Fluid Mechanics and Navier-Stokes Equations

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
January 2017
Gazimuğusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

Navier-Stokes differential equations constitute the most prominent mathematical model for explaining and describing the behavior of fluids. These equations found the basis of fluid mechanics and have countless applications in numerous fields. Besides their wide applications, the problem of existence and smoothness of solutions of Navier-Stokes equations, remains of the most challenging open problem of mathematics. Regarding the practical and scientific significance of these equations, this thesis discusses the details of derivation of Navier-Stokes equations and other related issues of fluid mechanics. Furthermore, some of the key mathematical theorems and techniques used for derivation and analysis of Navier-Stokes equations are explained and reviewed.

Keywords: Navier-Stokes equations, fluid dynamics, fluid mechanics

ÖZ

Navier-Stokes diferansiyel denklemleri akışkanlar mekaniği için çok önemlidir ve birçok farklı alanda uygulamalara sahiptir. Bu denklemler akışkanların hareketini açıklamaktadır. Bu tez, akışkanların temel mekaniklerini incelemekte ve Navier-Stokes diferansiyel denklemlerini ele almaktadır.

Anahtar Kelimeler: Navier-Stokes denklemleri, Akışkanlar dinamiği, akışkanlar mekaniği

ACKNOWLEDGMENT

I would like to express my deepest gratitude to my supervisor, Prof. Dr. Aghamirza Bashirov, for his kind advices and wise guidance, without which this thesis could not have been written. I would also like specifically to thank the Vice Chair of EMU Mathematics Department, Prof. Dr. Sonuç Oğurlu, for her warm and valuable help. And as always, I am sincerely grateful to all my teachers, from whom I have learned what I know.

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Chapter 1

INTRODUCTION

Navier-Stokes equations, named after the Irish mathematician and physicist George Stokes and French physicist Claude-Louis Navier, are the most fundamental equations at the core of fluid mechanics and provide a mathematical model for describing the governing dynamics of fluids. These equations are of significant scientific and practical importance in numerous fields, including but not limited to fluid mechanics, mechanical engineering, geophysics, aerodynamics, astrophysics, thermal engineering, ocean engineering, and biology. Navier-Stokes equations are extensively used in weather forecasts, aircraft design, engineering of piping systems, design of power stations, study of blood circulation in body, and many other areas.

Apart from their numerous practical applications, Navier-Stokes equations have also been amongst the major mathematical challenges of the 21th century, and have generated a large body of purely mathematical research. The problem of existence and smoothness of solution of Navier-Stokes equations for three and higher dimensions, remains one of the open millennial problems in mathematics.

Given the indisputable importance of Navier-Stokes equations, the aim of the current thesis is to provide a review of Navier-Stokes equations in fluid mechanics and an in-depth discussion of their derivation. The thesis is divided into seven chapters, starting with the introduction and moving to the second chapter, which provides a brief review of key required material from vector calculus and fluid mechanics. Some important theorems of vector calculus including Stokes and divergence theorems together with some basic notions of fluid mechanics such as viscosity and pressure, are provided and discussed.

The third chapter focuses in more details on the notions of pressure and pressure gradients and discusses of how the gradient of the pressure field relates to the force per unit volume produced by it. Furthermore, Pascal's Law for isotropic fluids is reviewed and other properties of pressure fields are discussed. The fourth chapter introduces the notions of internal stresses in fluids and briefly presents the mathematical methods used for analyzing stresses in fluids.

The fifth chapter deals with the kinematics of fluid flow, introducing the Lagrangian and Eulerian coordinates and material derivative. The chapter continues with a discussion about control volumes and physical interpretations of the gradient of the velocity field. The last section of the chapter, states and proves the Reynolds transport theorem for both fixed and deformable control volumes. Chapter 6 mainly focuses on derivation of mass and linear momentum conservation equations for both compressible and incompressible fluids. The chapter also provides a detailed discussion about the applications of Reynolds transport theorem and Cauchy stress theorem in deriving the conservation of linear momentum equation.

The last chapter of the current thesis, discusses the derivation of Navier-Stokes equations for compressible and incompressible fluids. Decomposition of stress tensor into volumetric and deviatoric tensors is explained and Stokes constitutive equations are employed to calculate the deviatoric tensor. Finally, Navier-Stokes equations are driven for both compressible and incompressible flows and the details are discussed.

Chapter 2

PRELIMINARIES

The first section of this chapter reviews some of the key notions and theorems of vector calculus that are required for later chapters. It starts with an introduction to vector and scalar fields, and continues with explaining the gradient, divergence, and curl operators and their properties. Subsequently, the section reviews the details of line, surface, and volume integration in vector spaces and explains the Gauss, divergence and Stokes theorems. The second section of the chapter provides an introduction to the basics of fluid mechanics and some of the essential thermodynamic notions, such as density and pressure, that are used in the formulation of Navier-Stokes equations, are briefly explained. This chapter is written based on the textbooks of Rutherford [1989] and Fay [1998].

2.1 Vector Calculus Review

A scalar field on $\mathbb{R}^3$ could be defined as a function $g:\mathbb{R}^3 \to \mathbb{R}$ and a vector field on $\mathbb{R}^3$ could be defined as a function $f:\mathbb{R}^3 \to \mathbb{R}^3$ that has the following general form:

$$f(x,y,z) = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$$

where f_1, f_2, and f_3 are scalar fields on $\mathbb{R}^3$. Examples of scalar and vector fields include;

$$\psi(x,y,z) = x^2 + z^2$$
$$\vec{V}(x,y,z) = x\sin^2(y)\vec{i} + y\cos^2(x)\vec{j} + z\vec{k}$$
$$\mathrm{E}(x,y,z) = e^{-z^2} + xe^{-x^2}$$
$$\vec{T}(x,y,z) = \frac{\partial^2 \mathrm{E}}{\partial x^2}\vec{i} + \mu x^2 \frac{\partial^2 \mathrm{E}}{\partial y^2}\vec{j} + \sinh(y+z)\frac{\partial^2 \mathrm{E}}{\partial z^2}\vec{k}$$

2.1.1 Gradient, Divergence, and Curl Operators

The gradient of a scalar field, S, is defined as:

$$Grad\ S \equiv \frac{\partial S}{\partial x}\vec{i} + \frac{\partial S}{\partial y}\vec{j} + \frac{\partial S}{\partial z}\vec{k}$$

The gradient of a scalar field, is a vector field such that for every point, the gradient of the scalar field at that point, heads to the direction with the maximum rate of increase in the field. By using the ∇ differential operator, defined as:

$$\nabla \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

The gradient of S can be written as:

$$\nabla \mathrm{S} = \frac{\partial S}{\partial x}\vec{i} + \frac{\partial S}{\partial y}\vec{j} + \frac{\partial S}{\partial z}\vec{k}$$

The divergence of a vector field V is a scalar field that is defined as:

$$div\, V \equiv \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

The divergence of a vector field can be written as its inner product with the ∇ operator;

$$\nabla \, . \, \mathrm{V} = (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}) \, . \, (V_1\vec{i} + V_2\vec{j} + V_3\vec{k}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

The Curl of a vector field is another vector field that is defined as:

$$Curl\, V \equiv (\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z})\vec{i} + (\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x})\vec{j} + (\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y})\vec{k}$$

The curl of a vector field can be written as its cross product with the ∇ operator;

$$\nabla \times V = (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}) \times (\mathrm{V}_1\vec{i} + \mathrm{V}_2\vec{j} + \mathrm{V}_3\vec{k}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{V}_1 & \mathrm{V}_2 & \mathrm{V}_3 \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} =$$

$$(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z})\vec{i} + (\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x})\vec{j} + (\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y})\vec{k}$$

Another important differential operator that is commonly used in vector calculus is the Laplacian operator that is defined as the inner product of the *del* operator with itself;

$$\Delta \equiv \nabla \,.\, \nabla = (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}) \,.\, (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In this regard, the Laplacian of a scalar field is a scalar field, and the Laplacian of a vector field is a vector field.

Some of the important relations among the gradient, divergence and curl operators are summarized in the next theorem.

Theorem 2.1

For any scalar fields S and S', and vector fields V and V', the following equations hold:

2.1.1) $curl\ (grad\ S) = 0$
2.1.2) $div\ (curl\ V) = 0$
2.1.3) $\nabla(SS') = S\nabla S' + S'\nabla S$
2.1.4) $\nabla \, . \, (SV) = \nabla S \, . \, V + S\nabla \, . \, V$
2.1.5) $\nabla \times (SV) = \nabla S \times V + S\nabla \times V$
2.1.6) $\nabla \, . \, (V \times V') = V' \, . \, \nabla \times V - V \, . \, \nabla \times V'$

2.1.2 Line, Surface, and Volume integrals

Let $u(r) = u_1(r)\vec{i} + u_2(r)\vec{j} + u_3(r)\vec{k}$, be a piecewise smooth curve restricted to $a \le r \le b$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be an arbitrary Riemann integrable function. Then the line integral of f along u is defined as:

$$\int_u f(u_1, u_2, u_3)ds = \int_a^b f(u_1(r), u_2(r), u_3(r))\sqrt{(\frac{du_1}{dr})^2 + (\frac{du_2}{dr})^2 + (\frac{du_3}{dr})^2}dr$$

Line integration can also be defined for vector fields in the following way. Let $F = F_1(u_1, u_2, u_3)\vec{i} + F_2(u_1, u_2, u_3)\vec{j} + F_3(u_1, u_2, u_3)\vec{k}$ be a real vector field on $\mathbb{R}^3$, and let $F_1, F_2,$ and F_3 be Riemann integrable functions. Furthermore, let $C(r)$ be a piecewise smooth curve in $\mathbb{R}^3$, restricted to $a \le r \le b.$ Then the line integral of F along C, is defined as:

$$\int_C F \, . \, d\vec{S} = \int_C F_1 dx + \int_C F_2 dy + \int_C F_3 dz$$

Note that here $d\vec{S}$ is the vector curve element, and is defined as $d\vec{S} = \vec{n}\, ds$, where $\vec{n}$ is normal unit vector of the curve. Let $u(r,s) = u_1(r,s)\vec{i} + u_2(r,s)\vec{j} + u_3(r,s)\vec{k}$ be a smooth surface in $\mathbb{R}^3$, restricted to $a \le r \le b$ and $c \le s \le d$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be an arbitrary Riemann integrable function. Then the surface integral of f over u, is defined as:

$$\iint_u f(u_1,u_2,u_3)dA = \int_a^b \int_c^d f(u_1(r,s),u_2(r,s),u_3(r,s)) \| \frac{\partial u}{\partial r} \times \frac{\partial u}{\partial s} \| dsdr$$

Surface integration for vector fields can be defined in a similar way. Let $u(r,s)$ be a 2-parameter oriented smooth surface over the region $a \le r \le b$ and $c \le s \le d$, and let $F = F_1(u_1,u_2,u_3)\vec{i} + F_2(u_1,u_2,u_3)\vec{j} + F_3(u_1,u_2,u_3)\vec{k}$ be a vector field on $\mathbb{R}^3$, where $F_1, F_2,$ and F_3 are Riemann integrable functions. Then the surface integral of F over u is defined as:

$$\iint_u F \,.\, d\vec{A} = \iint_u F_1 dA + \iint_u F_2 dA + \iint_u F_3 dA$$

Note that here $d\vec{A}$ is the vector surface element, and is defined as $d\vec{A} = \vec{n}\, dA$, where $\vec{n}$ is the outward normal unit vector of the surface.

2.1.4 Divergence and Stokes Theorems

Divergence theorem is one of the fundamental theorems in vector calculus that relates the surface integral of a vector field to the volume integral of its divergence.

Theorem 2.3 (Divergence Theorem)

Let F be a vector field on $\mathbb{R}^3$, let ∂V be a smooth completely closed surface in $\mathbb{R}^3$, and let V be the volume enclosed by ∂V. Then the following integral equation holds:

$$\iiint_V \nabla \,.\, F dV = \oiint_{\partial V} F \,.\, d\vec{S} \qquad (2.1)$$

Stokes theorem relates the surface integral of the curl of a vector field on a closed smooth surface to its line integral on the boundary of that surface.

Theorem 2.4 (Stokes Theorem)

Let $F:\mathbb{R}^3 \to V(\mathbb{R}^3)$ be a vector field and let ∂S be a closed smooth 1-parameter curve in $\mathbb{R}^3$ let S be the surface enclosed by ∂S. Then

$$\iint_S (\nabla \times F)\,.\,\mathrm{d}\vec{A} = \oint_{\partial S} F\,.\,d\vec{S} \tag{2.2}$$

2.2 Basic Notions of Fluid Mechanics

2.2.1 Density

Density of a fluid, generally denoted by ρ, is defined as the mass contained in its unit volume, and is one of the fundamental physical properties of fluids. The density of a fluid determines its mass per unit volume and hence determines how the fluid reacts to the forces acting upon it. Fluids with lower density accelerate more rapidly in presence of a force and have generally weaker viscous stresses, however fluids with higher density are harder to accelerate and usually have stronger viscous stresses.

As such, low density fluids are in general more turbulent than high density fluids. Moreover, density is more likely to change with the thermodynamic conditions of the fluid such as its temperature or pressure, when its level is low. Theoretically, density is assumed to be the mass per unit volume contained in a fluid element and hence can be treated as a continuous scalar field.

2.2.2 Pressure

The pressure on an element of a hydrostatic fluid is the compressive force that acts on the boundary of the fluid element, pressing it inwards in all directions. The compressive pressure force is in fact produced through the molecular interactions on the boundary of the fluid element. The pressure on a fluid element can significantly change, relative to the behavior of the velocity field, external forces and thermodynamic properties of the fluid, such as its temperature and density. In fact, pressure and velocity fields are coupled in all fluids and any change in one could produce a change in the other. By and large, pressure gradients produce motion in fluids, with the flow heading from high pressure regions to low pressure ones.

2.2.3 Viscosity

Viscosity of a fluid is a measure of how the fluid resists shear deformations caused by its internal stresses. In fact, viscous fluids are more likely to resist the flow and preserve their shape against the shear forces acting upon them, compared to fluids with low viscosity, that are more susceptible to strain. The above definition illustrates the notion of viscosity in a plain manner, but does not provide a quantitative method for measuring viscosity. So as to find such a quantitative definition, a simple experimental setting can be assumed, where a layer of fluid is placed between two parallel plates.

Now assume that the lower plate is fixed and the upper plates moves at a constant velocity. Then a reasonable measure of fluid’s resistance to flow could be the amount of force per unit area, required for moving the upper plate. Evidently, the required force per unit area would to be higher for fluids with higher viscosity that are more resistant to strain, and lower for fluids with lower viscosity. In practice, experimental data has indicated that the required force is proportional to the velocity with which the plate is moving, and inversely proportional to the distance between the plates.

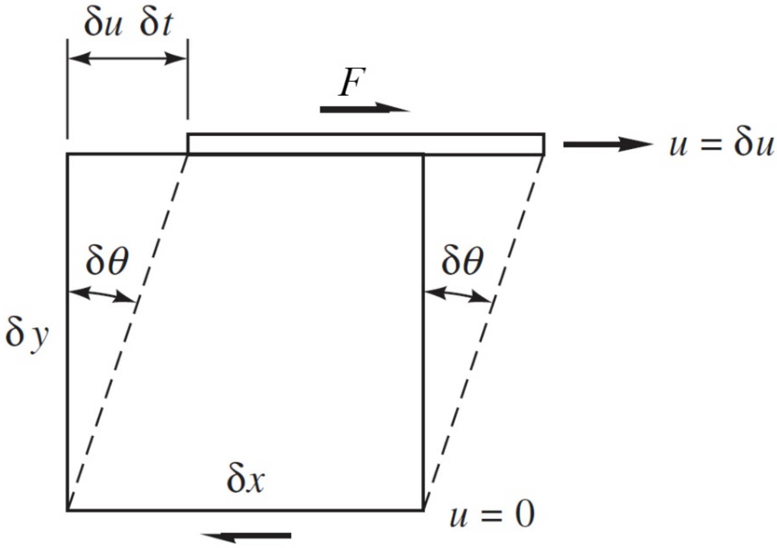

Figure 2.1: A layer of viscous fluid is placed between two plates. The upper plate is moving with a constant velocity, while the other plate is fixed.

If we let F denote the force per unit area required for moving the plate with the constant velocity Δu, and let Δy denote the distance between the two plates. Then based on numerous experimental observations in many known fluids, the following equations holds:

$$F = \mu \frac{\Delta u}{\Delta y} \tag{2.3}$$

where μ is the constant of proportionality. Note that this observation was first made by Sir Isaac Newton, and fluids for which the equation (2.3) holds, are called Newtonian Fluids. As it could be seen in Figure 2.1, $\Delta\theta$ indicates the angular deformation of the fluid due to the shear stress, in the time interval Δt, and therefore, $\frac{\Delta\theta}{\Delta t}$ is the average time-rate of shear deformation in the fluid. Evidently,

$$\tan(\Delta\theta) = \frac{\Delta u \Delta t}{\Delta y} \quad \Rightarrow \quad \frac{\tan(\Delta\theta)}{\Delta t} = \frac{\Delta u}{\Delta y}$$

Taking the limit of both sides as the time interval approaches zero yields;

$$\left.\begin{aligned} \lim_{\Delta t \to 0} \frac{\tan(\Delta\theta)}{\Delta t} = \frac{d\theta}{dt} \\ \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y} = \frac{du}{dy} \end{aligned}\right\} \Rightarrow \quad \frac{d\theta}{dt} = \frac{du}{dy} \tag{2.4}$$

The equation (2.5) relates the rate of deformation in the fluid with the rate of change in its velocity with respect to y. Then putting equations (2.3) and (2.4) together we obtain:

$$F = \mu \frac{du}{dy} = \mu \frac{d\theta}{dt}$$

Note that the constant of proportionality in this equation, does in fact provide a measure of fluid's resistance to shear strain, which was the intuitive definition of viscosity. Hence, μ can be reasonably used as a quantitative measure of viscosity for Newtonian fluids, and is called the viscosity constant. Furthermore, experimental data have revealed that in most known Newtonian fluids, thermodynamic conditions do not significantly change the viscosity of the fluid, and therefor it can be assumed constant.

Chapter 3

PRESSURE FIELDS

The first section of the following chapter introduces the notion of pressure gradient and provides a thorough investigation of how the gradient of the pressure field relates to the force per unit volume produced by it. Pascal's Law for isotropic fluids is stated and proved, and further properties of pressure fields are explored. The second section of the chapter generalizes the findings to an arbitrary fixed control volume, and shows how the total pressure force acting on the surface of the control volume can be obtained by integrating the gradient of the pressure field. This chapter is organized and written based on the books of Fay [1998] and White [2011] on fluid mechanics.

3.1 Pressure Gradient

It is known that shear stresses are zero in hydrostatic fluids, and therefore stress analysis at any point of the fluid would be restricted to that point. That is to say, the Mohr circle for any fluid element becomes a point, and the only force to consider would be the pressure. This evidently implies that pressure in a hydrostatic fluid is a point property that can be fully described by a scalar field. This scalar field associates with each point in the fluid, the magnitude of the force, compressing the fluid element at that point, equally in all directions.

For a more in-depth investigation of how the pressure field acts in a fluid, assume an infinitesimally small right triangular prism, where Δx and Δz are the perpendicular legs of the lower base, Δs is the hypotenuse, and Δy is the distance between the bases. Moreover, let σ_x and σ_z denote the pressure forces acting on the volume, respectively in x and z directions, and let σ denote the pressure force in the direction of the normal unit vector of the hypotenuse. Then since the fluid is hydrostatic, all fluid elements in the volume have zero acceleration, and hence the sum of forces acting on the volume must be zero.

$$\begin{cases} 0 = \sum F_x = \sigma_x \Delta y \Delta z - \Delta s \Delta y \sigma \sin\phi \\ 0 = \sum F_z = \sigma_z \Delta y \Delta x - \Delta s \Delta y \sigma \cos\phi - \dfrac{\rho g \Delta x \Delta y \Delta z}{2} \end{cases} \tag{3.1}$$

where ϕ denotes the angle facing Δz, g is the standard gravity constant, and the gravity force is assumed to be acting in the z direction, pressing the fluid downwards.

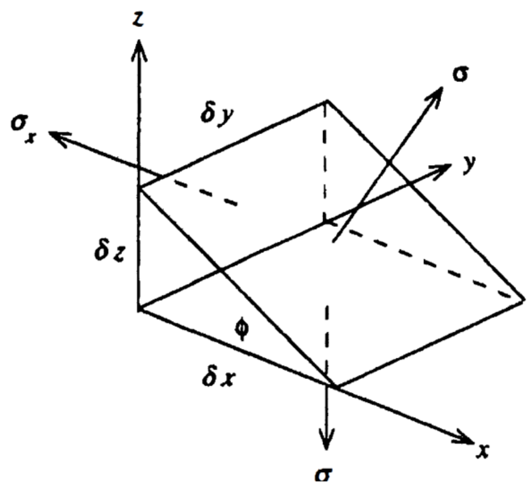

Figure 3.1: A differentially small right triangular prism, containing hydrostatic fluid

Considering Figure 3.1, evidently $\Delta z = \Delta s \sin\phi$ and $\Delta x = \Delta s \cos\phi$. Substituting these into equation (3.1), yields:

$$\begin{cases} \sigma_x = \sigma \\ \sigma_z = \sigma + \dfrac{\rho g \Delta z}{2} \end{cases} \tag{3.2}$$

These equations indicate that the pressure field is constant in x and y directions, and changes in the z direction, proportional to Δz. This property is in fact shared by all hydrostatic fluids, and is of critical importance in hydrostatics. Using equations (3.1) and taking their limit as Δz approaches to zero, yields:

$$\sigma_x = \sigma_z = \sigma \tag{3.4}$$

Note that when $\Delta z \to 0$, the triangular prism shrinks into a point, and since the angle θ can be chosen arbitrarily, equation (3.4) proves that at any point of a hydrostatic fluid, the pressure force acts equally in all directions. This result is known as the Pascal's Law and is one of the fundamental properties of isotropic hydrostatic fluids.

Now, let P denote the pressure field in an isotropic fluid, and consider the fluid element illustrated in Figure 3.2. Knowing that pressure is the compressive force per unit area that is normal to the surface of the fluid element, the force pressing the left $dydz$ side of the fluid element in the x direction, would be $P(x_0, y_0, z_0)dydz$, where (x_0, y_0, z_0) is the mid-point of the left $dydz$ face. Likewise, the pressure force acting on the right $dydz$ face of the fluid element in the x-direction is:

$$-(P(x_0, y_0, z_0) + \frac{\partial}{\partial x} P(x_0, y_0, z_0)dx)dydz$$

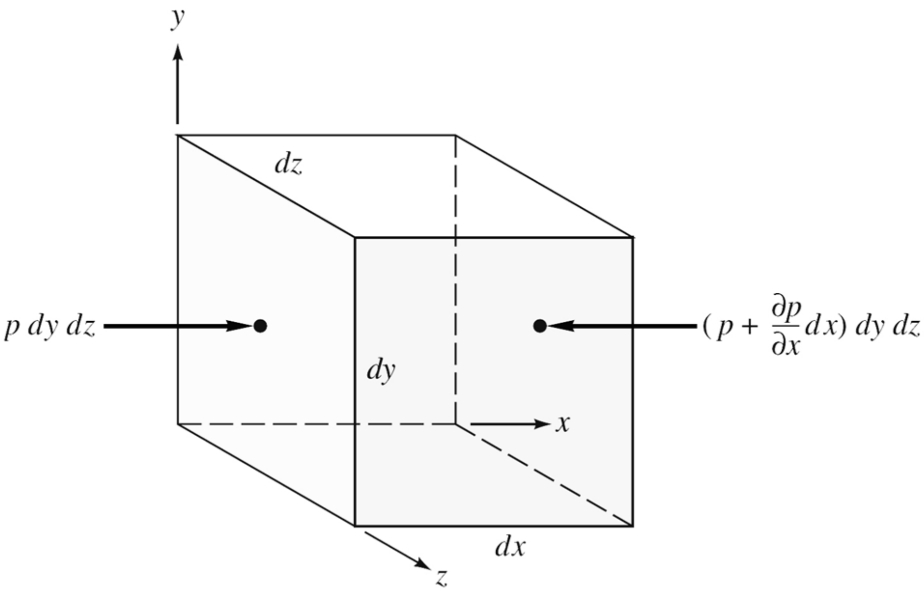

Figure 3.2: The pressure force acting on the surface of a fluid element, in the x-direction.

Therefore the net force produced by the pressure field, acting on the fluid element in the x-direction is:

$$dF_x = Pdydz - (P + \frac{\partial}{\partial x} Pdx)dydz = -\frac{\partial}{\partial x} Pdxdydz$$

In a similar manner, it can be shown that the net forces produced by pressure field in the y and z-directions, are respectively:

$$dF_y = Pdxdz - (P + \frac{\partial}{\partial y} Pdy)dxdz = -\frac{\partial}{\partial y} Pdxdydz$$

$$dF_z = Pdxdy - (P + \frac{\partial}{\partial z} Pdz)dxdy = -\frac{\partial}{\partial z} Pdxdydz$$

Hence, the net force acting on the surface of the element fluid can be written as:

$$dF = dF_x \vec{i} + dF_y \vec{j} + dF_z \vec{k} = -(\frac{\partial}{\partial x} P\vec{i} + \frac{\partial}{\partial y} P\vec{j} + \frac{\partial}{\partial z} P\vec{k})dv$$

$$= -(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k})Pdv = -\nabla Pdv$$

Therefore

$$dF_{total} = -\nabla P dv \quad \Rightarrow \quad -\nabla P = \frac{dF_{total}}{dv} = dF_{(per\ unite\ volume)} \tag{3.5}$$

Equation (3.5) clearly shows that the net force per unit area is in fact produced by the gradient of the pressure field, and not the pressure field itself.

3.2 Pressure force on an Arbitrary Volume

Evidently, to obtain the pressure forces acting upon the boundary of an arbitrary control volume V the pressure field must be integrated over ∂V. It is trivial that at any point on the boundary of V, the pressure force acts in the direction of the inward normal unit vector at that point. Hence, at each point (x, y, z) on ∂V, the pressure force can be written as $-P(x, y, z)\vec{n}$, and the total pressure force acting on ∂V can be obtained by integrating $-P\vec{n}$ over ∂V.

$$F_{total} = \oiint_{\partial V} -P\vec{n}\, dS = \oiint_{\partial V} -P d\vec{S} \tag{3.6}$$

Applying the Gauss theorem, to equation (3.6) yields:

$$F_{total} = \oiint_{\partial V} -P d\vec{S} = \iiint_{V} -\nabla P dv$$

Moreover, note that:

$$F_{total} = \iiint_{V} F_{(per\ unite\ volume)}\, dV = \iiint_{V} -\nabla P dV \quad \Rightarrow$$
$$\iiint_{V} (F_{(per\ unite\ volume)} + \nabla P) dv = 0 \quad \Rightarrow$$
$$F_{(per\ unite\ volume)} = -\nabla P$$

Chapter 4

STRESS ANALYSIS

This chapter explains internal stresses in fluids and provides the required mathematical background for describing and analyzing such stresses. The stress tensor and its derivation are briefly explained and reviewed. The last section of this chapter provides a detailed explanation and the proof of the Cauchy's stress theorem, that is one of the fundamental and theoretical basis of stress analysis in fluid mechanics and plays a key role in derivation of Navier-Stokes differential equations. The current chapter is prepared and written based on the books of Rutherford [1989], White [2011], and Fay [1998].

4.1 Stress in Fluids

The action of a surface force on the boundary of a control volume in a fluid, condenses the layer of fluid molecules at the boundary and reduces the distance of the boundary and the adjacent molecules inside the volume. Consequently, the molecules inside the contracted layer repel each other and impose a compressing force on the next adjacent layer, and as a result of these molecular interactions, the force is conveyed through the fluid. Such internal forces, that are produced and transmitted through the molecular interactions within the fluid, are called stresses.

To obtain a quantitative definition of internal stress, assume a control volume, V , that is contained in a fluid. The molecules situated right outside the control volume surface, exert a force per unit area on the fluid elements on the boundary. This force per unit area is called the stress acting on the surface of control volume, and is denoted by σ.

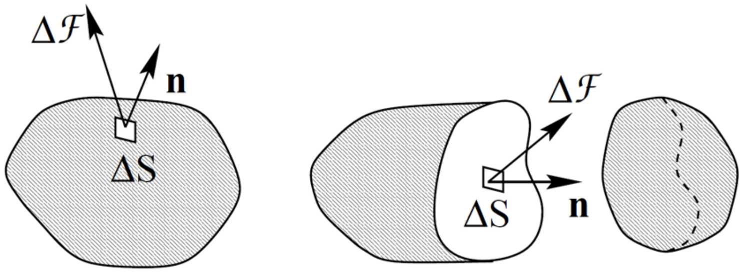

Figure 4.1: An arbitrary surface element on the boundary of a control volume and the net surface force acting on it.

More precisely, let ΔS denote the surface element of the control volume at point A, and let ΔF be the net surface force acting upon ΔS. Then the stress vector at point A can be defined as:

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}$$

In general, surface forces can be categorized as shear and normal stresses. The shear component at any point is tangent to the surface at that point, and the normal stress is perpendicular to the surface. That is to say, the normal stress is always parallel to the normal unit vector of the surface and the shear stress is always perpendicular to it.

4.2 Stress Tensor

As will be shown in the next section, in order to determine normal and shear stresses on any surface element with arbitrary orientation, its sufficient to have the stresses on the three standard surface elements, $dxdy$, $dxdz$, and $dydz$. The stress force acting on $dxdz$ composes of one normal and two shear stress vectors, denoted by σ_{yy}, σ_{yx}, and σ_{yz} respectively. The normal stress vector, σ_{yy}, is perpendicular to the $dxdz$ surface in the y direction. The shear stresses σ_{yx} is tangent to $dxdz$ in the x direction, and σ_{yz} is tangent to $dxdz$ in the z direction.

Hence, using the vector notation, the stress force acting on $dxdz$ surface element, can be written as:

$$T_{xz} = (\sigma_{yx}, \sigma_{yy}, \sigma_{yz})$$

Similarly, stress forces acting on the $dxdy$ and $dydz$, can be written as:

$$T_{yz} = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz})$$
$$T_{xy} = (\sigma_{zx}, \sigma_{zy}, \sigma_{zz})$$

These three stress vectors can be summarized in a second-rank tensor, called the stress tensor, that is defined as:

$$T = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{4.1}$$

4.3 Cauchy Stress Theorem

This section deals with the Cauchy's stress theorem, that states the stress vectors on a surface element with arbitrary orientation can be written as a linear combination of the stress vectors on $dxdy$, $dydz$, and $dxdz$ surface elements.

More precisely, if T is the stress tensor at a given point, and $\vec{n}$ is the normal unit vector of an arbitrary surface at that point, then

$$T_{(n)} = \vec{n} \, . T$$

where $T_{(n)}$ is the stress vector for the given surface at the given point. To prove Cauchy's theorem, let D be an imaginary tetrahedron volume in a fluid. As is shown in Figure 4.2, faces ΔS_1, ΔS_2, and ΔS_3 are parallel to the xy, xz, and yz planes respectively, and the oblique surface, ΔS, has an arbitrary orientation.

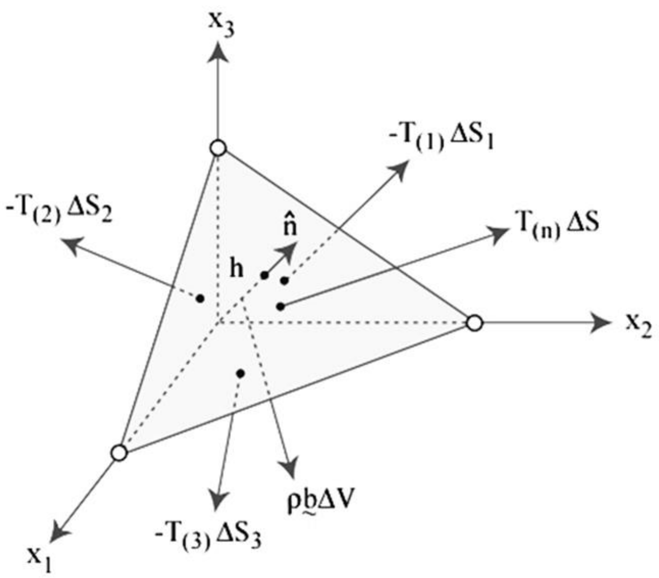

Figure 4.2: Cauchy tetrahedron.

Regarding the fact that the mass of the volume D, can be written as $\frac{h\rho g}{3}\Delta S$, The Newton's second law of motion states that:

$$\sum F = \frac{h\rho g}{3}\Delta S\vec{A} \qquad (4.2)$$

where $\vec{A}$ denotes the acceleration of D . Note that the sum of forces acting on the volume D can be written as:

$$\sum F = T_{(n)}\Delta S - T_{(1)}\Delta S_1 - T_{(2)}\Delta S_2 - T_{(3)}\Delta S_3 \tag{4.3}$$

Putting equations (4.2) and (4.3) together, we obtain:

$$\frac{h\rho g}{3}\Delta S\vec{A} = T_{(n)}\Delta S - T_{(1)}\Delta S_1 - T_{(2)}\Delta S_2 - T_{(3)}\Delta S_3 \tag{4.4}$$

Now assuming that, $\vec{n} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, the areas of the three perpendicular faces of the tetrahedron, can be written as the projections of the oblique surface on xy , xz , and yz planes, in the following manner:

$$\begin{cases} \Delta S_1 = (\vec{i}\,.\,\vec{n})\Delta S = a_1\Delta S \\ \Delta S_2 = (\vec{j}\,.\,\vec{n})\Delta S = a_2\Delta S \\ \Delta S_3 = (\vec{k}\,.\,\vec{n})\Delta S = a_3\Delta S \end{cases}$$

Substituting these equalities into equation (4.4), yields:

$$\frac{h\rho g}{3}\Delta S\vec{A} = T_{(n)}\Delta S - T_{(1)}a_1\Delta S - T_{(2)}a_2\Delta S - T_{(3)}a_3\Delta S$$

$$\Rightarrow \quad T_{(n)} - a_1T_{(1)} - a_2T_{(2)} - a_3T_{(3)} = \frac{h\rho g}{3}\vec{A}$$

Taking the limit of both sides, as the height of tetrahedron goes to zero, will make the right-hand side of the equation zero, and we obtain:

$$T_{(n)} = a_1T_{(1)} + a_2T_{(2)} + a_3T_{(3)}$$

This clearly proves that the stress vector for an arbitrary surface element, can be written as a linear combination of the stress vectors of $dxdy$, $dxdz$, and $dydz$.

By expanding the stress vectors of $dxdy$, $dxdz$, and $dydz$, we obtain:

$$\begin{aligned} T_{(1)} &= T_1^1\vec{i} + T_1^2\vec{j} + T_1^3\vec{k} \\ T_{(2)} &= T_2^1\vec{i} + T_2^2\vec{j} + T_2^3\vec{k} \\ T_{(3)} &= T_3^1\vec{i} + T_3^2\vec{j} + T_3^3\vec{k} \end{aligned} \tag{4.5}$$

Note that, T_i^j's in equations (4.5), are in fact the elements of the stress tensor, such that $T_i^j = \sigma_{ij}$, and hence we can write:

$$\begin{aligned} T_{(1)} &= \sigma_{xx}\,\vec{i} + \sigma_{xy}\,\vec{j} + \sigma_{xz}\,\vec{k} \\ T_{(2)} &= \sigma_{yx}\,\vec{i} + \sigma_{yy}\,\vec{j} + \sigma_{yz}\,\vec{k} \\ T_{(3)} &= \sigma_{zx}\,\vec{i} + \sigma_{zy}\,\vec{j} + \sigma_{zz}\,\vec{k} \end{aligned}$$

Summarizing these equations in the matrix form, yields the Cauchy stress theorem:

$$(T_{(n)}^1, T_{(n)}^2, T_{(n)}^3) = (n_1, n_2, n_3)\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \Rightarrow$$

$$T_{(n)} = \vec{n}\,.\,T \tag{4.6}$$

In the next chapter, it will be shown that the stress tensor is in fact symmetric and can be represented with 6 variables at each point.

Chapter 5

KINEMATICS OF FLUID FLOW

The first section of this chapter introduces the concept of Eulerian and Lagrangian coordinate systems. The two reference system are explained and contrasted in details, so as to provide a firm theoretical ground for the subsequent material that are based on these coordinates. The second section of the chapter, explains the material derivative of a field and its physical interpretation. Furthermore, local and convective derivatives are described and their relation with material derivative is explained. In the third section of the chapter, the gradient of the velocity field is explained in more details and a thorough review of both its physical and mathematical interpretations is provided. The last section, deals with the Reynolds transport theorem and explains how conservation laws can be formulated for control volumes. Reynolds transport theorem is stated and proved for both fixed and deformable control volumes. This chapter is based on the textbooks of Batchelor [1998], Fay [1998], and White [2011].

5.1 Lagrangian versus Eulerian Coordinates

Eulerian and Lagrangian coordinates are two of the main reference systems used for formulation of physical laws. The Eulerian coordinate system could be conceived of as a static reference system, where the laws of physics are observed from a fixed point of view.

In a Eulerian system, physical properties such as temperature, velocity, acceleration, and pressure, are assigned to fix coordinate points, and not to the dynamic material moving in the coordinate system. As an instance, the motion of a fluid is represented by a velocity vector field, that assigns to each point (x, y, z, t) in space-time, a velocity vector, $\mathrm{V}(x, y, z, t)$, that represents the velocity of the fluid element that happens to be at point (x, y, z) and at time t.

Therefore, the values of the velocity vector field at a point, at different times, represent the velocity of different fluid elements, that happened to be at that point, at the given times. In fact, the Eulerian coordinate system does not keep track of fixed fluid elements over time, but rather observes the behavior of the fluid in a fixed coordinate, through time. The Eulerian coordinate is indeed the most appropriate choice of reference system in fluid mechanics, since most of what is needed to be known about the behavior of a fluid, could be obtained from its behavior at certain fixed spatial points, and the data concerning the behavior of single fluid elements, is both irrelevant and extremely complicated to analyze.

On the other hand, formulation of physical laws would considerably easier, if the coordinate system could move together with the fluid elements, where only local changes in physical properties would be observable. This type of dynamic reference system is called the Lagrangian coordinate, which travels with the flow and observes the physical properties of fixed fluid element at different points in space and time. As such, in Lagrangian system, the velocity vectors are assigned to fixed fluid elements, and not fixed coordinate points.

The flow in Lagrangian coordinate system at each point in time, can be mathematically described as a 1-parameter family of smooth mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$, where the domain represents the location of fluid elements at time $t = 0$, and the range represents the location of fluid elements at time t. Let this 1-parameter family of mappings be denoted by $M_t(x, y, z) : \mathbb{R}^3 \to \mathbb{R}^3$, where (x, y, z) labels a fluid element at time $t = 0$ and $M_t(x, y, z)$ specifies the location of that fluid element at time t.

Note that using the Lagrangian coordinate, the velocity of a given fluid element, (x, y, z), can be readily calculated as:

$$\widehat{V}(x, y, z, t) = \frac{\partial}{\partial t} M_t(x, y, z)$$

Regarding the fact that M_t is bijective, the inverse mapping, $M_t^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ can be defined, and its plain to show that the following equations hold:

$$\begin{aligned} V(x, y, z, t) &= \widehat{V}(M_t^{-1}(x, y, z), t) \\ \widehat{V}(x, y, z, t) &= V(M_t(x, y, z), t) \end{aligned} \tag{5.1}$$

5.2 Material Derivative

Fig. 6. shows the motion of a fluid element through a path in $\mathbb{R}^3$, starting at point (A, t_1) and ending at point (B, t_2). Let (x_1, y_1, z_1) and (x_2, y_2, z_2) denote the spatial coordinates of A and B, respectively, and let $V = \mathrm{V}_1 \vec{i} + \mathrm{V}_2 \vec{j} + \mathrm{V}_3 \vec{k}$ denote the velocity vector field, where

$$\begin{cases} \mathrm{V}_1 = \mathrm{V}_1(x, y, z, t) \\ \mathrm{V}_2 = \mathrm{V}_2(x, y, z, t) \\ \mathrm{V}_3 = \mathrm{V}_3(x, y, z, t) \end{cases}$$

Note that since the x, y, and z components of the velocity field are both space and time variable, the described motion is an unsteady flow.

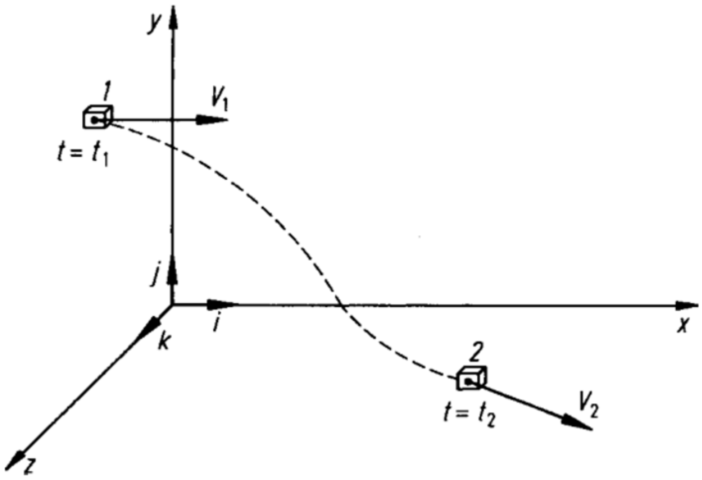

Figure 5.1: The path of a fluid element in an unsteady flow

Now, let $T = T(x, y, z, t)$ denote the temperature of the fluid at point (x, y, z, t). Then the temperature of the fluid element at points A and B are given by $T(A, t_1)$ and $T(B, t_2)$, respectively. Using Taylor expansion of $T(B, t_2)$ around (A, t_1), would yield:

$$T(B,t_2) = T(A,t_1) + \frac{\partial}{\partial x}T(A,t_1)\Delta X + \frac{\partial}{\partial y}T(A,t_1)\Delta Y + \frac{\partial}{\partial z}T(A,t_1)\Delta Z$$
$$+ \frac{\partial}{\partial t}T(A,t_1)\Delta t + \mathrm{O}(\Delta X^2 + \Delta Y^2 + \Delta Z^2 + \Delta t^2)$$

$$\Rightarrow \quad \frac{\Delta T}{\Delta t} = \frac{\partial T}{\partial x}\frac{\Delta X}{\Delta t} + \frac{\partial T}{\partial y}\frac{\Delta Y}{\Delta t} + \frac{\partial T}{\partial z}\frac{\Delta Z}{\Delta t} + \frac{\partial}{\partial t}T + \mathrm{O}(\Delta X + \Delta Y + \Delta Z + \Delta t) \qquad (5.2)$$

where,

$$\begin{cases} \Delta X = x_2 - x_1 \\ \Delta Y = y_2 - y_1 \\ \Delta Z = z_2 - z_1 \\ \Delta T = T(B, t_2) - T(A, t_1) \end{cases}$$

Note that left-hand side of the equation (5.2) is the average time-rate of change in the temperature of the fluid element, between points A and B. Let $\frac{DT}{Dt}$ denote the limit of the left-hand side of the equation (5.2), as the time interval approaches to zero.

$$\frac{DT}{Dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta T}{\Delta t}$$

Then the right-hand side of the equation (5.2) can be written as:

$$\lim_{\Delta t \to 0} \frac{\partial T}{\partial x}\frac{\Delta X}{\Delta t} + \frac{\partial T}{\partial y}\frac{\Delta Y}{\Delta t} + \frac{\partial T}{\partial z}\frac{\Delta Z}{\Delta t} + \frac{\partial}{\partial t}T + \mathrm{O}(\Delta X + \Delta Y + \Delta Z + \Delta t)$$
$$= \frac{\partial T}{\partial x}\frac{dX}{dt} + \frac{\partial T}{\partial y}\frac{dY}{dt} + \frac{\partial T}{\partial z}\frac{dZ}{dt} + \frac{\partial}{\partial t}T$$

and hence, $\frac{DT}{Dt} = \frac{\partial T}{\partial x}\frac{dX}{dt} + \frac{\partial T}{\partial y}\frac{dY}{dt} + \frac{\partial T}{\partial z}\frac{dZ}{dt} + \frac{\partial}{\partial t}T$

Note that here, ΔT indicates the change in the temperature of the fluid element that moves from A to B, which implies the Lagrangian coordinate is being used. Indeed, $\frac{DT}{Dt}$ indicates the time-rate of change in the temperature of the fluid element, which is clearly different from $\frac{\partial T}{\partial t}$, which indicates the time-rate of change of the temperature field. Indeed, $\frac{\partial T}{\partial t}$ is the time-rate of change in temperature, served in the

Eulerian coordinates. Furthermore, since $\frac{dX}{dt}$, $\frac{dY}{dt}$, and $\frac{dZ}{dt}$ indicate the instantaneous time-rate of change in the location of the fluid element, in the x, y, and z directions, evidently:

$$\frac{dX}{dt} = V_1 \qquad \frac{dY}{dt} = V_2 \qquad \frac{dZ}{dt} = V_3$$

and therefore we have:

$$\frac{DT}{Dt} = \left(\frac{\partial T}{\partial x}V_1 + \frac{\partial T}{\partial y}V_2 + \frac{\partial T}{\partial z}V_3\right) + \frac{\partial T}{\partial t} = (\mathrm{V} \,.\, \nabla)T + \frac{\partial T}{\partial t}$$

The operator $\frac{D}{Dt}$ is called the material or substantial derivative and is defined as:

$$\frac{D}{Dt} \equiv (\mathrm{V} \,.\, \nabla) + \frac{\partial}{\partial t} = V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial}{\partial y} + V_3 \frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

Material derivative of any physical property is the total time-rate of change in that property, which is the sum of the local instantaneous changes in that property and the changes due to the instantaneous change in the location. Indeed, the term $\frac{\partial}{\partial t}$ captures the local changes that occur at a fixed point, where the fluid element happens to be, and is called the local derivative. Local derivative is in fact the normal time-derivative of the considered physical property in Eulerian coordinate. The term $\mathrm{V} \,.\, \nabla$ captures the changes of the physical property due to the change in the location of the fluid element and is called the convective derivative.

A key application of the material derivative appears in the calculation of the acceleration field in a flow. As stated earlier, taking the time-derivative of the velocity field would only yield the local time-rate of change in the velocity field, which is not the same as the acceleration field experienced by moving fluid elements. As such calculating the acceleration field for fixed coordinate points would require the material derivative of the velocity field, where both local and convective changes are taken into account. Then in the light of the previous discussion, the acceleration field can be written as:

$$Convective\ acceleration = \frac{DV}{Dt} = (\mathrm{V}\ .\ \nabla)\,\mathrm{V} + \frac{\partial V}{\partial t} \tag{5.3}$$

Equation (5.3) is called the convective acceleration and plays a key role in all continuum equations, such as the Navier-Stokes differential equations. Holding the assumption that the density of the fluid is constant, the convective momentum of a fluid element can be written as:

$$Convective\ momentum = \rho\frac{DV}{Dt}d\nu = \rho((\mathrm{V}\ .\ \nabla)\,\mathrm{V} + \frac{\partial V}{\partial t})d\nu \tag{5.4}$$

5.3 Control Volumes and the Divergence of Velocity Field

In general, physical problems are mainly formulated in terms of systems that interacts with their surroundings, and conservation laws are applied to such systems. The most notable of such conservation laws are the conservation of mass, linear momentum, angular momentum, and energy. The conservation of a physical property plainly means that the net change in that property is zero over any time interval, in the given system.

Conservation laws in fluid mechanics are commonly formulated on a particular type of physical systems, called the control volumes. A control volume is an imaginary closed surface within the fluid, that the flow can pass through with no resistance. In this regard, conservation laws can be formulated based on the net quantity of certain physical properties, carried by the flow passing through the surface of a control volume and the changes in those physical properties within the control volume. In fluid mechanics, such physical properties typically include mass, energy, and linear momentum.

Nonetheless, in order to calculate the net quantity of any physical property, carried by the flow that passes through a given surface, the volume of the passing flow must be calculated first. The later could be accomplished by calculating the flow passing through an arbitrary surface element of the boundary of the control volume, and integrate it over the whole boundary.

Let CV be a control volume, let dA be an arbitrary surface element of ∂CV, and let $\vec{n}$ denote the outward unit vector normal to dA. Then the volume of the flow passing through dA, in the time interval Δt, can be written as:

$$d\Lambda = \|V\| \Delta t dA \cos\phi$$

where ϕ is the angel between the velocity vector, V and $\vec{n}$. Note that since $\|\vec{n}\| = 1$, we can write;

$\|V\| \cos\phi = \|V\| . \|\overrightarrow{n}\| \cos\phi = V . \vec{n}$, and hence:

$$d\Lambda = (\mathrm{V} \,.\, \vec{n})\Delta t\, dA \tag{5.5}$$

Note that since $\vec{n}$ is assumed to be the outward normal vector of dA, positive values of $V \,.\, \vec{n}$ indicate the outflow of the fluid, and negative values indicate the inflow.

Hence, the net volume of the flow passed through ∂CV in the time interval Δt, can be written as:

$$\Lambda = \iint_{\partial CV} d\Lambda = \Delta t \iint_{\partial CV} (\mathrm{V} \,.\, \vec{n}) dA = \Delta t \iint_{\partial CV} \mathrm{V} \,.\, d\vec{A} \tag{5.6}$$

where $d\vec{A} = \vec{n}\, dA$, is the vector surface element. Applying the divergence theorem to the equation (5.6), we obtain:

$$\Lambda = \Delta t \iint_{\partial CV} V \,.\, d\vec{A} = \Delta t \iiint_{CV} (\nabla \,.\, \mathrm{V}) dv$$

The time-rate of change in the net volume of the flow passing through the surface of the control volume, can be obtain by letting $\Delta t \to 0$:

$$\frac{d\Lambda}{dt} = \iiint_{CV} (\nabla \,.\, \mathrm{V}) dv \tag{5.7}$$

Likewise, the net flow of mass through the boundary of the control volume in the time interval Δt, can be written as:

$$m = \Delta t \iiint_{CV} \rho (\nabla \,.\, \mathrm{V}) dv$$

And the time-rate of change in the net mass of the flow passing through the surface of the control volume can be written as:

$$\frac{dm}{dt} = \iiint_{CV} \rho(\nabla \,.\, \mathrm{V})dv \tag{5.8}$$

In case of an incompressible fluid, where the density is constant, equation (5.8), can be further simplified into:

$$\frac{dm}{dt} = \rho \iiint_{CV} (\nabla \,.\, \mathrm{V})dv$$

Another approach for formulating conservation laws, is to use a deformable control volume, that can travel with the flow and change shape. Evidently, this approach uses a Lagrangian coordinate and material derivative would be required for calculating the instantaneous changes in the physical properties of the fluid contained in the control volume.

To apply this method, let CV' be a deformable control volume, and let dA be an arbitrary surface element on the boundary of CV', that moves as the control volume changes shape. Note that the movement of dA in the infinitesimal time interval Δt, would change the volume of CV, by

$$d\Lambda = \| V \| \, dA \Delta t \cos\phi$$

Where, V is the velocity vector of dA, $\vec{n}$ is the outward normal vector of dA, and ϕ is the angle between them.

Then similar to previous section, $d\Lambda$ can be written as:

$$d\Lambda = (V\ .\ \vec{n})\Delta t\ dA$$

and the time-rate of change in the volume of CV' can be obtain by taking the limit of $\frac{\Delta\Lambda}{\Delta t}$ as $\Delta t \to 0$.

$$\frac{D\Lambda}{Dt} = \iint\limits_{\partial CV'} (V\ .\ \vec{n})dA = \iint\limits_{\partial CV'} V\ .\ d\vec{A} = \iiint\limits_{CV'} (\nabla\ .\ \mathrm{V})d\nu$$

Note that here, the rate of change in the volume of CV', must be written as the material derivative, since CV' is moving the flow.

Now if we let the size of the control volume approach zero, for the obtained infinitesimally small control volume, the gradient of the velocity field can be assumed constant over CV', and hence:

$$\iiint\limits_{CV'} (\nabla\ .\ \mathrm{V})d\nu = (\nabla\ .\ \mathrm{V})\iiint\limits_{CV'} d\nu = \Lambda(\nabla\ .\ \mathrm{V})$$

Then evidently

$$\frac{D\Lambda}{Dt} = \Lambda(\nabla\ .\ \mathrm{V}) \quad \Rightarrow \quad \nabla\ .\ \mathrm{V} = \frac{1}{\Lambda}\frac{D\Lambda}{Dt} \tag{5.9}$$

Indeed, the term $\nabla\ .\ \mathrm{V}$ repeatedly appears in fluid mechanics and is one of the key terms in the formulation of Navier-Stokes equations as well.

Note that since the size of the moving control volume is differentially small, it can be considered as a fluid element, and in this regard, the exact physical interpretation of $\nabla . \mathrm{V}$ could be obtained from the equation (5.9), as the time-rate of change in the volume of the fluid element, per unit volume.

5.4 Reynolds Transport Theorem

As was mentioned earlier, the conservation laws in fluid mechanics are formulated for control volumes. That is, the system under consideration is restricted to an imaginary volume of the fluid and its physical properties are investigated, and subsequently generalized to the whole system. As such, certain mathematical tools are required to relate the findings in a control volume to the whole system. Reynolds transport theorem is one of the fundamental mathematical tools, commonly used in this regard.

In the following section, Reynolds transport theorem will be driven and explained for both fixed and deformable control volumes.

5.3.1 Fixed Control Volumes

Let G denote a fixed control volume, and let dA be a surface element on its boundary. Then based on equation (5.5), the volume of the flow passing through dA in the time interval Δt can be written as:

$$d\Lambda = (V \, . \, \vec{n})\Delta t dA$$

Now, let ζ denote a physical property of the system, such as its pressure, temperature, energy, or linear momentum, and let $\bar{\zeta}$, denote the intensive form of ζ, that is the amount of ζ per unit mass of the fluid:

$$\bar{\zeta} \equiv \frac{d\zeta}{dm}$$

Then the total amount of ζ in the control volume can be written as:

$$\zeta_G = \iiint_G d\zeta = \iiint_G \bar{\zeta}\, dm = \iiint_G \rho\bar{\zeta}\, dv$$

In essence, Reynolds transport theorem states that:

$$\begin{pmatrix}\text{The change in the total} \\ \text{amount of } \zeta \text{ in the whole} \\ \text{system, in the time} \\ \text{interval } \Delta t\end{pmatrix} = \begin{pmatrix}\text{The change in the total} \\ \text{amount of } \zeta \text{ in the control} \\ \text{volume in the time} \\ \text{interval } \Delta t\end{pmatrix} + \begin{pmatrix}\text{The amount of } \zeta \text{ in} \\ \text{the fluid leaving the} \\ \text{control volume, in the} \\ \text{time interval } \Delta t\end{pmatrix} - \begin{pmatrix}\text{The amount of } \zeta \text{ in} \\ \text{the fluid entering the} \\ \text{control volume, in the} \\ \text{time interval } \Delta t\end{pmatrix}$$

That in short can be written as:

$$\zeta_{system} = \zeta_{control\ volume} + \zeta_{out} - \zeta_{in}$$

Note that the net amount of ζ entering and leaving the control volume in the time interval Δt, can be written as:

$$\zeta_{out} - \zeta_{in} = \Delta t \iint_{\partial G} \rho\bar{\zeta}(V \,.\, \vec{n})dA$$

Hence,

$$\zeta_{system} = \iiint_G \rho\bar{\zeta}\, dv + \Delta t \iint_{\partial G} \rho\bar{\zeta}(V \,.\, \vec{n})dA \quad \Rightarrow$$

$$\frac{d\zeta_{system}}{dt} = \frac{d}{dt}\iiint_G \rho\bar{\zeta}\, dv + \iint_{\partial G} \rho\bar{\zeta}(V \,.\, \vec{n})dA$$

Note that, since the shape and volume of that control volume does not vary with time, the time-derivative can be moved inside the integral and we obtain the Reynolds transport theorem for fixed control volumes:

$$\frac{d\zeta_{system}}{dt} = \iiint_G \frac{\partial}{\partial t}(\rho\bar{\zeta})dv + \iint_{\partial G} \rho\bar{\zeta}(V \, . \, \vec{n})dA \tag{5.10}$$

5.3.2 Deformable Control Volumes

A major difficulty of formulating conservation laws for arbitrary deformable control volumes, is that the surface of the control volume varies with time. Let D_0 denote the deformable control volume at time 0 and note that since the control volume is moving with the flow, the Lagrangian coordinate system must be used. Then, the control volume at time t can be written as:

$$D(t) = \{(x', y', z') \mid (x', y', z') = M_t(x, y, z), (x, y, z) \in D_0\} = M_t(D_0)$$

Where M_t denotes the Lagrangian flow map, that takes each fluid element to its location at time t. Then the time-rate of change in the amount of ζ in the fluid contained in $D(t)$, can be obtain by taking the time-derivative of $\iiint_{D(t)} \rho\bar{\zeta}\, dv$, which clearly is the total amount of ζ , contained in $D(t)$.

Let $J(M_t, t)$ be the Jacobian matrix of M_t, and $\Lambda_D(t)$ denote the volume of $D(t)$. Then evidently,

$$\Lambda_D(t) = \iiint_{D(t)} dv = \iiint_{D_0} \det(J)dM_t \tag{5.11}$$

It is plain to show that the change in $\Lambda_D(t)$ in the time interval Δt, can be written as:

$$\Lambda_D(t+\Delta t)-\Lambda_D(t)=\Delta t \iint_{\partial D(t)} (V\,.\,\vec{n})dA \quad \Rightarrow \tag{5.12}$$

$$\frac{d\Lambda_D}{dt}=\lim_{\Delta t\to 0}\frac{\Lambda_D(t+\Delta t)-\Lambda_D(t)}{\Delta t}=\iint_{\partial D(t)} (V\,.\,\vec{n})dA \tag{5.13}$$

Putting equations (5.11) and (5.12) together, we obtain:

$$\iiint_{D_0}\frac{\partial}{\partial t}\det(J)dM_t=\frac{d\Lambda_D}{dt}=\iint_{\partial D(t)} (V\,.\,\vec{n})\,dA=\iiint_{D(t)}(\nabla\,.\,\mathrm{V})\,dv=$$
$$\iiint_{D_0}(\nabla\,.\,\mathrm{V})\det(J)dM_t$$

Note that since the choice of D_0 is arbitrary,

$$\begin{aligned}&\iiint_{D_0}\frac{\partial}{\partial t}\det(J)dM_t=\iiint_{D_0}(\nabla\,.\,\mathrm{V})\det(J)dM_t \quad \Rightarrow\\ &\frac{\partial}{\partial t}\det(J)=(\nabla\,.\,\mathrm{V})\det(J)\end{aligned} \tag{5.14}$$

By applying equation (5.14), we can write:

$$\frac{d}{dt}\iiint_{D(t)} \rho\overline{\zeta}(x,y,z,t)dv = \frac{d}{dt}\iiint_{D_0} \rho\overline{\zeta}(M_t(x,y,z),t)\det(J)dM_t$$

$$= \iiint_{D_0} \frac{\partial}{\partial t}(\rho\overline{\zeta}(M_t(x,y,z),t)\det(J))dM_t$$

$$= \iiint_{D_0} \frac{D}{Dt}(\rho\overline{\zeta}(M_t(x,y,z),t))\det(J)dM_t$$

$$+ \iiint_{D_0} \rho\overline{\zeta}(M_t(x,y,z),t)\frac{\partial}{\partial t}\det(J)dM_t$$

$$= \iiint_{D_0} \frac{D}{Dt}(\rho\overline{\zeta}(M_t(x,y,z),t))\det(J)dM_t$$

$$+ \iiint_{D_0} \rho\overline{\zeta}(M_t(x,y,z),t)(\nabla\,.\,\mathrm{V})\det(J)dM_t$$

$$= \iiint_{D_0} (\frac{D}{Dt}(\rho\overline{\zeta}(M_t(x,y,z),t)) + \rho\overline{\zeta}(M_t(x,y,z),t)(\nabla\,.\,\mathrm{V}))\det(J)dM_t$$

$$= \iiint_{D(t)} \frac{D}{Dt}(\rho\overline{\zeta}(x,y,z,t)) + \rho\overline{\zeta}(x,y,z,t)(\nabla\,.\,\mathrm{V})dv$$

$$= \iiint_{D(t)} \frac{D\rho\overline{\zeta}}{Dt} + \rho\overline{\zeta}(\nabla\,.\,\mathrm{V})dv$$

$$= \iiint_{D(t)} \frac{\partial\rho\overline{\zeta}}{\partial t} + \mathrm{V}\,.\,\nabla\rho\overline{\zeta} + \rho\overline{\zeta}(\nabla\,.\,\mathrm{V})dv$$

$$= \iiint_{D(t)} \frac{\partial\rho\overline{\zeta}}{\partial t} + \nabla\,.\,(\rho\overline{\zeta}\,\mathrm{V})dv$$

Hence, we obtain the general form of the Reynolds transport theorem:

$$\frac{d}{dt}\iiint_{D(t)} \rho\overline{\zeta}dv = \iiint_{D(t)} \frac{D\rho\overline{\zeta}}{Dt} + \rho\overline{\zeta}(\nabla\,.\,\mathrm{V}))dv = \iiint_{D(t)} \frac{\partial\rho\overline{\zeta}}{\partial t} + \nabla\,.\,(\rho\overline{\zeta}\,\mathrm{V})dv \qquad (5.15)$$

Chapter 6

CONSERVATION LAWS IN FLUID MECHANICS

Building upon previous material, this chapter deals with the derivation of conservation laws for fluids. The first section of this chapter explains the conservation of mass equation in both integral and differential forms, and explores the details of deriving the equation by using the control volume technique. The second section, focuses on the conservation of linear momentum, and shows how the conservation equation can be driven by equating the sum of body and surface forces acting on the control volume, to the time-rate of change in its total momentum. Applications of Reynolds transport theorem and Cauchy stress theorem in deriving the conservation of linear momentum equation are discussed in details. This chapter is prepared and written based on the books of Batchelor [1998], Fay [1998], and White [2011]

6.1 Conservation of Mass

The integral form of the conservation of mass for a fluid, could be obtained by using the Reynolds transport theorem for the mass contained in an arbitrary control volume. Note that in classical mechanics, the conservation of mass principle states that the time-rate of change in the mass of a physical system, is always zero. In this regard, let G be an arbitrary fixed control volume, and let m denote its mass. Then the conservation of mass equation can be written as:

$$\frac{dm}{dt} = 0 \tag{6.1}$$

By applying the Reynolds transport theorem (5.10) for the mass contained in G , we obtain:

$$\frac{dm}{dt} = \iiint_G \frac{\partial}{\partial t} \rho dv + \iint_{\partial G} \rho (V \cdot \vec{n}) dA$$

and hence using the equation (6.1), the integral form of the conservation of mass principle, can be written as:

$$\iiint_G \frac{\partial}{\partial t} \rho dv + \iint_{\partial G} \rho (V \cdot \vec{n}) dA = 0 \tag{6.2}$$

Note that in case of an incompressible fluid, for which the density field could be assumed constant, further simplification of the above equation is possible. In fact, the majority of known liquids and some gases under normal thermodynamic conditions and low velocity, could be treated as incompressible fluids, since the variations in their density field is practically insignificant.

For an incompressible fluid, where ρ is a constant,

$$\frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \iiint_G \frac{\partial}{\partial t} \rho dv = 0 \tag{6.3}$$

Moreover,

$$\iint_{\partial G} \rho (V \cdot \vec{n}) dA = \rho \iint_{\partial G} (V \cdot \vec{n}) dA = \rho \iint_{\partial G} V \cdot d\vec{A}$$

and by applying the divergence theorem we obtain:

$$\rho\iint_{\partial G} V \, . \, d\vec{A} = \iiint_{G} (\nabla . \mathrm{V}) d\nu \tag{6.4}$$

Putting equations (6.2), (6.3), and (6.4) together, we obtain:

$$0 = \iiint_{G} \frac{\partial}{\partial t} \rho d\nu + \iint_{\partial G} \rho (V \, . \, \vec{n}) dA = 0 + \iiint_{G} (\nabla . \mathrm{V}) d\nu = \iiint_{G} (\nabla . \mathrm{V}) d\nu$$

But since the integral is taken over an arbitrary fixed control volume, and the integral is always zero, the integrand must be zero, hence:

$$\iiint_{G} (\nabla . \mathrm{V}) d\nu = 0 \quad \Rightarrow \quad \nabla . \mathrm{V} = 0 \tag{6.5}$$

Equation (6.5) describes the conservation of mass for incompressible fluids, and is also known as the continuity equation. Remembering, that the divergence of the velocity field, indicates the time-rate of change in the volume of fluid elements per unit volume, the continuity equation implies that in an incompressible fluid, where the density of fluid elements cannot change, the volume of fluid elements must not change as well.

In other words, let Λ denote the volume of a fluid element, then as was shown previously:

$$\nabla \, . \, \mathrm{V} = \frac{1}{\Lambda} \frac{D\Lambda}{Dt}$$

and the continuity equation can be written as:

$$\frac{D\Lambda}{Dt} = 0 \tag{6.6}$$

The conservation of mass principle can also be formulated for a deformable control volume. In case of a deformable control volume, note that the control volume would always contain the same fluid elements, and hence according to the conservation of mass principle, the time-rate of change in the mass of the control volume must be zero.

Let $D(t)$ be a deformable control volume, then the mass contained in $D(t)$ can be obtained from the following integral:

$$mass\ contained\ \ in\ D(t)\ \ = \iiint_{D(t)} \rho dv$$

and the conservation of mass principle can be written as:

$$\frac{d}{dt}(mass\ contained\ \ in\ D(t)) = 0 \quad \Rightarrow \quad \frac{d}{dt}\iiint_{D(t)} \rho dv = 0$$

Applying the Reynolds transport theorem (5.15), to the above equation yields:

$$0 = \frac{d}{dt}\iiint_{D(t)} \rho dv = \iiint_{D(t)} \frac{\partial \rho}{\partial t} + \nabla \,.\, (\rho \mathrm{V}) dv$$

Note that since, the above integral is zero for any arbitrary choice of $D(t)$, then the integrand must be zero, and we obtain the general form of the continuity equation, that holds for both compressible and incompressible fluids:

$$0 = \frac{\partial \rho}{\partial t} + \nabla \,.\, (\rho \mathrm{V}) = \frac{\partial \rho}{\partial t} + \rho \nabla \,.\, \mathrm{V} + \mathrm{V} \,.\, \nabla \rho \quad \Rightarrow$$

$$\frac{\partial \rho}{\partial t} + \mathrm{V} \,.\, \nabla \rho + \rho \nabla \,.\, \mathrm{V} = 0 \qquad (6.7)$$

As can be seen, the first part of the equation (6.7) is in fact the material derivative of the density field, and the equation can be in short written as:

$$\frac{D\rho}{Dt} + \rho\nabla \,.\, \mathrm{V} = 0 \tag{6.8}$$

6.2 Conservation of Linear Momentum

The conservation of linear momentum principle, is in fact the reformulation of Newton's second law of motion for fluids. The main principle is that the sum of forces acting on the control volume, such as the gravity, internal stresses, and pressure, must be equal to the time-rate of change in the net momentum of the control volume. Therefore, to formulate the conservation of linear momentum principle, the time-rate of change in the momentum of the control volume must be obtained, using the Reynolds transport theorem, and equated to the sum of forces. Note that similar to previous section, the control volume can be assumed fixed or deformable.

Let G be a fixed control volume, and let $U \equiv mV$ denote momentum. Then the intensive momentum would be:

$$\bar{U} = \frac{dU}{dm} = \frac{d(mV)}{dm} = V\frac{dm}{dm} + m\frac{dV}{dm} = V + 0 = V$$

Hence, the Reynolds transport theorem (5.10) for the intensive momentum can be written as:

$$\frac{d}{dt}U_{total} = \iiint_G \frac{\partial}{\partial t}(\rho\bar{U})dv + \iint_{\partial G} \rho\bar{U}(V\,.\,\vec{n})dA = \iiint_G \frac{\partial}{\partial t}(\rho V)dv + \iint_{\partial G} \rho V(V\,.\,\vec{n})dA$$

Then by equating the time-rate of change the in momentum of the control volume to the sum of forces acting on it, we obtain:

$$\sum_{G} F = \iiint_{G} \frac{\partial}{\partial t}(\rho V) dv + \iint_{\partial G} \rho V (V \,.\, \vec{n}) dA \tag{6.9}$$

The above equation might be further simplified, for incompressible fluids, where the scalar field, ρ, could be assumed constant and removed outside the integrals.

In case of considering a deformable control volume, the general form of the Reynolds transport theorem (5.15) must be applied. In this regard, let $D(t)$ denote a deformable control volume at time t. Then the total momentum of $D(t)$ can be written as:

$$U_{total} = \iiint_{D(t)} \rho V(x, y, z, t) dv$$

The time-rate of change in the total momentum of $D(t)$, is simply the time-derivative of the above integral, which must be equal to the sum of forces acting on $D(t)$, and we can write:

$$\sum_{D(t)} F = \frac{dU_{total}}{dt} = \frac{d}{dt} \iiint_{D(t)} \rho V(x, y, z, t) dv \tag{6.10}$$

Note that, the time-derivative of the above integral could be calculated using the Reynolds transport theorem (5.15), as follows:

$$\begin{aligned}
\frac{d}{dt} \iiint_{D(t)} \rho V(x, y, z, t) dv &= \iiint_{D(t)} \frac{D\rho V}{Dt} + \rho V (\nabla \,.\, \mathrm{V}) dv \\
&= \iiint_{D(t)} \frac{D\rho V}{Dt} dv + \iiint_{D(t)} \rho V (\nabla \,.\, \mathrm{V}) dv \\
&= \iiint_{D(t)} \rho \frac{DV}{Dt} dv + \iiint_{D(t)} V \frac{D\rho}{Dt} dv + \iiint_{D(t)} \rho V (\nabla \,.\, \mathrm{V}) dv \\
&= \iiint_{D(t)} \rho \frac{DV}{Dt} dv + \iiint_{D(t)} V \left(\frac{D\rho}{Dt} + \rho (\nabla \,.\, \mathrm{V})\right) dv
\end{aligned}$$

Note that, due to the conservation of mass principle (6.7), we have:

$$\frac{D\rho}{Dt} + \rho(\nabla \, . \, \mathrm{V}) = 0$$

which evidently implies that:

$$\iiint_{D(t)} V(\frac{D\rho}{Dt} + \rho(\nabla \, . \, \mathrm{V}))dv = 0$$

and hence the equation (6.10) can be simplified into:

$$\sum_{D(t)} F = \frac{dU_{total}}{dt} = \iiint_{D(t)} \rho \frac{DV}{Dt} dv + \iiint_{D(t)} V(\frac{D\rho}{Dt} + \rho(\nabla \, . \, \mathrm{V}))dv = \iiint_{D(t)} \rho \frac{DV}{Dt} dv \quad \Rightarrow$$

$$\sum_{D(t)} F = \iiint_{D(t)} \rho \frac{DV}{Dt} dv \tag{6.11}$$

Note that the material derivative of the velocity field is in fact, the acceleration field experienced by fluid elements as they travel with the flow, which agrees with the intuitive meaning of the above equation as the integral form of the Newton's second law of motion. Moreover, using the definition of material derivative, we can write:

$$\frac{DV}{Dt} = (\mathrm{V} \, . \, \nabla) \, \mathrm{V} + \frac{\partial V}{\partial t}$$

and the above integral can be written as:

$$\iiint_{D(t)} \rho \frac{DV}{Dt} dv = \iiint_{D(t)} \rho((\mathrm{V} \, . \, \nabla) \, \mathrm{V} + \frac{\partial V}{\partial t})dv$$

and hence equation (6.11) becomes:

$$\sum_{D(t)} F = \iiint_{D(t)} \rho((\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \frac{\partial V}{\partial t}) dv \tag{6.12}$$

Now, if we let the size of the control volume approach zero, $D(t)$ would become a fluid element, and on the differential level, the term $\rho((\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \frac{\partial V}{\partial t})$ could be assumed to have the same value over $D(t)$, and hence the above integral reduces to the differential form of its integrand, that is:

$$((\mathrm{V}\,.\,\nabla)\rho\,\mathrm{V} + \rho\frac{\partial V}{\partial t})\; dv$$

Note that the forces acting on $D(t)$ would also reduce to the differential size, and the equation (6.12) can be written as:

$$\sum dF = ((\mathrm{V}\,.\,\nabla)\rho\,\mathrm{V} + \rho\frac{\partial V}{\partial t})\; dv \tag{6.13}$$

This partial differential equation is in essence, the conservation of linear momentum principle.

The next step in completing the equation is to calculate the sum of forces acting on an arbitrary fluid element. In fact, the forces acting on a fluid element can be categorized in two classes, namely the surface and body forces. Surface forces refer to the type of forces that are experienced on the boundary of the fluid element. These forces could arise from hydrostatic pressure and stresses produced by neighboring fluid elements or external objects such as walls or barriers that meet the surface of the fluid element. Body forces refer to the type of forces that act of the whole fluid element without any actual material interactions. Such forces for the most part include but are not limited to gravity and electromagnetic fields.

Let F_g denote the gravity field acting on the fluid. Then the differential gravity force acting on the body of a fluid element, can be written as:

$$dF_g = \rho g dv$$

where g is the standard gravity constant.

Furthermore, let $\overline{f}$ denote the sum of other body forces per unit volume, acting on a fluid element. Then

$$\sum F_B = \rho(g + \overline{f})dv$$

Surface forces acting on the boundary of the fluid element are composed of the stresses produced by the movement of neighboring fluid elements and the hydrostatic pressure produced by the compressive force imposed on the boundary of the fluid element by the fluid surrounding it.

Note that, the hydrostatic pressure force is always normal to the surface of the fluid element and presses it inwards. As was shown in chapter 4. surface forces acting on the boundary of a fluid element, can be represented by the following stress tensor (4.1);

$$T = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

where σ_{ij} denotes the force per unit area in the j-direction acting on the face of the fluid element, perpendicular to the i-axis.

In this regard, the sum of stresses acting on the surface of the fluid element in the x-direction, can be calculated in the following way.

If we consider the point sitting at the center of the fluid element, then the force per unit area acting on the right $dydz$ face of the fluid element, in the x-direction would be:

$$\sigma_{xx} + \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x} dx$$

and the force per unit area acting on the left $dydz$ face, in the x-direction would be:

$$-\sigma_{xx} + \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x} dx$$

That is since, σ_{xx} is the normal stress acting on the $dydz$ faces of the fluid element and is always assumed to be pointing outwards and positive in the x-direction, by convention.

Therefore, the sum of forces acting on the $dydz$ faces, in the x-direction would be:

$$(\sigma_{xx} + \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x} dx)dydz + (-\sigma_{xx} + \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x} dx)dydz = \frac{\partial \sigma_{xx}}{\partial x} dxdydz = \frac{\partial \sigma_{xx}}{\partial x} dv \qquad (6.14)$$

In the same manner, it can be shown that the shear forces per unit area acting on the $dxdz$ faces of the fluid elements are:

$$\sigma_{yx} + \frac{1}{2}\frac{\partial \sigma_{yx}}{\partial y} dy$$

$$-\sigma_{yx} + \frac{1}{2}\frac{\partial \sigma_{yx}}{\partial y} dy$$

and the sum of forces in the x-direction acting on the $dxdz$ faces of the fluid element would be:

$$(\sigma_{yx} + \frac{1}{2}\frac{\partial \sigma_{yx}}{\partial y} dy)dxdz + (-\sigma_{yx} + \frac{1}{2}\frac{\partial \sigma_{yx}}{\partial y} dy)dxdz = \frac{\partial \sigma_{yx}}{\partial y} dv \tag{6.15}$$

Finally, the shear forces per unit area for the $dxdy$ faces can be written as:

$$\sigma_{zx} + \frac{1}{2}\frac{\partial \sigma_{zx}}{\partial z} dz$$

$$-\sigma_{zx} + \frac{1}{2}\frac{\partial \sigma_{zx}}{\partial z} dz$$

and likewise, the sum of forces acting on the $dxdy$ faces, in the x-direction, would be:

$$(\sigma_{zx} + \frac{1}{2}\frac{\partial \sigma_{zx}}{\partial z} dz)dxdy + (-\sigma_{zx} + \frac{1}{2}\frac{\partial \sigma_{zx}}{\partial z} dz)dxdy = \frac{\partial \sigma_{zx}}{\partial z} dv \tag{6.16}$$

Putting equations (6.14), (6.15) and (6.16) together, the total sum of forces acting on the fluid element in the x-direction can be written as:

$$\sum dF_x = (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z})dv$$

In the same way, it is easy to show that the sum of forces acting on the boundary of the fluid element in the y and z-directions, would be:

$$\sum dF_y = (\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z})dv$$

$$\sum dF_z = (\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z})dv$$

Then the total sum of surface forces acting on the fluid element can be written as:

$$\begin{aligned}\sum dF_s = \sum dF_x\vec{i} + dF_y\vec{j} + dF_z\vec{k} &= (\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z})\vec{i}\,dv \\ &+ (\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z})\vec{j}\,dv \\ &+ (\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z})\vec{k}\,dv\end{aligned}$$

$$= \left[(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z})\vec{i} + (\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z})\vec{j} + (\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z})\vec{k}\right] dv$$

Note that using the matrix notations, we can write:

$$(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z})\vec{i} + (\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z})\vec{j} + (\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z})\vec{k}$$

$$= (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k})\begin{pmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{pmatrix} = \nabla\,.\,T$$

and the above equation can be summarized in the tensor form as:

$$\sum dF_s = (\nabla\,.\,T)dv$$

Hence, the total sum of forces acting on the fluid element, can be written as the sum of body and surface forces, as follows:

$$\sum dF = \sum dF_B + dF_s = (\rho(g + \bar{f}) + (\nabla . T))d\nu$$

and the conservation of linear momentum principle, can be written as:

$$(\rho(g + \bar{f}) + (\nabla . T))d\nu = ((\mathrm{V} . \nabla)\rho \mathrm{V} + \rho \frac{\partial V}{\partial t})\, d\nu \quad \Rightarrow$$

$$\rho(g + \bar{f}) + (\nabla . T) = (\mathrm{V} . \nabla)\rho \mathrm{V} + \rho \frac{\partial V}{\partial t} \qquad (6.17)$$

Note that by using the material derivative, the equation (6.17) can be further simplified into:

$$\rho \frac{DV}{Dt} = \rho(g + \bar{f}) + (\nabla . T)$$

This vector equation indeed summarizes the following equations:

$$\rho(V_x \frac{\partial}{\partial x} V_x + V_y \frac{\partial}{\partial y} V_x + V_z \frac{\partial}{\partial z} V_x + \frac{\partial}{\partial t} V_x) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho(g_x + \bar{f}_x)$$

$$\rho(V_x \frac{\partial}{\partial x} V_y + V_y \frac{\partial}{\partial y} V_y + V_z \frac{\partial}{\partial z} V_y + \frac{\partial}{\partial t} V_y) = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho(g_y + \bar{f}_y)$$

$$\rho(V_x \frac{\partial}{\partial x} V_z + V_y \frac{\partial}{\partial y} V_z + V_z \frac{\partial}{\partial z} V_z + \frac{\partial}{\partial t} V_z) = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho(g_z + \bar{f}_z)$$

This result could also be found a mathematically shorter and more rigorous way, using the Cauchy stress theorem. Note that from the equation (6.12) we have:

$$\sum_{D(t)} F = \iiint_{D(t)} \rho((\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \frac{\partial V}{\partial t})dv$$

and that the forces acting on the fluid element can be written as the sum of surface and body forces;

$$\sum_{D(t)} F = \sum_{D(t)} F_S + \sum_{D(t)} F_B$$

The action of body forces on the control volume can be written as the volume integral of differential body forces acting on a fluid element;

$$\sum_{D(t)} F_B = \iiint_{D(t)} \rho(g + \overline{f})dv$$

and the action of surface forces on the control volume can be written as the surface integral of differential surface forces acting on a fluid element. But note that according to Cauchy stress theorem (4.6), surface forces acting on an arbitrary surface element dA can be written as $\vec{n}\,.\,T$, where $\vec{n}$ denotes the normal unit vector of dA, and T is the stress tensor.

Hence we can write, $\sum_{D(t)} F_S = \iint_{\partial D(t)} (\vec{n}\,.\,T)\,dA$, and the total sum of forces acting on $D(t)$ can be written as:

$$\sum_{D(t)} F = \iiint_{D(t)} \rho(g + \overline{f})dv + \iint_{\partial D(t)} (\vec{n}\,.\,T)dA \qquad (6.18)$$

Applying the divergence theorem to the surface integral in the equation (6.18), yields:

$$\begin{aligned}\sum_{D(t)} F &= \iiint_{D(t)} \rho(g+\overline{f})dv + \iint_{\partial D(t)} (\vec{n}\,.\,T)dA \\ &= \iiint_{D(t)} \rho(g+\overline{f})dv + \iiint_{D(t)} (\nabla\,.\,T)dv \\ &= \iiint_{D(t)} \rho(g+\overline{f}) + (\nabla\,.\,T)dv\end{aligned}$$

By equating the above equation with the time-rate of change in the total momentum of the control volume, we obtain:

$$\iiint_{D(t)} \rho(g+\overline{f}) + (\nabla\,.\,T)dv = \sum_{D(t)} F = \iiint_{D(t)} \rho((\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \frac{\partial V}{\partial t})dv \quad \Rightarrow$$

$$\iiint_{D(t)} [\rho(g+\overline{f}) + (\nabla\,.\,T) - \rho((\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \frac{\partial V}{\partial t})]dv = 0$$

Note that since the choice of the control volume is arbitrary, and the integral is always zero, the integrand must be zero, and hence;

$$\rho((\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \frac{\partial V}{\partial t}) = \rho(g+\overline{f}) + (\nabla\,.\,T) \tag{6.19}$$

which is the conservation of linear momentum equation for fluids. Note that the left-hand side of the equation (6.19) is indeed the material derivative of the velocity field, which is the convective acceleration of the flow, and can physically be interpreted as the acceleration field experienced by the flow depending on its location, and independent of time. An example of convective acceleration of a steady flow moving in a cylindrical duct with decreasing diameter. Note that in this example, the velocity field is constant at each coordinate point over time, but the speed of fluid elements increases as they move along the duct. As such, the acceleration field experienced by the flow, depends on the location of the flow in the duct and is constant over time at each coordinate point.

Chapter 7

NAVIER-STOKES EQUATIONS

This chapter provides an in-depth discussion about the derivation of Navier-Stokes equations for both compressible and incompressible fluids. It explains how the stress tensor can be decomposed into volumetric and deviatoric stress tensors and uses the Stokes constitutive equations to calculate the deviatoric tensor based on the velocity field. Subsequently, Navier-Stokes equations for compressible and incompressible fluids are driven and discussed. This chapter is written based on the books of Ladyzhenskaya [1969], Rutherford [1989], and White [2011].

7.1 Decomposition of Stress Tensor

As was shown in the previous chapter, mass and linear momentum conservation laws for a fluid can be written as:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{V}\,.\,(\rho \mathrm{V}) = 0 \\ \rho(\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \rho \dfrac{\partial V}{\partial t} = \rho(g + \overline{f}) + (\nabla\,.\,T) \end{cases}$$

which in case of an incompressible fluid, can be reduced to:

$$\begin{cases} \nabla\,.\,\mathrm{V} = 0 \\ \rho(\mathrm{V}\,.\,\nabla)\,\mathrm{V} + \rho \dfrac{\partial V}{\partial t} = \rho(g + \overline{f}) + (\nabla\,.\,T) \end{cases}$$

Note that the above system of partial differential equations, is unsolvable due to the number of unknown variables contained in the stress tensor, that exceed the number of equations. As such, a constitutive equation is required to relate the stress variables to the velocity field. The stress tensor in fact contains all the normal and shear stresses acting on the surface on an arbitrary control volume, say $D(t)$, and can be decomposed into two simpler tensors. The first tensor, called the volumetric stress tensor, contains all the stresses that tend to change the volume of $D(t)$, and the second tensor, called the deviatoric stress tensor, includes all the viscous stresses that tend to deform $D(t)$.

As was explained in chapter 2, in Newtonian fluids, the only volumetric stress acting on the surface of control volumes, is the hydrostatic pressure that compresses the surface in the direction of its inward normal vector. Moreover, in Newtonian Fluids hydrostatic pressure has no preference in direction and acts equally in all directions.

Hence, the volumetric tensor would be a diagonal matrix, where all the diagonal entries are $-P$. The negative sign is due to the fact that by convention, the outward normal unit vector of the surface is assumed positive.

Then the stress tensor can be written as:

$$T = -PI + \Upsilon \tag{7.1}$$

where I is the 3×3 identity matrix, and Υ denotes the deviatoric stress tensor.

A major assumption about the nature of viscous stresses in Newtonian fluids, is that such stresses are always proportional to the time-rate of deformation, where the constant of proportionality is called the viscosity constant.

7.2 Derivation of Navier-Stokes Equations

As was shown in chapter 1, in case of a simple steady flow in the x-direction, with the constant speed V_x, the shear stress acting in the x-direction on the surface perpendicular to y, can be written as:

$$\tau = \mu \frac{dV_x}{dy}$$

This phenomenon was initially found by Isaac Newton and eventually generalized by Stokes in 1845. The Stokes equations relate the viscous stresses acting on the surface of a control volume, with the time-rate of deformation of that surface.

Let τ_{ij} denote the viscous stress acting in the j-direction on the surface perpendicular to the i-axis. Then the deviatoric stress tensor can be written as:

$$\Upsilon = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{7.2}$$

and stokes constitutive equations can be written as:

$$
\begin{cases}
\tau_{xx} = 2\mu(-\frac{1}{3}\nabla \, . \, \mathrm{V} + \frac{\partial}{\partial x}V_x) \\
\tau_{xy} = \mu(\frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x) \\
\tau_{xz} = \mu(\frac{\partial}{\partial x}V_z + \frac{\partial}{\partial z}V_x) \\
\tau_{yx} = \mu(\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y) \\
\tau_{yy} = 2\mu(-\frac{1}{3}\nabla \, . \, \mathrm{V} + \frac{\partial}{\partial y}V_y) \\
\tau_{yz} = \mu(\frac{\partial}{\partial y}V_z + \frac{\partial}{\partial z}V_y) \\
\tau_{zx} = \mu(\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z) \\
\tau_{zy} = \mu(\frac{\partial}{\partial z}V_y + \frac{\partial}{\partial y}V_z) \\
\tau_{zz} = 2\mu(-\frac{1}{3}\nabla \, . \, \mathrm{V} + \frac{\partial}{\partial z}V_z)
\end{cases}
$$

In case of an incompressible fluid, the deviatoric stress tensor can be simplified into:

$$
\Upsilon = \mu \begin{pmatrix}
2\frac{\partial V_x}{\partial x} & \frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x & \frac{\partial}{\partial x}V_z + \frac{\partial}{\partial z}V_x \\
\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y & 2\frac{\partial V_y}{\partial y} & \frac{\partial}{\partial y}V_z + \frac{\partial}{\partial z}V_y \\
\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z & \frac{\partial}{\partial z}V_y + \frac{\partial}{\partial y}V_z & 2\frac{\partial V_z}{\partial z}
\end{pmatrix} \tag{7.3}
$$

Equation (7.3) can be summarized in the matrix form, in the following way:

$$
\begin{pmatrix}
2\frac{\partial}{\partial x}V_x & \frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x & \frac{\partial}{\partial x}V_z + \frac{\partial}{\partial z}V_x \\
\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y & 2\frac{\partial}{\partial y}V_y & \frac{\partial}{\partial y}V_z + \frac{\partial}{\partial z}V_y \\
\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z & \frac{\partial}{\partial z}V_y + \frac{\partial}{\partial y}V_z & 2\frac{\partial}{\partial z}V_z
\end{pmatrix} =
$$

$$\begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial}{\partial y}V_x & \frac{\partial}{\partial z}V_x \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_y}{\partial x} & \frac{\partial V_z}{\partial x} \\ \frac{\partial V_x}{\partial y} & \frac{\partial V_y}{\partial y} & \frac{\partial V_z}{\partial y} \\ \frac{\partial V_x}{\partial z} & \frac{\partial V_y}{\partial z} & \frac{\partial V_z}{\partial z} \end{pmatrix} = \nabla V + \nabla V^{\mathrm{T}}$$

and hence, the deviatoric stress tensor can be written as:

$$\Upsilon = \mu(\nabla V + \nabla V^{\mathrm{T}}) \tag{7.4}$$

Finally, the general form of the Navier-Stokes equations for Newtonian fluids can be acquired by substituting the volumetric and deviatoric stress tensors, into the conservation of linear momentum equation (6.19).

Note that by using the equation (7.1) and Stokes constitutive equations, entries of the stress tensor can be written as:

$$
\begin{cases}
\sigma_{xx} = -P + 2\mu(-\frac{1}{3}\nabla . \mathrm{V} + \frac{\partial}{\partial x}V_x) \\
\sigma_{xy} = \mu(\frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x) \\
\sigma_{xz} = \mu(\frac{\partial}{\partial x}V_z + \frac{\partial}{\partial z}V_x) \\
\sigma_{yx} = \mu(\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y) \\
\sigma_{yy} = -P + 2\mu(-\frac{1}{3}\nabla . \mathrm{V} + \frac{\partial}{\partial y}V_y) \\
\sigma_{yz} = \mu(\frac{\partial}{\partial y}V_z + \frac{\partial}{\partial z}V_y) \\
\sigma_{zx} = \mu(\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z) \\
\sigma_{zy} = \mu(\frac{\partial}{\partial z}V_y + \frac{\partial}{\partial y}V_z) \\
\sigma_{zz} = -P + 2\mu(-\frac{1}{3}\nabla . \mathrm{V} + \frac{\partial}{\partial z}V_z)
\end{cases}
$$

and by substituting them into the equation (6.19), we obtain:

$$
\begin{cases}
\rho(V_x \frac{\partial}{\partial x}V_x + V_y \frac{\partial}{\partial y}V_x + V_z \frac{\partial}{\partial z}V_x + \frac{\partial}{\partial t}V_x) = \frac{\partial}{\partial x}(-P - \frac{2}{3}\mu\nabla\ .\ \mathrm{V} + 2\mu \frac{\partial}{\partial x}V_x) \\
\qquad + \mu \frac{\partial}{\partial y}(\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y) \\
\qquad + \mu \frac{\partial}{\partial z}(\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z) + \rho(g_x + \overline{f}_x) \\
\rho(V_x \frac{\partial}{\partial x}V_y + V_y \frac{\partial}{\partial y}V_y + V_z \frac{\partial}{\partial z}V_y + \frac{\partial}{\partial t}V_y) = \frac{\partial}{\partial y}(-P - \frac{2}{3}\mu\nabla\ .\ \mathrm{V} + 2\mu \frac{\partial}{\partial y}V_y) \\
\qquad + \mu \frac{\partial}{\partial x}(\frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x) \\
\qquad + \mu \frac{\partial}{\partial z}(\frac{\partial}{\partial z}V_y + \frac{\partial}{\partial y}V_z) + \rho(g_y + \overline{f}_y) \\
\rho(V_x \frac{\partial}{\partial x}V_z + V_y \frac{\partial}{\partial y}V_z + V_z \frac{\partial}{\partial z}V_z + \frac{\partial}{\partial t}V_z) = \frac{\partial}{\partial z}(-P - \frac{2}{3}\mu\nabla\ .\ \mathrm{V} + 2\mu \frac{\partial}{\partial z}V_z) \\
\qquad + \mu \frac{\partial}{\partial x}(\frac{\partial}{\partial x}V_z + \frac{\partial}{\partial z}V_x) \\
\qquad + \mu \frac{\partial}{\partial y}(\frac{\partial}{\partial y}V_z + \frac{\partial}{\partial z}V_y) + \rho(g_z + \overline{f}_z)
\end{cases}
$$

These equations, together with the continuity equation, are known as the Navier-Stokes equations for Newtonian Fluids in conservation form.

These equations can be further simplified for incompressible fluids, where the divergence of the velocity field is zero. In that case, the $\nabla . \mathrm{V}$ can be removed from the above equations, and we obtain:

$$
\begin{cases}
\rho(V_x \frac{\partial}{\partial x}V_x + V_y \frac{\partial}{\partial y}V_x + V_z \frac{\partial}{\partial z}V_x + \frac{\partial}{\partial t}V_x) = \frac{\partial}{\partial x}(-P + 2\mu \frac{\partial}{\partial x}V_x) \\
\qquad + \mu \frac{\partial}{\partial y}(\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y) \\
\qquad + \mu \frac{\partial}{\partial z}(\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z) + \rho(g_x + \overline{f}_x) \\
\rho(V_x \frac{\partial}{\partial x}V_y + V_y \frac{\partial}{\partial y}V_y + V_z \frac{\partial}{\partial z}V_y + \frac{\partial}{\partial t}V_y) = \frac{\partial}{\partial y}(-P + 2\mu \frac{\partial}{\partial y}V_y) \\
\qquad + \mu \frac{\partial}{\partial x}(\frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x) \\
\qquad + \mu \frac{\partial}{\partial z}(\frac{\partial}{\partial z}V_y + \frac{\partial}{\partial y}V_z) + \rho(g_y + \overline{f}_y) \\
\rho(V_x \frac{\partial}{\partial x}V_z + V_y \frac{\partial}{\partial y}V_z + V_z \frac{\partial}{\partial z}V_z + \frac{\partial}{\partial t}V_z) = \frac{\partial}{\partial z}(-P + 2\mu \frac{\partial}{\partial z}V_z) \\
\qquad + \mu \frac{\partial}{\partial x}(\frac{\partial}{\partial x}V_z + \frac{\partial}{\partial z}V_x) \\
\qquad + \mu \frac{\partial}{\partial y}(\frac{\partial}{\partial y}V_z + \frac{\partial}{\partial z}V_y) + \rho(g_z + \overline{f}_z)
\end{cases}
$$

7.2 Navier-Stokes Equations in Vector Form

To write Navier-Stokes equations in vector form, note that the stress tensor for incompressible fluids can be written as:

$$
T = \begin{pmatrix}
-P + 2\mu \frac{\partial V_x}{\partial x} & \mu(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}) & \mu(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}) \\
\mu(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}) & -P + 2\mu \frac{\partial V_y}{\partial y} & \mu(\frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}) \\
\mu(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}) & \mu(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}) & -P + 2\mu \frac{\partial V_z}{\partial z}
\end{pmatrix}
$$

Let $(\nabla . T)_x$ denote the x component of the divergence of the stress tensor. Then we can write:

$$
\begin{aligned}
(\nabla\,.\,T)_x &= -\frac{\partial}{\partial x}P + 2\mu\frac{\partial}{\partial x}(\frac{\partial}{\partial x}V_x) + \mu\frac{\partial}{\partial y}(\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y) + \mu\frac{\partial}{\partial z}(\frac{\partial}{\partial z}V_x + \frac{\partial}{\partial x}V_z) \\
&= -\frac{\partial}{\partial x}P + 2\mu\frac{\partial^2 V_x}{\partial x^2} + \mu\frac{\partial^2 V_x}{\partial y^2} + \mu\frac{\partial^2 V_y}{\partial y\partial x} + \mu\frac{\partial^2 V_x}{\partial z^2} + \mu\frac{\partial^2 V_z}{\partial z\partial x} \\
&= -\frac{\partial}{\partial x}P + 2\mu\frac{\partial^2 V_x}{\partial x^2} + \mu\frac{\partial^2 V_x}{\partial y^2} + \mu\frac{\partial^2 V_y}{\partial y\partial x} + \mu\frac{\partial^2 V_x}{\partial z^2} + \mu\frac{\partial^2 V_z}{\partial z\partial x} \\
&= -\frac{\partial}{\partial x}P + \mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \mu\frac{\partial^2 V_x}{\partial x^2} + \mu\frac{\partial^2 V_y}{\partial y\partial x} + \mu\frac{\partial^2 V_z}{\partial z\partial x} \\
&= -\frac{\partial}{\partial x}P + \mu\nabla^2 V_x + \mu\frac{\partial}{\partial x}(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}) \\
&= -\frac{\partial}{\partial x}P + \mu\nabla^2 V_x + \mu\frac{\partial}{\partial x}(\nabla\,.\,\mathrm{V}) \\
&= -\frac{\partial}{\partial x}P + \mu\nabla^2 V_x
\end{aligned}
$$

Likewise, other components of $\nabla\,.\,T$ can be written as:

$$
\begin{aligned}
(\nabla\,.\,T)_y &= -\frac{\partial}{\partial y}P + \mu\nabla^2 V_y \\
(\nabla\,.\,T)_z &= -\frac{\partial}{\partial z}P + \mu\nabla^2 V_z
\end{aligned}
$$

and hence

$$
\begin{aligned}
\nabla\,.\,T &= (\nabla\,.\,T)_x\vec{i} + (\nabla\,.\,T)_y\vec{j} + (\nabla\,.\,T)_z\vec{k} \\
&= [-\frac{\partial}{\partial x}P + \mu\nabla^2 V_x]\vec{i} + [-\frac{\partial}{\partial y}P + \mu\nabla^2 V_y]\vec{j} + [-\frac{\partial}{\partial z}P + \mu\nabla^2 V_z]\vec{k} \\
&= -(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k})P + \mu\nabla^2(\mathrm{V}_x\vec{i} + \mathrm{V}_y\vec{j} + \mathrm{V}_z\vec{k}) \\
&= -\nabla P + \mu\nabla^2 V
\end{aligned}
$$

By substituting $\nabla\,.\,T$ into equation (6.19), we obtain:

$$
\rho\frac{\partial V}{\partial t} + \rho(\mathrm{V}\,.\,\nabla)\,\mathrm{V} = -\nabla P + \mu\nabla^2 V + \rho(g + \overline{f})
$$

This equation can be divided by the density constant, and the convective form of incompressible Navier-Stokes equations can be written as:

$$\begin{cases} \nabla \,.\, \mathrm{V} = 0 \\ \dfrac{\partial V}{\partial t} + (\mathrm{V} \,.\, \nabla)\,\mathrm{V} - \nu \nabla^2 V = -\dfrac{1}{\rho} \nabla P + F \end{cases}$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity constant, and F denotes the sum of body forces per unit volume.

To obtain the vector form of the Navier-Stokes equations for compressible fluids, note that the deviatoric stress tensor, can be written as:

$$\mu \begin{pmatrix} 2\dfrac{\partial}{\partial x}V_x - \dfrac{2}{3}\nabla \,.\, \mathrm{V} & \dfrac{\partial}{\partial x}V_y + \dfrac{\partial}{\partial y}V_x & \dfrac{\partial}{\partial x}V_z + \dfrac{\partial}{\partial z}V_x \\ \dfrac{\partial}{\partial y}V_x + \dfrac{\partial}{\partial x}V_y & 2\dfrac{\partial}{\partial y}V_y - \dfrac{2}{3}\nabla \,.\, \mathrm{V} & \dfrac{\partial}{\partial y}V_z + \dfrac{\partial}{\partial z}V_y \\ \dfrac{\partial}{\partial z}V_x + \dfrac{\partial}{\partial x}V_z & \dfrac{\partial}{\partial z}V_y + \dfrac{\partial}{\partial y}V_z & 2\dfrac{\partial}{\partial z}V_z - \dfrac{2}{3}\nabla \,.\, \mathrm{V} \end{pmatrix} =$$

$$\mu \begin{pmatrix} \dfrac{\partial V_x}{\partial x} & \dfrac{\partial}{\partial y}V_x & \dfrac{\partial}{\partial z}V_x \\ \dfrac{\partial V_y}{\partial x} & \dfrac{\partial V_y}{\partial y} & \dfrac{\partial V_y}{\partial z} \\ \dfrac{\partial V_z}{\partial x} & \dfrac{\partial V_z}{\partial y} & \dfrac{\partial V_z}{\partial z} \end{pmatrix} + \mu \begin{pmatrix} \dfrac{\partial V_x}{\partial x} & \dfrac{\partial V_y}{\partial x} & \dfrac{\partial V_z}{\partial x} \\ \dfrac{\partial V_x}{\partial y} & \dfrac{\partial V_y}{\partial y} & \dfrac{\partial V_z}{\partial y} \\ \dfrac{\partial V_x}{\partial z} & \dfrac{\partial V_y}{\partial z} & \dfrac{\partial V_z}{\partial z} \end{pmatrix} - \dfrac{2}{3}\mu \nabla \,.\, \mathrm{V} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\mu(\nabla V + \nabla V^{\mathrm{T}}) - \frac{2}{3}\mu(\nabla \,.\, \mathrm{V})\,\mathrm{I}$$

Then the stress tensor can be written as:

$$T = -P\,\mathrm{I} + \mu(\nabla V + \nabla V^{\mathrm{T}}) - \frac{2}{3}\mu(\nabla \,.\, \mathrm{V})\,\mathrm{I} \quad \Rightarrow$$

$$\nabla \,.\, T = \nabla \,.\, (-P\,\mathrm{I} + \mu(\nabla V + \nabla V^{\mathrm{T}}) - \frac{2}{3}\mu(\nabla \,.\, \mathrm{V})\,\mathrm{I})$$

$$= -P\nabla \,.\, \mathrm{I} + \mu\nabla \,.\, (\nabla V + \nabla V^{\mathrm{T}}) - \frac{2}{3}\mu\nabla \,.\, (\nabla \,.\, \mathrm{V})\,\mathrm{I}$$

$$= -\nabla P + \mu\nabla^2 V - \frac{2}{3}\mu\nabla(\nabla \,.\, \mathrm{V})$$

and by substituting $\nabla \,.\, T$ into equation (6.19), we obtain:

$$\rho\frac{\partial V}{\partial t} + \rho(\mathrm{V} \,.\, \nabla)\,\mathrm{V} = -\nabla P + \mu\nabla^2 V - \frac{2}{3}\mu\nabla(\nabla \,.\, \mathrm{V}) + \rho(g + \overline{f})$$

Hence, compressible Navier-Stocks equations can be written as:

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \mathrm{V} \,.\, (\rho\,\mathrm{V}) = 0 \\ \dfrac{\partial V}{\partial t} + (\mathrm{V} \,.\, \nabla)\,\mathrm{V} - \nu\,\nabla^2 V + \dfrac{2}{3}\nu\,\nabla(\nabla \,.\, \mathrm{V}) = -\dfrac{1}{\rho}\nabla P + F \end{cases}$$

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