Incomplete Pochhammer Ratios and Related Special Functions

Ceren Ustaoğlu

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Assoc. Prof. Dr. Ali Hakan Ulusoy
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy in Mathematics.

Prof. Dr. Mehmet Ali Özarslan
Supervisor

Examining Committee

1. Prof. Dr. Esra Erkuş Duman
2. Prof. Dr. Sonuç Zorlu Oğurlu
3. Prof. Dr. Mehmet Ali Özarslan
4. Prof. Dr. Yılmaz Şimşek
5. Asst. Prof. Dr. Mustafa Kara

ABSTRACT

This thesis includes five chapters. In the first chapter, general information and some preliminaries that used throughout the thesis are given.

In Chapter 2, the incomplete Pochhammer ratios are defined in terms of the incomplete beta function $B_y(x,z)$. With the help of these incomplete Pochhammer ratios, we introduce new incomplete Gauss hypergeometric, confluent hypergeometric and Appell's functions and investigate several properties of them such as integral representations, derivative formulas, transformation formulas and recurrence relation. Furthermore, an incomplete Riemann-Liouville fractional derivative operators are introduced. This definition plays a key role for our understanding of the linear and bilinear generating relations for the new incomplete Gauss hypergeometric functions.

In Chapter 3, we give the definitions of Caputo fractional derivative operators and show their use in the special function theory. For this purpose, new types of incomplete hypergeometric functions are introduced and their integral representations are obtained. Furthermore, we define incomplete Caputo fractional derivative operators and show that the images of some elementary functions under the action of incomplete Caputo fractional derivative operators give a new type of incomplete hypergeometric functions. For the new type incomplete hypergeometric functions linear and bilinear generating relations are obtained.

In Chapter 4, generalizations of incomplete gamma, beta, Gauss, confluent and Appell's hypergeometric functions are introduced. Also, Mellin transforms, transformation formulas, differentiation and difference formulas and fractional calculus formulas are obtained for these functions.

In Chapter 5, the extended incomplete Mittag-Leffler functions are introduced by using the extended incomplete beta functions and we investigate several properties of these functions. The Mellin transform of these functions is presented with regards to the incomplete Wright hypergeometric functions. Furthermore, we obtain the images of the extended incomplete Mittag-Leffler functions under the actions of Riemann-Liouville fractional integral and derivative operator. Some miscellaneous properties of these funtions are also given.

Keywords: incomplete Pochhammer ratios, incomplete hypergeometric functions, incomplete Riemann-Liouville fractional derivative operators, Generating functions, incomplete Caputo fractional derivative operators, generalized incomplete gamma and beta functions, generalized incomplete hypergeometric functions, incomplete Mittag-Leffler functions, extended incomplete Mittag-Leffler functions

ÖZ

Bu tez 5 bölümden oluşmaktadır. Birinci bölümde tez ile ilgili genel bilgiler ve tezde kullanılan tanımlar hakkında bilgiler verilmiştir.

İkinci bölümde, tamamlanmamış Pochhammer oranları tamamlanmamış beta fonksiyonlanları cinsinden tanımlanmıştır. Tamamlanmamış Pochhammer oranları yardımı ile tamamlanmamış Gauss, konfluent ve Appell hipergeometrik fonksiyonları tanımlanmıştır. Bu hipergeometrik fonksiyonlar için integral gösterimleri, türev formülleri, dönüşüm formülleri ve rekürans bağıntıları elde edilmiştir. Ayrıca, tamamlanmamış Riemann-Liouville kesirli türev operatörleri tanımlanmıştır. Bu tanım, tamamlanmamış Gauss hipergeometrik fonksiyonları için lineer ve lineer olmayan doğrucu fonksiyonları elde etmemize yardımcı olur.

Üçüncü bölümde, Caputo kesirli türev operatörlerinin tanımlarını verdik ve bunların özel fonksiyon teorisinde kullanımını gösterdik. Bu amaçla yeni bir tür tamamlanmamış hipergeometrik fonksiyonlar tanıtılmış ve integral gösterimleri elde edilmiştir. Ayrıca, tamamlanmamış Caputo kesirli türev operatörleri tanımlamakta ve tamamlanmamış Caputo fraksiyonel türev operatörlerinin etkisi altında bazı temel fonksiyonların görüntülerinin yeni tür hipergeometrik fonksiyonlar verdiğini göstermekteyiz. Yeni tür eksik hipergeometrik fonksiyonlar için lineer ve lineer olmayan doğrucu fonksiyonlar elde edilir..

Dördüncü bölümde, tamamlanmamış gamma, beta, Gauss ve confluent hipergeometrik fonksiyonlarının genelleştirilmiş formları tanımlanmıştır. Son olarak ise, bu fonksiyonlar ile ilgili bir takım özellikler gösterilmiştir.

Beşinci bölümde, genişletilmiş Mittag-Leffler fonksiyonları, genişletilmiş tamamlanmamış beta fonksiyonları kullanılarak tanıtılmış ve bu fonksiyonların çeşitli özelliklerini gösterilmiştir. Bu fonksiyonların Mellin dönüşümleri, tamamlanmamış Wright hipergeometrik fonksiyonları cinsinden verilmiştir. Ayrıca, Riemann-Liouville kesirli integrali ve türev operatörü etkisi altında genişletilmiş tamamlanmamış Mittag-Leffler fonksiyonlarının görüntülerini elde ediyoruz.

Anahtar Kelimeler: tamamlanmamış Pochhammer oranları, tamamlanmamış hipergeometrik fonksiyonlar, tamamlanmamış Riemann-Liouville kesirli türev operatörleri, doğrucu fonksiyonlar, tamamlanmamış Caputo kesirli türev operatörleri, genelleştirilmiş tamamlanmamış gamma ve beta fonksiyonları, genelleştirilmiş tamamlanmamış hipergeometrik fonksiyonlar, tamamlanmamış Mittag-Leffler fonksiyonları, genişletilmiş tamamlanmamış Mittag-Leffler fonksiyonları

To My Lovely Family

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TABLE OF CONTENTS

LIST OF SYMBOLS

$\Gamma(z)$	Gamma Function
$\Gamma_p(x)$	Extension of Gamma Function
B(x,z)	Beta Function
$B_p(x,y)$	Extension of Euler's Beta Function
$F_p(a,b;c;z)$	Extended Gauss Hypergeometric Function
$\phi_p(b;c;z)$	Extended Confluent Hypergeometric Function
$B_y(x,z)$	Incomplete Beta Function
$[b,c;y]_n$	Incomplete Pochhammer Ratio
${}_2F_1(a,b;c;z)$	Gauss Hypergeometric Function
${}_2F_1(a,[b,c;y];z)$	Incomplete Gauss Hypergeometric Function
${}_1F_1([a,b;y];z)$	Incomplete Confluent Hypergeometric Function
${}_1F_1(a;b;z)$	Confluent Hypergeometric Function
$M\{f(t):s\}$	Mellin transform of f
H(t)	Heaviside Unit Function
$\delta(t-t_0)$	Dirac Delta Function
$F_1[a,b,c;d,x,y]$	Incomplete First Appell's Hypergeometric Functions
$F_2[a,b,c;d,e;x,y]$	Incomplete Second Appell's Hypergeometric Functions
$D_z^{\mu}\{f(z)\}$	Riemann-Liouville Fractional Derivative of Order μ
$D_z^{\mu}[f(z);y]$	Incomplete Riemann-Liouville Fractional Derivative of Order μ
$F_{D,y}^{3}[a,b,c,d;e;x,w,z;y]$	Incomplete Lauricella's Hypergeometric Functions in Three Variables

$\mathrm{D}^{\alpha} f(z)$	Caputo fractional derivative operator
$\mathrm{C}_z^{\alpha}[f(z);y]$	Incomplete Caputo Fractional Derivative Operator
$\Gamma_p^{(\alpha,\beta;y)}(x,z)$	Generalized Incomplete Gamma Function
$\mathrm{B}_p^{(\alpha,\beta;y)}(x,z)$	Generalized Incomplete Beta Function
$\mathrm{F}_p^{(\alpha,\beta;y)}(a,b;c;z)$	Generalized Incomplete Gauss Hypergeometric Function
$\mathrm{F}_p^{\alpha,\beta}(a,[b,c;y];z)$	Extension of Generalized Incomplete Gauss Hypergeometric Function
${}_1F_1^{((\alpha,\beta;y);p)}(a;b;z)$	Generalized Incomplete Confluent Hypergeometric Function
$\mathrm{F}_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p)$	Extension of Generalized Incomplete Second Appell's Hypergeometric functions in Two Variables
$\mathrm{D}_z^{\mu,p}\{f(z)\}$	Extended Riemann-Liouville Fractional Derivative of Order μ
$\mathrm{E}_{\alpha}(z)$	Mittag-Leffler function
$\mathrm{E}_{\alpha,\beta}^{\gamma}(z)$	Generalized Mittag-Leffler function
$\mathrm{E}_{\alpha,\beta}^{\gamma;c}(z;p)$	Extended Mittag-Leffler function
$\mathrm{E}_{\alpha,\beta}^{[\gamma,c;y]}(x;p)$	Extended Incomplete Mittag-Leffler function
$\mathrm{B}_y(x,z;p)$	Extended Incomplete Beta Function
$\mathrm{E}_{\alpha,\beta}^{\gamma;c}[x;y]$	Incomplete Mittag-Leffler function
$(\mathrm{I}_{a^+}^{\lambda}f)(x)$	Right-Sided Riemann-Liouville Fractional Integral Operator
$(\mathrm{D}_{a^+}^{\lambda}f)(x)$	Right-Sided Riemann-Liouville Fractional Derivative Operator

Chapter 1

INTRODUCTION

The gamma function is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1} \exp(-t)\, dt, \;\; Re(z) > 0.$$

In 1994, the following extension of gamma function was introduced by Chaudhry and Zubair [13] as

$$\Gamma_p(x) := \int_0^\infty t^{x-1} \exp\left[-t - \frac{p}{t}\right] dt, \tag{1.0.1}$$

where $Re(p) > 0$.

The familiar incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ are defined by

$$\gamma(s,x) := \int_0^x t^{s-1} e^{-t} dt, \;\; Re(s) > 0;\; x \geqq 0 \tag{1.0.2}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1} e^{-t} dt, \;\; x \geqq 0;\; Re(s) > 0 \text{ when } x = 0, \tag{1.0.3}$$

respectively. These functions satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s), \;\; Re(s) > 0. \tag{1.0.4}$$

The function $\Gamma(s)$ and its incomplete versions $\gamma(s,x)$ and $\Gamma(s,x)$ are crucial in the study for analytical solutions of various problems including different branches of science and engineering [27].

The widely used Pochhammer symbol $(\lambda)_\nu\ (\lambda,\nu\in\mathbb{C})$ is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1 & \nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}, \\ \lambda(\lambda+1)\ldots(\lambda+\nu-1) & \nu\in\mathbb{N};\ \lambda\in\mathbb{C}. \end{array} \right\} \quad (1.0.5)$$

In terms of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, the incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ $(\lambda;\nu\in\mathbb{C};\ x\geqq 0)$ were defined as follows [12]:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)},\ \ \lambda,\nu\in\mathbb{C}\,;\ x\geqq 0$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)},\ \ \lambda,\nu\in\mathbb{C}\,;\ x\geqq 0.$$

In view of (1.0.4), these incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu+[\lambda;x]_\nu=(\lambda)_\nu,\ \ \lambda,\nu\in\mathbb{C}\,;\ x\geqq 0$$

where $(\lambda)_\nu$ is the Pochhammer symbol given by (1.0.5).

With the help of the incomplete gamma functions, the incomplete Gauss hypergeometric functions were defined as follows [12]:

$${}_2\gamma_1\left[\begin{array}{ccc} (a,x) & . & b\ ; \\ & & c\ ; \end{array} z\right] := \sum_{n=0}^{\infty}\frac{(a;x)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

and

$$ {}_2\Gamma_1\left[\begin{array}{ccc} (a,x) & . \; b & ; \\ & & z \\ & c & ; \end{array}\right] := \sum_{n=0}^{\infty} \frac{[a;x]_n (b)_n}{(c)_n} \frac{z^n}{n!}. $$

In view of (1.0.4), these incomplete Gauss hypergeometric functions are satisfy the following decomposition relation:

$$ {}_2\gamma_1\left[\begin{array}{ccc} (a,x) & . \; b & ; \\ & & z \\ & c & ; \end{array}\right] + {}_2\Gamma_1\left[\begin{array}{ccc} (a,x) & . \; b & ; \\ & & z \\ & c & ; \end{array}\right] = {}_2F_1\left[\begin{array}{ccc} a & . \; b & ; \\ & & z \\ & c & ; \end{array}\right]. $$

Euler's beta function is defined by

$$ B\left(x,y\right) := \int_0^1 t^{x-1} \left(1-t\right)^{y-1} dt, \;\; Re\left(x\right) > 0, \; Re\left(y\right) > 0. $$

In 1997, the following extension of Euler's beta function was introduced by Chaudhry [10] as

$$ B_p(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \tag{1.0.6} $$

where $Re\left(p\right) > 0, \; Re\left(x\right) > 0, \; Re\left(y\right) > 0$. Afterwards, Chaudhry [11] used $B_p(x,y)$ to extend the hypergeometric functions (and confluent hypergeometric functions) as follows:

$$ F_p(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \;\; p \geq 0; \; Re(c) > Re(b) > 0, \tag{1.0.7} $$

and

$$\phi_p(b;c;z) := \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \;\; p \geq 0; \; Re(c) > Re(b) > 0, \tag{1.0.8}$$

Organization of the thesis is as follows:

In Chapter 2, the incomplete Pochhammer ratios are introduced by using the incomplete beta function. The new incomplete functions such as Gauss hypergeometric, confluent hypergeometric and Appell's function can be introduced by the help of these incomplete Pochhammer ratios. Several properties of these functions are investigated. The incomplete Riemann-Liouville fractional derivative operator are defined and we show that the incomplete Riemann-Liouville fractional derivative of some elementary functions give the new incomplete functions.

In Chapter 3, incomplete Caputo fractional derivative operators are defined and it has been shown that the image of the elementary functions under the action of incomplete Caputo fractional operators give the new type incomplete hypergeometric functions.

In Chapter 4, we introduce the generalizations of incomplete gamma, beta, Gauss hypergeometric, confluent hypergeometric and Appell's hypergeometric functions. Several properties of these functions are investigated such as integral representations, Mellin transforms, fractional calculus formulas etc.

In Chapter 5, we introduce the extended incomplete Mittag-Leffler functions by with the help of the extended incomplete beta functions. Moreover, we obtain the images of the extended incomplete Mittag-Leffler functions under the actions of Riemann-Liouville fractional integral and derivative operator. Some miscellaneous properties of these functions are also given.

Chapter 2

INCOMPLETE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE OPERATORS AND INCOMPLETE HYPERGEOMETRIC FUNCTIONS

2.1 Introduction

In recent years, the gamma function $\Gamma(z)$ and the Pochhammer symbol $(\lambda)_v$ were used to extend the generalized hypergeometric functions and their multivariate versions. After these works, incomplete hypergeometric functions have become one of the hot topics of recent years [1], [17], [18], [19], [20], [29], [44], [47], [48], [50], [54], [55], [56], [57].

On the other hand, applications in many diverse areas of mathematical, physical and engineering problems have been found by fractional derivative operators. Because of this reason these operators have been an active research in recent years [7], [23], [24], [25], [26], [27], [62]. The use of fractional derivative operators in obtaining generating relations for some special functions can be found in [34],[52]. In this chapter, we are aimed to introduce new incomplete hypergeometric functions with the aid of incomplete Pochhammer ratios and investigate their certain properties. Moreover, we introduce incomplete Riemann-Liouville fractional derivative operators and we obtain some generating relations for these new incomplete hypergeometric function with the aid of these new defined operators. The organization of this chapter is as follows:

In Section 2.2, the incomplete Pochhammer ratios are introduced by using the incomplete beta function and some derivative formulas involving these new incomplete Pochhammer ratios are investigated. In Section 2.3, new incomplete Gauss hypergeometric functions and confluent hypergeometric functions are introduced with the help of these incomplete Pochhammer ratios ; also integral representations, derivative formulas, transformation formulas and recurrence relation are obtained for them. In Section 2.4, we define new incomplete Appell's functions $F_1[a,b,c;d;x,z;y]$, $F_1\{a,b,c;d;x,z;y\}$, $F_2[a,b,c;d,e;x,z;y]$ and $F_2\{a,b,c;d,e;x,z;y\}$ and obtain their integral representations. In Section 2.5, incomplete Riemann- Liouville fractional derivative operators are introduced and it is shown that the incomplete Riemann-Liouville fractional derivative of some elementary fuctions give the new incomplete functions that were defined in Sections 2.3 and 2.4. In the last section, we obtain linear and bilinear generating relations for incomplete hypergeometric functions.

2.2 The incomplete Pochhammer Ratio

The incomplete beta function is defined by

$$B_y(x,z) := \int_0^y t^{x-1}(1-t)^{z-1}dt,\ Re(x) > Re(z) > 0,\ 0 \leq y < 1 \tag{2.2.1}$$

and can be expressed in terms of the Gauss hypergeometric function

$$B_y(x,z) := \frac{y^x}{x}(1-y)^z\ {}_2F_1(1,x+z;1+x;y). \tag{2.2.2}$$

The incomplete beta function satisfy the following relation:

$$B_y(b+n,c-b) + B_{1-y}(c-b,b+n) = B(b+n,c-b), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{2.2.3}$$

The incomplete Pochhammer ratios $[b,c;y]_n$ and $\{b,c;y\}_n$ are introduced in terms of the incomplete beta function $B_y(x,z)$ as follows [39]:

$$[b,c;y]_n := \frac{B_y(b+n,c-b)}{B(b,c-b)} \tag{2.2.4}$$

and

$$\{b,c;y\}_n := \frac{B_{1-y}(c-b,b+n)}{B(b,c-b)} \tag{2.2.5}$$

where $0 \leq y < 1$. It is clear from (2.2.3) that

$$[b,c;y]_n + \{b,c;y\}_n = \frac{(b)_n}{(c)_n}. \tag{2.2.6}$$

In view of (2.2.2) , we have the following relations

$$[b,c;y]_n := \frac{1}{B(b,c-b)} \frac{y^{b+n}}{b+n}(1-y)^{c-b} {}_2F_1(1,c+n;b+n+1;y) \tag{2.2.7}$$

and

$$\{b,c;y\}_n := \frac{1}{B(b,c-b)} \frac{(1-y)^{c-b}}{c-b} y^{b+n} {}_2F_1(1,c+n;1+c-b;1-y). \tag{2.2.8}$$

In the following theorem, the $n-th$ derivatives of the incomplete beta fuction by means of incomplete Pochhammer ratios are investigated.

Theorem 2.2.1 *([39]) The following derivative formulas hold true:*

$$[b,c;y]_n = \frac{(-1)^n \Gamma(c)}{\Gamma(c-b+n)\Gamma(b)} y^{b+n} \frac{d^n}{dy^n}\left[y^{-b}B_y(b,c-b+n)\right], \tag{2.2.9}$$

and

$$\{b,c;y\}_n = \frac{\Gamma(b+n)}{\Gamma(b+2n)} \frac{1}{B(b,c-b)} (1-y)^{c-b} \frac{d^n}{dy^n}((1-y)^{-c+b+n}B_{1-y}(c-b-n,b+2n)). \tag{2.2.10}$$

Proof. Using (2.2.1) and (2.2.4), we immediately obtain the following equation:

$$[b,c;y]_n = \frac{y^{b+n}}{B(b,c-b)} \int_0^1 u^{b+n-1}(1-uy)^{c-b-1}du.$$

On the other hand, we have

$$y^{-b}B_y(b,c-b+n) = \int_0^1 u^{b-1}(1-uy)^{c-b+n-1}du. \tag{2.2.11}$$

Taking derivatives n times on both sides of (2.2.11) with respect to y, we can obtain a derivative formula for the incomplete beta function $[b,c;y]_n$ asserted by (2.2.9). Formula (2.2.10) can be proved in a similar way. ■

2.3 The new incomplete Gauss and confluent hypergeometric functions

In this section, we introduce new incomplete Gauss and confluent hypergeometric functions by [39]

$${}_2F_1(a,[b,c;y];x) := \sum_{n=0}^{\infty} (a)_n [b,c;y]_n \frac{x^n}{n!}, \tag{2.3.1}$$

$$ {}_2F_1(a,\{b,c;y\};x) := \sum_{n=0}^{\infty} (a)_n \{b,c;y\}_n \frac{x^n}{n!}, \tag{2.3.2} $$

$$ {}_1F_1([a,b;y];x) := \sum_{n=0}^{\infty} [a,b;y]_n \frac{x^n}{n!}, \tag{2.3.3} $$

and

$$ {}_1F_1(\{a,b;y\};x) := \sum_{n=0}^{\infty} \{a,b;y\}_n \frac{x^n}{n!} \tag{2.3.4} $$

where $0 \leq y < 1$.

An immediate consequence of (2.2.6) and the definitions (2.3.1), (2.3.2), (2.3.3) and (2.3.4) are the following decomposition formulas

$$ {}_2F_1(a,[b,c;y];x) + {}_2F_1(a,\{b,c;y\};x) = {}_2F_1(a,b;c;x) \tag{2.3.5} $$

and

$$ {}_1F_1([a,b;y];x) + {}_1F_1(\{a,b;y\};x) = {}_1F_1(a;b;x). \tag{2.3.6} $$

Theorem 2.3.1 *([39]) The following integral representation holds true:*

$$ \begin{aligned} {}_2F_1(a,[b,c;y],x) &= \frac{y^b}{B(b,c-b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-xuy)^{-a}du, \quad (2.3.7) \\ Re(c) &> Re(b) > 0, \ |\arg(1-x)| < \pi). \end{aligned} $$

Proof. Replacing the incomplete Pochhammer ratio $[b,c;y]_n$ in the definition (2.3.1) by its integral representation given by (2.2.1) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis of the

Theorem, we find

$$ {}_2F_1(a,[b,c;y],x) = \frac{1}{B(b,c-b)} \int_0^y t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt, \tag{2.3.8} $$

which can be written as follows:

$$ {}_2F_1(a,[b,c;y],x) = \frac{y^b}{B(b,c-b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-xuy)^{-a}du. \tag{2.3.9} $$

■

In a similar way, we have the following theorem:

Theorem 2.3.2 *([39]) The following integral representation holds true:*

$$ \begin{aligned} {}_2F_1(a,\{b,c;y\},x) &= \frac{(1-y)^{c-b}}{B(b,c-b)} \int_0^1 u^{c-b-1}(1-u(1-y))^{b-1}(1-x+xu(1-y))^{-a}du, \\ Re(c) &> Re(b) > 0, |\arg(1-x)| < \pi. \end{aligned} \tag{2.3.10} $$

Theorem 2.3.3 *([39]) The following result holds true:*

$$ \begin{aligned} {}_2F_1(a,[b,c;y],1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - \frac{(1-y)^{c-b-a}y^b}{B(b,c-b)(c-a-b)} \\ &\quad \times {}_2F_1(c-a,1;1+c-b-a;1-y). \end{aligned} \tag{2.3.11} $$

Proof. Putting $x=1$ in (2.3.5), we obtain

$$ \begin{aligned} {}_2F_1(a,[b,c;y],1) &= {}_2F_1(a,b;c;1) - {}_2F_1(a,\{b,c;1-y\},1) \qquad (2.3.12) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - \frac{(1-y)^{c-b-a}}{B(b,c-b)} \int_0^1 u^{c-b-a-1}(1-u(1-y))^{b-1}du. \end{aligned} $$

Using the Euler's integral representation for (2.3.12), we have

$$
{}_2F_1(a,[b,c;y],1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-\frac{(1-y)^{c-b-a}}{B(b,c-b)(c-b-a)}\,{}_2F_1(1-b,c-b-a;1+c-b-a;1-y). \tag{2.3.13}
$$

Using transformation formula

$$
{}_2F_1(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\beta-\alpha}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z), \tag{2.3.14}
$$

in (2.3.13), we obtain

$$
{}_2F_1(1-b,c-b-a;1+c-b-a;1-y)=y^b\,{}_2F_1(c-a,1;1+c-b-a;1-y). \tag{2.3.15}
$$

Considering (2.3.15) in (2.3.13), we get

$$
\begin{aligned}
{}_2F_1(a,[b,c;y],1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-\frac{(1-y)^{c-b-a}y^b}{B(b,c-b)(c-b-a)} \\
&\quad\times\,{}_2F_1(c-a,1;1+c-b-a;1-y).
\end{aligned} \tag{2.3.16}
$$

■

Theorem 2.3.4 *([39]) The following result holds true:*

$$
{}_2F_1(a,\{b,c;y\},1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-\frac{(1-y)^{c-b-a}y^b}{B(b,c-b)b}\,{}_2F_1(c-a,1;b+1;y)\,. \tag{2.3.17}
$$

Theorem 2.3.5 *([39]) The following integral representations hold true:*

$$
\begin{aligned}
{}_1F_1([a,b;y],x) &= \frac{y^a}{B(a,b-a)}\int_0^1 u^{a-1}(1-uy)^{b-a-1}e^{xuy}du, \\
&\quad Re(b) > Re(a) > 0 \qquad (2.3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_1F_1(\{a,b;y\},x) &= \frac{(1-y)^{b-a}}{B(a,b-a)}\int_0^1 u^{b-a-1}(1-u(1-y))^{a-1}e^{(1-u(1-y))x}du, \\
&\quad Re(b) > Re(a) > 0. \qquad (2.3.19)
\end{aligned}
$$

Proof. Replacing the incomplete Pochhammer ratio $[a,b;y]_n$ in the definition (2.3.3) by its integral representation given by (2.2.1), we are led to the desired result (2.3.18). Formula (2.3.19) can be proved in a similar way. ■

Theorem 2.3.6 *([39]) The following integral representation holds true:*

$$
\begin{aligned}
\int_0^1 y^{k-1}\,{}_2F_1(a,[b,c-k;y];x)dy &= \frac{1}{k}\left[{}_2F_1(a,b;c-k;x) - \frac{\Gamma(c-k)\Gamma(b+k)}{\Gamma(b)\Gamma(c)}\,{}_2F_1(a,b+k;c;x)\right], \\
&\quad k \in \mathbb{N}. \qquad (2.3.20)
\end{aligned}
$$

Proof. It is known that from the Euler's formula

$$
{}_2F_1(a,b+k;c;x) = \frac{1}{B(b+k,c-b-k)}\int_0^1 y^{b+k-1}(1-y)^{c-b-k-1}(1-xy)^{-a}dy, \;\; k \in \mathbb{N}.
$$

Taking $u = y^k$ and the remaining part as dv and applying the integration by parts, we get

$$ {}_2F_1(a,b+k;c;x) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(c-k)\Gamma(b+k)} \left[{}_2F_1(a,b;c-k;x) - k \int_0^1 y^{k-1}\, {}_2F_1(a,[b,c-k;y],x)dy \right]. $$

By rearranging the terms we get the result. ■

Corollary 2.3.7 *([39]) Taking $k = 1$ in Theorem 2.3.6, we get the following result:*

$$ \int_0^1 {}_2F_1(a,[b,c-1;y],x)dy = {}_2F_1(a,b;c-1;x) - \frac{b}{c-1}\, {}_2F_1(a,b+1;c;x). \quad (2.3.21) $$

Theorem 2.3.8 *([39]) The following integral representation holds true:*

$$ \int_0^1 y^{k-1}\, {}_2F_1(a,[b,c;y],x)dy = \frac{1}{k}\frac{\Gamma(c)\Gamma(c-b+k)}{\Gamma(c-b)\Gamma(c+k)}\, {}_2F_1(a,b;c+k;x). \quad (2.3.22) $$

Proof. It is known that

$$ {}_2F_1(a,b;c+k;x) = \frac{1}{B(b,c-b+k)} \int_0^1 y^{b-1}(1-y)^{c-b+k-1}(1-xy)^{-a}dy. $$

Taking $u = (1-y)^k$ and the rest as dv and using integration by parts, we get the result.

■

Corollary 2.3.9 *([39]) Taking $k = 1$ in Theorem 2.3.8, we get the following result:*

$$ {}_2F_1(a,b;c+1;x) = \frac{c}{c-b} \int_0^1 {}_2F_1(a,[b,c;y],x)dy. \quad (2.3.23) $$

Theorem 2.3.10 *([39]) The following derivative formula holds true:*

$$\frac{d^n}{dx^n}({}_2F_1(a,[b,c;y];x)) = \frac{(a)_n(b)_n}{(c)_n}\ {}_2F_1(a+n,[b+n,c+n;y];x). \qquad (2.3.24)$$

Proof. Using (2.3.8), differentiating on both sides with respect to x, we obtain

$$\begin{aligned}
\frac{d}{dx}({}_2F_1(a,[b,c;y];x)) &= \frac{a}{B(b,c-b)}\int_0^y t^b(1-t)^{c-b-1}(1-xt)^{-a-1}dt \\
&= \frac{a}{B(b,c-b)}\int_0^y t^{(b+1)-1}(1-t)^{(c+1)-(b+1)-1}(1-xt)^{-(a+1)}dt \\
&= \frac{ab}{c}\frac{1}{B(b+1,c-b)}\int_0^y t^{(b+1)-1}(1-t)^{(c+1)-(b+1)-1}(1-xt)^{-(a+1)}dt \\
&= \frac{ab}{c}\ {}_2F_1(a+1,[b+1,c+1;y];x
\end{aligned}$$

which is (2.3.24) for $n=1$. The general result follows by the principle of mathematical induction on n. ■

Theorem 2.3.11 *([39]) The following derivative formula holds true:*

$$\frac{d^n}{dx^n}({}_1F_1([a,b;y];x)) = \frac{(a)_n}{(b)_n}\ {}_1F_1([a+n,b+n;y];x). \qquad (2.3.25)$$

Theorem 2.3.12 *([39]) We have the following difference formula for* ${}_2F_1(a,[b,b+h;y];x)$ *:*

$$\frac{b+h-1}{B(b,h)}y^{b-1}(1-y)^{h-1}(1-xy)^{-a} = {}_2F_1(a,[b,b+h-1;y];x)+$$

$${}_2F_1(a,[b-1,b+h-1;y];x) - ax(b+h-1)\ {}_2F_1(a+1,[b,b+h;y];x). \qquad (2.3.26)$$

Proof. Recalling that the Mellin transform operator is defined by

$$\mathfrak{M}\{f(t):s\} := \int_0^\infty t^{s-1} f(t)dt,\ Re(s) > 0,$$

we observe that ${}_2F_1(a,[b,b+h;y];x)$ is the Mellin transform of the function

$$f(t:x;y,a;h) = H(y-t)(1-t)^{h-1}(1-xt)^{-a},$$

where

$$H(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases},$$

is the Heaviside unit function. Observing the fact that

$${}_2F_1(a,[b,b+h;y];x) := \frac{\mathfrak{M}\{f(t:x;y,a;h):b\}}{B(b,h)}, \tag{2.3.27}$$

we can write that

$$\frac{\partial}{\partial t}(f(t \quad : \quad x;y,a;h)) = -[(y-t)(1-t)^{h-1}(1-xt)^{-a} + (h-1)H(y-t)(1-t)^{h-2}(1-xt)^{-a}]$$
$$+ax(1-xt)^{-a-1}H(y-t)(1-t)^{h-1}, \tag{2.3.28}$$

where $\frac{\partial}{\partial t}(H(t)) = \delta(t-t_0)$,

$$\delta(t-t_0) = \begin{cases} \infty & \text{if} \quad t = t_0 \\ 0 & \text{if} \quad t \neq t_0 \end{cases},$$

is the Dirac delta function. Applying Mellin transform on both sides (2.3.28) and using (2.3.27) and the fact that

$$\mathfrak{M}\left\{f'(t):x\right\} = (1-x)\mathfrak{M}\{f(t):x-1\},$$

we have

$$\frac{b+h-1}{B(b,h)}y^{b-1}(1-y)^{h-1}(1-xy)^{-a} = \,_2F_1(a,[b,b+h-1;y];x)$$
$$+\,_2F_1(a,[b-1,b+h-1;y];x) - ax(b+h-1)\,_2F_1(a+1,[b,b+h;y];x).$$

This completes the proof. ■

In the following theorems, we give transformation formulas:

Theorem 2.3.13 *The following transformation formula holds true:*

$$_2F_1(a,[\beta,\gamma;y];z) = (1-z)^{-a}\,_2F_1(a,\{\gamma-\beta,\gamma;1-y\};\frac{z}{z-1}),\ \ |\arg(1-z)| < \pi. \tag{2.3.29}$$

Proof. Using (2.3.8), we obtain

$$_2F_1(a,[\beta,\gamma;y];z) = \frac{(1-z)^{-a}}{B(\beta,\gamma-\beta)}\int_{1-y}^{1}(1-s)^{\beta-1}s^{\gamma-\beta-1}\left(1-\frac{z}{z-1}s\right)^{-a}ds. \tag{2.3.30}$$

The substitution $s = 1-t$ in (2.3.30) leads to

$$\begin{aligned} {}_2F_1(a,[\beta,\gamma;y];z) &= \frac{(1-z)^{-a}}{B(\beta,\gamma-\beta)} \int_0^y t^{\beta-1}(1-t)^{\gamma-\beta-1} \left(1-\frac{z(1-t)}{z-1}\right)^{-a} dt \\ &= (1-z)^{-a}\, {}_2F_1(a,\{\gamma-\beta,\gamma;1-y\};\frac{z}{z-1}). \end{aligned}$$

■

Theorem 2.3.14 *([39]) The following transformation formula holds true:*

$$ {}_2F_1(a,\{\beta,\gamma;y\};z) = (1-z)^{-a}\, {}_2F_1(a,[\gamma-\beta,\gamma;1-y];\frac{z}{z-1}), \quad |\arg(1-z)| < \pi. \tag{2.3.31}$$

Theorem 2.3.15 *([39]) The following transformation formulas hold true:*

$$ {}_1F_1(\{\alpha,\beta;1-y\};z) = e^z\, {}_1F_1([\beta-\alpha,\beta;y];-z) \tag{2.3.32}$$

and

$$ {}_1F_1([\alpha,\beta;y];z) = e^z\, {}_1F_1(\{\beta-\alpha,\beta;1-y\};-z). \tag{2.3.33}$$

Proof. The proofs of (2.3.32) and (2.3.33) are direct consequences of Theorem 2.3.5.

■

2.4 The incomplete Appell's functions

In this section, we introduce the incomplete Appell's functions $F_1[a,b,c;d;x,z;y]$, $F_1\{a,b,c;d;x,z;y\}$, $F_2[a,b,c;d,e;x,z;y]$ and $F_2\{a,b,c;d,e;x,z;y\}$ by

$$F_1[a,b,c;d;x,z;y] := \sum_{m,n=0}^{\infty} [a,d;y]_{m+n}(b)_m(c)_n \frac{x^m}{m!}\frac{z^n}{n!}, \quad \max\{|x|,|z|\} < 1 \tag{2.4.1}$$

and

$$F_1\{a,b,c;d;x,z;y\} := \sum_{m,n=0}^{\infty} \{a,d;y\}_{m+n}(b)_m(c)_n \frac{x^m}{m!}\frac{z^n}{n!}, \quad \max\{|x|,|z|\} < 1 \tag{2.4.2}$$

and

$$F_2[a,b,c;d,e;x,z;y] := \sum_{m,n=0}^{\infty} (a)_{m+n}[b,d;y]_m[c,e;y]_n \frac{x^m}{m!}\frac{z^n}{n!}, \quad |x|+|z| < 1 \tag{2.4.3}$$

and

$$F_2\{a,b,c;d,e;x,z;y\} := \sum_{m,n=0}^{\infty} (a)_{m+n}\{b,d;y\}_m\{c,e;y\}_n \frac{x^m}{m!}\frac{z^n}{n!}, \quad |x|+|z| < 1. \tag{2.4.4}$$

We proceed by obtaining the integral representations of the functions $F_1[a,b,c;d;x,z;y]$, $F_1\{a,b,c;d;x,z;y\}$, $F_2[a,b,c;d,e;x,z;y]$ and $F_2\{a,b,c;d,e;x,z;y\}$.

Theorem 2.4.1 *([39]) For the incomplete Appell's functions $F_1[a,b,c;d;x,z;y]$ and $F_1\{a,b,c;d;x,z;y\}$, we have the following integral representation:*

$$\begin{aligned} F_1[a,b,c;d;x,z;y] &= \frac{y^a}{B(a,d-a)}\int_0^1 u^{a-1}(1-uy)^{d-a-1}(1-xuy)^{-b}(1-zuy)^{-c}du, \\ &\quad Re(d)>0,\ Re(a)>0,\ Re(b)>0,\ Re(c)>0, \\ &\quad |\arg(1-x)|<\pi,\ |\arg(1-z)|<\pi. \end{aligned} \tag{2.4.5}$$

and

$$F_1\{a,b,c;d;x,z;y\} = \frac{(1-y)^{d-a}}{B(a,d-a)}\int_0^1 u^{d-a-1}(1-u(1-y))^{a-1}$$
$$\times(1-x(1-u(1-y)))^{-b}(1-z(1-u(1-y)))^{-c}du,$$
$$Re(d)>0,\ Re(a)>0,\ Re(b)>0,\ Re(c)>0,$$
$$|\arg(1-x)|<\pi,\ \ |\arg(1-z)|<\pi. \qquad (2.4.6)$$

Proof. Replacing the integral representation of incomplete beta function which is given by (2.2.1), we find that

$$F_1[a,b,c;d;x,z;y] = \frac{1}{B(a,d-a)}\int_0^y t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-zt)^{-c}dt,$$

which can be written as

$$F_1[a,b,c;d;x,z;y] = \frac{y^a}{B(a,d-a)}\int_0^1 u^{a-1}(1-uy)^{d-a-1}(1-xuy)^{-b}(1-zuy)^{-c}du.$$

Whence the result. Formula (2.4.6) can be proved in a similar way. ■

Theorem 2.4.2 *([39]) For the incomplete Appell's functions* $F_2[a,b,c;d,e;x,z;y]$ *and* $F_2\{a,b,c;d,e;x,z;y\}$, *we have the following integral representation:*

$$F_2[a,b,c;d,e;x,z;y] = \frac{y^{b+c}}{B(b,d-b)B(c,e-c)}$$
$$\times\int_0^1\int_0^1 u^{b-1}(1-uy)^{d-b-1}v^{c-1}(1-vy)^{e-c-1}(1-xuy-zvy)^{-a}dudv,$$
$$Re(d)>Re(a)>Re(b)>Re(c)>Re(m)>0,\ \ |\arg(1-x-z)|<\pi. \ (2.4.7)$$

and

$$
\begin{aligned}
& F_2\{a,b,c;d,e;x,z;y\} \\
= \; & \frac{(1-y)^{d-b+e-c}}{B(b,d-b)B(c,e-c)} \int_0^1 \int_0^1 u^{d-b-1}(1-u(1-y))^{b-1} v^{e-c-1}(1-v(1-y))^{c-1} \\
& (1-x(1-u(1-y))-z(1-v(1-y)))^{-a} dudv, \qquad (2.4.8)
\end{aligned}
$$

$Re(d) > 0,\ Re(a) > 0,\ Re(b) > 0,\ Re(c) > 0,\ Re(e) > 0,\ |\arg(1-x-z)| < \pi.$

Proof. Replacing the integral representation of incomplete beta function which is given by (2.2.1), we get

$$
\begin{aligned}
F_2[a,b,c;d,e;x,z;y] &= \frac{1}{B(b,d-b)B(c,e-c)} \sum_{m,n=0}^{\infty} \int_0^y \int_0^y (a)_{m+n} \\
& \times t^{b+m-1}(1-t)^{d-b-1} s^{c+n-1}(1-s)^{e-c-1} \frac{x^m}{m!}\frac{z^n}{n!} dtds.
\end{aligned}
$$

Considering the fact that the series involved are uniformly convergent and we have a right to interchange the order of summation and integration, we get

$$
\begin{aligned}
F_2[a,b,c;d,e;x,z;y] &= \frac{1}{B(b,d-b)B(c,e-c)} \\
& \times \int_0^y \int_0^y t^{b-1}(1-t)^{d-b-1} s^{c-1}(1-s)^{e-c-1}(1-xt-zs)^{-a} dtds, \\
&= \frac{y^{b+c}}{B(b,d-b)B(c,e-c)} \\
& \times \int_0^1 \int_0^1 u^{b-1}(1-uy)^{d-b-1} v^{c-1}(1-vy)^{e-c-1} \\
& \times (1-xuy-zvy)^{-a} dudv.
\end{aligned}
$$

Formula (2.4.8) can be proved in a similar way. ■

2.5 Incomplete Riemann-Liouville fractional derivative operator

In this section, the incomplete Riemann-Liouville fractional derivative operators are introduced and investigated. The Riemann-Liouville fractional derivative of order μ is defined by

$$D_z^{\mu}\{f(z)\} := \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} dt, \quad Re(\mu) < 0. \tag{2.5.1}$$

Now, the incomplete Riemann-Liouville fractional derivative operators $D_z^{\mu}[f(z);y]$ and $D_z^{\mu}\{f(z);y\}$ are defined by

$$\begin{aligned} D_z^{\mu}[f(z);y] \quad &: \; = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1-u)^{-\mu-1} du \\ &: \; = \frac{z^{-\mu} y}{\Gamma(-\mu)} \int_0^1 f(ywz)(1-wy)^{-\mu-1} dw, \; Re(\mu) < 0. \end{aligned} \tag{2.5.2}$$

and its counterpart is defined by

$$\begin{aligned} D_z^{\mu}\{f(z);y\} \quad &: \; = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_y^1 f(uz)(1-u)^{-\mu-1} du \\ &: \; = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^{1-y} f((1-t)z) t^{-\mu-1} dt, \;\; Re(\mu) < 0. \end{aligned} \tag{2.5.3}$$

The investigation started by calculation of the incomplete fractional derivatives of some elementary functions.

Theorem 2.5.1 *([39]) Let $Re(\lambda) > -1$, $Re(\mu) < 0$. Then*

$$D_z^{\mu}[z^{\lambda};y] = \frac{B_y(\lambda+1,-\mu)}{\Gamma(-\mu)} z^{\lambda-\mu}. \tag{2.5.4}$$

Proof. Using (2.5.2) and (2.2.1), gives

$$\begin{aligned} D_z^{\mu}[z^{\lambda};y] &= \frac{z^{-\mu}}{\Gamma(-\mu)}\int_0^y (uz)^{\lambda}(1-u)^{-\mu-1}du \\ &= \frac{B_y(\lambda+1,-\mu)}{\Gamma(-\mu)}z^{\lambda-\mu}. \end{aligned}$$

Whence the result. ■

Theorem 2.5.2 *([39]) Let $Re(\lambda) > -1,\ Re(\mu) < 0$. Then*

$$D_z^{\mu}\{z^{\lambda};y\} = \frac{B_{1-y}(-\mu,\lambda+1)}{\Gamma(-\mu)}z^{-\mu+\lambda}. \tag{2.5.5}$$

Theorem 2.5.3 *([39]) Let $Re(\lambda) > 0,\ Re(\alpha) > 0,\ Re(\mu) < 0$ and $|z| < 1$. Then*

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}\ {}_2F_1(\alpha,[\lambda,\mu;y];z), \tag{2.5.6}$$

and

$$D_z^{\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\alpha};y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}\ {}_2F_1(\alpha,\{\lambda,\mu;y\};z). \tag{2.5.7}$$

Proof. Direct calculations yield

$$\begin{aligned} D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha};y] &= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1}du \\ &= \frac{z^{\mu-\lambda}y}{\Gamma(\mu-\lambda)}\int_0^1 (yz)^{\lambda-1}w^{\lambda-1}(1-ywz)^{-\alpha}(1-wy)^{\mu-\lambda-1}dw \\ &= \frac{z^{\mu-1}y^{\lambda}}{\Gamma(\mu-\lambda)}\int_0^1 w^{\lambda-1}(1-ywz)^{-\alpha}(1-wy)^{\mu-\lambda-1}dw. \end{aligned}$$

By (2.3.7),

$$\begin{aligned} D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha};y] &= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)_2F_1(\alpha,[\lambda,\mu;y];z) \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}\,{}_2F_1(\alpha,[\lambda,\mu;y];z). \end{aligned}$$

can be written. Hence the proof is completed. Formula (2.5.7) can be proved in a similar way. ■

Theorem 2.5.4 *([39]) Let* $Re(\lambda) > Re(\mu) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$ *;* $|az| < 1$ *and* $|bz| < 1$. *Then*

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1[\lambda,\alpha,\beta;\mu;az,bz;y], \quad (2.5.8)$$

and

$$D_z^{\lambda-\mu}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1\{\lambda,\alpha,\beta;\mu;az,bz;y\}. \quad (2.5.9)$$

Proof. The function shows that

$$\begin{aligned} & D_z^{\lambda-\mu}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};y] \\ = & \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}(1-auz)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1}du \\ = & \frac{z^{\mu-\lambda}y}{\Gamma(\mu-\lambda)}\int_0^1 (yw)^{\lambda-1}(z)^{\lambda-1}(1-aywz)^{-\alpha}(1-bywz)^{-\beta}(1-wy)^{\mu-\lambda-1}dw \\ = & \frac{z^{\mu-1}y^{\lambda}}{\Gamma(\mu-\lambda)}\int_0^1 w^{\lambda-1}(1-aywz)^{-\alpha}(1-bywz)^{-\beta}(1-wy)^{\mu-\lambda-1}dw. \end{aligned}$$

By (2.4.5), the following can be written

$$\begin{aligned} D_z^{\lambda-\mu}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};y] &= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)F_1[\lambda,\alpha,\beta;\mu;az,bz;y] \\ &= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1[\lambda,\alpha,\beta;\mu;az,bz;y]. \end{aligned}$$

Whence the result. Formula (2.5.9), can be proved in a similar way. ■

Theorem 2.5.5 *([39]) Let* $Re(\lambda) > Re(\mu) > 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ *;* $\left|\frac{t}{1-z}\right| < 1$*and* $|t|+|z| < 1$ *we have*

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha}\ {}_2F_1(\alpha.[\beta,\gamma;y];\frac{t}{1-z});y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2[\alpha,\beta,\lambda;\gamma,\mu;t,z;y], \tag{2.5.10}$$

and

$$D_z^{\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\alpha}\ {}_2F_1(\alpha.[\beta,\gamma;y];\frac{t}{1-z});y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2\{\alpha,\beta,\lambda;\gamma,\mu;t,z;y\}. \tag{2.5.11}$$

Proof. Using Theorem 2.5.1 and (2.4.3), gives

$$\begin{aligned} & D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha}\ {}_2F_1(\alpha.[\beta,\gamma;y];\frac{t}{1-z});y] \\ = \ & D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n B_y(\beta+n,\gamma-\beta)}{n!}\left(\frac{t}{1-z}\right)^n;y] \\ = \ & \frac{1}{B(\beta,\gamma-\beta)}D_z^{\lambda-\mu}[z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n B_y(\beta+n,\gamma-\beta)\frac{t^n}{n!}(1-z)^{-\alpha-n};y] \\ = \ & \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B_y(\beta+n,\gamma-\beta)\frac{t^n}{n!}\frac{(\alpha)_n(\alpha+n)_m}{m!}D_z^{\lambda-\mu}[z^{\lambda-1+m};y] \\ = \ & \frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B_y(\beta+n,\gamma-\beta)\frac{t^n}{n!}\frac{(\alpha)_{n+m}}{m!}\frac{B_y(\lambda+m,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu+m-1} \\ = \ & \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2[\alpha,\beta,\lambda;\gamma,\mu;t,z;y]. \end{aligned}$$

Hence proof is completed. Formula (2.5.11), can be proved in a similar way. ■

2.6 Generating Functions

Now, linear and bilinear generating relations are obtained for the incomplete hypergeometric functions ${}_2F_1(a,[b,c;y];x)$ by the help of the methods described in [11]. The following theorem is used to begin:

Theorem 2.6.1 *([39]) For the incomplete hypergeometric functions we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1(\lambda+n,[\alpha,\beta;y];z)t^n=(1-t)^{-\lambda}\,{}_2F_1(\lambda,[\alpha,\beta;y];\frac{z}{1-t}) \tag{2.6.1}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1(\lambda+n,\{\alpha,\beta;y\};z)t^n=(1-t)^{-\lambda}\,{}_2F_1(\lambda,\{\alpha,\beta;y\};\frac{z}{1-t}) \tag{2.6.2}$$

where $|z|<\min\{1,|1-t|\}$ *and* $Re(\lambda)>0$, $Re(\beta)>Re(\alpha)>0$.

Proof. Considering the elementary identity

$$[(1-z)-t]^{-\lambda}=(1-t)^{-\lambda}\left[1-\frac{z}{1-t}\right]^{-\lambda}$$

and expanding the left hand side, we have for $|t|<|1-z|$ that

$$(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{t}{1-z}\right)^n=(1-t)^{-\lambda}\left[1-\frac{z}{1-t}\right]^{-\lambda}.$$

Now, multiplying both sides of the above equality by $z^{\alpha-1}$ and applying the incomplete fractional derivative operator $D_z^{\alpha-\beta}[f(z);y]$ on both sides ,

$$D_z^{\alpha-\beta}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z)^{-\lambda}\left(\frac{t}{1-z}\right)^n z^{\alpha-1};y\right]=(1-t)^{-\lambda}D_z^{\alpha-\beta}\left[z^{\alpha-1}\left[1-\frac{z}{1-t}\right]^{-\lambda};y\right].$$

can be written. Interchanging the order, which is valid for $Re(\alpha)>0$ and $|t|<|1-z|$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_z^{\alpha-\beta}\left[z^{\alpha-1}(1-z)^{-\lambda-n};y\right]t^n=(1-t)^{-\lambda}D_z^{\alpha-\beta}\left[z^{\alpha-1}\left[1-\frac{z}{1-t}\right]^{-\lambda};y\right].$$

Using Theorem 2.5.3, the desired result is achieved. Formula (2.6.2), can be proved in a similar way. ■

The following theorem gives another linear generating relation for the incomplete hypergeometric functions.

Theorem 2.6.2 *([39]) For the incomplete hypergeometric functions we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1(\rho-n,[\alpha,\beta;y];z)t^n=(1-t)^{-\lambda}F_1[\alpha,\rho,\lambda;\beta;z;\frac{-zt}{1-t};y] \qquad (2.6.3)$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1(\rho-n,\{\alpha,\beta;y\};z)t^n=(1-t)^{-\lambda}F_1\{\alpha,\rho,\lambda;\beta;z;\frac{-zt}{1-t};y\} \qquad (2.6.4)$$

where $Re(\lambda)>0,\ Re(\rho)>0,\ Re(\beta)>Re(\alpha)>0;\ |t|<\frac{1}{1+|z|}$.

Proof. Considering

$$[1-(1-z)t]^{-\lambda}=(1-t)^{-\lambda}\left[1+\frac{zt}{1-t}\right]^{-\lambda}$$

and expanding the left hand side, we have for $|t|<|1-z|$ that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z)^n t^n=(1-t)^{-\lambda}\left[1-\frac{-zt}{1-t}\right]^{-\lambda}.$$

Now, multiplying both sides of the above equality by $z^{\alpha-1}(1-z)^{-\rho}$ and applying the fractional derivative operator $D_z^{\alpha-\beta}[f(z);y]$ on both sides, we get

$$D_z^{\alpha-\beta}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^{\alpha-1}(1-z)^{-\rho+n}t^n;y\right]=(1-t)^{-\lambda}D_z^{\alpha-\beta}\left[z^{\alpha-1}(1-z)^{-\rho}\left[1-\frac{-zt}{1-t}\right]^{-\lambda};y\right].$$

Interchanging the order, which is valid for $Re(\alpha)>0$ and $|zt|<|1-t|$, gives

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_z^{\alpha-\beta}\left[z^{\alpha-1}(1-z)^{-(\rho-n)};y\right]t^n=(1-t)^{-\lambda}D_z^{\alpha-\beta}\left[z^{\alpha-1}(1-z)^{-\rho}\left[1-\frac{-zt}{1-t}\right]^{-\lambda};y\right].$$

Using Theorem 2.5.3 and 2.5.4, the desired result is achieved. Generating relation (2.6.4), can be proved in a similar way. ■

Finally, the following bilinear generating relation for the incomplete hypergeometric functions are shown.

Theorem 2.6.3 *([39]) For the incomplete hypergeometric functions we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1(\gamma,[-n,\delta;y];x)\,{}_2F_1(\gamma,[\lambda+n,\beta;y];z)t^n=(1-t)^{-\lambda}F_2[\lambda,\alpha,\gamma;\beta,\delta;\frac{z}{1-t};\frac{-xt}{1-t};y] \tag{2.6.5}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1(\gamma,\{-n,\delta;y\};x)\,{}_2F_1(\gamma,\{\lambda+n,\beta;y\};z)t^n=(1-t)^{-\lambda}F_2\{\lambda,\alpha,\gamma;\beta,\delta;\frac{z}{1-t};\frac{-xt}{1-t};y\} \tag{2.6.6}$$

where $Re(\lambda)>0$, $Re(\gamma)>0$, $Re(\beta)>0$, $Re(\delta)>0$, $Re(\alpha)>0$; $|t|<\frac{1-|z|}{1+|x|}$ *and* $|z|<1$.

Proof. *Replacing t by* $(1-x)t$ *in (2.6.1), multiplying the resulting equality by* $x^{\gamma-1}$ *and then applying the incomplete fractional derivative operator* $D_x^{\gamma-\delta}[f(x);y]$, *we get*

$$\begin{aligned} & D_x^{\gamma-\delta}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^{\gamma-1}\,{}_2F_1(\lambda+n,[\alpha,\beta;y];z)(1-x)^n t^n;y\right] \\ = \; & D_x^{\gamma-\delta}\left[(1-(1-x)t)^{-\lambda}x^{\gamma-1}\,{}_2F_1(\lambda,[\alpha,\beta;y];\frac{z}{1-(1-x)t});y\right]. \end{aligned}$$

Interchanging the order, which is valid for $|z|<1,\left|\frac{1-x}{1-z}t\right|<1$ *and* $\left|\frac{z}{1-t}\right|+\left|\frac{xt}{1-t}\right|<1$, *we can write that*

$$\begin{aligned} & \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\gamma-\delta}\left[x^{\gamma-1}(1-x)^n;y\right]\,{}_2F_1(\lambda+n,[\alpha,\beta;y];z) \\ = \; & (1-t)^{-\lambda}D_x^{\gamma-\delta}\left[x^{\gamma-1}(1-\frac{-xt}{1-t})\,{}_2F_1(\lambda,[\alpha,\beta;y];\frac{\frac{z}{1-t}}{1-\frac{-xt}{1-t}});y\right]. \end{aligned}$$

Using Theorems 2.5.3 and 2.5.5, we get (2.6.5). Generating relation (2.6.6), can be proved in a similar way. ■

Chapter 3

INCOMPLETE CAPUTO FRACTIONAL DERIVATIVE OPERATORS

3.1 Introduction

This chapter will give clear definitions of Caputo fractional derivative operators and show its use in the special function theory. Next section will introduce new type versions of incomplete Gauss hypergeometric functions ${}_2F_1$, the Appell hypergeometric functions F_1, F_2 and the Lauricella hypergeometric functions $F^3_{D,y}$ and we obtain their integral representations. Incomplete Caputo fractional derivative operators are discussed in detail, Section 3.3 and show that the incomplete Caputo fractional derivative operators of some elementary functions give the new type incomplete hypergeometric functions defined in Section 3.2. The main results can be seen in Section 3.4.

3.2 New type incomplete hypergeometric functions

This section introduces new types of incomplete Gauss hypergeometric functions ${}_2F_1$, the incomplete Appell's hypergeometric functions F_1, F_2 and the incomplete Lauricella hypergeometric functions $F^3_{D,y}$. Throughout this chapter we assume that $0 \leq y < 1$ and $m \in \mathbb{N}$.

Definition 3.2.1 *([37]) New type incomplete hypergeometric functions are defined by*

$$ {}_2\digamma_1(a,[b,c;y];x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(b-m)_n} [b-m,c;y]_n \frac{x^n}{n!}, \tag{3.2.1}$$

and

$$ {}_2F_1(a,\{b,c;y\};x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(b-m)_n} \{b-m,c;y\}_n \frac{x^n}{n!} \tag{3.2.2} $$

for all $|x| < 1$ *where* $m < Re(b) < Re(c)$.

Definition 3.2.2 *([37]) New type incomplete Appell hypergeometric functions* F_1 *are defined by*

$$ F_1[a,b,c;d;x,z;y] := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(a-m)_{n+k}} [a-m,d;y]_{n+k} \frac{x^n}{n!} \frac{z^k}{k!}, \tag{3.2.3} $$

and

$$ F_1\{a,b,c;d;x,z;y\} := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(a-m)_{n+k}} \{a-m,d;y\}_{n+k} \frac{x^n}{n!} \frac{z^k}{k!} \tag{3.2.4} $$

for all $|x| < 1,\ |z| < 1$ *where* $m < Re(a) < Re(d)$.

Definition 3.2.3 *([37]) New type incomplete Appell hypergeometric functions* F_2 *are defined by*

$$ F_2[a,b,c;d,e;x,z;y] := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(b-m)_n(c-m)_k} [b-m,d;y]_n [c-m,e;y]_k \frac{x^n}{n!} \frac{z^k}{k!}, \tag{3.2.5} $$

and

$$ F_2\{a,b,c;d,e;x,z;y\} := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(b-m)_n(c-m)_k} \{b-m,d;y\}_n \{c-m,e;y\}_k \frac{x^n}{n!} \frac{z^k}{k!}, \tag{3.2.6} $$

for all $|x|+|z|<1$ *where* $m<Re(b)<Re(d)$ *and* $m<Re(c)<Re(e)$.

Definition 3.2.4 *([37]) New type incomplete Lauricella hypergeometric functions* $F^3_{D,y}$ *are defined by*

$$F^3_{D,y}[a,b,c,d;e;x,w,z;y] := \sum_{n,k,r=0}^{\infty} \frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(a-m)_{n+k+r}} [a-m,e;y]_{n+k+r} \frac{x^n}{n!}\frac{w^k}{k!}\frac{z^r}{r!}, \tag{3.2.7}$$

and

$$F^3_{D,y}\{a,b,c,d;e;x,w,z;y\} := \sum_{n,k,r=0}^{\infty} \frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(a-m)_{n+k+r}} \{a-m,e;y\}_{n+k+r} \frac{x^n}{n!}\frac{w^k}{k!}\frac{z^r}{r!} \tag{3.2.8}$$

for all $\sqrt{x}+\sqrt{w}+\sqrt{z}<1$ *where* $m<Re(a)<Re(e)$.

Note that when $m=0$, these functions reduce to the corresponding versions given by (2.3.1), (2.3.2), (2.4.1) - (2.4.4), respectively. On the other hand, in the case $y\to 1^-$, the functions in (3.2.1), (3.2.3), (3.2.5) and (3.2.7) are reduced to their usual versions (similarly, when $y\to 0^+$, the functions in (3.2.2), (3.2.4), (3.2.6) and (3.2.8) are reduced to their usual versions).

Now start by obtaining the integral representations of the functions given in definition (3.2.1) - (3.2.4).

Theorem 3.2.5 *([37]) The following integral representations hold true:*

$${}_2F_1(a,[b,c;y];x) = \frac{y^{b-m}}{B(b-m,c-b+m)} \int_0^1 u^{b-m-1}(1-uy)^{c-b+m-1} {}_2F_1(a,b;b-m;xuy)du, \tag{3.2.9}$$

and

$$
\begin{aligned}
{}_2F_1(a,\{b,c;y\};x) &= \frac{(1-y)^{c-b+m}}{B(b-m,c-b+m)}\int_0^1 u^{c-b+m-1}(1-u(1-y))^{b-m-1} \\
&\quad \times {}_2F_1(a,b;b-m;x(1-u(1-y)))du \qquad (3.2.10)
\end{aligned}
$$

for all $|x|<1$ *where* $m<Re(b)<Re(c)$.

Proof. Replacing the incomplete beta function $B_y(b-m+n,c-b+m)$ in the definition (3.2.1) by its integral representation given by (2.2.1) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis, we reach

$$
{}_2F_1(a,[b,c;y];x) = \frac{1}{B(b-m,c-b+m)}\int_0^y t^{b-m-1}(1-t)^{c-b+m-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(b-m)_n}\frac{(xt)^n}{n!}dt,
$$

which can be given as

$$
{}_2F_1(a,[b,c;y];x) = \frac{y^{b-m}}{B(b-m,c-b+m)}\int_0^1 u^{b-m-1}(1-uy)^{c-b+m-1}{}_2F_1(a,b;b-m;xuy)du.
$$

Hence the proof is completed. Formula (3.2.10) can be proved in a similar way. ■

Theorem 3.2.6 *([37]) The following integral representations hold true:*

$$
\begin{aligned}
F_1[a,b,c;d;x,z;y] &= \frac{y^{a-m}}{B(a-m,d-a+m)}\int_0^1 u^{a-m-1}(1-uy)^{d-a+m-1} \\
&\quad \times F_1(a;b,c;a-m;xuy,zuy)du, \qquad (3.2.11)
\end{aligned}
$$

and

$$\digamma_1\{a,b,c;d;x,z;y\} = \frac{y^{d-a+m}}{B(a-m,d-a+m)}\int_0^1 u^{d-a+m-1}(1-u(1-y))^{a-m-1}$$
$$\times F_1(a;b,c;a-m;x(1-u(1-y)),z(1-u(1-y))du, \tag{3.2.12}$$

for all $|x|<1,\ |z|<1$ *where* $m<Re(a)<Re(d)$.

Proof. Replacing the incomplete beta function $B_y(a-m+n+k,d-a+m)$ in the definition (3.2.3) by its integral representation given by (2.2.1), it results in

$$\begin{aligned}\digamma_1[a,b,c;d;x,z;y] &= \frac{1}{B(a-m,d-a+m)}\int_0^y t^{a-m-1}(1-t)^{d-a+m-1}\\ &\quad\times\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_n(c)_k}{(a-m)_{n+k}}\frac{(xt)^n}{n!}\frac{(zt)^k}{k!}dt,\\ &= \frac{y^{a-m}}{B(a-m,d-a+m)}\int_0^1 u^{a-m-1}(1-uy)^{d-a+m-1}\\ &\quad\times F_1(a;b,c;a-m;xuy,zuy)du.\end{aligned}$$

Whence the result. Formula (3.2.12) can be identified in a similar way. ■

Theorem 3.2.7 *([37]) The following integral representations hold true:*

$$\begin{aligned}\digamma_2[a,b,c;d,e;x,z;y] &= \frac{y^{b+c-2m}}{B(b-m,d-b+m)B(c-m,e-c+m)}\\ &\quad\times\int_0^1\int_0^1 u^{b-m-1}(1-uy)^{d-b+m-1}v^{c-m-1}(1-vy)^{e-c+m-1}\\ &\quad\times F_2(a;b,c;b-m,c-m;xuy,zvy)dudv,\end{aligned} \tag{3.2.13}$$

and

$$\mathcal{F}_2\{a,b,c;d,e;x,z;y\} = \frac{(1-y)^{d-b+e-c+2m}}{B(b-m,d-b+m)B(c-m,e-c+m)} \times \int_0^1\int_0^1 u^{d-b+m-1}(1-u(1-y))^{b-m-1}v^{e-c+m-1}(1-v(1-y))^{c-m-1} \times F_2(a;b,c;b-m,c-m;x(1-u(1-y)),z(1-v(1-y)))dudv, \tag{3.2.14}$$

for all $|x|+|z|<1$ *where* $m<Re(b)<Re(d)$ *and* $m<Re(c)<Re(e)$.

Proof. Replacing the integral representations of $B_y(b-m+n,d-b+m)$ and $B_y(c-m+k,e-c+m)$ in (3.2.5), we get

$$\mathcal{F}_2[a,b,c;d,e;x,z;y] = \frac{1}{B(b-m,d-b+m)B(c-m,e-c+m)} \times \int_0^y\int_0^y t^{b-m-1}(1-t)^{d-b+m-1}s^{c-m-1}(1-s)^{e-c+m-1} \times \sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_n(c)_k}{(b-m)_n(c-m)_k}\frac{(xt)^n}{n!}\frac{(zs)^k}{k!}dtds,$$

which is equivalent to

$$\mathcal{F}_2[a,b,c;d,e;x,z;y] = \frac{y^{b+c-2m}}{B(b-m,d-b+m)B(c-m,e-c+m)} \times \int_0^1\int_0^1 u^{b-m-1}(1-uy)^{d-b+m-1}v^{c-m-1}(1-vy)^{e-c+m-1} \times F_2(a;b,c;b-m,c-m;xuy,zvy)dudv.$$

Hence the proof is completed. Formula (3.2.14) can be proved in a similar way. ■

Theorem 3.2.8 *([37]) The following integral representation holds true:*

$$\digamma^{3}_{D,y}[a,b,c,d;e;x,w,z;y] = \frac{y^{a-m}}{B(a-m,e-a+m)} \int_0^1 u^{a-m-1}(1-uy)^{e-a+m-1} \times F^3_D(a,b,c,d;a-m;xuy,wuy,zuy)du, \qquad (3.2.15)$$

and

$$\digamma^{3}_{D,y}\{a,b,c,d;e;x,w,z;y\} = \frac{(1-y)^{e-a+m}}{B(a-m,e-a+m)} \int_0^1 u^{e-a+m-1}(1-u(1-y))^{a-m-1} \times F^3_D(a,b,c,d;a-m;xu(1-y),wu(1-y),zu(1-y))du, \qquad (3.2.16)$$

for all $\sqrt{x}+\sqrt{w}+\sqrt{z}<1$ *where* $m<Re(a)<Re(e)$.

Proof. Replacing the incomplete beta function $B_y(a-m+n+k+r,e-a+m)$ in the definition (3.2.7) by its integral representation given by (2.2.1), it can be observed that

$$\digamma^{3}_{D,y}[a,b,c,d;e;x,w,z;y] = \frac{1}{B(a-m,e-a+m)} \int_0^y t^{a-m-1}(1-t)^{e-a+m-1} \times \sum_{n,k,r=0}^{\infty} \frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(a-m)_{n+k+r}} \frac{(xt)^n}{n!} \frac{(wt)^k}{k!} \frac{(zt)^r}{r!} dt$$

which can be seen as

$$\digamma^{3}_{D,y}[a,b,c,d;e;x,w,z;y] = \frac{y^{a-m}}{B(a-m,e-a+m)} \int_0^1 u^{a-m-1}(1-uy)^{e-a+m-1} \times F^3_D(a,b,c,d;a-m;xuy,wuy,zuy)du.$$

Whence the result. Formula (3.2.16) can be proved in a similar way. ■

3.3 Incomplete Caputo fractional derivative operators

The classical Caputo fractional derivative is defined by

$$D^{\alpha}f(z) := \frac{1}{\Gamma(m-\alpha)}\int_0^z (z-v)^{m-\alpha-1}\frac{d^m}{dv^m}f(v)dv,$$

where $m-1 < Re(\alpha) < m,\ m \in \mathbb{N}$.

Now, we introduce the incomplete Caputo fractional derivatives as

$$C_z^{\alpha}[f(z);y] := \frac{1}{\Gamma(m-\alpha)}\int_0^{yz} (z-v)^{m-\alpha-1}\frac{d^m}{dv^m}f(v)dv \tag{3.3.1}$$

and

$$C_z^{\alpha}\{f(z);y\} := \frac{1}{\Gamma(m-\alpha)}\int_{yz}^{z} (z-v)^{m-\alpha-1}\frac{d^m}{dv^m}f(v)dv, \tag{3.3.2}$$

where $0 \leq y < 1$ and $m-1 < Re(\alpha) < m,\ m \in \mathbb{N}$. In the case $y \to 1^-$, (3.3.1) reduces to classical Caputo fractional derivative (similarly, when $y \to 0^+$, (3.3.2) reduces to classical Caputo fractional derivative).

The investigation begins with calculating the incomplete Caputo fractional derivatives of some elementary functions.

Theorem 3.3.1 *([37]) Let $m-1 < Re(\alpha) < m$ and $Re(\alpha) < Re(\lambda)$ then*

$$C_z^{\alpha}[z^{\lambda};y] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}\frac{B_y(\lambda-m+1,m-\alpha)}{B(\lambda-m+1,m-\alpha)}z^{\lambda-\alpha} \tag{3.3.3}$$

and

$$C_z^{\alpha}\{z^{\lambda};y\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} \frac{B_{1-y}(m-\alpha,\lambda-m+1)}{B(\lambda-m+1,m-\alpha)} z^{\lambda-\alpha}. \quad (3.3.4)$$

Proof. Theorem 3.3.1 gives

$$\begin{aligned} C_z^{\alpha}[z^{\lambda};y] &= \frac{1}{\Gamma(m-\alpha)} \int_0^{yz} (z-v)^{m-\alpha-1} \frac{d^m}{dv^m} v^{\lambda} dv \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} \int_0^{yz} (z-v)^{m-\alpha-1} v^{\lambda-m} dv \\ &= \frac{z^{\lambda-\alpha}}{\Gamma(m-\alpha)} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} y^{\lambda-m+1} \int_0^1 u^{\lambda-m}(1-uy)^{m-\alpha-1} du \\ &= .\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} \frac{B_y(\lambda-m+1,m-\alpha)}{B(\lambda-m+1,m-\alpha)} z^{\lambda-\alpha}. \end{aligned}$$

Hence the proof is completed. Formula (3.3.4) can be proved in a similar way. ■

The next theorem express the incomplete Caputo fractional derivative of an analytic function.

Theorem 3.3.2 *([37]) If $f(z)$ is an analytic function on the disk $|z| < \rho$ and has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then*

$$C_z^{\alpha}[z^{\lambda-1}f(z);y] = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} z^{\lambda-\alpha-1} \sum_{n=0}^{\infty} a_n \frac{(\lambda)_n}{(\lambda-m)_n} \frac{B_y(\lambda-m+n,m-\alpha)}{B(\lambda-m,m-\alpha)} z^n, \quad (3.3.5)$$

and

$$C_z^{\alpha}\{z^{\lambda-1}f(z);y\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} z^{\lambda-\alpha-1} \sum_{n=0}^{\infty} a_n \frac{(\lambda)_n}{(\lambda-m)_n} \frac{B_{1-y}(m-\alpha,\lambda-m+n)}{B(\lambda-m,m-\alpha)} z^n, \quad (3.3.6)$$

where $m-1 < Re(\alpha) < m$.

Proof. Using Theorem 3.3.1, results in

$$\begin{aligned}
C_z^{\alpha}[z^{\lambda-1}f(z);y] &= \sum_{n=0}^{\infty} a_n C_z^{\alpha}[z^{\lambda+n-1};y] \\
&= \sum_{n=0}^{\infty} a_n \left(\frac{1}{\Gamma(m-\alpha)} \int_0^{yz} (z-v)^{m-\alpha-1} \frac{d^m}{dv^m} v^{\lambda+n-1} dv \right) \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} z^{\lambda-\alpha-1} \sum_{n=0}^{\infty} a_n \frac{(\lambda)_n}{(\lambda-\alpha)_n} \frac{B_y(\lambda-m+n, m-\alpha)}{B(\lambda-m+n, m-\alpha)} z^n \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} z^{\lambda-\alpha-1} \sum_{n=0}^{\infty} a_n \frac{(\lambda)_n}{(\lambda-m)_n} \frac{B_y(\lambda-m+n, m-\alpha)}{B(\lambda-m, m-\alpha)} z^n.
\end{aligned}$$

Whence the result. Formula (3.3.6) can be proved in a similar way. ■

The following theorems will be useful for finding the generating function relations.

Theorem 3.3.3 *([37]) Let* $m-1 < Re(\lambda-\alpha) < m < Re(\lambda)$, *then*

$$C_z^{\lambda-\alpha}[z^{\lambda-1}(1-z)^{-\mu};y] = \frac{\Gamma(\lambda)}{\Gamma(\alpha)} z^{\alpha-1} \, {}_2F_1(\mu, [\lambda, \alpha; y]; z) \tag{3.3.7}$$

and

$$C_z^{\lambda-\alpha}\{z^{\lambda-1}(1-z)^{-\mu};y\} = \frac{\Gamma(\lambda)}{\Gamma(\alpha)} z^{\alpha-1} \, {}_2F_1(\mu, \{\lambda, \alpha; y\}; z) \tag{3.3.8}$$

for $|z| < 1$.

Proof. If the power series expansion of $(1-z)^{-\mu}$ is used, it shows

$$\begin{aligned}
C_z^{\lambda-\alpha}[z^{\lambda-1}(1-z)^{-\mu};y] &= C_z^{\lambda-\alpha}\left[z^{\lambda-1}\sum_{n=0}^{\infty}(\mu)_n\frac{z^n}{n!};y\right] \\
&= \sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}C_z^{\lambda-\alpha}[z^{\lambda+n-1};y] \\
&= \sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\alpha+n)}\frac{B_y(\lambda-m+n,m-\lambda+\mu)}{B(\lambda-m+n,m-\lambda+\mu)}z^{\alpha+n-1} \\
&= \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\mu)_n(\lambda)_n}{(\alpha)_n}\frac{B_y(\lambda-m+n,m-\lambda+\mu)}{B(\lambda-m+n,m-\lambda+\mu)}\frac{z^n}{n!} \\
&= \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\mu)_n(\lambda)_n}{(\lambda-m)_n}\frac{B_y(\lambda-m+n,m-\lambda+\mu)}{B(\lambda-m,\alpha-\lambda+m)}\frac{z^n}{n!}.
\end{aligned}$$

Using (3.2.1), we get the result. Formula (3.3.8) can be proved in a similar way. ■

Theorem 3.3.4 *([37]) Let $m-1<Re(\lambda-\alpha)<m<Re(\lambda)$, then*

$$C_z^{\lambda-\alpha}[z^{\lambda-1}(1-az)^{-\gamma}(1-bz)^{-\beta};y] = \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}\,F_1[\lambda,\gamma,\beta;\alpha;az;bz;y] \qquad (3.3.9)$$

and

$$C_z^{\lambda-\alpha}\{z^{\lambda-1}(1-az)^{-\gamma}(1-bz)^{-\beta};y\} = \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}\,F_1\{\lambda,\gamma,\beta;\alpha;az;bz;y\} \qquad (3.3.10)$$

for $|az|<1$ and $|bz|<1$.

Proof. Using the power series expansion of $(1-az)^{-\gamma},\ (1-bz)^{-\beta}$ gives

$$\begin{aligned}
& C_z^{\lambda-\alpha}[z^{\lambda-1}(1-az)^{-\gamma}(1-bz)^{-\beta};y] \\
= \ & C_z^{\lambda-\alpha}\left[\sum_{n,k=0}^{\infty}\frac{(\gamma)_n}{n!}\frac{(\beta)_k}{k!}a^n b^k z^{\lambda+n+k-1};y\right] \\
= \ & \sum_{n,k=0}^{\infty}\frac{(\gamma)_n}{n!}\frac{(\beta)_k}{k!}a^n b^k C_z^{\lambda-\alpha}[z^{\lambda+n+k-1};y] \\
= \ & \sum_{n,k=0}^{\infty}\frac{(\gamma)_n}{n!}\frac{(\beta)_k}{k!}a^n b^k \frac{\Gamma(\lambda+n+k)B_y(\lambda-m+n+k,m-\lambda+\alpha)}{\Gamma(\lambda-m+n+k)\Gamma(m-\lambda+\alpha)} z^{\alpha+n+k-1} \\
= \ & \frac{\Gamma(\lambda)}{\Gamma(\alpha)} z^{\alpha-1}\sum_{n,k=0}^{\infty}\frac{(\lambda)_{n+k}(\gamma)_n(\beta)_k}{(\lambda-m)_{n+k}}\frac{B_y(\lambda-m+n+k,m-\lambda+\alpha)}{B(\lambda-m,m-\lambda+\alpha)}\frac{(az)^n}{n!}\frac{(bz)^k}{k!}.
\end{aligned}$$

Using (3.2.3), we get the result. Formula (3.3.10) can be proved in a similar way. ■

Theorem 3.3.5 *([37]) Let $m-1 < Re(\lambda-\alpha) < m < Re(\lambda)$, then*

$$C_z^{\lambda-\alpha}[z^{\lambda-1}(1-az)^{-\gamma}(1-bz)^{-\beta}(1-cz)^{-\mu};y] = \frac{\Gamma(\lambda)}{\Gamma(\alpha)} z^{\alpha-1}\, F_{D,y}^{3}[\lambda,\gamma,\beta,\mu;\alpha;az;bz;cz;y] \tag{3.3.11}$$

and

$$C_z^{\lambda-\alpha}\{z^{\lambda-1}(1-az)^{-\gamma}(1-bz)^{-\beta}(1-cz)^{-\mu};y\} = \frac{\Gamma(\lambda)}{\Gamma(\alpha)} z^{\alpha-1}\, F_{D,y}^{3}\{\lambda,\gamma,\beta,\mu;\alpha;az;bz;cz;y\} \tag{3.3.12}$$

for $|az|<1,\ |bz|<1$ and $|cz|<1$.

Proof. Using the power series expansion of $(1-az)^{-\gamma},\ (1-bz)^{-\beta},(1-cz)^{-\mu}$ ends in,

$$
\begin{aligned}
& C_z^{\lambda-\alpha}[z^{\lambda-1}(1-az)^{-\gamma}(1-bz)^{-\beta}(1-cz)^{-\mu};y] \\
= \; & C_z^{\lambda-\alpha}\left[\sum_{n,k,r=0}^{\infty}\frac{(\gamma)_n}{n!}\frac{(\beta)_k}{k!}\frac{(\mu)_r}{r!}a^n b^k c^r z^{\lambda+n+k+r-1};y\right] \\
= \; & \sum_{n,k,r=0}^{\infty}\frac{(\gamma)_n}{n!}\frac{(\beta)_k}{k!}\frac{(\mu)_r}{r!}a^n b^k c^r C_z^{\lambda-\alpha}[z^{\lambda+n+k+r-1};y] \\
= \; & \sum_{n,k,r=0}^{\infty}\frac{(\gamma)_n}{n!}\frac{(\beta)_k}{k!}\frac{(\mu)_r}{r!}a^n b^k c^r \frac{\Gamma(\lambda+n+k+r)B_y(\lambda-m+n+k+r,m-\lambda+\alpha)}{\Gamma(\lambda-m+n+k+r)\Gamma(m-\lambda+\alpha)}z^{\alpha+n+k+r-1} \\
= \; & \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}\sum_{n,k,r=0}^{\infty}\frac{(\lambda)_{n+k+r}(\gamma)_n(\beta)_k(\mu)_r}{(\lambda-m)_{n+k+r}}\frac{B_y(\lambda-m+n+k+r,m-\lambda+\alpha)}{B(\lambda-m,m-\lambda+\alpha)}\frac{(az)^n}{n!}\frac{(bz)^k}{k!}\frac{(cz)^r}{r!}.
\end{aligned}
$$

Using (3.2.7), we get the result. Formula (3.3.12) can be shown in a similar way. ■

Theorem 3.3.6 *([37]) Let $m-1 < Re(\lambda-\alpha) < m < Re(\lambda)$ and $m < Re(\beta) < Re(\gamma)$, then*

$$
C_z^{\lambda-\alpha}\left[z^{\lambda-1}(1-z)^{-\mu}{}_2F_1\left(\mu,[\beta,\gamma;y];\frac{x}{1-z}\right);y\right] = \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}F_2[\mu,\beta,\lambda;\gamma,\alpha;x,z;y] \tag{3.3.13}
$$

and

$$
C_z^{\lambda-\alpha}\left\{z^{\lambda-1}(1-z)^{-\mu}{}_2F_1\left(\mu,\{\beta,\gamma;y\};\frac{x}{1-z}\right);y\right\} = \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}F_2\{\mu,\beta,\lambda;\gamma,\alpha;x,z;y\} \tag{3.3.14}
$$

for $|x|+|z| < 1$.

Proof. Using the power series expansion of $(1-z)^{-\mu}$, we get

$$
\begin{aligned}
& C_z^{\lambda-\alpha}\left[z^{\lambda-1}(1-z)^{-\mu}{}_2F_1\left(\mu,[\beta,\gamma;y];\frac{x}{1-z}\right);y\right] \\
= \; & C_z^{\lambda-\alpha}\left[z^{\lambda-1}(1-z)^{-\mu}\sum_{n=0}^{\infty}\frac{(\mu)_n(\beta)_n}{(\beta-m)_n n!}\frac{B_y(\beta-m+n,\gamma-\beta+m)}{B(\beta-m,\gamma-\beta+m)}\left(\frac{x}{1-z}\right)^n;y\right] \\
= \; & C_z^{\lambda-\alpha}\left[z^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\mu)_n(\beta)_n}{(\beta-m)_n}\frac{B_y(\beta-m+n,\gamma-\beta+m)}{B(\beta-m,\gamma-\beta+m)}\frac{x^n}{n!}(1-z)^{-\mu-n};y\right] \\
= \; & \sum_{n=0}^{\infty}\frac{(\mu)_n(\beta)_n}{(\beta-m)_n}\frac{B_y(\beta-m+n,\gamma-\beta+m)}{B(\beta-m,\gamma-\beta+m)}\frac{x^n}{n!}C_z^{\lambda-\alpha}[z^{\lambda-1}(1-z)^{-\mu-n};y] \\
= \; & \frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}\sum_{n,k=0}^{\infty}\left[\frac{(\mu)_{n+k}(\beta)_n(\lambda)_k}{(\beta-m)_n(\lambda-m)_k}\frac{B_y(\beta-m+n,\gamma-\beta+m)}{B(\beta-m,\gamma-\beta+m)}\right. \\
& \left.\times\frac{B_y(\lambda-m+k,\mu-\lambda+m)}{B(\lambda-m,\mu-\lambda+m)}\frac{x^n}{n!}\frac{z^k}{k!}\right].
\end{aligned}
$$

Using (3.2.5), we get the result. Formula (3.3.14) can be proved in a similar way. ■

3.4 Main Results

In this part of the chapter, linear and bilinear generating relations were given for the new type incomplete Gauss hypergeometric functions by using the relations obtained in (3.3.7), (3.3.8), (3.3.9), (3.3.10), (3.3.13) and (3.3.14).

Theorem 3.4.1 *([37]) Let $m-1 < Re(\lambda-\alpha) < m < Re(\lambda)$, then*

$$\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,[\lambda,\alpha;y];z)t^n = (1-t)^{-\mu}{}_2F_1\left(\mu,[\lambda,\alpha;y];\frac{z}{1-t}\right) \quad (3.4.1)$$

and

$$\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,\{\lambda,\alpha;y\};z)t^n = (1-t)^{-\mu}{}_2F_1\left(\mu,\{\lambda,\alpha;y\};\frac{z}{1-t}\right), \quad (3.4.2)$$

where $|z| < \min\{1,|1-t|\}$.

Proof. Taking into account the identity,

$$[(1-z)-t]^{-\mu} = (1-t)^{-\mu}(1-\frac{z}{1-t})^{-\mu}$$

and expanding the left hand side, we get for $|t| < |1-z|$ that

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}(1-z)^{-\mu}\left(\frac{t}{1-z}\right)^n = (1-t)^{-\mu}\left(1-\frac{z}{1-t}\right)^{-\mu}.$$

If we multiply both sides with $z^{\lambda-1}$ and apply the incomplete Caputo fractional derivative operator $C_z^{\lambda-\alpha}$, we get

$$C_z^{\lambda-\alpha}\left[\sum_{n=0}^{\infty} \frac{(\mu)_n t^n}{n!} z^{\lambda-1}(1-z)^{-\mu-n};y\right] = C_z^{\lambda-\alpha}\left[(1-t)^{-\mu} z^{\lambda-1}\left(1-\frac{z}{1-t}\right)^{-\mu};y\right].$$

Since $|t| < |1-z|$ and $Re(\lambda) > Re(\mu) > 0$, it is possible to change the order of the summation and derivative therefore,

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} C_z^{\lambda-\alpha}\left[z^{\lambda-1}(1-z)^{-\mu-n};y\right] t^n = (1-t)^{-\mu} C_z^{\lambda-\alpha}\left[z^{\lambda-1}\left(1-\frac{z}{1-t}\right)^{-\mu};y\right].$$

and after using Theorem 3.3.3, results can be identified on both sides. Formula (3.4.2) can be proved in a similar way. ■

Theorem 3.4.2 *([37]) Let $m-1 < Re(\lambda-\alpha) < m < Re(\lambda)$, then*

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}\, {}_2F_1(\rho-n,[\lambda,\alpha;y];z)t^n = (1-t)^{-\mu} F_1\left[\rho,\mu,\lambda;\alpha;z;\frac{-zt}{1-t};y\right] \tag{3.4.3}$$

and

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}\, {}_2F_1(\rho-n,\{\lambda,\alpha;y\};z)t^n = (1-t)^{-\mu} F_1\left\{\rho,\mu,\lambda;\alpha;z;\frac{-zt}{1-t};y\right\}, \quad (3.4.4)$$

where $|t| < \frac{1}{1+|z|}$.

Proof. Considering

$$[1-(1-z)t]^{-\mu} = (1-t)^{-\mu}\left(1+\frac{zt}{1-t}\right)^{-\mu}$$

and expanding the left hand side, we get

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}(1-z)^n t^n = (1-t)^{-\mu}\left(1-\frac{-zt}{1-t}\right)^{-\mu},$$

when $|t| < |1-z|$. If we multiply the both sides with $z^{\lambda-1}(1-z)^{-\rho}$ and apply the incomplete Caputo fractional derivative operator $C_z^{\lambda-\alpha}$, we get

$$C_z^{\lambda-\alpha}\left[\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} z^{\lambda-1}(1-z)^{-\rho+n}t^n;y\right] = C_z^{\lambda-\alpha}\left[(1-t)^{-\mu} z^{\lambda-1}(1-z)^{-\rho}\left(1-\frac{-zt}{1-t}\right)^{-\mu};y\right].$$

Since $|zt| < |1-t|$ and $Re(\lambda) > Re(\alpha) > 0$, it is possible to change the order of the summation and the derivative as

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} C_z^{\lambda-\alpha}\left[z^{\lambda-1}(1-z)^{-\rho+n};y\right] t^n = (1-t)^{-\mu} C_z^{\lambda-\alpha}\left[z^{\lambda-1}(1-z)^{-\rho}\left(1-\frac{-zt}{1-t}\right)^{-\mu};y\right].$$

So the result can be achieved after using Theorem 3.3.3 and Theorem 3.3.4. Formula (3.4.4) can be proved in a similar way. ■

Theorem 3.4.3 *([37]) Let $m-1<Re(\rho-\gamma)<m<Re(\rho)$ and $m<Re(\lambda)<Re(\alpha)$, then*

$$\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,[\lambda,\alpha;y];z)\,{}_2F_1(-n,[\rho,\gamma;y];x)t^n=(1-t)^{-\mu}\,F_2\left[\mu,\lambda,\rho;\alpha,\gamma;z,\frac{-xt}{1-t};y\right] \tag{3.4.5}$$

and

$$\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,\{\lambda,\alpha;y\};z)\,{}_2F_1(-n,\{\rho,\gamma;y\};x)t^n=(1-t)^{-\mu}\,F_2\left\{\mu,\lambda,\rho;\alpha,\gamma;z,\frac{-xt}{1-t};y\right\}. \tag{3.4.6}$$

Proof. If t is replaced by $(1-x)t$ in (3.4.1) and then multiply on both sides with $x^{\rho-1}$, it gives

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,[\lambda,\alpha;y];z)\,x^{\rho-1}(1-x)^n t^n\\ =\;&x^{\rho-1}[1-(1-x)t]^{-\mu}\,{}_2F_1\left(\mu,[\lambda,\alpha;y];\frac{z}{1-(1-x)t}\right).\end{aligned}$$

Applying the fractional derivative $C_x^{\rho-\gamma}$ on both sides and changing the order show

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,[\lambda,\alpha;y];z)C_x^{\rho-\gamma}\left[x^{\rho-1}(1-x)^n;y\right]\,t^n\\ =\;&C_x^{\rho-\gamma}\left[x^{\rho-1}[1-(1-x)t]^{-\mu}\,{}_2F_1\left(\mu,[\lambda,\alpha;y];\frac{z}{1-(1-x)t}\right);y\right]\end{aligned}$$

where $|z|<1$, $\left|\frac{1-x}{1-z}t\right|<1$ and $\left|\frac{z}{1-t}\right|+\left|\frac{xt}{1-t}\right|<1$. If the equality was given as

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,{}_2F_1(\mu+n,[\lambda,\alpha;y];z)C_x^{\rho-\gamma}\left[x^{\rho-1}(1-x)^n;y\right]\,t^n\\ =\;&(1-t)^{-\mu}\,C_x^{\rho-\gamma}\left[x^{\rho-1}\left[1-\frac{-xt}{1-t}\right]^{-\mu}\,{}_2F_1\left(\mu,[\lambda,\alpha;y];\frac{z}{1-\frac{-xt}{1-t}}\right);y\right]\end{aligned}$$

and using Theorem 3.3.3 and Theorem 3.3.6, delivers the expected result. Formula (3.4.6) can be proved in a similar way. ■

Chapter 4

EXTENSION OF INCOMPLETE GAMMA, BETA AND HYPERGEOMETRIC FUNCTIONS

4.1 Introduction

Recently, some generalizations of the generalized gamma, beta, Gauss hypergeometric and confluent hypergeometric functions have been introduced in [42]. Several properties for these functions are investigated such as integral representations, Mellin transforms, transformation formulas and differentiation formulas. Structure of this chapter is as follows:

Generalizations of incomplete gamma and Euler's function are elaborated in Section 4.2. Later, different integral representations and some properties of generalized incomplete Euler's beta function can be obtained. Furthermore, relations of generalized incomplete gamma and beta functions are also examined in this study. Section 4.3 introduces the generalized incomplete Gauss, confluent and Appell's hypergeometric functions. The study proceeds by obtaining some integral representations of these functions. Mellin's transform representation of the generalized incomplete Gauss hypergeometric and confluent hypergeometric functions are also investigated. Differentiation and transformation formulas of the above mentioned functions are presented. In addition, fractional calculus formula for generalized incomplete Gauss hypergeometric function can be seen given in terms of the generalized incomplete Appell's hypergeometric function.

4.2 Generalizations of incomplete gamma and Euler's beta function

This part of the study duells on the following generalizations of incomplete gamma and Euler's beta functions [38]

$$\Gamma_p^{(\alpha,\beta;y)}(x) \quad : \quad = \int_0^\infty t^{x-1}\ {}_1F_1\left([\alpha,\beta;y];-t-\frac{p}{t}\right)dt, \tag{4.2.1}$$
$$(Re(\alpha) \quad > \quad 0,\ Re(\beta)>0,\ Re(p)>0,\ Re(x)>0,\ 0\leq y<1)$$

and

$$B_p^{(\alpha,\beta;y)}(x,z) \quad : \quad = \int_0^1 t^{x-1}(1-t)^{z-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt, \tag{4.2.2}$$
$$(Re(\alpha) \quad > \quad 0,\ Re(\beta)>0,\ Re(p)>0,\ Re(x)>0,\ Re(z)>0,\ 0\leq y<1)$$

respectively.

In the case $p=0$, (4.2.1) reduces to

$$\Gamma^{(\alpha,\beta;y)}(x) := \int_0^\infty t^{x-1}\ {}_1F_1\left([\alpha,\beta;y];-t\right)dt. \tag{4.2.3}$$

The study continuous by obtaining the integral representations of the functions given in the above definition.

Theorem 4.2.1 *For the generalized incomplete gamma function, we have*

$$\Gamma_p^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)}\int_0^1 \Gamma_{\mu^2y^2p}(s)\mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1}d\mu,$$

where $\Gamma_p(s)$ *is given in (1.0.1).*

Proof. Using the integral representation of incomplete confluent hypergeometric function, we have

$$\Gamma_p^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha}}{B(\alpha,\beta-\alpha)} \int_0^{\infty}\int_0^1 t^{s-1} e^{-uyt-\frac{uyp}{t}} u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du dt.$$

Using a one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu = uyt,\ \mu = u$ in the above equality and considering that the Jacobian of the transformation is $J = \frac{1}{\mu y}$, we get

$$\Gamma_p^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)} \int_0^{\infty}\int_0^1 \nu^{s-1} e^{-\nu-\frac{\mu^2 y^2 p}{\nu}} d\nu \mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1} d\mu.$$

Focusing on the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$\begin{aligned}\Gamma_p^{(\alpha,\beta;y)}(s) &= \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)} \int_0^1 \left[\int_0^{\infty} \nu^{s-1} e^{-\nu-\frac{\mu^2 y^2 p}{\nu}} d\nu\right] \mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1} d\mu \\ &= \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)} \int_0^1 \Gamma_{\mu^2 y^2 p}(s) \mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1} d\mu.\end{aligned}$$

Whence the result. ■

Remark 4.2.2 *In the case* $p = 0$, *we have*

$$\Gamma^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)} \int_0^1 \Gamma(s) \mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1} d\mu = \frac{y^{\alpha-s}\Gamma(s) B_y(\alpha-s,\beta-\alpha)}{B(\alpha,\beta-\alpha)}. \tag{4.2.4}$$

This part of the study concentrates on, give the integral representation of $B_p^{(\alpha,\beta;y)}(x,z)$ by means of Chaudhry's extended beta function.

Theorem 4.2.3 *([38]) For the generalized incomplete Euler's beta function, we have*

$$B_p^{(\alpha,\beta;y)}(x,z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)} \int_0^1 B_{uyp}(x,z) u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du,$$

where $B_p(x,z)$ is given in (1.0.6).

Proof. Using the integral representation of incomplete confluent hypergeometric function, we have

$$B_p^{(\alpha,\beta;y)}(x,z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)} \int_0^1 \int_0^1 t^{x-1}(1-t)^{z-1} \exp\left(\frac{-uyp}{t(1-t)}\right) u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du dt.$$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$B_p^{(\alpha,\beta;y)}(x,z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)} \int_0^1 \left[\int_0^1 t^{x-1}(1-t)^{z-1} \exp\left(\frac{-uyp}{t(1-t)}\right) dt\right] u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du.$$

In view of (1.0.6), we get

$$B_p^{(\alpha,\beta;y)}(x,z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)} \int_0^1 B_{uyp}(x,z) u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du.$$

Hence the proof is completed. ■

Remark 4.2.4 *([38]) In the case $p = 0$ in the above Theorem gives*

$$B^{(\alpha,\beta;y)}(x,z) = \frac{y^{\alpha} B(x,z) B_y(\alpha, \beta-\alpha)}{B(\alpha, \beta-\alpha)}.$$

In the following theorem, we give the Mellin transform representation of the function $B_p^{(\alpha,\beta;y)}(x,z)$ in terms of the ordinary beta function and $\Gamma^{(\alpha,\beta;y)}(s)$.

Theorem 4.2.5 *([38]) Mellin transform representation of the generalized incomplete beta function is given by*

$$\int_0^{\infty} p^{s-1} B_p^{(\alpha,\beta;y)}(x,z)dp = B(x+s, z+s)\Gamma^{(\alpha,\beta;y)}(s), \tag{4.2.5}$$

where $Re(s) > 0$, $Re(x+s) > 0$, $Re(z+s) > 0$, $Re(p) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $0 \leq y < 1$.

Proof. Multiplying (4.2.2) by p^{s-1} and integrating with respect to p from $p = 0$ to $p = \infty$, we get

$$\int_0^{\infty} p^{s-1} B_p^{(\alpha,\beta;y)}(x,z)dp = \int_0^{\infty} p^{s-1} \int_0^1 t^{x-1}(1-t)^{z-1} \, {}_1F_1\left([\alpha,\beta;y]; \frac{-p}{t(1-t)}\right) dtdp. \tag{4.2.6}$$

From the uniform convergence of the integral, the order of integration in (4.2.6) can be interchanged. Therefore, we have

$$\int_0^{\infty} p^{s-1} B_p^{(\alpha,\beta;y)}(x,z)dp = \int_0^1 t^{x-1}(1-t)^{z-1} \int_0^{\infty} p^{s-1} \, {}_1F_1\left([\alpha,\beta;y]; \frac{-p}{t(1-t)}\right) dpdt. \tag{4.2.7}$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu=\frac{p}{t(1-t)},\ \mu=t$ in (4.2.7), we get

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=\int_0^1\mu^{(x+s)-1}(1-\mu)^{(z+s)-1}d\mu\int_0^\infty \nu^{s-1}\,{}_1F_1\left([\alpha,\beta;y];-\nu\right)d\nu.$$

Therefore, using (4.2.3), we have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=B(x+s,z+s)\Gamma^{(\alpha,\beta;y)}(s).$$

Whence the result. ■

Remark 4.2.6 *([38]) Putting $s=1$ and considering that $\Gamma^{(\alpha,\beta;y)}(1)=\frac{y^{\alpha-1}B_y(\alpha-1,\beta-\alpha)}{B(\alpha,\beta-\alpha)}$ in (4.2.5), we get*

$$\int_0^\infty B_p^{(\alpha,\beta;y)}(x,z)dp=B(x+1,z+1)\frac{y^{\alpha-1}B_y(\alpha-1,\beta-\alpha)}{B(\alpha,\beta-\alpha)}.$$

The following theorem can be proved in a similar manner.

Theorem 4.2.7 *([38]) Mellin transform representation of the generalized incomplete gamma function is given by*

$$\mathfrak{M}\left\{\Gamma_p^{(\alpha,\beta;y)}(x):s\right\}=\frac{\Gamma(s)\Gamma(x+s)B_y(\alpha-2s-x,\beta-\alpha)}{B(\alpha,\beta-\alpha)}. \tag{4.2.8}$$

Remark 4.2.8 *Putting $s=1$ in (4.2.8), we have*

$$\int_0^\infty \Gamma_p^{(\alpha,\beta;y)}(x)\,dp=\frac{\Gamma(x+1)B_y(\alpha-x-2,\beta-\alpha)}{B(\alpha,\beta-\alpha)}.$$

Theorem 4.2.9 *([38]) For the generalized incomplete beta function, we have the following integral representations:*

$$B_p^{(\alpha,\beta;y)}(x,z) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2z-1}\theta \; {}_1F_1([\alpha,\beta;y];-p\sec^2\theta - p\csc^2\theta)d\theta, \tag{4.2.9}$$

and

$$B_p^{(\alpha,\beta;y)}(x,z) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+z}} \; {}_1F_1\left([\alpha,\beta;y];-2p-p\left(u+\frac{1}{u}\right)\right)du. \tag{4.2.10}$$

Proof. The proofs of (4.2.9) and (4.2.10) are obtained from (4.2.2) with the substitution $t = \cos^2\theta$ and $t = \frac{u}{1+u}$, respectively. ■

Theorem 4.2.10 *([38]) For the generalized incomplete beta function, we have the following functional relation:*

$$B_p^{(\alpha,\beta;y)}(x,z+1) + B_p^{(\alpha,\beta;y)}(x+1,z) = B_p^{(\alpha,\beta;y)}(x,z).$$

Proof. Direct calculations yield,

$$\begin{aligned} B_p^{(\alpha,\beta;y)}(x,z+1) + B_p^{(\alpha,\beta;y)}(x+1,z) &= \int_0^1 t^{x-1}(1-t)^z \; {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\ &\quad + \int_0^1 t^{x}(1-t)^{z-1} \; {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\ &= \int_0^1 \left[t^{x-1}(1-t)^z + t^x(1-t)^{z-1}\right] \; {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\ &= \int_0^1 t^{x-1}(1-t)^{z-1} \; {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\ &= B_p^{(\alpha,\beta;y)}(x,z). \end{aligned}$$

Whence the result. ■

Theorem 4.2.11 *([38]) For the product of two generalized incomplete gamma function, we have the following integral representation:*

$$\begin{aligned}\Gamma_p^{(\alpha,\beta;y)}(x)\Gamma_p^{(\alpha,\beta;y)}(z) &= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+z)-1}\cos^{2x-1}\theta\sin^{2z-1}\theta \\ &\times {}_1F_1\left([\alpha,\beta;y];-r\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right) \\ &\times {}_1F_1\left([\alpha,\beta;y];-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)drd\theta.\end{aligned}$$

Proof. Substituting $t=\eta^2$ in (4.2.1), we get

$$\Gamma_p^{(\alpha,\beta;y)}(x)=2\int_0^{\infty}\eta^{2x-1}\,{}_1F_1\left([\alpha,\beta;y];-\eta^2-\frac{p}{\eta^2}\right)d\eta.$$

Therefore

$$\begin{aligned}\Gamma_p^{(\alpha,\beta;y)}(x)\Gamma_p^{(\alpha,\beta;y)}(z) &= 4\int_0^{\infty}\int_0^{\infty}\eta^{2x-1}\xi^{2z-1}\,{}_1F_1\left([\alpha,\beta;y];-\eta^2-\frac{p}{\eta^2}\right) \\ &\times {}_1F_1\left([\alpha,\beta;y];-\xi^2-\frac{p}{\xi^2}\right)d\eta d\xi.\end{aligned}$$

Letting $\eta=r\cos\theta$ and $\xi=r\sin\theta$ in the above equality,

$$\begin{aligned}\Gamma_p^{(\alpha,\beta;y)}(x)\Gamma_p^{(\alpha,\beta;y)}(z) &= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+z)-1}\cos^{2x-1}\theta\sin^{2z-1}\theta \\ &\times {}_1F_1\left([\alpha,\beta;y];-r\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right) \\ &\times {}_1F_1\left([\alpha,\beta;y];-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)drd\theta.\end{aligned}$$

Hence the proof is completed. ■

Theorem 4.2.12 *([38]) For the generalized incomplete beta function, we have the following summation relation:*

$$B_p^{(\alpha,\beta;y)}(x,1-z)=\sum_{n=0}^{\infty}\frac{(z)^n}{n!}B_p^{(\alpha,\beta;y)}(x+n,1)\,, \quad Re(p)>0,\ 0\leq y<1.$$

Proof. From the definition of the generalized incomplete beta function, we get

$$B_p^{(\alpha,\beta;y)}(x,1-z)=\int_0^1 t^{x-1}(1-t)^{-z}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.$$

Using the following binomial series expansion

$$(1-t)^{-z}=\sum_{n=0}^{\infty}(z)_n\frac{t^n}{n!},\quad |t|<1,$$

we obtain

$$B_p^{(\alpha,\beta;y)}(x,1-z)=\int_0^1\sum_{n=0}^{\infty}\frac{(z)_n}{n!}t^{x+n-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.$$

Therefore, interchanging the order of integration and summation and then using (4.2.2), we obtain

$$\begin{aligned}B_p^{(\alpha,\beta;y)}(x,1-z) &= \sum_{n=0}^{\infty}\frac{(z)_n}{n!}\int_0^1 t^{x+n-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt,\\ &= \sum_{n=0}^{\infty}\frac{(z)_n}{n!}B_p^{(\alpha,\beta;y)}(x+n,1).\end{aligned}$$

Whence the result. ■

In addition, we introduce another extension of the generalized incomplete gamma and beta functions

$$\Gamma_p^{(\alpha,\beta)}[x;y] := \int_0^y t^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{4.2.11}$$

$$\Gamma_p^{(\alpha,\beta)}\{x;y\} := \int_y^\infty t^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{4.2.12}$$

$$B_p^{(\alpha,\beta)}[x,z;y] := \int_0^y t^{x-1}(1-t)^{z-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,\ 0\le y<1,\ Re(x)>0, Re(z)>0, \tag{4.2.13}$$

and

$$B_p^{(\alpha,\beta)}\{x,z;y\} := \int_0^{1-y} t^{z-1}(1-t)^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,\ 0\le y<1,\ Re(x)>0, Re(z)>0, \tag{4.2.14}$$

respectively.

In the following theorem , we give the integral representations of these functions.

Theorem 4.2.13 *([38]) The following integral representation holds true:*

$$\Gamma_p^{(\alpha,\beta)}[x;y] = y^x \int_0^1 u^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)du. \tag{4.2.15}$$

Remark 4.2.14 *([38]) Using the integral representation of* ${}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)$

$$ {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)=\frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left(\frac{-pt}{uy(1-uy)}\right)dt, $$

in (4.2.15), we have

$$ \Gamma_p^{(\alpha,\beta)}[x;y]=\frac{y^x}{B(\alpha,\beta-\alpha)}\int_0^1\int_0^1 u^{x-1}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left(\frac{-pt}{uy(1-uy)}\right)dudt. $$

Theorem 4.2.15 *([38]) The following integral representation holds true:*

$$ B_p^{(\alpha,\beta)}[x,z;y]=y^x\int_0^1 u^{x-1}(1-uy)^{z-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)du \tag{4.2.16} $$

and

$$ B_p^{(\alpha,\beta)}\{x,z;y\}=y^{z-1}\int_0^1 u^{z-1}(1-u(1-y))^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{u(1-y)(1-u(1-y))}\right)du. \tag{4.2.17} $$

Remark 4.2.16 *([38]) Using the integral representation of confluent hypergeometric function* ${}_1F_1$ *in (4.2.16) and (4.2.17), we get*

$$ B_p^{(\alpha,\beta)}[x,z;y]=\frac{y^x}{B(\alpha,\beta-\alpha)}\int_0^1\int_0^1 u^{x-1}(1-uy)^{z-1}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left(\frac{-pt}{uy(1-uy)}\right)dudt \tag{4.2.18} $$

and

$$B_p^{(\alpha,\beta)}\{x,z;y\} = \frac{y^{z-1}}{B(\alpha,\beta-\alpha)}\int_0^1\int_0^1 u^{z-1}(1-u(1-y))^{x-1}t^{\alpha-1}(1-t)^{\beta-\alpha-1} \times \exp\left(\frac{-pt}{u(1-y)(1-u(1-y))}\right)dudt,$$

respectively.

4.3 Generalized incomplete Gauss hypergeometric and confluent hypergeometric functions

This part of the chapter examines the generalization (4.2.2) of incomplete beta functions to generalize the incomplete Gauss hypergeometric and confluent hypergeometric functions by

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) := \sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \tag{4.3.1}$$

and

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z) := \sum_{n=0}^{\infty}\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{4.3.2}$$

respectively. Furthermore, using (4.2.13) we define another extension of the generalized incomplete Gauss and Appell's hypergeometric functions as follows:

$$F_p^{\alpha,\beta}(a,[b,c;y];z) := \sum_{n=0}^{\infty}(a)_n\frac{B_p^{\alpha,\beta}[b+n,c-b;y]}{B(b,c-b)}\frac{z^n}{n!}, \tag{4.3.3}$$

$$Re(c) > Re(b) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0,$$

and

$$F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p):=\sum_{n,m=0}^{\infty}(\rho)_{n+m}\frac{B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]}{B(\nu,\gamma-\nu)}\frac{B_p(\lambda+m,\mu-\lambda)}{B(\lambda,\mu-\lambda)}\frac{x^n}{n!}\frac{z^m}{m!},$$
$$Re(\gamma)>Re(\nu)>0,\ Re(\mu)>Re(\lambda)>0,\ Re(\alpha)>0,\ Re(\beta)>0. \tag{4.3.4}$$

4.3.1 Integral representations

Theorem 4.3.1 *([38]) For the following integral representation holds true for the generalized incomplete Gauss hypergeometric function:*

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{y^\alpha}{B(\alpha,\beta-\alpha)}\int_0^1 F_{uyp}(a,b;c;z)\,u^{\alpha-1}(1-uy)^{\beta-\alpha-1}\,du, \\ Re(\beta) &> Re(\alpha)>0,\ Re(c)>Re(b)>0,\ |\arg(1-u)|<1, \end{aligned}$$

where $F_p(a,b;c;z)$ is given in (1.0.7).

Proof. Since

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

we have from the Theorem 4.2.3

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{y^\alpha}{B(\alpha,\beta-\alpha)}\sum_{n=0}^{\infty}\frac{(a)_n}{B(b,c-b)}\int_0^1 B_{uyp}(b+n,c-b)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du\frac{z^n}{n!}.$$

Taking into consideration, the uniform convergence of the series involved and the absolute convergence of the integral, interchanging the order of series and the integral, we get

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)} \int_0^1 \left\{ \sum_{n=0}^{\infty} (a)_n \frac{B_{uyp}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \right\} u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du.$$

Whence the result. ■

Remark 4.3.2 *([38]) Using (1.0.7), we have the following integral representation*

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{y^\alpha}{B(\alpha,\beta-\alpha)B(b,c-b)} \\ &\times \int_0^1 \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left(\frac{-uyp}{t(1-t)}\right) \\ &\times (1-zt)^{-a} u^{\alpha-1}(1-uy)^{\beta-\alpha-1} dt du. \end{aligned}$$

In a similar manner, we are led fairly easily to the theorem below:

Theorem 4.3.3 *([38]) For the generalized incomplete confluent hypergeometric function, we have the following integral representation:*

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)} \int_0^1 \phi_{uyp}(b;c;z) u^{\alpha-1}(1-uy)^{\beta-\alpha-1} du.$$

Remark 4.3.4 *([38]) Using (1.0.8), we have the following integral representation*

$$\begin{aligned} {}_1F_1^{((\alpha,\beta;y);p)}(b;c;z) &= \frac{y^\alpha}{B(\alpha,\beta-\alpha)B(b,c-b)} \\ &\times \int_0^1 \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left(\frac{-uyp}{t(1-t)} + zt\right) \\ &\times u^{\alpha-1}(1-uy)^{\beta-\alpha-1} dt du. \end{aligned}$$

Furthermore, the generalized incomplete Gauss hypergeometric functions can be provided with an integral representation by using the definition of the generalized incomplete beta functions (4.2.2) and (4.2.13), respectively. We get

Theorem 4.3.5 *([38]) For the generalized incomplete Gauss hypergeometric function, we have the following integral representations:*

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \\ &\quad \times {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt, \qquad (4.3.5) \\ Re(p) &> 0;\ 0\le y<1;\ Re(c)>Re(b)>0; \end{aligned}$$

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{1}{B(b,c-b)}\int_0^\infty u^{b-1}(1+u)^{a-c}\left[1+u(1-z)\right]^{-a} \\ &\quad \times {}_1F_1\left([\alpha,\beta;y];-2p-p\left(u+\frac{1}{u}\right)\right)du, \end{aligned}$$

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}} \sin^{2b-1}\nu\cos^{2c-2b-1}\nu(1-z\sin^2\nu) \\ &\quad \times {}_1F_1\left([\alpha,\beta;y];\frac{-p}{\sin^2\nu\cos^2\nu}\right)d\nu. \end{aligned}$$

Proof. Direct calculations yield

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \\ &= \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)\frac{z^n}{n!}dt \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}dt \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt. \end{aligned}$$

Substituting $u=\frac{t}{1-t}$ in (4.3.5), we get

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{1}{B(b,c-b)}\int_0^{\infty} u^{b-1}(1+u)^{a-c}\left[1+u(1-z)\right]^{-a} \\ &\quad \times {}_1F_1\left([\alpha,\beta;y];-2p-p\left(u+\frac{1}{u}\right)\right)du, \end{aligned}$$

On the other hand, substituting $t=\sin^2 \nu$ in (4.3.5), we have

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}} \sin^{2b-1}\nu \cos^{2c-2b-1}\nu(1-z\sin^2\nu) \\ &\quad \times {}_1F_1\left([\alpha,\beta;y];\frac{-p}{\sin^2\nu\cos^2\nu}\right)d\nu. \end{aligned}$$

Hence the proof is completed. ■

Theorem 4.3.6 *([38]) For the generalized incomplete Gauss hypergeometric function, we have the following integral representation:*

$$\begin{aligned} F_p^{\alpha,\beta}(a,[b,c;y];z) &= \frac{y^b}{B(b,c-b)}\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-uyz)^{-a} \\ &\quad \times {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)du, \qquad (4.3.6) \\ Re(c) &> Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ |\arg(1-z)|<\pi. \end{aligned}$$

Proof. Replacing the generalized incomplete beta function $B_p^{\alpha,\beta}[b+n,c-b;y]$ by its integral representation given by (4.2.13) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis of the Theorem, we find

$$F_p^{\alpha,\beta}(a,[b,c;y];z)=\frac{1}{B(b,c-b)}\int_0^y t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,$$

which can be written as follows:

$$F_p^{\alpha,\beta}(a,[b,c;y];z)=\frac{y^b}{B(b,c-b)}\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-uyz)^{-a}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)du.$$

■

Remark 4.3.7 *([38]) Using (4.2.18), we have the following integral representation*

$$\begin{aligned}F_p^{\alpha,\beta}(a,[b,c;y];z) &= \frac{y^b}{B(b,c-b)}\int_0^1\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-uyz)^{-a}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\\ &\quad\times\exp\left(\frac{-pt}{uy(1-uy)}\right)dudt.\end{aligned}$$

A similar procedure yields an integral representation of the generalized incomplete confluent hypergeometric function by using the definition of the generalized incomplete beta function.

Theorem 4.3.8 *([38]) For the generalized incomplete confluent hypergeometric function, we have the following integral representations:*

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt,$$

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 (1-u)^{b-1}u^{c-b-1}e^{z(1-u)}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{u(1-u)}\right)du,$$

$p\geq 0$ and $Re(c)>Re(b)>0$.

Theorem 4.3.9 *([38]) For the generalized incomplete Appell's hypergeometric function, we have the following integral representation:*

$$\begin{aligned} F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) &= \frac{y^{\nu}}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\int_0^1\int_0^1 u^{\nu-1}(1-uy)^{\gamma-\nu-1} \\ &\times {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)s^{\lambda-1}(1-s)^{\mu-\lambda-1} \\ &\times (1-xuy-zs)^{-\rho}\,dsdu. \end{aligned} \tag{4.3.7}$$

Proof. Replacing the generalized incomplete beta and extended beta functions by their integral representations (4.2.13) and (1.0.6), we get

$$\begin{aligned} F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) &= \frac{1}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\sum_{n,m=0}^{\infty}(\rho)_{n+m}\frac{x^n}{n!}\frac{z^m}{m!} \\ &\times \int_0^y\int_0^1 t^{\nu+n-1}(1-t)^{\gamma-\nu-1} \\ &\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)s^{\lambda+m-1}(1-s)^{\mu-\lambda-1} \\ &\times \exp\left(\frac{-p}{s(1-s)}\right)dsdt, \end{aligned}$$

Considering the fact that the series involved are uniformly convergent and we have a right to interchange the order of summation and integration, we get

$$\begin{aligned} F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) &= \frac{1}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\int_0^y\int_0^1 t^{\nu-1}(1-t)^{\gamma-\nu-1} \\ &\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)s^{\lambda-1}(1-s)^{\mu-\lambda-1} \\ &\times \exp\left(\frac{-p}{s(1-s)}\right)(1-xt-zs)^{-\rho}\,dsdt \\ &= \frac{y^{\nu}}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\int_0^1\int_0^1 u^{\nu-1}(1-uy)^{\gamma-\nu-1} \\ &\times {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)s^{\lambda-1}(1-s)^{\mu-\lambda-1} \\ &\times (1-xuy-zs)^{-\rho}\,dsdu. \end{aligned}$$

Whence the result. ■

4.3.2 Differentiation and difference formulas

In this subsection, using the formulas $B(b,c-b)=\frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1}=a(a+1)_n,$ we gain formulas including derivatives of generalized incomplete Gauss hypergeometric and confluent hypergeometric function with respect to the variable z.

Theorem 4.3.10 *([38]) For generalized incomplete Gauss hypergeometric function, we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\}=\frac{(b)_n(a)_n}{(c)_n}F_p^{(\alpha,\beta;y)}(a+n,b+n;c+n;z).$$

Proof. Taking derivative of $F_p^{(\alpha,\beta;y)}(a,b;c;z)$ with respect to z, we obtain

$$\begin{aligned}\frac{d}{dz}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\} &= \frac{d}{dz}\left\{\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}\\ &= \sum_{n=1}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}.\end{aligned}$$

Replacing $n\to n+1,$ we get

$$\begin{aligned}\frac{d}{dz}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\} &= \frac{ba}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_p^{(\alpha,\beta;y)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!}\\ &= \frac{ba}{c}F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z). \qquad (4.3.8)\end{aligned}$$

Recursive application of this procedure gives us the general form:

$$\frac{d^n}{dz^n}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\}=\frac{(b)_n(a)_n}{(c)_n}F_p^{(\alpha,\beta;y)}(a+n,b+n;c+n;z).$$

Whence the result. ■

Theorem 4.3.11 *([38]) For the generalized incomplete confluent hypergeometric function, we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\left\{ {}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)\right\} = \frac{(b)_n}{(c)_n}\, {}_1F_1^{((\alpha,\beta;y);p)}(b+n;c+n;z).$$

Theorem 4.3.12 *([38]) For the generalized incomplete Gauss hypergeometric function, we have the following recurrence relation:*

$$\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z) = \frac{bz}{c} F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z).$$

Proof. From the integral representation (4.3.5) of the generalized incomplete Gauss hypergeometric function, we find that

$$\begin{aligned}\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z) &= F_p^{(\alpha,\beta;y)}(a+1,b;c;z) - F_p^{(\alpha,\beta;y)}(a,b;c;z)\\ &= \frac{z}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a-1}\\ &\quad \times {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt. \end{aligned} \tag{4.3.9}$$

Now, changing the arguments in the generalized incomplete Gauss hypergeometric function by raising each parameter by one, we also find from (4.3.5) that

$$\begin{aligned}F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z) &= \frac{1}{B(b+1,c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a-1}\\ &\quad \times {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt. \end{aligned} \tag{4.3.10}$$

Finally, by comparing (4.3.9) and (4.3.10), we get

$$\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z) = \frac{bz}{c} F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z), \tag{4.3.11}$$

which is our first recurrence relation for the generalized incomplete Gauss hypergeometric function. ■

Remark 4.3.13 *([38]) Further, by using the differentiation formula (4.3.8), we obtain the following differential difference equation:*

$$\frac{d}{dz}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\} = \frac{a}{z}\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z),$$

where, just as in (4.3.9), Δ_a is the shift operator with respect to a.

4.3.3 Mellin Transform Representation

In this subsection, we obtain the Mellin transform representations of the generalized incomplete Gauss hypergeometric and confluent hypergeometric function.

Theorem 4.3.14 *([38]) For the generalized incomplete Gauss hypergeometric function, we have the following Mellin transform representation:*

$$\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):s\right\} = \frac{\Gamma^{(\alpha,\beta;y)}(s)\,B(b+s,c-b+s)}{B(b,c-b)}\ {}_2F_1\left(a,b+s;c+2s;z\right).$$

Proof. To obtain the Mellin transform, we multiply both sides of (4.3.5) by p^{s-1} and integrate with respect to p over the interval $[0,\infty)$. Thus we get

$$
\begin{aligned}
\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):s\right\} &= \int_0^\infty p^{s-1}F_p^{(\alpha,\beta;y)}(a,b;c;z)\,dp \\
&= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \\
&\quad\times\left[\int_0^\infty p^{s-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dp\right]dt
\end{aligned}
\tag{4.3.12}
$$

Since substituting $u=\frac{p}{t(1-t)}$ in (4.3.12),

$$
\begin{aligned}
\int_0^\infty p^{s-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dp &= \int_0^\infty u^{s-1}t^s(1-t)^s\,{}_1F_1([\alpha,\beta;y];-u)\,du \\
&= t^s(1-t)^s\int_0^\infty u^{s-1}\,{}_1F_1([\alpha,\beta;y];-u)\,du \\
&= t^s(1-t)^s\Gamma^{(\alpha,\beta;y)}(s).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):s\right\} &= \frac{1}{B(b,c-b)}\int_0^1 t^{b+s-1}(1-t)^{c-b+s-1}(1-zt)^{-a}\Gamma^{(\alpha,\beta;y)}(s)dt \\
&= \frac{\Gamma^{(\alpha,\beta;y)}(s)B(b+s,c-b+s)}{B(b,c-b)} \\
&\quad\times\frac{1}{B(b+s,c-b+s)}\int_0^1 t^{b+s-1}(1-t)^{c+2s-(b+s)-1}(1-zt)^{-a}\,dt \\
&= \frac{\Gamma^{(\alpha,\beta;y)}(s)B(b+s,c-b+s)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z).
\end{aligned}
$$

■

Theorem 4.3.15 *([38]) For the generalized incomplete confluent hypergeometric function, we have the following Mellin transform representation:*

$$
\mathfrak{M}\left\{{}_1F_1^{((\alpha,\beta;y);p)}(b;c;z):s\right\} = \frac{\Gamma^{(\alpha,\beta;y)}(s)B(b+s,c-b+s)}{B(b,c-b)}\,{}_1F_1(b+s;c+2s;z).
$$

4.3.4 Transformation formulas

Theorem 4.3.16 *([38]) For the generalized incomplete Gauss hypergeometric function, we have the following transformation formula:*

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = (1-z)^{-a} F_p^{(\alpha,\beta;y)}\left(a, c-b; c; \frac{z}{z-1}\right), |\arg(1-z)| < \pi.$$

Proof. By writing

$$[1-z(1-t)]^{-a} = (1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a}$$

and replacing $t \to 1-t$ in (4.3.5), we obtain

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 (1-t)^{b-1}t^{c-b-1}\left(1-\frac{z}{z-1}t\right)^{-a} {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.$$

Hence,

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = (1-z)^{-a} F_p^{(\alpha,\beta;y)}\left(a, c-b; c; \frac{z}{z-1}\right).$$

■

Remark 4.3.17 *([38]) Note that,replacing z by $1-\frac{1}{z}$ in Theorem 4.3.16, one easily obtains the following transformation formula*

$$F_p^{(\alpha,\beta;y)}\left(a,b;c;1-\frac{1}{z}\right) = z^a F_p^{(\alpha,\beta;y)}(a, c-b; c; 1-z), \quad |\arg(z)| < \pi.$$

Furthermore, replacing z by $\frac{z}{1+z}$ in Theorem 4.3.16, we get the following transforma-

tion formula

$$F_p^{(\alpha,\beta;y)}\left(a,b;c;\frac{z}{1+z}\right)=(1+z)^a F_p^{(\alpha,\beta;y)}(a,c-b;c;-z), \quad |\arg(1+z)<\pi|.$$

Theorem 4.3.18 *([38]) For the generalized incomplete confluent hypergeometric function, we have the following transformation formula:*

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=e^z\,{}_1F_1^{((\alpha,\beta;y);p)}(c-b;c;z).$$

Remark 4.3.19 *([38]) Setting $z=1$ in (4.3.5), we have the following relation between generalized incomplete Gauss hypergeometric and beta functions:*

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;1) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\ &= \frac{B_p^{(\alpha,\beta;y)}(b,c-a-b)}{B(b,c-b)}. \end{aligned}$$

4.3.5 Fractional calculus formulas

This section identifies extended Riemann-Liouville fractional derivative of the generalized incomplete Gauss hypergeometric function shows the generalization of the incomplete Appell's hypergeometric function. The extended Riemann-Liouville fractional derivative operator is defined by [34]

$$D_z^{\mu,p}\{f(z)\}=\frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt,\ Re(\mu)<0,\ Re(p)>0.$$

It is well known [34] that

$$D_z^{\mu,p}\left\{z^{\lambda}\right\}=\frac{B_p(\lambda+1,-\mu)}{\Gamma(-\mu)}z^{\lambda-\mu},\ Re(\lambda)>-1,\ Re(\mu)<0. \tag{4.3.13}$$

Theorem 4.3.20 *([38]) For $Re(\mu)>Re(\lambda)>0,\ Re(\rho)>0,\ Re(\nu)>0,\ Re(\gamma)>0;\ \left|\frac{x}{1-z}\right|<1$ and $|x|+|z|<1,$ we have*

$$D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\rho}F_p^{\alpha,\beta}\left(\rho,[\nu,\gamma;y],\frac{x}{1-z}\right)\right\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p).$$

Proof. Using (4.3.13) and (4.3.7), we get

$$\begin{aligned}
& D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\rho}F_p^{\alpha,\beta}\left(\rho,[\nu,\gamma;y],\frac{x}{1-z}\right)\right\} \\
= \;& D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\rho}\sum_{n=0}^{\infty}(\rho)_n\frac{B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]}{B(\nu,\gamma-\nu)}\frac{\left(\frac{x}{1-z}\right)^n}{n!}\right\} \\
= \;& \frac{1}{B(\nu,\gamma-\nu)}D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\rho)_nB_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]\frac{x^n}{n!}(1-z)^{-\rho-n}\right\} \\
= \;& \frac{1}{B(\nu,\gamma-\nu)}\sum_{n,m=0}^{\infty}B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]\frac{x^n}{n!}\frac{(\rho)_n(\rho+n)_m}{m!}D_z^{\lambda-\mu,p}\{z^{\lambda-1+m}\} \\
= \;& \frac{1}{B(\nu,\gamma-\nu)}\sum_{n,m=0}^{\infty}B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]\frac{x^n}{n!}\frac{(\rho)_{n+m}}{m!}\frac{B_p(\lambda+m,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu+m-1} \\
= \;& \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p).
\end{aligned}$$

Hence the proof is completed. ■

Chapter 5

EXTENDED INCOMPLETE MITTAG-LEFFLER FUNCTIONS

5.1 Introduction

Recently, an extended Mittag-Leffler function has been introduced by the authors [35]. Some properties of the extended Mittag-Leffler function such as integral representation, Mellin transform and the extended fractional derivative of the usual Mittag-Leffler function were given in [35]. Organization of this chapter is as follows:

In Section 5.2, incomplete and extended incomplete Mittag-Leffler functions are defined and the integral representation of incomplete Mittag-Leffler functions in terms of the Mittag-Leffler function is given. Furthermore, we introduce the incomplete Wright hypergeometric functions and integral representations for these functions are given. In Section 5.3, the integral representations of extended incomplete Mittag-Leffler functions are obtained in terms of both Prabhakar's Mittag-Leffler function and known elementary functions. The Mellin transform of the extended incomplete Mittag-Leffler functions are given in terms of the incomplete Wright hypergeometric functions. Also, some derivative formulas for this function are provided. In Section 5.4, the images of the extended incomplete Mittag-Leffler function under the actions of Riemann-Liouville fractional integral and derivative operator are achieved. The relationship between the extended incomplete Mittag-Leffler function and simple Laguerre polynomials is gained in the last section.

5.2 Extended incomplete Mittag-Leffler functions

The function $E_{\alpha}(z)$,

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}, \tag{5.2.1}$$

was defined and studied by Mittag-Leffler in [31], [32]. It is a direct generalization of the exponential series, since, for $\alpha=1$, we have the exponential function. The function defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)} \tag{5.2.2}$$

gives a generalization of equation (5.2.1) in [60], [61]. Afterward, Prabhakar [45] introduced the generalized Mittag-Leffler function by

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}}{\Gamma(\alpha n+\beta)}\frac{z^{n}}{n!} \tag{5.2.3}$$

where $\alpha,\ \beta,\ \gamma\in\mathbb{C}$ with $Re(\alpha)>0$. For $\gamma=1$, it reduces to the Mittag-Leffler function given in equation (5.2.2). An extended Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,c}(z;p)$ [35], is introduced as follows:

$$E_{\alpha,\beta}^{\gamma;c}(z;p)=\sum_{n=0}^{\infty}\frac{B_{p}(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{n}}{\Gamma(\alpha n+\beta)}\frac{z^{n}}{n!},\ \ p\geq 0,\ Re(c)>Re(\gamma)>0,$$

where $B_{p}(x;y)$ is an extension of beta function defined in (1.0.6). In this section, the extended incomplete Mittag-Leffler functions are introduced as follows:

$$\begin{aligned} E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) &= \sum_{n=0}^{\infty}\frac{B_{y}(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_{n}}{\Gamma(\alpha n+\beta)}\frac{x^{n}}{n!}, \\ p &\geq 0,\ 0\leq y<1,\ Re(c)>Re(\gamma)>0 \end{aligned} \tag{5.2.4}$$

and

$$E_{\alpha,\beta}^{\{\gamma,c;y\}}(x;p) = \sum_{n=0}^{\infty} \frac{B_{1-y}(c-\gamma,\gamma+n;p)}{B(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!}, \qquad (5.2.5)$$
$$p \geq 0,\ 0 \leq y < 1,\ Re(c) > Re(\gamma) > 0$$

where for $B_y(x,z;p)$ we have

$$B_y(x,z;p) = \int_0^y t^{x-1}(1-t)^{z-1} \exp\left(\frac{-p}{t(1-t)}\right) dt,\ Re(p) > 0,\ Re(x) > 0,\ Re(z) > 0, \qquad (5.2.6)$$

the extended incomplete beta function defined in [10].

Corollary 5.2.1 *([40]) Note that, when $p = 0$ in equation (5.2.4) and (5.2.5) we introduce the incomplete Mittag-Leffler functions*

$$E_{\alpha,\beta}^{\gamma;c}[x;y] = \sum_{n=0}^{\infty} \frac{B_y(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!}, \qquad (5.2.7)$$
$$0 \leq y < 1,\ Re(c) > Re(\gamma) > 0$$

and

$$E_{\alpha,\beta}^{\gamma;c}\{x;y\} = \sum_{n=0}^{\infty} \frac{B_{1-y}(c-\gamma,\gamma+n)}{B(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!}, \qquad (5.2.8)$$
$$0 \leq y < 1,\ Re(c) > Re(\gamma) > 0$$

respectively.

Theorem 5.2.2 *([40]) For the incomplete Mittag-Leffler functions, we have the fol-*

lowing integral representations:

$$E_{\alpha,\beta}^{\gamma;c}[x;y]=\frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_{0}^{1}u^{\gamma-1}(1-uy)^{c-\gamma-1}E_{\alpha,\beta}^{c}(xuy)\,du, \tag{5.2.9}$$

and

$$E_{\alpha,\beta}^{\gamma;c}\{x;y\}=\frac{(1-y)^{c-\gamma}}{B(\gamma,c-\gamma)}\int_{0}^{1}u^{c-\gamma-1}(1-u(1-y))^{\gamma-1}E_{\alpha,\beta}^{c}(x(1-u(1-y)))\,du \tag{5.2.10}$$

where $Re(c)>Re(\gamma)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ 0\leq y<1.$

Proof. Replacing the incomplete beta function $B_y(\gamma+n,c-\gamma)$ in the definition (5.2.7) by its integral representation given by (2.2.1), we have that

$$\begin{aligned}E_{\alpha,\beta}^{\gamma;c}[x;y] &= \frac{1}{B(\gamma,c-\gamma)}\int_{0}^{y}t^{\gamma-1}(1-t)^{c-\gamma-1} \\ &\quad\times\sum_{n=0}^{\infty}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{(xt)^n}{n!}dt, \\ &= \frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_{0}^{1}u^{\gamma-1}(1-uy)^{c-\gamma-1}E_{\alpha,\beta}^{c}(xuy)\,du.\end{aligned}$$

Whence the result. Formula (5.2.10) can be proved in a similar manner. ■

On the other hand, the Wright generalized hypergeometric function ${}_p\psi_q(z)$ [52], called also Fox-Wright function is defined as

$${}_p\psi_q\left[\begin{array}{c}(\alpha_1,\beta_1),\ldots(\alpha_p,\beta_p)\\ \\ (\rho_1,\mu_1),\ldots(\rho_q,\mu_q)\end{array},z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+\beta_1 n)\ldots\Gamma(\alpha_p+\beta_p n)}{\Gamma(\rho_1+\mu_1 n)\ldots\Gamma(\rho_q+\mu_q n)}\frac{z^n}{n!},$$

where the coefficients $\beta_i\ (i=1,\ldots,p)$ and $\mu_j\ (j=1,\ldots,q)$ are positive real numbers such that

$$1+\sum_{j=1}^{q}\mu_j-\sum_{i=1}^{p}\beta_i\geq 0.$$

Introducing the incomplete Wright hypergeometric functions are as follows. Choosing $p=q=2,\ \beta_1=1,\ \beta_2=1$ and $\mu_1=1,$ since

$$\begin{aligned}{}_2\psi_2\left[\begin{array}{c}(\alpha_1,1),(\alpha_2,1)\\ \\(\rho_1,1),(\rho_2,\mu_2)\end{array};z\right] &= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+n)}{\Gamma(\rho_1+n)\Gamma(\rho_2+\mu_2 n)}\frac{z^n}{n!}\\ &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n}{(\rho_1)_n\Gamma(\rho_2+\mu_2 n)}\frac{z^n}{n!},\end{aligned}$$

using the fact that

$$\frac{(\alpha_1)_n}{(\rho_1)_n}=\frac{B(\alpha_1+n,\rho_1-\alpha_1)}{B(\alpha_1,\rho_1-\alpha_1)},$$

then using the following relation

$$B_y(\alpha_1+n,\rho_1-\alpha_1)+B_{1-y}(\rho_1-\alpha_1,\alpha_1+n)=B(\alpha_1+n,\rho_1-\alpha_1),\ 0\leq y<1 \tag{5.2.11}$$

and we get

$${}_2\psi_2^*[\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z]:=\sum_{n=0}^{\infty}[\alpha_1,\rho_1;y]_n\frac{(\alpha_2)_n}{\Gamma(\rho_2+\mu_2 n)}\frac{z^n}{n!},\ 0\leq y<1 \tag{5.2.12}$$

and

$${}_2\psi_2^*\{\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z\}:=\sum_{n=0}^{\infty}\{\alpha_1,\rho_1;y\}_n\frac{(\alpha_2)_n}{\Gamma(\rho_2+\mu_2 n)}\frac{z^n}{n!},\ \ 0\leq y<1. \tag{5.2.13}$$

An immediate consequence of (5.2.11) and the definitions (5.2.12) and (5.2.13) satisfy the following decomposition formula

$$\begin{aligned} & {}_2\psi_2^*\left[\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z\right] + {}_2\psi_2^*\left\{\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z\right\} \\ = & \frac{\Gamma(\rho_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\,{}_2\psi_2\left[\begin{array}{c}(\alpha_1,1),(\alpha_2,1) \\ \\ (\rho_1,1),(\rho_2,\mu_2)\end{array};z\right]. \end{aligned}$$

Theorem 5.2.3 *([40]) For the incomplete Wright hypergeometric functions, we have the following integral representations:*

$$ {}_2\psi_2^*\left[\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z\right] = \frac{y^{\alpha_1}}{B(\alpha_1,\rho_1-\alpha_1)}\int_0^1 u^{\alpha_1}(1-uy)^{\rho_1-\alpha_1-1}E_{\rho_2,\mu_2}^{\alpha_2}(zuy)\,du, \tag{5.2.14}$$

and

$$\begin{aligned} {}_2\psi_2^*\left\{\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z\right\} &= \frac{(1-y)^{\rho_1-\alpha_1}}{B(\alpha_1,\rho_1-\alpha_1)}\int_0^1 u^{\rho_1-\alpha_1-1}(1-u(1-y))^{\alpha_1-1} \\ & \times E_{\rho_2,\mu_2}^{\alpha_2}(z(1-u(1-y)))\,du \end{aligned} \tag{5.2.15}$$

where $Re(\rho_1) > Re(\alpha_1) > 0,\ 0 \le y < 1.$

Proof. Replacing the incomplete Pochhammer ratio $[\alpha_1,\rho_1;y]_n$ in the definition (2.2.4) by its integral representation given by (2.2.1) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis, we find

$$ {}_2\psi_2^*\left[\alpha_1,\alpha_2;\rho_1,(\rho_2,\mu_2);z\right] = \frac{1}{B(\alpha_1,\rho_1-\alpha_1)}\int_0^y t^{\alpha_1-1}(1-t)^{\rho_1-\alpha_1-1}E_{\rho_2,\mu_2}^{\alpha_2}(zt)\,dt,$$

which can be shown as

$$_{2}\psi_{2}^{*}\left[\alpha_{1},\alpha_{2};\rho_{1},(\rho_{2},\mu_{2});z\right]=\frac{y^{\alpha_{1}}}{B(\alpha_{1},\rho_{1}-\alpha_{1})}\int_{0}^{1}u^{\alpha_{1}}(1-uy)^{\rho_{1}-\alpha_{1}-1}E_{\rho_{2},\mu_{2}}^{\alpha_{2}}(zuy)\,du.$$

Whence the result. Formula (5.2.15) can be proved in a similar way. ■

5.3 Some properties of the extended incomplete Mittag-Leffler functions

To begin with, the following theorem is used, which gives the integral representations of the extended incomplete incomplete Mittag-Leffler functions.

Theorem 5.3.1 *([40]) For the extended incomplete Mittag-Leffler functions, we have the following integral representations:*

$$E_{\alpha,\beta}^{[\gamma,c;y]}(x;p)=\frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_{0}^{1}u^{\gamma-1}(1-uy)^{c-\gamma-1}\exp\left(\frac{-p}{uy(1-uy)}\right)E_{\alpha,\beta}^{c}(xuy)du, \tag{5.3.1}$$

and

$$\begin{aligned}E_{\alpha,\beta}^{\{\gamma,c;y\}}(x;p) &= \frac{(1-y)^{c-\gamma}}{B(\gamma,c-\gamma)}\int_{0}^{1}u^{c-\gamma-1}(1-u(1-y))^{\gamma-1}\exp\left(\frac{-p}{u(1-y)(1-u(1-y))}\right)\\ &\quad\times E_{\alpha,\beta}^{c}(x(1-u(1-y)))du, \end{aligned}\tag{5.3.2}$$

where $0\leq y<1,\ Re(p)>0,\ Re(c)>Re(\gamma)>0.$

Proof. Using equation (5.2.6) in equation (5.2.4), we get

$$E_{\alpha,\beta}^{[\gamma,c;y]}(x;p)=\sum_{n=0}^{\infty}\left\{\int_{0}^{y}t^{\gamma+n-1}(1-t)^{c-\gamma-1}\exp\left(\frac{-p}{t(1-t)}\right)dt\right\}\frac{(c)_{n}}{B(\gamma,c-\gamma)}\frac{x^{n}}{\Gamma(\alpha n+\beta)n!}. \tag{5.3.3}$$

Interchanging the order of summation and integration in (5.3.3), which is guaranteed under the assumptions given in the statement of the theorem, we get

$$
\begin{aligned}
& E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) \quad (5.3.4)\\
= \; & \frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_0^1 u^{\gamma-1}(1-uy)^{c-\gamma-1}\exp\left(\frac{-p}{uy(1-uy)}\right)\sum_{n=0}^{\infty}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{(xuy)^n}{n!}du.
\end{aligned}
$$

Using equation (5.2.3) in equation (5.3.4), we get the desired result. Formula (5.3.2) can be proved in a similar way. ■

Corollary 5.3.2 *([40]) Note that, taking* $u=\frac{t}{1+t}$ *in (5.3.1), gives*

$$
\begin{aligned}
E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) \; = \; & \frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_0^{\infty}\frac{t^{\gamma-1}}{(1+t)^c}(1+t(1-y))^{c-\gamma-1}\exp\left(\frac{-p(1+t)^2}{ty(1+t(1-y))}\right)\\
& \times E_{\alpha,\beta}^{c}\left(\frac{xty}{1+t}\right)dt.
\end{aligned}
$$

Using the definition of Prabhakar's Mittag-Leffler's function, Bayram and Kurulay obtained the recurrence formula [28]

$$
E_{\alpha,\beta}^{c}(tz)=\beta E_{\alpha,\beta+1}^{c}(tz)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{c}(tz).
$$

Inserting the above recurrence relation into equations (5.3.1) and (5.3.2), the following recurrence relations for the extended incomplete Mittag-Leffler's functions are achieved.

Theorem 5.3.3 *([40]) The following recurrence relations hold true for the extended incomplete Mittag-Leffler functions:*

$$
E_{\alpha,\beta}^{[\gamma,c;y]}(x;p)=\beta E_{\alpha,\beta+1}^{[\gamma,c;y]}(x;p)+\alpha x\frac{d}{dx}E_{\alpha,\beta+1}^{[\gamma,c;y]}(x;p)
$$

and

$$E_{\alpha,\beta}^{\{\gamma,c;y\}}(x;p) = \beta E_{\alpha,\beta+1}^{\{\gamma,c;y\}}(x;p) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}^{\{\gamma,c;y\}}(x;p)$$

In the next theorem, we give the Mellin transform of the extended incomplete Mittag-Leffler function in terms of the incomplete Wright generalized hypergeometric function.

Theorem 5.3.4 *([40]) The Mellin transform of the extended incomplete Mittag-Leffler functions are given by*

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{\Gamma(s)B(\gamma+s,c-\gamma+s)}{B(\gamma,c-\gamma)}\,{}_2\psi_2^*\left[\gamma+s,c;c+2s,(\alpha,\beta);x\right], \quad (5.3.5)$$

and

$$M\{E_{\alpha,\beta}^{\{\gamma,c;y\}}(x;p);s\} = \frac{\Gamma(s)B(\gamma+s,c-\gamma+s)}{B(\gamma,c-\gamma)}\,{}_2\psi_2^*\{\gamma+s,c;c+2s,(\alpha,\beta);x\}, \quad (5.3.6)$$

where $p \geq 0,\ Re(c) > Re(\gamma) > 0,\ Re(s) > 0,\ Re(\alpha) > 0$ *and* $Re(\beta) > 0.$

Proof. Taking the Mellin transform of the extended incomplete Mittag-Leffler function, we have

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \int_0^\infty p^{s-1} E_{\alpha,\beta}^{\gamma,c;y}(x;p)dp. \quad (5.3.7)$$

Using equation (5.3.1) in equation (5.3.7), we get

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_0^{\infty} p^{s-1} \times \left[\int_0^1 u^{\gamma-1}(1-uy)^{c-\gamma-1}\exp\left(\frac{-p}{uy(1-uy)}\right)\right] E_{\alpha,\beta}^{c}(xuy)\,du\,dp. \quad (5.3.8)$$

Interchanging the order of integrals in equation (5.3.8), which is valid because of the conditions in the statement of the Theorem, we get

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_0^1 \left[u^{\gamma-1}(1-uy)^{c-\gamma-1}E_{\alpha,\beta}^{c}(xuy)\right] \times \int_0^{\infty} p^{s-1}\exp\left(\frac{-p}{uy(1-uy)}\right)dp\,du. \quad (5.3.9)$$

Now taking $t = \frac{p}{uy(1-uy)}$ in equation (5.3.9) and using the fact that $\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt$, we get

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{y^{\gamma+s}\Gamma(s)}{B(\gamma,c-\gamma)}\int_0^1 u^{\gamma+s-1}(1-uy)^{c-\gamma+s-1}E_{\alpha,\beta}^{c}(xuy)\,du. \quad (5.3.10)$$

Using the definition of Prabhakar's generalized Mittag-Leffler function $E_{\alpha,\beta}^{c}(xuy)$ in equation (5.3.10), we get

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{y^{\gamma+s}\Gamma(s)}{B(\gamma,c-\gamma)}\int_0^1 u^{\gamma+s-1}(1-uy)^{c-\gamma+s-1}\sum_{n=0}^{\infty}\frac{(c)_n(xuy)^n}{\Gamma(\alpha n+\beta)n!}du.$$

Interchanging the order of summation and integration, which is valid for $Re(c) > Re(\gamma) > 0,\ Re(s) > 0,\ Re(c-\gamma+s) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0,$ we get

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{\Gamma(s)}{B(\gamma,c-\gamma)} \times \sum_{n=0}^{\infty} \frac{(c)_n x^n}{\Gamma(\alpha n+\beta)\,n!} y^{\gamma+s+n} \int_0^1 u^{\gamma+s+n-1} (1-uy)^{c-\gamma+s-1}\, du. \qquad (5.3.11)$$

Using the incomplete beta function in equation (5.3.11), we have

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{\Gamma(s)}{B(\gamma,c-\gamma)} \sum_{n=0}^{\infty} B_y(\gamma+s+n, c-\gamma+s) \frac{(c)_n x^n}{\Gamma(\alpha n+\beta)\,n!},$$

which can be written as follows:

$$M\{E_{\alpha,\beta}^{[\gamma,c;y]}(x;p);s\} = \frac{B(\gamma+s, c-\gamma+s)\,\Gamma(s)}{B(\gamma,c-\gamma)}\, {}_2\psi_2^{*}\left[\gamma+s, c; c+2s, (\alpha,\beta); x\right].$$

Hence the proof is completed. Formula (5.3.6) can be proved in a similar way. ■

Theorem 5.3.5 *([40]) The following derivative formula holds true for the extended incomplete Mittag-Leffler function:*

$$\frac{d^n}{dx^n}\left(E_{\alpha,\beta}^{[\gamma,c;y]}(x;p)\right) = (c)_n E_{\alpha,\beta+n\alpha}^{[\gamma+n,c+n;y]}(x;p),\ n\in\mathbb{N}. \qquad (5.3.12)$$

Proof. Using (5.3.1), differentiating on both sides with respect to x, we obtain

$$\frac{d}{dx}\left(E_{\alpha,\beta}^{[\gamma,c;y]}(x;p)\right) = cE_{\alpha,\beta+\alpha}^{[\gamma+1,c+1;y]}(x;p),$$

which is (5.3.12) for $n=1$. The general result follows by the principle of mathematical induction on n. ■

Theorem 5.3.6 *([40]) The following differentiation formula holds true for the extended incomplete Mittag-Leffler function:*

$$\frac{d^n}{dx^n}\left(x^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}(\lambda x^{\alpha};p)\right)=x^{\beta-n-1}E_{\alpha,\beta-n}^{[\gamma,c;y]}(\lambda x^{\alpha};p). \tag{5.3.13}$$

Proof. In equation (5.3.12), replace x by λx^{α} and multiply $x^{\beta-1}$, then taking the $x-$derivative n times, we get the desired result. ■

Theorem 5.3.7 *([40]) The following derivative formula holds true for the extended incomplete Mittag-Leffler function:*

$$\frac{d^n}{dp^n}\left(E_{\alpha,\beta}^{[\gamma,c;y]}(x;p)\right)=(-1)^n\frac{B(\gamma-n,c-\gamma-n)}{B(\gamma,c-\gamma)}E_{\alpha,\beta}^{[\gamma-n,c-2n;y]}(x;p).$$

Proof. Taking the $p-$derivative n times in equation (5.3.1), we get the result. ■

5.4 Fractional calculus formulas

In this section, we first recall the definitions of the right-sided Riemann-Liouville fractional integral operator $I_{a^+}^{\lambda} f$ and the right-sided Riemann-Liouville fractional derivative operator $D_{a^+}^{\lambda} f$, which are [23]

$$\left(I_{a^+}^{\lambda} f\right)(x)=\frac{1}{\Gamma(\lambda)}\int_a^x\frac{f(t)}{(x-t)^{1-\lambda}}dt,\ \ Re(\lambda)>0,\ x>a \tag{5.4.1}$$

and

$$\left(D_{a^+}^{\lambda} f\right)(x)=\left(\frac{d}{dx}\right)^n\left(I_{a^+}^{n-\lambda} f\right),\ n=[Re(\lambda)]+1,\ x>a. \tag{5.4.2}$$

The authors [23] have defined the generalized form of the Riemann-Liouville fractinal derivative operator $D_{a^+}^{\lambda}$ in (5.4.2) by introducing the right-sided fractional derivative operator $D_{a^+}^{\lambda,\delta}$ of order $0<\lambda<1$ and type $0\leqq\delta\leqq 1$ with respect to x as follows:

$$\left(D_{a^+}^{\lambda,\delta}f\right)(x)=\left(I_{a^+}^{\delta(1-\lambda)}\frac{d}{dx}\left(I_{a^+}^{(1-\delta)(1-\lambda)}f\right)\right)(x). \tag{5.4.3}$$

Obviously, when $\delta=0$ then (5.4.3) reduces to the operator $D_{a^+}^{\lambda}$ defined in (5.4.2).

It is well known [23] that

$$\left(I_{a^+}^{\lambda}(\tau-a)^{\mu-1}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)}(x-a)^{\lambda+\mu-1},\ \ Re(\lambda)>0,\ Re(\mu)>0, \tag{5.4.4}$$

and

$$\begin{aligned}\left(D_{a^+}^{\lambda,\delta}(\tau-a)^{\mu-1}\right)(x) &= \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}(x-a)^{\mu-\lambda-1},\ \ Re(\lambda)>0,\ Re(\mu)>0,\\ 0 &< \lambda<1, 0\leqq\delta\leqq 1. \end{aligned} \tag{5.4.5}$$

Theorem 5.4.1 *([40]) Suppose* $x>a\,(a\in\mathbb{R}_+=[0,\infty))$, $s=\tau-a$ *and* $\nu=x-a$, *then*

$$\left(I_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x)=\nu^{\lambda+\beta-1}E_{\alpha,\lambda+\beta}^{[\gamma,c;y]}\left(\omega(\nu)^{\alpha};p\right), \tag{5.4.6}$$

$$\left(D_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x)=\nu^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}^{[\gamma,c;y]}\left(\omega(\nu)^{\alpha};p\right), \tag{5.4.7}$$

and

$$\left(D_{a^+}^{\lambda,\delta}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x)=\nu^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}^{[\gamma,c;y]}\left(\omega(\nu)^{\alpha};p\right). \qquad (5.4.8)$$

Proof. Direct calculations yield

$$\begin{aligned}\left(I_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x) &= \frac{1}{\Gamma(\lambda)}\int_a^x \frac{s^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)}{(x-\tau)^{1-\lambda}}d\tau \\ &= \frac{1}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n\,\omega^n}{\Gamma(\alpha n+\beta)\,n!} \\ &\quad\times\int_a^x s^{\alpha n+\beta-1}(x-\tau)^{\lambda-1}d\tau \\ &= \sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n\,\omega^n}{\Gamma(\alpha n+\beta)\,n!}\left(I_{a^+}^{\lambda}\left[s^{\alpha n+\beta-1}\right]\right)(x).\end{aligned}$$

By the use of (5.4.4), we have

$$\begin{aligned}\left(I_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x) &= \sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n\,\omega^n}{\Gamma(\alpha n+\beta)\,n!}\nu^{\alpha n+\lambda+\beta-1} \\ &\quad\times\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\lambda+\beta)} \\ &= \nu^{\lambda+\beta-1}\sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{\left(\omega(\nu)^{\alpha}\right)^n}{n!} \\ &= \nu^{\lambda+\beta-1}E_{\alpha,\lambda+\beta}^{[\gamma,c;y]}\left(\omega(\nu)^{\alpha};p\right),\end{aligned}$$

which completes the proof of result (5.4.6).

Now, we have

$$\left(D_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x)=\left(\frac{d}{dx}\right)^n\left(I_{a^+}^{n-\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x),$$

which on using (5.4.6) takes the following form:

$$\left(D_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{\gamma,c}\left[s^{\alpha};y\right]\right)(x)=\left(\frac{d}{dx}\right)^{n}\left\{\nu^{\beta+n-\lambda-1}E_{\alpha,\beta+n-\lambda}^{\gamma,c}\left[\nu^{\alpha};y\right]\right\}(x).$$

Applying (5.3.13), we have

$$\left(D_{a^+}^{\lambda}(s)^{\beta-1}E_{\alpha,\beta}^{\gamma,c}\left[s^{\alpha};y\right]\right)(x)=\nu^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}^{[\gamma,c;y]}\left(\omega(\nu)^{\alpha};p\right).$$

This completes the desired proof.

To prove (5.4.7), we have

$$\begin{aligned}
&\left(D_{a^+}^{\lambda,\delta}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x)\\
=&\left(D_{a^+}^{\lambda,\delta}\left[\sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{\omega^n}{n!}(s)^{\alpha n+\beta-1}\right]\right)(x)\\
=&\sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n\,\omega^n}{\Gamma(\alpha n+\beta)\,n!}\\
&\times\left(D_{a^+}^{\lambda,\delta}(s)^{\alpha n+\beta-1}\right)(x).
\end{aligned}$$

By applying (5.4.5), we get

$$\begin{aligned}
\left(D_{a^+}^{\lambda,\delta}(s)^{\beta-1}E_{\alpha,\beta}^{[\gamma,c;y]}\left(\omega(s)^{\alpha};p\right)\right)(x) &= \sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma;p)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{\omega^n}{n!}\\
&\quad\times\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta-\lambda)}(\nu)^{\alpha n+\beta-\lambda-1}\\
&= \nu^{\beta-\lambda-1}\sum_{n=0}^{\infty}\frac{B_y(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{\left(\omega(\nu)^{\alpha}\right)^n}{n!}\\
&= \nu^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}^{[\gamma,c;y]}\left(\omega(\nu)^{\alpha};p\right),
\end{aligned}$$

which completes the required proof. ■

In the following theorem, the images of the Prabhakar's Mittag-Leffler function under the actions of incomplete Riemann-Liouville fractional derivative operator are obtained.

Theorem 5.4.2 *([40]) Let* $0 \leq y < 1,\ Re(\mu) > Re(\lambda) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0$. *Then*

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}E_{\alpha,\beta}^{\gamma}(z);y\right] = \frac{z^{\mu-1}B(\lambda,\gamma-\lambda)}{\Gamma(\mu-\lambda)}E_{\alpha,\beta}^{\lambda,\mu}[z;y].$$

Proof. Replacing μ by $\lambda-\mu$ in the definition of the incomplete Riemann-Liouville fractional derivative operator (2.5.2), we get

$$\begin{aligned} & D_z^{\lambda-\mu}\left[z^{\lambda-1}E_{\alpha,\beta}^{\gamma}(z);y\right] \\ = & \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}E_{\alpha,\beta}^{\gamma}(uz)(1-u)^{\mu-\lambda-1}du \\ = & \frac{z^{\mu-1}y^{\lambda}}{\Gamma(\mu-\lambda)}\int_0^1 t^{\lambda-1}(1-ty)^{\mu-\lambda-1}E_{\alpha,\beta}^{\gamma}(zty)\,dt. \end{aligned}$$

By (5.3.1), we can write

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}E_{\alpha,\beta}^{\gamma}(z);y\right] = \frac{z^{\mu-1}B(\lambda,\gamma-\lambda)}{\Gamma(\mu-\lambda)}E_{\alpha,\beta}^{\lambda,\mu}[z;y].$$

Hence the proof is completed. ■

5.5 Relations between the extended incomplete Mittag-Leffler function with Laguerre polynomials

In this section, a representation is given on the extended incomplete Mittag-Leffler function in terms of Laguerre polynomials.

Theorem 5.5.1 *([40]) For the extended incomplete Mittag-Leffler function, we have*

$$\exp(2p)\,E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) = \frac{1}{B(\gamma,c-\gamma)}\sum_{m,n,k=0}^{\infty}\frac{L_m(p)L_n(p)(c)_k}{\Gamma(\alpha k+\beta)}\frac{x^k}{k!} \times B_y(\gamma+m+k+1,c-\gamma+n+1), \quad (5.5.1)$$

and

$$\exp(2p)\,E_{\alpha,\beta}^{\{\gamma,c;y\}}(x;p) = \frac{1}{B(\gamma,c-\gamma)}\sum_{m,n,k=0}^{\infty}\frac{L_m(p)L_n(p)(c)_k}{\Gamma(\alpha k+\beta)}\frac{x^k}{k!} \times B_{1-y}(c-\gamma+m+1,\gamma+n+k+1), \quad (5.5.2)$$

where $Re(c) > Re(\gamma) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0.$

Proof. We start by recalling the useful identity used in [34]

$$\exp\left(\frac{-p}{t(1-t)}\right) = \exp(-2p)\sum_{m,n=0}^{\infty} L_n(p)\,L_m(p)\,t^{m+1}(1-t)^{n+1};\ 0<t<1. \quad (5.5.3)$$

Used equation (5.5.3) in equation (5.3.1), we get

$$E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) = \frac{y^{\gamma}}{B(\gamma,c-\gamma)}\int_0^1 u^{\gamma-1}(1-uy)^{c-\gamma-1}\exp(-2p) \times \sum_{m,n=0}^{\infty} L_n(p)\,L_m(p)\,(uy)^{m+1}(1-uy)^{n+1}E_{\alpha,\beta}^{c}(xuy)\,du. \quad (5.5.4)$$

Now, taking into account the series expansion of Prabhakar's generalized Mittag-Leffler's function $E_{\alpha,\beta}^{c}(xuy)$ in equation (5.5.4), we have

$$
\begin{aligned}
E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) &= \frac{y^{\gamma}\exp(-2p)}{B(\gamma,c-\gamma)}\int_0^1 u^{\gamma-1}(1-uy)^{c-\gamma-1} \\
&\quad\times\sum_{m,n=0}^{\infty} L_n(p)L_m(p)(uy)^{m+1}(1-uy)^{n+1}\sum_{k=0}^{\infty}\frac{(c)_k(xuy)^k}{\Gamma(\alpha k+\beta)k!}du \\
&= \frac{y^{\gamma}\exp(-2p)}{B(\gamma,c-\gamma)}\int_0^1 u^{\gamma-1}(1-uy)^{c-\gamma-1} \\
&\quad\times\sum_{m,n,k=0}^{\infty}\frac{L_n(p)L_m(p)(c)_k}{\Gamma(\alpha k+\beta)k!}(uy)^{m+k+1}(1-uy)^{n+1}x^k du. \quad (5.5.5)
\end{aligned}
$$

Interchanging the order of integration and summation in equation (5.5.5), which can be done under the assumptions of the theorem, we have

$$
\begin{aligned}
E_{\alpha,\beta}^{[\gamma,c;y]}(x;p) &= \frac{\exp(-2p)}{B(\gamma,c-\gamma)}\sum_{m,n,k=0}^{\infty}\frac{L_n(p)L_m(p)(c)_k}{\Gamma(\alpha k+\beta)k!}x^k \\
&\quad\times B_y(\gamma+m+k+1,c-\gamma+n+1). \quad (5.5.6)
\end{aligned}
$$

Multiplying both sides of equation (5.5.6) by $\exp(2p)$, we get the result. Formula (5.5.2) can be proved in a similar way. ■

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