Matrix Summability Methods

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ABSTRACT

The purpose of this study is to prepare a brief summary of some matrix summability methods. This thesis consists of four chapters. In the first chapter, a general introduction to the matrix summability method is mentioned.

In Chapter 2, deals with the basic definitions and theorems of sequences associated with matrix summability methods. In this thesis contains some proper examples and definitions that are related to sequences, subsequences, bounded sequences, monotone sequences, convergent sequences, divergent sequences and Cauchy sequences. In addition to this, some basic theorems about sequences and basic properties of infinite matrices are given. Some of properties of infinite matrices; product of matrices, triangle matrices, inverse of matrices, triangular matrices.

In Chapter 3, contains the theory of matrix summability methods. In the first part, basic definitions and theorems of matrix summability methods are examined. Then, these two theorems and their examples are given which enable us to learn whether the matrices are conservative matrix (Schur-Kojima Theorem) or regular matrix (Silverman-Teoblitz Theorem). Furthermore, comparable matrix methods (stronger or weaker matrix) are included. In the last section, zero preserving matrix, multiplicative matrix and some related theorems are examined.

In the last chapter, some matrix summability methods are discussed together with general definitions, theorems, and examples. Also the proofs of the theorems are given. These methods are: Cesaro methods, Hölder methods, Euler Knopp methods, and Hausdorff matrix methods.

Keywords: matrix summability methods, conservative matrix, regular matrix, Cesaro methods, Hausdorff methods, Riesz methods, Hölder methods, Euler methods.

ÖZ

Bu çalışmanın amacı bazı matris toplanabilme yöntemlerinin kısa bir özetini hazırlamaktır. Bu tez dört bölümden oluşmaktadır. Birinci bölümde, matris toplanabilme yöntemine genel bir girişten bahsedilmektedir.

Bölüm 2'de, matris toplanabilme yöntemleri ile ilişkili olan dizilerin temel tanımları ve teoremleri ele alınmaktadır. Bu bölümde, diziler, alt diziler, sınırlı diziler, monoton diziler, yakınsak diziler, ıraksak diziler ve Cauchy dizileri ile ilgili tanımlar ve uygun örnekleri bulunmaktadır. Buna ek olarak, sonsuz matrislerin temel özellikleri verilmiştir. Sonsuz matrislerin bazı özellikleri; matrislerin çarpımı, üçgen matrisler, matrislerin tersidir.

Bölüm 3, matris toplanabilme yöntemleri teorisini içermektedir. İlk kısmında, matris toplanabilirlik yöntemlerinin temel tanımları ve teoremleri incelenmiştir. Daha sonra, matrislerin konservatif matris (Schur-Kojima Teoremi) veya düzenli matris (Silverman-Teoblitz Teoremi) olup olmadığını öğrenmemizi sağlayan bu iki teorem ve örnekleri verilmiştir. Buna ek olarak, karşılaştırılabilir matris yöntemlerine (daha güçlü veya daha zayıf matris) yer verilmiştir. Son kısımda, sıfır koruyucu matris ve çarpımsal matris yöntemleri ile ilgili bazı teoremler incelenmiştir.

Son bölümde, bazı matris toplanabilme yöntemleri genel tanımlar, teoremler ve örneklerle birlikte tartışılmaktadır. Ayrıca teoremlerin kanıtları da verilmiştir. Bu yöntemler: Cesaro metodu, Hölder metodu, Euler metodu ve Hausdorff matris metodudur.

Anahtar Kelimeler: matris toplanabilme yöntemi, konservatif matris, regular matris, Cesaro metodu, Hausdorff metodu, Riesz metodu, Hölder metodu, Euler metodu.

I am dedicating this thesis to who always believe me my mother Sevda Dağbaşı, my father Ahmet Dağbaşı and my dear sisters Ayşe and Naziyet.

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Chapter 1

INTRODUCTION

The main idea of this thesis is to prepare a brief summary about matrix summability methods. This brief summary will include some basic definitions and theorems related with matrix summability methods such that a non-familiar reader will get an idea about what is a matrix summability method. Moreover, since a matrix summability method is a mapping on a subspace of the space of sequences a summary of the basic theory of sequences and related topics will be given. To prepare such a summary, we shall start with the basic definitions and properties of infinite matrices. After that basic definitions and theorems about matrix summability methods will be discussed. Two important concepts conservative and regular matrix methods will be explained by using different sources. A matrix method A is called conservative if it preserves the convergence of a sequence x. An important subclass of conservative matrices is the class of regular matrices. A regular matrix is conservative matrix method so that it preserves the limit of a convergent sequence x as well. In this thesis after basic theory of matrix methods we focus on some well-known matrix summability methods such as Cesaro Matrix method, Euler Method, Hölder Method and in more general perspective Hausdorff methods. All these matrix methods will be discussed with details including the inclusion relations between these methods.

In Chapter 2, we introduce a short summary of the theory of sequences which includes the basic theory about sequences, subsequences, bounded sequences, monotone sequences, convergent and Cauchy sequences. Moreover, we give some theorems related with sequences, subsequences, bounded sequences, monotone sequences, convergent and Cauchy sequences. All definitions and theorems given in this chapter are illustrated by appropriate examples. In the last part of Chapter 2, we discuss infinite matrices and operations such as matrix addition, matrix multiplication and multiplication of a matrix by a scalar on infinite matrices.

Chapter 3 is devoted to the theory of matrix summability methods. We start to this chapter with basic definitions and properties of matrix summability methods. Then we give basic definitions related with matrix summability methods such as application domain, convergence domain, stronger or weaker matrix summability methods. Also we explain what is a comparable matrix method. In the last part of this chapter we discuss two basic theorems of matrix summability methods namely, Schur-Kojima and Silvermen-Teoblitz theorems. The Schur-Kojima theorem gives the necessary and sufficient condition for a matrix to be a conservative matrix. The second theorem which is Silvermen-Teoblitz theorem gives necessary and sufficient conditions for a matrix to be a regular matrix. In the last part of this chapter we study zero preserving and multiplicative matrix methods and some related theorems. To make it more clear to the readers, all definitions and theorems are explained on suitable examples.

In Chapter 4, some well-known regular matrix summability methods such as Cesaro matrix methods, Euler Matrix methods, Hölder matrix methods and Hausdorff

Matrix methods are introduced and studied. Some well-known properties of these matrix methods are given. Related theorems are given with proofs. In the last part of this chapter we study inclusion properties of these matrix methods. Similar to other chapters, all definitions and theorems are illusrated by examples.

Finally we would like to mention that, this thesis consisting of four chapters, The first two chapters, Chapter 1 and Chapter 2 are used to give some basic theory of the sequences and infinite matrices. Of course, the whole theory of sequences and infinite matrices are not given here, we just give a summary of then including the part that will be needed to dicsuss the main chapters of the thesis. Main chapters of the thesis are Chapter 3 and Chapter 4, and these to chapters are used to explain the main part of the thesis as we mentioned above.

Chapter 2

PRELIMINARIES

In this chapter, some basic topics that we need in the thesis will be included. Firstly, some basic features and theorems related to sequences will be briefly summarized. Then, we shall introduce basic properties of infinite matrices. To prepare this chapter we use the following references ([2], [3], [4], [6], [9], [10], [13], [14]).

2.1 Sequences

In this section, we briefly discuss sequences, sub-sequences, convergent sequences, divergent sequences, bounded sequences, monotone sequences, Cauchy sequences and basic theorems related to these concepts.

Definition 2.1: Any function defined from the set of natural number is called a sequence and denoted by the $\mathbb{N}$. The sequences take various names according to their codomains. If the codomain of the sequence is a subset of real number $\mathbb{R}$, then the sequence is called real number sequence. If the codomain is a subset of $\mathbb{Q}$ (Rational numbers), the sequence is called a rational number sequence. In the case that codomain is a subset of $\mathbb{C}$ (complex numbers), it is called complex valued sequences. A sequence is shown in many sources by $(s_1, s_2, s_3, ...s_n, ...)$ or shortly (s_n). Usually s_n, (the n^{th} term) is called the general term of the sequence.

The space of all sequence is denoted by ω **.**

$$\omega := \left\{ x = (x_k), x : \mathbb{N} \cup \{0\} \to \mathrm{K} \right\}$$

$K = \mathbb{R}$ *or* $K = \mathbb{C}$.

Example 2.1: The function $f(n) = s_n := \left(\frac{1}{4n}\right)$ is a rational number sequence with n^{th} term $\left(\frac{1}{4n}\right)$.

So,

$$s_n := \left(\frac{1}{4n}\right) = \left(1, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \ldots\right).$$

Example 2.2: The function $f(n) = s_n := \left(\sqrt{n}\right)$ is a real number sequence with n^{th} term $s_n = \left(\sqrt{n}\right)$, with

$$s_n := \left(\sqrt{n}\right) = \left(1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\right).$$

Example 2.3: The function $f(n) = s_n := \left(3n^2 i\right)$ gives a complex valued sequence, with,

$$s_n := \left(3n^2 i\right) = \left(0, 3i, 12i, 27i, 48i, \ldots\right).$$

Definition 2.2: Let (s_n) be any sequence and let (n_k) be a strictly increasing $(n_1 < n_2 < n_3 < \ldots < n_k < n_{k+1} < \ldots)$ sequence of natural numbers. If $\left(s_{n_k}\right) \subset (s_n)$ then $\left(s_{n_k}\right)$ is called a subsequence of (s_n).

Example 2.4: Let $x_n := \left(\frac{1}{3n}\right) = \left(1, \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18}, \frac{1}{21}, \ldots\right)$ then

$$x := \frac{1}{6n} = \left(\frac{1}{6}, \frac{1}{12}, \frac{1,}{18}, \ldots\right)$$

is a subsequence of x_n where $n_k := \left(\frac{1}{6k}\right) \quad k = 1, 2, 3, \ldots$

Definition 2.3: If all terms of the sequence x_n are less than or equal than a real number L_1, x_n is called bounded above and L_1 is called an upper bound. That is x_n is bounded above if ,

$$\exists\, L_1 \in \mathbb{R} \text{ such that } x_n \leq L_1 \text{ for all } n \in \mathbb{N}.$$

Definition 2.4: If all terms of the sequence x_n are greater than or equal to a real number L_2, it is called bounded below and L_2 is called a lower bound for x_n. That is x_n is bounded below if,

$$\exists\, L_2 \in \mathbb{R} \text{ such that } x_n \geq L_2 \text{ for all } n \in \mathbb{N}.$$

Definition 2.5: Let x_n be a sequence. If there is a number $L > 0$ that provides the condition $-L \leq x_n \leq L$,the sequence x_n is called bounded sequence. In other words x_n is bounded if and only if,

$$\exists\, L > 0 \text{ such that } |x_n| \leq L \text{ for all } n \in \mathbb{N}.$$

Obviously, a bounded sequence is a sequence which is bounded above and bounded below.

The space of all bounded sequences is represented by the notation ℓ^∞ *or* m.

$$\ell^\infty = m := \left\{ x = (x_k) : \|x\|_\infty = \sup_k |x_k| < \infty \right\}$$

where,

$$\|\,.\,\|_\infty : \ell^\infty \to \mathbb{R}^+ \cup \{0\}$$

is called the supremum norm.

Example 2.5: The sequence $x_n := \left(-3^n\right)$ is not a bounded sequence.

Example 2.6: The sequence $x_n := \left(\frac{(-1)^n}{n^2}\right)$ is a bounded sequence from both the above and below.

Example 2.7: The sequence,

$$x_n := (-n) = (0, -1, -2, -3, -4, \ldots)$$

is bounded above , but it is not bounded below. Hence (x_n) is not a bounded sequence.

Example 2.8: The sequence,

$$x_n := \left(n^2\right) = (0, 1, 4, 9, 16, 25, \ldots)$$

is bounded below, but it is not bounded above. Therefore (x_n) is not a bounded sequence.

Definition 2.6: If a sequence (x_n) is bounded above, the smallest upper bound is called the supremum of the sequence and indicated by the notation $\sup x_n$. Similarly if a sequence is bounded below, the greatest lower bound is called the infimum of the sequence (x_n) and denoted by $\inf x_n$.

Definition 2.7 *(ε - Neighbourhood)*: Let $\varepsilon > 0$ *and* $b \in \mathbb{R}$. Then,

$$K := \{x : |x - b| < \varepsilon,\ x \in R\}$$

is a called the ε - neighbourhood of b.

The ε - neighbourhood of b is the interval

$$\left(b-\varepsilon, b+\varepsilon\right).$$

Definition 2.8 (*Convergent Sequence*): Let s_n be a real number sequence and $s \in \mathbb{R}$. The sequence s_n is convergent to s if and only if every ε - neighbourhood of s excludes only finitely many terms of s_n. This is shown as

$$\lim s_n = s \quad \text{or} \quad (s_n) \to s$$

In other words s_n converges to s, if

$$\forall \varepsilon > 0,\ \exists \mathrm{N} = \mathrm{N}\left(\varepsilon\right) \ \textit{such that}\ \forall n \geq \mathrm{N}\left(\varepsilon\right) \quad \left|s_n - s\right| < \varepsilon .$$

The space of all convergent sequence is generally represented by the notation c,

$$c := \left\{ s = (s_n) \in w \mid \lim_{n\to\infty} s_n \ \text{ exist} \right\}.$$

Example 2.9: Given $s_n := \left(\dfrac{4n^2+2}{5n^2+7}\right)$.

Then,

$$\lim_{n\to\infty} \frac{4n^2+1}{5n^2+4} = \frac{4}{5}$$

Thus, the sequence is convergent to $\dfrac{4}{5}$.

Bounded sequences need not be convergent, this is illusrated in the following example.

Example 2.10: The sequence $s_n := \left((-1)^n + 1\right)$ is bounded, but since,

$$s_n := (0, 2, 0, 2, ...) \text{ and } \lim_{n\to\infty}\left((-1)^n + 1\right)$$

does not exists. That is s_n is not a convergent sequence.

As a consequence of the definition, we can conclude that an unbounded sequence is not convergent.

Example 2.11: The sequence $x_n := \left(\sqrt{n}\right)$ is not a bounded sequence. Also x_n is not convergent sequence.

Definition 2.9: The sequence which is convergent to 0, is called a null sequence. The space of all null sequences is denoted by c_0.

$$c_0 := \left\{ s = (s_n) \in w \mid \lim_{n\to\infty} s_n = 0 \right\}.$$

Example 2.12: The sequence $s_n := \left(\dfrac{1}{n^2 + n}\right)$ is converges to 0. So it is a null sequence.

Definition 2.10: A sequence which is not convergent is called divergent sequence.

Example 2.13: The sequences $s_n := (-1)^n$ and $x_n := (n)$ are divergent sequences.

We have the following inclusion relation for the space of sequences c_0, c, ℓ^∞ and w.

$$c_0 \subset c \subset \ell^\infty \subset w.$$

Theorem 2.1: Let $\{s_n\}$ *and* $\{d_n\}$ be two sequences and let $\alpha \in \mathbb{R}$. If $\{s_n\}$ *and* $\{d_n\}$ are convergent, then

a. $\lim\limits_{n\to\infty}(s_n + d_n) = \lim\limits_{n\to\infty} s_n + \lim\limits_{n\to\infty} d_n$

b. $\lim\limits_{n\to\infty}(s_n\, d_n) = \left(\lim\limits_{n\to\infty} s_n\right)\cdot\left(\lim\limits_{n\to\infty} d_n\right)$

c. $\lim\limits_{n\to\infty}(\alpha s_n) = \alpha \lim\limits_{n\to\infty} s_n$

d. $\lim\limits_{n\to\infty} \dfrac{s_n}{d_n} = \dfrac{\lim\limits_{n\to\infty} s_n}{\lim\limits_{n\to\infty} d_n} \quad if \quad \lim\limits_{n\to\infty} d_n \neq 0.$

Theorem 2.2: If the sequence $\{s_n\}$ is convergent, then each sub-sequence of $\{s_{n_k}\}$ is also convergent and it converges to the same limit.

Proof: Let $\{s_{n_k}\}$ be a sub-sequences of $\{s_n\}$, where $\lim\limits_{n\to\infty} s_n = s.$ Then,

$$\forall \varepsilon >, \exists N \text{ such that } \forall n > N \quad |s_n - s| < \varepsilon.$$

Since $\{n_k\}$ is strictly increasing, $\exists K\ s.t.\ \ \forall k > K,\ \ n_k > N$. Then

$$\forall k > K,\ |s_{n_k} - s| < \varepsilon.$$

This means that, $\lim\limits_{k\to\infty} s_{n_k} = s.$

Example 2.14: Consider $x_n : \left(\dfrac{1}{n}\right)$ which is convergent to 0. So every sub-sequence of x_n converges to 0.

The converse of the above theorem is not true, that is if a sequence is divergent this does not mean that its all subsequences are divergent. This is illusrated in the following example.

Example 2.15: Consider $x_n := (-1)^n = (-1,1,-1,1,....)$, then $x_{2n} := (1,1,1,1,....)$ and $x_{2n+1} := (-1,-1,-1,-1,....)$ are two convergent subsequences of x_n. The subsequence x_{2n} converges to 1 and x_{2n+1} converges to -1, but x_n is not convergent.

Theorem 2.3: Let s_n be a convergent sequence. Then limit of x_n is unique.

Proof: Suppose that s_n has two limits, x_1 *and* x_2. Then,

$$\lim_{n\to\infty} s_n = x_1 \quad \text{and} \quad \lim_{n\to\infty} s_n = x_2 .$$

Given $\varepsilon > 0$, $\exists \mathrm{N} = \mathrm{N}_1(\varepsilon)$ *such that* $\forall n \geq \mathrm{N}_1(\varepsilon)$

$$|s_n - x_1| < \frac{\varepsilon}{2}.$$

Likewise, $\exists \mathrm{N} = \mathrm{N}_2(\varepsilon)$ *such that* $\forall n \geq \mathrm{N}_2(\varepsilon)$

$$|s_n - x_2| < \frac{\varepsilon}{2}.$$

Let $N := \max\{N_1, N_2\}$, than $\forall n \geq N$, we have,

$$|x_1 - x_2| = |x_1 - s_n + s_n - x_2| \leq |s_n - x_1| + |s_n - x_2|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

which implies that,

$$x_1 = x_2.$$

Theorem 2.4: A convergent sequence s_n is bounded.

Proof: Let s_n be a convergent sequence with $\lim s_n = s.$ Then,

$$\forall \varepsilon > 0,\ \exists \mathrm{N} = \mathrm{N}(\varepsilon) \text{ such that } \forall n \geq \mathrm{N}(\varepsilon) \quad |s_n - s| < \varepsilon .$$

Now, take $\varepsilon = 1$, then

$$\exists \mathrm{N} \ \ \textit{such that} \ \forall n \geq \mathrm{N} \ \ |s_n - s| < 1.$$

That is $\forall n \geq N$,

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s|.$$

So if,

$$M = \max\left(|s_1|, |s_2|, \ldots, |s_N|, 1 + |s|\right)$$

then,

$$\forall n, |s_n| \leq M.$$

So (s_n) is bounded.

Theorem 2.5: Any convergent sequence is bounded, but every bounded sequence may not be convergent.

Example 2.16: The sequence $(-1)^n$ is a bounded but it is not a convergent sequence.

Definition 2.11: Let x_n be a sequence,

- **a.** If $\forall n \in \mathbb{N},\ x_n \leq x_{n+1}$, x_n is called an increasing sequence
- **b.** If $\forall n \in \mathbb{N},\ x_n \geq x_{n+1}$, x_n is called a decreasing sequence
- **c.** If $\forall n \in \mathbb{N},\ x_n < x_{n+1}$, x_n is called a strictly increasing sequence
- **d.** If $\forall n \in \mathbb{N},\ x_n > x_{n+1}$, x_n is called a strictly decreasing sequence
- **e.** An increasing or decreasing sequence is called a monotone sequence.

Example 2.17: Given $x_n := \left(1 - \frac{1}{n}\right)$.

Then,

$$x_n := \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right)$$

x_n is increasing, strictly increasing and monotone.

Example 2.18: Let $x_n := \left(1 + \frac{1}{n}\right)$.

Then,

$$x_n := \left(2, \frac{3}{2}, \frac{4}{3}, \frac{4}{4}, \ldots\right)$$

x_n is decreasing, strictly decreasing and monotone.

Remark: (x_n) is increasing (strictly increasing) if the sequence $(-x_n)$ is decreasing, (strictly decreasing).

Theorem 2.6 (*Monotone Convergence Theorem*): Let (s_n) be a monotone and bounded sequence , then $\lim s_n$ is exist.

Proof: We will prove that, an increasing bounded sequence converges to its supremum and decreasing bounded sequence converges to its infimum.

a. Suppose that, (s_n) is increasing and bounded sequence and let $s = \sup\{s_1, s_2, s_3 \ldots\}$.

Since (s_n) is bounded above , $s = \sup\{s_n\}$ exists . We need to show that $\lim_{n \to \infty} s_n = s$.

By the properties of supremum $\forall \varepsilon > 0,\ \exists s_N$ *such that* $s - \varepsilon < s_N$.

Therefore, $\forall n > N$

$$s-\varepsilon < s_N \le s_n \le s < s+\varepsilon$$

$$-\varepsilon < s_n - s < \varepsilon$$

$$|s_n - s| < \varepsilon$$

(s_n) is an increasing and $\lim_{n\to\infty} s_n = s$.

b. Suppose that, (s_n) is a decreasing sequence and bounded. Then $s = \inf\{s_1, s_2, ...\}$.

Let $\forall \varepsilon > 0$ $\exists s_N$ such that $s_N < s+\varepsilon$. Since (s_n) is a decreasing sequence.

Then $\forall n > N$,

$$s-\varepsilon < s \le s_n \le s_N < s+\varepsilon.$$

We see that, $\lim_{n\to\infty} s_n = s.$

Therefore, (s_n) is a convergent.

Definition 2.12: A sequence x_n is said to be a Cauchy sequence, if

$$\forall \varepsilon > 0,\ \exists N(\varepsilon) \in \mathbb{N}\ \ such\ that\ \ \forall n,m \ge N(\varepsilon), |s_n - s_m| < \varepsilon.$$

Theorem 2.7: A convergent sequence is a Cauchy sequence.

Proof: Let s_n be a convergent sequence with $\lim s_n = s$. Then,

$$\forall \varepsilon > 0,\ \exists N\ \ such\ that\ \forall n > N\ \ |s_n - s| < \frac{\varepsilon}{2}.$$

Therefore, $\forall n,m > N$

$$|s_n - s_m| \le |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $\{s_n\}$ is a Cauchy.

Theorem 2.8: Each Cauchy sequence is bounded.

Proof: Let $\{s_n\}$ be a Cauchy sequence. Then,

$$\forall \varepsilon > 0,\ \exists N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n, m \geq N(\varepsilon),$$

$$|s_n - s_m| < \varepsilon.$$

Take $\varepsilon = 1$, then, $\exists N$ such that $\forall n, m > N,\quad |s_n - s_m| < 1.$

Fix $m_0 > N. Then\ \forall n > N,$

$$|s_n| \leq |s_n - s_{m_0}| + |s_{m_0}| < 1 + |s_{m_0}|$$

So, if

$$M = \max\{|s_1|, |s_2|, ..., |s_N|, |s_{m_0}| + 1\}$$

then, $\forall n,\ |s_n| \leq M$. Therefore $\{s_n\}$ is bounded.

Theorem 2.9: Every Cauchy sequence is convergent.

Theorem 2.10 *(Then Balzano-Weierstrass Theorem)*: Every bounded real number sequence has at least one convergent sub-sequence.

Proof: Asuume that $\{s_n\}$ is bounded:

1. $\{s_n\}$ has finite number of distinct term. After that one of terms must have infinite repeatitions. The subsequence that is the consisting of repation is a constant sequence and therefore converges. So this implies that sequence is converges.

2. Suppose that, $\{s_n\}$ has a infinite number distinct terms. Since $\{s_n\}$ is bounded then we have,

$$c \le s_n \le d\ ,$$

For some real numbers c and d. We divide the interval of $[c,d]$ into two sub-intervals. These are $\left[c,\ \frac{c+d}{2}\right]$ and $\left[\frac{c+d}{2}, d\right]$. At least one of them contains infinite many distinct term of $\{s_n\}$. Assume that sub-interval $[c_1, d_1]$ contains ∞ many terms. So, we acquire a sequence of intervals $\{[c_k, d_k]\}$ with properties.

a. $\forall k,\ [c_k, d_k]$ contains ∞ many distinct terms of $\{s_n\}$.

b. $\forall k,\ [c_{k+1}, d_{k+1}] \subseteq [c_k, d_k]$.

c. $\forall k,\ d_k - c_k = \frac{d-c}{2^k}$.

Let sellect a subsequence $\{s_{n_k}\}$ *of* $\{s_n\}$. Take any term s_{n_1} in $[c_1, d_1]$. Since $[c_2, d_2]$ contains ∞ many distinct terms of $\{s_n\}, \exists n_2 > n_1\ \ s.t.\ \ s_{n_2} \in [c_2, d_2]$. In this way and obtain subsequence $\{s_{n_k}\}$ *of* $\{s_n\}$. This subsequence has the following property,

$$\forall k, m > K,\ \left|s_{n_k} - s_{n_m}\right| \le d_K - c_k = \frac{d-c}{2^K}.$$

If any $\varepsilon > 0$ is given, we can find K such that $\frac{d-c}{2^K} < \varepsilon$. *Then* $\forall k, m > K,$

$$\left|s_{n_k} - s_{n_m}\right| < \frac{d-c}{2^K} < \varepsilon\ .$$

We say that $\{s_{n_k}\}$ is cauchy sequence and therefore, the sequence is converges.

Definition 2.13: Let $\{s_n\}$ be a sequence, then

a. $\{s_n\}$ is divergent to $+\infty$ means that,

$$\forall A_1 \in \mathbb{R} \ \exists n_1 \in N, \ \forall n \geq n_1 \ \ s_n > A_1.$$

In this case we write ,

$$\lim_{n \to \infty} s_n = +\infty.$$

b. $\{s_n\}$ is a divergent to $-\infty$ so,

$$\forall A_2 \in \mathbb{R} \ \exists n_2 \in N, \ \forall n \geq n_2 \ \ s_n < A_2.$$

In this case we write,

$$\lim_{n \to \infty} s_n = -\infty.$$

Example 2.19: The sequence $s_n := \sqrt{n}$ diverges to infinity. That is,

$$\lim_{n \to \infty} \sqrt{n} = +\infty.$$

Example 2.20: The sequence $s_n := (-n)$ diverges to $-\infty$. That is,

$$\lim_{n \to \infty} (-n) = -\infty.$$

Theorem 2.11:

a. Assume that (s_n) *and* (d_n) are sequences and, $s_n \to +\infty$.

1. If d_n is bounded below. Then,

$$\lim_{n \to \infty} (s_n + d_n) = +\infty.$$

2. If $\alpha > 0$,

$$\lim_{n \to \infty} (\alpha s_n) = +\infty.$$

3. Let $\{d_n\}$ is bounded and $s_n \neq 0$, so

$$\lim_{n\to\infty} \frac{d_n}{s_n} = 0.$$

b. Assume that (s_n) *and* (d_n) are sequences then, $s_n \to -\infty$.

1. If d_n is bounded above. Then,

$$\lim_{n\to\infty} (s_n + d_n) = -\infty.$$

2. If $\alpha > 0$,

$$\lim_{n\to\infty} (\alpha x_n) = -\infty.$$

3. Let $\{d_n\}$ is bounded and $s_n \neq 0$, so

$$\lim_{n\to\infty} \frac{d_n}{s_n} = 0.$$

2.2 Infinite Matrices

This section is devoted to the infinite matrices and their properties. In this section we are aiming to discuss infinite matrices and basic operations, such as addition, subtraction, scalar multiplication and multiplication on infinite matrices. These properties of infinite matrices will be used in the later chapters.

Definition 2.14: An infinite matrix $A = (a_{nk})$ is a matrix that has infinitely many rows and columns.

$$\begin{array}{cc} A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \ldots\ldots a_{0k} \ \cdots \\ a_{10} & a_{11} & a_{12} \ldots\ldots a_{1k} \ \cdots \\ \vdots & \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1} & a_{n2} & a_{n3} \ldots\ldots a_{nk} \ \cdots \\ \vdots & \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \end{bmatrix} & \to n^{th} \ row \\ \uparrow k^{th} \ column & \end{array}$$

Definition 2.15: Let $X=(x_{nk})$ and $Y=(y_{nk})$ be any two infinite matrices and let λ be a scalar. Then,

a. $X+Y=(x_{nk})+(y_{nk})$

b. $\lambda X=(\lambda x_{nk})$.

Definition 2.16: An infinite matrix $A=(a_{nk})$ is called a non-negative infinite matrix if $a_{nk}\geq 0$ for all $n,k\in\mathbb{N}^0$.

Example 2.21: The following matrices are infinite

$$B=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 2 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

The infinite matrix B is non-negative.

$$C=\begin{pmatrix} -1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & -1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Definition 2.17: Let $x=(x_k)$ *and* $y=(y_k)$ be any sequences and C,D be infinite matrices. Then,

a. $xy=(x_k y_k)\quad (scalar\ product)$

b. $Cx := (Cx_k) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \ldots\ldots & c_{1k} & \ldots \\ c_{21} & c_{22} & c_{23} & \ldots\ldots & c_{2k} & \ldots \\ . & . & . & \ldots\ldots & . & \ldots \\ . & . & . & \ldots\ldots & . & \ldots \\ c_{n1} & c_{n2} & c_{n3} & \ldots\ldots & c_{nk} & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum c_{1k} x_k \\ \sum c_{2k} x_k \\ \vdots \\ \sum c_{nk} x_k \\ \vdots \end{pmatrix}$

Example 2.22: Let $d = d_n = (1,2,3,4,5\ldots)$ be sequences, and $A = (a_{nk})$ be an infinite matrices, with,

$$A := (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

then

$$Ad = (a_{nk} \cdot d_n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{3}{2} \\ \frac{1}{3} + \frac{3}{3} + \frac{5}{3} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{pmatrix}.$$

c. $yD = \left(\sum y_n d_{nk}\right)$

Example 2.23: Let $s_n := (0,2,4,6,8...)$ be a sequence and

$E = e_{nk} := \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots \end{pmatrix}$ is infinite matrix, then

$$\left(s_n \cdot e_{nk}\right) = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 2 & 0 & 2 & \cdots \\ 4 & 0 & 4 & 0 & 4 & \cdots \\ 6 & 0 & 6 & 0 & 6 & \cdots \\ 8 & 0 & 8 & 0 & 8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

d. $C.D = \left(b_{nk}\right)$ *where* $b_{nk} = \left(\sum c_{nv} d_{vk}\right) \left(n, k \in \mathbb{N}^0\right)$

Example 2.24: Let

$$C = c_{nk} = \begin{pmatrix} 3 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad and \quad D = d_{nk} = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

then,

$$C \cdot D = c_{nk} \cdot d_{nk} := \begin{pmatrix} 3 & 0 & 3 & 0 & \cdots \\ 0 & 3 & 0 & 3 & \cdots \\ 3 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = b_{nk}$$

Definition 2.18: Let A and B two infinite matrices. Then,

a. If $A \cdot B = \mathrm{I}$ then A is called left inverse of B, and B is called the right inverse of A.

Example 2.25: Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

then

$$A \cdot B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = \mathrm{I}$$

b. If $A \cdot B = B \cdot A = \mathrm{I}$ then B is called the inverse of A and denoted by $B = A^{-1}\left(or\ A = B^{-1}\right)$.

Example 2.26 $A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$

then,

$$A \cdot B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = \mathrm{I}$$

and

$$B \cdot A := \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = \mathrm{I}.$$

Therefore A and B are inverses of each other.

Definition 2.19: The matrix $A = (a_{nk})$ is called a triangular matrix if $a_{nk} = 0$ for $k > n$. A triangular matrix with $a_{nn} \neq 0$ for all n is called a triangle matrix.

Example 2.27: The matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 7 & 0 & 0 & 0 & \cdots \\ 4 & 9 & 6 & 0 & 0 & \cdots \\ 5 & 2 & 8 & 9 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is a triangular matrix.

Example 2.28: The matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & & \\ 2 & 2 & 0 & 0 & \cdots & \cdots & & \\ 3 & 3 & 3 & 0 & \cdots & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ n & n & n & n & \cdots & n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \end{pmatrix}.$$

is a triangle matrix.

Chapter 3

MATRIX METHODS

So far, we discuss some concepts that will be needed to study Matrix Summability theory which is the main subject of this thesis. In the present chapter, we shall discuss the general theory of matrix summability methods. We shall start with matrix transformations, and then, some basic definitions like, convergence and application domains, conservative and regular matrices, Kojima Schur and Silverman Teoblitz theorems will be considered. Later operations on matrices that preserves regularity and conservative property will be studied. In the last part of this chapter we shall focus on multiplicative matrix methods, zero preserving matrix methods and their properties.

Definition 3.1 [5]**:** Let $A=(a_{nk})$ be an infinite matrix and $x=(x_n)$ be a sequence then the A-transform of $x:=(x_k)$ is denoted by $Ax:=((Ax)_n)$ and defined as

$$(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k\ ,$$

if it converges for each n.

A matrix summability method consisting of three parts;

1. An infinite matrix $A=(a_{nk})$
2. Convergence Domain
3. Limit operator.

If $A=(a_{nk})$ is an infinite matrix then, the following set

$$w_A=\left\{x=(x_k)\in w, Ax:=\sum_k a_{nk}x_k \textit{ converges for every } n\geq 0\right\}$$

is called the **application domain** of A. In other words application domain w_A of an infinite matrix $A=(a_{nk})$ is a subset of w, such that for all $x\in w_A$, Ax is defined.

The following subset c_A of w_A

$$c_A:=\{x\in w,\ Ax\in c\}$$

is called the **convergence domain** of A. In other words, $x\in c_A$, if and only if Ax is convergent.

Also, for $A=(a_{nk})$ we can define the following limit operator,

$$A-\lim x=\lim_{k-\infty}(Ax).$$

Therefore, $(A, c_A, \lim A)$ is called a matrix summability method.

Example 3.1: The infinite Identify Matrix $(\mathrm{I}, c, \lim)$ is a matrix summability method where

$$\mathrm{I}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is the Identity Matrix.

Example 3.2: Consider the following matrix

$$Z_{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & & \end{pmatrix}$$

which is called Zweir matrix.

The matrix method $\left(Z_{\frac{1}{2}}, c_{Z_{\frac{1}{2}}}, Z_{\frac{1}{2}} - \lim \right)$ is called the Zweir summability method.

Definition 3.2 *(Conservative Matrix)*[1]**:** An infinite matrix A is called conservative matrix if and only if the convergence of the sequence x implies the convergence of Ax.

In other words, A is conservative if and only if

$$\forall x \in c \quad \Rightarrow \quad Ax \in c \quad \text{(limit of } x \text{ and } Ax \text{ can be different).}$$

The space of all conservative matrices will be denoted by (M_{con}).

Example 3.3: Let,

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

For the sequnece,

$$x_n = \left(\frac{1}{2}\right)^n$$

we have

$$\lim_{n\to\infty} x_n = 0.$$

On the other hand,

$$Ax_n = \begin{pmatrix} 2 \\ 1+\frac{1}{4} \\ \frac{1}{2}+\frac{1}{8} \\ \frac{1}{4}+\frac{1}{16} \\ \vdots \end{pmatrix}$$

$\lim_{n\to\infty} Ax_n = 0$.

Example 3.4: Let,

$$0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

For any sequnece, $x_n = (x_1, x_2, x_3, ...)$, $\lim_{n\to\infty} Ax_n = 0$. Therefore, zero matrix is conservative. Moreover, consider the convergent sequence $x_n = (3,3,3,.....)$ which converges to 3. Ax_n is convergent but the limit of Ax_n is 0. So, for the conservative matrices limits need not be the same.

The following Theorem of Kojima and Schur gives necessary and sufficient conditions for conservative matrices.

Theorem 3.3 *(Kojima and Schur)* [7]**:**

Let $A=\left(a_{nk}\right)$ $n,k=0,1,2,3,...$ is an infinite matrix. Then, $A=\left(a_{nk}\right)$ is conservative matrix if and only if,

1. $\lim\limits_{n\to\infty} a_{nk} = \lambda_k \qquad \forall k = 0,1,2,...$

2. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty} a_{nk} = \lambda$

3. $\sup\left|\sum\limits_{k=1}^{\infty} a_{nk}\right| \le H < \infty$ *for all* $H > 0.$

Example 3.5: Let,

$$Z_{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

1. $\lim\limits_{n\to\infty} Z_{\frac{1}{2}} = 0 \qquad$ for all $k = 0,1,2,...$
2. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty} Z_{\frac{1}{2}} = 1$
3. $\sup\left|\sum\limits_{k=1}^{\infty} Z_{\frac{1}{2}}\right| \to 1 < \infty$

Zweir matrix is a conservative matrix. That is for any

$$x \in c \Rightarrow Z_{\frac{1}{2}} x \in c.$$

Example 3.6: Let

$$D = d_{nk} := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

be an infinite matrix.

By Kojima and Schur theorem;

1. $\lim_{n\to\infty} d_{nk} = 0$ for all $k = 0,1,2,...$

2. $\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{nk} = 1$

3. $\sup \left| \sum_{k=1}^{\infty} d_{nk} \right| \to 1 < \infty$

D is a conservative matrix.

Theorem 3.4 [5]**:** Let n be any positive integer and E be a conservative matrix. Then E^n is also conservative.

Proof: We need to show that $E^n x$ is also convergent. We can prove this theorem by using mathematical induction. Let, E be a conservative matrix. Then,

$n = 1 \quad \to \quad E^n x = Ex$. Here, Ex is a convergent.

$n = k \quad \to$ Assume that E^k is a conservative. Then, $E^n x = E^k x$ is convergent.

That is $E^k x \to L_k$.

$n = k+1 \quad \to \quad E^n x = E^{k+1} x$. Here, Ex is a conservative and E^k is a conservative.

That is $E^{k+1} x = E^k (Ex) \to L_{k+1}$.

Therefore, $E^n x$ is a conservative.

Theorem 3.5 [5]**:** If B and D are two conservative matrix. Then

a. $D+B$ is conservative

b. $DB \text{ and } BD$ are conservative

c. λD where λ any scalar is conservative.

Proof:

a. Taking a convergent sequence $x=(x_k)$ such that $\lim\limits_{x\to\infty} x_k = L.$ Then,

$$D \text{ is conservative} \Rightarrow Dx = L_1$$

$$B \text{ is conservative} \Rightarrow Bx = L_2$$

$(D+B)x = Dx+Bx \to L_1+L_2$. Therefore $D+B$ is conservative.

b. Let $x=(x_k)$ be a convergent sequence with $\lim\limits_{n-\infty} x_n = L$. Then, Dx is convergent sequence. So,

Bx is a convergent. That is $Bx \to L_1$ and D is a conservative matrix.

$$(D.B)(x) = D(Bx) \to L_2$$

- The proof that DB is conservative. Can be obtained in a similar way.

c. For any scalar λ and for any convergent sequence $x=(x_k)$, with $\lim\limits_{n-\infty} x_n = L$

$$(\lambda D)(x) = \lambda(Dx) \to \lambda L.$$

So, λD is conservative.

Remark: If A is a conservative matrix, then

$$nA = A+A+A+....+A$$

is also conservative matrix.

Definition 3.3 (*Regular Matrix*) [1]**:** An infinite matrix $A=\left(a_{nk}\right)$ is called regular matrix if for any convergent sequence, x_n with $x_n \to L$, Ax_n is convergent and $Ax_n \to L$. (The limit values of x_n *and* Ax_n must be the same).

The space of conservative matrix will be denoted by M_{reg}.

Theorem 3.6 (Silverman *and Teoblitz*) [7]**:**

An infinite matrix $A=\left(a_{nk}\right)$ is regular if and only if,

1. $\lim\limits_{n\to\infty}, a_{nk} = 0 \quad \textit{for each } k=0,1,2,...$

2. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty} a_{nk} = 1$

3. $\sup\left|\sum\limits_{k=1}^{\infty} a_{nk}\right| \le H < \infty \quad \textit{for all} \ \ H>0.$

Example 3.7: Consider the following infinite matrix,

$$B=b_{nk} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{n} & 0 & \frac{1}{n} & 0\cdots & \frac{1}{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Since,

1. $\lim\limits_{n\to\infty} b_{nk} = 0 \quad$ for all $k=0,1,2,...$

2. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty} b_{nk} = 1$

3. $\sup\left|\sum\limits_{k=1}^{\infty} b_{nk}\right| \to 1 < \infty$

Therefore, for any $x \in c$ where $x \to L$, $Bx \in c$ and $Bx \to L$.

Example 3.8: Let D be an infinite matrix given by

$$D = d_{nk} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\cdots \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0\cdots \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & 0 & 0\cdots \\ 0 & 0 & 0 & \frac{5}{4} & -\frac{1}{4} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

By Silverman and Teoblitz theorem,

1. $\lim_{n\to\infty} d_{nk} = 0$ for all $k = 0,1,2,...$

2. $\lim_{n\to\infty} \sum_{k=1}^{\infty} d_{nk} = 1$

3. $\sup \left| \sum_{k=1}^{\infty} d_{nk} \right| \to 1 < \infty.$

D is a regular matrix. For any $x \in c$ where $x \to L$, $Dx \in c$ and $Dx \to L$.

Theorem 3.7 [5]**:** If A is any regular matrix, Then A^n is also regular.

Proof: Let $x \in c$ be any convergent sequence with $x \to L$. If $n = 1$,

$$Ax \to L \quad \text{(A is regular)}.$$

Suppose that A^k is regular for $k > 1$. That is $(A^k x) \to L$, then

$$A^{k+1}x = A(A^k x) \to L.$$

So, A^{k+1} is a regular. A^n is also regular.

Theorem 3.8 [5]**:** If D and B are two regular matrices, then

a. $\frac{1}{2}(D + B)$ is regular.

b. $DB \; and \; BD$ are regular.

Proof: Taking any convergent sequence x with $x \to L$, then obviously, $Dx \to L$ and $Bx \to L$. Now,

a. $\frac{1}{2}(D+B)x = \left(\frac{1}{2}Dx + \frac{1}{2}Bx\right) \to L$, since x is arbitrary $\frac{(D+B)}{2}$ is regular.

b. Similarly, $(DB)x = D(Bx)$, but Bx is a sequence that converges to L, and D is regular implies then, $D(Bx) \to L$. Since x is arbitrary DB is regular. Similarly one can show that BD is also regular.

Theorem 3.9 [5]**:** Let $D_1, D_2, ...D_n \; be$ n regular matrices. Then

a. $\frac{1}{n}(D_1 + D_2 + ... + D_n)$ is regular.

b. $D_1 \cdot D_2 \cdots D_n$ is regular.

Proof: Let x be any convergent sequence, with $x \to L$ then,

a. Since $D_1, D_2, ..., D_n$ are regular matrices, $D_i x \to L,\; i = 1, 2, ..., n$. Then,

$$\frac{1}{n}(D_1 + D_2 + ... + D_n)(x) = \frac{D_1(x)}{n} + \frac{D_2(x)}{n} + ... + \frac{D_n(x)}{n}$$

$$= \frac{1}{n}(L + L + L + ... + L) = L \; .$$

Therefore,

$$\frac{1}{n}(D_1 + D_2 + ... + D_n)$$

is a regular.

b. Taking any convergent sequence x with $x \to L$ and let $S = D_1 \cdot D_2 \cdots D_n$. We need to show that $Sx \to L$.

$$Sx = (D_1 \cdot D_2 \cdot D_3 \cdots D_n)(x) = D_1 \cdot D_2 \cdot D_3 \cdots D_{n-1}(D_n x)$$

Here, $D_n x \to L$

Say $D_n x = y \to L$. Then,

$$Sx = D_1 \cdot D_2 \cdots D_{n-1} y .$$

Similarly,

$$D_{n-1} x = y_1 \to L ,$$

and

$$Sx = D_1 \cdot D_2 \cdots D_{n-2} y_1$$

If we continue in this way, we get,

$$Sx \to L$$

which implies that $D_1 \cdot D_2 \cdots D_n$ is a regular matrix.

Remark: As a consequence of the definitions, we have, $M_{reg} \subset M_{con}$.

Definition 3.4 [5]**:** Let A be an infinite matrix. Then, A is a zero preserving matrix if

$$\text{for all } x \in c_0, \; Ax \in c_0.$$

The space of all zero preserving matrices will be denoted by M_{c_0} *or* M_0 .

Theorem 3.10 [5]**:** Let E be a zero preserving matrix then,

1. $\lim\limits_{n\to\infty} e_{nk} = 0 \qquad$ for all $k = 0,1,2,...$
2. $\sup\limits_{k} \left|\sum e_{nk}\right| < H < \infty$.

Example 3.9: The following matrix,

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

satisfies the conditions,

$$\lim_{n\to\infty} d_{nk} = 0 \quad for\ each \ \ k = 0,1,2,...$$

and

$$\sup_k \left|\sum d_{nk}\right| = 2 < \infty .$$

So, D is a zero preserving matrix.

Theorem 3.11 [5]**:** Taking two zero preserving matrices D and K. Then**,**

- **a.** $K + D$ is zero preserving matrix.
- **b.** KD and DK are zero preserving matrix.
- **c.** For any scalar λ, λA is zero preserving matrix.

Proof: Suppose that $s \in c_0$ is arbitrary. Then,

- **a.** $(K + D)s = Ks + Ds \to 0.$
- **b.** $DKs = D(Ks)$, but $K \in M_0$ implies that $Ks \to 0$ and $D \in M_0$ implies that $D(Ks) \to 0$, therefore $DK \in M_0$.

  Similarly, we can show that $KD \in M_0$.
- **c.** $(\lambda D)(s) = \lambda(Ds)$, but $D \in M_0$ implies that $Ds \to 0$ and $\lambda(Ds) \to 0$, which implies that $\lambda D \in M_0$.

Lemma 3.1 [5]**:** Let $A_1, A_2, ..., A_n \in M_0$. Then,.

1. $A_1 + A_2 + ... + A_n \in M_0$.
2. $\lambda_1 A_1 + \lambda_2 A_2 + ... + \lambda_n A_n \in M_0$ where λi is an arbitrary scalars $i = 1, 2, ..., n$.
3. $A_1 . A_2 ... A_n \in M_0$.

Proof: Let $x \in c_0$ be any sequence, then $A_i x \to 0,\ i = 1, 2, ..., n$.

1. Obviously, $\left(A_1 + A_2 + ... + A_n\right)(x) = \left(A_1 x + A_2 x + ... + A_n x\right) \to 0$. Therefore,

$$A_1 + A_2 + ... + A_n \in M_0.$$

2. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be any scalars and $x \in c_0$. Then,

$\left(\lambda_1 A_1 + \lambda_2 A_2 + ... + \lambda_n A_n\right)(x) = \lambda_1 \left(A_1 x\right) + \lambda_2 \left(A_2 x\right) + ... + \lambda_n \left(A_n x\right) \to 0$ and

$$\lambda_1 A_1 + \lambda_2 A_2 + ... + \lambda_n A_n \in M_0.$$

3. Let $x \in c_0$ be any sequence. Then,

$$\left(A_1 . A_2 .... A_n\right)(x) = A_1 . A_2 \cdots A_{n-1} \underset{\in c_0}{\left(A_n x\right)} = A_1 \cdot A_2 \cdots A_{n-2} \underbrace{\left(A_{n-1}\left(A_n x\right)\right)}_{\in c_0}$$

If we continue in this way,

$$\left(A_1 \cdot A_2 \cdots A_n\right)(x) \to 0$$

So,

$$A_1 \cdot A_2 \cdots A_n \in M_0.$$

Definition 3.5 [5]**:** Let E be an infinite matrix, and λ be a scalar then E is called multiplicative matrix with multiplier λ, if

$$\lim\left(Ex\right) = \lambda \lim x \quad \textit{for all } x \in c.$$

The space of all multiplicative matrices with multiplier λ is denoted by M_λ.

Theorem 3.12 [5]**:** Let E be an infinite matrix. Then E is multiplicative with multiplier λ if and only if,

1. $\lim\limits_{n\to\infty}\sum e_{nk} = \lambda$
2. $\lim\limits_{n\to\infty} e_{nk} = 0$
3. $\sup\limits_{k}\left|\sum e_{nk}\right| < \infty$.

Example 3. 10: Let $E = \left(e_{nk}\right)$ be an infinite matrix where

$$E = e_{nk} := \begin{cases} 3 + \frac{1}{n}, & k = n \\ -\frac{1}{n}, & k = n+1 \\ 0, & otherwise \end{cases}$$

then,

$$E = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0\cdots \\ 0 & \frac{7}{2} & -\frac{1}{2} & 0 & 0 & 0\cdots \\ 0 & 0 & \frac{10}{3} & -\frac{1}{3} & 0 & 0\cdots \\ 0 & 0 & 0 & \frac{13}{4} & -\frac{1}{4} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

1. $\lim\limits_{n\to\infty}\sum e_{nk} = 3$
2. $\lim\limits_{n\to\infty} e_{nk} = 0$
3. $\sup\limits_{k}\left|\sum e_{nk}\right| = 3 < \infty$.

So, E is multiplicative matrix with multiplier $\lambda = 3$.

Lemma 3.2 [5]**:** If $A_1, A_2, ..., A_n \in M_\lambda$. Then

1. $\frac{1}{n}(A_1 + A_2 + ... + A_n) \in M_\lambda$.

2. $A_1 \cdot A_2 \in M_{\lambda^2}$.

3. $A_1^n \in M_{\lambda^n}$.

Proof: Taking a convergent sequence x such that $\lim_{x\to\infty} x = L$. Then,

1. $\frac{1}{n}(A_1 + A_2 + ... + A_n)x = \frac{1}{n}(A_1 x + A_2 x + ... A_n x)$

$$\frac{1}{n}(n\lambda L) = \lambda L$$

2. $(A_1 \cdot A_2)(x) = A_1(A_2 x)$ but $A_2 x \to \lambda L$ *and*

$$A_1(A_2 x) \to \lambda(\lambda L) = \lambda^2 L \quad \Rightarrow A_1 \cdot A_2 \in M_{\lambda^2}$$

3. By mathematical induction:

If $n = 1 \quad \Rightarrow \quad A_1 \in M_\lambda$

Suppose that $(A_1)^k \in M_{\lambda^k}$. For any convergent sequence x with limit L, we have,

$$(A_1)_x^{\ k} \to \lambda^k L.$$

Then, we need to show that $(A_1)^{k+1} \in M_{\lambda^{k+1}}$.

Let,

$$(A_1)^{k+1} x = A_1 \underbrace{(A_1^k x)}_{\lambda^k L} \to \lambda(\lambda^k L) \to \lambda^{k+1} L.$$

Thus,

$$A_1^n \in M_{\lambda^n}.$$

Remark: As a consequence of the definitions**,** we have

a. $M_{reg} = M_1$

b. $M_\lambda \subset M_0$.

Lemma 3.3 [7]: If $D \in M_{reg}$ and $K \in M_0$.

a. $DK \in M_0$

b. $KD \in M_0$.

Lemma 3.4 [7]: Let $D \in M_{reg}$ and $K \in M_\lambda$. Then

a. $DK \in M_\lambda$

b. $KD \in M_\lambda$

c. $\lambda D \in M_\lambda$ for all λ

d. $\frac{1}{\lambda} K \in M_{reg}$ *if* $\lambda \neq 0$.

In the following definition we give some new definitions such as conservative matrix for c_0 and coercive methods, also some definitions that are given above are define by using a different way.

Definition 3.6 [5]**:** Let D be an infinite matrix and $(D, c_D, D-\lim)$ be a matrix summability method then,

a. D is called conservative for c_0, if $c_0 \subset c_D$ that is

$$s \in c_0 \rightarrow s \in c_D \text{ , } Ds \in c$$

b. D is conservative if $c \subset c_D$, that is,

$$s \in c \;\to\; Ds \in c$$

c. D is regular if $c \subset c_D$ and $D - \lim s = \lim s \quad \forall s \in c$.

d. D is coercive if, $\ell^{\infty} \subset c_D$.

Example : Consider the zero matrix,

$$B^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

is coercive, for every bounded sequence x. That is $B^0 x \to 0$.

Example 3.12: The Cesaro matrix of order 1,

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n}\cdots\cdots & \frac{1}{n} & & 0\cdots \\ \vdots & \vdots & \vdots\cdots\cdots & \vdots & \vdots & \end{pmatrix}$$

is not coercive.

Consider the sequence $x = \left(0,1,0,0,\underbrace{1,1,1,1}_{4^1-times},\underbrace{0,0,\ldots\ldots,0}_{8\,times},\underbrace{1,\ldots\ldots 1}_{4^2-times},0,\ldots\ldots \right)$. Then,

$$[C_1 x]_{4^n-1} = \frac{1}{4^n}\sum_{k=0}^{n-1} 4^k = \frac{4^n - 1}{3\cdot 4^n} = \frac{4^n\left(1-\frac{1}{4^n}\right)}{3\cdot 4^n} \to \frac{1}{3}$$

$$[C_1 x]_{2\cdot 4^n - 1} = \frac{1}{2\cdot 4^n}\sum_{k=0}^{n-1} 4^k = \frac{4^{n+1}-1}{3\cdot 2\cdot 4^n} \to \frac{2}{3}$$

The sequence x is a bounded. But $x \notin c_{C_1}$. Therefore, C_1 is not coercive.

Let $D = (D, c_D, D - \lim)$ *and* $K = (K, c_K, K - \lim)$ be two matrix summability methods. Furthermore, let $L \subset w$ be any space of sequence, then we can give the following definitions (see [1] and [5]).

Definition 3.7: K is stronger than D, or D is weaker than K, if and only if

$$c_D \subset c_K .$$

On the other hand, K is called stronger than D relative to L if

$$C_D \cap L \subset C_K .$$

Definition 3.8: We say that matrix methods D and K are equivalent. If,

$$c_D = c_K .$$

Moreover, D and K are equivalent relative to L if D and K are equivalent on

$$L \cap C_D \cap C_K .$$

Definition 3.9: Let $D = (D, c_D, D - \lim)$ *and* $K = (K, c_K, K - \lim)$ be two matrix summability methods. Then D and K are called consistent if,

$$D - \lim x = K - \lim x \text{ for all } x \in c_D \cap c_K .$$

Moreover, D and K are called consistent relative to $L \subset w$ if

$$D - \lim x = K - \lim x,\ \forall x \in C_D \cap C_K \cap L$$

Example 3.12: Consider the following Zweir matrix and Cesaro matrix of order 1.

$$Z_{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \cdots\cdots & \frac{1}{n} & 0 \cdots & \\ \vdots & \vdots & \vdots \cdots\cdots & \vdots & \vdots & \end{pmatrix}$$

Then, taking $x = (1, 0, -1, 1, 0, -1, 1, 0, -1, ...)$ then,

$$Z_{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 0 \\ \vdots \end{pmatrix} \qquad x \notin c_{Z_{\frac{1}{2}}}$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \cdots\cdots & \frac{1}{n} & 0 \cdots & \\ \vdots & \vdots & \vdots \cdots\cdots & \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \\ 1/4 \\ 0 \\ 1/6 \\ \vdots \end{pmatrix} \qquad x \in c_{C_1}$$

Since $c_{Z_{\frac{1}{2}}} \subset c_{C_1}$, C_1 is stronger than $Z_{\frac{1}{2}}$. In other words $Z_{\frac{1}{2}}$ is weaker than C_1.

Moreover, the above example shows that, $c_{C_1} \not\subset c_{Z_{\frac{1}{2}}}$ $\left(c_{C_1} \neq c_{Z_{\frac{1}{2}}} \right)$, which means that C_1 and $Z_{\frac{1}{2}}$ are not equivalent (C_1 and $Z_{\frac{1}{2}}$ are consistent).

Example 3.14: Let E and $Z_{\frac{1}{2}}$ be an infinite matrix. Then $x=(-1)^n$ and $y_n=(n)$ are any two sequence.

a. $x=(-1)^n$

$$Ex=\begin{pmatrix} 1 & 0 & 0 & 0\cdots \\ -1 & 1 & 0 & 0\cdots \\ 0 & -1 & 1 & 0\cdots \\ 0 & 0 & -1 & 1\cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 2 \\ -2 \\ \vdots \end{pmatrix} \rightarrow \quad x\notin c_E$$

$$Z_{\frac{1}{2}}x=\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0\cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ \vdots \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \rightarrow x\in c_{z_{\frac{1}{2}}}$$

$$x:=(-1)^n\in\left(c_{Z_{\frac{1}{2}}}\setminus c_A\right).$$

b. $y_n=(n)$

$$Ey=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\cdots \\ -1 & 1 & 0 & 0 & 0\cdots \\ 0 & -1 & 1 & 0 & 0\cdots \\ 0 & 0 & -1 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \qquad y\in c_E.$$

$$Z_{\frac{1}{2}} y = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 3/2 \\ 5/2 \\ \vdots \end{pmatrix} \qquad y \notin c_{Z_{\frac{1}{2}}}.$$

$$y := (n) \in \left(c_E \setminus c_{Z_{\frac{1}{2}}} \right).$$

Thus, we can see that $E \textit{ and } Z_{\frac{1}{2}}$ are not comparable methods.

Theorem 3.13 *(Comparison by using a transition matrix)* [5]**:** Taking three infinite matrix these are $D, K \textit{ and } E$ with $K = ED$ such that $(ED)x \textit{ and } E(Dx)$ are defined $(ED)x = E(Dx)$ holds for all $x \in c_D$.

1. K is stronger than D if E is conservative $(c_D \subset c_K)$.

Proof: Assume that E is conservative. We need to show that $(c_D \subset c_K)$. Let x be any elements of c_D.

$$x \in c_D \to Dx \text{ is a convergent.}$$

$Kx = (EDx) = E(Dx)$. Here, E is conservative and Dx is a convergent.

Therefore,

$$E(Dx) \to \text{convergent and } x \in c_K.$$

$c_D \subset c_K \Rightarrow K$ is stronger than D.

2. K is stronger than and consistent with D if E is regular $(c_D \subset c_K)$ and $(D - \lim = K - \lim)$.

Proof: Suppose that E is regular $\left(c_D \subset c_K\right)$.Then $\forall x \in c_D$.

$$\lim_K x = \lim_E Dx = \lim_D x$$

$$\lim_K = \lim_D .$$

Theorem 3.14 (*Comparison, Consistency*) [5]**:** Let D be a row-finite and E be a triangle. Then, $C := DE^{-1}$. This theorem has the following statements.

a. D is stronger than E if and only if C is conservative .

b. D is stronger than and consistent with E if and only if C is regular.

Proof: We will prove $"\Rightarrow"$ for a and b.

a. Suppose that C is not conservative, $\exists\, z \in c$ such that $z \notin c_C$.

Then for all

$$x = E^{-1}z \quad \Rightarrow \; Ex = z \;\; is \text{ a convergent } \left(x \in c_E\right).$$

$Dx = \left(CE\right)x = C\left(Ex\right) = Cz$ does not convergent $\left(Cz \notin c\right)$.

Thus, D is not stronger than E .

b. Assume that C is conservative but not regular. After that,

$\exists\, z \in c$ such that $\lim_c z = \lim z$.

So that, if $x = E^{-1}z$ we have D is stronger than E

$$x \in c_E \subset c_D \;.$$

$$\lim_D x = \lim_C Ex = \lim_C z \neq \lim z = \lim_E x.$$

Thus,

$$\lim_D x = \lim_E x.$$

D is stronger than and consistent with E if and only if C is regular.

Chapter 4

SOME MATRIX METHODS

This chapter is devoted to some well-known matrix summability methods such as Cesaro Matrix methods, Hölder Methods, Reisz Method and Euler- Knopp Methods. In this chapter, we shall discuss some basic properties of these matrix methods.

4.1 Cesaro Methods

Definition 4.1 [5]**:** Let $\alpha \in \mathbb{R}$ with $\left(-\alpha \notin \mathbb{N}\right)$. The Cesaro matrix $C_\alpha := \left(c_{nk}^{\alpha}\right)$ defined by

$$C_{nk}^{\alpha} := \begin{cases} \dfrac{\dbinom{n-k+\alpha-1}{n-k}}{\dbinom{n+\alpha}{n}} & ; \quad k \le n \\ 0 & ; \quad k > n \end{cases}$$

is called the Cesaro matrix of order α or Cesaro method and it is denoted by C_α .

Example 4.1: The Cesaro matrix of order 0 ($\alpha = 0$) is the identity matrix I.

Example 4.2: The C_1 (Cesaro matrix of order 1)

If we choose $\alpha = 1$ in the above definition of $C_\alpha = (c^{\alpha}{}_{nk})$. We obtain C_1, the Cesaro matrix of order 1, where

$$C^1_{nk} := \begin{cases} \dfrac{\binom{n-k}{n-k}}{\binom{n+1}{n}} & ; k \le n \\ 0 & ; k > n \end{cases} = \begin{cases} \dfrac{1}{n} & ; k \le n \\ 0 & ; k > n \end{cases}$$

and

$$C_1 = (c_{nk}) = \frac{1}{n} \quad k \le n.$$

Let $x_n = (x_1, x_2, x_3, ...)$ be any sequence. Then,

$$C_1 x_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \cdots & \frac{1}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots \cdots & \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ \vdots \\ x_n \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{x_1 + x_2}{2} \\ \frac{x_1 + x_2 + x_3}{3} \\ \vdots \\ \frac{x_1 + x_2 + x_3 \cdots x_n}{n} \\ \vdots \end{pmatrix} = \left(\frac{1}{n} \sum_{k=0}^{n} x_k \right)$$

The matrix method $(C_1, c_{C_1}, C_1 - \lim)$ is called the Cesaro Summability method of order 1.

Example 4.3: The second order Cesaro matrix, C_2 is defined by

$$C^2_{nk} := \begin{cases} \dfrac{\binom{n-k+1}{n-k}}{\binom{n+2}{n}} & ; k \le n \\ 0 & ; k > n \end{cases}$$

or

$$C_{nk}^{2} := \begin{cases} \dfrac{2(n-k+1)}{(n+1)(n+2)} & ;k \le n \\ \\ 0 & ;k > n \end{cases}$$

the matrix form of C_2 is the following matrix

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\cdots \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0\cdots \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0\cdots \\ \frac{4}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and the other Cesaro matrices can be defined accordingly when needed.

Theorem 4.1 [5]**:** Let $n \in \mathbb{N}^0$ and $\alpha \ge 0$ then C_α *is* regular.

Proof: It is enough to show that, C_α satisfies the conditions of Silverman – Teoblitz Theorem. Recall that,

$$\sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} = \sum_{k=0}^{n} \binom{k+\alpha-1}{k} = \binom{n+\alpha}{n}.$$

Therefore,

$$\sum_{k=0}^{n} c^{\alpha}{}_{nk} = 1.$$

On the other hand,

$$\frac{\dbinom{n-k+\alpha-1}{n-k}}{\dbinom{n+\alpha}{n}} = \frac{\dfrac{(n-k+\alpha-1)(\mathrm{n}-\mathrm{k}+\alpha-2)\cdots(\alpha)}{(n-k)!}}{\dfrac{(\mathrm{n}+\alpha)(\mathrm{n}+\alpha-1)\cdots(\alpha+1)}{n!}}$$

$$= \frac{n}{n+\alpha-1}\frac{n-1}{n+\alpha-2}\cdots\frac{n-k+1}{n+\alpha-k}\frac{\alpha}{n+\alpha} \to 0.$$

Finally, we need to prove that

$\sup\left|\sum_{k=1}^{\infty} c^{\alpha}{}_{nk}\right| = M < \infty$.

If $\alpha \geq 0$ then, $c^{\alpha}{}_{nk} \geq 0$ and

$$\sup\left|\sum_{k=1}^{\infty} c^{\alpha}{}_{nk}\right| = M < \infty,$$

follows from

$$\sum_{k=0}^{n} c^{\alpha}{}_{nk} = 1.$$

Remark : If $\alpha < 0$, C_{α} is not conservative and regular.

Theorem 4.2 [5]**:** Let $-1 < \alpha < \beta$. Then C_{β} is stronger than and consistent with C_{α}.

That is

$$c_{C_{\alpha}} \subset c_{C_{\beta}} \text{ and } \lim_{C_{\alpha}} x = \lim_{C_{\beta}} x.$$

Proof : Let α and β be two real numbers satisfying $-1 < \alpha \leq \beta$ and $x = (x_k)$ and $y = (y_v)$ and $z = (z_l)$ be sequences, such that

$$y = (y_v) := \mathrm{C}_{\alpha}\, x, \qquad \text{and} \qquad z = (z_l) := \mathrm{C}_{\beta}\, x.$$

In this case, the transition matrix $A = (\mathrm{a}_{lv})$ which is defined by;

$$a_{l\nu} := \begin{cases} \dfrac{\binom{l-\nu+\beta-\alpha-1}{l-\nu}}{\binom{l+\beta}{l}} & if\ \ \nu \le l \\ \\ 0 & if\ \nu > l, \end{cases}$$

for $l, \nu \in \mathbb{N}^0$, satisfies the condition,

$$C_\beta = AC_\alpha .$$

Now, we shall prove that the transition matrix A is regular. It is obvious that,

$$\begin{aligned} \sum_\nu |a_{l\nu}| = \sum_{\nu=0}^{l} a_{l\nu} &= \binom{l+\beta}{l}^{-1} \sum_{\nu=0}^{l} \binom{l-\nu+\beta-\alpha-1}{l-\nu}\binom{\nu+\alpha}{\nu} \\ &= \binom{l+\beta}{l}^{-1} \binom{l-0+\beta-\alpha+(\alpha+1)-1}{l-0} \\ &= 1 \end{aligned}$$

This means that the first and the third conditions of the Silverman Teoblitz Theorem are satisfied. Now we need to prove limit condition of the Silverman Teoblitz Theorem.

As a first step we shall consider the case $\nu = 0$ .Taking, $\tau = \beta - \alpha - 1$, we obtain that

$$\begin{aligned} \frac{\binom{l+\tau}{l}}{\binom{l+\beta}{l}} &= \frac{(l+\tau)(l+\tau-1)\cdots(\tau+1)}{(l+\beta)(l+\beta-1)\cdots(\beta+1)} \\ &= \left(1-\frac{\beta-\tau}{l+\beta}\right)\left(1-\frac{\beta-\tau}{l+\beta-1}\right)\cdots\left(1-\frac{\beta-\tau}{\beta+1}\right). \end{aligned}$$

If we use the fact that, $1+r \le e^r \ \left(r \in \mathbb{R}\right)$,

$$0 \le a_{l0} \le e^{-\frac{\beta-\tau}{l+\beta}} e^{-\frac{\beta-\tau}{l+\beta-1}} \cdots e^{-\frac{\beta-\tau}{\beta+1}}$$

$$= \exp\left(-(\beta-\tau)\sum_{k=1}^{l} \frac{1}{k+\beta} \right) \to 0, \ (l \to \infty),$$

since $\tau = \beta - \alpha - 1$ and $\sum_{\nu} \frac{1}{\nu+1} \to \infty.$ Therefore, $(a_{l0})_l \to 0.$ On the other hand,

for the case, $\nu \ge 1,$ and $l \ge \nu$ we have,

$$\frac{a_{l,\nu}}{a_{l,\nu-1}} = \frac{\binom{l-\nu+\tau}{l-\nu}}{\binom{l-\nu+1+\tau}{l-\nu+1}} \frac{\binom{\nu+\alpha}{\nu}}{\binom{\nu-1+\alpha}{\nu-1}} = \frac{(l-\nu+1)}{(l-\nu+1+\tau)} \frac{(\nu+\alpha)}{\nu}$$

$$= \frac{\nu+\alpha}{\nu} \quad (l \to \infty),$$

which implies that,

$$\lim_{l} a_{l0} = \lim_{l} a_{l1} = \cdots = \lim_{l} a_{l,\nu-1} = \lim_{l} a_{l,\nu} = 0.$$

Since A is regular then C_β is stronger than and consistent with C_α .

Definition 4.2 *(Type M)* [5]**:** A matrix A with bounded columns is called of Type M if

$$tA = 0 \quad \text{implies that} \quad t = 0$$

for all, absolutely summable sequences $t,$ ($t \in l$, that is $\sum_{k=0}^{\infty} |t_k| < \infty$).

Theorem 4.3 [5]**:** A regular triangle, $A = (a_{nk})$ is of type M if A^{-1} has bounded columns.

Proof: Assume that $t \in l$ and $tA = 0$ then, since A^{-1} has bounded columns we have,

$t = tAA^{-1} = (tA)A^{-1} = 0.$

Theorem 4.4 [5]**:** For any $\alpha \geq 0$ the matrix C_α is of M type .

Proof: For any $\alpha \geq 0$ the matrix C_α is regular triangle. Its inverse satisfies the limit condition of the regular matrices so it has bounded columns. So it is of Type M .

Theorem 4.5 *(Knopp)* [5]**:** For each $\alpha \in \mathbb{N}$, the matrix methods C_α and $C_1 \cdot C_{\alpha-1}$ are equivalent and consistent.

4.2 Hölder Methods

The Hölder matrices are derived by the Cesaro matrix of order 1 by iteration. Therefore, many properties of Hölder methods can be obtained from the Cesaro matrix of order 1. This is the most important advantages for the Hölder methods. On the other hand, since the product of two Cesaro matrix is not a Cesaro matrix, this will cause to handle Hölder matrices in an easy way.

Definition 4.3 [5]**:** Let C_1 be the Cesaro matrix of order 1 where $\alpha \in \mathbb{N}^0 \left(\alpha \geq 1\right)$. The Hölder matrix (or Hölder method) of order α is denoted by H^α and defined as

$$H^\alpha := \left(C_1\right)^\alpha \quad .$$

Lemma 4.1 [5]**:** The Hölder method has the following properties:

a. $H^0 = \mathrm{I}$

b. $H = H^1 = C_1$

c. $H^\alpha = H.H^{\alpha-1} = C_1 H^{\alpha-1} \quad \left(\alpha \geq 1\right)$

d. $H^{\alpha+\beta} = H^\alpha . H^\beta$

e. H^α is well defined and a triangle as a product of triangles.

Example 4.4: Let's find H *and* H^2. Obviously, $H = C_1$. The second order Hölder method is $H^2 = C_1 H = C_1 C_1$. So,

$$H^2 = C_1 \cdot C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$H^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{11}{18} & \frac{5}{18} & \frac{2}{18} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Example 4.5: $H^3 = H \cdot H^2 = C_1 \cdot H^2$. Then,

$$C_1 \cdot H^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{11}{18} & \frac{5}{18} & \frac{2}{18} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$H^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{7}{8} & \frac{1}{8} & 0 & 0 & \cdots \\ \frac{85}{108} & \frac{19}{108} & \frac{4}{108} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Theorem 4.6 [5]**:** H^{α} is regular.

Proof: From the Definition 4.3, $H^{\alpha} := (C_1)^{\alpha}$ **,** if we take $\alpha = 1$ we have, $H = H^1 = C_1$ and it is regular. As a product of regular matrices H^{α} is regular.

Theorem 4.7 [5]**:** For each $\alpha \in \mathbb{N}$, the method $H^{\alpha+1}$ is strictly stronger than and consistent with H^{α}. That is to say

$$c_{H^{\alpha}} \subseteq c_{H^{\alpha+1}} \text{ and } \lim_{H^{\alpha}} x = \lim_{H^{\alpha+1}} x \quad \left(\forall x \in c_{H^{\alpha}}\right).$$

Proof: By *comparison, consistency* theorem,

"If $E\ and\ D$ are row finite and $C = E^{-1}D$. Then C is regular if and only if D is stronger than and consistent E ".

$$H^{\alpha+1} = HH^{\alpha} = C_1 H^{\alpha}.$$

So, since C_1 is regular, $H^{\alpha+1}$ is stronger than and consistent with H^{α} **.**

Moreover, we know that

$$c \subset c_{C_1} \text{ but } c \neq c_{C_1}.$$

Therefore this implies that $H^{\alpha+1}$ is strictly stronger than H^{α} **.**

Theorem 4.8 *(M type)* [5]**:** If $\alpha \in \mathbb{N}^0$ then the matrix H^{α} is of type M .

Proof: Let $\alpha \in \mathbb{N}^0$. It is clear that $\left(H^{\alpha}\right)^{-1} = \left(C_1^{-1}\right)^{\alpha}$ and it is column finite triangle.

Therefore H^{α} is of type M .

4.3 Riesz Methods (Weighted Means)

The Riesz method or weighted mean represents a class of regular matrices. It is known as the generalization of C_1, the Cesaro matrix of order 1. The most important advantage of this method is to be a simple method to define a regular matrix and its inverses by using a sequence of numbers with some conditions. That is if you have a sequence (p_n) then you can create a regular matrix from this sequence.

Definition 4.4 [5]**:** Let $p=(p_k)$ be a sequence with $p_0>0$ *and* $p_k \geq 0$ *for* $k \in \mathbb{N}$.

Then define P_n by ,

$$P_n := \sum_{k=0}^{n} p_k = p_0 + p_1 + ...p_n.$$

Definition 4.5 [1]**:** Let $p=(p_k)$ be a sequence with $p_k \geq 0, \quad k \in \mathbb{N}$ and $P_n := \sum_{k=0}^{n} p_k$.

Then the Riesz matrix or method corresponding to $p=(p_k)$ is denoted by

(R_p) *or* R_p and defined by,

$$R_p := (R, p) := (R, p_n) := (r_{nk})$$

with

$$r_{nk} := \begin{cases} \dfrac{p_k}{P_n} & if \quad k \leq n \\ 0 & otherwise \end{cases} \qquad (k, n \in \mathbb{N}^0).$$

Example 4.6: Let $p=\left(p_k\right)=e=\left(1,1,1,....1,.....\right)$. Then $P_n=n+1$.

Then,

$$\left(R,e\right)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots\cdots & \frac{1}{n+1} & 0\cdots \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots \end{pmatrix}=C_1$$

which is the Cesaro matrix of order 1. In other words the Cesaro matrix of order one is a Reisz Method (or Weighted mean) generated by the sequence $p=\left(p_k\right)=\left(1,1,1,....1,....\right)$.

Definition 4.6 [1]**:** Let $p_n>0,\quad \forall n\in\mathbb{N}^0$, the inverse $R_p^{-1}:=\left(r_{nk}\right) of\ R_p$ is given by

$$\hat{r}_{nk}:=\begin{cases} n, & k=n \\ -(n-1), & k=n-1 \\ 0, & otherwise \end{cases}\qquad \left(n,k\in\mathbb{N}^0\right).$$

Example 4.7: The inverse of C_1 is the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 0 & 0 & 0 & \cdots \\ 0 & -2 & 3 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -\left(n-1\right) & n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & & \end{pmatrix}$$

Theorem 4.8 [5]**:** Let R_p be a Riesz method associated with p. Then, R_p is conservative.

Proof: Let R_p be a Riesz method with

$$r_{nk} := \begin{cases} \dfrac{p_k}{P_n}, & k \le n \\ 0, & otherwise \end{cases}$$

To show that this is a conservative matrix it is enough to show that it satisfies the conditions of Kojima-Schur Theorem.

a. $\sum_{k=0}^{\infty} r_{nk} = \sum_{k=0}^{n} r_{nk} = \sum_{k=0}^{n} \frac{p_k}{P_n} = \frac{1}{P_n}\sum_{k=0}^{n} p_k = 1$,therefore row sum condition is satisfied.

b. $\sup \sum_k |r_{nk}| = \sup \sum_k r_{nk} \le 1$

c. For fixed k, r_{nk} is decreasing and bounded sequence, so it is convergent.

Therefore as a consequence of (a), (b) and (c), the Riesz Method is conservative.

Theorem 4.9 [5]**:** Let R_p be a Riesz method associated with p. Then, R_p is regular if and only if $p_n \to \infty$.

Proof: To show that this is a regular matrix it is enough to show that it satisfies the conditions of Silverman-Teoblitz Theorem Let R_p be a Riesz method, then

a. $\sum_{k=0}^{\infty} r_{nk} = \sum_{k=0}^{n} r_{nk} = \sum_{k=0}^{n} \frac{p_k}{P_n} = \frac{1}{P_n}\sum_{k=0}^{n} p_k = 1$ therefore row sum condition is satisfied.

b. $\sup \sum_k |r_{nk}| = \sup \sum_k r_{nk} \le 1 \le \infty$,

c. If $P_n \to \infty \; as \; n \to \infty \;\; r_{nk} = \frac{p_k}{P_n} \to 0 \;\; n \to \infty$. This means that

$\lim_{n\to\infty} r_{nk} = 0 \quad \forall k \in \mathbb{N}.\cdot$

So, as a consequence of Silverman-Teoblitz Theorem Riesz method is regular.

Theorem 4.10 *(M type)* [1]**:** If $p_n > 0 \ \forall n \in \mathbb{N}^0$, the matrix R_p is of type M .

Theorem 4.11 (*Comparison*) [1]: Let R_p be a regular Riesz method generated by (p_k) with $p_k > 0$ and let $B = (b_{nk})$ be any conservative matrix. B is stronger than R_p if and only if;

1. $\lim_{k\to\infty} \dfrac{b_{nk}}{p_k} = 0$

and

2. $\sup_n \sum_k P_k \left| \dfrac{b_{nk}}{p_k} - \dfrac{b_{n,k+1}}{p_{k+1}} \right| < 0.$

4.4 Hausdorff Methods

The class of Hausdorff Methods is a class of regular matrices that includes the Hölder and Cesaro matrix methods. Basically, a Hausdorff method is based on differences of a sequence or more generally on difference matrix. The representation of the Hölder matrices as a Hausdorff matrix enable us to extend the definition of Hölder matrices for $\alpha \in \mathbb{N}^0$ to $\alpha \in \mathbb{C}$ (the set of) complex numbers.

Definition 4.7 (*Difference Operator*) [5]& [7]**:** Let $x = (x_k)$ be a sequence $k \in \mathbb{N}^0$.

Define the following operator;

$$\Delta_{x_k}^0 := x_k \qquad \text{and} \qquad \Delta_{x_k}^n := \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1} \quad (n \geq 1).$$

Is called the difference operator. By induction,

$$\Delta_{x_k}^n := \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v} .$$

and if take $k = 0$ we get,

$$\Delta_{x_0}^{n} := \sum_{v=0}^{n} (-1)^{v} \binom{n}{v} x_v \,.$$

The above equation defines the following matrix which is called difference matrix.

Definition 4.8 (*Difference matrix*): The following matrix Δ,

$$\Delta = \left(\Delta_{nv}\right) := \begin{cases} (-1)^{v} \binom{n}{v} & if \quad 0 \le v \le n \\ 0 & if \quad v > n \end{cases}$$

Or equivalently,

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is called the difference matrix.

Remark ***(****Inverse of* Δ ***):*** Since $\Delta\Delta = I$, then the inverse of the difference matrix is itself, that is $\Delta^{-1} = \Delta$.

Definition 4.9 (Hausdorff *matrix*) [5]**:** Let $p = \left(p_n\right)$ be a sequence where $n \in \mathbb{N}^0$.

The Hausdorff matrix generated by the sequence $p = \left(p_n\right)$ is denoted by H_p and defined by

$$H_p := \left(H, p\right) := \left(H, p_n\right) := \Delta diag\left(p_n\right)\Delta \,\textbf{.}$$

Here, $diag\left(p_n\right)$ is the diagonal matrix with diagonal elements p_n.

Example 4.8: Let $H_{\left(\frac{1}{n+1}\right)}{}^{\alpha} := \left(H : \left(\frac{1}{n+1}\right)^{\alpha}\right) = C_1{}^{\alpha} = H^{\alpha} = C_1$

$$\Delta \cdot diag\left(\frac{1}{n+1}\right) \cdot \Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = C_1$$

hence, C_1 is a Hausdorff method.

Example 4.9: Let $p_n := \left(2^n\right)$. Then $H_{2^n} := \left(H : \left(2^n\right)\right) = \Delta diag\left(2^n\right)\Delta$

$$\Delta \cdot diag\left(2^n\right) \cdot \Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 0 & \cdots \\ 0 & 0 & 0 & 8 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{2^n} := \left(H : \left(2^n\right)\right) := \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 2 & 0 & 0 & \cdots \\ 1 & -4 & 4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Lemma 4.2 [5]**:** Let H_p *and* H_q be two Hausdorff matrices generated by the sequences $p=(p_n)$ *and* $q=(q_n)$ respectively. Then,

1. $H_p := (h_{nk}) := \begin{cases} \binom{n}{k} \Delta^{n-k} p_k & if\ 0 \le k \le n \\ 0 & if\ k > n \end{cases} \quad (k, n \in \mathbb{N}^0).$

2. $H_{p+q} = H_p + H_q$

3. The matrices H_p is a triangle, if and only if $p_n \neq 0 \quad \forall (n \in \mathbb{N}^0)$.

4. $H_p H_q = H_{pq}$

5. Inverse: $(H_p)^{-1}$ exists if and only if $p_n \neq 0\ \forall n \in \mathbb{N}.$ Furthermore,

$$(H_p)^{-1} = (H, p_n^{-1}) \ .$$

Proof:

1. Let $\Delta diag(p_n) = (\Delta_{nv} p_v)$, we get for all $k \le n$ the equalities

$$h_{nk} = \sum_v \Delta_{nv} p_v \Delta_{vk} = \sum_{v=k}^{n} (-1)^v \binom{n}{v} p_v (-1)^k \binom{v}{k}$$

$$= \sum_{v=k}^{n} (-1)^{v+k} \ p_v \binom{n}{k}\binom{n-k}{v-k}$$

$$= \binom{n}{k} \sum_{v=0}^{n-k} (-1)^v \ p_{v+k} \binom{n-k}{v} = \binom{n}{k} \Delta^{n-k} p_k.$$

2. It comes from Lemma 4.2 (1). Since Δ^{n-k} is linear for $n \ge k$

$$H_{p+q} = \left(\Delta diag(p_n + q_n)\Delta\right)$$

$$= \Delta diag(p_n + q_n)\Delta = (H, p_n) + (H, q_n).$$

$$= H_p + H_q.$$

3. Since p_n is the coefficient of H_p in the n^{th} position of its diagonal $\Leftrightarrow p_n \neq 0$.

4. Since Hausdorff matrices are row finite. We get,

$$H_p H_q = \left(\Delta diag\left(p_n\right)\Delta\right)\left(\Delta diag\left(q_n\right)\Delta\right)$$

$$= \Delta diag\left(p_n\right) diag\left(q_n\right)\Delta$$

$$= \Delta diag\left(p_n q_n\right)\Delta = \left(H, p_n q_n\right).$$

5. Let H_p is a triangle necessary and sufficient condition $p_n \neq 0 \ \ \forall n \in \mathbb{N}$.

In this way $q = \left(q_n\right)$ with $q = p_n^{-1} \quad \forall n \in \mathbb{N}^0$. From it comes Lemma 4.2 (4),

$$H_p H_q = \left(H, p_n q_n\right) = \left(H, e\right) = \mathrm{I}$$

Therefore H_q is the inverse of H_p.

Theorem 4.12 [5]: If H_p and H_q are Hausdorff matrices and H_p is triangle. Then the following statements hold:

- **a.** H_q is stronger than H_p if and only if $\left(H, \dfrac{q_n}{p_n}\right)$ is conservative.
- **b.** H_q is stronger than and consistent with H_p if and only if $\left(H, \dfrac{q_n}{p_n}\right)$ is regular.

Theorem 4.13 (*Characterization*) [5]: Let $p = (p_n)$ be a sequence with $p_n \neq p_k \left(n \neq k\right)$. And let $B = \left(b_{nk}\right)$ be lower triangular matrix. Then B is a Hausdorff matrix if and only if

$$BH_P = H_P B \quad \left(B = H_p B H_p^{-1}\right).$$

Proof: Let $p=(p_n)$ be a sequence and $B=(b_{nk})$ be a lower triangular matrix. Assume that B is a Hausdorff matrix, then $B=H_q$. In this case

$BH_P=H_qH_P=H_{qp}=(\mathrm{H},\mathrm{q}_n\mathrm{p}_n)=(\mathrm{H},\mathrm{p}_n\mathrm{q}_n)=H_PH_q=H_PB.$

Now, suppose that $BH_P=H_PB.$ Multiplying both sides of $BH_P=H_PB$ by Δ, we get,

$$\Delta BH_P\Delta=\Delta H_PB\Delta\ ,$$

or

$$\Delta B\Delta diag(p_n)=diag(p_n)\Delta B\Delta\ .$$

Now if we put, $C=(c_{nk})=\Delta B\Delta$, we get $c_{nk}\,p_k=p_nc_{nk}$, which means that, $c_{nk}=0$ for $k\neq n$ since $p_n\neq p_k$. But this means that C is a diagonal matrix $C=diag(c_{nn})=\Delta B\Delta$ and this proves that $B=\Delta diag(b_{nn})\Delta.$

Theorem 4.14 (*Consistency*) [5]**:**Regular Hausdorff matrices are pairwise consistent.

Proof: Let $s\in c_{H_p}\cap c_{H_q}$. Then,

$$\lim_{H_p} s=\lim H_ps=\lim_{H_q} H_ps=\lim\left(H_qH_p\right)s$$

$$\lim\left(H_pH_q\right)s=\lim_{H_p} H_qs=\lim H_qs=\lim_{H_q} s.$$

From Lemma 4.2 (4), we have,

$$H_ps\in c\ \ and\ \ H_qs\in c.$$

Lemma 4.3 [5]**:** H_{α^n} and $\mathrm{H}_{\left(\frac{1}{\alpha}\right)^n}$ are inverse of each other.

Proof: By Lemma 4.2 (4) $H_pH_q=H_{pq}$

$$H_{\alpha^n}\cdot H_{\left(\frac{1}{\alpha}\right)^n} = H_{\left(\frac{1}{\alpha}\right)^n}\cdot H_{\alpha^n} = H_{\left(\alpha\frac{1}{\alpha}\right)^n} = H_{(1)^n} = H_1 = \mathrm{I}.$$

Example 4.10: H_{2^n} *and* $H_{\left(\frac{1}{2}\right)^n}$ are inverse of each other.

$$H_{2^n} := \left(H:(2)^n\right)$$

$$\Delta\cdot diag\left(2^n\right)\cdot\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 4 & 0 & \cdots \\ 0 & 0 & 0 & 8 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{2^n} := \left(H:\left(2^n\right)\right) := \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -1 & 2 & 0 & 0 & \cdots \\ 1 & -4 & 4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{\frac{1}{2}^n} := \left(H:\left(\frac{1}{2}\right)^n\right)$$

$$\Delta\cdot diag\left(\left(\frac{1}{2}\right)^n\right)\cdot\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{4} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{8} & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{\frac{1}{2}^n} := \left(H:\left(\frac{1}{2}\right)^n\right) := \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$H_{2^n}\cdot H_{\left(\frac{1}{2}\right)^n}=\begin{pmatrix}1&0&0&0&\dots\\-1&2&0&0&\cdots\\1&-4&4&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0&\dots\\\frac{1}{2}&\frac{1}{2}&0&0&\cdots\\\frac{1}{4}&\frac{1}{2}&\frac{1}{4}&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0&\dots\\0&1&0&0&\cdots\\0&0&1&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}=\mathrm{I}.$$

Theorem 4.15 [5]**:** Let $H=\left(H:p_n\right)$ be any Hausdorff matrix generated by $p=(p_n)$. If $\left(H:=\left(h_{nk}\right)\right)$, then

$$\left(\sum_{k=0}^{\infty}h_{nk}\right)_{n\in\mathbb{N}^0}=p_0.$$

Proof: Let

$$\left(\Delta diag\left(p_n\right)\Delta\right)e=\Delta diag\left(p_n\right)\Delta e$$

$$\Delta e=\begin{pmatrix}1&0&0&0&\dots\\1&-1&0&0&\cdots\\1&-2&1&0&\cdots\\1&-3&3&1&0\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\\\vdots\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\\\vdots\end{pmatrix}=e_0$$

$$p_ne_0=\begin{pmatrix}p_0&0&0&0&\cdots\\0&p_1&0&0&\cdots\\0&0&p_2&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\cdots\\0&0&0&p_n&\cdots\\\vdots&\vdots&\vdots&\vdots&\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}p_0\\0\\0\\0\\0\\\vdots\end{pmatrix}=p_0e_0$$

$$\Delta e_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = e.$$

4.5 Euler- Knopp Methods

The Euler-Knopp methods are special case of Hausdorff methods. Let $\alpha \in \mathbb{C}$ be any complex number then the Hausdorff matrix, $H := (H : \alpha^n)$ generated by the sequence (α^n) is called an Euler matrix of order α .

Definition 4.10 [7]**:** *If* $E = \left(e_{nk}^{(\alpha)}\right)$ is an Euler matrix generated by α . Then,

$$e_{nk}^{(\alpha)} = \begin{cases} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} & , 0 \le k \le n \\ 0 & , otherwise \end{cases}$$

more specifically, the Euler matrix E_α is:

$$E_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (1-\alpha) & \alpha & 0 & 0 & 0 & 0 \\ (1-\alpha)^2 & 2\alpha(1-\alpha) & \alpha^2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\alpha)^n & n\alpha(1-\alpha)^{n-1} & \frac{n(n-1)}{2}\alpha^2(1-\alpha)^{n-2} & \cdots & \alpha^n & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \end{bmatrix}.$$

Lemma 4.4 ***(Properties of Euler Matrix)*** [5]: Let $\alpha, \beta \in \mathbb{C}$. The Euler method has the following properties:

1. $E_\alpha E_\beta = E_{\alpha\beta}$.
2. $E_\alpha^{-1} = E_{\alpha^{-1}} \qquad \alpha \neq 0.$

Proof: 1. Let, $a = (a_j)$ be any sequence and let

$$e_{nk}^{\alpha} := \binom{k}{j} \alpha^j (1-\alpha)^{k-j} \text{ and } e_{sk}^{\beta} := \binom{j}{t} \beta^t (1-\beta)^{j-t}.$$

$$E_\alpha \cdot E_\beta = \sum_{j=0}^{k} \binom{k}{j} \alpha^j (1-\alpha)^{k-j} \sum_{t=0}^{j} \binom{j}{t} \beta^t (1-\beta)^{j-t} a_j$$

$$= \sum_{j=0}^{k} \binom{k}{t} (\alpha\beta)^t \sum_{j=t}^{k} \binom{k-t}{j-t} \beta^t (\alpha - \alpha\beta)^{j-t} (1-\alpha)^{k-j} a_j$$

$$= \sum_{t=0}^{k} \binom{k}{t} (\alpha\beta)^t \sum_{j=0}^{k-t} \binom{k-t}{j} (\alpha - \alpha\beta)^j (1-\alpha)^{k-t-j} a_j$$

$$= \sum_{t=0}^{k} \binom{k}{t} (\alpha\beta)^t (1-\alpha\beta)^{k-t} a_j = E_{\alpha\beta}.$$

Lemma 4.5 [5]**:** The inverse of E_α is $E_{\frac{1}{\alpha}}$.

Proof: By Lemma 4.4 (1) $E_{\alpha\beta} = E_\alpha E_\beta$.

$$E_{\frac{1}{\alpha}} \cdot E_\alpha \cdot = E_\alpha \cdot E_{\frac{1}{\alpha}} = E_{\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha}} = E_1 = \mathrm{I}$$

Theorem 4.16 [5] **:** The Euler Method E_α generated by $\alpha \in \mathbb{C}$ is a conservative method if $0 \le \alpha \le 1$.

Proof: We need to prove that the matrix E_α satisfies the conditions of Kojima-Schur Theorem. By the definition,

$$\sum_{k=0}^{n} e_{nk}^{\alpha} = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} = (\alpha + (1-\alpha))^n = 1.$$

On the other hand,

$$Sup \sum \left| e_{nk}^{\alpha} \right| = \sup \sum \left| \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \right| = 1 < \infty .$$

Finally, for $0 \le \alpha \le 1$ each column of a Euler matrix is convergent therefore E_α is conservative.

Theorem 4.17 [5]**:** E_α is regular if $0 < \alpha \le 1$.

Proof: We need to prove that the matrix E_α satisfies the conditions of Silverman-Teoblitz Theorem. By the definition,

$$\sum_{k=0}^{n} e_{nk}^{\alpha} = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} = (\alpha + (1-\alpha))^n = 1.$$

On the other hand,

$$Sup \sum \left| e_{nk}^{\alpha} \right| = \sup \sum \left| \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \right| = 1 < \infty .$$

Finally, for $0 < \alpha \le 1$ each column of a Euler matrix is convergent to zero therefore E_α is conservative.

Theorem 4.18 [5] **:**If $0 < \beta \le \alpha$. Then, E_β is stronger than and consistent with E_α .

Proof: By Theorem 4.17, E_α is regular if and only if $0 < \alpha \le 1$. Let $0 < \beta \le \alpha$. We need to prove that E_β is stronger than and consistent with E_α .

By lemma 4.4 (1) $E_{\alpha\beta} = E_\alpha E_\beta$. Then $E_\beta = E_{\frac{\beta}{\alpha}} E_\alpha$. Here $E_{\frac{\beta}{\alpha}}$ is regular. If we apply Theorem 4.12 (b), completes the proof.

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