Thin-shell Wormholes in $f(R)$-Gravity

Khashayar Kianfar

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Physics

Eastern Mediterranean University
September 2019
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Acting Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. İzzet Sakallı
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Physics.

Prof. Dr. S. Habib Mazharimousavi
Supervisor

Examining Committee

1. Prof. Dr. Ayhan Bilsel
2. Prof. Dr. Mustafa Halilsoy
3. Prof. Dr. S. Habib Mazharimousavi

ABSTRACT

We study the possibility of constructing \ thin-shell wormhole (TSW) in a particular $f(R)$-gravity model coupled minimally with nonlinear electromagnetic fields. In doing so, first we give a new static spherically symmetric solution of the theory. Then we apply the cut-and-paste method to construct the TSW. As (the third order derivative with respect to R) $f'''(R) \neq 0$ we use the specific junction conditions to match the two spacetimes. We find the exact equilibrium radius of the shell from non-black hole solution and show that a linear perturbation leaves the TSW stable.

Keywords: $f(R)$ gravity, thin-shell wormhole, junction conditions, non-linear electrodynamics, stability.

ÖZ

Doğrusal olmayan elektromagnetik alanla minimal kuplajlı özel bir $f(R)$ yerçekim alan modelinde İnce Zar Solucen Deliklerı (İZSD) incelenmektedir. Önce yeni bir statik, Küresel simetrık çözüm buluyoruz. Bunu kullanarak kesip- yapıştırma yöntemi ile İZSD tanımlanıyor. Üçüncü derece türev $f'''(R) \neq 0$, durumunda bağlanma değerleri ile iki uzay-zaman birleştiriliyor. Kesin denge durumu yarıçapı etrafında karadelik içermeyen çözüm kullanılarak doğrusal sarsıma kararlı bır İZSD elde ediyoruz.

Anahtar Kelimeler: $f(R)$ yerçekimi, ince-zar solucan deliği, bağlantı şartları, doğrusal olmayan elektrodinamik, kararlılık.

DEDICATION

To My Family

&

Teacher

ACKNOWLEDGMENTS

Before everything I have to be thankful of my supervisor Prof. Dr. S. Habib Mazharimousavi for all of these years that he always has been a mentor, teacher, a big brother to me, for his guidance during my master studying, a teacher who inspired me to continue and helping me in all aspect of my life.

I would like to record my gratitude to Prof. Dr. M. Halilsoy who also helped me out with his comments about my thesis.

On the other hand, I would like to acknowledge the members of my graduate committee, our head of department Prof. Dr. İzzet Sakallı and also Dr. Mustafa Riza for their advice and guidance. I would like to acknowledge Prof. Dr. A. Bilsel the member jury.

My thanks go to my friends who helped and encouraged me during the period of my studies and this thesis such as Dr. Danial Forghani

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

In Einstein's general theory of relativity described by the Einstein-Hilbert (EH) action supplemented by an energy-momentum, in general exotic, construction of thin-shell 1wormholes (TSWs) [1-3], has turned almost into a routine process. The original idea by Visser [4] was to localize the non-physical source on a thin-layer, leaving the rest of the bulk with a physical source. Similar constructions of TSWs in modified, highly non-linear theories have also been attempted with considerable handicaps [5-7]. Amongst those modified theories $f(R)$ theory [8] has already been much popular during recent decades. In this approach, the action of EH is modified into an arbitrary junction of the Ricci scalar denoted by the $f(R)$ [9-10] theory. In general, such a theory may attain the EH limit or not. For physical requirements, however, the $f(R)$ theory must reproduce all the experimental tests that Einstein's theory has successfully passed. Besides, the stability criterion, as well as the absence of ghosts conditions, must be satisfied before the $f(R)$ theory is considered feasible [11].

In this thesis our aim is restricted by construction of TSWs in a particular $f(R)$ theory given by $f(R) = R + 2\alpha\sqrt{R+R_0} + R_1$ [12]. in which α, R_0 and R_1 are dimensionful constant, parameters of the theory. For $\alpha = 0$, the theory reduces to the EH form in which R_1 acts as a cosmological constant. Our choice of $f(R)$ relies on an exact solution in the presence of non-linear electromagnetism. The extended source of our $f(R)$ is provided by a Lagrangian of non-linear electrodynamics (NED) of the form

$\mathscr{L} = -\frac{1}{4\pi}\left(\mathscr{F} + 2\beta\sqrt{-\mathscr{F}}\right)$, in which $\mathscr{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the electromagnetic invariant and β is a constant parameter. Let us add that the same model of $f(R)$ was considered previously within the context of linear Maxwell electrodynamics [13] and Yang-Mills fields [14]. In this approach, our NED is powered by a pure electric field without a magnetic component so that from the outset our problem is assumed static. Once obtain an exact solution of the model our next step is to search for the proper junction conditions required in the construction of TSWs. We reviewed the junction conditions valid for general relativity proposed long ago by Israel [15-16] do not apply in the present problem without modifications. We search for an extension of those conditions and arrive at the conditions [17-18] applicable whenever we have $f''' = \frac{d^3 f}{dR^3} \neq 0$. To construct TSWs we employ the exact solutions for black holes or non-black holes. Our finding in the present problem is that although we obtain black hole solution it's event horizon lies outside the possible radius/ throat of the TSW. Since this is not admissible for such passage through the throat from one universe to the other we had to abandon the black hole solution and be satisfied only with the non-black hole solution with a naked singularity. As a matter of fact, this is the case that we encountered first: usually, in other models, it was possible to choose the radius of the shell arbitrary outside the event horizon of the available black hole. Now we face a situation that the thin-shell can not be located arbitrarily. The possible location of the shell which is determined by the theory contains a naked singularity at the center instead of a black hole. Once we fix our thin-shell appropriately to serve as a throat our next task is to perturb the resulting TSW. We do and find out that for stability to be effective a non-barotropic equation of state must be imposed at the throat after the perturbation. This implies that the pressure (p) and energy density (σ) on the shell are related by $p = \mathscr{P}(a, \sigma)$

where a stands for the time-dependent radius of the shell. Such a type of variable equation of state [19] was proposed a priori but here it arises in a natural way which can be considered an interesting result. Naturally, if our TSW was not stable it would collapse at the slightest perturbation to the central naked singularity. Fortunately, this does not happen, for tuned parameters we obtain a spherical harmonic oscillation about the equilibrium throat of our TSW.

The thesis is organized as follows. In chapter 2 we introduce our moded and derive exact solutions. TSW construction in the model and its stability is analyzed in chapter 3. We summarize our results in a conclusion which appears in chapter 4.

Chapter 2

JUNCTION CONDITIONS

In this Chapter we start with a brief calculation that leads us to the Israel conditions. Then, we review Visser's works to describe the stability of thin-shell wormholes(TSWs).

2.1 Israel Junction Conditions

Let's define a general diagonal metric as

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \tag{1}$$

or we can say that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{2}$$

in which $g_{\mu\nu} = diag\left[g_{tt}, g_{rr}, g_{\theta\theta}, g_{\phi\phi}\right]$. We may represent the metric in Gaussian normal coordinates such that

$$ds^2 = \varepsilon dw^2 + \gamma_{ij}dx^i dx^j, \tag{3}$$

in which $\varepsilon = +1, -1$ for spacelike and timelike respectivley. Furthermore assume a hypersurface $F(x^a) := \omega = 0$ in which $\frac{\partial F}{\partial w} = 1$ and $\frac{\partial F}{\partial x^i} = 0$. The normal 4-vector to the hyperplane F is defined by

$$n_\mu = \frac{1}{\sqrt{\Delta}}\frac{\partial F}{\partial x^\mu}, \tag{4}$$

in which $\Delta = \frac{\partial F}{\partial x^\mu}\frac{\partial F}{\partial x^\nu}g^{\mu\nu}$ such that $n_\mu n^\mu = \varepsilon$. Applying (4) one finds $n_t = \frac{1}{\sqrt{\Delta}}$, $n_r = 0$,

$n_\theta = n_\phi = 0$, $\frac{1}{\Delta} g^{tt} n_t^2 = 1 \Rightarrow \Delta = g^{tt} \Rightarrow$

$$n_\mu = \frac{1}{\sqrt{g_{tt}}}(1,0,0,0). \tag{5}$$

The first fundamental form of the hyperplane is defined to be

$$h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j}, \tag{6}$$

in which ξ^i are the coordinates on the hyperplane. The second fundamental form is also defined as

$$K_{ij} = -n_\rho \left(\frac{\partial^2 x^\rho}{\partial \xi^i \partial \xi^j} + \Gamma^\rho_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j} \right). \tag{7}$$

in which $\Gamma^\rho_{\mu\nu}$ are the Christofell's symbols, defined by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda}). \tag{8}$$

Furthermore, the Guass-Codazzi [20] equations are given by

$$R_{wiwj} = K_{ij,w} + K_{mj} K_i^m, \tag{9}$$

$$R_{wijm} = \nabla_j K_{im} - \nabla_m K_{ij}, \tag{10}$$

and

$$R_{lijm} = {}^3R_{lijm} + \varepsilon \left[K_{ij} K_{lm} - K_{im} K_{lj} \right]. \tag{11}$$

Herein, R stands for the Riemann tensor of the original bulk spacetime while ${}^3R_{lijm}$ stands for the same tensor for three dimensional hypersurface.

On the other hand, Einstein's field equations are given by

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{12}$$

in which $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and $R = {}^4R$ is the Ricci scalar of the bulk spacetime. Herein $\kappa = 8\pi G$ is the Einstein's constant with G the Newton's gravitational constant.

Our aim is to apply the Einstein's field equations and in the limit, find the junction conditions. Hence, we start with calculating the Einstein's tensor. To do so, first, we need to know the Ricci tensor's components; which may be found as follows.

The first component is R_{ww} to be found as

$$R_{ww} = R^i_{wiw} = g^{ij} R_{jwiw} = g^{ij} R_{wjwi}. \tag{13}$$

We note that the Riemann tensor is skew symmetry which means that $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha}$. This property was used in the last step Eq.(13). Applying (9) in (13) one finds

$$R_{ww} = g^{ij} \left[K_{ij,w} + K_{mi} K^m_j \right] = K^i_{i,w} + K_{mi} K^{mi} = K_{,w} + tr(K^2), \tag{14}$$

where $K = K^i_i$ is the trace of K^j_i and $tr(K^2) = K_{mi} K^{mi}$. Please note the $tr(K^2) \neq tr\left(K^j_i\right)$. Similarly, we obtain

$$R_{wi} = R^j_{wji} = g^{lj} R_{lwji} = -g^{lj} R_{wlji}. \tag{15}$$

Again, applying (10) in (15) we find

$$R_{wi} = -g^{lj}\left(K_{li;j} - K_{lj;i}\right) = -K^{j}_{i;j} + K^{j}_{j;i} = K_{;i} - K^{j}_{i;j}, \tag{16}$$

$$R_{wi} = -\nabla_j K^j_i + \nabla_j K^j_w. \tag{17}$$

Let's add that K is a scalar and $K_{;\mu} = K_{,\mu}$. Also for the second step we used the property of metric tensor $g^{\mu\nu}_{;\lambda} = 0$. Finally, we write

$$R_{ij} = R^{\mu}_{i\mu j} = R^{m}_{imj} + R^{w}_{iwj} = g^{ml}R_{limj} + g^{ww}R_{wiwj} = -g^{ml}R_{lijm} + \varepsilon R_{wiwj}. \tag{18}$$

Applying (9) and (11) in (17) one obtains

$$\begin{gathered} R_{ij} = -g^{mj}\left({}^3R_{lijm} + \varepsilon\left[K_{ij}K_{lm} - K_{im}K_{lj}\right]\right) \\ \varepsilon\left(K_{ij,w} + K_{mj}K^m_i\right) = +{}^3R_{ij} + \varepsilon\left[-K_{ij}K^m_m + K^l_i K_{lj}\right] + \varepsilon\left(K_{ij,w} + K_{mj}K^m_i\right). \end{gathered} \tag{19}$$

After some manipulation we find

$$\begin{aligned} R_{ij} &= {}^3R_{ij} + \varepsilon\left[-K_{ij}K + K^l_i K_{jl} + K_{ij,w} + K^m_i K_{jm}\right] = \\ &= {}^3R_{ij} + \varepsilon\left[2K_{il}K^l_j + K_{ij,w} - KK_{ij}\right]. \end{aligned} \tag{20}$$

Please note that due to the symmetry property of the second fundamental form K_{ij}; i.e., $K_{ij} = K_{ji}$; the two terms $K^l_i K_{jl}$ and $K^m_i K_{jm}$ are identical.

In the Einstein's tensor, in addition to $R_{\mu\nu}$ we need also $R = R^{\mu}_{\mu}$. Hence we write,

$$
\begin{aligned}
R = R^{\mu}_{\mu} = R^{w}_{w} + R^{i}_{i} = \\
= g^{ww}R_{ww} + g^{\mu i}R_{\mu i} = g^{ww}R_{ww} = g^{ji}R_{ji} + \\
\varepsilon R_{ww} + g^{ij}\left({}^{3}R_{ij} + \varepsilon\left[2K_{il}K + K_{ij,w} - KK_{ij}\right]\right).
\end{aligned}
\tag{21}
$$

Considering the terms, one finds

$$
\begin{aligned}
R = \varepsilon\left(K_{,w} + tr(K^2)\right) + {}^{3}R + \varepsilon\left[2K_{il}K^{il} + K^{i}_{i,w} - KK^{i}_{i}\right] \\
= {}^{3}R + \varepsilon\left[3tr(K^2) + 2K_{,w} - K^2\right].
\end{aligned}
\tag{22}
$$

Finally,

$$
R = {}^{3}R + \varepsilon\left[3tr(K^2) + 2K_{,w} - K^2\right]. \tag{23}
$$

Having all components of the Ricci tensor and the Ricci scalar found, we write the components of the Einsteins's field equations as follows

$$
\begin{aligned}
G_{ww} = \kappa T_{ww} \\
R_{ww} - \frac{1}{2}R = \kappa T_{ww} \\
K_{,w} + tr(K^2) - \frac{1}{2}\left[{}^{3}R + \varepsilon\left(3tr(K^2) + 2K_{,w} - K^2\right)\right] = \kappa T_{ww}.
\end{aligned}
\tag{24}
$$

This can be simplified as

$$
-\frac{1}{2}tr(K^2) - \frac{1}{2}{}^{3}R + \frac{1}{2}K^2 = \kappa T_{ww}. \tag{25}
$$

Next, we write,

$$
G_{w\mu} = \kappa T_{w\mu}, \tag{26}
$$

or equivalently

$$R_{w\mu} - \frac{1}{2} g_{w\mu} R = \kappa T_{w\mu}, \tag{27}$$

which knowing that $g_{wi} = 0$, implies

$$R_{wi} = \kappa T_{wi}, \tag{28}$$

and consequently

$$\nabla_i K - \nabla_j K_i^j = \kappa T_{ij}. \tag{29}$$

Finally, the last component of the Einstein's equations gives ;

$$G_{ij} = \kappa T_{ij}, \tag{30}$$

or equivalently

$$R_{ij} - \frac{1}{2} R g_{ij} = \kappa T_{ij}. \tag{31}$$

From (19) and (22) we find

$$\begin{gathered} {}^3R_{ij} + \varepsilon\left[2K_{il}K_j^l + K_{ij,w} - KK_{ij}\right] - \\ \frac{1}{2} g_{ij}\left[{}^3R + \varepsilon\left(3tr(K^2) + 2K_{,w} - K^2\right)\right] = \kappa T_{ij}. \end{gathered} \tag{32}$$

With simplification , we get

$$\left({}^3R_{ij} - \frac{1}{2}{}^3R g_{ij}\right) + \varepsilon\left[2K_{il}K_l^l + K_{ij,w} - KK_{ij} - \frac{3}{2} g_{ij} tr(K^2) - g_{ij}K_{,w} + \frac{1}{2} g_{ij} K^2\right]. \tag{33}$$

Next, we integrate all components of the Einstein's equation and find the limit $w \to 0$

as follows :

$$\lim_{w\to 0}\int_{-w}^{w} dwG_{ww} = \lim_{w\to 0}\int_{-w}^{w} dwT_{ww}, \tag{34}$$

and so on . Knowing the second fundamental form, Ricci tensors and Ricci scalars are regular functions implies

$$\lim_{w\to 0}\int_{-w}^{w} G_{ww}dw = 0, \quad \lim_{w\to 0}\int_{-w}^{w} G_{wi}dw = 0, \tag{35}$$

but

$$\lim_{w\to 0}\int_{-w}^{w} G_{ij}dw = \varepsilon\left[-g_{ij}K + K_{ij}\right]\Bigg|_{0^-}^{0^+} \neq 0. \tag{36}$$

On the other hand, if we assume $T_{\mu\nu} = T^{-}_{\mu\nu}\Theta(-w) + T^{+}_{AB}\Theta(+w) + S_{\mu\nu}\delta(w)$ we find

$$\lim_{w\to 0}\int_{-w}^{w} T_{ww}dw = S_{ww} = 0, \tag{37}$$

$$\lim_{w\to 0}\int_{-w}^{w} T_{wi}dw = S_{wi} = 0. \tag{38}$$

In summary, the junction conditions reduce to

$$\varepsilon\Bigg[K_{ij} - Kg_{ij}\Bigg]_{-}^{+} = \kappa S_{ij}, \tag{39}$$

in which $\big[X\big]_{-}^{+}$ implies $\big[X\big]^{+} - \big[X\big]^{-}$.

2.2 Junction Conditions For $f(R)$ Gravity

For a general spherically symmetric spacetime in $f(R)$ modified theory of gravity, let's assume Σ to be a timelike hyperplane. As of the first chapter, we consider the line element of the spacetime to be as

$$ds_B^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{40}$$

with the induced metric on the timelike hyperplane Σ given by (See Fig.1)

$$ds_\Sigma^2 = h_{ij}d\xi^i d\xi^j, \tag{41}$$

in which $h_{ij} = g_{\mu\nu}\frac{dx^\mu}{d\xi^i}\frac{dx^\nu}{d\xi^j}$. Furthermore, we assume that $n_\mu = \frac{1}{\sqrt{\Delta}}\frac{dF}{dx^\mu}$ to be the spacelike 4-normal to the hyperplane Σ. In [11], it was proved that in any general nonlinear $f(R)$ gravity such that $f(R) \neq R$, K_i^i and R must be continuous i.e.

$$\left[K_i^i\right] = K_i^{i+} - K_i^{i-} = 0, \tag{42}$$

and

$$\left[R\right] = R^+ - R^- = 0. \tag{43}$$

Note that K_i^i is the trace of K_i^j, the second fundamental form of the hyperplane , while R is the Ricci scalar of the bulk spacetime.

Let's also note that while condition (42) is required for all nonlinear $f(R)$ gravity, the condition (43) is avoidable if $f'''(R) = 0$. For instance R^2 gravity satisfies $f'''(R) = 0$. Following [11], we also divide the junction conditions in $f(R) \neq R$ gravity into two sections

i)$f'''(R) \neq 0$

For this wide class of $f(R)$ gravity the generalized junction conditions are given by the

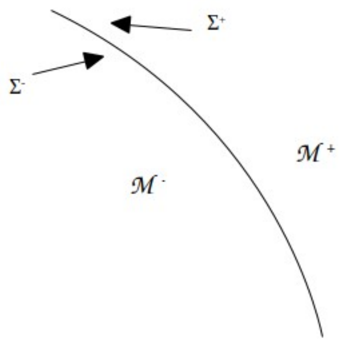

Figure 1: A plot of the timelike hyperplanes $\Sigma^{\pm}$ within the bulk $\mathscr{M} = \mathscr{M}^{+} \cup \mathscr{M}^{-}$

conditions (42) and (43) together with the following ;

$$\kappa S_i^j = \Big(-f'(R)[K_i^j] + f''(R) n^{\mu} [\nabla_{\mu} R] \delta_i^j \Big)_{\Sigma}, \tag{44}$$

in which all functions must be evaluated at Σ . Furthermore one can show that (44) implies

$$n^i S_{ij} = 0. \tag{45}$$

ii) $f'''(R) = 0$

In this rather special case, in addition to the condition (42) the following condition must be satisfied

$$\kappa S_i^j = -\{1 + \alpha(R^+ + R^-\}[K_i^j] + \alpha\{2a\delta_i^j - [R](K_i^{j+} + K_i^{j-})\}, \tag{46}$$

in which $\alpha = \frac{1}{2} f''(R)$, is a constant and $a = n^i[n_i R]$ defined on Σ.

Chapter 3

THIN-SHELL WORMHOLES IN $f(R)$-GRAVITY

3.1 The bulk solution in $f(R)$ theory of gravity

The action of the $f(R)$ modified theory of gravity coupled with a nonlinear Maxwell Lagrangian is given by

$$I = \int \sqrt{-g}d^4x\left(\frac{f(R)}{2\kappa} + \mathscr{L}(\mathscr{F})\right), \tag{47}$$

in which $\mathscr{L}(\mathscr{F})$ is the nonlinear Maxwell Lagrangian given by

$$\mathscr{L}(\mathscr{F}) = -\frac{1}{4\pi}\left(\mathscr{F} + 2\beta\sqrt{-\mathscr{F}}\right), \tag{48}$$

where $\mathscr{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant and

$$f(R) = R + 2\alpha\sqrt{R+R_0} + R_1. \tag{49}$$

Herein, R_0, α and R_1 are theory constants and $F_{\mu\nu}$ is the electromagnetic tensor defined through

$$\mathbf{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu. \tag{50}$$

Variation of the action (47) with respect to the metric tensor $g_{\mu\nu}$ yields the Einstein's field equations given by

$$f'(R)R_{\mu}^{\nu}+\left(\Box f'(R)-\frac{1}{2}f(R)\right)\delta_{\mu}^{\nu}-\nabla^{\nu}\nabla_{\mu}f'(R)=\kappa T_{\mu}^{\nu}, \tag{51}$$

in which $f'(R)=\frac{df}{dR}$, and $\Box\psi=\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\partial^{\mu})\psi$.

Furthermore, the energy momentum tensor T_{μ}^{ν} is given by

$$T_{\mu}^{\nu}=\mathscr{L}(\mathscr{F})\delta_{\mu}^{\nu}-F_{\mu\lambda}F^{\nu\lambda}\frac{\partial\mathscr{L}(\mathscr{F})}{\partial\mathscr{F}}. \tag{52}$$

In this study we choose the spacetime to be spherically symmetric and static whose line element is given by

$$ds^2=-\psi(r)dt^2+\frac{dr^2}{\psi(r)}+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right). \tag{53}$$

The nonlinear Maxwell's field equations are also found by the variation of the action with respect to the vector potential A_{μ} and is given by

$$d\left({}^{\star}\mathbf{F}\frac{\partial\mathscr{L}(\mathscr{F})}{\partial\mathscr{F}}\right)=0, \tag{54}$$

in which the dual field ${}^{\star}\mathbf{F}$ is defined by

$${}^{\star}\mathbf{F}=\frac{1}{2}{}^{\star}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}, \tag{55}$$

where ${}^{\star}F_{\mu\nu}=\frac{1}{2}\varepsilon^{\alpha\beta}{}_{\mu\nu}F_{\alpha\beta}$, in which $\varepsilon_{\mu\nu}^{\alpha\beta}=g^{\sigma\alpha}g^{\lambda\beta}\varepsilon_{\sigma\lambda\mu\nu}$ such that $\varepsilon_{\sigma\lambda\mu\nu}$ is the Levi-Civita tensor.

The Maxwell field used in this study is a pure electric field given by

$$\mathbf{F} = E(r)dt \wedge dr, \tag{56}$$

with its dual field

$$^{\star}\mathbf{F} = {}^{\star}(E(r)dt \wedge dr) = E^{\star}(dt \wedge dr). \tag{57}$$

To find the dual of $dt \wedge dr$ we use the line element (53) which gives

$$^{\star}\left(\sqrt{\psi}dt \wedge \frac{dr}{\sqrt{\psi}}\right) = rd\theta \wedge r\sin\theta d\phi, \tag{58}$$

after simplification we obtain

$$^{\star}(dt \wedge dr) = r^2 \sin\theta d\theta \wedge d\phi. \tag{59}$$

The nonlinear Maxwell's equation, then becomes

$$d\left(E\mathscr{L}_{\mathscr{F}}r^2 \sin\theta d\theta \wedge d\phi\right) = 0, \tag{60}$$

which upon the fact that $E = E(r)$ it yields

$$E(r)\mathscr{L}_{\mathscr{F}}r^2 = constant = C. \tag{61}$$

Here, C is an integration constant. On the other hand , $\mathscr{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}F_{tr}F^{tr}$. Explicitly one gets

$$\mathscr{F} = \frac{1}{2}g^{tt}g^{rr}F_{tr}^2 = -\frac{1}{2}E^2. \tag{62}$$

Combining (62), (61) and (48) one finds

$$\mathscr{L}_{\mathscr{F}} = -\frac{1}{4\pi}(1 - \frac{\beta}{\sqrt{-\mathscr{F}}}), \tag{63}$$

and

$$-E\frac{1}{4\pi}(1 - \frac{\beta}{\frac{1}{\sqrt{2}}|E|})r^2 = C. \tag{64}$$

This equation implies $(E > 0)$

$$E - \sqrt{2}\beta = -4\pi\frac{\tilde{C}}{r^2} \qquad\qquad E > 0, \tag{65}$$

or after redifining the constant $\tilde{C} = -4\pi C$ it gives .

$$E = \sqrt{2}\beta + \frac{\tilde{C}}{r^2}. \tag{66}$$

Let's note that the nonlinear Maxwell's Lagrangian (48), in the limit $\beta \to 0$, reduces to the linear Maxwell Lagrangian i.e,

$$\lim_{\beta\to 0}\mathscr{L}(\mathscr{F}) = -\frac{1}{4\pi}\mathscr{F}. \tag{67}$$

Hence, we expect the electeric field found in (66) to reduce to the classical electric field $\frac{Q}{r^2}$ in the limit $\beta \to 0$. This, however, reveals the nature of $\tilde{C}$ to be the electric charge i .e, $\tilde{C} = Q$. As a result the electric field of the nonlinear Maxwell theory is

$$E = \sqrt{2}\beta + \frac{Q}{r^2}. \tag{68}$$

Having the closed form of the electric field and consequently the Maxwell's invariant

$\mathscr{F} = -\frac{1}{2}E^2$ one finds from (52)

$$T_r^r = T_t^t = \mathscr{L} - F_{tr}F^{tr}\mathscr{L}_{\mathscr{F}} = \mathscr{L} - 2\mathscr{F}\mathscr{L}_{\mathscr{F}}, \tag{69}$$

and

$$T_\theta^\theta = T_\phi^\phi = \mathscr{L}. \tag{70}$$

Explicitly, one finds

$$\begin{gathered} T_t^t = T_r^r = -\frac{1}{4\pi}(\mathscr{F} + 2\beta\sqrt{-\mathscr{F}}) - 2\mathscr{F}\{-\frac{1}{4\pi}(1 - \frac{\beta}{\sqrt{-\mathscr{F}}})\} = \frac{\mathscr{F}}{4\pi} = \\ \frac{-\frac{1}{2}E^2}{4\pi} = -\frac{1}{8\pi}(\sqrt{2}\beta + \frac{Q}{r^2})^2, \end{gathered} \tag{71}$$

and

$$T_\theta^\theta = T_\phi^\phi = -\frac{1}{4\pi}(\mathscr{F} + 2\beta\sqrt{-\mathscr{F}}) = -\frac{1}{4\pi}(-\frac{1}{2}E^2 + 2\frac{\beta}{\sqrt{2}}E) = \frac{-E}{8\pi}(\sqrt{2}\beta - \frac{Q}{r^2})^2. \tag{72}$$

3.1.2 Solution of the Einstein's field equations

In this section we use the energy momentum tensor's components found in the previous section to solve the Einstein's field equaitions (51).

We start with the tt, and rr component of the Einstein's field equations, which read

$$\begin{aligned} f'(R)R_t^t + \left(\Box f'(R) - \frac{1}{2}f\right) - \nabla^t\nabla_t f'(R) = \kappa T_{t,r}^t. \\ f'(R)R_r^r + \left(\Box f'(R) - \frac{1}{2}f\right) - \nabla^r\nabla_r f'(R) = \kappa T_r^r. \end{aligned} \tag{73}$$

where

$$R_t^t = R_r^r = -\frac{r\psi'' + 2\psi'}{2r}. \tag{74}$$

Subtracting components of (73) one gets

$$\nabla^t \nabla_t f'(R) = \nabla^r \nabla_r f'(R). \tag{75}$$

This is because , $R^t_t = R^r_r$ and $T^t_t = T^r_r$. Next, we find $\nabla^t \nabla_t f'$ and $\nabla^r \nabla_r f'$ bu applying the definition of the covariant derivative; i.e. ,

$$\nabla^\mu F_\nu = g^{\mu\alpha} \nabla_\alpha F_\nu = g^{\mu\alpha}(F_{\nu;\alpha}) = g^{\mu\alpha}(F_{\nu,\alpha} - \Gamma^\lambda_{\alpha\nu} F_\lambda). \tag{76}$$

Here $F_\nu = \nabla_\nu f'$ and $\Gamma^\lambda_{\alpha\nu}$ is the Christoffel symbols .Explicitly

$$\nabla^t \nabla_t f' = g^{t\alpha}\left((\partial_t f')_{,\alpha} - \Gamma^\lambda_{\alpha t} \partial_\lambda f'\right) = g^{tt}\left[f'_{,t,t} - \Gamma^\lambda_{tt} \partial_\lambda f'\right] = -g^{tt} \Gamma^\lambda_{tt} f'_{,\lambda}. \tag{77}$$

As , f' is a function of R only, with R given by

$$R = R^\mu_\mu = 2R^t_t + 2R^\theta_\theta = -\frac{r^2\psi'' + 5r\psi' + 2(\psi - 1)}{r^2}, \tag{78}$$

then,$f'_{,t,t} = f'_{,t} = 0$ and only $f'_{,r} \neq 0$. Hence,

$$\nabla^t \nabla_t f' = -g^{tt} \Gamma^r_{tt} f'_{,r}. \tag{79}$$

Here, $\Gamma^r_{tt} = \frac{1}{2}\psi\psi'$ and consequently

$$\nabla^t \nabla_t f' = \frac{1}{\psi}\frac{1}{2}\psi\psi' f'_{,r} = \frac{1}{2}\psi' f'_{,r}. \tag{80}$$

A similiar calculation leads to

$$\nabla^r \nabla_r f' = g^{r\alpha}\left[f'_{,r,\alpha} - \Gamma^{\lambda}_{\alpha r} f'_{,\lambda}\right] = g^{rr}\left(f'_{,r,r} - \Gamma^{\lambda}_{rr} f'_{,\lambda}\right) = \psi f'_{,r,r} - \psi \Gamma^{\lambda}_{rr} f'_{,\lambda}. \tag{81}$$

One finds, $\Gamma^{\lambda}_{rr} = -\frac{1}{2}\frac{\psi'}{\psi}$ only nonzero component of Γ^{λ}_{rr} which implies ;

$$\nabla^r \nabla_r f' = \psi f'_{,r,r} - \psi(-\frac{1}{2}\frac{\psi'}{\psi}) f'_{,r} = \psi f'_{,r,r} + \frac{1}{2}\psi' f'_{,r}. \tag{82}$$

Finally (75) becomess :

$$\frac{1}{2}\psi' f'_{,r} = \psi f'_{,r,r} + \frac{1}{2}\psi' f'_{,r}, \tag{83}$$

which implies

$$\psi f'_{,r,r} = 0, \tag{84}$$

or $f'_{,r,r} = 0$. This means that the second derivative of $f'(R)$ with respect to r must be zero. Integration gives

$$f'(R) = c_0 + c_1 r, \tag{85}$$

in which c_0 and c_1 are two integration constants. On the other hand , $f(R)$ is given by (49) , therefore

$$\frac{d}{dR}(R + 2\alpha\sqrt{R+R_0} + R_1) = c_0 + c_1 r. \tag{86}$$

Once we can solve this equation to find R , i.e.

$$R = -R_0 + \frac{\alpha^2}{(c_1 r + c_0 - 1)^2}. \tag{87}$$

By setting $c_0 = 1$, the solution for R becomes

$$R = -R_0 + \frac{\alpha^2}{c_1^2 r^2}. \tag{88}$$

In terms of the metric function ψ we have R to be given in (78) and (88) one finds

$$-\frac{r^2\psi'' + 5r\psi' + 2(\psi - 1)}{r^2} = -R_0 + \frac{\alpha^2}{c_1^2 r^2}, \tag{89}$$

which can be solved for ψ , the solution is given by

$$\psi = 1 - \frac{6\alpha^2}{c_1^2} + \frac{R_0}{12}r^2 + \frac{c_2}{r} + \frac{c_3}{r^2}, \tag{90}$$

in which c_2 and c_3 are the integration constants. Let's add that the metric function found in (90) has to satisfy all field equations and through that we find the nature of the parameters in this solution.

Once more we look at the tt and rr components of the Einstein's field equations. Knowing $f'(R) = 1 + c_1 r$ one finds

$$\nabla^t \nabla_t f' = \frac{1}{2}\psi' f'_{,r} = \frac{1}{2}\psi'(c_1) = \frac{c_1}{2}\psi', \tag{91}$$

and

$$\Box f' = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu f' = \frac{1}{r^2}\partial_r\left[r^2 g^{rr}\partial_r\right](1 + c_1 r) = \frac{1}{r^2}\partial_r\left[r^2\psi c_1\right] = \frac{c_1}{r^2}(2r\psi + r^2\psi'). \tag{92}$$

Finally the tt or rr components of the field equations implies

$$(1+c_1 r)R_t^t + \left(\frac{c_1}{r^2}(2r\psi + r^2\psi') = \frac{1}{2}f\right) - \frac{c_1}{2}\psi' = \kappa(-\frac{1}{8\pi})(\sqrt{2}\beta + \frac{Q}{r^2})^2. \tag{93}$$

Here, f is found to be ;

$$\begin{aligned} f'(R) = \frac{df}{dR} &= 1 + c_1 r \\ \Rightarrow \frac{df}{dr} &= (1+c_1 r)\frac{dR}{dr} \\ \Rightarrow f &= \int (1+c_1 r)\frac{dR}{dr}dr + c_4 \\ &= \int (1+c_1 r)(-\frac{2\alpha^2}{c_1^2 r^3})dr + c_4 \\ \Rightarrow f &= -\frac{2\alpha^2}{c_1^2}\left(\frac{r^{-3+1}}{-3+1} + c_1\frac{r^{-2+1}}{-2+1}\right) + c_4. \end{aligned} \tag{94}$$

In short

$$f = \frac{\alpha^2}{c_1^2 r^2} + \frac{2\alpha^2}{c_1 r} + c_4, \tag{95}$$

in which c_4 is an integration constant. Finally eq. (93) reveals

$$2\beta^2 - \frac{R_0}{4} - \frac{c_4}{2} = 0, \tag{96}$$

$$c_1^2 - \alpha^2 = 0, \tag{97}$$

$$4\sqrt{2}Q\beta c_1^2 + 3c_2 c_1^3 - \alpha^2 = 0, \tag{98}$$

and

$$Q^2 - c_3 = 0. \tag{99}$$

Imposing ,$c_1 = \alpha$, $c_2 = \frac{1-4\sqrt{2}\beta Q}{3\alpha}$, $c_3 = Q$ and $c_4 = 4\beta^2 - \frac{R_0}{2}$ yields

$$\psi(r) = \frac{1}{2} + \frac{R_0}{12}r^2 - \frac{4\sqrt{2}\beta Q - 1}{3\alpha r} + \frac{Q^2}{r^2}, \tag{100}$$

and

$$f(R) = \frac{1}{r^2} + \frac{2\alpha}{r} + 4\beta^2 - \frac{R_0}{2}. \tag{101}$$

Rewriting $f(R)$ in terms of R , one finds

$$f(R) = R + 2\alpha\sqrt{R + R_0} + 4\beta^2 + \frac{R_0}{2}. \tag{102}$$

Comparig this with the original form given in (49) we find

$$R_1 = 4\beta^2 + \frac{R_0}{2}. \tag{103}$$

The last two equations to be checked are the $\theta\theta$ and $\phi\phi$ components of the Einstein's field equation.

Due to the symmetry $\theta\theta$ and $\phi\phi$ components of the Einstein's equation are identical . Let's concentrate on $\theta\theta$ component only .

From (51) we find

$$f' R^{\theta}_{\theta} + (\Box f' - \frac{1}{2} f) - \nabla^{\theta}\nabla_{\theta} f' = \kappa T^{\theta}_{\theta}. \tag{104}$$

To continue we need to find

$$\nabla^{\theta}\nabla_{\theta} f' = g^{\theta\theta}[F_{\theta\theta} - \Gamma^{\lambda}_{\theta\theta} F_{\lambda}], \tag{105}$$

in which $F_\theta = \partial_\theta f' = 0$ and $F_\lambda = \partial_\lambda f'$ is nonzero only for $\lambda = r$. Hence,

$$\nabla^\theta \nabla_\theta f' = g^{\theta\theta}(-\Gamma^r_{\theta\theta} f'_{,r}) = -\frac{1}{r^2}(-\psi r) f'_{,r} = \frac{\psi}{r} f'_{,r}. \tag{106}$$

Finally we find ($c_1 = \alpha$, $\kappa = 1, G = 1$)

$$(1 + c_1 r) R^\theta_\theta + \left[\frac{c_1}{r}(2\psi + r\psi') - \frac{1}{2} f\right] - \frac{\psi}{r}(c_1) = \kappa(\frac{1}{4\pi})(-\frac{E^2}{2} + \sqrt{2}\beta E), \tag{107}$$

in which $E = \sqrt{2}\beta + \frac{Q}{r^2}$. Putting ψ and R^θ_θ gives

$$R^\theta_\theta = -\frac{r\psi' - 1 + \psi}{r^2}. \tag{108}$$

Together with f and E, one finds this equation satisfied.

In summary, we have found a solution for $f(R)$ modified theory of gravity given by (102) coupled with a nonlinear Maxwell electric field given by (47) in the form of line element (52) with the metric function (99).

This solution is a singular solution whose Ricci invariant is given by

$$R = \frac{1}{r^2} - R_0. \tag{109}$$

By setting $R_0 = 0$ the solution becomes

$$\psi = \frac{1}{2} - \frac{\mu}{3\alpha r} + \frac{Q^2}{r^2}, \tag{110}$$

in which $\mu = 4Q\sqrt{2}\beta - 1$ and

$$f(R) = R + 2\alpha\sqrt{R} + 4\beta^2, \tag{111}$$

with the nonlinear Maxwell Lagrangian

$$\mathscr{L} = -\frac{1}{4\pi}(\mathscr{F} + 2\beta\sqrt{-\mathscr{F}}). \tag{112}$$

Please note that μ in (110) plays the role of the mass of the solution which is clearly from nonlinear gravity $f(R)$ and Maxwell Lagrangian $\mathscr{L}(\mathscr{F})$. This solution admits, black hole with two horizons or a single double horizon , and naked singularity , depending on $\frac{Q}{\sqrt{2}\mu}$ less than , equal or greater than one, respectively. The two horizons are given by

$$r_{\pm} = \frac{\mu}{3\alpha}\left(1 \pm \sqrt{1 - \frac{18\alpha^2 Q^2}{\mu^2}}\right), \tag{113}$$

while the double horizon is found to be

$$r_D = \frac{1}{3}\frac{\mu}{\alpha}. \tag{114}$$

Please note also that if $\mu < \sqrt{18}\alpha Q$ there is no horizon at all. and the solution is naked singular.

3.2 Thin-Shell Wormholes in $f(R)$-Gravity coupled with Nonlinear Electromagnetism

Let's start with the line element (53) with $\psi(r)$ given by (100) the solution of the $f(R)$-gravity (102) coupled with the nonlinear electrodynamics (48).

Using the standart method of cut and paste introduced by Matt Visser in [4] and applying the generalized Israel junction conditions [16] which we reviwed in chapter (2), in this chapter we construct a thin-shell in the $f(R)$- gravity (102).

First we cut-out the region $r < a(\tau)$ from the bulk spacetime (53) and make two identical copies of the rest of the manifold and call them $\mathscr{M}^+$ and $\mathscr{M}^-$. $\mathscr{M}^+$ and $\mathscr{M}^-$ are incomplete individually but if we paste them at their boundaries $r = a(\tau)$ then the resultant Manifold i.e. , $\mathscr{M} = \mathscr{M}^+ \cup \mathscr{M}^-$ is a complete manifold, (See Fig.2).

An embedded diagram of the resultant manifold is plotted in Fig.1b . As it is seen, the two submanifolds $\mathscr{M}^+$ and $\mathscr{M}^-$ are connected with a thin-shell (time-like) $r = a(\tau)$. This spherical timelike thin-shell will be called the throat between the two submanifolds, In other words, the traveler going toward the center (let's say) of the space time $\mathscr{M}^+$, when she reaches the surface $r = a$ without realizing, enters the second spacetime $\mathscr{M}^-$. Hence $r = a(\tau)$ which is a timelike hypersurface plays the role of a gate or throat. In principle $a(\tau) > r_h$ in which r_h is the event horizon of the bulk spacetime. Therefore, the traveler never encounters a horizon in her journey from $\mathscr{M}^+$ and $\mathscr{M}^-$ or $\mathscr{M}^-$ to $\mathscr{M}^+$. The hypersurface $\Sigma^\pm := r^\pm - a(\tau) = 0$, is one of the boundary of each submanifold and we glue them at $\Sigma = r - a(\tau) = 0$. In other words, in each submanifold i.e. , $\mathscr{M}^+$, one may write

$$ds_\pm^2 = -\psi(r_\pm)dt^2 + \frac{dr_\pm^2}{\psi(r_\pm)} + r_\pm^2\left(d\theta_\pm^2 + \sin^2\theta_\pm d\phi_\pm^2\right) \tag{115}$$

(See Fig.3).

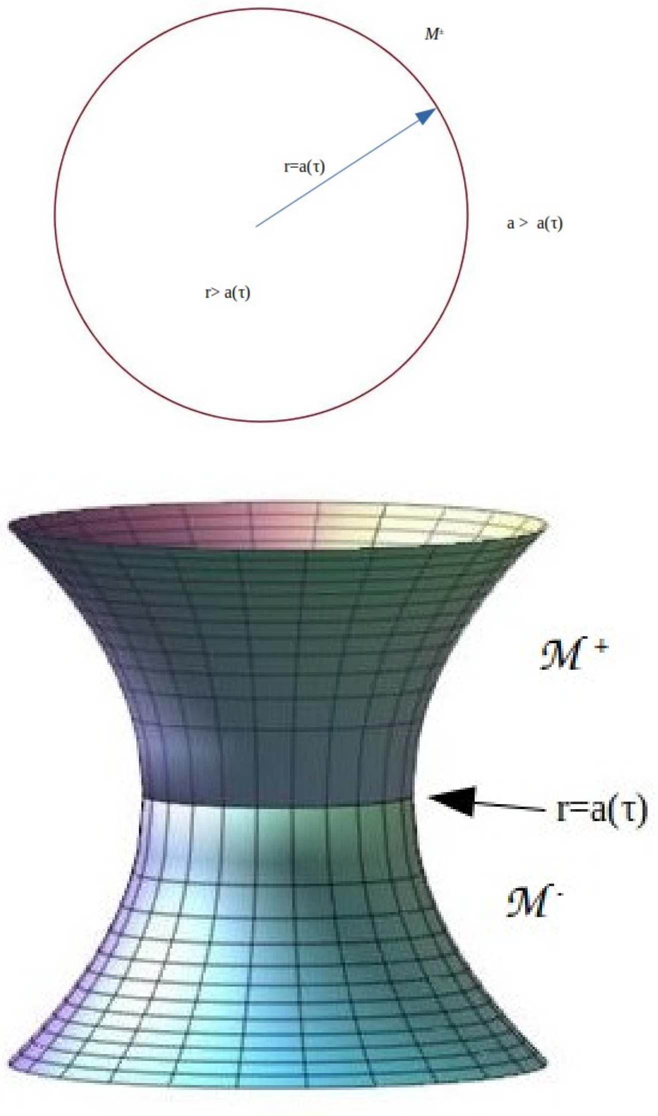

Figure 2: a) Two geometry of the throat, b) An embeded plot of the thin-shell wormhole.

The first boundary condition is to have the induced metric continuous across the throat. Using the definition of the induced metric for $\mathscr{M}^+$ and $\mathscr{M}^-$ one finds

$$h_{ij}^{\pm} = g_{\alpha\beta}^{\pm} \frac{\partial x_{\pm}^{\alpha}}{\partial \xi_{\pm}^{i}} \frac{\partial x_{\pm}^{\beta}}{\partial \xi_{\pm}^{j}} \quad , \tag{116}$$

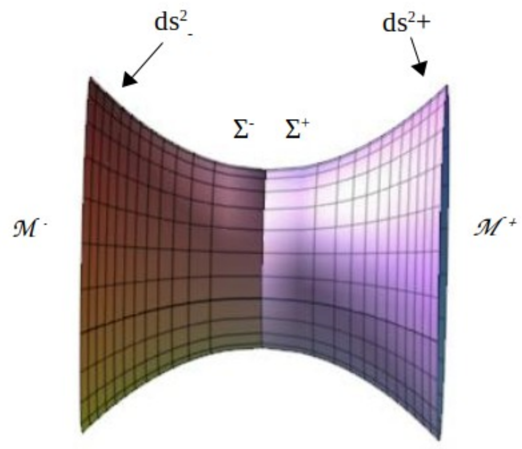

Figure 3: A plot of the geometry of the throat in the thin-shell wormhole

in which $\alpha,\beta = \{t,r,\theta,\phi\}^{\pm}$ while $i,j = \{t,\theta,\phi\}$. Explicitly

$$
\begin{aligned}
h^{\pm}_{tt} &= g^{\pm}_{\alpha\beta}\frac{dx^{\alpha}_{\pm}}{dt_{\pm}}\frac{dx\pm^{\beta}}{dt_{\pm}} \\
h^{\pm}_{tt} &= g^{\pm}_{rr}\frac{dr}{dt_{\pm}}\frac{dr}{dt_{\pm}} = g^{\pm}_{rr}\left(\frac{da(\tau)}{dt_{\pm}}\right)^2 \\
&= \frac{1}{\psi(a)}\left(\frac{da(\tau)/d\tau}{dt_{\pm}/d\tau}\right)^2 = \frac{1}{\psi(a)}\left(\frac{\dot{a}}{\dot{t}_{\pm}}\right)^2.
\end{aligned}
\tag{117}
$$

Herein , $\dot{a} = \frac{da}{d\tau}$ and $\dot{t}_{\pm} = \frac{dt_{\pm}}{d\tau}$. Having h_{tt} continuous across the thin-shell implies $\dot{t}_{+} = \dot{t}_{-}$. Eq (117) shows that $h^{+}_{\tau\tau} = h^{-}_{\tau\tau}$. Next, we find $h^{\pm}_{\theta\theta}$ as follows

$$
\begin{aligned}
h^{\pm}_{\theta\theta} &= \left(g^{\pm}_{\alpha\beta}\frac{dx^{\alpha}_{\pm}}{d\theta_{\pm}}\frac{dx^{\beta}_{\pm}}{d\theta_{\pm}}\right)_{\Sigma^{\pm}} \\
\left(g^{\pm}_{\alpha\beta}(\frac{d\theta_{\pm}}{d\theta_{\pm}})^2\right)_{\Sigma^{\pm}} &= (g^{\pm}_{\theta\theta})_{\Sigma^{\pm}} \\
&= a^2.
\end{aligned}
\tag{118}
$$

Again we see that $h^{+}_{\theta\theta} = h^{-}_{\theta\theta}$. Finally

$$
h^{\pm}_{\phi\phi} = g^{\pm}_{\phi\phi} = a^2\sin^2(\theta_{\pm}). \tag{119}
$$

Eq (119) implies that , in order to have $h_{\phi\phi}$ continuous , i.e, $h^{+}_{\phi\phi} = h^{-}_{\phi\phi}$, one should assume $\theta_{+} = \theta_{-}$. Next, we write the induced metric for each submanifold, given by

$$ds^2_{\pm} = -\psi(a)dt^2_{\pm} + \frac{\dot{a}^2 d\tau^2}{\psi(a)} + a^2(d\theta^2 + \sin^2\theta d\phi^2_{\pm}). \tag{120}$$

or after considering $\dot{t}_{+} = \dot{t}_{-}$ one may write it as

$$ds^2_{\pm} = (-\psi(a)\dot{t}^2 + \frac{\dot{a}^2}{\psi(a)})d\tau^2 + a^2(d\theta^2 + \sin^2 d\phi^2_{\pm}). \tag{121}$$

There are two points which should be clarified ; 1) ϕ_{+} may not be equal to ϕ_{-}.

2) $-\psi(a)\dot{t}^2 + \frac{\dot{a}^2}{\psi(a)}$ may be set to -1 .

Actually, physically $\phi_{+} \in [0, 2\pi]$ and $\phi_{-} \in [0, 2\pi]$. Any translation does not change $d\phi_{+}$ and $d\phi_{-}$ which implies that one can set them identical. The second point yields

$$ds^2_{\pm} = -d\tau^2 + \dot{a}^2(d\theta^2 + \sin^2 d\phi^2), \tag{122}$$

in which

$$\dot{t}^2_{\pm} = \dot{t}^2 = \frac{1}{\psi(a)}(1 + \frac{\dot{a}^2}{\psi(a)}). \tag{123}$$

From now on we refer to (122) , as the induced metric of the throat in which τ stands for the proper time on the throat . To apply the other boundary condition including (38) and (39) we must introduce the normal 4- vectors on the throat from both manifold perspective ; i.e, (See Fig. 4 and 5.) The standard definition of $n^{\pm}_{\gamma}$ are given by

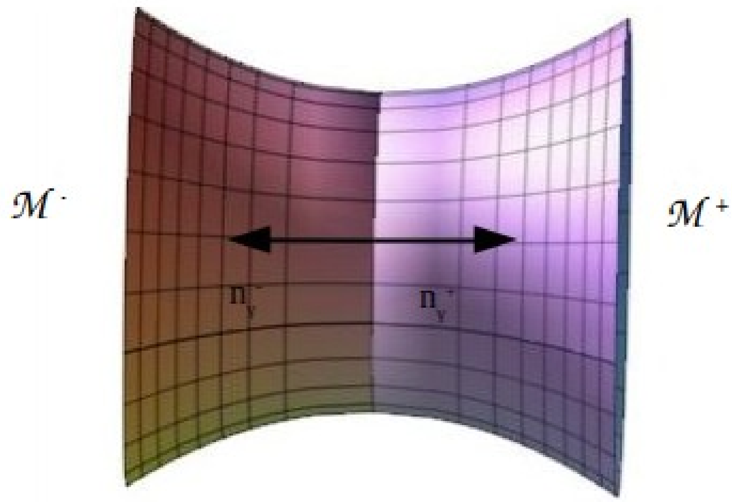

Figure 4: A plot of the Normal 4-vectors to the either side of the throat

$$n_\gamma^\pm = \frac{\pm 1}{\sqrt{\Delta^\pm}} \frac{\partial \Sigma^\pm}{\partial x^\gamma}, \tag{124}$$

in which $\Sigma^\pm := r^\pm - a(\tau) = 0$ and $\Delta^\pm$is the coefficient makes $n_\gamma^\pm n^{\pm\gamma} = 1$. The positive direction is chosen to be from the throat toward $\mathscr{M}^+$ which makes the negative direction from the throat toward $\mathscr{M}^-$.

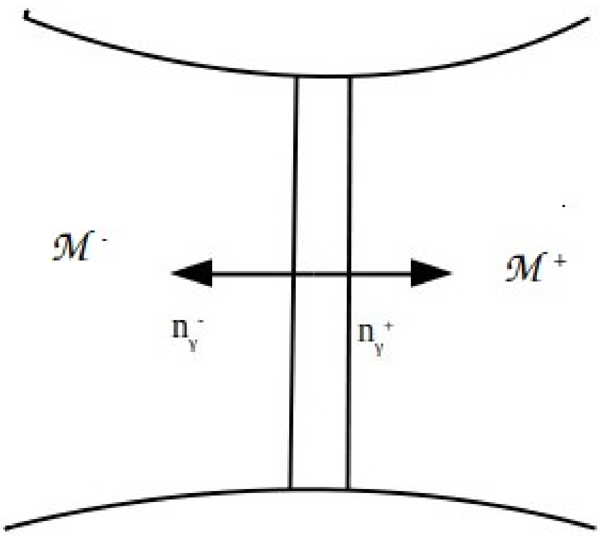

Figure 5: A plot of the direction of the normal 4-vectors $n_\gamma^\pm$

Therefore , One finds

$$n_t^{\pm} = \frac{\pm 1}{\sqrt{\Delta^{\pm}}}\Big(-\frac{da(\tau)}{dt^{\pm}}\Big) = \frac{-1}{\sqrt{\Delta^{\pm}}}\frac{\dot{a}(\tau)}{\dot{t}^{\pm}},$$
$$n_r^{\pm} = \frac{\pm 1}{\sqrt{\Delta^{\pm}}}(1) = \frac{1}{\sqrt{\Delta^{\pm}}} \tag{125}$$
$$n_{\theta}^{\pm} = n_{\phi}^{\pm} = 0.$$

Imposing $n_{\gamma}^{\pm} n^{\pm\gamma} = 1$ yields

$$g_{\pm}^{\alpha\gamma} n_{\gamma}^{\pm} n_{\alpha}^{\pm} = 1 \Rightarrow g^{tt\pm}\frac{\dot{a}^2}{\Delta^{\pm}\dot{t}^2} + \frac{rr\pm}{\Delta^{\pm}} = 1. \tag{126}$$

One finds

$$\Delta^{\pm} = g^{tt}\frac{\dot{a}^2}{\dot{t}^2} + g^{rr}, \tag{127}$$

or explicitly

$$\Delta^{\pm} = -\frac{1}{\psi}\frac{\dot{a}^2}{\frac{1}{\psi}(1+\frac{\dot{a}^2}{\psi})} + \psi = \psi - \frac{\dot{a}^2\psi}{\psi+\dot{a}^2} = \frac{\psi^2}{\psi+\dot{a}^2}. \tag{128}$$

Finally ,

$$n_{\gamma}^{\pm} = \pm\frac{\sqrt{\psi+\dot{a}^2}}{\psi}\Big(-\frac{\dot{a}}{\dot{t}}, 1, 0, 0\Big), \tag{129}$$

or using $\dot{t} = \frac{\sqrt{\psi+\dot{a}^2}}{\psi}$ one may write

$$n_{\gamma}^{\pm} = \pm\big(-\dot{a}, \dot{t}, 0, 0\big). \tag{130}$$

The next quantity to be calculated is the second fundamental form $K_{ij}^{\pm}$. Accroding to the definition one writes

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm}\left(\frac{\partial^2 x^{\gamma}}{\partial\xi^i\partial\xi^j} + \Gamma_{\alpha\beta}^{\gamma}\frac{\partial x^{\alpha}}{\partial\xi^i}\frac{\partial X^{\beta}}{\partial\xi^i}\right)\Bigg|_{\pm}, \tag{131}$$

Without loss of generality , we drop the sub/super index $\pm$ and write

$$
\begin{aligned}
K_{\tau\tau} &= -n_\gamma\left(\frac{\partial^2 x^\gamma}{\partial\tau\partial\tau} + \Gamma^\gamma_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau}\right)_\Sigma \\
&= -n_t\left(\frac{\partial^2 t}{\partial\tau^2} + 2\Gamma^t_{rt}\dot{a}\dot{t} + \Gamma^t_{rr}\dot{a}^2\right)_\Sigma - n_r\left(\frac{\partial^2 r}{\partial\tau^2 + \Gamma^r_{rr}\dot{a}^2}\right)_\Sigma,
\end{aligned}
\tag{132}
$$

which after knowing $\Gamma^r_{rr} = -\frac{\psi'}{2\psi}, \Gamma^t_{rt} = \frac{\psi'}{2\psi}$ and $\Gamma^t_{rr} = 0$ it becomes :

$$
\begin{aligned}
K_{\tau\tau} &= -(-\dot{a})(\ddot{t} + \frac{\psi'}{\psi}\dot{t}\dot{a}) - \dot{t}(\ddot{a} - \frac{\psi}{2\psi}\dot{a}^2) \\
\Rightarrow K_{\tau\tau} &= \dot{a}\ddot{t} + \frac{\psi'}{\psi}\dot{t}\dot{a} - \dot{t}(\ddot{a} - \frac{\psi'(a)}{2\psi}\dot{a}^2).
\end{aligned}
\tag{133}
$$

Furthermore,

$$
K_{\theta\theta} = -n_t\left(\frac{\partial^2 t}{\partial\theta^2} + \Gamma^t_{\theta\theta}\right) - n_r\left(\frac{\partial^2 r}{\partial\theta^2} + \Gamma^r_{\theta\theta}\right) = \left(-(-\dot{a})(0) - \dot{t}(0 + \psi a)\right) = \dot{t}a\psi(a),
\tag{134}
$$

and

$$
K_{\phi\phi} = \dot{t}a\psi(a)\sin^2\theta.
\tag{135}
$$

Explicitly we obtained

$$
K^\pm_{ij} = \pm diag\left[\dot{a}\ddot{t} + \frac{\psi'}{\psi}\dot{t}\dot{a} - \dot{t}(\ddot{a} - \frac{\psi'}{2\psi}\dot{a}^2), \dot{t}a\psi(a), \dot{t}a\psi(a)\sin^2\theta\right],
\tag{136}
$$

or in a more convenient form

$$
K^{j\,\pm}_i = \pm diag\left[-\dot{a}\ddot{t} - \frac{\psi'\dot{t}\dot{a}}{\psi} + \dot{t}(\ddot{a} - \frac{\psi'}{2\psi}\dot{a}^2), \frac{\dot{t}\psi}{a}, \frac{\dot{t}\psi}{a}\right].
\tag{137}
$$

To simplify more we remember

$$\dot{t}^2 = \frac{\psi + \dot{a}^2}{\psi^2} = \frac{1}{\psi} + \frac{\dot{a}^2}{\psi^2}, \tag{138}$$

which yields,

$$2\dot{t}\ddot{t} = -\frac{\psi' \dot{a}}{\psi^2} + \frac{2\dot{a}\ddot{a}}{\psi^2} - \frac{2\psi' \dot{a}^2 \ddot{a}}{\psi^3}, \tag{139}$$

and consequently

$$\ddot{t} = \frac{\dot{a}}{2\dot{t}} \left(\frac{2\ddot{a}}{\psi^2} - \frac{\psi'}{\psi^2} - \frac{2\psi' \dot{a}^2}{\psi^3} \right). \tag{140}$$

Note that the chain-rule has been used to write $\frac{d}{d\tau}\psi = \frac{d\psi}{da}\frac{da}{d\tau} = \psi' \dot{a}$. Finally after the simplification one finds

$$K_i^{j\,\pm} = \pm diag\left[+ \frac{\psi' + 2\ddot{a}}{2\sqrt{\psi + \dot{a}^2}}, \frac{\sqrt{\psi + \dot{a}^2}}{a}, \frac{\sqrt{\psi + \dot{a}^2}}{a} \right]. \tag{141}$$

In $f(R)$ -gravity when $f''' \neq 0$, the following two condiditions should be satisfied;

$$\left[K_i^i\right] = \left[K\right] = 0, \tag{142}$$

and

$$\left[R\right] = 0. \tag{143}$$

The second one is trivially satisfied because the two submanifolds M^+ and M^- are identical and therefore $R^+ = R^-$. The first condition however, implies

$$\frac{\psi' + 2\ddot{a}}{2\sqrt{\psi + \dot{a}^2}} + \frac{2}{a}\sqrt{\psi + \dot{a}^2} = 0. \tag{144}$$

This condition effectively gives a dynamic equation for the throat's radius. If we assume an equilibrium radius for the throat such that $\dot{a}=\ddot{a}=0$ and $a=a_0$ the (144) leads to

$$\frac{\psi'(a_0)}{2\sqrt{\psi(a_0)}}+\frac{2}{a_0}\sqrt{\psi(a_0)}=0, \tag{145}$$

or

$$a_0\psi'(a_0)+4\psi(a_0)=0. \tag{146}$$

The last junction condition to be satisfied is

$$\kappa S_i^j=-f'(R)[K_i^j]+f''(R)\left[n^\gamma\nabla_\gamma R\right]\delta_i^j, \tag{147}$$

in which $S_i^j=(-\sigma,p,p)$ is the matter energy-momentum tensor on the throat. Practically , $[n^\gamma\nabla_\gamma R]$ reduces to

$$\begin{aligned} n^\gamma\nabla_\gamma R &= (n^\gamma\nabla_\gamma R)_{\Sigma^+}-(n^\gamma\nabla_\gamma R)_{\Sigma^-}=(n^\gamma R')_{\Sigma^+}-(n^\gamma R')_{\Sigma^-}=(2n^\gamma R')_{\Sigma^+} \\ &= (2g^{rr}n_rR')_{\Sigma^+}-(2\psi(a)t\dot{R}')_{\Sigma^+}=2\sqrt{\psi+\dot{a}^2}R'. \end{aligned} \tag{148}$$

Upon (148) , one finds from (147)

$$\kappa(-\sigma)=-F'\frac{\psi'+2\ddot{a}}{2\sqrt{\psi+\dot{a}^2}}+F''2\sqrt{\psi+\dot{a}^2}R', \tag{149}$$

and

$$\kappa p=-2F'\frac{\sqrt{\psi+\dot{a}^2}}{a}+F''2\sqrt{\psi+\dot{a}^2}R'. \tag{150}$$

For the specific $f(R)$-gravity we study here (102) one finds

$$f' = 1 + \frac{\alpha}{\sqrt{R+R_0}} \quad and \quad f'' = \frac{-\alpha}{2(R+R_0)^{\frac{3}{2}}}. \tag{151}$$

with $R = -R_0 + \frac{1}{a^2}$ one finds

$$R' = \frac{-2}{a^3}, \quad f' = 1 + \alpha a, \quad f'' = -\frac{\alpha}{2}a^3, \tag{152}$$

and consequently

$$-\kappa\sigma = -(1+\alpha a)\frac{\psi' + 2\ddot{a}}{\sqrt{\psi + \dot{a}^2}} + 2(-\frac{\alpha}{2}a^3)\sqrt{\psi + \dot{a}^2}(\frac{-2}{a^3}), \tag{153}$$

and

$$\kappa p = -2(1+\alpha a)\frac{\sqrt{\psi + \dot{a}^2}}{a} + 2(-\frac{\alpha}{2}a^3)\sqrt{\psi + \dot{a}^2}(\frac{-2}{a^3}). \tag{154}$$

Finally,

$$\sigma = \frac{1}{\kappa}\frac{(1+\alpha a)(\psi' + 2\ddot{a})}{\sqrt{\psi + \dot{a}^2}} - \frac{2\alpha}{\kappa}\sqrt{\psi + \dot{a}^2}, \tag{155}$$

and

$$p = \frac{-2(1+\alpha a)}{\kappa a}\sqrt{\psi + \dot{a}^2} + \frac{2\alpha}{\kappa}\sqrt{\psi + \dot{a}^2}. \tag{156}$$

Furthermore (144) implies

$$\begin{aligned} \sigma = \frac{1+\alpha a}{\kappa}\left(-\frac{4}{a}\sqrt{\psi + \dot{a}^2}\right) - \frac{2\alpha}{\kappa}\sqrt{\psi + \dot{a}^2} \\ = -\frac{2}{\kappa a}\left(2 + 3\alpha a\right)\sqrt{\psi + \dot{a}^2} \end{aligned}. \tag{157}$$

and

$$p = -\frac{2}{\kappa a}\sqrt{\psi + \dot{a}^2}. \tag{158}$$

Therefore the equation of state of the matter on the shell is found to be

$$\frac{p}{\sigma} = \frac{1}{2+3\alpha a} = \omega, \tag{159}$$

or

$$p = \frac{1}{2+3\alpha a}\sigma = \psi(a,\sigma). \tag{160}$$

This is quite interesting that naturally the equation of state has formed to be of non-barotropic [19] . At the equilibrium state where $a = a_0$ and $\dot{a} = \ddot{a} = 0$ one finds

$$\sigma_0 = -\frac{2}{\kappa a_0}\left(2+3\alpha a_0\right)\sqrt{\psi_0}, \tag{161}$$

and

$$p_0 = -\frac{2}{\kappa a_0}\sqrt{\psi_0}. \tag{162}$$

Results :

As we have found, the metric function is given by

$$\psi(a) = \frac{1}{2} + \frac{R_0 a^2}{12} - \frac{4\sqrt{2}\beta Q - 1}{3\alpha a} + \frac{Q^2}{a^2}, \tag{163}$$

and it's first dervative is obtained

$$\psi'(a) = +\frac{4\sqrt{2}\beta Q - 1}{3\alpha a^2} - \frac{2Q^2}{a^3} + \frac{R_0}{6}a. \tag{164}$$

The condition (146) yields

$$R_0\alpha(a_0)^4 + 4\alpha(a_0)^2 + (2 - 8Q\beta\sqrt{2})a_0 + 4Q^2\alpha = 0. \tag{165}$$

This is a fourth order equation which can not be resolved analytically. However, for our interest, we set $R_0 = 0$ which simpifies significantly. Upon this, (165) becomes

$$a_0^2 + \frac{1}{2\alpha}\left(2 - 4Q\beta\sqrt{2}\right)a_0 + Q^2 = 0, \tag{166}$$

with roots at

$$a_0^{\pm} = \frac{\mu}{4\alpha}\left[1 \pm \sqrt{1 - (\frac{4\alpha Q}{\mu})^2}\right], \tag{167}$$

in which $\mu = 1 - 4Q\beta\sqrt{2}$.

On the other hand, the horizon of the solution (163) is given by

$$\psi(r) = 0. \tag{168}$$

The corresponding horizons are obtained to be

$$r_h^{\pm} = \frac{\mu}{3\alpha}\left(1 \pm \sqrt{1 - (\frac{\sqrt{18}\alpha Q}{\mu})^2}\right). \tag{169}$$

It is not difficult to see that both $a_0^{\pm}$ are smaller than $r_h^{\pm}$ which is the event horizon. The only alternative left for this solution is the non-black hole case i.e. $\psi(a) \neq 0$.

The latter implies

$$1 - (\frac{\sqrt{18}\alpha Q}{\mu})^2 \quad < 0, \tag{170}$$

but at the same time it must give a real solution for (166) . Hence,

$$0 < \quad 1 - (\frac{4\alpha Q}{\mu})^2. \tag{171}$$

Combining (71) and (169) leads to

$$16\left(\frac{\alpha Q}{\mu}\right)^2 \quad < 1 < \quad 18\left(\frac{\alpha Q}{\mu}\right)^2, \tag{172}$$

or more conveniently

$$16 < \quad \left(\frac{\mu}{\alpha Q}\right)^2 \quad < 18. \tag{173}$$

Inserting μ one finally finds

$$16 < \quad \left(\frac{1-4Q\sqrt{2}\beta}{\alpha Q}\right)^2 \quad < 18. \tag{174}$$

There are two distinct cases;

1) $-1+4Q\sqrt{2}\beta\ > 0, \alpha\ > 0\,, \beta\ > 0$,

which implies

$$4 < \quad \frac{-1+4Q\sqrt{2}\beta}{\alpha Q} \quad < \sqrt{18}. \tag{175}$$

2) $-1+4Q\sqrt{2}\beta\ < 0$, $\alpha\ > 0\,, \beta\ > 0$

$$4 < \quad \frac{1-4Q\sqrt{2}\beta}{\alpha Q} \quad < \sqrt{18}. \tag{176}$$

The First case i.e, $-1+4Q\sqrt{2}\beta\ < 0$, means $\mu < 0$ and therefore a_0 becomes negative which is not desired.

Hence, the only case left is $-1+4Q\sqrt{2}\beta\ > 0$ which implies

$$a_0 = \frac{\mu}{4\alpha}\left(1+\sqrt{1-\left(\frac{4Q\alpha}{\mu}\right)^2}\right), \tag{177}$$

both are acceptable. In Fig.6 we plot the metric function $\psi(r)$ in terms of r for $Q = 1$,

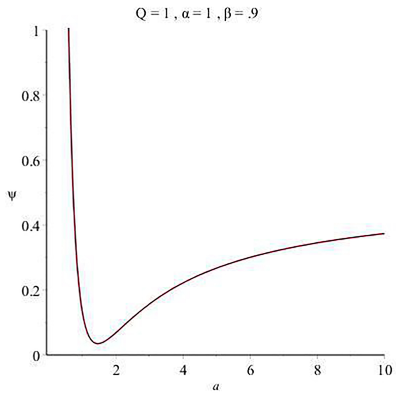

Figure 6: A plot of the Metric Function $\psi(r)$ in terms of radius r for $Q=1$, $\alpha=1$ and $\beta=0.9$.

$\alpha=1$ and $\beta=0.9$ such that $-1+4Q\sqrt{2}\beta>0$ is satisfied.

3.3 Stability Analysis

In this section we study the dynamic stability of the thin-shell wormhole solution, constructed in the previous section. In doing so, we apply a linear-radial perturbation to the TSW and upon that $\dot{a}$ and $\ddot{a}$ are not zero.

The radius of the throat $a(\tau)$ after the perturbation should satisfy (144). One may rewrite (166) as

$$\psi'+2\ddot{a}=\frac{-4}{a}(\psi+\dot{a}^2), \tag{178}$$

which after applying the chain-rule, i.e, $\ddot{a}=\frac{d\dot{a}}{da}\dot{a}$ one simply finds

$$\psi'+2\dot{a}\frac{d\dot{a}}{da}=\frac{-4}{a}(\psi+\dot{a}^2). \tag{179}$$

This equation, after multiplying by a^4 and simplifying $2\dot{a}\frac{d\dot{a}}{da}=\frac{d}{da}(\dot{a}^2)$ implies

$$a^4\psi'+4a^3\psi=a^4\frac{d}{da}(\dot{a}^2)+4a^3(\dot{a}^2). \tag{180}$$

Both sides are total derivatives, i.e,

$$\frac{d}{da}(a^4\psi) = \frac{d}{da}(a^4(\dot{a}^2)), \tag{181}$$

which after an integration yields

$$a^4\psi = a^4(\dot{a}^2) + c, \tag{182}$$

in which c in the integration constant. To find c, we recall that at $a = a_0$, $\dot{a} = 0$. Hence, we obtain

$$c = a_0^4\psi(a_0) = a_0^4\psi_0. \tag{183}$$

Taking back c into (182) we get finally

$$\dot{a}^2 + \frac{a_0^4}{a^4}\psi_0 - \psi = 0. \tag{184}$$

This is the equation of motion of the throat after the perturbation . Writing this equation as

$$\dot{a}^2 + V(a) = 0, \tag{185}$$

in which

$$V(a) = \frac{a_0^4}{a^4}\psi_0 - \psi, \tag{186}$$

one can study the stability status of the TSW as follows. As we already assumed, let's keep the equilibrium point to be at $a = a_0$ where $\dot{a}_0 = \ddot{a}_0 = 0$.

The Taylor expansion of the potential $V(a)$ about $a = a_0$ is obtained to be

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + \mathscr{O}((a - a_0)^3). \tag{187}$$

Herein

$$V(a_0) = \frac{a_0^4}{a_0^4}\psi_0 - \psi_0 = 0, \tag{188}$$

and

$$V'(a_0) = \frac{d}{da}\left(\frac{a_0^4}{a^4}\psi_0 - \psi\right)_{a=a_0} = \left(-\frac{4a_0^4}{a^5}\psi_0 - \psi'\right)_{a=a_0} = -\left(4\psi_0 + a_0\psi_0'\right)\frac{1}{a_0} \tag{189}$$

$V'(a_0)$ also vanishes due to the condition (149). Finally one obtains

$$V''(a_0) = \left[+\frac{20a_0^4}{a^6}\psi_0 - \psi''\right]_{a=a_0} = \frac{20}{a_0^2}\psi_0 - \psi_0''. \tag{190}$$

Nevertheless, the linear equation of motion of the TSW after the perturbation becomes

$$\dot{a}^2 + \frac{1}{2}\left(\frac{20}{a_0^2}\psi_0 - \psi_0''\right)(a - a_0)^2 \simeq 0. \tag{191}$$

Therefore, the nature of the solution of (191) depends on the sign of $\omega^2 = \frac{1}{2}(\frac{20}{a_0^2}\psi_0 - \psi_0'')$. If $\omega^2 > 0$ the solution after the perturbation is oscillatory which is an indication for the stability. On the other hand, if $\omega^2 < 0$ the motion becomes of exponential type which implies the TSW is unstable.

In summary, we shall look for the possible values for the parameters such that the expression for $V''(a_0)$, i.e,

$$V''(a_0) = \frac{20}{a_0^2}\psi_0 - \psi_0'', \tag{192}$$

becomes positive. Considering ψ given in (169) with $R_0 = 0$ and $-1+4\sqrt{2}\beta Q = \mu$ one finds

$$V''(a_0) = \frac{6\mu a_0 - 14Q^2\alpha - 10\alpha a_0^2}{\alpha a_0^4}. \tag{193}$$

We recall that, a_0 is given by Eq. (177) which upon a substitution in (193) we find

$$V_0''(a_0) = \frac{256\alpha^3}{(\mu+\sqrt{\xi})^4}\left(\frac{7\mu^2}{8\alpha} + \frac{\mu\sqrt{\xi}}{4\alpha} - 14Q^2\alpha - \frac{5\xi}{8\alpha}\right), \tag{194}$$

in which $\xi = \mu^2 - 16Q^2\alpha^2 > 0$.

We recall that μ has been bounded due to other conditions such that

$$\sqrt{16}\alpha Q < \mu < \sqrt{18}\alpha Q. \tag{195}$$

In Figure 7 we plot $V_0''(a_0)$ in terms of μ for $\alpha = 1$ and $Q = 1$. Figure 6 shows clearly that in the domain of μ, $V_0'' > 0$. Finally, we conclude that our TSW in $f(R)$-gravity

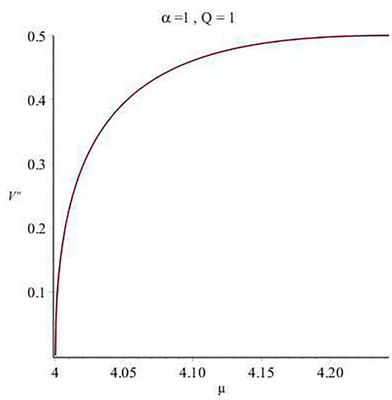

Figure 7: A plot of V_0'' in terms of μ for $\alpha = 1$ and $Q = 1$.

is stable against a linear perturbation.

Chapter 4

CONCLUSION

Due to the junction conditions construction of TSWs in $f(R)$-gravity in contrast to Einstein's general relativity, is a rather difficult operation. This originates from the tough conditions imposed on the first and second fundamental forms. We overcome the difficulty by considering a class of $f(R)$ model coupled with NED whose third derivative i.e. $f'''(R) \neq 0$ so that it satisfies the generalized junction conditions. An exact solution is obtained which is supported by an external static field within the context of NED. It admits electric black holes which, however, does not serve our purpose of TSWs. The reason is simple: the existence of the event horizon is not compatible with the radius of the thin-shell. We follow therefore a different route. We choose the non-black hole branch of the solution which allows us to locate the shell. The shell's radius becomes founded from above which is stable against linear radial perturbations. The fluid energy-momentum emerging of the throat upon perturbation satisfies naturally a non-barotropic equation of state. If the shell was not stable then it would collapse at the slightest perturbation to the naked singularity at the center.

REFERENCES

[1] A. Einstein and N. Rosen, *The Particle Problem in the General Theory of Relativity*, Phys. Rev. **48**, 73 (1935).

[2] J. Wheeler, *Geometrodynamics*, New York, Academic Press (1962).

[3] M. Visser, *Lorentzian wormholes* , AIP Press, New York (1996).

[4] M. Visser, *Traversable wormholes: Some simple examples,* Phys. Rev. D **39**, 3182 (1989).

[5] H. G. Ellis, *Ether flow through a drainhole - a particle model in general relativity*, J. Math. Phys. **14**, 104-118 (1973).

[6] M. S. Morris, K. S. Thorne, *Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity*, Am. J. Phys. **56**, 395 (1988).

[7] C. Misner, K. Thorne and J. A. Wheeler, *Gravitation*, San Francisco, W. H. Freeman and Co. (1973).

[8] T. P. Sotiriou and V. Faraoni, $f(R)$ *theories of gravity* Rev. Mod. Phys. **82**, 451 (2010).

[9] S. Nojiri and S. D. Odintsov, *Unified cosmic history in modified gravity: from* $f(R)$ *theory to Lorentz non-invariant models*, Phys. Reprts. **505**, 59 (2011).

[10] N. Deruelle, M. Sasaki, and Y. Sendoudam, *Junction Conditions in $f(R)$ Theories of Gravity* , Prog. Theor. Phys. **119**, 237 (2008).

[11] P. P. Fiziev, *Withholding Potentials, Absence of Ghosts and Relationship between Minimal Dilatonic Gravity and $f(R)$ Theories*, Phys. Rev. D. **87**, 044053 (2013).

[12] S. H. Mazharimousavi and M. Halilsoy, *Black hole solutions in $f(R)$ gravity coupled with nonlinear Yang-Mills field*, Phys. Rev. D. **84**, 064032 (2011).

[13] S. H. Mazharimousavi, M. Halilsoy and T. Tahamtan, *Solutions for f(R) gravity coupled with electromagnetic field*, Eur. Phys. J. C. **72**, 1851 (2012).

[14] S. H. Mazharimousavi, M. Halilsoy and T. Tahamtan, *Constant curvature f(R) gravity minimally coupled with Yang-Mills field*, Eur. Phys. J. C **72**, 1958 (2012).

[15] G. Darmois, *Mèmorial des Sciences Màthematiques,* Fascicule XXV (Gauthier-Villars, Paris), Chap V (1927).

[16] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, Nuovo Cim. B **44**, 1 (1966).

[17] J. M. M. Senovilla. *Junction conditions for F (R) gravity and their consequences,* Phys. Rev. D. **88**, 064015 (2012).

[18] M. Mars and J. M. M. Senovilla, *Geometry of general hypersurfaces in spacetime: junction conditions*, Class. Quantum Grav. **10**, 1865 (1993).

[19] V. Varela, *Note on linearized stability of Schwarzschild thin-shell wormholes with variable equations of state*, Phys. Rev. D **92**, 044002 (2015).

[20] A. Gray, *The Peterson-Mainardi-Codazzi Equations. §28.3 in Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd ed.* Boca Raton, FL: CRC Press, pp. 649-652, (1997).