{(A)}(n,t)=\mathrm{B}_q(n,t), \tag{2.3.13}$$

Proof (a):

$$\gamma_q^{(A)}(n)=\beta_q^{(A)}(n,\infty)\frac{1}{(1-q)^n},$$

Now multiply both sides by $\mathrm{K}(A;n)$ then we get

$$\gamma_q^{(A)}(n)\mathrm{K}(A;n)=\frac{\beta_q^{(A)}(n,\infty)}{(1-q)^n}\mathrm{K}(A;n),$$

but

$$\beta_q^{(A)}(n,\infty)\mathrm{K}(A;n)=\mathrm{B}_q(n,\infty),$$

So,

$$\gamma_q^{(A)}(n)\mathrm{K}(A;n)=\frac{1}{(1-q)^n}\beta_q^{(A)}(n,\infty)=\Gamma_q(n),\quad \text{(by Theorem 2.2.1)}$$

Proof (b): It’s enough to show that $\mathrm{K}(A;n)\beta_q^{(A)}(n,t)$ enable to be composed as a FPS in q with coefficients rational functions in $a=q^t$and$b=q^n$. Firstly, we have

$$\mathrm{K}(A;n)\beta_q^{(A)}(n,t)=\frac{1}{(1+A)}A^n\left(1+\frac{1}{A}\right)_q^n(1+A)_q^{1-n}\int_0^{\infty/A}\frac{x^{n-1}}{(1+x)_q^{n+t}}d_qx.$$

Now let $y=Ax, d_qy=Ad_qx$.

$$\mathrm{K}(A;n)\beta_q^{(A)}(n,t)=\frac{1}{(1+A)}A^n\left(1+\frac{1}{A}\right)_q^n(1+A)_q^{1-n}\int_0^{\infty/1}\frac{(y/A)^{n-1}}{A}\frac{1}{\left(1+\dfrac{y}{A}\right)_q^{n+t}}d_qy.$$

$$=(1+A)_q^{1-n}\left(1+\frac{1}{A}\right)_q^n\frac{1}{(1+A)}\int_0^{\infty/1}y^{n-1}\frac{1}{\left(1+\frac{y}{A}\right)_q^{n+t}}d_qy. \tag{2.3.14}$$

Fix $A>0$. Here we want to rewrite the factor in the forward of the integral after letting $b=q^n$as the following notation

$$\frac{(1+A)_q^{\infty}}{\left(1+\frac{qA}{b}\right)_q^{\infty}}\frac{\left(1+\frac{1}{A}\right)_q^{\infty}}{\left(1+\frac{b}{A}\right)_q^{\infty}}\frac{1}{(1+A)}$$

It's obviously a FPS in q with CRF in a and b. After that we want to explain the integral term in the expression (2.3.14) which can be written as

$$\int_0^1\frac{y^{n-1}}{\left(1+\frac{y}{A}\right)_q^{n+t}}d_qy+\int_1^{\infty/1}\frac{y^{n-1}}{\left(1+\frac{y}{A}\right)_q^{n+t}}d_qy. \tag{2.3.15}$$

Now use q-integral in the first term in (2.3.15) and after that let $a=q^t$ and $b=q^n$.

$$\int_0^1\frac{1}{\left(1+\frac{y}{A}\right)_q^{n+t}}y^{n-1}d_qy=(1-q)\sum_{k\geq 0}q^k\left(q^k\right)^{n-1}\frac{1}{\left(1+\frac{q^k}{A}\right)^{n+t}},$$

$$=(1-q)\sum_{k\geq 0}b^k\frac{1}{\left(1+\frac{q^k}{A}\right)_q^{\infty}}\left(1+\frac{q^{k+n+t}}{A}\right)_q^{\infty},$$

$$=(1-q)\sum_{k\geq 0}b^k\frac{1}{\left(1+q^k/A\right)_q^{\infty}}\left(1+ab\frac{q^k}{A}\right)_q^{\infty}.$$

This is manifestly a FPS in q with CRF in a and b. Now rewrite the other term of (2.3.15) as

$$\int_0^q\frac{x^{t-1}}{x^{n+t}\left(1+\frac{1}{Ax}\right)_q^{n+t}}d_qx. \tag{2.3.16}$$

First, using the relation (2.1.2) to find the integral. We see that in the following expression

$$\int_1^{\infty} \frac{y^{n-1}}{\left(1+\frac{y}{A}\right)_q^{n+t}} d_q y = \int_0^q \frac{1}{x^2} \frac{\left(\frac{1}{x}\right)^{n-1}}{\left(1+\frac{1}{Ax}\right)_q^{n+t}} d_q x,$$

$$= \int_0^q \frac{x^{-n-1}}{\left(1+\frac{1}{Ax}\right)_q^{n+t}} d_q x,$$

$$= \int_0^q \frac{x^{t-1}}{x^{n+t}\left(1+\frac{1}{Ax}\right)_q^{n+t}} d_q x.$$ (here we multiplied by x^{t-t} to get the result)

Now we make recall for the definition of $\mathrm{K}(x;n)$ to get the last result

$$\mathrm{K}(Ax;n+t) = (Ax)^{n+t} (1+Ax)_q^{1-t-n} \frac{1}{(1+Ax)} \left(1+\frac{1}{Ax}\right)_q^{t+n},$$

$$x^{n+t}\left(1+\frac{1}{Ax}\right)_q^{n+t} = \frac{\mathrm{K}(Ax;n+t)(1+Ax)}{A^{n+t}(1+Ax)_q^{1-n-t}},$$

$$\frac{1}{x^{n+t}\left(1+\frac{1}{Ax}\right)_q^{n+t}} = \frac{1}{(1+Ax)} (1+Ax)_q^{1-n-t} \frac{A^{n+t}}{\mathrm{K}(Ax;n+t)}.$$ (The result)

The most important note is that, although $K(Ax;n+t)$ is "not constant" in x, $\mathrm{K}(Aq^k;n+t) = \mathrm{K}(A;n+t), \forall k \in \mathbb{Z},$ by lemma (2.3.1), so the inside of Jackson integral able to be consider as a "constant" so we can rewrite (2.3.16) like

$$\frac{A^{n+t}}{\mathrm{K}(A;n+t)} \int_0^q \frac{1}{(1+Ax)} x^{t-1} (1+Ax)_q^{1-n-t} d_q x. \tag{2.3.17}$$

Finally we want to rewrite the first factor in (2.3.17)

$$\frac{A^{n+t}}{\mathrm{K}\left(A;n+t\right)}=\left(1+A\right)\frac{A^{n+t}}{\left(1+\frac{1}{A}\right)_q^{n+t}\left(1+A\right)_q^{1-n-t}A^{n+t}}.$$

Now by $\left(1+a\right)_q^n=\frac{1}{\left(1+q^n a\right)_q^\infty}\left(1+a\right)_q^\infty$, then we get

$$=\frac{1}{\left(1+\frac{1}{A}\right)_q^\infty\left(1+A\right)_q^\infty}\left(1+\frac{ab}{A}\right)_q^\infty\left(1+\frac{qA}{ab}\right)_q^\infty\left(1+A\right).\text{(Where } b=q^n \text{ and } =q^t\text{)} \qquad (2.3.18)$$

and we rewrite the integral in (2.3.17) by using q-integral

$$\int_0^q x^{t-1}\frac{1}{\left(1+Ax\right)}\left(1+Ax\right)_q^{1-n-t}d_q x=\left(1-q\right)\sum_{j\geq 0}q^{j+1}\left(q^{j+1}\right)^{t-1}\left(1+Aq^{j+1}\right)_q^{1-n-t}\frac{1}{1+Aq^{j+1}},$$

$$=\left(1-q\right)\sum_{j\geq 0}\left(q^{j+1}\right)^t\left(1+Aqq^{j+1}\right)_q^\infty\left(1+Aq^{j+1}\right)_q^\infty\frac{1}{\left(1+Aq^{j+1}\right)_q^\infty\left(1+Aq^{j+1}q^{1-n-t}\right)_q^\infty},$$

$$=\left(1-q\right)\sum_{j\geq 0}a^{j+1}\left(1+Aq^{j+2}\right)_q^\infty\frac{1}{\left(1+\frac{Aq^{j+2}}{ab}\right)_q^\infty},\text{(where } b=q^n \text{ and } a=q^t\text{)} \qquad (2.3.19)$$

Obviously the two expressions (2.3.18) and (2.3.19) are a FPS in q with CRF in b and a. □

Chapter 3

APPLICATIONS

3.1 Integral expression which is symmetric in *n* and *t* for *q*–Beta Function

We can see from theorem (2.2.1) the q-analouge of BF symmetric in n and t, but from the integral term in (2.2.7) this is not clear. Now we will apply theorem (2.3.1) then we obtain the integral equation for q-analouge of BF be "symmetric" under the interchange of n and t.

If $A > 0$ then by theorem (2.3.1) we obtain that,

$$\mathrm{B}_q(n,t) = \mathrm{K}(A;n)\int_0^{\infty/A} \frac{x^{n-1}}{(1+x)_q^{n+t}} d_q x. \tag{3.1.1}$$

By using the results of Lemma 2.1.1 and the definition of "$\mathrm{K}(x;n)$"we obtain

$$\frac{1}{x^n}(1+x)_q^{n+t} = \mathrm{K}\left(\frac{1}{x};n\right)\left(1+\frac{q}{q^n x}\right)_q^n (1+q^n x)_q^t. \tag{3.1.2}$$

Proof:

$$\frac{1}{x^n}(1+x)_q^{n+t} = \frac{1}{x^n}(1+x)_q^n (1+q^n x)_q^t,$$

$$= \frac{(1+x)_q^n \left(1+\frac{1}{x}\right)_q^{1-n}}{x^n\left(1+\frac{1}{x}\right)} \frac{(1+q^n x)_q^t \left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)_q^{1-n}},$$

$$= \frac{1}{\left(1+\frac{1}{x}\right)_q^{1-n}} \mathrm{K}\left(\frac{1}{x};n\right)\left(1+q^n x\right)_q^t \left(1+\frac{1}{x}\right),$$

$$= \frac{1}{\left(1+\frac{1}{x}q\right)_q^{-n}} \mathrm{K}\left(\frac{1}{x};n\right)\left(1+q^n x\right)_q^t,$$

$$= \left(1+q^n x\right)_q^t \mathrm{K}\left(\frac{1}{x};n\right)\left(1+\frac{q}{x}q^{-n}\right)_q^n.$$

Hence by Lemma 2.3.1 we obtain

$$\mathrm{K}\left(\frac{1}{x};n\right) = \mathrm{K}\left(A;n\right),\ \forall x = \frac{q^t}{A},\ t \in \mathbb{Z}\ .$$

After we substitute (3.1.2) back into (3.1.1) and then alteration the variable $y = q^n x$, we get

$$\mathrm{B}_q\left(n,t\right) = \int_0^{\infty/\alpha} \frac{1}{y\left(1+\frac{q}{y}\right)_q^n \left(1+y\right)_q^t} d_q y,\quad \forall \alpha > 0\ . \tag{3.1.3}$$

Proof:

$$\mathrm{K}\left(A;n\right) = \frac{\left(1+x\right)_q^{n+t}}{x^n\left(1+\frac{q}{q^n x}\right)_q^n \left(1+q^n x\right)_q^t},$$

Now substitute $\mathrm{K}\left(A;n\right)$ in

$$\mathrm{B}_q\left(n,t\right) = \mathrm{K}\left(A;n\right)\int_0^{\infty/A} \frac{x^{n-1}}{\left(1+x\right)_q^{n+t}} d_q x,$$

then change of variable $x = \frac{y}{q^n}, d_q x = \frac{d_q y}{q^n}$.

After that we get:

$$\mathrm{B}_q(n,t)=\frac{\left(1+\frac{y}{q^n}\right)_q^{n+t}}{\left(\frac{y}{q^n}\right)^n\left(1+\frac{q}{y}\right)_q^n(1+y)_q^t}\int_0^{\infty/A}\frac{1}{q^n}\frac{1}{\left(1+\frac{y}{q^n}\right)_q^{n+t}}\left(\frac{y}{q^n}\right)^{n-1}d_q y,$$

$$=\int_0^{\infty/\alpha}\frac{1}{\left(1+\frac{q}{y}\right)_q^n(1+y)_q^t}\frac{1}{y}d_q y,\quad \forall\alpha>0\ .$$

Remark: The integral equation of q-analouge of BF is clearly "symmetric" in n and t, after changing the variable ($x=\frac{q}{y}$).

3.2 Interpretation invariance of a specific kind of improper integrals

Jackson's integral failed because there is no analogue for the TI identity

$$\int_0^b g(x)dx=\int_c^{b+c}g(x-c)dx.$$

That is clearly true for "classical" integrals. We qualified to obtain a q-analogue of TI for improper integrals for specific type function which is $x^\beta/(1+x)_q^\alpha$ by using Theorem 2.3.1.

In particular we will prove the following corollary

Corollary 3.2.1. For $\beta>0$ and $\alpha>\beta+1$ then

$$\int_0^{\infty/\beta}\frac{1}{(1+x)_q^\alpha}x^\beta d_q x=\frac{q}{K(A,\beta)q^\alpha}\int_0^{\infty/1}\frac{1}{x^\alpha}x^\beta\left(1-\frac{1}{x}\right)_q^\beta d_q x. \qquad (3.2.1)$$

Remark: We changed $x\to x-1$ in the "classical" limit $q=1$ in the left–hand side of (3.2.1) to get the right–hand side

Proof:

$$\mathrm{B}_q(n,t)=\int_0^1 x^{t-1}(1-qx)_q^{n-1}\,d_qx,$$

$$=\int_q^{\infty/1}\frac{1}{x^{t+1}}\left(1-\frac{q}{x}\right)_q^{n-1}d_qx,\ \text{(by : } \int_0^A f(x)\,d_qx=\int_{q/A}^{\infty/A}\frac{1}{x^2}f\left(\frac{1}{x}\right)d_qx.\text{)}$$

$$=\int_1^{\infty/1}\frac{1}{(qy)^{t+1}}\left(1-\frac{q}{qy}\right)_q^{n-1}qd_qy,\ \text{(by letting } y=\frac{x}{q},d_qx=qd_qy.\text{)}$$

$$=\frac{1}{q^t}\int_1^{\infty/1}\frac{1}{y^{t+n-n+1}}\left(1-\frac{1}{y}\right)_q^{n-1}d_qy,$$

$$=\frac{1}{q^t}\int_1^{\infty/1}\frac{y^{n-1}(1-1/y)_q^{n-1}}{y^{t+n}}d_qy. \tag{3.2.2}$$

and from Theorem 2.3.1 we have

$$\mathrm{B}_q(n,t)=\mathrm{K}(A;n)\int_0^{\infty/A}\frac{x^{n-1}}{(1+x)_q^{n+t}}d_qx. \tag{3.2.3}$$

Now make comparison between (3.2.2) and (3.2.3) and letting $\beta=n-1,\alpha=n+t.$

We get

$$\mathrm{K}(A;\beta+1)\int_0^{\infty/\beta}\frac{1}{(1+x)_q^{\alpha}}x^{\beta}d_qx=\frac{1}{q^{\alpha-\beta-1}}\int_1^{\infty/1}\frac{1}{x^{\alpha}}\left(1-\frac{1}{x}\right)_q^{\beta}x^{\beta}d_qx.$$

$$\int_0^{\infty/\beta}\frac{1}{(1+x)_q^{\alpha}}x^{\beta}d_qx=\frac{1}{q^{\alpha-\beta-1}\mathrm{K}(A;\beta+1)}\int_1^{\infty/1}\frac{1}{x^{\alpha}}\left(1-\frac{1}{x}\right)_q^{\beta}x^{\beta}\,d_qx.$$

$$=\frac{1}{\mathrm{K}(\mathrm{A};\beta)q^{\beta}q^{\alpha-\beta-1}}\int_1^{\infty/1}\frac{1}{x^{\alpha}}\left(1-\frac{1}{x}\right)_q^{\beta}x^{\beta}d_qx.\ \text{(by the fact } \mathrm{K}(A;\beta+1)=q^{\beta}\mathrm{K}(A;\beta).\text{)}$$

$$=\frac{1}{\mathrm{K}(\mathrm{A};\beta)}\frac{q}{q^{\alpha}}\int_1^{\infty/1}\frac{1}{x^{\alpha}}\left(1-\frac{1}{x}\right)_q^{\beta}x^{\beta}d_qx.$$

3.3 Identitie

When we rewrite the equations (2.3.12), (2.3.13) and (3.1.3) by utilizing the definition of improper integrals, they give us a several interesting identities including q–BS (Bilateral Series).

The equation (2.3.12) can be considered as

$$\left(1-q\right)_q^{\infty}\left(1+q^n/A\right)_q^{\infty}\left(1+qA/q^n\right)_q^{\infty}=\left(1+qA\right)_q^{\infty}\left(1-q^n\right)_q^{\infty}\sum_{k=-\infty}^{\infty}q^{nk}\left(1+1/A\right)_q^{k}. \quad (3.3.1)$$

Proof:

$$\Gamma_q\left(n\right)=\mathrm{K}\left(A;n\right)\gamma_q^{(A)}\left(n\right),$$

$$\Gamma_q\left(n\right)=\mathrm{K}\left(A;n\right)\int_0^{\infty/A(1-q)}x^{n-1}e_q^{-x}d_qx.$$

$$\Gamma_q\left(n\right)=\mathrm{K}\left(A;n\right)\int_0^{\infty/A(1-q)}\frac{1}{\left(1+\left(1-q\right)x\right)_q^{\infty}}x^{n-1}d_qx.$$

$$A^n\frac{1}{\left(1+A\right)}\left(1+1/A\right)_q^{n}\left(1+A\right)_q^{1-n}\left(1-q\right)\sum_{k=-\infty}^{\infty}q^k\frac{1}{\left(1-q\right)A}\frac{\left(q^k/A\left(1-q\right)\right)^{n-1}}{\left(1+q^k/A\right)_q^{\infty}}=\frac{(1-q)_q^{n-1}}{\left(1-q\right)^{n-1}}.$$

$$\frac{1}{\left(1+A\right)}\sum_{k=-\infty}^{\infty}\frac{1}{\left(1+q^k/A\right)_q^{\infty}}\left(1+1/A\right)_q^{n}\left(1+A\right)_q^{1-n}q^{kn}=(1-q)_q^{n-1}.$$

$$\frac{1}{\left(1+A\right)}\sum_{k=-\infty}^{\infty}\frac{1}{\left(1+q^k/A\right)_q^{\infty}}\frac{\left(1+A\right)}{\left(1+q^n/A\right)_q^{\infty}}\left(1+1/A\right)_q^{\infty}\left(1+qA\right)_q^{-n}q^{kn}=\frac{(1-q)_q^{\infty}}{\left(1-q^n\right)_q^{\infty}}.$$

$$\left(1+qA\right)_q^{\infty}\left(1-q^n\right)_q^{\infty}\sum_{k=-\infty}^{\infty}\frac{\left(1+1/A\right)_q^{\infty}}{\left(1+q^k/A\right)_q^{\infty}}q^{kn}=\left(1+q^n/A\right)_q^{\infty}\left(1+qA/q^n\right)_q^{\infty}(1-q)_q^{\infty}.$$

$$\sum_{k=-\infty}^{\infty}q^{kn}\left(1+1/A\right)_q^{k}\left(1+qA\right)_q^{\infty}\left(1-q^n\right)_q^{\infty}=(1-q)_q^{\infty}\left(1+q^n/A\right)_q^{\infty}\left(1+qA/q^n\right)_q^{\infty}.$$

Letting$x=-q^n/A$ in equation (3.3.1) we get

$$(1-q)_q^{\infty}(1-x)_q^{\infty}(1-q/x)_q^{\infty}=(1+qA)_q^{\infty}(1+Ax)_q^{\infty}\sum_{k=-\infty}^{\infty}(-x)^k(1+1/A)_q^k. \qquad (3.3.2)$$

Note that

$$\lim_{A\to\infty}A^k(1+1/A)_q^k=q^{k(k-1)/2}.$$

It means that, if $A=0$, then the equation (3.3.2) becomes the famous Jacobi triple product identity (1.17).

We can rewrite the equation (2.3.13) as

$$\sum_{k=-\infty}^{\infty}\frac{(1+1/A)_q^k}{(1+q^{n+t}/A)_q^k}q^{nk}=\frac{(1-q)_q^{\infty}(1-q^{n+t})_q^{\infty}(1+q^n/A)_q^{\infty}(1+qA/q^n)_q^{\infty}}{(1+q^{n+t}/A)_q^{\infty}(1+qA)_q^{\infty}(1-q^n)_q^{\infty}(1-q^t)_q^{\infty}}. \qquad (3.3.3)$$

Proof:

$$\beta_q^{(A)}(n,t)\mathrm{K}(A;n)=\mathrm{B}_q(n,t),$$

but $\mathrm{B}_q(n,t)=\dfrac{\Gamma_q(t)\Gamma_q(n)}{\Gamma_q(n+t)}=\dfrac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}(1-q)$

So

$$\beta_q^{(A)}(n,t)\mathrm{K}(A;n)=\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}(1-q),$$

$$\mathrm{K}(A;n)\int_0^{\infty/A}\frac{x^{n-1}}{(1+x)_q^{n+t}}d_qx=\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}(1-q),$$

$$A^n(1+1/A)_q^n(1+A)_q^{1-n}\frac{1}{(1+A)}(1-q)\sum_{k=-\infty}^{\infty}\frac{q^k}{A}\frac{(q^k/A)^{n-1}}{(1+q^k/A)_q^{n+t}}=\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}(1-q).$$

$$(1+1/A)_q^n(1+A)_q^{1-n}\frac{1}{(1+A)}\sum_{k=-\infty}^{\infty}\frac{1}{(1+q^k/A)_q^{n+t}}q^{nk}=\frac{(1-q)_q^{t-1}(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}$$

$$\sum_{k=-\infty}^{\infty} q^{nk} \frac{1}{\left(1+q^k / A\right)_q^{n+t}} \left(1+qA\right)_q^{-n} \left(1+1/A\right)_q^{n} = \left(1-q\right)_q^{t-1} \left(1-q\right)_q^{n-1} \frac{1}{\left(1-q\right)_q^{n+t-1}},$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left(1+q^k / A\right)_q^{t+n}} q^{kn} \frac{\left(1+1/A\right)_q^{\infty}}{\left(1+q^n / A\right)_q^{\infty}} \frac{\left(1+qA\right)_q^{\infty}}{\left(1+qA/q^n\right)_q^{\infty}} = \frac{1}{\left(1-q^n\right)_q^{\infty} (1-q^t)_q^{\infty}} (1-q)_q^{\infty} (1-q^{n+t})_q^{\infty}.$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left(1+q^k / A\right)_q^{\infty}} q^{kn} \left(1+1/A\right)_q^{\infty} \left(1+q^k q^{n+t} / A\right)_q^{\infty} = \frac{(1-q)_q^{\infty} \left(1+q^n / A\right)_q^{\infty} \left(1+qA/q^n\right)_q^{\infty} (1-q^{n+t})_q^{\infty}.}{\left(1-q^n\right)_q^{\infty} (1-q^t)_q^{\infty} \left(1+qA\right)_q^{\infty}}$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left(1+q^{n+t} / A\right)_q^{k}} q^{kn} \left(1+1/A\right)_q^{k} \left(1+q^{n+t} / A\right)_q^{\infty} = \frac{(1-q)_q^{\infty} (1-q^{n+t})_q^{\infty} \left(1+q^n / A\right)_q^{\infty} \left(1+qA/q^n\right)_q^{\infty}}{\left(1-q^n\right)_q^{\infty} (1-q^t)_q^{\infty} \left(1+qA\right)_q^{\infty}}.$$

$$\sum_{k=-\infty}^{\infty} \left(1+1/A\right)_q^{k} q^{kn} \frac{1}{\left(1+q^{n+t} / A\right)_q^{k}} = \frac{(1-q^{n+t})_q^{\infty} \left(1+q^n / A\right)_q^{\infty} \left(1+qA/q^n\right)_q^{\infty} (1-q)_q^{\infty}}{\left(1-q^n\right)_q^{\infty} (1-q^t)_q^{\infty} \left(1+q^{n+t} / A\right)_q^{\infty} \left(1+qA\right)_q^{\infty}}.$$

Now, letting: $a=-1/A, b=-q^{n+t}/A, x=q^n.$ in this formula we have:

$$\sum_{k=-\infty}^{\infty} \frac{\left(1-a\right)_q^{k}}{\left(1-b\right)_q^{k}} x^k = \frac{\left(1-q\right)_q^{\infty} \left(1-\frac{b}{a}\right)_q^{\infty} \left(1-ax\right)_q^{\infty} \left(1-q/ax\right)_q^{\infty}}{\left(1-b\right)_q^{\infty} \left(1-q/a\right)_q^{\infty} \left(1-x\right)_q^{\infty} \left(1-b/ax\right)_q^{\infty}}$$

Clearly, we get Ramanujan's identity (1.18). We conclude that, proof of theorem (2.3.1) it is also a proof for Ramanujan's identity.

At the end of this section we able to rewrite equation (3.1.3) as

$$\sum_{k=-\infty}^{\infty} \frac{\left(1+1/\alpha\right)_q^{k} \left(1+q\alpha\right)_q^{-k}}{\left(1+q^t/\alpha\right)_q^{k} \left(1+q^{n+1}\alpha\right)_q^{-k}} = \frac{\left(1-q\right)_q^{\infty} \left(1-q^{n+t}\right)_q^{\infty}}{\left(1-q^t\right)_q^{\infty} \left(1-q^n\right)_q^{\infty}} \frac{\left(1+1/\alpha\right)_q^{\infty} \left(1+q\alpha\right)_q^{\infty}}{\left(1+q^t/\alpha\right)_q^{\infty} \left(1+q^{n+1}\alpha\right)_q^{\infty}}. \quad (3.3.4)$$

Proof:

$$\mathrm{B}_q(n,t)=\int_0^{\infty/\alpha}\frac{d_q y}{\left(1+\frac{q}{y}\right)_q^n(1+y)_q^t\, y}=\frac{(1-q)_q^{t-1}(1-q)(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}.$$

$$(1-q)\sum_{k=-\infty}^{\infty}\frac{1}{(q^k/\alpha)(1+q^k/\alpha)_q^t(1+q\alpha/q^k)_q^n}(q^k/\alpha)=\frac{(1-q)_q^{t-1}(1-q)(1-q)_q^{n-1}}{(1-q)_q^{n+t-1}}.$$

$$\sum_{k=-\infty}^{\infty}\frac{1}{(1+q^k/\alpha)_q^t(1+q\alpha/q^k)_q^n}=\frac{1}{(1-q)_q^{t+n-1}}(1-q)_q^{n-1}(1-q)_q^{t-1}.$$

$$\sum_{k=-\infty}^{\infty}\frac{(1+q^kq^t/\alpha)_q^{\infty}}{(1+q\alpha q^{-k})_q^n(1+q^k/\alpha)_q^{\infty}}(1+q^{1+n}q^{-k}\alpha)_q^{\infty}=\frac{(1-q)_q^{\infty}(1-q^{n+t})_q^{\infty}}{(1-q^n)_q^{\infty}(1-q^t)_q^{\infty}}.$$

$$\sum_{k=-\infty}^{\infty}\frac{(1+q^{1+n}\alpha)_q^{\infty}(1+q\alpha)_q^{-k}(1+q^t/\alpha)_q^{\infty}(1+1/\alpha)_q^{k}}{(1+q^{1+n}\alpha)_q^{-k}(1+q\alpha)_q^{\infty}(1+q^t/\alpha)_q^{k}(1+1/\alpha)_q^{\infty}}=(\frac{1}{(1-q^n)_q^{\infty}(1-q^t)_q^{\infty}}1-q)_q^{\infty}(1-q^{n+t})_q^{\infty}.$$

$$\sum_{k=-\infty}^{\infty}(1+1/\alpha)_q^k(1+q\alpha)_q^{-k}\frac{1}{(1+q^t/\alpha)_q^k(1+q^{1+n}\alpha)_q^{-k}}=\frac{(1+q\alpha)_q^{\infty}(1+1/\alpha)_q^{\infty}(1-q)_q^{\infty}(1-q^{n+t})_q^{\infty}}{(1-q^n)_q^{\infty}(1-q^t)_q^{\infty}(1+q^{1+n}\alpha)_q^{\infty}(1+q^t/\alpha)_q^{\infty}}.$$

Finally, letting: $a=-\frac{1}{\alpha}, b=-\frac{q^t}{\alpha}, c=-q^{n+1}\alpha$ in this equation we get equation (1.19)

$$\sum_{k=-\infty}^{\infty}\frac{1}{(1-b)_q^k(1-c)_q^{-k}}(1-a)_q^k(1-q/a)_q^{-k}=\frac{(1-q/a)_q^{\infty}(1-q)_q^{\infty}(1-a)_q^{\infty}(1-bc/q)_q^{\infty}}{(1-b)_q^{\infty}(1-c)_q^{\infty}(1-b/a)_q^{\infty}(1-ac/q)_q^{\infty}}$$

We conclude that equation (3.3.4) reduces to equation (1.19).

Chapter 4

THE (p,q)-GAMMA FUNCTION AND (p,q)-BETA FUNCTION

4.1 Notations and miscellaneous relations

Before we study the new generalization of GF and BF, which is (p,q)-analogue for each one. As of now some notations and definitions of (p,q)-calculus will be introduced in this section.

Definition1: for any positive integer number, the (p,q)-numbers define as

$$[t]_{p,q} = p^{t-1} + p^{t-2}q + p^{t-3}q^2 + \ldots + pq^{t-2} + q^{t-1} = \frac{p^t - q^t}{p-q}. \qquad (4.1.1)$$

Since $[t]_{p,q} = \frac{p^t - q^t}{p-q}$.

$$= p^{t-1}\frac{\left(p - p^{1-t}q^t\right)}{\left(p-q\right)} = p^{t-1}\frac{\left(p^{1-t}\frac{q^t}{p} - 1\right)}{\left(\frac{q}{p} - 1\right)},$$

$$= p^{t-1}\frac{\left((q/p)^t - 1\right)}{\left((q/p) - 1\right)} = p^{t-1}[t]_{q/p},$$

Clearly, we can also write $[t]_{p,q}$ as: $[t]_{p,q} = p^{t-1}[t]_{q/p}$

Definition2: For $t \in \mathbb{N}$ the (p,q)-factorial is given by:

$$[t]_{p,q}! = \prod_{n=1}^{t} [n]_{p,q}!, \qquad t \geq 1, \ [0]_{p,q}! = 1. \tag{4.1.2}$$

Definition3: The (p,q) -binomial coefficient is defined as

$$\begin{bmatrix} a \\ b \end{bmatrix}_{p,q} = \frac{[a]_{p,q}!}{[a-b]_{p,q}![b]_{p,q}!}, \qquad 0 \leq b \leq a \ . \tag{4.1.3}$$

Clearly, we can see by definition that

$$\begin{bmatrix} a \\ b \end{bmatrix}_{p,q} = \begin{bmatrix} a \\ a-b \end{bmatrix}_{p,q}, \qquad 0 \leq b \leq a \ .$$

Definition 4: The (p,q) -powers is given by

$$(z \oplus b)_{p,q}^{k} = (z+b)(pz+bq)\ldots(zp^{k-1}+bq^{k-1}) = \prod_{j=0}^{k-1}(zp^j + q^j b) \tag{4.1.4}$$

$$(z \ominus b)_{p,q}^{k} = (z-b)(pz-bq)\ldots(zp^{k-1}-bq^{k-1}) = \prod_{j=0}^{k-1}(zp^j - q^j b) \tag{4.1.5}$$

These definitions are extended to the following expressions

$$(z \oplus b)_{p,q}^{\infty} = (z+b)(pz+bq)(zp^2+bq^2)\ldots = \prod_{j=0}^{\infty}(zp^j + q^j b) \tag{4.1.6}$$

$$(z \ominus b)_{p,q}^{\infty} = (z-b)(pz-bq)(zp^2-bq^2)\ldots = \prod_{j=0}^{\infty}(zp^j - q^j b) \tag{4.1.7}$$

Note that, the convergence is required in these equations.

Definition 5: the (p,q) -derivative of the function g is given by

$$D_{p,q}g(z) = \frac{g(pz) - g(qz)}{(p-q)z}, \qquad z \neq 0 \ . \tag{4.1.8}$$

and if g is differentiable at 0, then $D_{p,q}g(0) = g'(0)$. Also notice that for p=1, the (p,q) -derivative minimize to the q-derivative.

The following is the product rules for the (p,q)-derivative

$$D_{p,q}\left(g(z)f(z)\right)=g(pz)D_{p,q}f(z)+f(qz)D_{p,q}g(z), \tag{4.1.9}$$

$$D_{p,q}\left(g(z)f(z)\right)=f(pz)D_{p,q}g(z)+g(qz)D_{p,q}f(z), \tag{4.1.10}$$

Proposition 4.1.1: For any integer $t \geq 1$, we have

(1) $D_{p,q}(z \ominus b)_{p,q}^{t} = [t]_{p,q}(pz \ominus b)_{p,q}^{t-1}$, (4.1.11)

(2) $D_{p,q}(b \ominus z)_{p,q}^{t} = -[t]_{p,q}(b \ominus qz)_{p,q}^{t-1}$, (4.1.12)

Note that, $D_{p,q}(z \ominus b)_{p,q}^{0} = 0$,

Proof (1):

By using the (p,q)-derivative definition (4.1.8) we have

$$D_{p,q}(z \ominus b)_{p,q}^{t} = \frac{(pz \ominus b)_{p,q}^{t} - (qz \ominus b)_{p,q}^{t}}{(p-q)z},$$

$$= \frac{(pz-b)(p^2z-qb)\ldots(p^tz-q^{t-1}b)-(qz-b)(qpz-qb)\ldots(qp^{t-1}z-q^{t-1}b)}{(p-q)z},$$

$$= \frac{(pz-b)(p^2z-qb)\ldots(p^{t-1}z-q^{t-2}b)(p^tz-q^{t-1}b)-(qz-b)q(pz-b)q(p^2z-qb)\ldots q(p^{t-1}z-q^{t-2}b)}{(p-q)z}$$

$$= \frac{(pz-b)(p^2z-qb)\ldots(p^{t-1}z-q^{t-2}b)\big(p^tz-q^{t-1}b-q^{t-1}(qz-b)\big)}{(p-q)z},$$

$$= \frac{(pz-b)(p^2z-qb)\ldots(p^{t-1}z-q^{t-2}b)(p^tz-q^{t-1}b-q^tz+q^{t-1}b)}{(p-q)z},$$

$$= (pz-b)(p^2z-qb)\ldots(p^{t-1}z-q^{t-2}b)\frac{(p^tz-q^tz)}{(p-q)z},$$

$$= (pz \ominus b)_{p,q}^{n-1}\frac{(p^t-q^t)}{(p-q)} = [t](pz \ominus b)_{p,q}^{n-1},$$

Proof (2):

$$D_{p,q}(b \ominus z)_{p,q}^{t} = \frac{(b \ominus pz)_{p,q}^{t} - (b \ominus qz)_{p,q}^{t}}{(p-q)z},$$

$$= \frac{(b-pz)(bp-pqz)(bp^2-pq^2z)\dots(bp^{t-1}-pq^{t-1}z)-(b-qz)(bp-q^2z)(bp^2-q^3z)\dots(bp^{t-1}-q^tz)}{(p-q)z},$$

$$= \frac{(qz-b)(q^2z-bp)\dots(q^{t-1}z-bp^{t-2})(q^tz-bp^{t-1})-(pz-b)p(qz-b)p(q^2z-bp)\dots p(q^{t-1}z-bp^{t-2})}{(p-q)z},$$

$$= \frac{(qz-b)(q^2z-bp)\dots(q^{t-1}z-bp^{t-2})(q^tz-bp^{t-1}-(pz-b)p^{t-1})}{(p-q)z},$$

$$= \frac{(qz-b)(q^2z-bp)\dots(q^{t-1}z-bp^{t-2})(q^t-p^t)}{(p-q)};$$

$$= -\frac{(p^t-q^t)}{(p-q)}(b-qz)(bp-q^2z)\dots(bp^{t-2}-q^{t-1}z),$$

$$= -[t]_{p,q}(b \ominus qz)_{p,q}^{t-1}$$

Proposition 4.1.2: It's easily to verify the following identities

(1) $(s \ominus t)_{p,q}^{k} = \frac{(s\ominus t)_{p,q}^{\infty}}{(sp^k \ominus tq^k)_{p,q}^{\infty}}$,

(2) $(s \ominus t)_{p,q}^{k+j} = (s \ominus t)_{p,q}^{k}(sp^k \ominus tq^k)_{p,q}^{j}$,

(3) $\left(sp^k \ominus q^k t\right)_{p,q}^{j} = \frac{(s\ominus t)_{p,q}^{j}\left(sp^j \ominus tq^j\right)_{p,q}^{k}}{(s\ominus t)_{p,q}^{k}}$,

(4) $\left(sp^j \ominus q^j t\right)_{p,q}^{k-j} = \frac{(s\ominus t)_{p,q}^{k}}{(s\ominus t)_{p,q}^{j}}$,

Proof (1):

$$\frac{(s \ominus t)_{p,q}^{\infty}}{(sp^k \ominus tq^k)_{p,q}^{\infty}} = \frac{(s-t)(sp-qt)\dots(sp^{k-1}-q^{k-1}t)(sp^k \ominus tq^k)_{p,q}^{\infty}}{(sp^k \ominus tq^k)_{p,q}^{\infty}}$$

$$= (s \ominus t)_{p,q}^{k},$$

Proof (2):

$$(s \ominus t)_{p,q}^{k+j} = (s-t)(sp-qt)\dots\left(sp^{k-1}-q^{k-1}t\right)(sp^k-tq^k)\dots\left(sp^{k+j-1} - tq^{k+j-1}\right),$$

$$= (s \ominus t)_{p,q}^{k}(sp^k-tq^k)\dots\left(sp^{k+j-1}-tq^{k+j-1}\right),$$

$$= (s \ominus t)_{p,q}^{k} (sp^k \ominus tq^k)_{p,q}^{j},$$

Proof (3):

$$(s \ominus t)_{p,q}^{j+k} = (s \ominus t)_{p,q}^{j} \left(sp^j \ominus tq^j\right)_{p,q}^{k},$$

$$(s \ominus t)_{p,q}^{k} (sp^k \ominus tq^k)_{p,q}^{j} = (s \ominus t)_{p,q}^{j} \left(sp^j \ominus tq^j\right)_{p,q}^{k},$$

$$(sp^k \ominus tq^k)_{p,q}^{j} = \frac{(s \ominus t)_{p,q}^{\mathrm{j}} \left(sp^j \ominus tq^j\right)_{p,q}^{k}}{(s \ominus t)_{p,q}^{\mathrm{k}}},$$

Proof (4):

$$\left(sp^j \ominus q^j t\right)_{p,q}^{k-j} = \left(sp^j \ominus q^j t\right)_{p,q}^{k} \left(sp^{j+k} \ominus q^{j+k} t\right)_{p,q}^{-j},$$

$$= \frac{\left(sp^j \ominus q^j t\right)_{p,q}^{k}}{(sp^k \ominus q^k t)_{p,q}^{j}} = \frac{(s \ominus t)_{p,q}^{j+k} (s \ominus t)_{p,q}^{k}}{(s \ominus t)_{p,q}^{j} (s \ominus t)_{p,q}^{j+k}} = \frac{(s \ominus t)_{p,q}^{k}}{(s \ominus t)_{p,q}^{j}},$$

Definition 6: The (p, q)-integral of $g(x)$ on $[0, b]$ is given by

$$\int_0^b g(x) d_{p,q} x = (p - q) b \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} g\left(\frac{q^j}{p^{j+1}} b\right), \quad 0 < q < p < 1$$

Definition 7: The (p, q)-IBP is given by

$$\int_b^c f(pz) D_{p,q} g(z) d_{p,q} z = f(c) g(c) - f(b) g(b) - \int_b^c g(qz) D_{p,q} f(z) d_{p,q} z, \quad (4.1.13)$$

4.2 The Definitons of (p, q)-Gamma and (p, q)-Beta Functions

Definition 1: The (p, q)-GF for a nonnegative integer z is given by

$$\Gamma_{p,q}(z) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^z \ominus q^z)_{p,q}^{\infty}} (p - q)^{1-z} = \frac{(p \ominus q)_{p,q}^{z-1}}{(p - q)}, \quad 0 < q < p. \quad (4.2.1)$$

Remark: Notice that if $p = 1$, then $\Gamma_{p,q}(z)$ is reduces to $\Gamma_q(z)$.

Lemma 4.2.1. For all $z \in \mathbb{N}$, the (p,q)- GF obtain the fundamental relation

$$\Gamma_{p,q}(z+1) = [z]_{p,q}\Gamma_{p,q}(z), \tag{4.2.2}$$

Proof:

$$\Gamma_{p,q}(z+1) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^{z+1} \ominus q^{z+1})_{p,q}^{\infty}}(p-q)^{-z} \quad .$$

$$= (p-q)^{1-z}\frac{(p \ominus q)_{p,q}^{\infty}}{(p^{z} \ominus q^{z})_{p,q}^{\infty}}\frac{(p^{z} \ominus q^{z})_{p,q}^{\infty}}{(p^{z+1} \ominus q^{z+1})_{p,q}^{\infty}(p-q)} \quad .$$

$$= \Gamma_{p,q}(z)\frac{(p^{z} \ominus q^{z})_{p,q}^{\infty}}{(p^{z+1} \ominus q^{z+1})_{p,q}^{\infty}(p-q)} = \Gamma_{p,q}(z)\frac{(p^{z}-q^{z})(p^{z+1} \ominus q^{z+1})_{p,q}^{\infty}}{(p^{z+1} \ominus q^{z+1})_{p,q}^{\infty}(p-q)} \quad .$$

$$= \Gamma_{p,q}(z)\frac{(p^{z}-q^{z})}{p-q} = [z]_{p,q}\Gamma_{p,q}(z).$$

We obtain also

$$\Gamma_{p,q}(z+1) = \frac{(p \ominus q)_{p,q}^{z}}{(p-q)^{z}} = [z][z-1]\Gamma_{\mathrm{p,q}}(z) = [z]_{p,q}!, \quad 0 < q < p, \tag{4.2.3}$$

Definition 2: The (p,q)-BF for $s,t \in \mathbb{N}$ is given by

$$\mathrm{B}_{p,q}(s,t) = \int_0^1 (pz)^{s-1}(p \ominus pqz)_{p,q}^{t-1}\, d_{p,q}z. \tag{4.2.4}$$

Theorem 4.2.1. The relation between (p,q)-GF and (p,q)-BF for $s,t \in \mathbb{N}$ is defined by

$$\mathrm{B}_{p,q}(s,t) = p^{[t(2s+t-2)+t-2]/2}\frac{\Gamma_{\mathrm{p,q}}(\mathrm{s})\Gamma_{\mathrm{p,q}}(\mathrm{t})}{\Gamma_{\mathrm{p,q}}(\mathrm{s+t})} \tag{4.2.5}$$

Proof:

First, for $s,t \in \mathbb{N}$ we have:

$$\mathrm{B}_{p,q}(s,t) = \int_0^1 (pz)^{s-1}(p \ominus pqz)_{p,q}^{t-1}\, d_{p,q}z,$$

Now, apply (p,q)-integral by parts for $f(x) = z^{s-1}$ and $g(x) = -\frac{(p \ominus pz)_{p,q}^{t}}{p[t]_{p,q}}$, where $D_{p,q}(p \ominus pz)_{p,q}^{t} = -[t]_{p,q} p(p \ominus pqz)_{p,q}^{t-1}$,

then we get

$$\mathrm{B}_{p,q}(s,t) = \frac{[s-1]_{p,q}}{p^{s-1}[t]_{p,q}} \int_0^1 (pz)^{s-2}(p \ominus pqz)_{p,q}^{t}\, d_{p,q}z,$$

$$= \frac{[s-1]_{p,q}}{p^{s-1}[t]_{p,q}} \mathrm{B}_{p,q}(s-1, t+1), \tag{4.2.6}$$

We can write, for positive integer :

$$\mathrm{B}_{p,q}(s,t+1) = \int_0^1 (pz)^{s-1}(p \ominus pqz)_{p,q}^{t-1+1}\, d_{p,q}z,$$

$$= \int_0^1 (pz)^{s-1}(p \ominus pqz)_{p,q}^{t-1}\,(p^t \ominus pq^t z) d_{p,q}z,$$

$$= p^t \int_0^1 (pz)^{s-1}(p \ominus pqz)_{p,q}^{t-1} d_{p,q}z - q^t \int_0^1 (pz)^{s}(p \ominus pqz)_{p,q}^{t-1} d_{p,q}z$$

$$= p^t \mathrm{B}_{p,q}(s,t) - q^t \mathrm{B}_{p,q}(s+1,t) \tag{4.2.7}$$

After that, using (4.2.6) to get

$$\mathrm{B}_{p,q}(s,t+1) = p^t \mathrm{B}_{p,q}(s,t) - q^t \frac{[s]_{p,q}}{p^s[t]_{p,q}} \mathrm{B}_{p,q}(s,t+1), \tag{4.2.8}$$

which mean

$$\mathrm{B}_{p,q}(s,t+1)\left(1 + q^t \frac{[s]_{p,q}}{p^s[t]_{p,q}}\right) = p^t \mathrm{B}_{p,q}(s,t),$$

$$\mathrm{B}_{p,q}(s,t+1)\left(\frac{p^s(p^t-q^t) + q^t(p^s-q^s)}{p^s(p^t-q^t)}\right) = p^t \mathrm{B}_{p,q}(s,t),$$

$$\mathrm{B}_{p,q}(s,t+1) = p^{s+t} \frac{(p^t-q^t)}{(p^{t+s}-q^{t+s})} \mathrm{B}_{p,q}(s,t), \tag{4.2.9}$$

We know

$$\mathrm{B}_{p,q}(s,1)=\int_0^1 (pz)^{s-1}\, d_{p,q}z=\frac{p^{s-1}}{[s]_{p,q}}, \tag{4.2.10}$$

For t positive integer we get

$$\mathrm{B}_{p,q}(s,t)=p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}\mathrm{B}_{p,q}(s,t-1),$$

$$=p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}p^{s+t-2}\frac{(p^{t-2}-q^{t-2})}{(p^{t+s-2}-q^{t+s-2})}\mathrm{B}_{p,q}(s,t-2),$$

$$=p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}p^{s+t-2}\frac{(p^{t-2}-q^{t-2})}{(p^{t+s-2}-q^{t+s-2})}\dots p^{s+1}\frac{(p-q)}{(p^{s+1}-q^{s+1})}\mathrm{B}_{p,q}(s,1),$$

$$=p^{s+t-1}\frac{(p^{t-1}-q^{t-1})}{(p^{t+s-1}-q^{t+s-1})}p^{s+t-2}\frac{(p^{t-2}-q^{t-2})}{(p^{t+s-2}-q^{t+s-2})}\dots p^{s+1}\frac{(p-q)}{(p^{s+1}-q^{s+1})}\frac{p^{s-1}}{[s]_{p,q}},$$

$$=\frac{p^{(s-1)+(s)+(s+1)+\cdots+(s+t-1)}}{p^{s}}\frac{(p\ominus q)_{p,q}^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q),$$

It mean that

$$\mathrm{B}_{p,q}(s,t)=p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q), \tag{4.2.11}$$

Letting $m=[t(2s+t-2)+t-2]/2.$

By using proposition (4.1.2) part (2) and (4.2.11) we have

$$\mathrm{B}_{p,q}(s,t)=p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q),$$

$$=p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p-q)^{t-1}}\frac{(p\ominus q)_{p,q}^{s-1}}{(p-q)^{s-1}}\frac{(p-q)^{s-1}}{(p\ominus q)_{p,q}^{s-1}}\frac{(p-q)^{t-1}}{(p^{s}\ominus q^{s})_{p,q}^{t}}(p-q),$$

$$=p^{m}\frac{(p\ominus q)_{p,q}^{t-1}}{(p-q)^{t-1}}\frac{(p\ominus q)_{p,q}^{s-1}}{(p-q)^{s-1}}\frac{(p-q)^{s+t-1}}{(p\ominus q)_{p,q}^{s+t-1}}=p^{m}\frac{\Gamma_{\mathrm{p,q}}(\mathrm{t})\Gamma_{\mathrm{p,q}}(\mathrm{s})}{\Gamma_{\mathrm{p,q}}(\mathrm{s+t})}.$$

4.3 The analogous definition of (p,q)-Gamma and (p,q)-Beta Functions

We defined the (p,q)-analogue for GF and BF and we found the relation between them in the previous section. In this section we want to study a new functions (namely $\tilde{\Gamma}_{p,q}(z)=\Gamma_{q/p}(z)$ and $\tilde{B}_{p,q}(s,t)=B_{q/p}(s,t)$) and we will show how these functions are relevant to (p,q)-GF and (p,q)-BF.

Definition 1: For $(0<q<p<1)$ we get

$$D_{p,q}g(z)=D_{q/p}g(z/p), \tag{4.3.1}$$

Proof: By definition of the (p,q)-derivative we know

$$D_{p,q}g(z)=\frac{g(pz)-g(qz)}{(p-q)z}, \qquad z\neq 0 \ .$$

$$=\frac{g\left(\frac{q}{p}pz\right)-g(pz)}{\left(\frac{q}{p}-1\right)pz}=D_{q/p}g(pz),$$

Definition 2: For $(0<q<p<1)$ we get

$$\int_0^b g(z)d_{p,q}z=\int_0^b g(z/p)d_{q/p}z.$$

Proof:

$$\int_0^b g(z)d_{p,q}z=(p-q)b\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}g\left(b\frac{q^j}{p^{j+1}}\right),$$

$$=(1-q/p)b\sum_{j=0}^{\infty}\left(\frac{q}{p}\right)^{j} g\left(b\left(\frac{q}{p}\right)^{j}\frac{1}{p}\right),$$

$$=\int_0^b g(z/p)d_{q/p}z.$$

Definition 3: The (q/p)-IBP is given by

$$\int_0^b g(z)D_{q/p}f(z)d_{q/p}z = g(z)f(z)\Big|_0^b - \int_0^b f(\frac{q}{p}z)D_{q/p}g(z)d_{q/p}z, \tag{4.3.2}$$

Definition 4: For $(0<q/p<1)$ we get

$$\tilde{\Gamma}_{p,q}(z)=\Gamma_{\frac{q}{p}}(z)=\frac{\left(1-\frac{q}{p}\right)_{q/p}^{\infty}}{\left(1-\left(\frac{q}{p}\right)^{z}\right)_{q/p}^{\infty}}\left(1-\frac{q}{p}\right)^{1-z}, \tag{4.3.3}$$

Proposition 4.3.1: For positive integer z we have

$$\tilde{\Gamma}_{p,q}(z)=p^{2z-1}\Gamma_{p,q}(z)\,,\quad 0<p<q<1. \tag{4.3.4}$$

Proof:

$$\tilde{\Gamma}_{p,q}(z)=\Gamma_{\frac{q}{p}}(z)=\frac{\left(1-\frac{q}{p}\right)_{\frac{q}{p}}^{\infty}}{\left(1-\left(\frac{q}{p}\right)^{z}\right)_{\frac{q}{p}}^{\infty}}\left(1-\frac{q}{p}\right)^{1-z}=\frac{\prod_{j=1}^{\infty}\left(1-(q/p)^{j}\right)}{\prod_{j=0}^{\infty}\left(1-(q/p)^{z+j}\right)}\left(1-\frac{q}{p}\right)^{1-z},$$

$$=\frac{\prod_{j=1}^{\infty}\left(\frac{p^{j}-q^{j}}{p^{j}}\right)}{\prod_{j=0}^{\infty}\left(\frac{p^{j+z}-q^{j+z}}{p^{j+z}}\right)}\frac{(p-q)^{1-z}}{p^{1-z}}=p^{2z-1}\frac{\prod_{j=1}^{\infty}\left(p^{j}-q^{j}\right)}{\prod_{j=0}^{\infty}\left(p^{j+z}-q^{j+z}\right)}(p-q)^{1-z},$$

$$=p^{2z-1}\frac{(p\ominus q)_{p,q}^{\infty}}{(p^{z}\ominus q^{z})_{p,q}^{\infty}}(p-q)^{1-z}=p^{2z-1}\Gamma_{p,q}(z),\qquad 0<\frac{q}{p}<1.$$

Definition 5: For $s,t>0$, then

$$\tilde{B}_{p,q}(s,t) = B_{\frac{q}{p}}(s,t) = \int_0^1 z^{s-1}\left(1-\frac{q}{p}\right)_{q/p}^{t-1} d_{q/p}z,$$

also we can write the previous expression as:

$$\tilde{B}_{p,q}(s,t) = B_{\frac{q}{p}}(s,t) = \frac{\Gamma_{\frac{q}{p}}(s)\Gamma_{\frac{q}{p}}(t)}{\Gamma_{\frac{q}{p}}(s+t)}, \tag{4.3.5}$$

Proposition 4.3.2: For $s, t > 0$, we have

$$\tilde{B}_{p,q}(s,t) = \frac{1}{p}B_{p,q}(s,t), \tag{4.3.6}$$

Proof: By using the equation (4.3.4) we get

$$\tilde{B}_{p,q}(s,t) = B_{\frac{q}{p}}(s,t) = \frac{\Gamma_{\frac{q}{p}}(s)\Gamma_{\frac{q}{p}}(t)}{\Gamma_{\frac{q}{p}}(s+t)} = \frac{p^{2s-1}\Gamma_{p,q}(s)p^{2t-1}\Gamma_{p,q}(t)}{p^{2s+2t-1}\Gamma_{p,q}(s+t)},$$

$$= \frac{1}{p}\frac{\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)} = \frac{1}{p}B_{p,q}(s,t),$$

Chapter 5

CONCLUSION

At the end of this sudy, we explained the integral representation of the q-analogue of two special functions, q-GF and q-BF which gave a very attractive q-constant.Also, we found the proof of the famous Jacobi triple product, containing the identity of Jacobi and after that we obtained a new proof for Ramanujan's equation. Furthermore, we introduced a new generalization of GF and BF that were: the (p,q)-GF and the (p,q)-BF. Finally, we obtained equivalent definitions for (p,q)-analogue for GF and BF. My contribution was to briefly explain and proof all the formulas in a simple way, where the readers can easily understand the final results and we obtained a new equivalent definitions for (p,q)-analogue for GF and BF.

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