# Computational Numerical Solution Algorithm for Fractional Differential Equations

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# ABSTRACT

This study focused on three main problems, firstly, a study on the existence of the solution for a coupled system of fractional differential equations with integral boundary conditions. The solution process for the existence and uniqueness of solutions for the proposed problem was obtained by using the contraction mapping principle, and then by using Leray–Schauder's alternative method.

Secondly, investigation and approximation of solutions of Caputo type fractional differential equation with nonlinear boundary conditions has been solved by using an appropriate parameterization technique, where nonlinear boundary conditions were transformed to linear boundary conditions by using vector parameters. To solve the transformed problem, a numerical-analytic scheme was constructed to find the relation between different type's two-point and multipoint linear boundary condition and nonlinear boundary conditions.

Finally, efficient numerical - analytical computational algorithm for solving systems of fractional differential equations (SFDEs) Nonlinear Point Boundary-Value Problem with Nonlinear Boundary Conditions were considered. The fractional derivative was described in the Caputo sense.

The method is based on numerical approximations of systems of fractional differential equations, where the properties of this method were utilized to reduce SFDEs to the system of algebraic equations. Special attention is given to study the convergence and estimate the error of the presented method. The methods introduce a promising tool for solving many systems of non-linear fractional differential equations. Numerical examples were presented to illustrate the validity and the great potential of both proposed techniques.

iv

**Keywords:** fractional differential equation, sequential, Caputo, nonlocal integral boundary conditions

# ÖZ

Bu çalışma, üç temel probleme odaklanmış, ilk olarak, integral sınır koşulları olan birleştirilmiş kesirli diferansiyel denklem sistemi için çözümün varlığına ilişkin bir çalışma. Önerilen problem için çözümlerin varlığı ve tekliği için çözüm süreci, büzülen dönüşüm özelliğini kullanarak ve sonra Leray-Schauder'in alternatif yöntemini kullanarak elde edildi.

İkincisi, Caputo tipi kesirli diferansiyel denklemin çözümlerinin doğrusal olmayan sınır koşullarıyla araştırılması ve yakınlaştırılması, doğrusal olmayan sınır koşullarının vektör parametreleri kullanılarak doğrusal sınır koşullarına dönüştürüldüğü uygun bir parametre belirleme tekniği kullanılarak çözülmüştür. Dönüştürülen problemi çözmek için, farklı türün iki noktalı ve çok noktalı doğrusal sınır koşulu ile doğrusal olmayan sınır şartları arasındaki ilişkiyi bulmak için sayısal-analitik bir şema oluşturulmuştur.

Son olarak, kesirli diferensiyel denklem sistemlerinin çözümü için etkin sayısal - analitik hesaplama algoritması, Lineer Olmayan Sınır Koşulları ile Lineer Olmayan Nokta Sınır Değer Problemi ele alınmıştır.

Yöntem, kesirli diferensiyel denklem sistemleri cebirsel denklem sistemine indirgemek için bu yöntemin özelliklerinin kullanıldığı kesirli diferansiyel denklem sistemlerinin sayısal yaklaşımlarına dayanmaktadır. Yakınsama çalışmalarına ve sunulan yöntemin hatasını tahmin etmeye özel önem verilir. Yöntemler, birçok doğrusal olmayan kesirli diferensiyel denklem sistemini çözmek için umut verici bir araç sunmaktadır. Her iki önerilen tekniğin geçerliliğini ve yüksek potansiyelini göstermek için sayısal örnekler sunulmuştur.

**Anahtar Kelimeler:** kesirli diferansiyel denklem, sıralı, Caputo, yerel olmayan integral sınır koşulları

# DEDICATION

*To My Lovely Wife, Children and Parents*

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# TABLE OF CONTENTS

# LIST OF TABLES

# LIST OF FIGURES

# LIST OF ABBREVIATIONS

| | |
|---|---|
| BC | Boundary condition. |
| FDE | Fractional Differential Equation. |
| FPT | Fixed Point Theory. |
| RL | Riemann-Liouville. |
| $\|\bullet\|$ | Norm of $\bullet$ |
| ${}_{RL}D^{(.)}$ | Riemann-Liouville Fractional Derivatives of order. |
| ${}_{RL}I^{(.)}$ | Riemann-Liouville Fractional integral of order. |
| $\mathbb{N}$ | The set of all natural numbers. |
| $\mathbb{R}$ | The set of all real numbers. |
| $J$ | Any given closed interval. |

# Chapter 1

# INTRODUCTION

Fractional calculus is one of the major topics in modern mathematics that has attracted many mathematicians and generated a notable body of research literature, mostly due to its wide and critical applications in various fields. In essence, fractional calculus is a generalization of normal calculus where the concepts of derivation and integration are generalized to non-integer orders [1, 2, 3].

The origins of fractional calculus can be traced back to Leibniz where he first described the concept of semi-derivatives in a letter to L'hopital [4, 5]. Since that time, contributions of many prominent figures in mathematics including Euler, Fourier, Holmgren, Lagrange, Caputo, Riemann, Heaviside, Liouville, and Laplace have laid the foundations of fractional calculus by creating the formal mathematical means required for dealing with fractional-order type of calculus.

In fact, the non-integer nature of orders in fractional calculus equips this field with great flexibility and efficiency in terms of modeling and explaining highly complex phenomena, where normal calculus is practically insufficient. And as such, fractional calculus has found numerous applications in various fields of natural and applied sciences, from quantum mechanics and computational fluid dynamics to biology, engineering, and chemistry [6, 7, 8, 9]. Apart from practical applications, fractional calculus has many pivotal theoretical advantageous over normal calculus as well. For instance, Riemann–Liouville fractional derivative can be employed on a function that is not continuous at the origin or differentiable in the normal calculus sense. Caputo derivate can directly incorporate the initial and boundary conditions into the problem formulation which makes it a strong differential operator for dealing with sophisticated systems and phenomena.

Wide applications of fractional calculus, has brought the topic of initial and boundary value problems for coupled systems of fractional differential equations under a sharp focus, and is one of the major focuses of the current thesis as well. The first main aim of this thesis is to prove the existence and uniqueness of solutions for a coupled system of fractional differential equations with four-point boundary conditions as well as providing a technique for investigation and approximation of solutions of Caputo type fractional differential equations with nonlinear boundary conditions. The second main aim of the thesis is to develop a program to utilize the aforementioned technique on different practical problems and present the results. The thesis is prepared in five chapters that are briefly reviewed in the following section.

Chapter 2 (Preliminaries and Definitions) this chapter provides the basic definitions, properties and terminology from fields of fractional calculus, special functions, and functional analysis, that are required for understanding the subsequent chapters. This chapter aims at providing the important preliminary concepts that other chapters are built upon, and can be skipped if the reader is already familiar with the field of fractional calculus.

Chapter 3 (On a Coupled System of Fractional Differential Equations with Four Point Integral Boundary Conditions) motivated by recent researches done on the SFDE's.

This chapter studies the existence and uniqueness of the solution for a coupled system of fractional differential equations with integral boundary conditions. The proof employs the Banach contraction mapping principle as well as Leray–Schauder's alternative theorem, and in the end, several examples and confirming results are provided.

Chapter 4 (On the Parametrization of Caputo type Fractional Differential Equation with two-point nonlinear boundary conditions) In this chapter, we apply a technique for investigation and approximation of solutions of Caputo type fractional differential equation with nonlinear boundary conditions. By using an appropriate parametrization technique, nonlinear boundary conditions are transformed to linear boundary conditions by using vector parameters. To study the transformed problem, a numerical scheme is developed, which has been successfully used in relation to different type's two-point and multipoint linear boundary condition and nonlinear boundary conditions.

Chapter 5 (Numerical Method and Algorithm for Solving Caputo type Fractional Differential Equations with Two Point Nonlinear Boundary Conditions). This chapter builds upon the findings of previous chapter and develops a numerical method and algorithm for solving Caputo type fractional systems with two-point nonlinear boundary conditions. The chapter initially focuses on the major computational challenges for solving fractional differential systems, and the complexities involved in such computations. One of the major challenges discussed in the chapter is extremely heavy computational costs for fractional systems that involve non-analytical integrals and the chapter provides a technique from Deep Reinforcement Learning paradigm to face this problem. The program and algorithm are explained in details and applied to an example, where numerical results are obtained and presented to illustrate the validity and possible applications of the method. Furthermore, the algorithm is also applied to a fractional system that involves analytical integrals, as a simpler special case, and results are demonstrated.

# Chapter 2

# PRELIMINARIES AND DEFINITIONS

In this chapter, we mention the most important mathematical tools including definitions, properties, propositions, lemmas and theorems.

## 2.1 Special Functions

**Definition 2.1.1** $\Gamma(\xi)=\int_0^\infty e^{-t}\,t^{\xi-1}dt, \forall\xi>0$, where $\Gamma(\,.\,)$ is the Gamma Function.

**Property 2.1.2**

i. $\Gamma(1)=1$.

ii. $\Gamma(\xi+1)=\xi\Gamma(\xi), \xi>0$.If $r\in\mathbb{N}$ then$\Gamma(\xi+1)=\xi!$ .

iii. The Incomplete Gamma function is given by

$$\gamma^*(u,\xi)=\frac{1}{u\Gamma(u)}\int_0^\xi x^{u-1}\,e^{-x}dx.$$

**Definition 2.1.3** $B(s,v)=\int_0^1 \eta^{s-1}\,(1-\eta)^{v-1}d\eta, \forall s,v>0$ where $B(\,.\,,.\,)$ is called the Beta Function.

**Property 2.1.4** $\forall s,v>0$,

i. $B(s,v)=B(v,s)$.

ii. $B(s,v)=\frac{\Gamma(s)\Gamma(v)}{\Gamma(s+v)}$.

iii. The Incomplete Beta function is given by $B_\psi(s,v)=\int_0^\psi \eta^{s-1}\,(1-\eta)^{v-1}d\eta, \quad 0<\psi<1$.

## 2.2 Function Spaces

Given the Banach space $C[a,b]$ that contains all continuous functions from $[a,b] \to \mathbb{R}$ with the norm $\|g\| = \underset{a \le t \le b}{Max}|g(t)|, \forall t \in [a,b], \forall \psi \ge 0.$

Assume $g_\psi(t) = (t-a)^\psi g(t)$, define the space $C_\psi[a,b]$ which is the space that contains $g$ such that $g_\psi \in C_\psi[a,b]$, where $g$ is any continuous function space $C_\psi[a,b]$ endowed with the norm $\|g\|_\psi = \underset{a \le t \le b}{Max}(t-a)^\psi|g(t)|$, is a Banach space as well.

Define $L^1([a,b],\mathbb{R})$ the space of measurable functions, which is also a Banach space, with the norm $\|g\|_{L^1} = \int_a^b |g(t)|dt$, $g: [a,b] \to \mathbb{R}$ is Lebesgue integrable function.

**Definition 2.2.1** Consider the interval $J \subseteq \mathbb{R}$. A function $g: J \to \mathbb{R}$ is absolutely continuous on $J$ if $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ such that for all finite set of pairwise disjoint subintervals $(s_k, v_k) \subset J$ satisfying $\sum v_k - s_k < \delta$ then $\sum|g(v_k) - g(s_k)| < \varepsilon$.
The collection of all absolutely continuous functions on $J$ is denoted by $AC(J)$.

**Remark 2.2.2** If $J = [a,b]$, then the following are equivalent

i. $g \in AC[a,b]$.

ii. $g$ has a derivative $g'$ almost everywhere, the derivative is Lebesegue integrable such that $g(\varphi) - g(a) = \int_a^\varphi g'(t)dt, \forall \varphi \in [a,b]$.

iii. There exists a lebesegue integrable function $h$ on $[a,b]$ such that $g(\varphi) - g(a) = \int_a^\varphi h(t)dt, \forall \varphi \in [a,b]$, so if $g \in AC[a,b]$, then $h = g'$ almost everywhere.

**Properties 2.2.3**

i. If $k_1 \in AC[a,b], k_1 \neq 0$ ,then $\frac{1}{k_1} \in AC[a,b]$.

ii. Absolutely continuous property implies uniformly continuous one.

iii. If $k_1$ is Lipschitz continuous function then $k_1$ is absolutely continuous.

**Definition 2.2.4** Given the function $g: J \to \mathbb{R}$ then $g \in AC^{\psi}(J), \psi = 1,2, \ldots$ if $g^{(\psi-1)} \in AC(J)$. Particularly, $AC^1(J) = AC(J)$.

**Definition 2.2.5** Let $(X,d)$ be a metric space. $Q: X \to X$ is said to be Lipschitzian if there is $k_Q \geq 0$ with $d\big(Q(x_1), Q(x_2)\big) \leq k_Q d(x_1, x_2), \forall x_1, x_2 \in X, x_1 \neq x_2$.

If $Q$ is Lipschitzian then it is continuous, when $k_Q < 1$ then is said to be contraction mapping. Whereas, if $k_Q = 1$ then $Q$ is said to be nonexpansive.

**Theorem 2.2.6 (Banach's Contraction mapping principle).**

Let $(X,d)$ be a complete metric space, $Q: X \to X$ is a contraction, then

i. $Q$ has a unique fixed point $v \in X$, that is $Q(v) = v$.

ii. $\forall v_0 \in X$, we have $\lim_{n\to\infty} Q^n(v_0) = v$ with

$$d(Q^n(v_0), v) \leq \frac{{k_Q}^n}{1-k_Q} d\big(v_0, Q^n(v_0)\big).$$

**Theorem 2.2.7 (Local version of Banach's Contraction mapping principle).**

Give the complete metric space $(X,d), Q: B(x_0,r) \to X$ is a contraction on this ball with $d\big(Q(x_1), Q(x_2)\big) \leq d(x_1, x_2), \forall x_1, x_2 \in B(x_0, r), 0 \leq k_Q < 1$ such that $d(Q(x_0), x_0) < \big(1 - k_Q\big)r$. Then unique fixed point for $Q$ in $B(x_0, r)$ holds true. Here $B(x_0, r)$ is a closed ball centered at $x_0$ of radius $r$.

**Theorem 2.2.8** Given the complete metric space $(X, d), Q: X \to X$ satisfying $d\big(Q(x_1), Q(x_2)\big) < \phi\big(d(x_1, x_2)\big), \forall x_1, x_2 \in X$, here $\phi: [0, \infty) \to [0, \infty)$ is any monotonic nondecreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$, for a fixed $t > 0$, then has a unique fixed point with $\lim_{n\to\infty} Q^n(x_0) = x, \forall x \in X$.

**Theorem 2.2.9 (Nonlinear alternative of Leray-Shauder type)**

Given the open subset $V$of a Banach space $S$, $0 \in V$ and $Q: \bar{V} \to S$ be a contraction such that $Q(\bar{V})$ is bounded and $\bar{V}$ is the closure of $V$ then either

i. $Q$ has a fixed point in $\bar{V}$, or

ii. $\exists \lambda \in (0,1)$ and $v \in \partial V$ such that $v = \lambda Q(v)$ holds.

**Theorem 2.2.10 (Arzela-Ascoli Theorem)**

$Q \subset C(S, \mathbb{R})$ is compact iff it is closed, bounded and equicontinuous.

**Theorem 2.2.11 (Krasnoselskii's Theorem)**

Give the Banach space $(E, \|\,.\,\|)$, closed convex $B \subset E, A$ is open, where $A \subset B$, and $p \in A$, assume that $Q: \bar{A} \to B$ can be written as $Q = Q_1 + Q_2$.

In addition, $Q(\bar{A})$ is bounded set in $B$ satisfying

**i.** $Q_1: \bar{A} \to B$ is continuous and completely continuous.

ii. $Q_2: \bar{A} \to B$ is a contraction, $\exists \tau$ a continuous nondecreasing function $\tau: [0, \infty) \to [0, \infty)$ with $\tau(a_1) > a_1, a_1 > 0$, such that $|Q_2(a_1) - Q_2(a_2)| \le \tau(\|a_1 - a_2\|)$, for any $a_1, a_2 \in \bar{A}$.

Then

i. $Q(a_0) = a_0, a_0 \in \bar{A}, or$

ii. $\exists a \in \partial A$ and $\mu \in (0,1)$ with $a = \mu Q(a) + (1 - \mu)p$.

## 2.3 Caputo Fractional Derivative

It is turn out that ${}_{RL}D^{(.)}$ has a weak points in some real models, indeed, a new definition of fractional derivatives has to be introduced. The Caputo fractional derivative of order $\Theta$ proposed by an Italian mathematician is an alternative fractional derivative to the RL- fractional derivative which given by

$$({}_cD_a^{\Theta}g)(s) = \begin{cases} \int_a^s \frac{(s-u)^{m-\Theta-1}h^{(m)}(u)du}{\Gamma(m-\Theta)}, m-1 < \Theta < m \in \mathbb{N}, \\ g^{(m)}(s) \quad , \quad \Theta \in \mathbb{N}, \end{cases}$$

Caputo 1967, it is important to note that the Caputo derivative is more restrictive than the RL-fractional derivative as it requires the $nth$ derivative of the function $g$ . Which leads to assume that it is exist whenever the ${}_cD^{(.)}$ is used, and fortunately in the most applications the used functions have the $nth$ derivative.

Consider the set of functions $g(t)$, continuous and integrable in any finite interval $(0, y), y \in \mathbb{R}$ For the ${}_cD^{(.)}$ it is required that the $n^{th}$ derivative of the function must integrable, Next in this study all functions are already assumed to satisfy this condition.

The following results are some main properties of the ${}_cD^{(.)}$:

i. ${}_cD_a^{\Theta}g(\zeta) = {}_{RL}I_a^{\mathrm{m}-\Theta}D^m g(\zeta)$, where $D^m$ is the standard differentiation operator$D^m = \frac{d^m}{d\zeta^m}$,

ii. $\lim_{\Theta \to m} {}_cD_a^{\Theta}g(\zeta) = g^{(m)}(\zeta)$,

$\lim_{\Theta \to m-1} {}_cD_a^{\Theta}g(\zeta) = g^{(m-1)}(\zeta) - g^{(m-1)}(0)$.

iii. ${}_cD_a^{\Theta}\big(\alpha g(\zeta) + \beta h(\zeta)\big) = \alpha\, {}_cD_a^{\Theta}g(\zeta) + \beta\, {}_cD_a^{\Theta}h(\zeta), \alpha, \beta \in \mathbb{R}$.

iv. ${}_cD_a^{\Theta}D^m g(\zeta) = {}_cD_a^{\Theta+\mathrm{m}}g(\zeta) \neq D^m\, {}_cD_a^{\Theta}g(\zeta)$,

v. $\quad {}_cD_a^{\Theta}\left(g(\zeta)h(\zeta)\right) = \sum_{i=0}^{\infty}\binom{\Theta}{i}\left(D^{\Theta-i}g(\zeta)\right)h^{(i)}(\zeta) -$

$\sum_{i=0}^{m-1}\frac{\zeta^{i-\Theta}}{\Gamma(i+1-\Theta)}\left(\left(g(\zeta)h(\zeta)\right)^{i}(0)\right).$

vi. $\quad {}_cD_a^{\Theta} b = 0, b$ is a constant.

vii. $\quad {}_cD_a^{\Theta}\zeta^{\omega} = \begin{cases}\frac{\Gamma(\omega+1)}{\Gamma(\omega+1-\Theta)}\zeta^{\omega-\Theta}, & m-1 < \Theta < m, \omega > m-1, \omega \in \mathbb{R}, \\ 0, & m-1 < \Theta < m, \omega \leq m-1\ ,\end{cases}$

# Chapter 3

# ON A COUPLED SYSTEM OF FRACTIONAL DIFFERNTIAL EQUATIONS WITH FOUR POINT INTEGRAL BOUNDARY CONDITIONS

This chapter is on the existence of the solution for a coupled system of fractional differential equations with integral boundary conditions. The first result will address the uniqueness of solutions for the proposed problem and it is based on the contraction mapping principle. Secondly, by using Leray–Schauder's alternative we manage to prove the existence of solutions. Finally, the conclusion is confirmed and supported by examples.

The following coupled system of fractional differential equations was studied:

$$\begin{cases} D^{\alpha}x(t) = f\big(t, x(t), y(t), D^{\gamma}y(t)\big), t \in [0,T], \\ \qquad 1 < \alpha \leq 2, \qquad 0 < \gamma < 1, \\ D^{\beta}y(t) = g\left(t, x(t), D^{\delta}x(t), y(t)\right), t \in [0,T], \\ \qquad 1 < \beta \leq 2, \qquad 0 < \delta < 1, \end{cases}$$

Supplemented with the coupled nonlocal and integral boundary conditions of the form

$$\begin{cases} x(0) = h(y), \quad \int_0^T y\,(s)ds = \mu_1 x(\eta), \\ y(0) = \varphi(x), \quad \int_0^T x\,(s)ds = \mu_2 y(\xi), \quad \eta, \xi \in (0,T), \end{cases}$$

where $D^i$ denotes the Caputo fractional derivatives of order $i = \alpha, \beta, \gamma, \delta$ and $f, g: [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, h, \varphi: C([0,T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and $\mu_1, \mu_2$ are real constants.

In [27], the authors investigated the existence and uniqueness of solutions for the coupled system of nonlinear fractional differential equations with three-point boundary conditions, given below:

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), 0 < t < 1, \\ D^{\beta}v(t) = g(t, u(t), D^{q}u(t)), 0 < t < 1, \\ u(0) = 0, u(1) = \gamma u(\eta), v(0) = 0, v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha,\ \beta < 2,\ p, q, \gamma > 0,\ 0 < \eta < 1,\ \alpha - q \geq 1,\ \beta - p \geq 1,\ \gamma\eta^{\alpha-1} < 1,$ $\gamma\eta^{\beta-1} < 1,$ and $D$ is the standard Riemann–Liouville fractional derivative and $f, g: [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. It is worth mentioning that the nonlinear terms in the coupled system contain the fractional derivatives of the unknown functions.

Moreover, in a study [28], the following coupled system of nonlinear fractional differential equations, with the given boundary conditions was studied:

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), 0 < t < 1, \\ D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)), 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where $1 < \alpha,\ \beta < 2,\ \mu, \nu > 0,\ \alpha - \nu \geq 1,\ \ \beta - \mu \geq 1,\ \ and\ f, g: [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and $D$ is the standard Riemann–Liouville differentiation.

This chapter is aimed to study a coupled system of nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}x(t) = f\big(t, x(t), y(t), D^{\gamma}y(t)\big), t \in [0, T], \\ \qquad 1 < \alpha \leq 2, \qquad 0 < \gamma < 1, \\ D^{\beta}y(t) = g\big(t, x(t), D^{\sigma}x(t), y(t)\big), t \in [0, T], \\ \qquad 1 < \beta \leq 2, \qquad 0 < \sigma < 1, \end{cases} \tag{1}$$

supported by integral boundary conditions of the form

$$\begin{cases} \int_0^T x(s)ds = \rho_1 y(\zeta_1), \int_0^T x'(s)ds = \rho_2 y'(\zeta_2), \\ \int_0^T y(s)ds = \mu x(\eta_1), \int_0^T y'(s)ds = \mu_2 x'(\eta_2), \\ \eta_1, \eta_2, \zeta_1, \zeta_2 \in [0,T], \end{cases} \tag{2}$$

where $D^k$ denote the Caputo fractional derivatives of order $k$, and $f, g: [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, are given continuous functions, and $\rho_1, \rho_2, \mu_1, \mu_2$are real constants.

The chapter is organized as follows. In Section 1, we recall some definitions from fractional calculus, and state and prove an auxiliary lemma, which gives an explicit formula for a solution of nonhomogeneous equation correspond to our problem. The main results for the coupled system of Caputo fractional differential equations with integral boundary conditions, using the Banach fixed point theorem and Leray-Schauder alternative, are presented in Section 2. The chapter concludes with concrete examples.

## 3.1 Preliminaries

Firstly, we recall definitions of fractional derivative and integral [1].

**Definition3.1.1** The Riemann-Liouville fractional integral of order $\alpha$ for a continuous function $h$ is given by

$$(I_0^\alpha h)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \frac{h(t)}{(s-t)^{1-\alpha}} dt, \alpha > 0,$$

provided that the integral exists on $\mathbb{R}^+$.

We use the following notations.

$$\Delta_1 = T^2 - \mu_1 \rho_1 \neq 0, \Delta_2 = T^2 - \mu_2 \rho_2 \neq 0,$$

$$\Theta_1(t) \quad := \frac{2T\rho_1\zeta_1\mu_2\rho_2 - T^4\rho_2 + 2T\rho_1\mu_1\eta_1\rho_2 - T^2\mu_2\rho_1\rho_2}{\Delta_1\Delta_2} + \frac{\rho_2 Tt}{\Delta_2}.$$

$$\Theta_2(t) \qquad := \frac{-2T^2\rho_1\zeta_1 + T^3\rho_2 - 2\rho_1\mu_1\eta_1\rho_2 + \rho_1 T^3}{\Delta_1\Delta_2} - \frac{\rho_2 t}{\Delta_2},$$

$$\Theta_3 := \frac{T\rho_1}{\Delta_1}, \Theta_4 := -\frac{\rho_1}{\Delta_1},$$

$$\Xi_1(t) \quad := \frac{2T^2\rho_1\zeta_1\mu_2 - T^3\rho_2\mu_2 + 2\rho_1\mu_1\eta_1\rho_2\mu_2 - \rho_1\mu_2 T^3}{\Delta_1\Delta_2} + \frac{t\mu_2\rho_2}{\Delta_2}$$

$$\Xi_2(t) \qquad := \frac{-2T\rho_1\zeta_1\mu_2 + T^4 - 2T\rho_1\mu_1\eta_1 + T^2\rho_1\mu_2}{\Delta_1\Delta_2} - \frac{Tt}{\Delta_2},$$

$$\Xi_3 := \frac{\rho_1\mu_1}{\Delta_1}, \Xi_4 := -\frac{T}{\Delta_1},$$

$$\hat{\Theta}_1(t) \quad := \frac{1}{\Delta_1}\left(\rho_2 T\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) + \mu_2\rho_2\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) + \frac{1}{\Delta_2}\mu_2\rho_2 t.$$

$$\hat{\Theta}_2(t) \qquad := \frac{1}{\Delta_1}\left(-\rho_2\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) - T\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) - \frac{1}{\Delta_2}Tt,$$

$$\hat{\Theta}_3 := \frac{1}{\Delta_1}\rho_1\mu_1, \hat{\Theta}_4 := \frac{-T}{\Delta_1},$$

$$\hat{\Xi}_1(t) \quad := \frac{1}{\Delta_1}\left(\mu_2\rho_2\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) + \mu_2 T\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) + \frac{1}{\Delta_2}\mu_2 Tt$$

$$\hat{\Xi}_2(t) \qquad := \frac{1}{\Delta_1}\left(-T\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) - \mu_2\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) - \frac{1}{\Delta_2}\mu_2 t,$$

$$\hat{\Xi}_3 := \frac{\mu_1 T}{\Delta_1}, \hat{\Xi}_4 := \frac{-\mu_1}{\Delta_1}.$$

To show that the problem (1) and (2) is equivalent to the problem of finding solutions to the Volterra integral equation, we need the following auxiliary lemma.

**Lemma 3.1.1** Let $w, z \in C([0,T], R)$. Then the unique solution for the problem

$$\begin{cases} D^\alpha x(t) = w(t), \quad t \in [0,T], \quad 1 < \alpha \leq 2, \\ D^\beta y(t) = z(t), \quad t \in [0,T], \quad 1 < \beta \leq 2, \\ \int_0^T x(s)ds = \rho_1 y(\zeta_1), \int_0^T x'(s)ds = \rho_2 y'(\zeta_2), \\ \int_0^T y(s)ds = \mu x(\eta_1), \int_0^T y'(s)ds = \mu_2 x'(\eta_2), \end{cases} \tag{3}$$

is

$$x(t) = \Theta_1(t)\left(I^{\beta-1}z\right)(\zeta_2) + \Theta_2(t)\int_0^T \left(I^{\beta-1}z\right)(s)ds + \Theta_3\left(I^{\beta}z\right)(\zeta_1) - \Theta_4\int_0^T \left(I^{\beta}z\right)(s)ds$$

$$+ \Xi_1(t)(I^{\alpha-1}w)(\eta_2) + \Xi_2(t)\int_0^T (I^{\alpha-1}w)(s)ds + \Xi_3(I^{\alpha}w)(\eta_1)$$

$$- \Xi_4\int_0^T (I^{\alpha}w)(s)ds$$

$$+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s)ds, \qquad (4)$$

and

$$y(t)$$

$$= \widehat{\Theta}_1(t)\left(I^{\beta-1}z\right)(\zeta_2) + \widehat{\Theta}_2(t)\int_0^T \left(I^{\beta-1}z\right)(s)ds + \widehat{\Theta}_3\left(I^{\beta}z\right)(\zeta_1) - \widehat{\Theta}_4\int_0^T \left(I^{\beta}z\right)(s)ds$$

$$+ \widehat{\Xi}_1(t)(I^{\alpha-1}w)(\eta_2) + \widehat{\Xi}_2(t)\int_0^T (I^{\alpha-1}w)(s)ds + \widehat{\Xi}_3(I^{\alpha}w)(\eta_1) - \widehat{\Xi}_4\int_0^T (I^{\alpha}w)(s)ds$$

$$+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} z(s)ds. \qquad (5)$$

**Proof.** We know that, see [1] Lemma 2.12, the general solutions for the FDE in (3) is defined as

$$\begin{aligned} x(t) &= c_1 t + c_2 + (I^{\alpha}w)(t) \\ y(t) &= d_1 t + d_2 + \left(I^{\beta}z\right)(t), \end{aligned} \qquad (6)$$

where $c_1, c_2, d_1, d_2 \in R$ are arbitrary constants. It follows that

$$\begin{aligned} x'(t) &= c_1 + (I^{\alpha-1}w)(t), \\ y'(t) &= d_1 + \left(I^{\beta-1}z\right)(t). \end{aligned}$$

Applying the conditions

$$\int_0^T x'(s)ds = \rho_2 y'(\zeta_2), \int_0^T y'(s)ds = \mu_2 x'(\eta_2),$$

we get

$$\begin{aligned} c_1 T + \int_0^T (I^{\alpha-1}w)(s)ds &= \rho_2 d_1 + \rho_2\left(I^{\beta-1}z\right)(\zeta_2), \\ d_1 T + \int_0^T \left(I^{\beta-1}z\right)(s)ds &= \mu_2 c_1 + \mu_2(I^{\alpha-1}w)(\eta_2). \end{aligned}$$

Solving the above equations together for $c_1$ and $d_1$ we get

$$c_1 = \frac{1}{\Delta_2}\Big(\rho_2 T\big(I^{\beta-1}z\big)(\zeta_2) - \rho_2\int_0^T\big(I^{\beta-1}z\big)(s)ds + \mu_2\rho_2(I^{\alpha-1}w)(\eta_2) - T\int_0^T(I^{\alpha-1}w)(s)ds\Big),$$

$$d_1 = \frac{1}{\Delta_2}\Big(\mu_2 T(I^{\alpha-1}w)(\eta_2) - \mu_2\int_0^T(I^{\alpha-1}w)(s)ds + \mu_2\rho_2\big(I^{\beta-1}z\big)(\zeta_2) - T\int_0^T\big(I^{\beta-1}z\big)(s)ds\Big).$$

Considering the following boundary conditions not involving derivatives

$$\int_0^T x\,(s)ds = \rho_1 y(\zeta_1), \int_0^T y\,(s)ds = \mu_1 x(\eta_1),$$

we get

$$c_2T - \rho_1 d_2 = \rho_1 d_1\zeta_1 + \rho_1\big(I^{\beta}z\big)(\zeta_1) - c_1\frac{T^2}{2} - \int_0^T(I^{\alpha}w)(s)ds\,,$$
$$d_2T - \mu_1 c_2 = \mu_1 c_1\eta_1 + \mu_1(I^{\alpha}w)(\eta_1) - d_1\frac{T^2}{2} - \int_0^T\big(I^{\beta}z\big)(s)ds\,.$$

This implies

$$c_2 = \frac{1}{\Delta_1}\Big(T\rho_1 d_1\zeta_1 + \rho_1 T\big(I^{\beta}z\big)(\zeta_1) - c_1\frac{T^3}{2} - T\int_0^T(I^{\alpha}w)(s)ds + \rho_1\mu_1 c_1\eta_1 + \rho_1\mu_1(I^{\alpha}w)(\eta_1) - \rho_1 d_1\frac{T^2}{2} - \rho_1\int_0^T\big(I^{\beta}z\big)(s)ds\Big),$$

$$d_2 = \frac{1}{\Delta_1}\Big(T\mu_1 c_1\eta_1 + \mu_1 T(I^{\alpha}w)(\eta_1) - d_1\frac{T^3}{2} - T\int_0^T\big(I^{\beta}z\big)(s)ds + \mu_1\rho_1 d_1\zeta_1 + \rho_1\mu_1\big(I^{\beta}z\big)(\zeta_1) - c_1\mu_1\frac{T^2}{2} - \mu_1\int_0^T(I^{\alpha}w)(s)ds\Big).$$

Inserting the values of $c_1$ and $d_1$we get

$$\begin{aligned} c_2 = {} & \frac{2T\rho_1\zeta_1\mu_2\rho_2 - T^4\rho_2 + 2T\rho_1\mu_1\eta_1\rho_2 - T^2\mu_2\rho_1\rho_2}{2\Delta_1\Delta_2}\big(I^{\beta-1}z\big)(\zeta_2) \\ & + \frac{-2T^2\rho_1\zeta_1 + T^3\rho_2 - 2\rho_1\mu_1\eta_1\rho_2 + \rho_1 T^3}{2\Delta_1\Delta_2}\int_0^T\big(I^{\beta-1}z\big)(s)ds \\ & + \frac{T\rho_1}{\Delta_1}\big(I^{\beta}z\big)(\zeta_1) - \frac{\rho_1}{\Delta_1}\int_0^T\big(I^{\beta}z\big)(s)ds \\ & + \frac{2T^2\rho_1\zeta_1\mu_2 - T^3\rho_2\mu_2 + 2\rho_1\mu_1\eta_1\rho_2\mu_2 - \rho_1\mu_2 T^3}{2\Delta_1\Delta_2}(I^{\alpha-1}w)(\eta_2) \\ & + \frac{-2T\rho_1\zeta_1\mu_2 + T^4 - 2T\rho_1\mu_1\eta_1 + T^2\rho_1\mu_2}{2\Delta_1\Delta_2}\int_0^T(I^{\alpha-1}w)\,(s)ds \\ & + \frac{\rho_1\mu_1}{\Delta_1}(I^{\alpha}w)(\eta_1) - \frac{T}{\Delta_1}\int_0^T(I^{\alpha}w)\,(s)ds, \end{aligned}$$

$$
\begin{aligned}
d_2 \quad &= \frac{2\rho_2 T^2 \mu_1 \eta_1 - \rho_2 \mu_1 T^3 + 2\mu_2 \rho_2 \mu_1 \rho_1 \zeta_1 - \mu_2 \rho_2 T^3}{2\Delta_1 \Delta_2} \left(I^{\beta-1} z\right)(\zeta_2) \\
&+ \frac{-2T\rho_2 \mu_1 \eta_1 + \rho_2 \mu_1 T^2 - 2T\mu_1 \rho_1 \zeta_1 - T^4}{2\Delta_1 \Delta_2} \int_0^T \left(I^{\beta-1} z\right)(s) ds \\
&+ \frac{1}{\Delta_1} \rho_1 \mu_1 \left(I^\beta z\right)(\zeta_1) - \frac{1}{\Delta_1} T \int_0^T \left(I^\beta z\right)(s) ds \\
&+ \frac{2T\mu_2 \rho_2 \mu_1 \eta_1 - \mu_2 \rho_2 \mu_1 T^2 + 2\mu_2 T \mu_1 \rho_1 \zeta_1 - \mu_2 T^4}{2\Delta_1 \Delta_2} (I^{\alpha-1} w)(\eta_2) \\
&+ \frac{-2T^2 \mu_1 \eta_1 + \mu_1 T^3 - 2\mu_2 \rho_2 \mu_1 \rho_1 \zeta_1 + \mu_2 T^3}{2\Delta_1 \Delta_2} \int_0^T (I^{\alpha-1} w)(s) ds \\
&+ \mu_1 T \frac{1}{\Delta_1} (I^\alpha w)(\eta_1) - \mu_1 \frac{1}{\Delta_1} \int_0^T (I^\alpha w)(s) ds.
\end{aligned}
$$

Substituting $c_1, c_2, d_1, d_2$ in (6) we get (4) and (5). ■

**Remark 3.1.1** In (4) and (5) $x(t)$ and $y(t)$ depend on $\eta_i, \zeta_i, \mu_i, \rho_i, i = 1{,}2$.

## 3.2 Existence Results

Consider the space

$$C_\gamma([0,T],\mathbb{R}) = \{x(t){:}x(t)\in C([0,T],\mathbb{R}) \text{ } and \text{ } D^\gamma x(t)\in C([0,T],\mathbb{R})\},$$

with the norm

$$\|x\|_\gamma = \|x\| + \| D^\gamma x \| = \max_{0\le t\le T}|x(t)| + \max_{0\le t\le T}|D^\gamma x(t)|.$$

It is clear that $\left(C_\gamma([0,T],\mathbb{R}),\|\cdot\|_\gamma\right)$ is a Banach space. Consequently, the product space $\left(C_\sigma([0,T],\mathbb{R})\times C_\gamma([0,T],\mathbb{R}),\|\cdot\|_{\sigma\times\gamma}\right)$ is a Banach Space with the $\|(x,y)\|_{\sigma\times\gamma} = \|x\|_\sigma + \|y\|_\gamma$ for $(x,y) \in C_\sigma([0,T],\mathbb{R}) \times C_\gamma([0,T],\mathbb{R})$.

Next, using Lemma 3.1.1, we define the operator $G{:}\, C_\sigma([0,T],\mathbb{R}) \times C_\gamma([0,T],\mathbb{R}) \to C_\sigma([0,T],\mathbb{R}) \times C_\gamma([0,T],\mathbb{R})$ as follows

$$G(x,y)(t) = (G_1(x,y)(t), G_2(x,y)(t)),$$

where

$$G_1(x,y)(t) = \Theta_1(t)\left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(\zeta_2) + \Theta_2(t)\int_0^T \left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(s)ds$$
$$+\Theta_3\left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(\zeta_1) - \Theta_4\int_0^T \left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(s)ds$$
$$+\Xi_1(t)\left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(\eta_2) + \Xi_2(t)\int_0^T \left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(s)ds$$
$$+\Xi_3\left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(\eta_1) - \Xi_4\int_0^T \left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(s)ds$$
$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s),y(s),D^{\gamma}y(s))ds,$$

and

$$G_2(x,y)(t) = \hat{\Theta}_1(t)\left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(\zeta_2) + \hat{\Theta}_2(t)\int_0^T \left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(s)ds$$
$$+\hat{\Theta}_3\left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(\zeta_1) - \hat{\Theta}_4\int_0^T \left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(s)ds$$
$$+\hat{\Xi}_1(t)\left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(\eta_2) + \hat{\Xi}_2(t)\int_0^T \left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(s)ds$$
$$+\hat{\Xi}_3\left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(\eta_1) - \hat{\Xi}_4\int_0^T \left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(s)ds$$
$$+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s,x(s),y(s),D^{\sigma}x(s))ds.$$

In what follows, we use the following notations.

$$\Theta = \|\Theta_1\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)} + \|\Theta_2\|\frac{T^{\beta-1}}{\Gamma(\beta)} + |\Theta_3|\frac{\zeta_1^{\beta}}{\Gamma(\beta+1)} + |\Theta_4|\frac{T^{\beta}}{\Gamma(\beta+1)} + \frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Theta_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\Theta_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right),$$
$$\Xi = \|\Xi_1\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)} + \|\Xi_2\|\frac{T^{\alpha-1}}{\Gamma(\alpha)} + |\Xi_3|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\Xi_4|\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Xi_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\Xi_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right).$$

$$\hat{\Theta} = \|\hat{\Theta}_1\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)} + \|\hat{\Theta}_2\|\frac{T^{\beta-1}}{\Gamma(\beta)} + |\hat{\Theta}_3|\frac{\zeta_1^{\beta}}{\Gamma(\beta+1)} + |\hat{\Theta}_4|\frac{T^{\beta}}{\Gamma(\beta+1)} + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\left(\|\hat{\Theta}_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\hat{\Theta}_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right),$$
$$\hat{\Xi} = \|\hat{\Xi}_1\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)} + \|\hat{\Xi}_2\|\frac{T^{\alpha-1}}{\Gamma(\alpha)} + |\hat{\Xi}_3|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)} + |\hat{\Xi}_4|\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\beta}}{\Gamma(\beta+1)}$$
$$+\frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\left(\|\hat{\Xi}_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\hat{\Xi}_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\beta-1}}{\Gamma(\beta)}\right),$$

where $\Theta_i(t), \hat{\Theta}_i(t), \Xi_i(t), \hat{\Xi}_i(t), i = 1, \ldots, 4$, are defined before Lemma 3.1.1.

Now we state and prove our first main result.

**Theorem 3.2.1** Let $f, g: [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ be jointly continuous functions. Assume that

(i) there exist constants $l_f > 0, l_g > 0 \forall t \in [0,T], and\ x_i, y_i \in \mathbb{R}, i = 1,2,3$

$$|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)| \leq l_f(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

$$|g(t,x_1,x_2,x_3) - g(t,y_1,y_2,y_3)| \leq l_g(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|).$$

(ii) $$1 - 2\left(\Theta l_g + \Xi l_f\right) > 0, 1 - 2\left(\hat{\Theta} l_g + \hat{\Xi} l_f\right) > 0.$$

Then the boundary value problem (1), (2) has a unique solution on $[0,T]$.

Proof. Assume that $\varepsilon > 0$ is a real number satisfying

$$\varepsilon \geq max\left(\frac{2(\Theta g_0 + \Xi f_0)}{1 - 2\left(\Theta l_g + \Xi l_f\right)}, \frac{2\left(\hat{\Theta} g_0 + \hat{\Xi} f_0\right)}{1 - 2\left(\hat{\Theta} l_g + \hat{\Xi} l_f\right)}\right),$$

where

$$\max_{0 \leq t \leq T} |f(t,0,0,0)| = f_0 < \infty, \max_{0 \leq t \leq T} |g(t,0,0,0)| = g_0 < \infty.$$

Define

$$\Omega_\varepsilon = \left\{(x,y) \in C_\sigma([0,T],\mathbb{R}) \times C_\gamma([0,T],\mathbb{R}) : \|(x,y)\|_{\sigma \times \gamma} \leq \varepsilon\right\}.$$

**Step 1:** Show that $G\Omega_\varepsilon \subset \Omega_\varepsilon$.

By our assumption, for $(x,y) \in \Omega_\varepsilon, t \in [0,T]$, we have

$$\begin{aligned} |f(t,x(t),y(t),D^\gamma y(t))| &\leq |f(t,x(t),y(t),D^\gamma y(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\ &\leq l_f(|x(t)| + |y(t)| + |D^\gamma y(t)|) + f_0 \\ &\leq l_f\left(\|x\|_\sigma + \|y\|_\gamma\right) + f_0 \leq l_f \varepsilon + f_0, \end{aligned} \tag{7}$$

similarly, we have

$$\left|g\left(t,x(t),D^\sigma x(t),y(t)\right)\right| \leq l_g \varepsilon + g_0. \tag{8}$$

Using these estimates, we get

$$
\begin{aligned}
|G_1(x,y)(t)| \quad &\leq |\Theta_1(t)|\big(I^{\beta-1}|g|\big)(\zeta_2) + |\Theta_2(t)|\int_0^T \big(I^{\beta-1}|g|\big)(s)ds \\
&+|\Theta_3|\big(I^{\beta}|g|\big)(\zeta_1) + |\Theta_4|\int_0^T \big(I^{\beta}|g|\big)(s)ds \\
&+|\Xi_1(t)|(I^{\alpha-1}|f|)(\eta_2) + |\Xi_2(t)|\int_0^T (I^{\alpha-1}|f|)(s)ds \\
&+|\Xi_3|(I^{\alpha}|f|)(\eta_1) + |\Xi_4|\int_0^T (I^{\alpha}|f|)(s)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s),D^{\gamma}y(s))|ds.
\end{aligned}
$$

We use the following type inequalities

$$
\begin{aligned}
\big(I^{\beta-1}|g|\big)(\zeta_2) &= \frac{1}{\Gamma(\beta-1)}\int_0^{\zeta_2} (t-s)^{\beta-2}|g(s)|ds, \\
&\leq \frac{1}{\Gamma(\beta-1)}\int_0^{\zeta_2} (t-s)^{\beta-2}ds\|g\| = \frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}\|g\|,
\end{aligned}
$$

to get

$|G_1(x,y)(t)|$

$$
\leq \Big(|\Theta_1(t)|\big(I^{\beta-1}1\big)(\zeta_2)+|\Theta_2(t)|\int_0^T \big(I^{\beta-1}1\big)(s)ds+|\Theta_3|\big(I^{\beta}1\big)(\zeta_1)+|\Theta_4|\int_0^T \big(I^{\beta}1\big)(s)ds\Big)\|g\|
$$

$$
+\Big(|\Xi_1(t)|(I^{\alpha-1}1)(\eta_2)+|\Xi_2(t)|\int_0^T (I^{\alpha-1}1)(s)ds+|\Xi_3|(I^{\alpha}1)(\eta_1)+|\Xi_4|\int_0^T (I^{\alpha}1)(s)ds\Big)\|f\|
$$

$$
+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\|f\|
$$

$$
\leq \left(\|\Theta_1\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\Theta_2\|\frac{T^{\beta-1}}{\Gamma(\beta)}+|\Theta_3|\frac{\zeta_1^{\beta}}{\Gamma(\beta+1)}+|\Theta_4|\frac{T^{\beta}}{\Gamma(\beta+1)}\right)\|g\|
$$

$$
+\left(\|\Xi_1\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\Xi_2\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+|\Xi_3|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)}+|\Xi_4|\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)\|f\| + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\|f\|. \tag{9}
$$

Hence, by (7) and (8) we have

$$
\| G_1(x,y) \| \leq \big(\Theta l_g+\Xi l_f\big)\varepsilon + (\Theta g_0+\Xi f_0).
$$

On the other hand,

$$
\begin{aligned}
\frac{d}{dt}G_1(x,y)(t) &= \Theta_1'(t)\big(I^{\beta-1}g\big)(\zeta_2) + \Theta_2'(t)\int_0^T \big(I^{\beta-1}g\big)(s)ds \\
&+\Xi_1'(t)(I^{\alpha-1}f)(\eta_2) + \Xi_2'(t)\int_0^T (I^{\alpha-1}f)(s)ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}f(s,x(s),y(s),D^{\gamma}y(s))ds.
\end{aligned}
$$

and

$$\left|\frac{d}{dt}G_1(x,y)(t)\right| \leq \left(\|\Theta_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\Theta_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right)\|g\|$$
$$+\left(\|\Xi_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\Xi_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)\|f\|.$$

It follows that

$$\begin{aligned}|D^\sigma G_1(x,y)(t)| &\leq \int_0^t \frac{(t-s)^{-\sigma}}{\Gamma(1-\sigma)}\left|\frac{d}{ds}G_1(x,y)(s)\right| ds \\ &\leq \frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Theta_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\Theta_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right)\|g\| \\ &+\frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Xi_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\Xi_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)\|f\|. \qquad (10)\end{aligned}$$

Thus by (7)–(10) we obtain

$$\begin{aligned}\| G_1(x,y) \|_\sigma &= \| G_1(x,y) \| + \| D^\sigma G_1(x,y) \| \\ &\leq \Theta\|g\| + \Xi\|f\| \\ &\leq \left(\Theta l_g+\Xi l_f\right)\varepsilon + (\Theta g_0+\Xi f_0) \leq \frac{\varepsilon}{2}.\end{aligned} \qquad (11)$$

In similar way we get

$$\begin{aligned}\| G_2(x,y) \|_\gamma &= \| G_2(x,y) \| + \| D^\gamma G_2(x,y) \| \\ &\leq \left(\hat\Theta l_g+\hat\Xi l_f\right)\varepsilon + \left(\hat\Theta g_0+\hat\Xi f_0\right) \leq \frac{\varepsilon}{2}.\end{aligned} \qquad (12)$$

From (11) and (12) we get

$$\| G_1(x,y) \|_\sigma + \| G_2(x,y) \|_\gamma \leq \varepsilon.$$

**Step 2:** Show that $G$ is a contraction.

Now for $x_1, x_2, y_1, y_2 \in \Omega_\varepsilon, \forall t \in [0,T]$ we have

$$\begin{aligned}&\| G_1(x_1,y_1)-G_1(x_2,y_2) \|_\sigma \\ &\qquad \leq \left(\Theta l_g+\Xi l_f\right)(\| x_1-x_2 \|+\| y_1-y_2 \|+\| D^\gamma y_1-D^\gamma y_2 \|), \\ &\| G_2(x_1,y_1)-G_2(x_2,y_2) \|_\gamma \\ &\qquad \leq \left(\hat\Theta l_g+\hat\Xi l_f\right)(\| x_1-x_2 \|+\| y_1-y_2 \|+\| D^\sigma x_1-D^\sigma x_2 \|).\end{aligned}$$

So we obtain

$$\| (G_1,G_2)(x_1,y_1)-(G_1,G_2)(x_2,y_2) \|_{\sigma\times\gamma}$$
$$\leq \left( (\Theta l_g+\Xi l_f)+(\hat{\Theta} l_g+\hat{\Xi} l_f) \right) \| (x_1,y_1)-(x_2,y_2) \|_{\sigma\times\gamma},$$

which shows that $G$ is a contraction. So, by the Banach fixed point theorem, the operator (($G_1$,$G_2$) has a unique fixed point in $\Omega_\varepsilon$.

The second result is based on the Leray-Schauder alternative. Now we formulate and prove the second existence result.

**Theorem 3.2.2** Let $f,g:[0,T]\times\mathbb{R}^3\to\mathbb{R}$ be continuous functions. Assume that (i) there exist a positive real constants $\theta_i,\lambda_i(i{=}0,1,2,3)$ such that $\forall x_i\in\mathbb{R},(i{=}1,2,3)$

$$|f(t,x_1,x_2,x_3)| \leq \theta_0+\theta_1|x_1|+\theta_2|x_2|+\theta_3|x_3|,$$

$$|g(t,x_1,x_2,x_3)| \leq \lambda_0+\lambda_1|x_1|+\lambda_2|x_2|+\lambda_3|x_3|.$$

(ii) $max\{A,B\}<1$ where

$$\begin{aligned} A &= (\Theta+\hat{\Theta})\lambda_1+(\Xi+\hat{\Xi})max(\theta_1,\theta_3),\\ B &= (\Theta+\hat{\Theta})max(\lambda_2,\lambda_3)+(\Xi+\hat{\Xi})\theta_2. \end{aligned}$$

Then there exists at least one solution for the problem (1), (2) on $[0,T]$.

**Proof.** The proof will be divided into several steps.

**Step1:** We show that $G: C_\sigma([0,T],\mathbb{R})\times C_\gamma([0,T],\mathbb{R})\to C_\sigma([0,T],\mathbb{R})\times C_\gamma([0,T],\mathbb{R})$ is completely continuous. The continuity of the operator holds true because of continuity of the function $f,g$. Let$\Omega\subset C_\sigma([0,T],\mathbb{R})\times C_\gamma([0,T],\mathbb{R})$ be bounded. Then there exist $k_f,k_g>0$ such that

$$\left|f(t,x(t),y(t),D^\gamma y(t))\right|\leq k_f,\left|g(t,x(t),D^\sigma x(t),y(t))\right|\leq k_g,\forall(x,y)\in\Omega,$$

also, from (11) it follows that

$$\begin{aligned} \| G_1(x,y) \|_\sigma &= \| G_1(x,y) \| + \| D^\sigma G_1(x,y) \|\\ &\leq \Theta\|g\|+\Xi\|f\| \\ &\leq \Theta k_g+\Xi k_f. \end{aligned} \tag{13}$$

Similarly, we obtain that

$$\begin{aligned}\| G_2(x,y) \|_\gamma &= \| G_2(x,y) \| + \| D^\gamma G_2(x,y) \| \\ &\leq \hat{\Theta}\|g\| + \hat{\Xi}\|f\| \\ &\leq \hat{\Theta}k_g + \hat{\Xi}k_f. \end{aligned} \quad (14)$$

So, from (13) and (14) we conclude that our operator G is uniformly bounded.

Now, let us show that $G$ is equicontinuous. Consider$t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then we have:

$$\begin{aligned}|G_1(x,y)(t_2) - G_1(x,y)(t_1)| \leq\ & |\Theta_1(t_1) - \Theta_1(t_2)|\left(I^{\beta-1}|g|\right)(\zeta_2) \\ & + |\Theta_2(t_1) - \Theta_2(t_2)| \int_0^T \left(I^{\beta-1}|g|\right)(s)ds \\ & + |\Xi_1(t_1) - \Xi_1(t_2)|(I^{\alpha-1}|f|)(\eta_2) \\ & + |\Xi_2(t_1) - \Xi_2(t_1)| \int_0^T (I^{\alpha-1}|f|)(s)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}||f|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} |t_2-s|^{\alpha-1}|f|ds,\end{aligned}$$

and

$$|G_1(x,y)'(t_2) - G_1(x,y)'(t_1)| \leq \frac{k_f}{\Gamma(\alpha)}[(t_2-t_1)^{\alpha-1} + |t_2{}^{\alpha-1} - t_1{}^{\alpha-1}|].$$

Thus

$$\begin{aligned}|D^\gamma G_1(x,y)(t_2) - D^\gamma G_1(x,y)(t_1)| &= \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}|G_1(x,y)'t_2) - G_1(x,y)'(t_1)|ds \\ &\leq \frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\frac{k_f}{\Gamma(\alpha)}[(t_2-t_1)^{\alpha-1} + |\, t_2{}^{\alpha-1} - t_1{}^{\alpha-1}|],\end{aligned}$$

which implies that$\| G_1(x,y)(t_2) - G_1(x,y)(t_1) \| \to 0$, independent of $(x,y) as\ t_2 \to t_1$. Similarly $\| G_2(x,y)(t_2) - G_2(x,y)(t_1) \| \to 0$, ,independent of $(x,y) t_2 \to t_1$.Thus, $G(x,y)$is equicontinuous, so by Arzela-Ascoli theorem $G\,(x,y)$ is completely continuous.

**Step 2:** Boundedness of

$$R = \left\{(x,y) \in C_\sigma([0,T],\mathbb{R}) \times C_\gamma([0,T],\mathbb{R}) : (x,y) = rG(x,y), r \in [0,1]\right\}.$$

Let

$$x(t) = rG_1(x,y)(t), y(t) = rG_2(x,y)(t),$$

then

$$|x(t)| = r|G_1(x,y)(t)|.$$

By using our assumption we can easily get

$$\begin{aligned}\|x\|_\sigma = r\| G_1(x,y) \|_\sigma &= \| G_1(x,y) \| + \| D^\sigma G_1(x,y) \| \\ &\leq \Theta\|g\| + \Xi\|f\| \\ &\leq \Theta\left(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|+\lambda_3\|y\|_\gamma\right) \\ &+\Xi(\theta_0+\theta_1\|x\|+\theta_2\|y\|+\theta_3\|x\|_\sigma),\end{aligned}$$

and in similar way, we can have

$$\begin{aligned}\|y\|_\gamma = r\| G_2(x,y) \|_\gamma &= \| G_2(x,y) \| + \| D^\gamma G_2(x,y) \| \\ &\leq \hat{\Theta}\|g\| + \hat{\Xi}\|f\| \\ &\leq \hat{\Theta}\left(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|+\lambda_3\|y\|_\gamma\right) \\ &+\hat{\Xi}(\theta_0+\theta_1\|x\|+\theta_2\|y\|+\theta_3\|x\|_\sigma).\end{aligned}$$

So

$$\|x\|_\sigma + \|y\|_\gamma \leq \left(\Theta+\hat{\Theta}\right)\lambda_0 + \left(\Xi+\hat{\Xi}\right)\theta_0 + max\{A,B\}\|(x,y)\|_{\sigma\times\gamma} \quad ,$$

where

$$\|(x,y)\|_{\sigma\times\gamma} \leq \frac{\left(\Theta+\hat{\Theta}\right)\lambda_0 + \left(\Xi+\hat{\Xi}\right)\theta_0}{1 - max\{A,B\}}.$$

As a result the set R is bounded. So, by Leray-Schauder alternative the operator $G$ has at least one fixed point, which is the solution for the problem (1) with the boundary conditions (2) on $[0,T]$. ■

## 3.3 Examples

**Example 3.3.1** Consider the following coupled system of fractional differential equation:

$$\begin{cases} {}^cD^{6/5}x(t) = \dfrac{e^{-3t}}{12\sqrt{6400+t^4}}\left(sin\left(x(t)\right)+cos\left(y(t)\right)+sin\left(D^{1/5}y(t)\right)\right) \\ {}^cD^{6/5}y(t) = \dfrac{1}{12\sqrt{3600+t^2}}\left(cos\left(x(t)\right)+\dfrac{|y(t)|}{2+|y(t)|}+\dfrac{\left|D^{1/3}x(t)\right|}{4+\left|D^{1/3}x(t)\right|}\right), t \in [0,1] \end{cases}$$

With the integral boundary conditions:

$$\int_0^1 x(s)ds = 3y(1/3), \int_0^1 x'(s)ds = -2y'(1/4),$$
$$\int_0^1 y(s)ds = x(1), \int_0^1 y'(s)ds = 2x'(1/2).$$

It is clear that

$$f(t,x,y,z) = \frac{e^{-3t}}{12\sqrt{6400+t^4}}(sinx+cosy+sinz),$$
$$g(t,x,y,z) = \frac{1}{12\sqrt{3600+t^2}}\left(cosx+\frac{|y|}{2+|y|}+\frac{|z|}{4+|z|}\right),$$

are jointly continuous and satisfy the Lipschitz condition with $l_f = 1/320, l_g = 1/240$.

$$T = 1, \rho_1 = 3, \zeta_1 = 1/3, \rho_2 = -2, \zeta_2 = 1/4, \mu_1 = 1, \eta_1 = 1, \mu_2 = 2, \eta_2 = 1/2, \gamma = 1/5, \sigma = 1/3,$$

and $\Theta, \Xi, \hat{\Theta}, \hat{\Xi}$ can be chosen as follows:

$$\Theta = 3.4959, \Xi = 6.4324, \hat{\Theta} = 5.1602, \hat{\Xi} = 4.6058.$$

Then we obtain:

$$\begin{aligned} 1-2\left(\Theta l_g+\Xi l_f\right) &= 1-0.0693 = 0.9307 > 0, \\ 1-2\left(\hat{\Theta} l_g+\hat{\Xi} l_f\right) &= 1-0.0718 = 0.9282 > 0. \end{aligned}$$

Obviously, all the condition of Theorem 3.2.1 are satisfied so there exists unique solution for this problem.

**Example 3.3.2** Consider the following system:

$$\begin{cases} {}^cD^{6/5}x(t) = \dfrac{1}{40+t^3} + \dfrac{y(t)}{115\left(1+x^2(t)\right)} + \dfrac{1}{3(100+t^2)} sin\left(D^{1/5}y(t)\right) + \dfrac{1}{3\sqrt{3600+t}} e^{-3t} sin\left(x(t)\right) \\ {}^cD^{6/5}y(t) = \dfrac{1}{\sqrt{9+t^2}} sint + \dfrac{1}{180} e^{-2t} sin\left(y(t)\right) + \dfrac{1}{150} x(t) + \dfrac{1}{3(180+t)} D^{1/3}x(t), t \in [0,1], \end{cases}$$

with the following boundary conditions:

$$\int_0^1 x\,(s)ds = 3y(1/3), \int_0^1 x'\,(s)ds = -2y'(1/4),$$
$$\int_0^1 y\,(s)ds = x(1), \int_0^1 y'\,(s)ds = 2x'(1/2),$$

$$T = 1, \rho_1 = 3, \zeta_1 = 1/3, \rho_2 = -2, \zeta_2 = 1/4, \mu_1 = 1, \eta_1 = 1, \mu_2 = 2, \eta_2 = 1/2, \gamma = 1/5, \sigma = 1/3,$$
$$\Theta = 3.4959, \Xi = 6.4324, \hat{\Theta} = 5.1602, \hat{\Xi} = 4.6058.$$

It is clear that:

$$|f(t,x_1,x_2,x_3)| \leq \frac{1}{40} + \frac{1}{180}|x_1| + \frac{1}{115}|x_2| + \frac{1}{300}|x_3|,$$
$$|g(t,x_1,x_2,x_3)| \leq \frac{1}{3} + \frac{1}{150}|x_1| + \frac{1}{180}|x_2| + \frac{1}{540}|x_3|.$$

Thus

$$\theta_0 = 1/40, \theta_1 = 1/180, \theta_2 = 1/115, \theta_3 = 1/300,$$
$$\lambda_0 = 1/3, \lambda_1 = 1/150, \lambda_2 = 1/180, \lambda_3 = 1/540.$$

We found A and B such that: $A = 0.1190, B = 0.1444$and that $max\{A,B\} = 0.1444 < 1$. Since the conditions of Theorem 3.2.2 is achieved. So, there exists a solution for this problem.

## 3.4 Conclusion

We studied the existence of solutions for a coupled system of fractional differential equations with integral boundary conditions. The first result was based on the Banach fixed point theorem. Secondly, by using Leray–Schauder's alternative, we proved the existence of solutions for Caputo fractional equations with integral boundary conditions. Finally, our results are supported by examples.

# Chapter 4

# ON THE PARAMETRIZATION OF CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATION WITH TWO POINT NONLINEAR BOUNDARY CONDITIONS

In this chapter, we apply the technique proposed in [2] for investigation and approximation of solutions of Caputo type fractional differential equation with nonlinear boundary conditions. By using an appropriate parametrization technique, nonlinear boundary conditions are transformed to linear boundary conditions by using vector parameters.

To study the transformed problem, we construct a numerical-analytic scheme which is successful in relation to different types two-point and multipoint linear boundary condition and nonlinear boundary conditions. According to the main idea of the numerical analytic technique, certain type of successive approximations constructed analytically. We give sufficient conditions or the uniform convergence of the successive approximations. Also, it is indicated that these successive approximations uniformly converge to a parametrized limit function and state the relationship of this limit function and exact solution. Finally, some results are illustrated by using defined conditions and techniques.

## 4.1 Identification and Parametrization of the Problem

Let us consider Caputo type fractional differential equation with nonlinear boundary conditions:

$$ {}^{c}D^{\alpha}x(t) = h\big(t, x(t)\big), t \in [0, T], \tag{1}$$

$$ Ax(0) + Bx(T) + g(x(0), x(T)) = d, d \in \mathbb{R}^{n}, \tag{2}$$

Where ${}^{c}D^{\alpha}$ is the Caputo derivative of order $\alpha \in (0,1]$, the functions $h: [0, T] \times D \to \mathbb{R}^{n}, and\ g: D \times D \to \mathbb{R}^{n}$ are continuous and the set $D \subset \mathbb{R}^{n}$ is closed and bounded domain. $A$ and $B$ are $n \times n$ matrices, $det\ B \neq 0$ and $d$ is a $n$ −dimensional vector.

By using appropriate parametrization technique [38], the given problem (1), (2) is reduced to certain parametrized two-point boundary conditions. To see that, we apply "freezing" technique which is similar to [38]: That means, we introduce the vectors of parameters

$$ \omega := x(0) = (\omega_1, \omega_2 \ldots \omega_n)^{T}, $$

$$ \varphi := x(T) = (\varphi_1, \varphi_2, \ldots, \varphi_n)^{T}, \tag{3}$$

$$ d(\omega, \varphi) := d - g(\omega, \varphi). \tag{4}$$

and by using (4) , the problem (1) ;(2) can be rewritten as follows:

$$ {}^{c}D^{\alpha}x(t) = h(t, x(t)) $$

$$ Ax(0) + Bx(T) = d(\omega, \varphi). \tag{5}$$

Parametrized boundary value problem (5) will be studied under the following condition:

A) The function $h: [0, T] \times \mathbb{R}^{n} \to \mathbb{R}^{n}$ satisfies the Lipschitz condition:

$$ \|h(t, u) - h(t, v)\| \leq L\|u - v\| \tag{6}$$

For all $t \in [0, T], u, v \in D$,where $L$ is a positive constant.

B) Let

$$k(t) = ((2t^{\alpha})/(\Gamma(\alpha+1)))(1-(t/T))^{\alpha}.$$

Then , $k(t)$ takes its maximum value at $t = \left(\frac{T}{2}\right)$ and

$$\|k\|_{\infty} = ((T^{\alpha})/(2^{2\alpha-1}\Gamma(\alpha+1))).$$

Define,

$$\|h\|_{\infty} = max_{(t,x)\in[0,T]\times D}\|h(t,x)\|$$

and a vector function $\delta: D \times D \to \mathbb{R}^n$ is

$$\delta(\omega,\varphi) := \|\kappa\|_{\infty}\|h\|_{\infty} + \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|,$$

where $I_n$ is the $n \times n$ identity matrix and $\omega, \varphi \in D$ of the form (3).

$\delta$ is the radius of a neighborhood $C$ of the point $\omega \in D$ is defined as follows:

$B(\omega, \delta(\omega,\varphi)) := \{x \in \mathbb{R}^n : \|x-\omega\| \leq \delta(\omega,\varphi)$ for all $\varphi \in D \subset \mathbb{R}^n\}$

the set

$$D_{\delta} := \left\{\omega \in D : B(\omega, \delta(\omega,\varphi)) \subset D \text{ for all } \varphi \in D\right\}$$

is nonempty

C) $L\|k\|_{\infty} < 1$,where $L$ is a positive constant and satisfies the inequality (6) . For studying of the solution of the parametrized boundary value problem (5), we consider the sequence of functions $\{x_m\}$ which is defined by the iterative formula as follows:

$$x_m(t,\omega,\varphi) = \omega + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} h(s, x_{m-1}(s,\omega,\varphi))ds - \left(\frac{t}{T}\right)^{\alpha}\int_0^T (T-s)^{\alpha-1} h(s, x_{m-1}(s,\omega,\varphi))ds\right] + \left(\frac{t}{T}\right)^{\alpha}[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega] \quad (7)$$

for $t \in [0,T], m = 1,2,3\ldots$

where

$$x_0(t,\omega,\varphi) = (x_{01}, x_{02}, .., x_{0n})^T = \omega \in D_{\delta}$$

$$x_m(t,\omega,\varphi) = (x_{m,1}(t,z,\varphi), x_{m,2}(t,z,\varphi)\ldots, x_{m,n}(t,z,\varphi))^T$$

and $\omega, \varphi$ are considered as parameters.

In addition, it is easily to see that the sequence of functions $x_m$ are satisfied linear parmetrized boundary conditions (5) for all $m \geq 1, \omega \in D_\delta, \varphi \in D$. Now, we prove that the sequence of the functions (7) is uniformly convergent and show the relationship between this sequence of the functions and the limit function.

**Theorem 4.1.1** Assume that the parametrized boundary value problem (5) satisfy the conditions (A) ,(B) and (C) .

Then for all $\varphi \in D$ and $\omega \in D_\delta$, the following assertions are true:

1. All functions of sequence (7) are continuous and satisfy the parametrized boundary conditions (5)

$$Ax_m(0,\omega,\varphi) + Bx_m(T,\omega,\varphi) = d(\omega,\varphi), m = 1,2,3\ldots \quad (8)$$

2. The sequence of functions (7) converges uniformly in $t \in [0,T]$ $as$ $m \to \infty$ to the limit function.

$$x^*(t,\omega,\varphi) = lim_{m\to\infty} x_m(t,\omega,\varphi). \quad (9)$$

3. The limit function $x^*$satisfies the initial condition

$$x^*(0,\omega,\varphi) = \omega$$

and

$$Ax^*(0,\omega,\varphi) + Bx^*(T,\omega,\varphi) = d(\omega,\varphi)$$

4. The limit function (9) is the unique continuous solution of the integral equation

$$x(t) := \omega + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} h\big(s, x(s)\big)ds - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h\big(s, x(s)\big)ds\right]$$

$$+ \left(\frac{t}{T}\right)^{\alpha} [B^{-1}d(\omega,\varphi) - (B^{-1}A$$

$$+ I_n)\omega] , \tag{10}$$

or $x(t)$ is the unique solution on $[0, T]$ of the Cauchy problem:

$$^{c}D^{\alpha}x(t) = h\big(t, x(t)\big) + {}^{\alpha}\Omega(\omega,\varphi), \quad x(0) = \omega, \tag{11}$$

where

$$^{\alpha}\Omega(\omega,\varphi) = \frac{-\alpha}{T^{\alpha}}\left[\int_0^T (T-s)^{\alpha-1} h(s, x^*(t,\omega,\varphi))ds - \Gamma(\alpha)[B^{-1}d(\omega,\varphi)\right.$$

$$\left. - (B^{-1}A + I_n)\omega]\right] , \tag{12}$$

5. Error estimation:

$$\|x^*(t,\omega,\varphi) - x_m(t,\omega,\varphi)\|$$

$$\leq (L\|\kappa\|_\infty)^m(\|h\|_\infty\|\kappa\|_\infty + [B^{-1}d(\omega,\varphi) - (B^{-1}A$$

$$+ I_n)\omega])\frac{1}{1 - L\|\kappa\|_\infty}.$$

**Proof.**

1. Continuity of the sequence $\{x_m\}$ defined by (7) follows directly from the construction of sequence and by direct computation, it is easy to show that the sequence $x_m$satisfies the parametrized boundary conditions (5) .

2. We prove that the sequence of functions is a Cauchy sequence in the Banach space $C([a, b], \mathbb{R}^n)$.

At first, we show that $x_m(t,\omega,\varphi) \in D$.for all $(t,\omega,\varphi) \in [0,T] \times D_\delta \times D, m \in \mathbb{N}$. We get the following equation from (7) for $m = 1$:

$$x_1(t,\omega,\varphi) = \omega + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\, h(s, x_0(s,\omega,\varphi))ds \right.$$

$$\left. - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1}\, h(s, x_0(s,\omega,\varphi))ds \right] + \left(\frac{t}{T}\right)^{\alpha} [B^{-1}d(\omega,\varphi)$$

$$- (B^{-1}A + I_n)\omega]\,, \tag{13}$$

Moreover, it can be written as follows:

$$\|x_1(t,\omega,\varphi) - \omega\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^t \left|(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1}\right| |h(s,\omega)|\, ds \right.$$

$$\left. - \left(\frac{t}{T}\right)^{\alpha} \int_t^T \left|\left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1}\right| |h(s,\omega)| ds \right]$$

$$+ \left\|\left(\frac{t}{T}\right)^{\alpha} [B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\right\| :$$

$$= I_1 + I_2 + I_3 \tag{14}$$

1. we estimate $I_1$ :

$$I_1 \leq \frac{1}{\Gamma(\alpha)}\left[\int_0^t \left|(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1}\right| \|h\|_\infty\, ds \right.$$

$$= \left(\frac{t}{T}\right)^{\alpha} \frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)} \|h\|_\infty, \tag{15}$$

where the expression under the absolute value is nonnegative

$$\frac{1}{(t-s)^{1-\alpha}} \geq \left(\frac{t}{T}\right)^{\alpha} \frac{1}{(t-s)^{1-\alpha}} \geq \left(\frac{t}{T}\right)^{\alpha} \frac{1}{(T-s)^{1-\alpha}}.$$

Then, we estimate $I_2$ and $I_3$:

$$I_2 \leq \frac{1}{\Gamma(\alpha)}\left[\int_t^T \left|\left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1}\right| \|h(s,\omega)\|\, ds \right.$$

$$= \left(\frac{t}{T}\right)^{\alpha} \frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)} \|h\|_\infty, \tag{16}$$

and

$$I_3 = \left(\frac{t}{T}\right)^{\alpha} \|B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega\| \tag{17}$$

Substituting (15) ,(16) and (17) into the relation (14) and we obtain the following result:

$$\begin{aligned}
&\|x_1(t,\omega,\varphi) - \omega\| \\
&\qquad \le \frac{2t^{\alpha}}{\Gamma(\alpha+1)}\left(1 - \frac{t}{T}\right)^{\alpha} \|h\|_{\infty} + \left(\frac{t}{T}\right)^{\alpha} \|B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega\| \\
&\qquad \le \frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}\|h\|_{\infty} + \|B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega\| \\
&\qquad = \|\kappa\|_{\infty}\|h\|_{\infty} + |\|B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega\|| \\
&\qquad = \delta(\omega,\varphi). \qquad (18)
\end{aligned}$$

Thus

$$x_1(t,\omega,\varphi) \in D$$

For

$$(t,\omega,\varphi) \in [0,T] \times D_{\delta} \times D.$$

By induction, it can be shown that all functions $x_m(t,\omega,\varphi)$ defined by (7) also belong to the set $D$ for all $m = 1,2,3,\ldots t \in [0,T], \omega \in D_{\delta}, \varphi \in D$.

2. we consider the difference

$$\begin{aligned}
&x_{m+1}(t,\omega,\varphi) - x_m(t,\omega,\varphi) \\
&\qquad = \frac{1}{\Gamma(\alpha)}\Bigg(\int_0^t (t-s)^{\alpha-1}[h(s,x_m(s,\omega,\varphi) - h(s,x_{m-1}(s,\omega,\varphi)]ds \\
&\qquad - \int_0^T \left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1} \\
&\qquad \times [h(s,x_m(s,\omega,\varphi) - h(s,x_{m-1}(s,\omega,\varphi)]ds\Bigg) \qquad (19)
\end{aligned}$$

for $m = 1,2,3,\ldots$.

Then the difference (19) is denoted by $r_m(t,\omega,\varphi)$ as follows:

$$r_m(t,\omega,\varphi) := \|x_m(t,\omega,\varphi) - x_{m-1}(t,\omega,\varphi)\|, \text{ for all } m = ,2,3,\dots \quad (20)$$

and the inequality (18) can be rewritten in the following form

$$r_1(t,\omega,\varphi) = \|x_1(t,\omega,\varphi) - \omega\|$$

$$\leq \|\kappa\|_\infty \|h\|_\infty$$

$$+ \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|. \quad (21)$$

Taking into account the Lipshitz condition $(A)$ and the relation (21) for $m = 2$,we get

$$r_2(t,\omega,\varphi) \leq \frac{L}{\Gamma(\alpha)}\left(\int_0^t \left[(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1}\right]\right.$$

$$\left. + \int_t^T \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1}\right) |x_1(t,\omega,\varphi) - \omega|$$

$$= \frac{L}{\Gamma(\alpha)}\left(\int_0^t \left[(t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1}\right]\right.$$

$$\left. + \int_t^T \left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1}\right) r_1(t,\omega,\varphi)$$

$$\leq \frac{2Lt^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{t}{T}\right)^\alpha [\|\kappa\|_\infty \|h\|_\infty$$

$$+ \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|]$$

$$\leq \mathrm{L}\|\kappa\|_\infty^2 \|h\|_\infty + \mathrm{L}\|\kappa\|_\infty |[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]|$$

Hence

$$r_2(t,\omega,\varphi) \leq L\|\kappa\|_\infty [\|\kappa\|_\infty \|h\|_\infty$$

$$+ \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|]. \quad (22)$$

Therefore, by using the mathematical induction we obtain the following equation:

$$r_{m+1}(t,\omega,\varphi) \leq (L\|\kappa\|_\infty)^m [\|\kappa\|_\infty \|h\|_\infty + \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|],$$

$$m = 0,1,2.. \quad (23)$$

In view of (23) and by using triangular inequality we get

$$\left\|x_{m+j}(t,\omega,\varphi) - x_m(t,\omega,\varphi)\right\|$$

$$\leq \left\|x_{m+j}(t,\omega,\varphi) - x_{m+j-1}(t,\omega,\varphi)\right\| + \left\|(x_{m+j-1}(t,\omega,\varphi) - x_{m+j-2}t,\omega,\varphi)\right\|$$

$+\ldots+\|x_{m+1}(t,\omega,\varphi)-x_m(t,\omega,\varphi)\|$

$= r_{m+j}(t,\omega,\varphi)+r_{m+j-1}(t,\omega,\varphi)+\ldots+r_{m+1}(t,\omega,\varphi)$

$=\sum_{i=1}^{j} r_{m+i}(t,\omega,\varphi)$

$$\begin{aligned}&\leq (L\|\kappa\|_\infty)^m(\|h\|_\infty\|\kappa\|_\infty\\ &+\|[B^{-1}d(\omega,\varphi)-(B^{-1}A\\ &+I_n)\omega]\|)\sum_{i=1}^{\infty} L^{i-1}\|\kappa\|_\infty^{i-1}\end{aligned} \tag{24}$$

From the assumption $(C)$, it follows that

$$lim_{m\to 0}(L\|\kappa\|)^m=0.$$

Hence, by (24), $\{x_m\}$ is Cauchy sequence and uniformly converges on $[0,T]\times D_\delta\times D$ to a certain limit $x^*$.

3. Taking limit in (8) as $m\to\infty$,we see that $x^*$satisfies the boundary conditions directly.

4. By using contradiction, the uniqueness of solution is shown. So, assume that there is two limit functions such as $x_1{}^*(t,\omega,\varphi)$ and $x_2{}^*(t,\omega,\varphi)$ .Then, estimating the difference between $x_1{}^*$and $x_2{}^*$

$$\|x_1{}^*(t,\omega,\varphi)-x_2{}^*(t,\omega,\varphi)\|$$

$$\leq \frac{L}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}|x_1{}^*(s,\omega,\varphi)-x_2{}^*(s,\omega,\varphi)|ds\right.$$

$$\left.+\int_0^T (T-s)^{\alpha-1}|x_1{}^*(s,\omega,\varphi)-x_2{}^*(s,\omega,\varphi)|ds\right]$$

$$\leq L\|\kappa\|_\infty\|x_1{}^*-x_2{}^*\|_\infty.$$

Thus

$$\|x_1{}^*-x_2{}^*\|_\infty\leq L\|\kappa\|_\infty\|x_1{}^*-x_2{}^*\|_\infty$$

It can be written

$$(1-L\|\kappa\|_\infty)\|x_1{}^*-x_2{}^*\|_\infty\leq 0$$

So,

$$\|x_1{}^* - x_2{}^*\| = 0 \Rightarrow x_1{}^* - x_2{}^* = 0 \Rightarrow x_1{}^* = x_2{}^*.$$

5. Passing to $j \to \infty$ in (24) we get

$$\|x_1^*(t,\omega,\varphi) - x_2^*(t,\omega,\varphi)\|$$

$$\leq (L\|\kappa\|_\infty)^m(\|h\|_\infty\|\kappa\|_\infty$$

$$+ \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|) \sum_{i=1}^{\infty} L^{i-1}L^{i-1}\|\kappa\|_\infty^{i-1}$$

$$= (L\|\kappa\|_\infty)^m(\|h\|_\infty\|\kappa\|_\infty$$

$$+ \|[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\|) \frac{1}{1 - L\|\kappa\|_\infty}$$

**Remark 4.1.1** If $A = I_n, B = -I_n, g(x(0), x(T)) = 0, d = 0$,boundary condition (2) becomes $x(0) = x(T)$. Note that, this problem was studied in [2].

## 4.2 Relationship between Limit Function and the Solution of the Nonlinear Boundary-Value Problem

We consider the following equation

$$^cD^\alpha x(t) = h(t,x) + \psi, t \in [0,T] \tag{25}$$

and

$$x(0) = \omega, \tag{26}$$

where $\psi = col(\psi_1 \ldots \psi_n)$is the parameter of control.

**Theorem 4.2.1** Let $\omega \in D_\delta, \varphi \in D$ be arbitrarily defined vectors. Suppose that all conditions of theorem 4.1.1 are satisfied. The solution $x = x(t,\omega,\varphi,\psi)$of the initial-value problem (25),(26) satisfies the boundary conditions (5) if and only if $x = (t,\omega,\varphi,\psi)$ coincides with the limit function $x^* = x^*(t,\omega,\varphi,\psi)$ of sequence (7). Moreover,

$$\psi = \psi_{\omega,\varphi} = \frac{-\alpha}{T^{\alpha}}\left[\int_0^T (T-s)^{\alpha-1} h(s, x^*(t,\omega,\varphi))ds - \Gamma(\alpha)[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\right]. \quad (27)$$

**Proof.** Sufficiency: The proof is similar to the proof of theorem in [7].

Necessity: Fixed an arbitrary value $\bar{\psi} \in \mathbb{R}^n$ and assume that the problem

$$^{c}D^{\alpha}x(t) = h(t,x) + \bar{\psi}, t \in [0,T]$$

with initial condition $x(0) = \omega$ $(26)$ has the solution $\bar{x} = \bar{x}(t)$ satisfying the two-point boundary conditions (5) :

$$A\bar{x}(0) + B\bar{x}(T) = d(\omega,\varphi).$$

Then $\bar{x}$ is a solution of the integral equation

$$\bar{x}(t) = \omega + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h\big(s,\bar{x}(s)\big)ds + \frac{t^{\alpha}\,\bar{\psi}}{\Gamma(\alpha+1)}. \quad (28)$$

When $t = T$ in (28) we get the following equation

$$\bar{x}(T) = \omega + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} h\big(s,\bar{x}(s)\big)ds + \frac{T^{\alpha}\,\bar{\psi}}{\Gamma(\alpha+1)}. \quad (29)$$

Moreover,

$$\bar{x}(0) = \omega$$

and

$$\bar{x}(T) = B^{-1}[d(\omega,\varphi) - A\omega]. \quad (30)$$

By using (29) and (30) we obtain

$$\bar{\psi} = \frac{-\alpha}{T^{\alpha}}\left[\int_0^T (T-s)^{\alpha-1} h\big(s,\bar{x}(s)\big)ds + \Gamma(\alpha)[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\right]. \quad (31)$$

Then, substituting (31) into the (28)

$$\bar{x}(T) := \omega + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} h\big(s, \bar{x}(s)\big) ds - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} h\big(s, \bar{x}(s)\big) ds\right]$$

$$+ \left(\frac{t}{T}\right)^{\alpha} [B^{-1} d(\omega, \varphi) - (B^{-1} A + I_n)\omega],$$

Moreover, the limit function $x^*$ is a solution of the (25), (26) for $\psi = \psi_{\omega,\varphi}$of the form (27) and satisfies the boundary conditions (5).

$$x^*(t, \omega, \varphi, \psi) = \omega + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h\big(s, x^*(t, \omega, \varphi, \psi)\big) ds + \frac{t^{\alpha}\psi}{\Gamma(\alpha+1)}. \quad (32)$$

By using the same steps before, we get

$$x^*(T, \omega, \varphi, \psi) = \omega + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h\big(s, x^*(T, \omega, \varphi, \psi)\big) ds + \frac{T^{\alpha}\psi}{\Gamma(\alpha+1)}. \quad (33)$$

The limit function $x^*$ satisfies the following boundary conditions.

$$A x^*(0, \omega, \varphi, \psi) + B x^*(T, \omega, \varphi, \psi) = d(\omega, \varphi)$$

with the boundary conditions

$$x^*(0, \omega, \varphi, \psi) = \omega$$

and

$$x^*(T, \omega, \varphi, \psi) = B^{-1}[d(\omega, \varphi) - A\omega]. \quad (34)$$

By using relations (33) and (34) we get

$$\psi_{\omega,\varphi} = \frac{-\alpha}{T^{\alpha}}\left[\int_0^T (T-s)^{\alpha-1})^{\alpha-1} h(s, x^*(s, \omega, \varphi, \psi)) ds + \Gamma(\alpha)[B^{-1} d(\omega, \varphi) - (B^{-1} A\right.$$

$$\left. + I_n)\omega]\right]. \quad (35)$$

After substituting relation (35) into (32) ,we have

$$x^*(t,\omega,\varphi,\psi):=\omega$$
$$+\frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}h(s,x^*(s,\omega,\varphi,\psi)ds-\left(\frac{t}{T}\right)^{\alpha}\int_0^T (T\right.$$
$$\left.-s)^{\alpha-1}h(s,x^*(s,\omega,\varphi,\psi)ds\right]+\left(\frac{t}{T}\right)^{\alpha}[B^{-1}d(\omega,\varphi)-(B^{-1}A$$
$$+I_n)\omega].$$

Taking the difference between $\bar{x}$ and $x^*$ , we get

$$x^*(t,\omega,\varphi,\psi)-\bar{x}(t)$$
$$=\frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}[h(s,x^*(s,\omega,\varphi,\psi))-h(s,\bar{x}(s))]\mathrm{ds}\right.$$
$$\left.-\left(\frac{t}{T}\right)^{\alpha}\int_0^T (T-s)^{\alpha-1}[h(s,x^*(s,\omega,\varphi,\psi))-h(s,\bar{x}(s))]ds\right],$$

then, by using Lipshitz condition the difference between $x^*$ and $\bar{x}$ will be the following integral inequalities

$$\|x^*(s,\omega,\varphi,\psi)-\bar{x}(t)\|$$
$$\leq\frac{L}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\|x^*(s,\omega,\varphi,\psi)-\bar{x}(s)\|\mathrm{ds}\right.$$
$$\left.+\int_0^T (T-s)^{\alpha-1}\|x^*(s,\omega,\varphi,\psi)-\bar{x}(s)\|ds\right]\leq L\|\kappa\|_\infty\|x^*-\bar{x}\|_\infty$$

Thus,

$$\|x^*-\bar{x}\|_\infty\leq L\|\kappa\|_\infty\|x^*-\bar{x}\|_\infty$$

It can be written

$$(1-L\|\kappa\|_\infty)\|x^*-\bar{x}\|_\infty\leq 0$$

So,

$$\|x^*-\bar{x}\|_\infty=0\Rightarrow x^*-\bar{x}=0\Rightarrow x^*=\bar{x}$$

The theorem is proved.

**Theorem 4.2.2** Assume that the conditions $(A)$ and $(B)$ are satisfied for the Caputo type fractional differential equation (1) with nonlinear boundary conditions (2).Then, $(x^*(\cdot,\omega^*,\varphi^*),\varphi^*)$ is a solution of the parametrized boundary-value problem (1) ,(5) if and only if $\omega^* = (\omega_1^*,\omega_2^*,\dots,\omega_n^*)$and $\varphi^* = ({\varphi_1}^*,{\varphi_2}^*,\dots,\varphi_n^*)$ satisfy the system of determining algebraic or transcendental equations

$$\Omega(\omega,\varphi) = \frac{-\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1} h\big(s,x^*(s,\omega,\varphi)\big)ds - \Gamma(\alpha)[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\right] = 0, \tag{36}$$

$$x^*(T,\omega,\varphi) = \varphi. \tag{37}$$

**Proof.** The result is obtained from theorem 4.1.1 and by observing that the differential equation  coincides with (1) if and only if the couple $(\omega^*,\varphi^*)$ satisfies the equation

$$(\omega^*,\varphi^*) = 0.$$

The following assertion indicates the determining system of equation (36) ,37)shows all possible solution of the Caputo type differential equation (1) with nonlinear boundary conditions (2).

**Remark 4.2.1** Assume that all conditions of Theorem 4.1.1 are satisfied and there exist vectors $\omega \in D_\delta$ and $\varphi \in D$satisfying the system of determining equations (36), (37). Then the Caputo type differential equation (1) with nonlinear boundary conditions (2) have the solution $x(\cdot)$ such that

$$x(0) = \omega,$$

$$x(T) = \varphi.$$

Also, this solution has the following form

$$x(t) = x^*(t,\omega,\varphi), t \in [0,T], \tag{38}$$

where $x^*$ is the limit function of sequence (7) .Conversely, if the Caputo type differential equation (1) with nonlinear boundary conditions (2) has a solution $x(.)$,then this solution necessarily has the form (38) and the system of determining equations (36) ,(37) is satisfied for

$$\omega = x(0),$$

$$\varphi = x(T).$$

**Remark 4.2.2** For some $m \geq 1$,a function $\Omega_m: D \times D \to \mathbb{R}^n$is defined by the formula

$$\Omega_m(\omega,\varphi) := \frac{-\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1} h(s, x_m(t,\omega,\varphi))ds - \Gamma(\alpha)[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\right],$$

where $\omega$ and $\varphi$ are given by (3) .To study the solvability of the parametrized boundary-value problem (5) ,we consider the approximate determining system of algebraic equations of the form

$$\Omega_m(\omega,\varphi) := \frac{-\alpha}{T^\alpha}\left[\int_0^T (T-s)^{\alpha-1})^{\alpha-1} h(s, x_m(t,\omega,\varphi))ds - \Gamma(\alpha)[B^{-1}d(\omega,\varphi) - (B^{-1}A + I_n)\omega]\right] = 0, \tag{39}$$

$$x_m(T,\omega,\varphi) = \varphi, \tag{40}$$

where $x_m$ is the vector function specified by the recurrence relation (7).

## 4.3 Example

Considering a system of Caputo type fractional differential equation

$${}^cD^\alpha x_1 = x_2 = f_1(t, x_1, x_2)$$

$${}^cD^\alpha x_2 = -\left(\frac{1}{2}\right)x_2{}^2 - \left(\frac{1}{2}\right)x_1 + \left(\frac{t}{8}\right)x_2 + \left(\frac{t^{1-\alpha}}{4\Gamma(2-\alpha)}\right) + \left(\frac{2t^{\alpha+1}+1}{16\Gamma(2+\alpha)}\right) = f_2(t, x_1, x_2) \tag{41}$$

with nonlinear boundary conditions

$$x_1(0) + x_1\left(\left(\frac{1}{2}\right)\right) - \left[x_2\left(\left(\frac{1}{2}\right)\right)\right]^2 = \left(\frac{2^{\alpha+1}+1}{2^{\alpha}8\Gamma(\alpha+2)}\right) - \left(\frac{1}{64}\right),$$

$$x_2(0) + x_1\left(\left(\frac{1}{2}\right)\right) - x_2\left(\left(\frac{1}{2}\right)\right) = \left(\frac{2^{\alpha}+1}{2^{\alpha}8\Gamma(\alpha+2)}\right) - \left(\frac{1}{8}\right) \tag{42}$$

The exact solution of the Caputo type fractional differential equation (41) with nonlinear boundary conditions (42) is

$$x_1{}^* = \frac{2\mathrm{t}^{\alpha+1}+1}{8\Gamma(\alpha+2)},$$

$$x_2{}^* = \frac{\mathrm{t}}{4}.$$

Then, the nonlinear boundary conditions can be represented in the matrix-vector form as follows

$$Ax(0) + Bx\left(\frac{1}{2}\right) + g\left(x(0), x\left(\frac{1}{2}\right)\right) = d, \tag{43}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

$$d = \begin{pmatrix} \dfrac{2^{\alpha+1}+1}{2^{\alpha}8\Gamma(\alpha+2)} - \dfrac{1}{64} \\ \dfrac{2^{\alpha}+1}{2^{\alpha}8\Gamma(\alpha+2)} - \dfrac{1}{8} \end{pmatrix}, g = \left(x(0), x\left(\frac{1}{2}\right)\right)^T = \begin{pmatrix} -\left[x_2\left(\frac{1}{2}\right)\right]^2 \\ 0 \end{pmatrix}.$$

Then, new parameters are introduced as follows

$$x(0) = \omega \coloneqq \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

$$x\left(\frac{1}{2}\right) = \varphi \coloneqq \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{44}$$

In view of parametrization (44), the boundary conditions (42) can be rewritten in the form of linear two-point parametrized boundary conditions

$$Ax(0) + Bx\left(\frac{1}{2}\right) = d - g(\omega, \varphi).$$

Moreover,

$$d(\omega, \varphi) = d - g(\omega, \varphi) = \begin{pmatrix} \frac{2^{\alpha+1}+1}{2^\alpha 8\Gamma(\alpha+2)} - \frac{1}{64} + \varphi_2^2 \\ \frac{2^\alpha + 1}{2^\alpha 8\Gamma(\alpha+2)} - \frac{1}{8} \end{pmatrix}. \quad (45)$$

At last, by using (45), the boundary conditions (42) can be rewritten in the form

$$Ax(0) + Bx\left(\frac{1}{2}\right) = d(\omega, \varphi). \quad (46)$$

Now, we check the conditions of convergence of successive approximations such as $(A)$ and $(B)$. At first, the domain $D$is defined as

$$D = \left\{(x_1, x_2) \colon |x_1| \leq 1, |x_2| \leq \frac{3}{4}\right\}, t \in [0,1].$$

Then, the first condition $A$ which is about the Lipschitz condition is satisfied as follows:

$$L = max(0,1,1/2,7/8)$$

Therefore,

$$L = 1.$$

For the second condition, at first, we find

$$\|\kappa\| = 0.7979$$

$$\|h\| = \begin{pmatrix} 0.7500 \\ 2.1731 \end{pmatrix}.$$

Then

$$\delta(\omega, \varphi) \colon= \|\kappa\| \|h\| + |[B^{-1} d(\omega, \varphi) - (B^{-1}A + I_n)\omega]|$$

$$\leq \begin{pmatrix} 1.1599 + \varphi_2^2 - 2\omega_1 \\ 0.2815 + \varphi_2^2 - \omega_1 \end{pmatrix}.$$

So, the condition of nonemptiness of the set $D_\delta$ is satisfied.

For the problem of Caputo type fractional differential equation (41) with nonlinear boundary conditions (46) ,the successive approximations (7) have the form

$$x_{m,1}(t,\omega,\varphi) := \omega_1$$

$$+\frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f_1(s, x_{m-1,1}(s,\omega,\varphi), x_{m-1,1}(s,\omega,\varphi))ds\right.$$

$$\left.-\left(\frac{t}{T}\right)^{\alpha}\int_0^T (T-s)^{\alpha-1} f_1(s, x_{m-1,1}(s,\omega,\varphi), x_{m-1,1}(s,\omega,\varphi))ds\right]$$

$$+\left(\frac{t}{T}\right)^{\alpha}\left[\frac{2^{\alpha+1}+1}{2^{\alpha}8\Gamma(\alpha+2)} - \frac{1}{64} + \varphi_2^2 - 2\omega_1\right],$$

$$x_{m,2}(t,\omega,\varphi) := \omega_2$$

$$+\frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f_2(s, x_{m-1,1}(s,\omega,\varphi), x_{m-1,1}(s,\omega,\varphi))ds\right.$$

$$\left.-\left(\frac{t}{T}\right)^{\alpha}\int_0^T (T-s)^{\alpha-1} f_2(s, x_{m-1,1}(s,\omega,\varphi), x_{m-1,1}(s,\omega,\varphi))ds\right]$$

$$+\left(\frac{t}{T}\right)^{\alpha}\left[\frac{1}{8\Gamma(\alpha+2)} + \frac{7}{64} + \varphi_2^2 - \omega_1\right],$$

Then, we choose randomly $\alpha = 0.9$ and next chapter will discuss the algorithm and result by using Mathematica.

# Chapter 5

# ALGHORITHM AND RESULT FOR COUPLED NONLINEAR FRACTIONAL DIFFERENTIAL SYSTEM WITH INTEGRAL BOUNDARY CONDITIONS

In this chapter, we will apply the theoretical method that was developed in the previous sections and solve a coupled nonlinear fractional differential system with integral boundary conditions. Here we must state that in general solving nonlinear fractional-order differential equations involves extremely heavy computations and hence optimization and efficiency of employed numerical methods are of critical importance.

Besides that, numerical root approximations that are used in this chapter require solving highly complex non-analytical integrals that are recursively determined, which drastically increases the computational costs of the method. As such, we have employed Deep Reinforcement Learning (DRL) techniques to optimize and boost the computational efficiency of our numerical method. Note that the employed DRL technique does not influence the stability, convergence, or accuracy of the method, but solely increases the computational efficiency by optimizing intermediary numerical approximations involved in the solution. All numerical calculations in this chapter have been done in Mathematica and the DRL optimization algorithm has been separately coded in C++ and used with Mathematica program conjunctively.

As mentioned above, root approximations play a critical role in our numerical method and a key focus of our DRL optimization algorithm, since they involve recursive non-analytical integrals that are extremely costly. The root approximation algorithm employs the Newton-Raphson method.

## 5.1 Deep Reinforcement Learning Paradigm

As discussed earlier solving nonlinear fractional differential systems involves extensive computational resources and hence employing high-level Artificial Intelligence ( AI ) optimizations are of great importance as the obstacle of extensive computational costs in such numerical approximations that be effectively reduced by using AI methods such as deep learning and reinforcement learning.

In the past decade, applications of DLR for dealing with numerical solution of partial differential equations have been comprehensively studied and three major approaches have been developed:

1) In this approach the network is trained by using the sampled data that is randomly selected from the solution domain together with the boundary and initial conditions. The efficiency of the solution is enhanced through each training session over the solution domain. In this approach the solution directly and continuously drives from the outputs of the DRL algorithm.

2) In this approach the trained network produces intermediary results. Solution of the differential system are then numerically computed from the outputs of the DRL algorithm.

3) In the last approach internal states and values of the network are indirectly used to numerically compute the solution of the fractional differential system, and as such the

computed solution solely depends on the internal parameters of the DRL algorithm and not on its final outputs.

Here we will adopt the last approach as it's more suitable for optimizing recursive root approximations for nonlinear systems. As a matter of fact, the main advantage of using a DRL technique is its flexibility it terms of representing nonlinear computations. As we will see later, the solution algorithm starts with an initial guessed-value for the roots. Subsequently in each iteration real roots are approximated and the fractional differential system is numerically solved and the outputs are fed to the next iteration, where numerical errors are reduced after each run of the loop.

A major point here is despite the nonlinear nature of the studied fractional system, a notable proportion of recursive calculations at each iteration are mathematically redundant and can be cut-off by employing a suitable adaptive update policy. Here the DRL technique comes into play as it can be trained to identify repetitive operations (operations on parameters with insignificant changes over iterations) and update approximate them directly to reduce the computational load of each iteration. Although this technique will reduce the accuracy of the numerical method in the short run, it dramatically increases the efficiency of the method in high iterations.

Deep reinforcement learning is an effective technique for dealing with this problem, since a solution of the fractional differential system might not be known beforehand, but the error of approximated solution can be calculated numerically. This is essentially a weak-label learning task by trail-error, that can also be considered a control problem. In this sense, the approximated solution at each iteration is the current state of the DRL algorithm, the action is cutting-off parameters with minimal changes over iterations, and the goal is to keep the cut-off error less than a specified threshold. Here the action policy can be calculated based on the approximated solution at each iteration and the critic (governing equation) by using a deterministic deep policy network.

In the next section we will review some of the important terminology used in Reinforcement Learning and explain the method that have been employed in our calculations in details.

To proceed with the details of our numerical method and algorithm, we will first introduce an example, explore our numerical method and algorithm based on the example, and finally present the obtained approximated solution, approximation errors, and other results.

## 5.2 Numerical Example of a Nonlinear Two-Point Boundary-Value Problem with Nonlinear Boundary Conditions

Consider the formulas for $x_{m,1}$ and $x_{m,2}$ from previous chapter and let:

$$x_{m,1} = Z + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f_1\big(s, x_{m-1,1}, x_{m-1,2}\big)ds - \left(\frac{t}{T}\right)^{\alpha}\int_0^T (T-s)^{\alpha-1} f_1\big(s, x_{m-1,1}, x_{m-1,2}\big)ds\right] + \left(\frac{t}{T}\right)^{\alpha}\left[\frac{(2^{\alpha+1}+1)}{2^{\alpha}8\Gamma(\alpha+2)} - \frac{1}{64} + L^2 - 2Z\right]$$

$$x_{m,2} = W + \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f_2\left(s, x_{m-1,1}, x_{m-1,2}\right) ds - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} f_2\left(s, x_{m-1,1}, x_{m-1,2}\right) ds\right] + \left(\frac{t}{T}\right)^{\alpha}\left[\frac{1}{8\Gamma(\alpha+2)} + \frac{7}{64} + L^2 - Z\right]$$

$$f_1\left(t, x_{m,1}, x_{m,2}\right) = x_{m,2}$$

$$f_2\left(t, x_{m,1}, x_{m,2}\right) = -\frac{1}{2}\left(x_{m,2}\right)^2 - \frac{1}{2}\left(x_{m,1}\right) + \frac{t}{8}\left(x_{m,2}\right) + \frac{t^{1-\alpha}}{4\Gamma(2-\alpha)} + \frac{(2t^{\alpha+1}+1)}{16\Gamma(\alpha+2)}.$$

It is worth mentioning that here we are defining $x_{m,1}$ and $x_{m,2}$ as iterative function, where m is the index of iteration, and $(W, Z, L, M)$ is the approximated root of $E_1, E_2, E_3$ $and$ $E_4$. In addition, considering the following differential system, we can write:

$$E_1 = -\frac{\alpha}{T^{\alpha}}\left[\int_0^T (T-s)^{\alpha-1} f_1\left(s, x_{1,1}, x_{1,2}\right) ds - \Gamma(\alpha)\left(\frac{2^{\alpha+1}+1}{2^{\alpha}8\Gamma(\alpha+2)} - \frac{1}{64} + L^2 - 2Z\right)\right] = 0$$

$$E_2 = -\frac{\alpha}{T^{\alpha}}\left[\int_0^T (T-s)^{\alpha-1} f_2\left(s, x_{1,1}, x_{1,2}\right) ds - \Gamma(\alpha)\left(\frac{1}{8\Gamma(\alpha+2)} + \frac{7}{64} + L^2 - Z\right)\right] = 0$$

$$E_3 = M + Z - L^2 + \frac{1}{64} - \frac{(2^{\alpha+1}+1)}{2^{\alpha}8\Gamma(\alpha+2)} = 0.$$

$$E_4 = L + Z - W - L^2 - \frac{7}{64} - \frac{1}{8\Gamma(\alpha+2)} = 0.$$

From the above nonlinear differential equations and boundary conditions, the exact solution is as follows:

$$x_1^* = \frac{(2t^{\alpha+1}+1)}{8\Gamma(\alpha+2)}$$

$$x_2^* = \frac{t}{4}.$$

As we can see in the above calculations, are required conditions are met and so we can apply our previously developed numerical method to this parametrized boundary-value problem, by using the recursively defined $x_{m,1}$ and $x_{m,2}$ functions. In the next section we will explain and discuss the application of our numerical method and algorithm to this problem. Please note that the method and its related algorithms are presented in terms of the notations used in the example, however the overall method and algorithm can be applied to any problem of a similar nature.

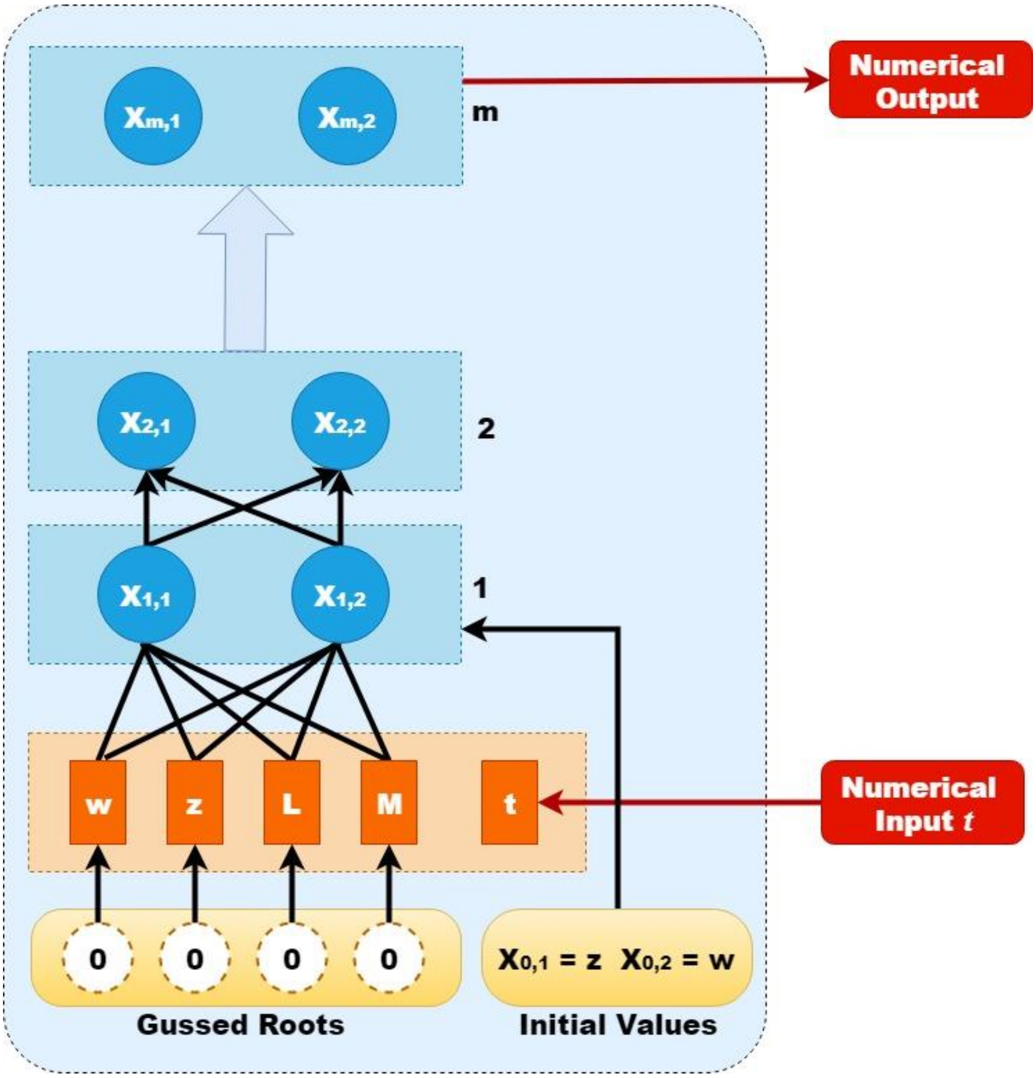


Figure 1: Initial Solution of $x_{m,1}$ and $x_{m,2}$ functions with guessed initial roots

To solve the fractional differential system in the example, initial values for the roots W, Z, L, and M are required. This algorithm starts the first iteration by using zero initial values as guessed roots and then refines the approximations at each iteration. Note that initial values for recursive functions $x_{m,1}$ and $x_{m,2}$ at point t = 0 are given by the problem and directly inserted in the first iteration.

As was mentioned earlier, the major computational challenge in this family of problems, is to recursively finding the roots of the system that involves highly complex non-analytical integrals. As can be seen in the example, to compute the roots for m iterations, 2^ (m+1) non-analytical integrals must be solved numerically. The following table shows the non-analytical integral operations and CPU time involved in solving the system based on the iteration index.

Table 1: Non-analytical integral operations and their CPU (INTEL i7-8700K 3.7GHZ) time

| m | Number of integral operations | CPU time (min) |
|---|---|---|
| 1 | 4.00 | 1.25 |
| 4 | 32.00 | 6.31 |
| 9 | 1,024.00 | 1643.41 |
| 19 | 1,048,576.00 | 54958.29 |

In fact, besides the computational challenge that will be dealt with DRL, this method poses a programming challenge as well. Since the integrals involved in finding $x_{m,1}$ and $x_{m,2}$functions at each iteration are non-analytical, only a numerical solution can be obtained. In this sense, at each step m, $x_{m,1}$ and $x_{m,2}$ can be numerically calculated at each given point t, but would not be obtained as functions. However, to calculate $x_{m+1,1}$ and $x_{m+1,2}$, the values of $x_{m,1}$ and $x_{m,2}$ must be inserted back to the equations as functions of t.

To overcome this problem, we have used an algorithmic definition for $x_{m,1}$ and $x_{m,2}$ functions. In this sense, the whole algorithmic operations involved in computing values of $x_{m,1}$ and $x_{m,2}$ based on t inputs, are treated as the definition of these functions for plotting the graphs, calculating approximations errors and most importantly calculating $x_{m,1}$ and $x_{m,2}$ for next iteration. To create these functional definitions a temporary numerical variable (register) is used that keeps a record of algorithmic operations used for $x_{m,1}$ and $x_{m,2}$ in each iteration and inserts them back into equations E1 and E2. In other words, at iteration m, equations E1 and E2 would contain algorithms that retrieves back to all previous iterations.

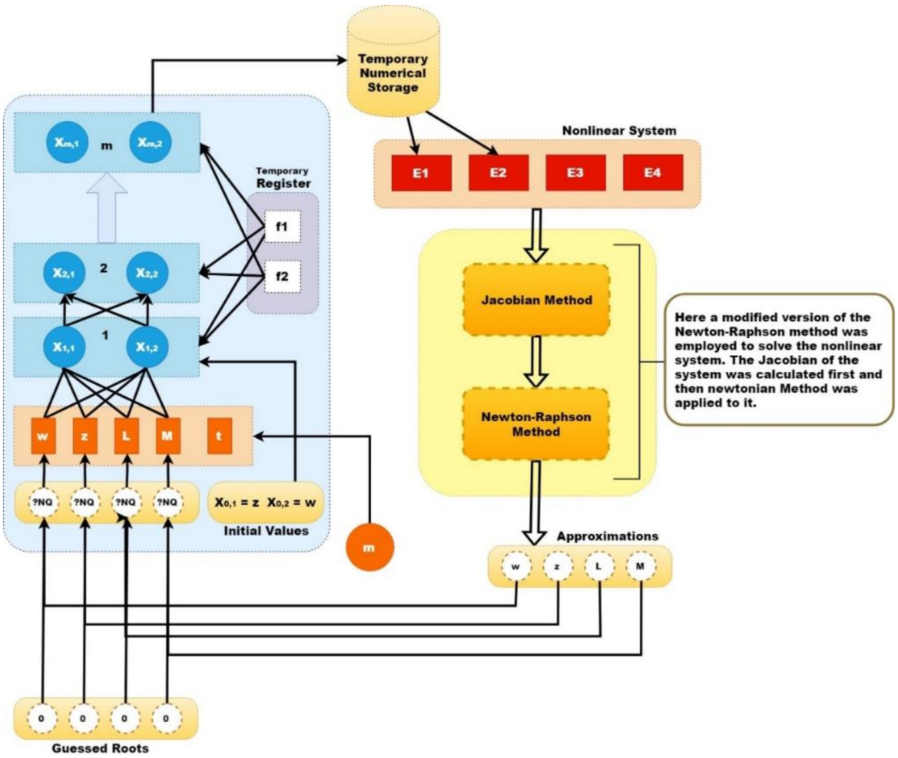


Figure 2: Using a temporary numerical variable to store the algorithmic definitions of $x_{m,1}$ and $x_{m,2}$ and recursively solve the system

Upon inserting algorithmic definitions of $x_{m,1}$ and $x_{m,2}$ into our system of nonlinear equations, the multivariate Newton-Raphson method (as was examined earlier) is applied and approximated roots $(W, Z, L, M)$ are updated and inserted back to the algorithm for next iteration. The complete flowchart of the algorithm can be seen in the next figure.

Note that the most computationally expensive part of the above algorithm is indeed the algorithmic definition of $x_1$ and $x_2$, since at each iteration all previously employed operations on $x_{m,1}$ and $x_{m,2}$ must be recursively recalculated and updated. Here the DRL technique comes to play as it is applied to the variable that stores the algorithmic definitions of our functions.

The algorithmic definitions are updated through a separate DRL algorithm, where the algorithm updated only a proportion of the definitions over iterations, based on the error discrepancies of previous iterations. The goal of the DRL algorithm is to learn which operational updates can be cut-off without introducing significant errors to the approximation. As of the above example, after 50 iterations, the number of updated operations in $x_{m,1}$ and $x_{m,2}$ definitions only retrieved back to 15 last iterations (on average) which is a dramatic improvement over the original algorithm where all operations had to retrieve all way down to the first iteration.

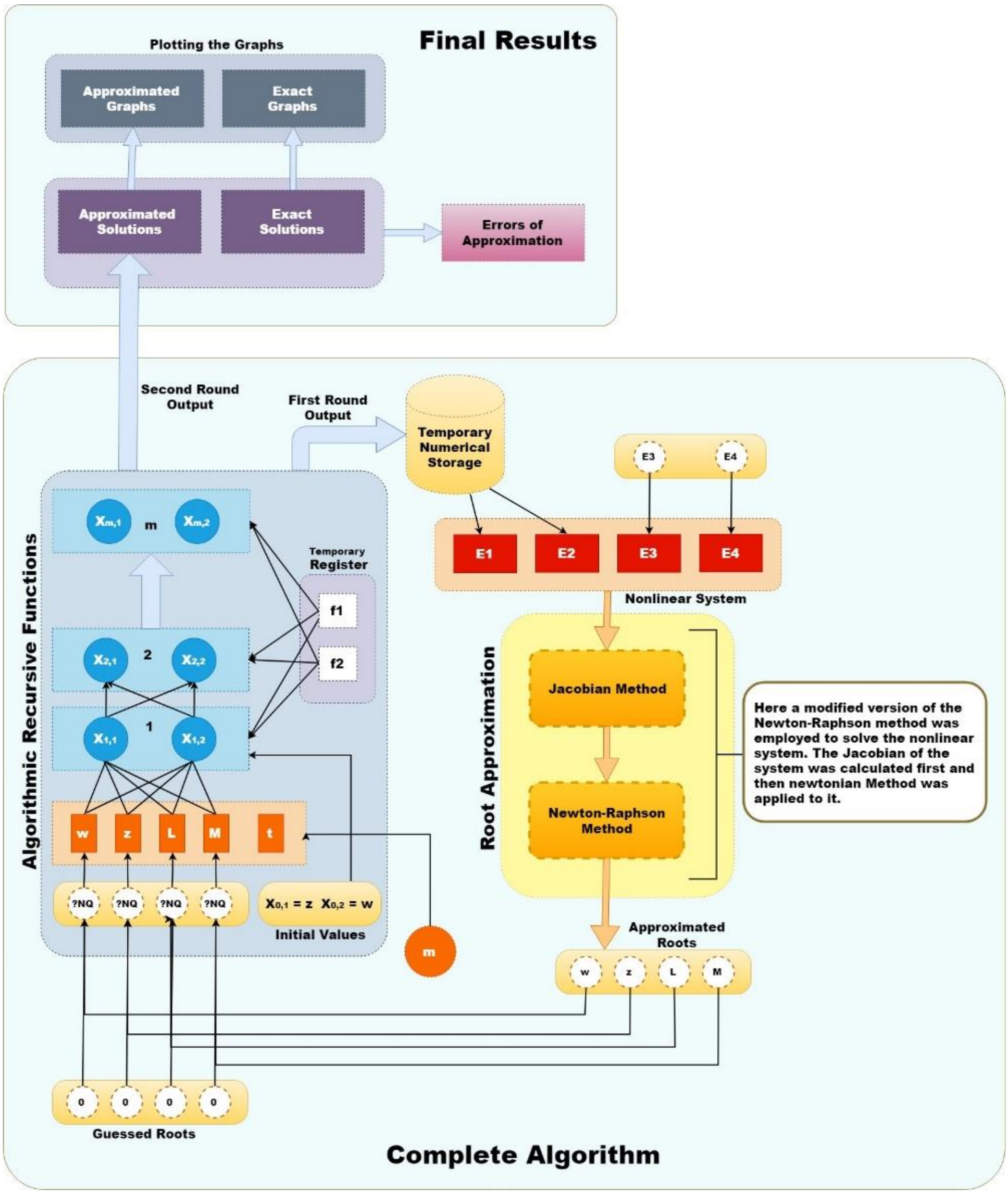


Figure 3: Complete algorithm which outputs the results of specified iterations

The results of the above algorithm, as was run in Mathematica are presented in the next section.

## 5.3 Results

In this section the output of programming algorithm will be discussed at some iterations, starting at the first iteration. The solution of the approximate system of determining equations on iteration number (1), is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error of the approximation is given below:

$\max_{0 \le t \le 1} |x_1(t) - x_{11}(t)| \le 0.02893$

$\max_{0 \le t \le 1} |x_2(t) - x_{12}(t)| \le 0.01547$

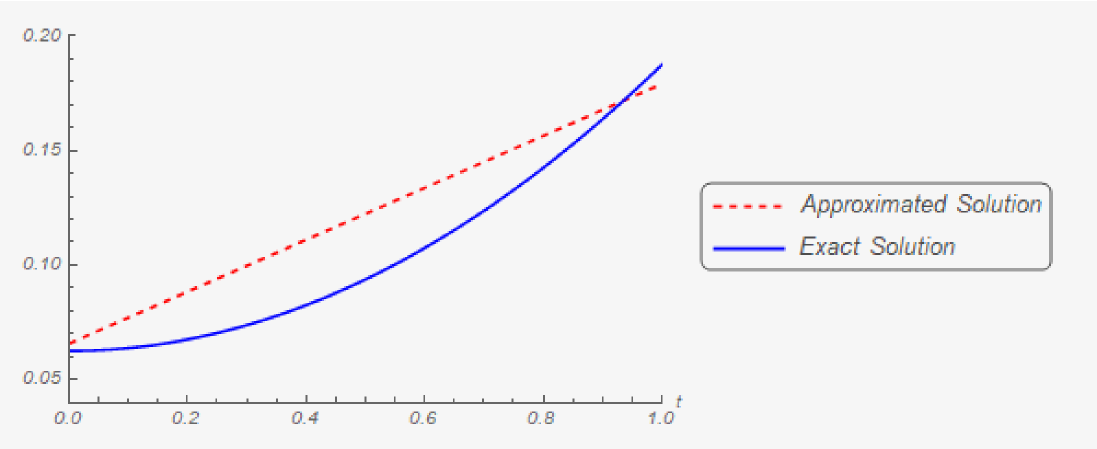


Figure 4: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0,1] at the first iteration of the numerical solution

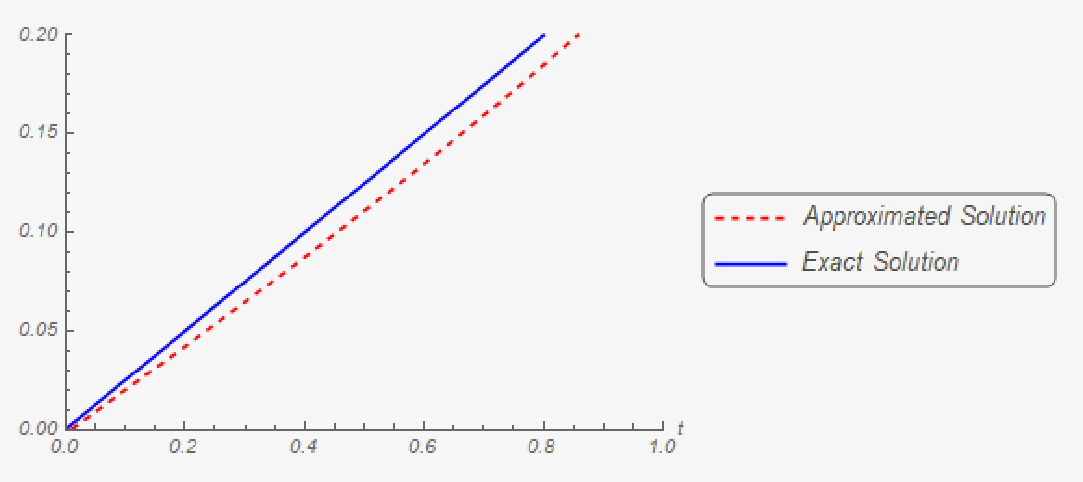


Figure 5: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0,1] at the first iteration of the numerical solution

The following tables show the difference between exact and approximated solutions (max norm approximation error) for the first iteration.

Table 2: Comparing $x_1(t)$ and $x_{1,1}(t)$ and their difference at certain points on the [0,1] interval.

| **t** | **Exact Solution** | **Approximated Solution** | **Error** |
|---|---|---|---|
| 0.1 | 0.06375 | 0.07704 | 0.01329 |
| 0.2 | 0.0675 | 0.08838 | 0.02088 |
| 0.3 | 0.07375 | 0.09973 | 0.02598 |
| 0.4 | 0.0825 | 0.1111 | 0.02857 |
| 0.5 | 0.09375 | 0.1224 | 0.02867 |
| 0.6 | 0.1075 | 0.1338 | 0.02626 |
| 0.7 | 0.1238 | 0.1451 | 0.02135 |
| 0.8 | 0.1425 | 0.1564 | 0.01395 |
| 0.9 | 0.1638 | 0.1678 | 0.00404 |
| 1 | 0.1875 | 0.1791 | 0.008367 |

Table 3: Comparing $x_2(t)$ and $x_{1,2}(t)$ and their difference at certain points on the [0,1] interval.

| t | Exact Solution | Approximated Solution | Error |
|---|---|---|---|
| 0.1 | 0.025 | 0.01992 | 0.005082 |
| 0.2 | 0.05 | 0.04215 | 0.007847 |
| 0.3 | 0.075 | 0.06464 | 0.01036 |
| 0.4 | 0.1 | 0.08749 | 0.01251 |
| 0.5 | 0.125 | 0.1108 | 0.01416 |
| 0.6 | 0.15 | 0.1348 | 0.01518 |
| 0.7 | 0.175 | 0.1595 | 0.01546 |
| 0.8 | 0.2 | 0.1851 | 0.01487 |
| 0.9 | 0.225 | 0.2117 | 0.01328 |
| 1 | 0.25 | 0.2394 | 0.01056 |

The solution of the approximate system on iteration number 50, is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error of approximation for 50th iteration is given below:

$\max_{0 \le t \le 1} x_1(t) - x_{50,1}(t) \le 0.02096$

$\max_{0 \le t \le 1} x_2(t) - x_{50,2}(t) \le 0.01744$

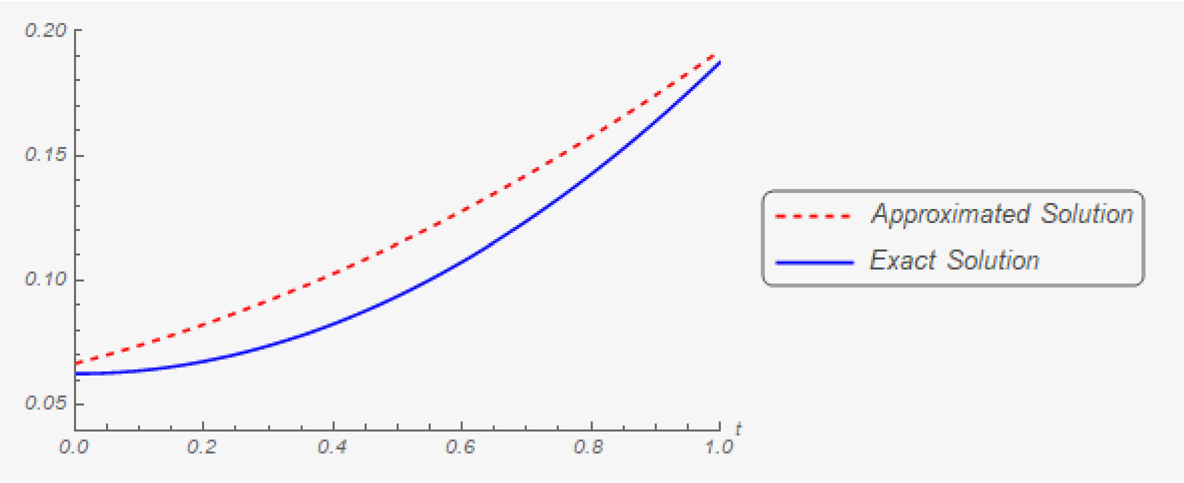


Figure 6: Graphs of $x_1(t)$ and $x_{50,1}(t)$ and their difference on the [0,1] at the iteration number 50 of the numerical solution

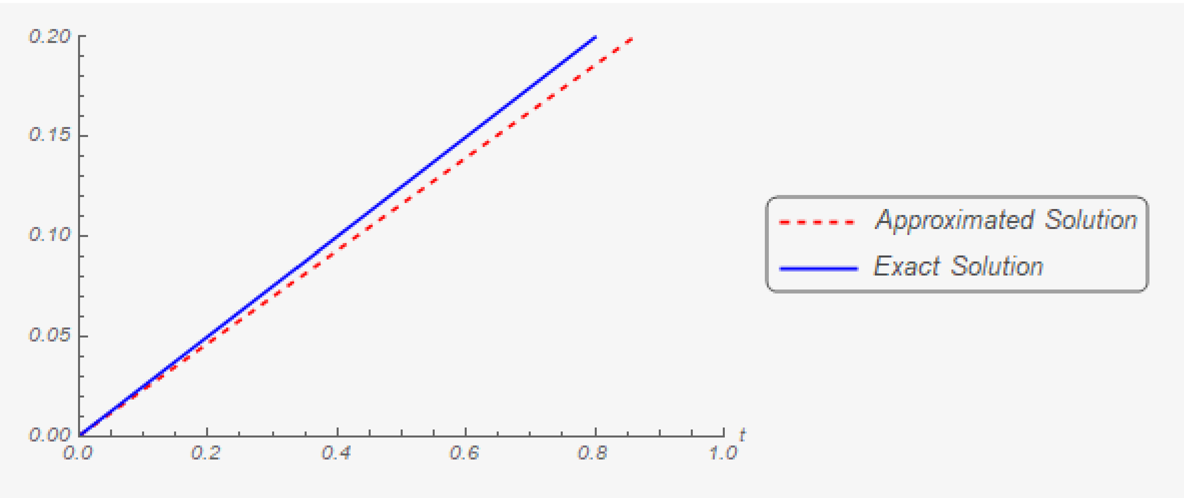


Figure 7: Graphs of $x_2(t)$ and $x_{50,2}(t)$ and their difference on the [0,1] at the iteration number 50 of the numerical solution

The solution of the approximate system on iteration number 100, is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error approximation at 100th iteration is as follows:

$\max_{0 \le t \le 1} |x_1(t) - x_{100,1}(t)| \le 0.01311$

$\max_{0 \le t \le 1} |x_2(t) - x_{100,2}(t)| \le 0.01471$

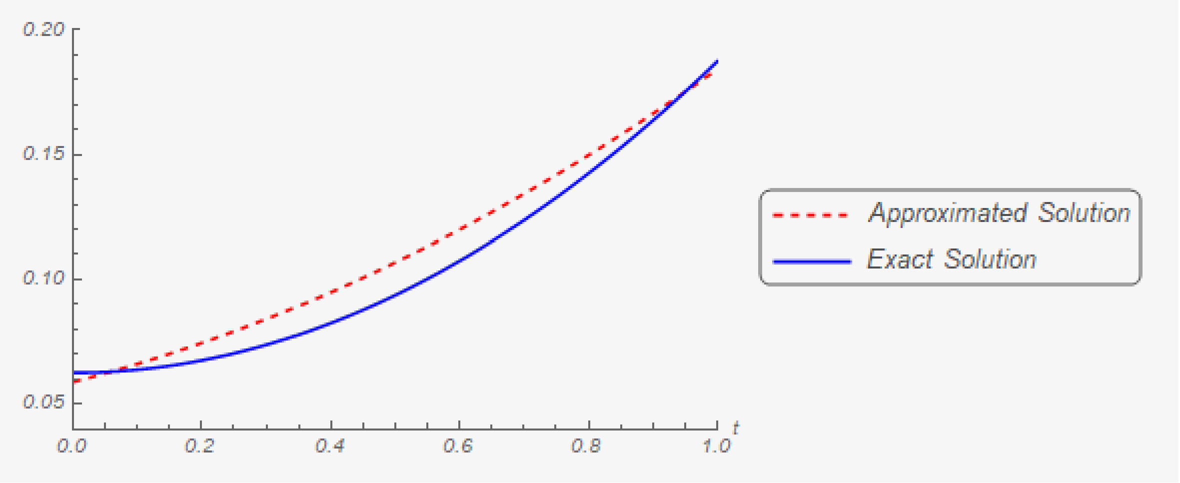


Figure 8: Graphs of $x_1(t)$ and $x_{100,1}(t)$ and their difference on the [0,1] at the iteration number 100 of the numerical solution

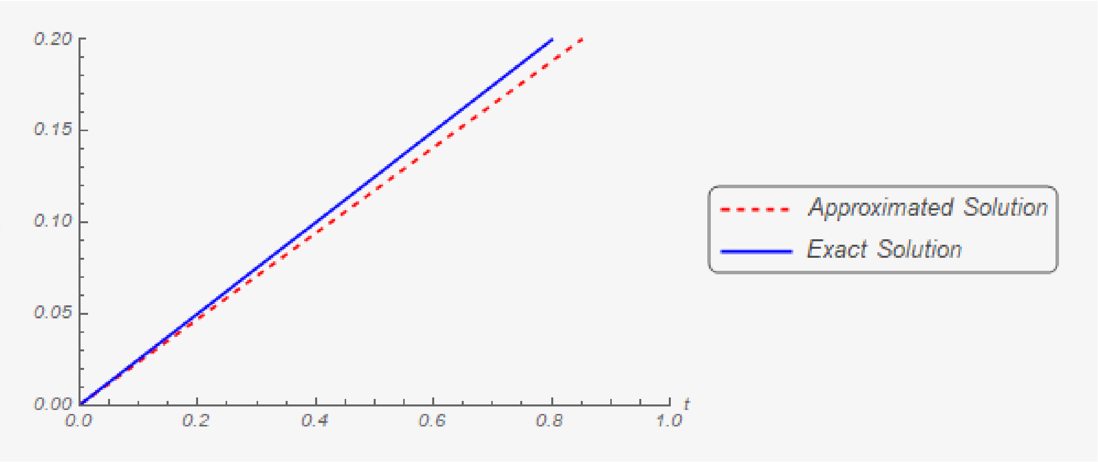


Figure 9: Graphs of $x_2(t)$ and $x_{100,2}(t)$ and their difference on the [0,1] at the iteration number 100 of the numerical solution

The solution of the approximate system on iteration number 150, is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error approximation for the 150th iteration is:

$\max_{0 \leq t \leq 1} |x_1(t) - x_{150,1}(t)| \leq 0.008333$

$\max_{0 \leq t \leq 1} |x_2(t) - x_{150,2}(t)| \leq 0.01077$

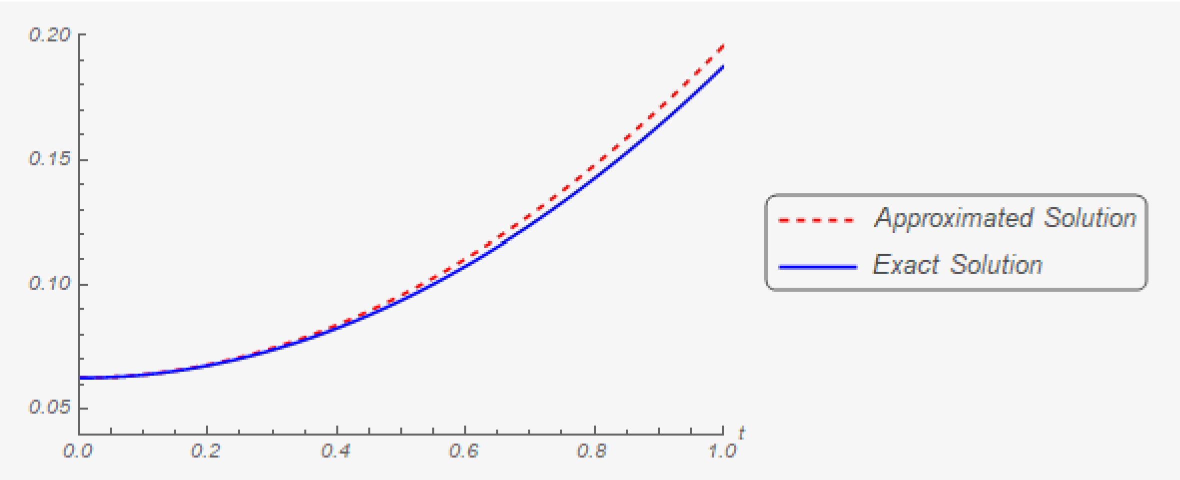


Figure 10: Graphs of $x_1(t)$ and $x_{150,1}(t)$ and their difference on the [0,1] at the iteration number 150 of the numerical solution

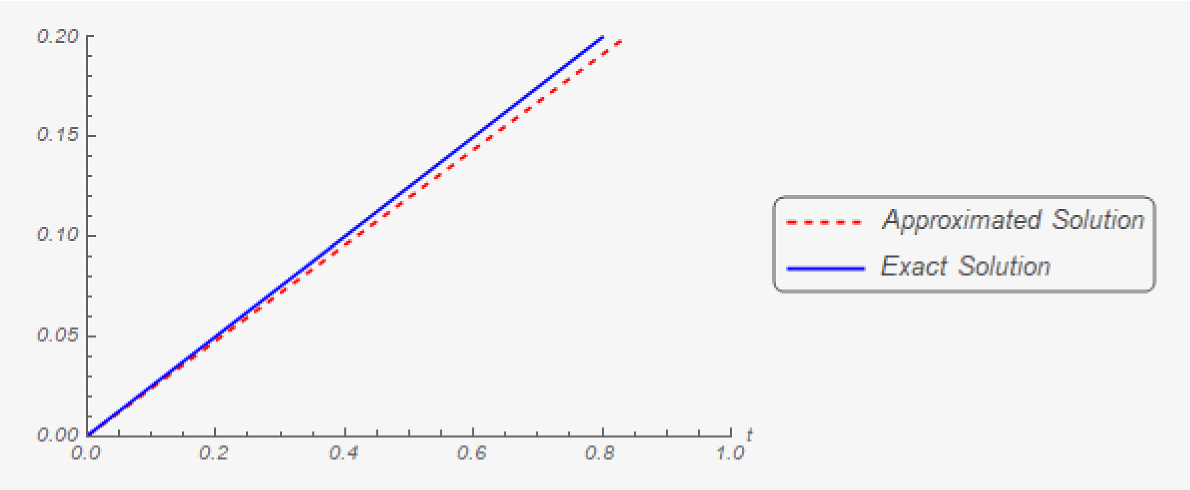


Figure 11: Graphs of $x_2(t)$ and $x_{150,2}(t)$ and their difference on the [0,1] at the iteration number 150 of the numerical solution

The solution of the approximate system on iteration number 200, is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error of approximation for the 200th iteration is:

$\max_{0\leq t\leq 1} |x_1(t) - x_{200,1}(t)| \leq 0.00487$

$\max_{0\leq t\leq 1} |x_2(t) - x_{200,2}(t)| \leq 0.006097$

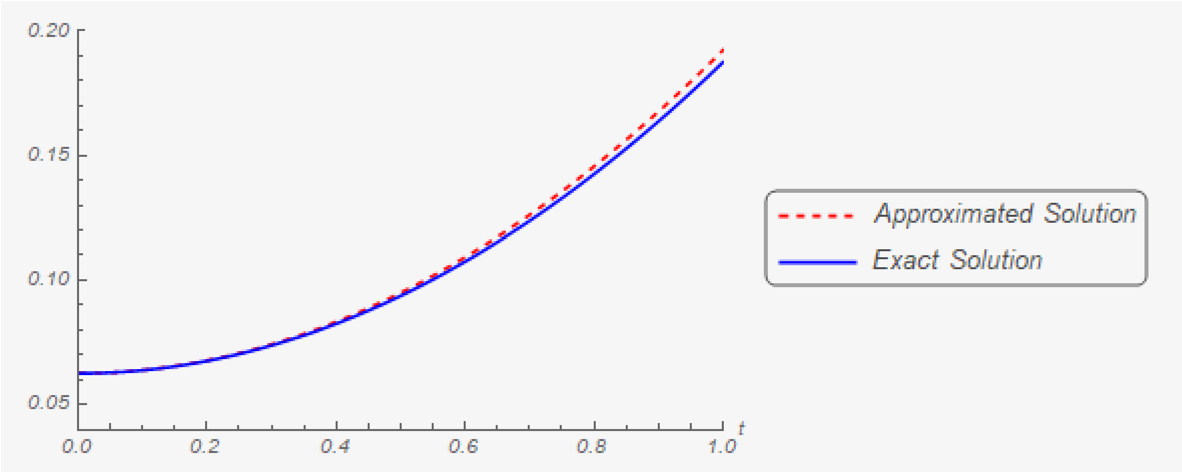


Figure 12: Graphs of $x_1(t)$ and $x_{200,1}(t)$ and their difference on the [0,1] at the iteration number 200 of the numerical solution

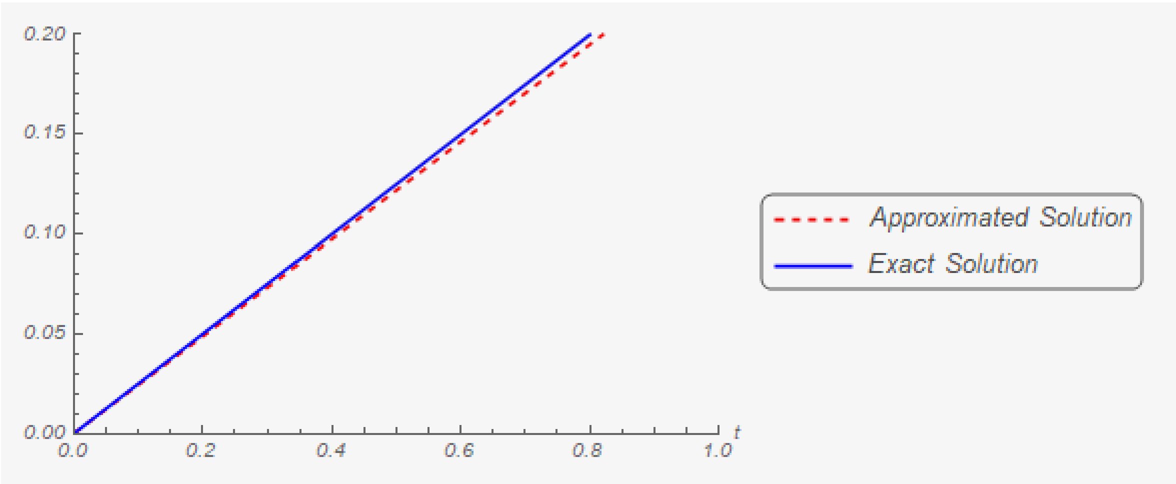


Figure 13: Graphs of $x_2(t)$ and $x_{200,2}(t)$ and their difference on the [0,1] at the iteration number 200 of the numerical solution

The solution of the approximate system on iteration number 250, is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error of approximation for the 250th iteration is:

$$\max_{0\le t\le 1}\left|x_1(t) - x_{250,1}(t)\right| \le 0.0004704$$

$$\max_{0\le t\le 1}\left|x_2(t) - x_{250,2}(t)\right| \le 0.0006233$$

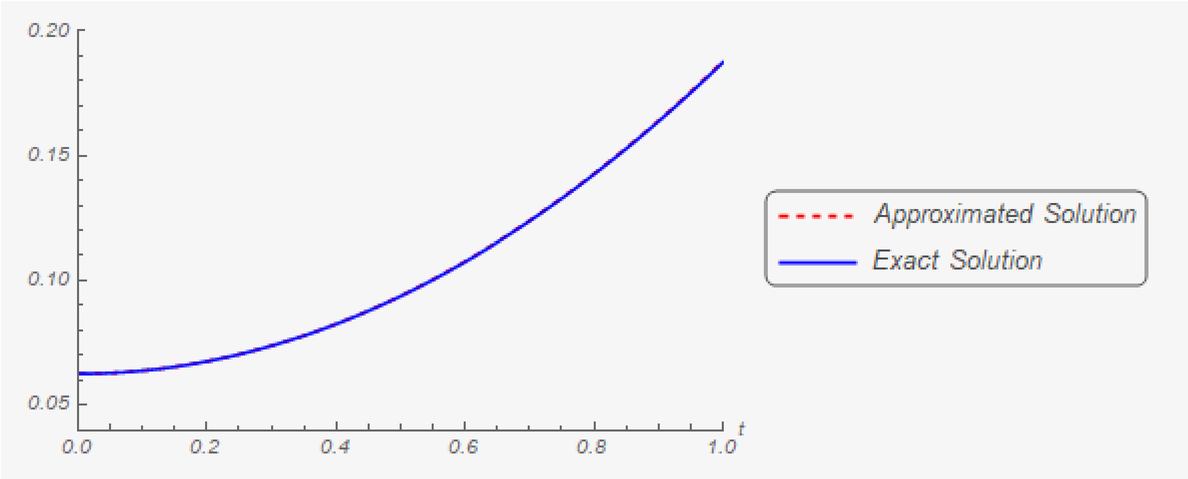


Figure 14: Graphs of $x_1(t)$ and $x_{250,1}(t)$ and their difference on the [0,1] at the iteration number 250 of the numerical solution

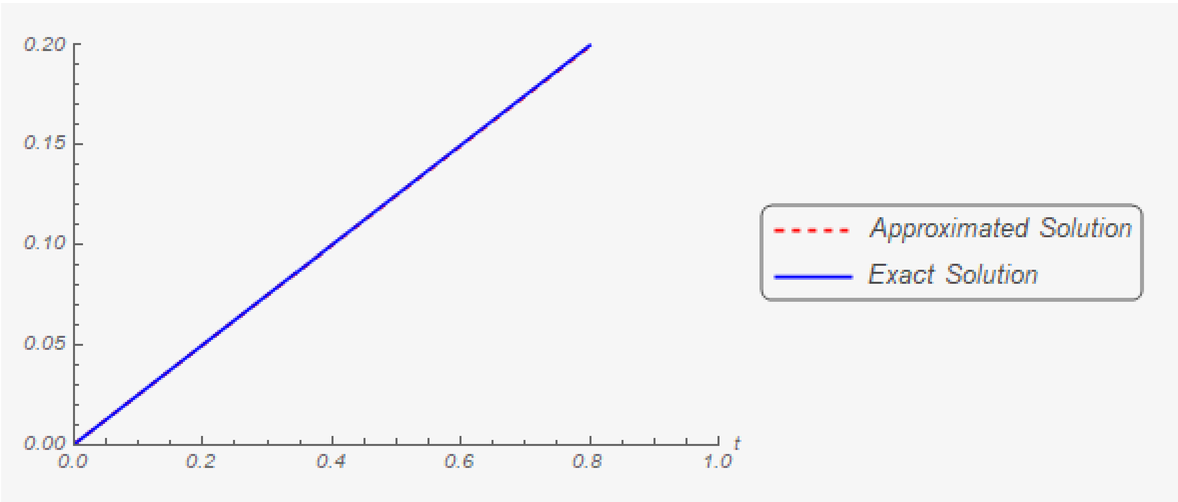


Figure 15: Graphs of $x_2(t)$ and $x_{250,2}(t)$ and their difference on the [0,1] at the iteration number 250 of the numerical solution

The solution of the approximate system on iteration number 300, is given by the following value of parameters.

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error of approximation for the 300th iteration is:

$\max_{0 \le t \le 1} |x_1(t) - x_{300,1}(t)| \le 0.00007809$

$\max_{0 \le t \le 1} |x_2(t) - x_{300,2}(t)| \le 0.00006241$

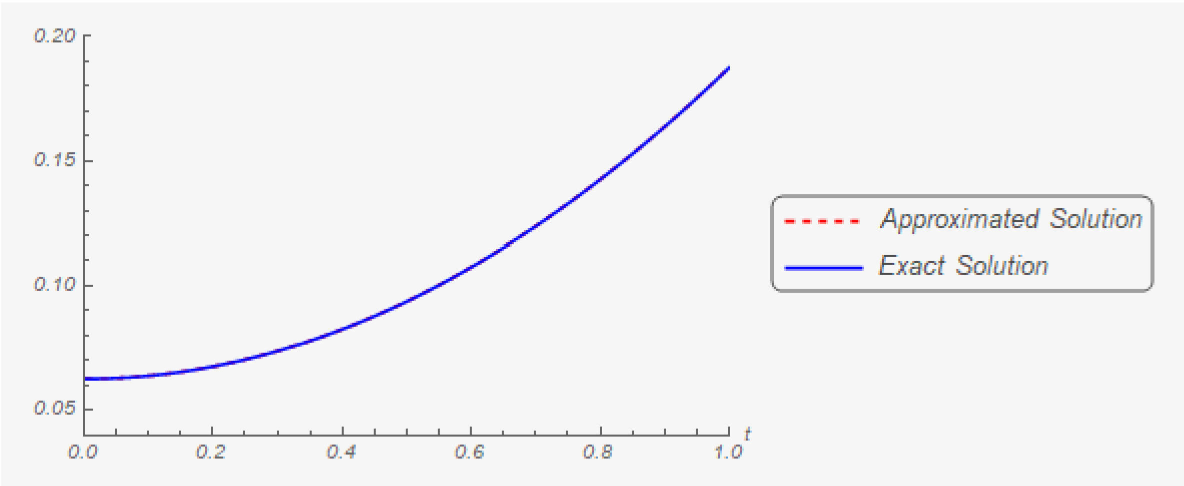


Figure 16: Graphs of $x_1(t)$ and $x_{300,1}(t)$ and their difference on the [0,1] at the iteration number 300 of the numerical solution

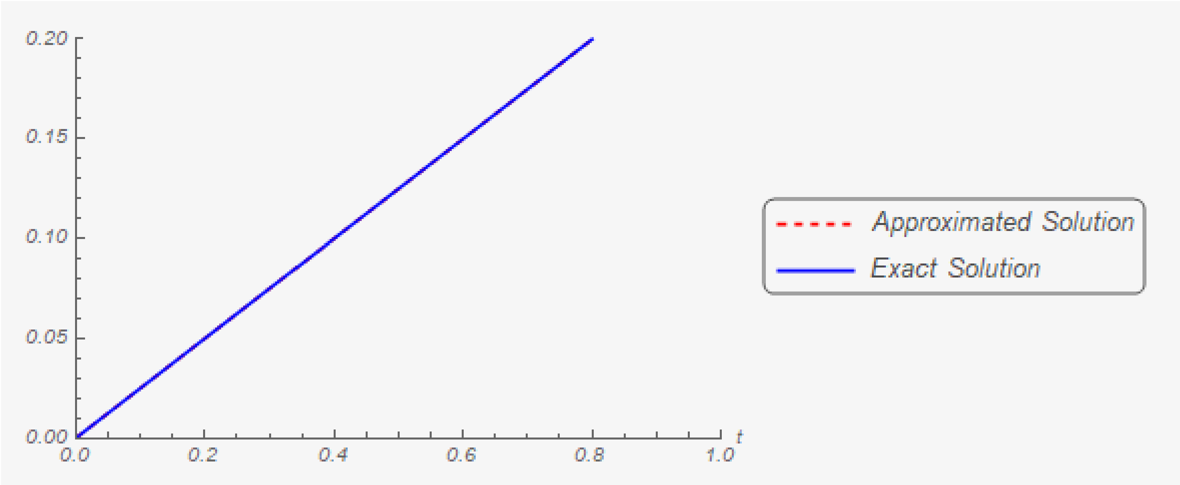


Figure 17: Graphs of $x_2(t)$ and $x_{300,2}(t)$ and their difference on the [0, 1] at the iteration number 300 of the numerical solution

The following tables show the difference between exact and approximated solutions (max norm approximation error) for the last iteration.

Table 4: Comparing $x_1(t)$ and $x_{300,1}(t)$ and their difference at certain points on the [0,1] interval

| t | Exact Solution | Approximated Solution | Error |
|---|---|---|---|
| 0.1 | 0.06375 | 0.06375 | 7.817E-07 |
| 0.2 | 0.0675 | 0.0675 | 0.000003127 |
| 0.3 | 0.07375 | 0.07376 | 7.036E-07 |
| 0.4 | 0.0825 | 0.08251 | 0.00001251 |
| 0.5 | 0.09375 | 0.09377 | 0.00001954 |
| 0.6 | 0.1075 | 0.1075 | 0.00002814 |
| 0.7 | 0.1238 | 0.1238 | 0.00003831 |
| 0.8 | 0.1425 | 0.1426 | 0.00005003 |
| 0.9 | 0.1638 | 0.1638 | 0.00006332 |
| 1 | 0.1875 | 0.1876 | 0.00007817 |

Table 5: Comparing $x_2(t)$ and $x_{300,2}(t)$ and their difference at certain points on the [0,1] interval

| t | Exact Solution | Approximated Solution | Error |
|---|---|---|---|
| 0.1 | 0.025 | 0.02499 | 0.000006248 |
| 0.2 | 0.05 | 0.04999 | 0.0000125 |
| 0.3 | 0.075 | 0.07498 | 0.00001875 |
| 0.4 | 0.1 | 0.09998 | 0.00002499 |
| 0.5 | 0.125 | 0.125 | 0.00003124 |
| 0.6 | 0.15 | 0.15 | 0.00003749 |
| 0.7 | 0.175 | 0.175 | 0.00004374 |
| 0.8 | 0.2 | 0.2 | 0.00004999 |
| 0.9 | 0.225 | 0.2249 | 0.00005624 |
| 1 | 0.25 | 0.2499 | 0.00006248 |

The algorithm stopped on the 364th iteration as the error of approximation for both $x_1(t)$ and $x_2(t)$ fell below the acceptable threshold of 10^-5, and the following final results were obtained:

$Z = 0.0656973365195$

$W = -0.00219529679272$

$M = 0.179133148137$

$L = 0.239437851344$

The error of approximation for the 364th iteration is:

$\max_{0 \le t \le 1} \left| x_1(t) - x_{364,1}(t) \right| \le 1.209 \times 10$^-6

$\max_{0 \le t \le 1} \left| x_2(t) - x_{364,2}(t) \right| \le 5.813 \times 10$^-6

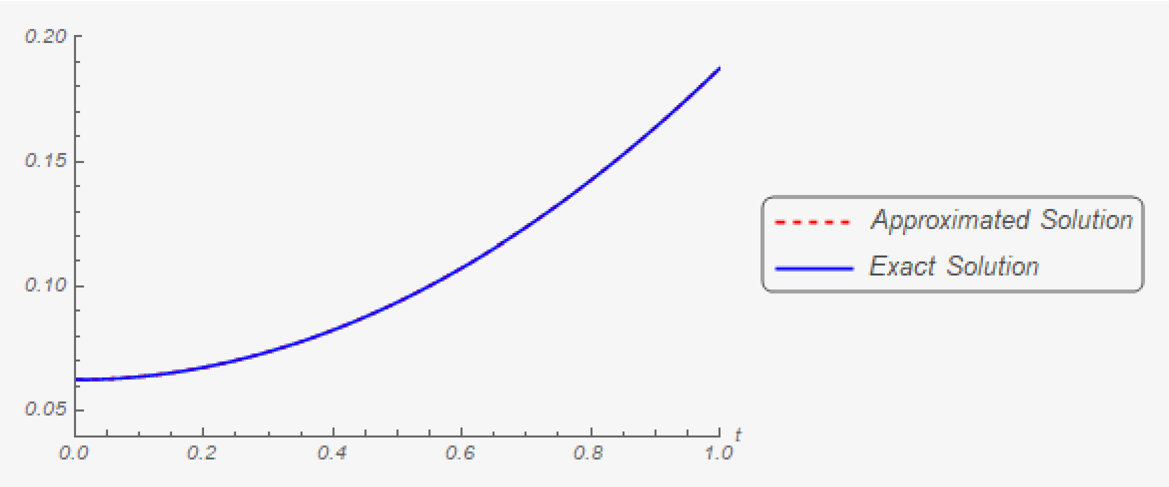


Figure 18: Graphs of $x_1(t)$ and $x_{364,1}(t)$ and their difference on the [0,1] at the iteration number 364 of the numerical solution

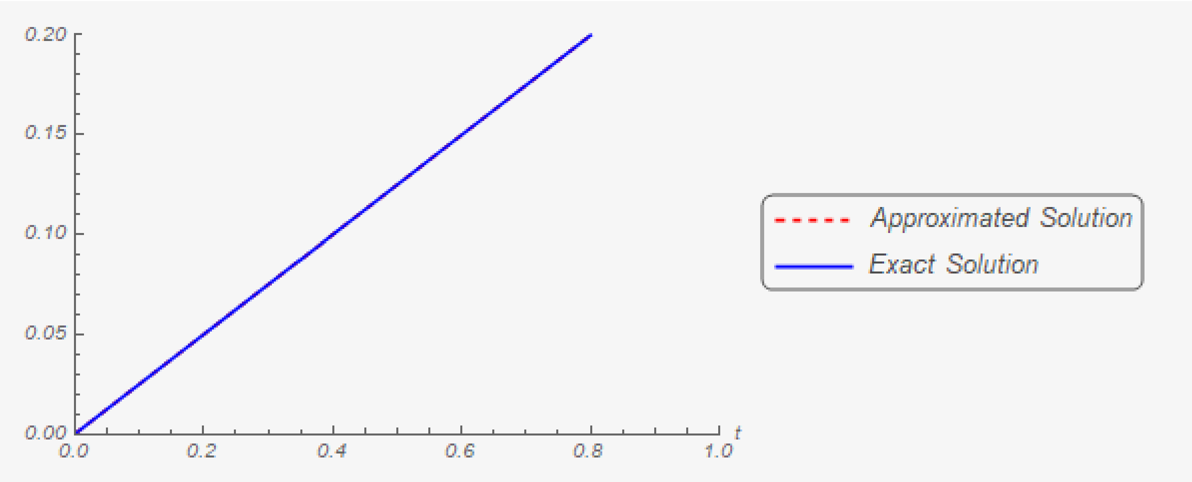


Figure 19: Graphs of $x_2(t)$ and $x_{364,2}(t)$ and their difference on the [0,1] at the iteration number 364 of the numerical solution

The following tables show the difference between exact and approximated solutions (max norm approximation error) for the last iteration.

Table 6: Comparing $x_1(t)$ and $x_{364,1}(t)$ and their difference at certain points on the [0,1] interval

| t | Exact Solution | Approximated Solution | Error |
|---|---|---|---|
| 0.1 | 0.06375 | 0.06375 | 1.563E-08 |

| | | | |
|---|---|---|---|
| 0.2 | 0.0675 | 0.0675 | 6.25E-08 |
| 0.3 | 0.07375 | 0.07375 | 1.406E-07 |
| 0.4 | 0.0825 | 0.0825 | 0.00000025 |
| 0.5 | 0.09375 | 0.09375 | 3.906E-07 |
| 0.6 | 0.1075 | 0.1075 | 5.625E-07 |
| 0.7 | 0.1238 | 0.1238 | 7.656E-07 |
| 0.8 | 0.1425 | 0.1425 | 0.000001 |
| 0.9 | 0.1638 | 0.1638 | 0.000001266 |
| 1 | 0.1875 | 0.1875 | 0.000001563 |

Table 7: Comparing $x_2(t)$ and $x_{364,2}(t)$ and their difference at certain points on the [0,1] interval

| **t** | **Exact Solution** | **Approximated Solution** | **Error** |
|---|---|---|---|
| 0.1 | 0.025 | 0.025 | 0.000000625 |
| 0.2 | 0.05 | 0.05 | 0.00000125 |
| 0.3 | 0.075 | 0.075 | 0.000001875 |
| 0.4 | 0.1 | 0.1 | 0.0000025 |
| 0.5 | 0.125 | 0.125 | 0.000003125 |
| 0.6 | 0.15 | 0.15 | 0.00000375 |
| 0.7 | 0.175 | 0.175 | 0.000004375 |
| 0.8 | 0.2 | 0.2 | 0.000005 |
| 0.9 | 0.225 | 0.225 | 0.000005625 |
| 1 | 0.25 | 0.25 | 0.00000625 |

Here we also present an application of our algorithm to solve an example from [38]. , where the integrals involved in root approximation and solving the system have analytical solution. In this case, since all integrals are analytically computable, no algorithmic definitions for X1 and X2 are required, and hence the DRL algorithm is not being applied as well.

Consider a system of nonlinear differential equations:

$$f_1(t, x_1, x_2) = x_2$$

$$f_2(t, x_1, x_2) = -\frac{1}{2}{x_2}^2 - \frac{1}{2}x_1 + \frac{t}{8}x_2 + \frac{t^2}{16} + \frac{9}{32}.$$

with nonlinear two-point boundary conditions of the form:

$$x_1(0) + x_1(1) - [x_2(1)]^2 = \frac{3}{16}$$

$$x_2(0) + x_1(1) - x_2(1) = -\frac{1}{16}$$

Moreover, we consider the following nonlinear system of equations:

$$E_1 = -\frac{1}{T}\int_0^T f_1(s, x_{m,1}, x_{m,2})ds + \frac{1}{T}\left[\frac{3}{16} + L^2 - 2Z\right] = 0.$$

$$E_2 = -\frac{1}{T}\int_0^T f_2(s, x_{m,1}, x_{m,2})ds + \frac{1}{T}\left[\frac{1}{4} + L^2 - Z\right] = 0.$$

$$E_3 = M + Z - L^2 - \frac{3}{16} = 0.$$

$$E_4 = L + Z - W - L^2 - \frac{1}{4} = 0.$$

From the above nonlinear differential equations and boundary conditions, possesses the exact solution

$$x_1^* = \frac{t^2}{8} + \frac{1}{16}$$

$$x_2^* = \frac{t}{4}$$

As we can see in the above calculations, are required conditions are met and so we can apply our previously developed numerical method to this parametrized boundary-value problem, by using the recursively defined $x_{m,1}$ and $x_{m,2}$ functions.

$$x_{m,1} = Z + \int_0^t f_1(s, x_{m-1,1}, x_{m-1,2})ds - \frac{t}{T}\int_0^T f_1(s, x_{m-1,1}, x_{m-1,2})ds + \frac{t}{T}\left(\frac{3}{16} + L^2 - 2Z\right), m = 1,2, \dots$$

$$x_{m,2} = Z + \int_0^t f_2(s, x_{m-1,1}, x_{m-1,2})ds - \frac{t}{T}\int_0^T f_2(s, x_{m-1,1}, x_{m-1,2})ds + \frac{t}{T}\left(\frac{1}{4} + L^2 - Z\right), m = 1,2, \dots$$

By using the computational algorithm, we obtain the following value of the components of the approximate solution as a result of the first iteration below are the analytically approximated functions:

$x_{11} = Z + t(0.1875 + L^2 - 2Z),$

$$x_{12} = W + 0.28125t + 0.02083333333t^3 + 0.0625Wt^2 - 0.5W^2t - 0.5Zt - t(0.3020833333 + 0.0625W - 0.5W^2 - 0.5Z) + t(0.25 + L^2 - Z)$$

where approximated roots are:

$Z \ = 0.0656973365195$

$W \ = -0.00219529679272$

$M \ = 0.179133148136$

$L \ = 0.239437851344Z$

approximation error for $x_1 = 0.02893270323401729$

approximation error for $x_2 = 0.015471619752078059$

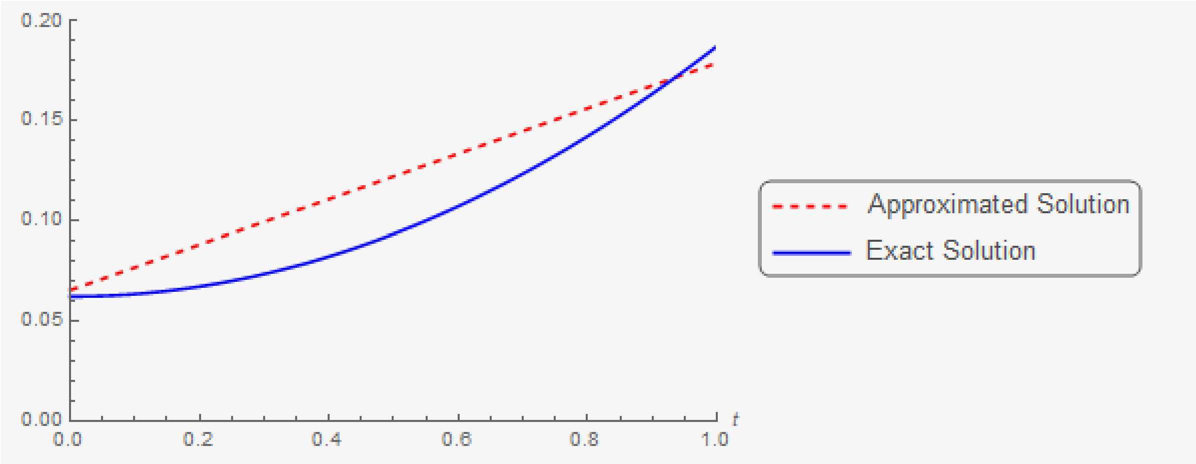


Figure 20: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1] at the first iteration of the analytical solution

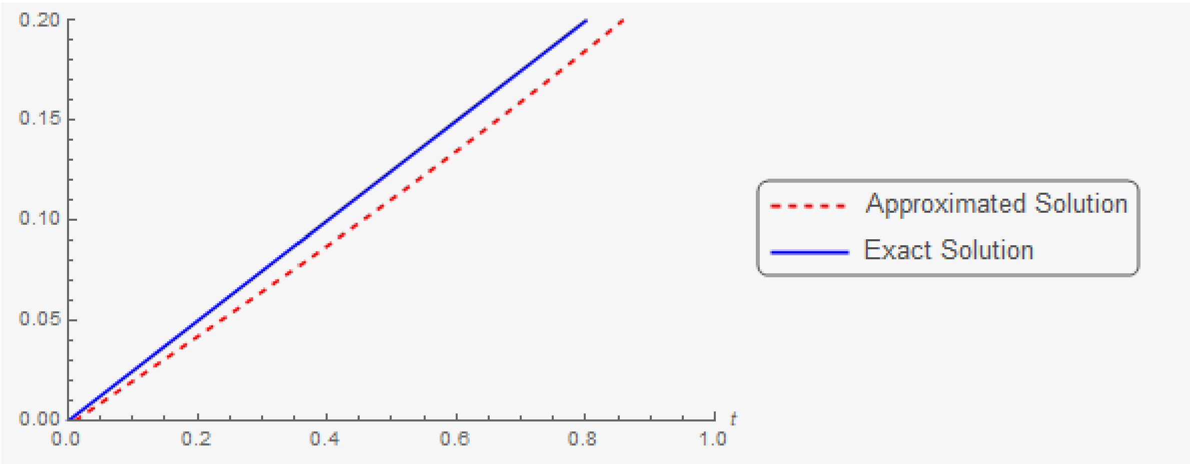


Figure 21: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1] at the first iteration of the analytical solution

At the second iteration we obtained the following analytical approximation functions:

$$x_{21} = 0.1134358116t + 0.06569733651,$$

$$x_{22} = 0.02083333333t^3 - 0.0001372060504t^2 + 0.2209370209t - 0.002195296806,$$

where approximated roots are:

$$Z = 0.0625247065424$$

$$W = -0.00000568825062154$$

$M = 0.187444906053$

$L = 0.2499392178022Z$

Approximation error for $x_1 = 0.0004205763744$

Approximation error for $x_2 = 0.001065024056$

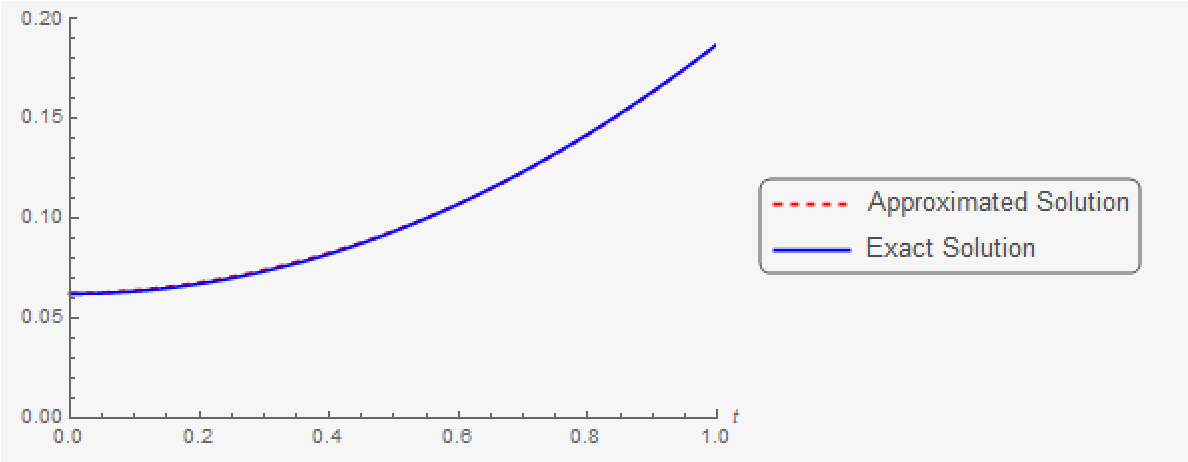


Figure 22: Graphs of $x_1(t)$ and $x_{2,1}(t)$ and their difference on the [0, 1] at the second iteration of the analytical solution

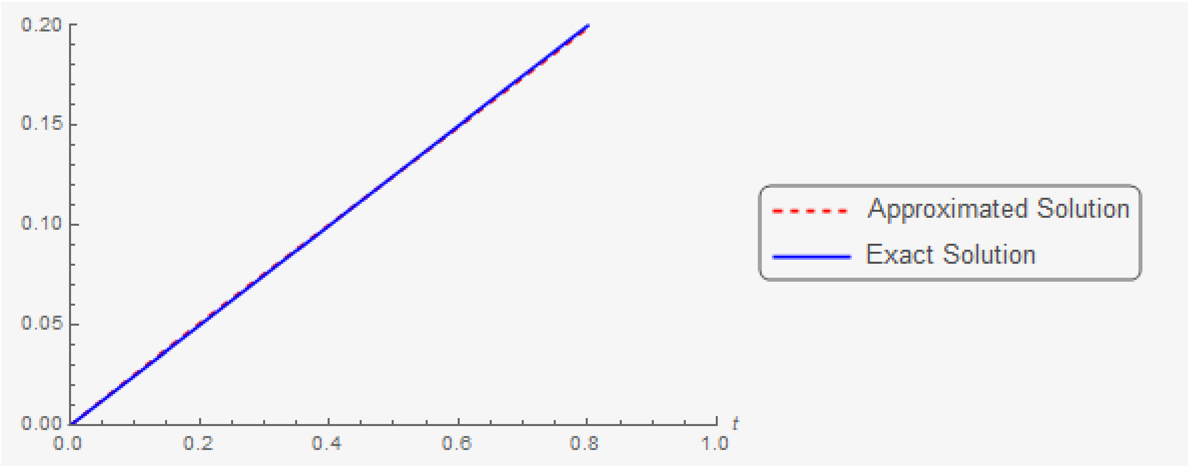


Figure 23: Graphs of $x_2(t)$ and $x_{2,2}(t)$ and their difference on the [0, 1] at the second iteration of the analytical solution

Finally, at the last iteration (third iteration) we obtained the following analytically approximated functions:

$$x_{31}(t) = 0.00208333332t^4 - 0.1185042241.10^{-6}t^3 + 0.1145559641t^2 + 0.0051560206t + 0.06252470657,$$

$$x_{32}(t) = -0.0003100198412t^7 + 0.1234419001.10^{-8}t^6 - 0.0004337997010t^5 + 0.3887933345.10^{-7}t^4 + 0.02163095107t^3 - 0.03122975378t^2 + 0.2600084702t - 0.5688202756.10^{-5},$$

*where approximated roots are:*

$Z = 0.0629636511074$

$W = -0.00141062814755$

$M$ = "0.185189759366

$L = 0.246279131219$

Approximation error for $x_1 = 0.002310136254657064$,

Approximation error for $x_2 = 0.0037207779674591625$

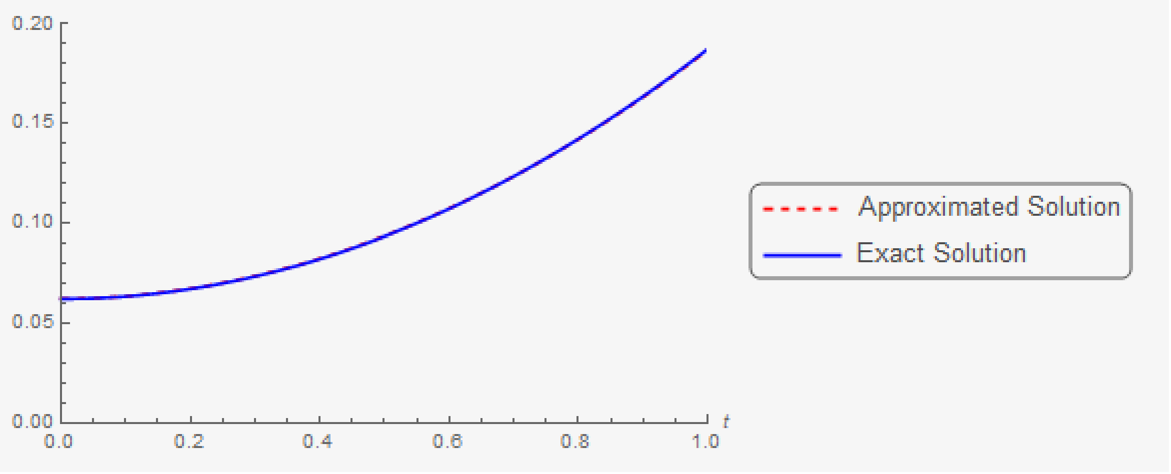


Figure 24: Graphs of $x_1(t)$ and $x_{3,1}(t)$ and their difference on the [0, 1] at the iteration third of the analytical solution

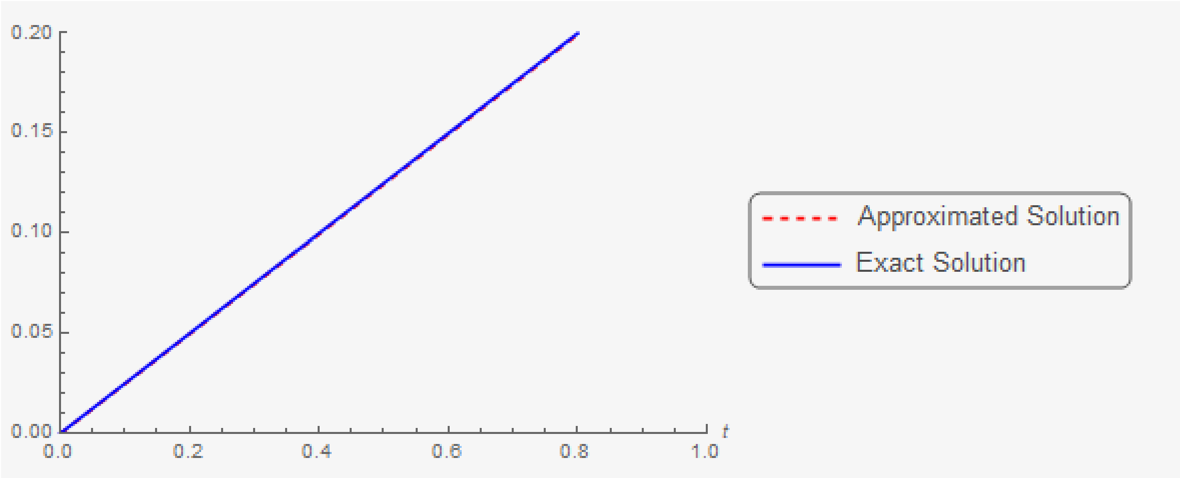


Figure 25: Graphs of $x_2(t)$ and $x_{3,2}(t)$ and their difference on the [0, 1] at the third iteration of the analytical solution

## 5.4 Stability Simulation of Three-Point Non-linear Boundary-Value Problems

We obtain some results concerning the solutions of certain types of three–point non–linear BVP, subject to non–linear boundary conditions. We make the stability simulation of three-point non-linear BVP. By the study, we justify our method, which is based upon a special type of approximations constructed in analytic form.

As an example of our algorithm, we chose solved example from Ref [53] to test our computational algorithm result.

Consider a system of nonlinear differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} = 0.05x_2 - 0.005t^2 + 0.1 = f_1(t, x_1, x_2), \\ \dfrac{dx_2}{dt} = -{x_2}^2 + 0.5x_1 + 0.01t^4 + 0.15t = f_2(t, x_1, x_2), \end{cases}$$

where $t \in \left[0, \frac{1}{2}\right]$ with nonlinear two-point boundary conditions of the form:

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1\left(\frac{1}{2}\right) + {x_2}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0. \end{cases}$$

Moreover, we consider the following nonlinear system of equations:

$$\Delta_{m,1}(z,\eta,L) = -2\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\eta,L), x_{m-1,2}(s,z,\eta,L)\right) ds$$

$$+ 2({z_2}^2 + \eta_1 + 0.025 - z_1) = 0,$$

$$\Delta_{m,2}(z,\eta,L) = -2\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\eta,L), x_{m-1,2}(s,z,\eta,L)\right) ds + 2(0.025 - z_1) = 0,$$

$$x_{m,1}\left(\frac{1}{4}, z, \eta, L\right) = \eta_1, \qquad x_{m,2}\left(\frac{1}{4}, z, \eta, L\right) = \eta_2,$$

$$x_{m,1}\left(\frac{1}{2}, z, \eta, L\right) = L_1, \qquad x_{m,2}\left(\frac{1}{2}, z, \eta, L\right) = L_2.$$

From the above nonlinear differential equations and boundary conditions, possesses the exact solution

$$\begin{cases} {x_1}^* = 0.1t, \\ {x_2}^* = 0.1t^2. \end{cases}$$

Thus, all conditions are satisfied for the analyzed problem. Hence, it is possible to apply the procedure of analytic algorithm. To study the solutions of the parametrized boundary-value problem, we introduce a sequence of functions $x_m$ defined by the recurrence relation

$$x_{m,1}(t,z,\eta,\lambda) \coloneqq z_1$$

$$+ \int_0^t f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds$$

$$- 2t\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds$$

$$+ 2t({z_2}^2 + \eta_1 + 0.025 - z_1),$$

$$x_{m,2}(t,z,\eta,\lambda) := z_2 + \int_0^t f_2\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds - 2t\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds + 2t(0.025 - z_1),$$

where $m = 1,2,3, \ldots,$

$$x_{0,1}(t,z,\eta,\lambda) = z_1 + \left(\frac{t}{T}\right)({z_2}^2 + \eta_1 + 0.025 - z_1),$$

$$x_{0,2}(t,z,\eta,\lambda) = z_2 + \left(\frac{t}{T}\right)(0.025 - z_1).$$

By using the computational algorithm, we obtain the following value of the components of the approximate solution as a result of the first iteration for different non-linear two point boundary condition.

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) := x_1\left(\frac{1}{2}\right) + {x_2}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) := x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0. \end{cases}$$

We obtain below result

Approximated Functions:

$$x_1 = 2.53276 \times 10^{-6} + 0.0998352\, t + 0.00124987\, t^2 - 0.00166667\, t^3$$

$$x_2 = -0.00329579 - 9.59585 \times 10^{-6}\, t + 0.100176\, t^2 - 0.000833164\, t^3 + 0.002\, t^5$$

Approximated roots:

$$z_1 = 2.53276 \times 10^{-6}, z_2 = -0.00329579,$$

$$n_1 = 0.0250134, n_2 = 0.00295172$$

$$l_1 = 0.0500243,\ l_2 = 0.0217017$$

The error for the first approximation is

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_1^*(t) - x_{11}(t)| \leq 0.0000303213$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \leq 0.00330445$$

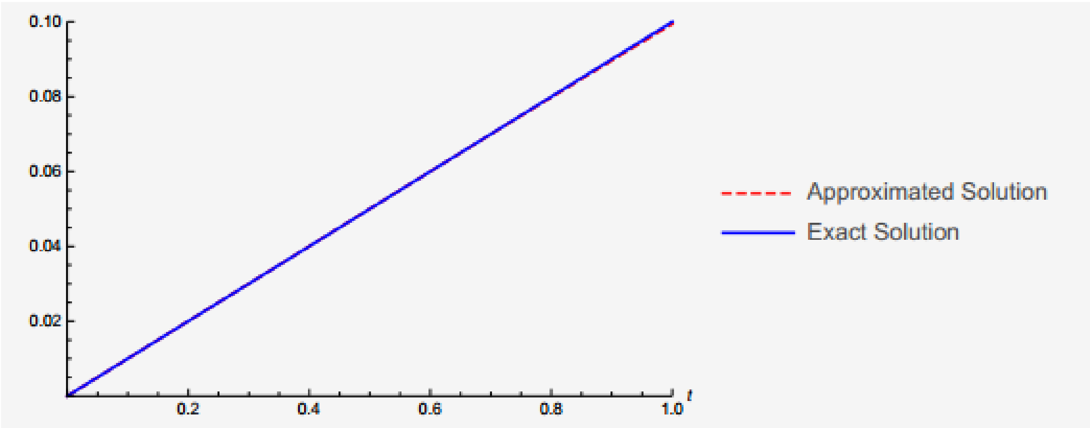


Figure 26: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1/2] at the first iteration

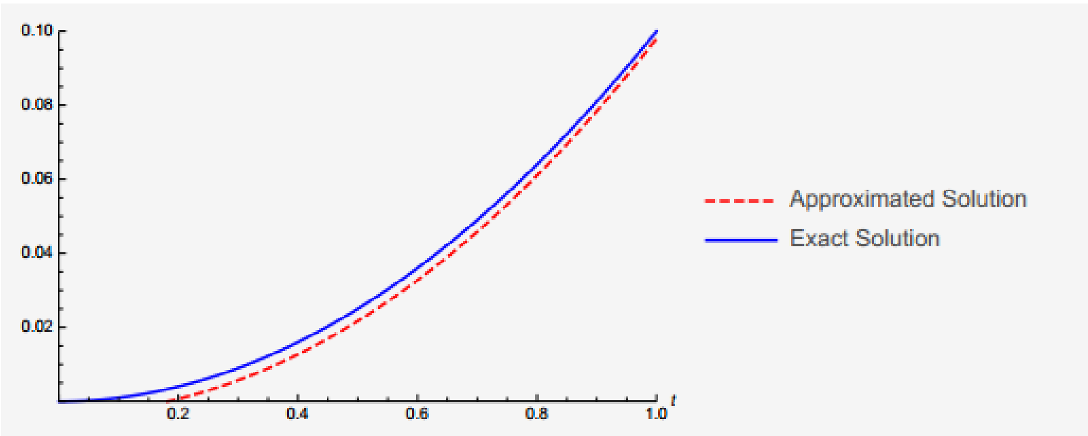


Figure 27: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1/2] at the first iteration

1st Iteration result where $S = 0.026$ .

Checking the stability on the first iteration where $t \in \left[0, \frac{1}{2}\right]$, with below non–linear two–point boundary conditions

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1\left(\frac{1}{2}\right) + {x_2}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0.001, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0.001. \end{cases}$$

We obtain below result

Approximated Functions:

$$X_1 = 0.000531149 + 0.102118\, t + 0.00127344\, t^2 - 0.00166667\, t^3$$

$$X_2 = 0.00619122 + 0.000894289\, t + 0.100269\, t^2 - 0.000864883\, t^3 + 0.002\, t^5$$

Approximated roots:

$$z_1 = 0.000531149, z_2 = 0.00619122$$

$$n_1 = 0.0256621, n_2 = 0.0126997$$

$$l_1 = 0.050796,\ l_2 = 0.0316601$$

The error for the first approximation is

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_1^*(t) - x_{11}(t)| \leq 0.00170032$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \leq 0.00665998$$

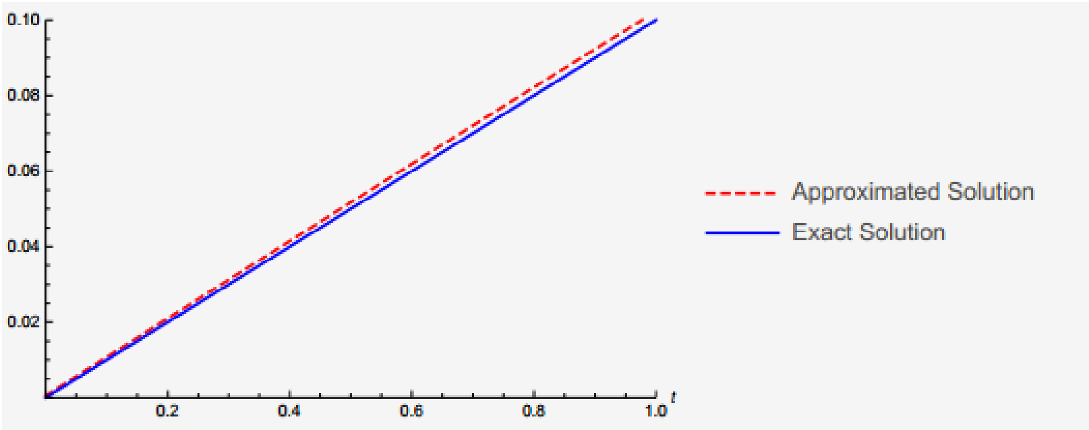


Figure 28: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1/2] at the first iteration

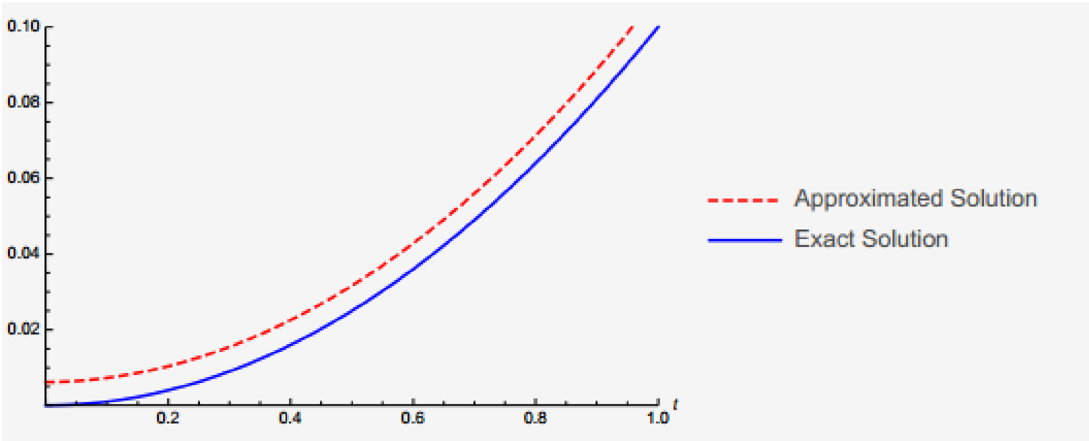


Figure 29: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1/2] at the first iteration

1st Iteration result where $S = 0.028$ .

Checking the stability on the first iteration where $t \in \left[0, \frac{1}{2}\right]$ , with below non–linear two–point boundary conditions

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) := x_1\left(\frac{1}{2}\right) + {x_2}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0.003, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) := x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0.003. \end{cases}$$

We obtain below result

Approximated Functions:

$$X_1 = 0.00133871 + 0.106096\, t + 0.00133306\, t^2 - 0.00166667\, t^3$$

$$X_2 = 0.00618905 + 0.00280629\, t + 0.101256t^2 - 0.000947766t^3 + 0.002t^5$$

Approximated roots:

$$z_1 = 0.00133871, z_2 = 0.00618905,$$

$$n_1 = 0.0264734, n_2 = 0.013296$$

$$l_1 = 0.0516184, l_2 = 0.0328503$$

The error for the first approximation is

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_1^*(t) - x_{11}(t)| \leq 0.00451156$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \leq 0.00785025$$

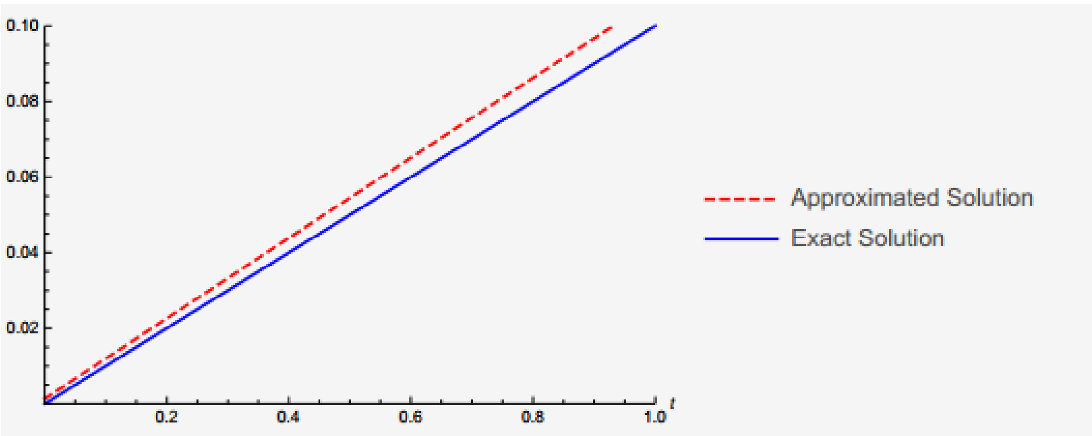


Figure 30: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1/2] at the first iteration

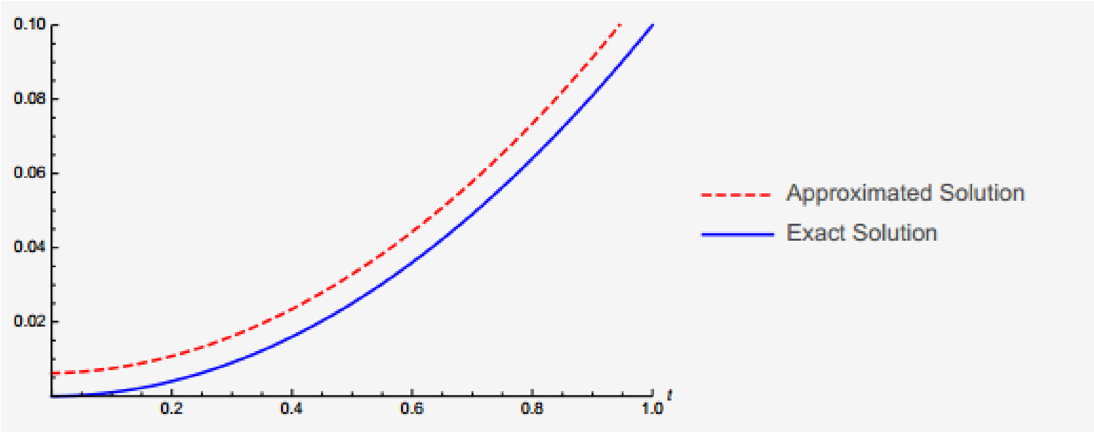


Figure 31: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1/2] at the first iteration

1st Iteration result where $S = 0.030$ .

Checking the stability on the first iteration where $t \in \left[0, \frac{1}{2}\right]$ , with below non–linear two–point boundary conditions

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) := x_1\left(\frac{1}{2}\right) + {x_2}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0.005, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) := x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0.005. \end{cases}$$

We obtain below result

Approximated Functions:

$$X_1 = 0.00218281 + 0.110074\, t + 0.00139086 t^2 - 0.00166667\, t^3$$

$$X_2 = 0.00618693 + 0.00464529\, t + 0.102244 t^2 - 0.00103173\, t^3 + 0.002\, t^5$$

Approximated roots:

$$z_1 = 0.00218281, z_2 = 0.00618693,$$

$$n_1 = 0.027321, \quad n_2 = 0.0138856$$

$$l_1 = 0.0524769, \quad l_2 = 0.0340041$$

The error for the first approximation is

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_1^*(t) - x_{11}(t)| \leq 0.00735921$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \leq 0.00900403$$

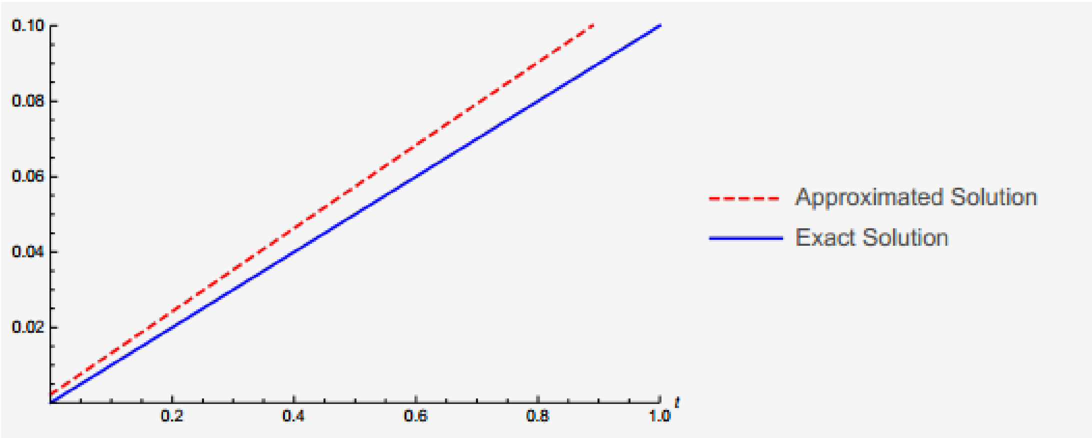


Figure 32: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1/2] at the first iteration

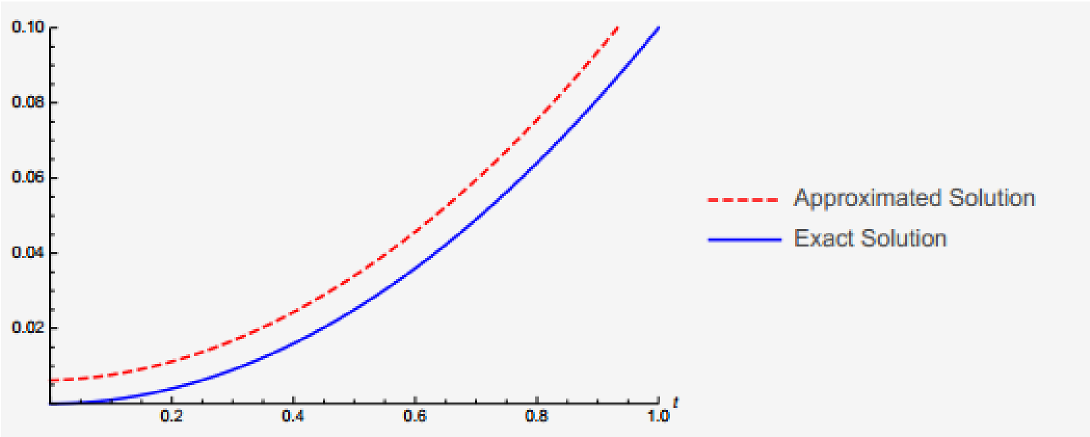


Figure 33: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1/2] at the first iteration

1st Iteration result where $S = 0.032$ .

Checking the stability on the first iteration where $t \in \left[0, \frac{1}{2}\right]$ , with below non–linear two–point boundary conditions

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1\left(\frac{1}{2}\right) + {x_2}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0.007, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0.007. \end{cases}$$

We obtain below result

Approximated Functions:

$$X_1 = 0.00295727 + 0.114051\, t + 0.00145214\, t^2 - 0.00166667\, t^3$$

$$X2 = 0.0061847 + 0.00662617\, t + 0.103231\ t^2 - 0.00112464 t^3 + 0.002 t^5$$

Approximated roots:

$$z_1 = 0.00295727,\ z_2 = 0.0061847$$

$$n_1 = 0.0280993, n_2 = 0.0144882$$

$$l_1 = 0.0532666,\ l_2 = 0.0352274$$

The error for the first approximation is

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_1^*(t) - x_{11}(t)| \leq 0.0101375$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \leq 0.0102273$$

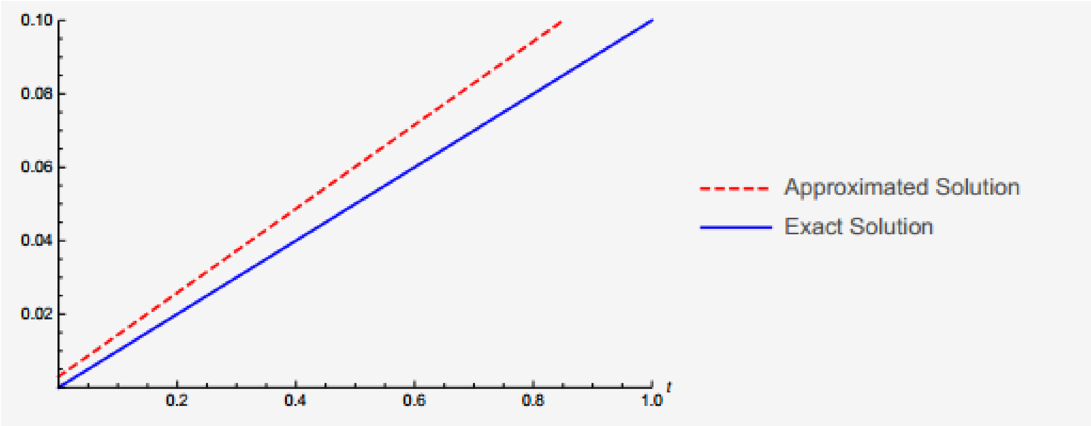


Figure 34: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1/2] at the first iteration

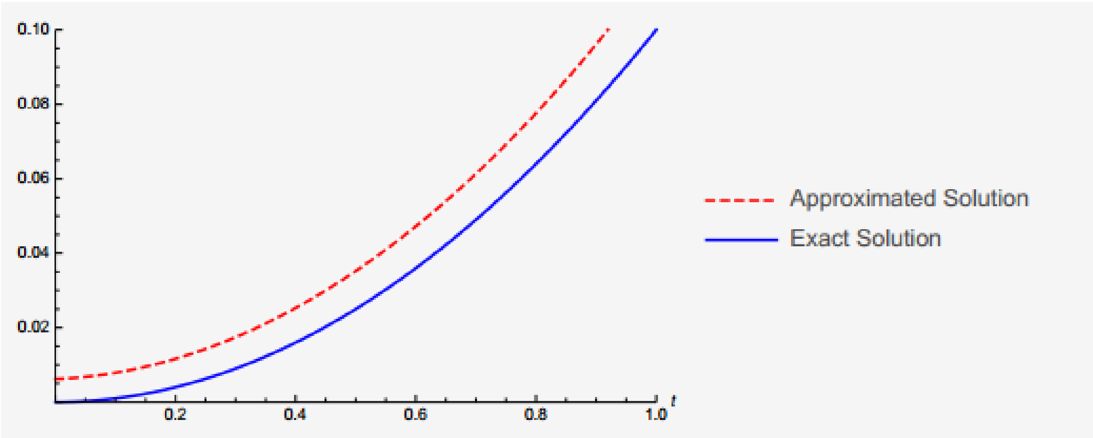


Figure 35: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1/2] at the first iteration

1st Iteration result where $S = 0.040$ .

Checking the stability on the first iteration where $t \in \left[0, \frac{1}{2}\right]$ , with below non–linear two–point boundary conditions

$$\begin{cases} g_1\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1\left(\frac{1}{2}\right) + x_2{}^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0.015, \\ g_2\left(x(0), x\left(\frac{1}{4}\right), x(1)\right) \coloneqq x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0.015. \end{cases}$$

We obtain the below result:

Approximated Functions:

$$X_1 = 0.00634268 + 0.129964\,t + 0.00168287\,t^2 - 0.00166667\,t^3$$

$$X_2 = 0.00617609 + 0.0139765\,t + 0.107181\,t^2 - 0.00151042\,t^3$$
$$+ 0.002\ t^5$$

Approximated roots:

$$z_1 = 0.00634268, z_2 = 0.00617609,$$

$$n_1 = 0.031499,\ n_2 = 0.0168454$$

$$l_1 = 0.0567095,\ l_2 = 0.0398334$$

The error for the first approximation is

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_1^*(t) - x_{11}(t)| \le 0.0215371$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \le 0.0148333$$

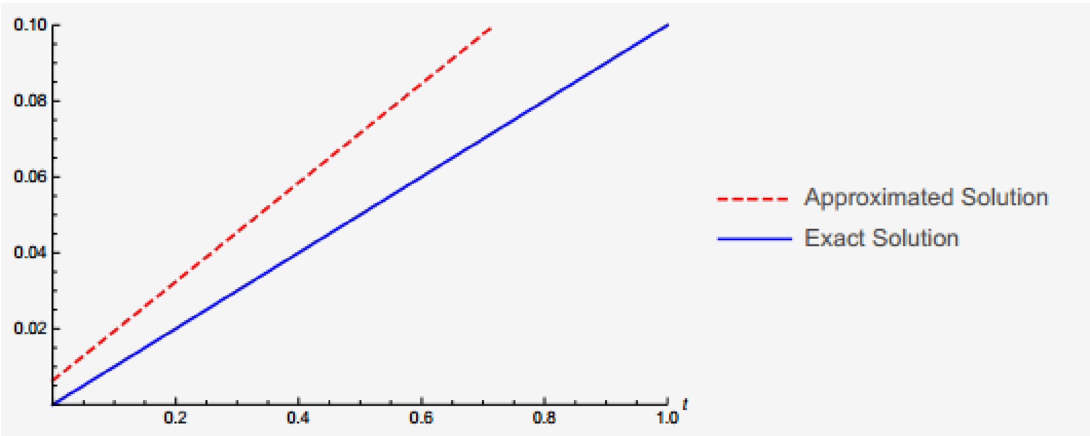


Figure 36: Graphs of $x_1(t)$ and $x_{1,1}(t)$ and their difference on the [0, 1/2] at the first iteration

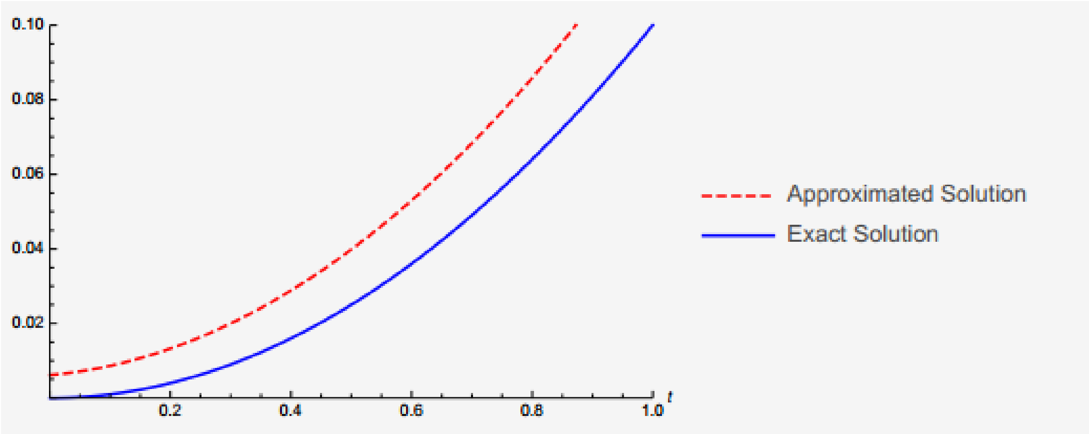

Figure 37: Graphs of $x_2(t)$ and $x_{1,2}(t)$ and their difference on the [0, 1/2] at the first iteration

## 5.5 Conclusion

Over the past decade, coupled systems of non-linear fractional differential equations has found numerous critical applications in many fields of applied science, and hence constitute a pivotal part of the modern mathematics. The current thesis has focused on proving the uniqueness and existence of solutions for a class of coupled nonlinear fractional differential equations with integral boundary conditions. Consequently, we also developed a parametrized two-point numerical scheme and its algorithm for solving such fractional systems.

However, in most applied problems, solving non-linear fractional differential systems poses a great computational challenge, due to the recursive nature of the solution and the non-analytical integrals involved. To over this problem, we applied a Deep Learning Reinforcement (DRL) technique to our parametrized numerical method, to identify and cut-off the recursive operations that had minimal contributions to reducing approximation error over iterations. This technique essentially used an algorithmic definition for the approximated function instead of a mathematical definition and kept a record of the mathematical operations involved in finding these functions at each iteration.

These operations were repetitively updated for each iteration based on the solution of the recursive system of equations obtained from the numerical scheme.

The DRL method was applied to these algorithmic definitions at each iteration to learn the contributions of each specific operation to error reduction and stop insignificant operations from being updated and hence reduce the computational load. The obtained numerical method and algorithm was applied to problem and obtained results were successful. The algorithm reached an approximation error below 10^-5 after 364 iterations.

Clearly the number of iterations required for reaching the acceptable error threshold was increased due to the use of DRL, since cutting-off a proportion of operations at each iteration and not updating them increases the approximation error. However, the applied DRL technique substantially increased the computational efficiency of the algorithm and reduced the calculation time.

It's worth mentioning that the Mathematica program that was used to solved the example problem, was specifically written for that problem, but the overall algorithm can be applied to any problem of a similar nature. Regarding the fact that solving most nonlinear fractional differential systems face a similar type of computational challenge, the current study can have important contributions to the field by introducing a parametrized numerical method that employs deep reinforcement learning to reduce the computational costs of finding approximated solutions. Moreover, future studies can improve and extend the findings of this study, by generalizing the numerical method to solve a wider class of nonlinear fractional differential systems with different initial and boundary conditions. Note that the overall technique of using DRL for reducing computational costs, applies to more general cases and is not specific to the initial or boundary conditions defined in this study.

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