Stability, Existence and Uniqueness of Boundary Value Problems for a Coupled System of Fractional Differential Equations

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ABSTRACT

The current thesis investigates four different nonlinear systems of fractional differential equations and deals with the existence, uniqueness, and stability of their solutions. The first studied problem is a coupled system of fractional differential equations with four-point integral boundary conditions. Existence and uniqueness of solutions are established by applying the contraction mapping principle and Leray–Schauder's alternative theorem. Finding and results are demonstrated and supported with numerical examples.

The second studied case is a boundary value problem for a coupled system of nonlinear fractional differential equations, where the existence and uniqueness of solutions is proven by using the Banach's fixed point theorem and Schauder's alternative. Furthermore, the Hyers-Ulam stability of solutions is discussed, sufficient stability conditions are drawn, and supporting numerical results are presented.

In the third problem, a coupled system of Caputo type sequential fractional differential equations with integral boundary conditions is studied. Similarly, existence and uniqueness of solutions are discussed and established by employing contraction mapping principle and Leray–Schauder's alternative theorem, and Hyers-Ulam stability of the boundary value problem is investigated.

The last problem is a nonlinear Caputo type sequential fractional differential equation with non-separated non-local integral fractional boundary conditions. Existence, uniqueness, and Hyers-Ulam stability of solutions are discussed and established, and theoretical findings are presented and supported by numerical examples.

Keywords: fractional differential equation, sequential, Caputo, integral boundary conditions, stability, Hyers-Ulam stability, existence and uniqueness of solutions.

ÖZ

Bu tezde, dört farklı doğrusal olmayan kesirli diferensiyel denklem sistemi araştırılmış ve çözümlerinin varlığı, benzersizliği ve kararlılığı çalışılmıştır. İlk çalışılan problem, dört noktalı integral sınır koşullarına sahip birleştirilmiş kesirli diferansiyel denklem sistemidir. Kasılma haritalama ilkesi ve Leray-Schauder'in alternatif teoremi uygulanarak çözümlerin varlığı ve benzersizliği sağlanmıştır. Elde edilen ve sonuçlar sayısal örneklerle gösterilmiş ve desteklenmiştir.

İkinci çalışılan durum, Banach sabit nokta teoremi ve Schauder alternatifi kullanılarak çözümlerin varlığı ve benzersizliğinin kanıtlandığı birleştirilmiş doğrusal olmayan kesirli diferansiyel denklemler sistemi için bir sınır değer problemidir. Ayrıca, çözümlerin Hyers-Ulam kararlılığı tartışılmış, yeterli kararlılık koşulları verilmiş ve sayısal sonuçlar desteklenmiştir.

Üçüncü problemde integral sınır koşullarına sahip eşleşmiş bir Caputo tipi ardışık kesirli diferansiyel denklem sistemi incelenmiştir. Benzer şekilde, daralma haritalama prensibi ve Leray-Schauder alternatif teoremi kullanılarak çözümlerin varlığı ve benzersizliği tartışılmış kurulmakta ve Sınır-Değer Probleminin Hyers-Ulam kararlılığı araştırılmıştır.

Son problem, ayrılmamış lokal olmayan integral kesirli sınır koşullarıyla doğrusal olmayan bir Caputo tipi sıralı kesirli diferansiyel denklemdir. Çözümlerin varlığı, tekliği ve Hyers-Ulam kararlılığı tartışılmış ve sayısal örnekler ile elde edilen sonuçlar desteklenmiştir.

Anahtar Kelimeler: kesirli diferansiyel denklem, sıralı, Caputo, integral sınır şartları, kararlılık, Hyers-Ulam kararlılığı, çözümlerin varlığı ve benzersizliği.

DEDICATION

To My Lovely Husband, Children and Parents

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I would like to express my thanks and special appreciation to my advisor Professor Dr. Nazım Mahmudov, you have been a wonderful mentor for me. I would like to thank you for encouraging my research and for allowing me to grow as a research scientist. Your advice on both research.

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LIST OF SYMBOLS AND ABBREVIATIONS

BC	Boundary condition,
BVP	Boundary Value Problem
FDE	Fractional Differential Equation
FPT	Fixed Point Theory
RL	Riemann-Liouville
$\|\sqcup\|$	Norm of $\sqcup$
$\mathbb{N}_0$	$\{0,1,2,3,\ldots\}$
$\mathbb{N}$	$\{1,2,3,\ldots\}$
$\mathbb{R}$	The set of all real numbers
J	Any given closed interval

Chapter 1

INTRODUCTION

Fractional differential equations constitute a pivotal domain of theoretical research with a wide range of applications in many fields of science and engineering. In fact, the non-integer nature of differential orders in fractional calculus equips this field with extensive flexibility in terms of modeling sophisticated nonlinear systems. Applications of fractional differential equations in such nonlinear complex models appear in numerous branches of science, including quantum physics, thermodynamics, biology, biomedicine, genetics, evolutionary biology, dynamical systems, chemistry, electronics, and more [1-6].

It’s worth mentioning that in majority of practical cases, boundary value problems of coupled fractional differential equations must be solved and hence investigating the existence, uniqueness, and stability of their solutions, is of great practical importance. In this regard, numerous research studies have examined the existence and uniqueness of solutions for coupled systems of differential equations with different boundary conditions, mostly by employing the contraction mapping principle and Schauder’s alternative theorem [7-28].

Furthermore, an important emphasis of many such studies have been on integral boundary conditions as most of the practically important applications of fractional differential equations involve integral boundaries. Examples of such applications include computational fluid dynamics, differential thermal analysis, biochemistry, and population dynamics [29-31]. Furthermore, movable integral boundary conditions play a key role in several subfields of engineering, structural mechanics, physics, and economics, as they can provide extensively more accurate models of the practical problems as compared to local boundary conditions [32-36]. Another critical issue related with applications of fractional differential equations is the stability of their approximated solutions. In fact, due to the highly complex and nonlinear nature of fractional systems, finding exact solutions is not generally liable and hence numerically approximated solutions must be obtained.

In this regard, the stability of these approximated solutions must be carefully investigated and established. One of the prominent methods for studying the stability of solutions for fractional differential equations, is the Hyers–Ulam method. This method has been used by many studies to establish the stability of their solutions for coupled fractional differential systems and has also been employed in applied studies in biology, physics, nonlinear optimization, and numerical analysis [37,38]. The rest of this thesis is organized in five chapters that are briefly explained and reviewed as follows:

Chapter 2 (Preliminaries and Definitions): This chapter provides basic required definitions, properties and the terminology of fractional calculus, fractional differential equations, and functional analysis, that will be used in subsequent chapters. This chapter aims at providing key preliminary concepts that other chapters are built upon, and can be skipped if the reader is familiar with the field of fractional differential equations.

Chapter 3 (Stability, Existence and Uniqueness of Boundary Value Problem for a Coupled System of Fractional Differential Equations): This chapter discusses a coupled system of fractional differential equations with boundary conditions. The existence and uniqueness of solutions for the given problem is established by applying contraction mapping principle and Leray-Schauder's alternative theorem, and sufficient conditions for the Hyers-Ulam stability of solutions are found. Obtained results are supported by examples and illustrated in the last section.

Chapter 4 (Existence and Stability of Coupled Sequential Fractional Differential System Boundary Conditions): This chapter focuses on the existence and uniqueness of solutions for a coupled system of Caputo type sequential fractional differential equations with integral boundary conditions. The existence of solutions is derived by applying Leray–Schauder's alternative, while the uniqueness of solution is established by Banach's contraction principle. Moreover, some necessary conditions for the Hyers-Ulam type stability of the solutions are developed and supporting examples are presented.

Chapter 5 (Existence and Ulam-Stability of a Coupled Sequential Fractional Differential Equations with Integral Boundary Conditions): This chapter studies the existence and uniqueness of solutions for a sequential fractional differential equations involving Caputo derivative of order $1 < \alpha \leq 2$ with integral boundary conditions. Moreover, the Hyers-Ulam stability of the solutions is discussed and supported by numerical results.

Chapter 2

PRELIMINARIES AND DEFINITIONS

In this chapter we briefly review and explain the required mathematical tools including definitions, properties, propositions, lemmas and theorems that will be used in later chapters [1,2].

2.1 Special Functions of Fractional Calculus

Definition 2.1.1 The Gamma Function is given by

$$\Gamma(\mu) = \int_0^\infty s^{\mu-1} e^{-s} ds, \qquad Re(\mu) > 0\,, \qquad \mu \in \mathbb{C}$$

Here $s^{\mu-1} = e^{(\mu-1)ln(s)}$.

The domain of Gamma function can be extended to the left half plane with $Re(\mu) \leq 0, \mu \neq 0, -1, -2, \ldots$, by analytic continuation.

Definition 2.1.2 (Pochhammer Symbol:$(\psi)_m,\ m \in \mathbb{N}$)

$$(\psi)_m = \psi(\psi+1)\ldots(\psi+m-1), \ \ (\psi)_0 = 1.$$

Obviously, $(\psi)_m = (-1)^m(1-\psi-m)_m$,moreover, $(\psi)_m = \frac{\Gamma(\psi+m)}{\Gamma(\psi)}$.

The Gamma function satisfies the following properties, where $\psi \in \mathbb{R}$:

i. $\Gamma(\psi+m) = (\psi)_m \Gamma(\psi)$,

$$\Gamma(\psi-m) = \frac{\Gamma(\psi)}{(\psi-m)_m} = \frac{(-1)^m}{(1-\psi)_m}\Gamma(\psi).$$

ii. $\frac{\Gamma(\psi)}{\Gamma(1-\psi)} = \frac{\pi}{sin\pi\psi}, \ \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

iii. Legendre duplication formula:

$$\Gamma(2\psi) = \frac{2^{2\psi-1}}{\sqrt{\pi}}\Gamma(\psi)\Gamma\left(\psi + \frac{1}{2}\right).$$

iv. $\Gamma(\psi + 1) = \psi\Gamma(\psi) = \psi(\psi - 1)\Gamma(\psi - 1) = \cdots =$

$\psi(\psi - 1) \ldots (2)(1)\Gamma(1) = \psi!\ if\ \psi \in \mathbb{N}.$

Definition 2.1.3 (Binomial Coefficients):

$$\binom{\xi}{m} = \frac{(-1)^m(-\xi)_m}{m!} = \frac{(-1)^{m-1}\xi\Gamma(m - \xi)}{\Gamma(1 - \xi)\Gamma(m + 1)}, \qquad m \in \mathbb{N}_0, \xi \in \mathbb{C}.$$

If $\xi = n \in \mathbb{N}$ then $\binom{n}{m} = \frac{(-1)^m(-n)_m}{m!} = \frac{n!}{(n-m)!m!}$.

For complex η, ψ with $\psi \neq -1, -2, \ldots$

$$\binom{\psi}{\eta} = \frac{\Gamma(\psi + 1)}{\Gamma(\eta + 1)\Gamma(\psi - \eta + 1)}.$$

Definition 2.1.4 (The Beta Function: $B(\psi, \eta)$)

$B(\psi, \eta) = \int_0^1 m^{\psi-1} (1 - m)^{\eta-1} dm, Re(\psi) > 0, Re(\eta) > 0$ where $B(.,.)$ is called the Beta Function .

For all $\psi, \eta > 0$, Beta function satisfy the following properties:

i. $B(\psi, \eta) = B(\eta, \psi)$.

ii. $B(\psi, \eta) = \frac{\Gamma(\psi)\Gamma(\eta)}{\Gamma(\psi+\eta)}$.

iii. The Incomplete Betta function is given by

$B_s(\psi, \eta) = \int_0^s m^{\psi-1} (1 - m)^{\eta-1} dm, 0 < s < 1.$

2.2 Function Spaces

Definition 2.2.1 A sequence $\{y_m\} \subset (\mathrm{X}, d)$ is said to be converge to $y \in \mathrm{X}$ if $\lim_{m\to\infty} d(y_m, y) = 0$ if and only if $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m > N(\varepsilon)$, $d(y_m, y) < \varepsilon$.

Definition 2.2.2 $\{y_m\} \subset (\mathrm{X}, d)$ is called a Caushy sequence if $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ such that $\forall m, n > N(\varepsilon), d(y_m, y_n) < \varepsilon$.

Definition 2.2.3 A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent to a point of X.

Definition 2.2.4 Let (X, d) be a metric space, $F: \mathrm{X} \to \mathrm{X}$ is said to be Lipschitzian if $\exists \beta \geq 0$ with $d\big(F(x), F(y)\big) \leq \beta d(x, y), \forall x, y \in \mathrm{X}$. (1)

Note that Lipschitzian map is continuous, and the smallest number β for which (1) holds and is said to be Lipschitz constant for F and is denoted by L. If $L < 1$, we say that F is a contraction. Whereas, if $L = 1$, then F is a nonexpansive.

Theorem 2.2.5 (Banach's Contraction mapping principle).

Let (X, d) be a complete metric space. Let $M: X \to X$ be a contraction$(0 \leq L < 1)$, then

i. M has a unique fixed point $t \in X : M(t) = t$.

ii. $\forall u \in X$, we have $\lim_{k \to \infty} M(u) = t$ with

$$d(M^k(u), t) \leq \frac{L^k}{1 - L} d\big(t, M(u)\big).$$

Where L is Lipschitz constant for F

Theorem 2.2.6 (Local version of Banach's Contraction mapping principle).

Assume that:

i. (X, d) is a complete metric space.

ii. $M: B(t_0, r) \to X$ is a contraction on $B(t_0, r) = \{t \in \mathrm{X}: d(t, t_0) < r\}, t_0 \in \mathrm{X}$, $\mathrm{r} > 0$, with $d(M(t_0), t_0) < (1 - \mathrm{L})r$.Then M has a unique fixed point in $B(t_0, r)$.

Theorem 2.2.7 Assume that:

i. (X, d) is a complete metric space.

ii. $d\big(M(x), M(y)\big) \leq \psi d(x, y), \forall x, y \in X$, where $\psi: [0, \infty) \to [0, \infty)$ is any nondecreasing function with $\lim_{m\to\infty} \psi^m(s) = 0$, for a fixed $s > 0$, then M has a unique fixed point $t \in X$ with $\lim_{m\to\infty} M^m(t_0) = t, \forall t_0 \in X$.

Theorem 2.2.8 (Schauder's Theorem) Assume that:

i. $C \neq \phi$, convex, closed subset of a normed linear space E.

ii. $M: C \to C$ is nonexpansive.

iii. $M(C)$ is a subset of a compact subset of C.

Then M has a fixed point.

Theorem 2.2.9 (Nonlinear alternative of Leray-Shauder type for contractive map)

Suppose that Z is an open subset of a Banach space X, $0 \in Z$ and $M: \bar{Z} \to X$ a contraction with $M(\bar{Z})$ is bounded then either

i. M has a fixed point in $\bar{Z}$,or

ii. $\exists \mu \in (0,1)$ and $z \in \partial Z$ with $z = \mu M(z)$ holds.

Theorem 2.2.10 (Arzela-Ascoli Theorem)

$M \subset C(X, \mathbb{R})$ is compact if and only if it is closed, bounded and equicontinuous.

Chapter 3

STABILITY, EXISTENCE AND UNIQUENESS OF BOUNDARY VALUE PROBLEM FOR A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

In this chapter, a coupled system of fractional differential equations with boundary conditions will be discussed. Currently we present three main result of this study: first the existence and uniqueness of solutions for the given problem is established by applying contraction mapping principle, then Leray-Schauder's alternative has been used to obtain the existence of solutions, and finally the Hyers-Ulam stability of solutions is discussed and sufficient conditions for the stability are developed. Obtained results are supported by examples and illustrated in the last section.

The following coupled system of fractional differential equations was studied by Ntouyas and Obaid [42]

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) = f\big(t, u(t), v(t)\big), t \in [0,1], \\ {}^{c}D_{0+}^{\beta}v(t) = g\big(t, u(t), v(t)\big), t \in [0,1], \\ u(0) = \gamma I^{p}u(\eta) = \gamma \displaystyle\int_{0}^{\eta} \frac{(\eta - s)^{p-1}}{\Gamma(p)} u(s)ds, 0 < \eta < 1, \\ v(0) = \delta I^{q}v(\zeta) = \delta \displaystyle\int_{0}^{\zeta} \frac{(\zeta - s)^{q-1}}{\Gamma(q)} v(s)ds, 0 < \zeta < 1. \end{cases}$$

Here ${}^{c}D_{0+}^{\alpha}$ and ${}^{c}D_{0+}^{\beta}$ are the Caputo fractional derivatives, $0 < \alpha, \beta \le 1$, $f, g \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$ and $p, q, \gamma, \delta \in \mathbb{R}$.

Ahmed and Ntouyas [43] employed Banach fixed point theorem and Schauder's fixed point theorem to prove the existence of the solutions of the following coupled fractional differential equations with coupled and uncoupled slit-strips-type integral boundary conditions:

$$\begin{cases} {}^{c}D^{q}x(t) = f\big(t, x(t), y(t)\big), & t \in [0,1], \ \ 1 < q \le 2, \\ {}^{c}D^{p}y(t) = g\big(t, x(t), y(t)\big), & t \in [0,1], \ \ 1 < q \le 2, \end{cases}$$

supplemented with coupled and uncoupled slit-strips-type integral boundary conditions, respectively, given by

$$\begin{cases} x(0) = 0, \ \ x(\zeta) = a\displaystyle\int_0^{\eta} y(s)ds + b\int_{\xi}^{1} y(s)ds, \ \ 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \ \ y(\zeta) = a\displaystyle\int_0^{\eta} x(s)ds + b\int_{\xi}^{1} x(s)ds, \ \ 0 < \eta < \zeta < \xi < 1, \end{cases}$$

and

$$\begin{cases} x(0) = 0, \ \ x(\zeta) = a\displaystyle\int_0^{\eta} x(s)ds + b\int_{\xi}^{1} x(s)ds, \ \ 0 < \eta < \zeta < \xi < 1, \\ y(0) = 0, \ \ y(\zeta) = a\displaystyle\int_0^{\eta} y(s)ds + b\int_{\xi}^{1} y(s)ds, \ \ 0 < \eta < \zeta < \xi < 1. \end{cases}$$

Furthermore, Alsulami et al. [44] investigated the following coupled fractional differential equations with non-separated coupled boundary conditions:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f\big(t, x(t), y(t)\big), t \in [0, T], 1 < \alpha \le 2, \\ {}^{c}D^{\beta}y(t) = g(t, x(t), y(t)), t \in [0, T], 1 < \beta \le 2, \end{cases}$$

subject to the following non-separated coupled boundary conditions:

$$\begin{cases} x(0) = \lambda_1 y(T), x'(0) = \lambda_2 y'(T), \\ y(0) = \mu_1 x(T), y'(0) = \mu_2 x'(T). \end{cases}$$

Note that ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ denote Caputo fractional derivatives of order α and β. Moreover, $\lambda_i, \mu_i, i = 1, 2$, are real constants with $\lambda_i\mu_i \neq 1$ and $f, g: [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are appropriately chosen functions.

We study a coupled system of nonlinear fractional differential equations in this chapter

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f\big(t, x(t), y(t)\big), & t \in [0,T], \ \ 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}y(t) = g\big(t, x(t), y(t)\big), & t \in [0,T], \ \ 1 < \beta \leq 2, \end{cases} \tag{1}$$

supplemented with integral boundary conditions of the form:

$$x(T) = \eta y'(\rho), \qquad y(T) = \zeta x'(\mu), \qquad x(0) = 0, \qquad y(0) = 0, \rho, \mu \in [0,T], \tag{2}$$

where ${}^{c}D^{k}$ denote the Caputo fractional derivatives of order $k, k = \alpha, \beta$, and $f, g \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$ are given continuous functions, and η, ζ are real constants.

3.1 Preliminaries

Firstly, we recall definitions of fractional derivative and integral [1].

Definition3.1.1 The Riemann-Liouville fractional integral of order α for a continuous function h is given by

$$(I^{\alpha}h)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \frac{h(t)}{(s-t)^{1-\alpha}} dt, \alpha > 0,$$

provided that the integral exists on $\mathbb{R}^{+}$.

To define the solution for the problem (1) and (2), we prove the following auxiliary lemma.

Lemma 3.1.1 Let $u, v \in C([0,T], \mathbb{R})$ then the unique solution for the problem

$$\begin{cases} {}^{c}D^{\alpha}x(t) = u(t), & t \in [0,T], \ \ 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}y(t) = v(t), & t \in [0,T], \ \ 1 < \beta \leq 2, \\ x(T) = \eta y'(\rho), \qquad y(T) = \zeta x'(\mu), & x(0) = 0, \qquad y(0) = 0, \rho, \mu \in [0,T] \end{cases} \tag{3}$$

is

$$x(t) = \frac{t}{\Delta}\left(\eta T \int_0^{\rho} \frac{(\rho - s)^{\beta-2}}{\Gamma(\beta - 1)} v(s)ds - T \int_0^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds + \eta\zeta \int_0^{\mu} \frac{(\mu - s)^{\alpha-2}}{\Gamma(\alpha - 1)} u(s)ds \right.$$

$$\left. - \eta \int_0^{T} \frac{(T - s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds \right) + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds, \qquad (4)$$

and

$$y(t) = \frac{t}{\Delta}\left(\eta\zeta \int_0^{\rho} \frac{(\rho - s)^{\beta-2}}{\Gamma(\beta - 1)} v(s)ds - \zeta \int_0^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds + T\zeta \int_0^{\mu} \frac{(\mu - s)^{\alpha-2}}{\Gamma(\alpha - 1)} u(s)ds \right.$$

$$\left. - T \int_0^{T} \frac{(T - s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds \right) + \int_0^{t} \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} v(s)ds, \qquad (5)$$

where $\Delta = T^2 - \eta\zeta \neq 0$.

Proof. General solutions of the fractional differential equations in (3) are known [41] as

$$x(t) = at + b + \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} u(s)ds, \qquad (6)$$

$$y(t) = ct + d + \frac{1}{\Gamma(\beta)} \int_0^{t} (t - s)^{\beta-1} v(s)ds, \qquad (7)$$

where a, b, c, d are arbitrary constants.

Apply conditions $x(0) = 0$ and $y(0) = 0$, and we obtain $b = d = 0$.

Here

$$x'(t) = a + \frac{1}{\Gamma(\alpha - 1)} \int_0^{t} (t - s)^{\alpha-2} u(s)ds,$$

$$y'(t) = c + \frac{1}{\Gamma(\beta - 1)} \int_0^{t} (t - s)^{\beta-2} v(s)ds.$$

Considering boundary conditions

$$x(T) = \eta y'(\rho), \qquad y(T) = \zeta x'(\mu)$$

we get

$$aT + \int_0^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds = \eta c + \eta \int_0^{\rho} \frac{(\rho - s)^{\beta-2}}{\Gamma(\beta - 1)} v(s)ds,$$

and

$$cT+\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds = a\zeta + \zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds,$$

so

$$a=\frac{1}{T}\left(\eta c+\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} v(s)ds-\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds\right),$$

$$c=\frac{1}{T}\left(a\zeta+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds-\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds\right).$$

Hence, by substituting the value of a into c, we obtain the final result for these constants as

$$c=\frac{1}{T}\left(\frac{\zeta}{T}\left[\eta c+\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} v(s)ds-\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds\right]\right.$$
$$\left.+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds-\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds\right),$$

$$c-\frac{\zeta\eta c}{T^2}=\frac{1}{T}\left(\frac{\zeta}{T}\left[\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} v(s)ds-\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds\right]\right.$$
$$\left.+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds-\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds\right),$$

$$c\left(\frac{T^2-\zeta\eta}{T^2}\right)=\frac{1}{T}\left(\frac{\zeta}{T}\left[\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} v(s)ds-\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds\right]\right.$$
$$\left.+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds-\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds\right),$$

$$c=\frac{T}{T^2-\zeta\eta}\left(\frac{\zeta}{T}\left[\eta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} v(s)ds-\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds\right]\right.$$
$$\left.+\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds-\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds\right),$$

$$c=\frac{1}{T^2-\zeta\eta}\left(\eta\zeta\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} v(s)ds-\zeta\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds\right.$$
$$\left.+T\zeta\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} u(s)ds-T\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} v(s)\,ds\right)$$

$$c=\frac{1}{\Delta}\left(\eta\zeta\int_0^{\rho}\frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-\zeta\int_0^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds+T\zeta\int_0^{\mu}\frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds\right.$$

$$\left.-T\int_0^{T}\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)\,ds\right),$$

and

$$a=\frac{1}{\Delta}\left(\eta T\int_0^{\rho}\frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}v(s)ds-T\int_0^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds+\eta\zeta\int_0^{\mu}\frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}u(s)ds\right.$$

$$\left.-\eta\int_0^{T}\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}v(s)\,ds\right).$$

Substituting the values of a, b, c, d in (6), (7) we get (4) and (5).The converse follows by direct computation. This completes the proof. ■

3.2 Existence Results

Let us consider the space

$$R=\{x(t),\qquad x(t)\in C([0,T]\},$$

$$S=\{y(t),\qquad y(t)\in C([0,T]\},$$

endowed with norm $\|x\|=\sup_{0\le t\le T}|x(t)|$ and $\|y\|=\sup_{0\le t\le T}|y(t)|$ respectively.

It is clear that both $(R,\|\,.\,\|)$ and $(S,\|\,.\,\|)$ are Banach Spaces.

Consequently, the product space $(R\times S,\|(x,y)\|)$ is a Banach Space as well (endowed with $\|(x,y)\|=\|x\|+\|y\|$).

In view of Lemma (3.1.1), we define the operator $G\colon R\times S\to R\times S$ as:

$$G(x,y)(t)=\big(G_1(x,y)(t),G_2(x,y)(t)\big),$$

where

$$G_1(x,y)(t) = \frac{t}{\Delta}\left(\eta T \int_0^{\rho} \frac{(\rho - s)^{\beta-2}}{\Gamma(\beta-1)} g\big(s, x(s), y(s)\big) ds\right.$$

$$- T \int_0^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), y(s)\big)\, ds$$

$$+ \eta\zeta \int_0^{\mu} \frac{(\mu - s)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(s, x(s), y(s)\big) ds$$

$$\left.- \eta \int_0^{T} \frac{(T - s)^{\beta-1}}{\Gamma(\beta)} g\big(s, x(s), y(s)\big)\, ds\right)$$

$$+ \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), y(s)\big) ds, \tag{8}$$

and

$$G_2(x,y)(t) = \frac{t}{\Delta}\left(\eta\zeta \int_0^{\rho} \frac{(\rho - s)^{\beta-2}}{\Gamma(\beta-1)} g\big(s, x(s), y(s)\big) ds\right.$$

$$- \zeta \int_0^{T} \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), y(s)\big)\, ds$$

$$+ T\zeta \int_0^{\mu} \frac{(\mu - s)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(s, x(s), y(s)\big) ds$$

$$\left.- T \int_0^{T} \frac{(T - s)^{\beta-1}}{\Gamma(\beta)} g\big(s, x(s), y(s)\big)\, ds\right)$$

$$+ \int_0^{t} \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} g\big(s, x(s), y(s)\big) ds. \tag{9}$$

In the first result, we establish the existence and the uniqueness of the solutions of the boundary value problem (1) and (2) by using Banach's contraction mapping principle.

Theorem 3.2.1 Assume $f, g: C([0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are jointly continuous functions and there exist constants $\phi, \psi \in \mathbb{R}$,such that$\forall\ x_1, x_2, y_1, y_2 \in \mathbb{R}, \forall t \in [0,T]$, we have

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \phi(|x_2 - x_1| + |y_2 - y_1|),$$

$$|g(t, x_1, x_2) - f(t, y_1, y_2)| \le \psi(|x_2 - x_1| + |y_2 - y_1|).$$

If

$$\phi(Q_1 + Q_3) + \psi(Q_2 + Q_4) < 1,$$

then the BVP (1) and (2) has a unique solution on $[0,T]$.

Where

$$Q_1=\frac{T}{|\Delta|}\left(\frac{T^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\ ,$$

$$Q_2=\frac{T}{|\Delta|}\left(\frac{|\eta|T\rho^{\beta-1}}{\Gamma(\beta)}+\frac{|\eta|T^{\beta}}{\Gamma(\beta+1)}\right),$$

$$Q_3=\frac{T}{|\Delta|}\left(\frac{|\zeta|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T|\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\right), \tag{10}$$

$$Q_4=\frac{T}{|\Delta|}\left(\frac{|\eta\zeta|\rho^{\beta-1}}{\Gamma(\beta)}+\frac{T^{\beta+1}}{\Gamma(\beta+1)}\right)+\frac{T^{\beta}}{\Gamma(\beta+1)}.$$

Proof. Define $\sup_{0\le t\le T}|f(t,0,0)|=f_0<\infty, \sup_{0\le t\le T}|g(t,0,0)|=g_0<\infty$ and $\Omega_\varepsilon=\{(x,y)\in R\times S:\|(x,y)\|\le\varepsilon\}$, and $\varepsilon>0$, such that

$$\varepsilon\ge\frac{(Q_1+Q_3)f_0+(Q_2+Q_4)g_0}{1-[\phi(Q_1+Q_3)+\psi(Q_2+Q_4)]}.$$

Firstly, we show that $G\Omega_\varepsilon\subseteq\Omega_\varepsilon$.

By our assumption, for $(x,y)\in\Omega_\varepsilon, t\in[0,T]$, we have

$$\left|f\big(t,x(t),y(t)\big)\right|\le\left|f\big(t,x(t),y(t)\big)-f(t,0,0)\right|+|f(t,0,0)|,$$

$$\le\phi(|x(t)|+|y(t)|)+f_0\le\phi(\|x\|+\|y\|)+f_0,$$

$$\le\phi\varepsilon+f_0,$$

and

$$\left|g\big(t,x(t),y(t)\big)\right|\le\psi(|x(t)|+|y(t)|)+g_0\le\psi(\|x\|+\|y\|)+g_0,$$

$$\le\psi\varepsilon+g_0,$$

which lead to

$$|G_1(x,y)(t)| \leq \frac{T}{|\Delta|}\left(|\eta|T\int_0^{\rho}\frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds(\psi(\|x\|+\|y\|)+g_0)\right.$$

$$+T\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds(\phi(\|x\|+\|y\|)+f_0)$$

$$+|\eta\zeta|\int_0^{\mu}\frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds(\phi(\|x\|+\|y\|)+f_0)$$

$$\left.+|\eta|\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\,(\psi(\|x\|+\|y\|)+g_0)\right)$$

$$+\sup_{0\leq t\leq T}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds(\phi(\|x\|+\|y\|)+f_0),$$

$$\leq (\phi(\|x\|+\|y\|)+f_0)\left[\frac{T}{|\Delta|}\left(\frac{T^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]$$

$$+(\psi(\|x\|+\|y\|)+g_0)\left[\frac{T}{|\Delta|}\left(\frac{|\eta|T\rho^{\beta-1}}{\Gamma(\beta)}+\frac{|\eta|T^{\beta}}{\Gamma(\beta+1)}\right)\right],$$

$$\leq (\phi(\|x\|+\|y\|)+f_0)Q_1+(\psi(\|x\|+\|y\|)+g_0)Q_2,$$

$$\leq (\phi\varepsilon+f_0)Q_1+(\psi\varepsilon+g_0)Q_2.$$

In a like manner

$$|G_2(x,y)(t)| \leq (\phi(\|x\|+\|y\|)+f_0)Q_3+(\psi(\|x\|+\|y\|)+g_0)Q_4$$

$$\leq (\phi\varepsilon+f_0)Q_3+(\psi\varepsilon+g_0)Q_4.$$

Hence

$$\|G_1(x,y)\| \leq (\phi\varepsilon+f_0)Q_1+(\psi\varepsilon+g_0)Q_2,$$

and

$$\|G_2(x,y)\| \leq (\phi\varepsilon+f_0)Q_3+(\psi\varepsilon+g_0)Q_4.$$

Consequently,

$$\|G(x,y)\| \leq (\phi\varepsilon+f_0)(Q_1+Q_3)+(\psi\varepsilon+g_0)(Q_2+Q_4) \leq \varepsilon\,.$$

So we get $\|G(x,y)\| \leq \varepsilon$ that is $G\Omega_{\varepsilon} \subseteq \Omega_{\varepsilon}$.

Now let $(x_1,y_1),(x_2,y_2) \in R\times S, \forall t\in[0,T]$ then we get

$$
\begin{aligned}
&|G_1(x_1,y_1)(t) - G_1(x_2,y_2)(t)| \\
&\quad \le \frac{T}{|\Delta|}\Bigg(|\eta| T \int_0^{\rho} \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)} ds \psi(\|x_2-x_1\| + \|y_2-y_1\|) \\
&\quad + T\int_0^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds \phi(\|x_2-x_1\| + \|y_2-y_1\|) \\
&\quad + |\eta\zeta| \int_0^{\mu} \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \phi(\|x_2-x_1\| + \|y_2-y_1\|) \\
&\quad + |\eta| \int_0^{T} \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} ds \psi(\|x_2-x_1\| + \|y_2-y_1\|) \Bigg) \\
&\quad + \sup_{0 \le t \le T} \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \phi(\|x_2-x_1\| \\
&\quad + \|y_2-y_1\|),
\end{aligned}
$$

$$
\begin{aligned}
&\|G_1(x_1,y_1) - G_1(x_2,y_2)\| \\
&\quad \le Q_1\phi(\|x_2-x_1\| + \|y_2-y_1\|) + Q_2\psi(\|x_2-x_1\| + \|y_2-y_1\|). \qquad (11)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\|G_2(x_1,y_1) - G_2(x_2,y_2)\| \\
&\quad \le Q_3\phi(\|x_2-x_1\| + \|y_2-y_1\|) + Q_4\psi(\|x_2-x_1\| + \|y_2-y_1\|). \qquad (12)
\end{aligned}
$$

From (11) and (12) we deduced that

$$
\|G(x_1,y_1) - G(x_2,y_2)\| \le \big(\phi(Q_1+Q_3) + \psi(Q_2+Q_4)\big)(\|x_2-x_1\| + \|y_2-y_1\|).
$$

Since $\phi(Q_1+Q_3) + \psi(Q_2+Q_4) < 1$, therefore, the operator G is a contraction operator. Hence, by Banach's fixed-point theorem, the operator G is has unique fixed point on, which is the unique solution of BVP (1) and (2) .This completes the proof.

The next result is based on the Leray-Schauder alternative.

Lemma 3.2.1 (Leray-Schauder alternative [45], p.4) Let $F\colon E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let

$E(F) = \{x \in E: x = \lambda F(x)\ for\ some\ 0 < \lambda < 1\}$. Then either the set $E(F)$ is unbounded or F has at least one fixed point).

Theorem 3.2.2 Assume $f, g: C([0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous function and there exist $\theta_1, \theta_2, \lambda_1, \lambda_2 \geq 0$ where $\theta_1, \theta_2, \lambda_1, \lambda_2$ are real constants and $\theta_0, \lambda_0 > 0$ such that $\forall x_i, y_i \in \mathbb{R}, (i = 1,2)$,we have

$$|f(t, x_1, x_2)| \leq \theta_0 + \theta_1 |x_1| + \theta_2 |x_2|,$$

$$|g(t, x_1, x_2)| \leq \lambda_0 + \lambda_1 |x_1| + \lambda_2 |x_2|,$$

If

$$(Q_1 + Q_3)\theta_1 + (Q_2 + Q_4)\lambda_1 < 1,$$

and

$$(Q_1 + Q_3)\theta_2 + (Q_2 + Q_4)\lambda_2 < 1,$$

where $Q_i, i = 1,2,3,4$ are defined in (10), then the problem (1) and (2) has at least one solution.

Proof. The proof will be divided into two steps

Step1: show that $G: R \times S \to R \times S$ is completely continuous .The continuity of the operator G holds by the continuity of the functions f, g .

Let $B \subseteq R \times S$ be a bounded. Then there exists positive constants k_1, k_2such that

$$\left|f\big(t, x(t), y(t)\big)\right| \leq k_1, \qquad \left|g\big(t, x(t), y(t)\big)\right| \leq k_2, \quad \forall t \in [0, T].$$

Then $\forall (x, y) \in B$, we have

$$|G_1(x, y)(t)| \leq Q_1 k_1 + Q_2 k_2,$$

Which implies that

$$\|G_1(x, y)\| \leq Q_1 k_1 + Q_2 k_2,$$

Similarly, we get

$$\|G_2(x, y)\| \leq Q_3 k_1 + Q_4 k_2,$$

Thus, from the above inequalities, it follows that the operator G is uniformly bounded, since

$$\|G(x,y)\| \leq (Q_1 + Q_3)k_1 + (Q_2 + Q_4)k_2.$$

Further, we show that the operator G is equicontinuous.Let $\omega_1, \omega_2 \in [0,T]$ with $\omega_1 < \omega_2$. This yields

$$
\begin{aligned}
&|G_1(x,y)(\omega_2) - G_1(x,y)(\omega_1)| \\
&\quad \leq \frac{\omega_2 - \omega_1}{|\Delta|}\Bigg(|\eta|T\int_0^\rho \frac{(\rho - s)^{\beta-2}}{\Gamma(\beta-1)}\big|g\big(s,x(s),y(s)\big)\big|ds \\
&\quad + T\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,x(s),y(s)\big)\big|ds \\
&\quad + |\eta\zeta|\int_0^\mu \frac{(\mu - s)^{\alpha-2}}{\Gamma(\alpha-1)}\big|f\big(s,x(s),y(s)\big)\big|ds \\
&\quad + |\eta|\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\big|g\big(s,x(s),y(s)\big)\big|ds\Bigg) \\
&\quad + \Bigg|\int_0^{\omega_2} \frac{(\omega_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds \\
&\quad - \int_0^{\omega_1} \frac{(\omega_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)ds\Bigg| \\
&\leq \frac{\omega_2 - \omega_1}{|\Delta|}\Bigg(|\eta|Tk_2\int_0^\rho \frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds + Tk_1\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + |\eta\zeta|k_1\int_0^\mu \frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + |\eta|k_2\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg) \\
&\quad + \Bigg|\int_0^{\omega_1}\left(\frac{(\omega_2 - s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\omega_1 - s)^{\alpha-1}}{\Gamma(\alpha)}\right) f\big(s,x(s),y(s)\big)\,ds\Bigg| \\
&\quad + \Bigg|\int_{\omega_1}^{\omega_2} \frac{(\omega_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)ds\Bigg|, \\
&\leq \frac{\omega_2 - \omega_1}{|\Delta|}\left(\frac{k_2|\eta|T\rho^{\beta-1}}{\Gamma(\beta)} + \frac{k_1T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{k_1|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)} + \frac{k_2|\eta|T^\beta}{\Gamma(\beta+1)}\right) \\
&\quad + \frac{k_1}{\Gamma(\alpha)}\left(\int_0^{\omega_1}((\omega_2 - s)^{\alpha-1} - (\omega_1 - s)^{\alpha-1})\,ds + \int_{\omega_1}^{\omega_2}(\omega_2 - s)^{\alpha-1}\,ds\right).
\end{aligned}
$$

And we obtain

$$|G_1(x,y)(\omega_2) - G_1(x,y)(\omega_1)|$$

$$\leq \frac{\omega_2 - \omega_1}{|\Delta|}\left(\frac{k_2|\eta|T\rho^{\beta-1}}{\Gamma(\beta)} + \frac{k_1 T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{k_1|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)} + \frac{k_2|\eta|T^{\beta}}{\Gamma(\beta+1)}\right)$$

$$+ \frac{k_1}{\Gamma(\alpha+1)}[{\omega_2}^{\alpha} - {\omega_1}^{\alpha}].$$

Hence we have $\|G_1(x,y)(\omega_2) - G_1(x,y)(\omega_1)\| \to 0$ independent of x and y as $\omega_2 \to \omega_1$. Furthermore, we obtain

$$|G_2(x,y)(\omega_2) - G_2(x,y)(\omega_1)|$$

$$\leq \frac{\omega_2 - \omega_1}{|\Delta|}\left(\frac{k_2|\eta\zeta|\rho^{\beta-1}}{\Gamma(\beta)} + \frac{k_1|\zeta|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{k_1 T|\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)} + \frac{k_2 T^{\beta+1}}{\Gamma(\beta+1)}\right)$$

$$+ \frac{k_2}{\Gamma(\beta+1)}\left[{\omega_2}^{\beta} - {\omega_1}^{\beta}\right],$$

which implies that $\|G_2(x,y)(\omega_2) - G_2(x,y)(\omega_1)\| \to 0$ independent of x and y as $\omega_2 \to \omega_1$.

Therefore, operator $G(x,y)$ is equicontinuous, and thus $G(x,y)$ is completely continuous.

Step 2: (Boundedness of operator)

Finally, show that $Z = \{(x,y) \in R \times S: (x,y) = hG(x,y), h \in [0,1]\}$ is bounded. Let $(x,y) \in \mathbb{R}$, with $(x,y) = hG(x,y)$ for any $t \in [0,T]$, we have

$$x(t) = hG_1(x,y)(t), \qquad y(t) = hG_2(x,y)(t).$$

Then

$$|x(t)| \leq Q_1(\theta_0 + \theta_1|x| + \theta_2|y|) + Q_2(\lambda_0 + \lambda_1|x| + \lambda_2|y|),$$

and

$$|y(t)| \leq Q_3(\theta_0 + \theta_1|x| + \theta_2|y|) + Q_4(\lambda_0 + \lambda_1|x| + \lambda_2|y|).$$

So we get

$$\|x\| \leq Q_1(\theta_0 + \theta_1\|x\| + \theta_2\|y\|) + Q_2(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|),$$

and

$$\|y\| \leq Q_3(\theta_0 + \theta_1\|x\| + \theta_2\|y\|) + Q_4(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|),$$

which imply that

$$\|x\| + \|y\| \leq (Q_1 + Q_3)\theta_0 + (Q_2 + Q_4)\lambda_0 + \big((Q_1 + Q_3)\theta_1 + (Q_2 + Q_4)\lambda_1\big)\|x\|$$
$$+ \big((Q_1 + Q_3)\theta_2 + (Q_2 + Q_4)\lambda_2\big)\|y\|.$$

Therefore,

$$\|(x, y)\| \leq \frac{(Q_1 + Q_3)\theta_0 + (Q_2 + Q_4)\lambda_0}{Q_0},$$

where $Q_0 = min\{1 - (Q_1 + Q_3)\theta_1 - (Q_2 + Q_4)\lambda_1, 1 - (Q_1 + Q_3)\theta_2 - (Q_2 + Q_4)\lambda_2\}$,

which proves that Z is bounded. By (Leray-Schauder theorem) the operator G has at least one fixed point. Therefore, the BVP (1) and (2) has at least one solution on $[0, T]$.The proof is complete. ■

3.3 Hyers-Ulam Stability

In this section, we will discuss the Hyers-Ulam stability of the solutions for the BVP (1) and (2) by means of integral representation of its solution given by

$$x(t) = G_1(x, y)(t), y(t) = G_2(x, y)(t),$$

where G_1 and G_2 are defined by (8) and (9).

Define the following nonlinear operators $N_1, N_2 \in C([0, \mathrm{T}], \mathbb{R}) \times C([0, \mathrm{T}], \mathbb{R}) \to C([0, \mathrm{T}], \mathbb{R})$;

$$^{c}D^{\alpha}x(t) - f\big(t, x(t), y(t)\big) = N_1(x, y)(t), \quad t \in [0, T],$$
$$^{c}D^{\beta}y(t) - g\big(t, x(t), y(t)\big) = N_2(x, y)(t), \quad t \in [0, T].$$

For some $\varepsilon_1, \varepsilon_2 > 0$, we consider the following inequality:

$$\|N_1(x, y)\| \leq \varepsilon_1, \quad \|N_2(x, y)\| \leq \varepsilon_2. \tag{13}$$

Definition 3.3.1 ([46,47]) The coupled system (1) and (2) is said to be Hyers-Ulam stable, if there exist $M_1, M_2 > 0$, such that for every solution$(x^*, y^*) \in C([0, T], \mathbb{R}) \times$

$C([0,T],\mathbb{R})$ of the inequality (13),there exists a unique solution $(x,y)\in C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ of problems (1) and (2) with

$$\|(x,y)-(x^*,y^*)\|\leq M_1\varepsilon_1+M_2\varepsilon_2.$$

Theorem 3.3.1 Let the assumptions of Theorem 3.2.1 hold. Then the BVP (1) and (2) is Hyers-Ulam stable.

Proof. Let $(x,y)\in C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ be the solution of the problems (1) and (2) satisfying (8) and (9). Let (x^*,y^*) be any solution satisfying (13):

$$\begin{aligned}{}^cD^\alpha x^*(t)&=f\big(t,x^*(t),y^*(t)\big)+N_1(x^*,y^*)(t),\quad t\in[0,T],\\ {}^cD^\beta y^*(t)&=g\big(t,x^*(t),y^*(t)\big)+N_2(x^*,y^*)(t),\quad t\in[0,T].\end{aligned}$$

So

$$\begin{aligned}x^*(t)&=G_1(x^*,y^*)(t)\\&\quad+\frac{t}{\Delta}\Bigg(\eta T\int_0^\rho\frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}N_2(x^*,y^*)(s)ds\\&\quad-T\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}N_1(x^*,y^*)(s)ds+\eta\zeta\int_0^\mu\frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}N_1(x^*,y^*)(s)ds\\&\quad-\eta\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}N_2(x^*,y^*)(s)ds\Bigg)\\&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}N_1(x^*,y^*)(s)ds,\end{aligned}$$

It follows that

$$\begin{aligned}&|G_1(x^*,y^*)(t)-x^*(t)|\\&\leq\frac{T}{|\Delta|}\Bigg(|\eta|T\int_0^\rho\frac{(\rho-s)^{\beta-2}}{\Gamma(\beta-1)}ds\varepsilon_2+T\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\,\varepsilon_1\\&\quad+|\eta\zeta|\int_0^\mu\frac{(\mu-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\varepsilon_1+|\eta|\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\,\varepsilon_2\Bigg)\\&\quad+\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\varepsilon_1,\end{aligned}$$

$$\leq \left[\frac{T}{|\Delta|}\left(\frac{T^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{|\eta\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\varepsilon_1+\frac{T}{|\Delta|}\left(\frac{|\eta|T\rho^{\beta-1}}{\Gamma(\beta)}+\frac{|\eta|T^{\beta}}{\Gamma(\beta+1)}\right)\varepsilon_2,$$

$$\leq Q_1\varepsilon_1+Q_2\varepsilon_{2.}$$

Similarly,

$$|G_2(x^*,y^*)(t)-y^*(t)|$$

$$\leq \frac{T}{|\Delta|}\left(\frac{|\zeta|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T|\zeta|\mu^{\alpha-1}}{\Gamma(\alpha)}\right)\varepsilon_1$$

$$+\left[\frac{T}{|\Delta|}\left(\frac{|\eta\zeta|\rho^{\beta-1}}{\Gamma(\beta)}+\frac{T^{\beta+1}}{\Gamma(\beta+1)}\right)+\frac{T^{\beta}}{\Gamma(\beta+1)}\right]\varepsilon_{2.}\leq Q_3\varepsilon_1+Q_4\varepsilon_2,$$

where $Q_i, i=1,2,3,4$ are defined in (10).

Therefore, we deduce by the fixed-point property of operator G, that is given by (8) and (9), which

$$|x(t)-x^*(t)|=|x(t)-G_1(x^*,y^*)(t)+G_1(x^*,y^*)(t)-x^*(t)|$$

$$\leq |G_1(x,y)(t)-G_1(x^*,y^*)(t)|+|G_1(x^*,y^*)(t)-x^*(t)|$$

$$\leq (Q_1\phi+Q_2\psi)\|(x,y)-(x^*,y^*)\|+Q_1\varepsilon_1+Q_2\varepsilon_{2,} \tag{14}$$

and similarly

$$|y(t)-y^*(t)|=|y(t)-G_2(x^*,y^*)(t)+G_2(x^*,y^*)(t)-y^*(t)|$$

$$\leq |G_2(x,y)(t)-G_2(x^*,y^*)(t)|+|G_2(x^*,y^*)(t)-y^*(t)|$$

$$\leq (Q_3\phi+Q_4\psi)\|(x,y)-(x^*,y^*)\|+Q_3\varepsilon_1+Q_4\varepsilon_{2,} \tag{15}$$

From (14) and (15) it follows that

$$\|(x,y)-(x^*,y^*)\|$$

$$\leq (Q_1\phi+Q_2\psi+Q_3\phi+Q_4\psi)\|(x,y)-(x^*,y^*)\|+(Q_1+Q_3)\varepsilon_1$$

$$+(Q_2+Q_4)\varepsilon_2,$$

$$\|(x,y)-(x^*,y^*)\|\leq \frac{(Q_1+Q_3)\varepsilon_1+(Q_2+Q_4)\varepsilon_2}{1-\big((Q_1+Q_3)\phi+(Q_2+Q_4)\psi\big)},$$

$$\leq M_1\varepsilon_1+M_2\varepsilon_2.$$

with

$$M_1 = \frac{(Q_1 + Q_3)}{1 - \left((Q_1 + Q_3)\phi + (Q_2 + Q_4)\psi\right)},$$

$$M_2 = \frac{(Q_2 + Q_4)}{1 - \left((Q_1 + Q_3)\phi + (Q_2 + Q_4)\psi\right)}.$$

Thus sufficient conditions for the Hyers-Ulam stability of the solutions is obtained.

■

3.4 Examples

Example 3.4.1 consider the following system of fractional differential equation

$$\begin{cases} {}^{c}D^{3/2}x(t) = \dfrac{1}{6\pi\sqrt{81 + t^2}}\left(\dfrac{|x(t)|}{3 + |x(t)|} + \dfrac{|y(t)|}{5 + |x(t)|}\right), \\ {}^{c}D^{7/4}y(t) = \dfrac{1}{12\pi\sqrt{64 + t^2}}\left(sin\big(x(t)\big) + sin\big(y(t)\big)\right), \\ \displaystyle\int_0^1 x'(s)ds = 2y'(1), \int_0^1 y'^{(s)}ds = -x'(1/2), \\ x(0) = 0, \qquad y(0) = 0, \end{cases} \tag{16}$$

$\alpha = \frac{3}{2}, \beta = \frac{7}{4}, T = 1, \eta = 2, \zeta = -1, \mu = \frac{1}{2}, \rho = 1.$

Using the given data , we find that $\Delta = 3, Q_1 = 1.269, Q_2 = 1.1398, Q_3 = 0.5167, Q_4 = 1.554, \phi = \frac{1}{54\pi}, \psi = \frac{1}{48\pi}$.

It's clear that

$$f(t, x, y) = \frac{1}{6\pi\sqrt{81 + t^2}}\left(\frac{|x(t)|}{3 + |x(t)|} + \frac{|y(t)|}{5 + |x(t)|}\right),$$

and

$$g(t, x, y) = \frac{1}{12\pi\sqrt{64 + t^2}}\left(sin\big(x(t)\big) + sin\big(y(t)\big)\right),$$

are jointly continuous functions and $\phi(Q_1 + Q_3) + \psi(Q_2 + Q_4) < 1$,

such that

$$\frac{1}{54\pi}(1.269 + 0.5167) + \frac{1}{48\pi}(1.1398 + 1.554) = 0.0283 < 1,$$

Thus all the conditions of Theorem 3.2.1 are satisfied, then problem (16) has a unique solution on [0,1].

Example 4.4.2 consider the following system of fractional differential equation

$$\begin{cases} {}^{c}D^{5/3}x(t) = \dfrac{1}{80+t^4} + \dfrac{|x(t)|}{120\left(1+y^2(t)\right)} + \dfrac{1}{4\sqrt{2500+t^2}}e^{-3t}\cos\left(y(t)\right), t \in [0,1] \\ {}^{c}D^{6/5}y(t) = \dfrac{1}{\sqrt{16+t^2}}\cos t + \dfrac{1}{150}e^{-3t}sin\left(y(t)\right) + \dfrac{1}{180}x(t), t \in [0,1] \\ \displaystyle\int_0^1 x'(s)ds = -3y'(1/3), \int_0^1 y'(s)ds = x'(1), \\ \qquad x(0) = 0, \qquad y(0) = 0, \end{cases} \tag{17}$$

$\alpha = \dfrac{5}{3}, \beta = \dfrac{6}{5}, T = 1, \eta = -3, \zeta = 1, \mu = 1, \rho = 1/3.$

Using the given data, we find that $\Delta = 3, Q_1 = 1.269, Q_2 = 1.1398, Q_3 = 0.5167, Q_4 = 1.554, \phi = \frac{1}{54\pi}, \psi = \frac{1}{48\pi}$.

It is clear that

$$|f(t,x,y)| \leq \frac{1}{80} + \frac{1}{120}|x| + \frac{1}{200}|y|,$$

$$|g(t,x,y)| \leq \frac{1}{4} + \frac{1}{180}|x| + \frac{1}{150}|y|.$$

Thus $\theta_0 = \frac{1}{80}, \theta_1 = \frac{1}{120}, \theta_2 = \frac{1}{200}, \lambda_0 = \frac{1}{4}, \lambda_1 = \frac{1}{180}, \lambda_2 = \frac{1}{150}$.

We found $(Q_1 + Q_3)\theta_1 + (Q_2 + Q_4)\lambda_1 = 0.0298 < 1$ and $(Q_1 + Q_3)\theta_2 + (Q_2 + Q_4)\lambda_2 = 0.0269 < 1$, then by Theorem 3.2.2 the problem (17) has at least one solution on $[0,1]$.

3.5 Conclusion

A coupled system of fractional differential equations with boundary conditions was investigated in this chapter. The existence and uniqueness of solutions for the given problem were discussed and proven by applying the contraction mapping principle and Leray-Schauder's alternative theorem. Furthermore, the stability of obtained solutions was discussed using the Hyers-Ulam method and sufficient conditions for the Hyers-Ulam stability were driven. Results of the study were successfully supported by numerical examples that were presented in the last section.

Chapter 4

EXISTENCE AND STABILITY OF COUPLED SEQUENTIAL FRACTIONAL DIFFERENTIAL SYSTEM BOUNDARY CONDITIONS

This chapter is concerned with the existence and uniqueness of solutions for a coupled system of Caputo type sequential fractional differential equations supplemented with integral boundary conditions. The existence of solutions is derived by applying Leray–Schauder's alternative, while the uniqueness of solution is established via Banach's contraction principle. Moreover, some necessary conditions for the Hyers-Ulam type stability to the solutions of the boundary value problem (BVPs) are developed. Finally the results are supported by example.

We study the following coupled system of Caputo type sequential fractional differential

$$\begin{cases} {}^{c}D^{\alpha-1}(D+k)x(t) = f\big(t, x(t), y(t)\big), & t \in [0,T], \quad 1 < \alpha \le 2, \qquad k > 0, \\ {}^{c}D^{\beta-1}(D+k)y(t) = g\big(t, x(t), y(t)\big), & t \in [0,T], \quad 1 < \beta \le 2, \qquad k > 0, \end{cases} \tag{1}$$

supplemented with integral boundary conditions of the form:

$$\begin{cases} \displaystyle\int_0^T x(s)ds = \rho_1 y(\zeta_1), \qquad \int_0^T x'(s)ds = \rho_2 y'(\zeta_2), \\ \displaystyle\int_0^T y(s)ds = \mu_1 x(\eta_1), \qquad \int_0^T y'(s)ds = \mu_2 x'(\eta_2), \quad \eta_1, \eta_2, \zeta_1, \zeta_2 \in [0,T], \end{cases} \tag{2}$$

where ${}^{c}D^{k}$ denote the Caputo fractional derivatives of order $k, k = \alpha, \beta,$ and $f, g: [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, are given continuous functions, and $\rho_1, \rho_2, \mu_1, \mu_2$ are real constants .

4.1 Preliminaries

Definition 4.1.1 Due to Miller-Ross [2] the sequential fractional derivative for a sufficiently smooth function $h(t)$ is defined as

$$D^{\sigma}h(t) = D^{\sigma_1}D^{\sigma_2} \dots D^{\sigma_n}h(t), \qquad (*)$$

where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$is a multi-index.

The operator D^{σ} in $(*)$ can either be Caputo or Riemann–Liouville or any other kind of integro-differential operator. For example

$${}^{c}D^{\beta}h(s) = D^{-(m-\beta)}\frac{d^m}{ds^m}h(s), \quad m-1 < \beta < m,$$

where $D^{-(m-\beta)}$ is the fractional integral operator of order $(m-\beta)$.On emphasizing that $D^{-q}h(s) = I^{q}h(s),\ q = m - \beta$; for more details, see [1](p. 87).

We prove the following auxiliary lemma to find the solution for the problem (1) and (2).

Lemma 4.1.1 Let $\psi, \phi \in C([0, T], \mathbb{R})$ then the unique solution of the problem:

$$\begin{cases} {}^{c}D^{\alpha-1}(D+k)x(t) = \psi(t), \quad 1 < \alpha \le 2, \\ {}^{c}D^{\beta-1}(D+k)y(t) = \phi(t), \quad 1 < \beta \le 2, \\ \int_0^T x(s)ds = \rho_1 y(\zeta_1), \qquad \int_0^T x'(s)ds = \rho_2 y'(\zeta_2) \\ \int_0^T y(s)ds = \mu_1 x(\eta_1), \qquad \int_0^T y'(s)ds = \mu_2 x'(\eta_2), \quad k > 0, t \in [0, T], \end{cases} \qquad (3)$$

is

$$
\begin{aligned}
x(t) = \Delta e^{-kt}\bigg[&\mu_2(I^{\alpha-1}\psi)(\eta_2) - k\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}\psi)(s)ds \\
&- \int_0^T \big(I^{\beta-1}\phi\big)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,\big(I^{\beta-1}\phi\big)(s)dsdx\bigg] \\
&+ \lambda e^{-kt}\bigg[\rho_2\big(I^{\beta-1}\phi\big)(\zeta_2) - k\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\,\big(I^{\beta-1}\phi\big)(s)ds \\
&- \int_0^T (I^{\alpha-1}\psi)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}\psi)(s)dsdx\bigg] \\
&+ \frac{1}{T^2-\mu_1\rho_1}\bigg[\frac{A\mu_2}{k}(I^{\alpha-1}\psi)(\eta_2) - A\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}\psi)(s)ds \\
&- \frac{A}{k}\int_0^T \big(I^{\beta-1}\phi\big)(s)ds + (A-\rho_1)\int_0^T\int_0^x e^{-k(x-s)}\,\big(I^{\beta-1}\phi\big)(s)dsdx \\
&+ \frac{B\rho_2}{k}\big(I^{\beta-1}\phi\big)(\zeta_2) - B\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\,\big(I^{\beta-1}\phi\big)(s)ds \\
&- \frac{B}{k}\int_0^T (I^{\alpha-1}\psi)(s)ds + (B-T)\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}\psi)(s)dsdx \\
&+ T\rho_1\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\,\big(I^{\beta-1}\phi\big)(s)ds + \mu_1\rho_1\int_0^{\eta_1} e^{-k(\eta_1-s)}\,(I^{\alpha-1}\psi)(s)ds\bigg] \\
&+ \int_0^t e^{-k(t-s)}\,(I^{\alpha-1}\psi)(s)ds, \qquad (4)
\end{aligned}
$$

and

$$y(t) = \theta e^{-kt}\left[\mu_2(I^{\alpha-1}\psi)(\eta_2) - k\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}\psi)(s)ds\right.$$

$$\left. - \int_0^T \left(I^{\beta-1}\phi\right)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}\phi\right)(s)dsdx\right]$$

$$+ \tau e^{-kt}\left[\rho_2\left(I^{\beta-1}\phi\right)(\zeta_2) - k\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}\phi\right)(s)ds\right.$$

$$\left. - \int_0^T (I^{\alpha-1}\psi)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}\psi)(s)dsdx\right]$$

$$+ \frac{1}{\omega}\left[\frac{C\mu_2}{k}(I^{\alpha-1}\psi)(\eta_2) - C\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}\psi)(s)ds\right.$$

$$- \frac{C}{k}\int_0^T \left(I^{\beta-1}\phi\right)(s)ds + (C-T)\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}\phi\right)(s)dsdx$$

$$+ \frac{D\rho_2}{k}\left(I^{\beta-1}\phi\right)(\zeta_2) - D\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}\phi\right)(s)ds$$

$$- \frac{D}{k}\int_0^T (I^{\alpha-1}\psi)(s)ds + (D-\mu_1)\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}\psi)(s)dsdx$$

$$\left. + \mu_1\rho_1\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\left(I^{\beta-1}\phi\right)(s)ds + T\mu_1\int_0^{\eta_1} e^{-k(\eta_1-s)}\,(I^{\alpha-1}\psi)(s)ds\right]$$

$$+ \int_0^t e^{-k(t-s)}\left(I^{\beta-1}\phi\right)(s)ds, \qquad (5)$$

where

$$\omega = T^2 - \mu_1\rho_1 \neq 0, \qquad \Delta = \frac{-k\rho_2 e^{-k\zeta_2}}{M}, \qquad \lambda = \frac{e^{-kT}-1}{M}, \qquad \sigma = \Delta k\mu_2 e^{-k\eta_2} - 1$$

$$M = \left(e^{-kT}-1\right)^2 - k^2\mu_2\rho_2 e^{-k(\zeta_2+\eta_2)} \neq 0, \qquad \theta = \frac{-\sigma}{e^{-kT}-1}, \qquad \tau = \frac{\lambda}{e^{-kT}-1},$$

$$A = \left[\theta kT\rho_1 e^{-k\zeta_1} + k\mu_1\rho_1\Delta e^{-k\eta_1} + \left(e^{-kT}-1\right)(\Delta T + \theta\rho_1)\right],$$

$$B = \left[\tau kT\rho_1 e^{-k\zeta_1} + k\mu_1\rho_1\lambda e^{-k\eta_1} + \left(e^{-kT}-1\right)(\lambda T + \tau\rho_1)\right],$$

$$C = \left[\Delta kT\mu_1 e^{-k\eta_1} + k\mu_1\rho_1\theta e^{-k\zeta_1} + \left(e^{-kT}-1\right)(\theta T + \Delta\mu_1)\right],$$

$$D = \left[\lambda kT\mu_1 e^{-k\eta_1} + k\mu_1\rho_1\tau e^{-k\zeta_1} + \left(e^{-kT}-1\right)(\tau T + \lambda\mu_1)\right],$$

Proof. The general solutions of the sequential fractional differential equations [41] in (3) is known as

$$x(t) = c_0 e^{-kt} + c_1 + \int_0^t e^{-k(t-s)} \, (I^{\alpha-1}\psi)(s)ds, \qquad (6)$$

$$y(t) = d_0 e^{-kt} + d_1 + \int_0^t e^{-k(t-s)} \, \big(I^{\beta-1}\phi\big)(s)ds, \qquad (7)$$

observe

$$x'(t) = -kc_0 e^{-kt} + (I^{\alpha-1}\psi)(t) - k\int_0^t e^{-k(t-s)} \, (I^{\alpha-1}\psi)(s)ds,$$

$$y'(t) = -kd_0 e^{-kt} + \big(I^{\beta-1}\phi\big)(t) - k\int_0^t e^{-k(t-s)} \, \big(I^{\beta-1}\phi\big)(s)ds,$$

where $c_i, d_i \in \mathbb{R}, i = 0,1$ are arbitrary constants.

Apply the conditions

$$\int_0^T x'(s)ds = \rho_2 y'(\zeta_2), \int_0^T y'(s)ds = \mu_2 x'(\eta_2)\,,$$

then we obtain:

$$\begin{aligned} c_0 = \Delta\Bigg[&\mu_2(I^{\alpha-1}\psi)(\eta_2) - k\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)} \, (I^{\alpha-1}\psi)(s)ds \\ &- \int_0^T \big(I^{\beta-1}\phi\big)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)} \, \big(I^{\beta-1}\phi\big)(s)dsdx\Bigg] \\ &+ \lambda\Bigg[\rho_2\big(I^{\beta-1}\phi\big)(\zeta_2) - k\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)} \, \big(I^{\beta-1}\phi\big)(s)ds \\ &- \int_0^T (I^{\alpha-1}\psi)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)} \, (I^{\alpha-1}\psi)(s)dsdx\Bigg], \end{aligned}$$

and

$$\begin{aligned} d_0 = \theta\Bigg[&\mu_2(I^{\alpha-1}\psi)(\eta_2) - k\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)} \, (I^{\alpha-1}\psi)(s)ds \\ &- \int_0^T \big(I^{\beta-1}\phi\big)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)} \, \big(I^{\beta-1}\phi\big)(s)dsdx\Bigg] \\ &+ \tau\Bigg[\rho_2\big(I^{\beta-1}\phi\big)(\zeta_2) - k\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)} \, \big(I^{\beta-1}\phi\big)(s)ds \\ &- \int_0^T (I^{\alpha-1}\psi)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)} \, (I^{\alpha-1}\psi)(s)dsdx\Bigg] \end{aligned}$$

In view of the conditions $\int_0^T x(s)ds = \rho_1 y(\zeta_1), \int_0^T y(s)ds = \mu_1 x(\eta_1)$,

we get

$$c_1 = \frac{1}{\omega}\left[\frac{A\mu_2}{k}(I^{\alpha-1}\psi)(\eta_2) - A\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}(I^{\alpha-1}\psi)(s)ds - \frac{A}{k}\int_0^T \left(I^{\beta-1}\phi\right)(s)ds\right.$$
$$+ (A-\rho_1)\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}\phi\right)(s)dsdx + \frac{B\rho_2}{k}\left(I^{\beta-1}\phi\right)(\zeta_2)$$
$$- B\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}\phi\right)(s)ds - \frac{B}{k}\int_0^T (I^{\alpha-1}\psi)(s)ds$$
$$+ (B-T)\int_0^T\int_0^x e^{-k(x-s)}(I^{\alpha-1}\psi)(s)dsdx$$
$$\left.+ T\rho_1\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\left(I^{\beta-1}\phi\right)(s)ds + \mu_1\rho_1\int_0^{\eta_1} e^{-k(\eta_1-s)}(I^{\alpha-1}\psi)(s)ds\right],$$

and

$$d_1 = \frac{1}{\omega}\left[\frac{C\mu_2}{k}(I^{\alpha-1}\psi)(\eta_2) - C\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}(I^{\alpha-1}\psi)(s)ds - \frac{C}{k}\int_0^T \left(I^{\beta-1}\phi\right)(s)ds\right.$$
$$+ (C-T)\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}\phi\right)(s)dsdx + \frac{D\rho_2}{k}\left(I^{\beta-1}\phi\right)(\zeta_2)$$
$$- D\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}\phi\right)(s)ds - \frac{D}{k}\int_0^T (I^{\alpha-1}\psi)(s)ds$$
$$+ (D-\mu_1)\int_0^T\int_0^x e^{-k(x-s)}(I^{\alpha-1}\psi)(s)dsdx$$
$$\left.+ \mu_1\rho_1\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\left(I^{\beta-1}\phi\right)(s)ds + T\mu_1\int_0^{\eta_1} e^{-k(\eta_1-s)}(I^{\alpha-1}\psi)(s)ds\right]$$

Substituting the values of c_0, c_1, d_0, d_1 in $(6), (7)$ we obtain (4) and (5), which is complete the proof. ■

4.2 Existence of Results

Let the space $Z = \{x(t) | x(t) \in C[0,T]\}$ endowed with the norm $\|x\| = max\{|x(t)|, t \in [0,T]\}$. It is clear that $(Z, \|.\|)$ is a Banach space. Also let $S = \{y(t) | y(t) \in C[0,T]\}$ endowed with the norm $\|y\| = max\{|y(t)|, t \in [0,T]\}$.The product space $(Z \times S, \|(x,y)\|)$ is also Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of lemma 4.1.1 we define the operator $Q: Z \times S \to Z \times S$ by

$$Q(x,y)(t) = \big(Q_1(x,y)(t), Q_2(x,y)(t)\big),$$

where

$$\begin{aligned}
Q_1(x,y)(t) = \Delta e^{-kt} \bigg[& \mu_2 (I^{\alpha-1} f)(\eta_2) - k\mu_2 \int_0^{\eta_2} e^{-k(\eta_2 - s)} (I^{\alpha-1} f)(s) ds \\
& - \int_0^T \big(I^{\beta-1} g\big)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} \big(I^{\beta-1} g\big)(s) ds dx \bigg] \\
& + \lambda e^{-kt} \bigg[\rho_2 \big(I^{\beta-1} g\big)(\zeta_2) - k\rho_2 \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} \big(I^{\beta-1} g\big)(s) ds \\
& - \int_0^T (I^{\alpha-1} f)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} (I^{\alpha-1} f)(s) ds dx \bigg] \\
& + \frac{1}{\omega} \bigg[\frac{A\mu_2}{k} (I^{\alpha-1} f)(\eta_2) - A\mu_2 \int_0^{\eta_2} e^{-k(\eta_2 - s)} (I^{\alpha-1} f)(s) ds \\
& - \frac{A}{k} \int_0^T \big(I^{\beta-1} g\big)(s) ds + (A - \rho_1) \int_0^T \int_0^x e^{-k(x-s)} \big(I^{\beta-1} g\big)(s) ds dx \\
& + \frac{B\rho_2}{k} \big(I^{\beta-1} g\big)(\zeta_2) - B\rho_2 \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} \big(I^{\beta-1} g\big)(s) ds \\
& - \frac{B}{k} \int_0^T (I^{\alpha-1} f)(s) ds + (B - T) \int_0^T \int_0^x e^{-k(x-s)} (I^{\alpha-1} f)(s) ds dx \\
& + T\rho_1 \int_0^{\zeta_1} e^{-k(\zeta_1 - s)} \big(I^{\beta-1} g\big)(s) ds + \mu_1 \rho_1 \int_0^{\eta_1} e^{-k(\eta_1 - s)} (I^{\alpha-1} f)(s) ds \bigg] \\
& + \int_0^t e^{-k(t-s)} (I^{\alpha-1} f)(s) ds,
\end{aligned}$$

and

$$Q_2(x,y)(t) = \theta e^{-kt}\Bigg[\mu_2(I^{\alpha-1}f)(\eta_2) - k\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}f)(s)ds$$

$$-\int_0^T \left(I^{\beta-1}g\right)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}g\right)(s)dsdx\Bigg]$$

$$+\tau e^{-kt}\Bigg[\rho_2\left(I^{\beta-1}g\right)(\zeta_2) - k\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}g\right)(s)ds$$

$$-\int_0^T (I^{\alpha-1}f)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}f)(s)dsdx\Bigg]$$

$$+\frac{1}{\omega}\Bigg[\frac{C\mu_2}{k}(I^{\alpha-1}f)(\eta_2) - C\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}f)(s)ds$$

$$-\frac{C}{k}\int_0^T\left(I^{\beta-1}g\right)(s)ds + (C-T)\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}g\right)(s)dsdx$$

$$+\frac{D\rho_2}{k}\left(I^{\beta-1}g\right)(\zeta_2) - D\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}g\right)(s)ds$$

$$-\frac{D}{k}\int_0^T(I^{\alpha-1}f)(s)ds + (D-\mu_1)\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}f)(s)dsdx$$

$$+\mu_1\rho_1\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\left(I^{\beta-1}g\right)(s)ds + T\mu_1\int_0^{\eta_1} e^{-k(\eta_1-s)}\,(I^{\alpha-1}f)(s)ds\Bigg]$$

$$+\int_0^t e^{-k(t-s)}\left(I^{\beta-1}g\right)(s)ds,$$

Theorem 4.2.1 Assume $f, g\colon C([0,T]\times\mathbb{R}^2)\to\mathbb{R}$ are jointly continuous functions and there exist constants $h_f, h_g \in \mathbb{R}$, such that $\forall\, x_1, x_2, y_1, y_2 \in \mathbb{R}, \forall t\in[0,T]$ we have

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le h_f(|x_2-x_1| + |y_2-y_1|),$$

$$|g(t,x_1,x_2) - f(t,y_1,y_2)| \le h_g(|x_2-x_1| + |y_2-y_1|).$$

If

$$h_f(N_1+N_3) + h_g(N_2+N_4) < 1,$$

then the boundary value problem (1),(2) has a unique solution on $[0,T]$, where

$$N_1 = \left[\frac{(\alpha+1)|\Delta\mu_2|e^{-kT}{\eta_2}^{\alpha}(\alpha{\eta_2}^{-1}+k)+|\lambda|e^{-kT}T^{\alpha}(\alpha+1+kT)}{\Gamma(\alpha+2)}\right]+\frac{T^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\ \left[\frac{|A\mu_2|{\eta_2}^{\alpha}(\alpha+1)(\alpha{\eta_2}^{-1}+k)+T^{\alpha}(|B|(\alpha+1)+kT|B-T|)+k(\alpha+1)|\mu_1\rho_1|{\eta_1}^{\alpha}}{k\Gamma(\alpha+2)|T^2-\mu_1\rho_1|}\right],$$

$$N_2 = \left[\frac{(\beta+1)T^{\beta}\big(|\Delta|e^{-kT}+kT\big)+|\lambda\rho_2|e^{-kT}{\zeta_2}^{\beta}(\beta+1)\big(\beta{\zeta_2}^{-1}+k\big)}{\Gamma(\beta+2)}\right]$$
$$+\ \left[\frac{T^{\beta-1}(|A|(\beta^2+\beta)+k|A-\rho_1|T^2)+(\beta+1)|B\rho_2|{\zeta_2}^{\beta}\big(\beta{\zeta_2}^{-1}+k\big)+k|T\rho_1|{\zeta_1}^{\beta}}{k\Gamma(\beta+2)|T^2-\mu_1\rho_1|}\right],$$

$$N_3 = \left[\frac{(\alpha+1)|\theta\mu_2|e^{-kT}{\eta_2}^{\alpha}(\alpha{\eta_2}^{-1}+k)+|\tau|e^{-kT}T^{\alpha}(\alpha+1+kT)}{\Gamma(\alpha+2)}\right]$$
$$+\ \left[\frac{|C\mu_2|{\eta_2}^{\alpha}(\alpha+1)(\alpha{\eta_2}^{-1}+k)+T^{\alpha}(|D|(\alpha+1)+kT|D-\mu_1|)+k(\alpha+1)|\mu_1T|{\eta_1}^{\alpha}}{k\Gamma(\alpha+2)|T^2-\mu_1\rho_1|}\right],$$

$$N_4 = \left[\frac{(\beta+1)T^{\beta}\big(|\theta|e^{-kT}+kT\big)+|\tau\rho_2|e^{-kT}{\zeta_2}^{\beta}(\beta+1)\big(\beta{\zeta_2}^{-1}+k\big)}{\Gamma(\beta+2)}\right]+\frac{T^{\beta}}{\Gamma(\beta+1)}$$
$$+\left[\frac{T^{\beta-1}(|C|(\beta^2+\beta)+k|C-T|T^2)+(\beta+1)|D\rho_2|{\zeta_2}^{\beta}\big(\beta{\zeta_2}^{-1}+k\big)+k|\mu_1\rho_1|{\zeta_1}^{\beta}}{k\Gamma(\beta+2)|T^2-\mu_1\rho_1|}\right],$$

Proof. Define $\sup_{0\le t\le T} f(t,0,0)=f_0<\infty$, $\sup_{0\le t\le T} g(t,0,0)=g_0<\infty$ and $\Omega_r=\{(x,y)\in Z\times S\colon \|(x,y)\|\le r\}$, and $r>0$, such that

$$r\ge\frac{(N_1+N_3)f_0+(N_2+N_4)g_0}{1-\big[h_f(N_1+N_3)+h_g(N_2+N_4)\big]}.$$

Firstly, show that $Q\Omega_r\subseteq\Omega_r$.

By our assumption, for $(x,y)\in\Omega_r, t\in[0,T]$, we have

$$\big|f\big(t,x(t),y(t)\big)\big|\le\big|f\big(t,x(t),y(t)\big)-f(t,0,0)\big|+|f(t,0,0)|,$$
$$\le h_f(|x(t)|+|y(t)|)+f_0\le h_f(\|x\|+\|y\|)+f_0,$$

$$\leq h_f r + f_0, \tag{8}$$

and

$$\left|g\big(t, x(t), y(t)\big)\right| \leq h_g(|x(t)| + |y(t)|) + g_0 \leq h_g(\|x\| + \|y\|) + g_0,$$

$$\leq h_g r + g_0, \tag{9}$$

which lead to

$$\begin{aligned}
|Q_1(x,y)(t)| \leq |\Delta| e^{-kT} \Bigg[& |\mu_2| (I^{\alpha-1}|f|)(\eta_2) + k|\mu_2| \int_0^{\eta_2} e^{-k(\eta_2 - s)} \, (I^{\alpha-1}|f|)(s) ds \\
& + \int_0^T \big(I^{\beta-1}|g|\big)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} \, \big(I^{\beta-1}|g|\big)(s) ds dx \Bigg] \\
& + |\lambda| e^{-kT} \Bigg[|\rho_2| \big(I^{\beta-1}|g|\big)(\zeta_2) + k|\rho_2| \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} \, \big(I^{\beta-1}|g|\big)(s) ds \\
& + \int_0^T (I^{\alpha-1}|f|)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} \, (I^{\alpha-1}|f|)(s) ds dx \Bigg] \\
& + \frac{1}{|\omega|} \Bigg[\frac{|A\mu_2|}{k} (I^{\alpha-1}|f|)(\eta_2) + |A\mu_2| \int_0^{\eta_2} e^{-k(\eta_2 - s)} \, (I^{\alpha-1}|f|)(s) ds \\
& + \frac{|A|}{k} \int_0^T \big(I^{\beta-1}|g|\big)(s) ds + |A - \rho_1| \int_0^T \int_0^x e^{-k(x-s)} \, \big(I^{\beta-1}|g|\big)(s) ds dx \\
& + \frac{|B\rho_2|}{k} \big(I^{\beta-1}|g|\big)(\zeta_2) + |B\rho_2| \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} \, \big(I^{\beta-1}|g|\big)(s) ds \\
& + \frac{|B|}{k} \int_0^T (I^{\alpha-1}|f|)(s) ds + |B - T| \int_0^T \int_0^x e^{-k(x-s)} \, (I^{\alpha-1}|f|)(s) ds dx \\
& + T|\rho_1| \int_0^{\zeta_1} e^{-k(\zeta_1 - s)} \, \big(I^{\beta-1}|g|\big)(s) ds \\
& + |\mu_1 \rho_1| \int_0^{\eta_1} e^{-k(\eta_1 - s)} \, (I^{\alpha-1}|f|)(s) ds \Bigg] \\
& + \sup_{0 \leq t \leq T} \int_0^t e^{-k(t-s)} \, (I^{\alpha-1}|f|)(s) ds.
\end{aligned}$$

Using (8) and (9) to get

$$|Q_1(x,y)(t)| \le \Bigg[|\Delta|e^{-kT}\mu_2\left((I^{\alpha-1}1)(\eta_2) + k\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}1)(s)ds\right)$$

$$+ |\lambda|e^{-kT}\left(\int_0^T (I^{\alpha-1}1)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}1)(s)dsdx\right)$$

$$+ \frac{1}{|\omega|}\Bigg(\frac{|A\mu_2|}{k}(I^{\alpha-1}1)(\eta_2) + |A\mu_2|\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}1)(s)ds$$

$$+ \frac{|B|}{k}\int_0^T (I^{\alpha-1}1)(s)ds + |B-T|\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}1)(s)dsdx$$

$$+ |\mu_1\rho_1|\int_0^{\eta_1} e^{-k(\eta_1-s)}\,(I^{\alpha-1}1)(s)ds\Bigg) + \int_0^T e^{-k(T-s)}\,(I^{\alpha-1}1)(s)ds\Bigg]\|f\|$$

$$+ \Bigg[|\Delta|e^{-kT}\left(\int_0^T \left(I^{\beta-1}1\right)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,\left(I^{\beta-1}1\right)(s)dsdx\right)$$

$$+ |\lambda|e^{-kT}\left(|\rho_2|\left(I^{\beta-1}1\right)(\zeta_2) + k|\rho_2|\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\,\left(I^{\beta-1}1\right)(s)ds\right)$$

$$+ \frac{1}{|\omega|}\Bigg(\frac{|A|}{k}\int_0^T \left(I^{\beta-1}1\right)(s)ds + |A-\rho_1|\int_0^T\int_0^x e^{-k(x-s)}\,\left(I^{\beta-1}1\right)(s)dsdx$$

$$+ \frac{|B\rho_2|}{k}\left(I^{\beta-1}1\right)(\zeta_2) + |B\rho_2|\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\,\left(I^{\beta-1}1\right)(s)ds\Bigg)\Bigg]\|g\|$$

$$\le \Bigg[\frac{(\alpha+1)|\Delta\mu_2|e^{-kT}{\eta_2}^{\alpha}(\alpha{\eta_2}^{-1}+k) + |\lambda|e^{-kT}T^{\alpha}(\alpha+1+kT)}{\Gamma(\alpha+2)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ \frac{|A\mu_2|{\eta_2}^{\alpha}(\alpha+1)(\alpha{\eta_2}^{-1}+k) + T^{\alpha}(|B|(\alpha+1) + kT|B-T|) + k(\alpha+1)|\mu_1\rho_1|{\eta_1}^{\alpha}}{k\Gamma(\alpha+2)|\omega|}\Bigg]\|f\|$$

$$+ \Bigg[\frac{(\beta+1)T^{\beta}\left(|\Delta|e^{-kT}+kT\right) + |\lambda\rho_2|e^{-kT}{\zeta_2}^{\beta}(\beta+1)\left(\beta{\zeta_2}^{-1}+k\right)}{\Gamma(\beta+2)}$$

$$+ \frac{T^{\beta-1}(|A|(\beta^2+\beta) + k|A-\rho_1|T^2) + (\beta+1)|B\rho_2|{\zeta_2}^{\beta}\left(\beta{\zeta_2}^{-1}+k\right) + k|T\rho_1|{\zeta_1}^{\beta}}{k\Gamma(\beta+2)|\omega|}\Bigg]\|g\|,$$

Hence, by (8) we have

$$\|Q_1(x,y)\| \le \left(h_f N_1 + h_g N_2\right)r + (N_1 f_0 + N_2 g_0) \le \frac{r}{2}\,, \qquad (10)$$

In similar way we get

$$\|Q_2(x,y)\| \le \left(h_f N_3 + h_g N_4\right)r + (N_3 f_0 + N_4 g_0) \le \frac{r}{2}\,, \qquad (11)$$

From (10) and (11) we obtain

$$\|Q(x,y)\| \leq r,$$

Now show that Q is a contraction.

Let$(x_1, y_1), (x_2, y_2) \in Z \times S, \forall t \in [0, T]$, then we get

$$\|Q_1(x_1, y_1) - Q_1(x_2, y_2)\|$$
$$\leq h_f N_1(\|x_1 - x_2\| + \|y_1 - y_2\|) + h_g N_2(\|x_1 - x_2\| + \|y_1 - y_2\|), \quad (12)$$

$$\|Q_2(x_1, y_1) - Q_2(x_2, y_2)\|$$
$$\leq h_f N_3(\|x_1 - x_2\| + \|y_1 - y_2\|) + h_g N_4(\|x_1 - x_2\| + \|y_1 - y_2\|). \quad (13)$$

From (12) and (13) we deduced that

$$\|Q(x_1, y_1) - Q(x_2, y_2)\| \leq \left(h_f(N_1 + N_3) + h_g(N_2 + N_4)\right)(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

Since $h_f(N_1 + N_3) + h_g(N_2 + N_4) < 1$, therefore, Q is a contraction operator. So, by Banach's fixed point theorem, the operator Q has a unique fixed point on$[0, T]$, which is the unique solution of the problem (1) and (2) ,which is complete the proof. ■

The second result is based on the Leray-Schauder alternative.

Theorem 4.2.2 Assume $f, g: C([0, T] \times \mathbb{R}^2) \to \mathbb{R}$ are continuous function and there exists a positive real constants $\delta_{\mathrm{i}}, \phi_{\mathrm{i}} (\mathrm{i} = 0,1,2)$ such that $\forall x_i \in \mathbb{R}, (i = 1,2)$we have

$$|f(t, x_1, x_2)| \leq \delta_0 + \delta_1 |x_1| + \delta_2 |x_2|,$$
$$|g(t, x_1, x_2)| \leq \phi_0 + \phi_1 |x_1| + \phi_2 |x_2|.$$

If

$$(Q_1 + Q_3)\delta_1 + (Q_2 + Q_4)\phi_1 < 1,$$

and

$$(Q_1 + Q_3)\delta_2 + (Q_2 + Q_4)\phi_2 < 1,$$

then the problem (1) and (2) has at least one solution.

Proof. The proof will be divided into several steps

Step1: Show that Q is completely continuous .The continuity of the operator holds true because of continuity of the function f, g.

Let R be a bounded set in $\Omega_r = \{(x,y) \in Z \times S : \|(x,y)\| \le r\}$, then there exists positive constants q_1, q_2 such that

$$\left|f\big(t, x(t), y(t)\big)\right| \le q_1, \qquad \left|g\big(t, x(t), y(t)\big)\right| \le q_2, \quad \forall t \in [0, T],$$

then for any$(x,y) \in R$ we have$|Q_1(x,y)(t)| \le N_1 q_1 + N_2 q_2$,

which implies that $\|Q_1(x,y)\| \le N_1 q_1 + N_2 q_2$.

Similarly, we get $\|Q_2(x,y)\| \le N_3 q_1 + N_4 q_2$.

Thus, it follows from the above inequalities that the operator Q is uniformly bounded, since

$$\|Q(x,y)\| \le (N_1 + N_3) q_1 + (N_2 + N_4) q_2.$$

Next, we show that the operator is equicontinuous.

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$., then we have

$$\begin{aligned}
&|Q_1(x,y)(t_2) - Q_1(x,y)(t_1)| \\
&\le |\Delta| e^{-k(t_2 - t_1)} \left[|\mu_2| (I^{\alpha-1}|f|)(\eta_2) + k|\mu_2| \int_0^{\eta_2} e^{-k(\eta_2 - s)} (I^{\alpha-1}|f|)(s) ds \right. \\
&\left. + \int_0^T \big(I^{\beta-1}|g|\big)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} \big(I^{\beta-1}|g|\big)(s) ds dx \right] \\
&+ |\lambda| e^{-k(t_2 - t_1)} \left[|\rho_2| \big(I^{\beta-1}|g|\big)(\zeta_2) + k|\rho_2| \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} \big(I^{\beta-1}|g|\big)(s) ds \right. \\
&\left. + \int_0^T (I^{\alpha-1}|f|)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} (I^{\alpha-1}|f|)(s) ds dx \right] \\
&+ \int_0^{t_2} e^{-k(t_2 - s)} (I^{\alpha-1}|f|)(s) ds + \int_0^{t_1} e^{-k(t_1 - s)} (I^{\alpha-1}|f|)(s) ds.
\end{aligned}$$

$$\le |\Delta| e^{-k(t_2-t_1)}\Bigg[|\mu_2 q_1|(I^{\alpha-1}1)(\eta_2) + k|\mu_2 q_1|\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}1)(s)ds$$

$$+ q_2\int_0^T \left(I^{\beta-1}1\right)(s)ds + kq_2\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}1\right)(s)dsdx\Bigg]$$

$$+ |\lambda| e^{-k(t_2-t_1)}\Bigg[|\rho_2 q_2|\left(I^{\beta-1}1\right)(\zeta_2) + k|\rho_2 q_2|\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}1\right)(s)ds$$

$$+ q_1\int_0^T (I^{\alpha-1}1)(s)ds + kq_1\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}1)(s)dsdx\Bigg]$$

$$+ q_1\left[\left(\int_0^{t_1} e^{-k(t_2-s)} - e^{-k(t_1-s)}\right)(I^{\alpha-1}1)(s)ds\right.$$

$$\left.+ \int_{t_1}^{t_2} e^{-k(t_2-s)}\,(I^{\alpha-1}1)(s)ds\right].$$

Hence we have $\|Q_1(x,y)(t_2) - Q_1(x,y)(t_1)\| \to 0$ independent of x and y as $t_2 \to t_1$. Similarly, $\|Q_2(x,y)(t_2) - Q_2(x,y)(t_1)\| \to 0$ independent of x and y as $t_2 \to t_1$. Therefore, the operator $Q(x,y)$ is equicontinuous, and thus the operator $Q(x,y)$ is completely continuous.

Step 2: (Boundedness of operator)

Finally, show that $\varepsilon = \{(x,y) \in Z \times S: (x,y) = NQ(x,y), N \in (0,1)\}$ is bounded.

Let

$$x(t) = NQ_1(x,y)(t), \qquad y(t) = NQ_2(x,y)(t),$$

then

$$|x(t)| \le N_1(\delta_0 + \delta_1|x| + \delta_2|y|) + N_2(\phi_0 + \phi_1|x| + \phi_2|y|),$$

and

$$|y(t)| \le N_3(\delta_0 + \delta_1|x| + \delta_2|y|) + N_4(\phi_0 + \phi_1|x| + \phi_2|y|).$$

So we get

$$\|x\| \le N_1(\delta_0 + \delta_1|x| + \delta_2|y|) + N_2(\phi_0 + \phi_1|x| + \phi_2|y|), \tag{14}$$

and

$$\|y\| \le N_3(\delta_0 + \delta_1|x| + \delta_2|y|) + N_4(\phi_0 + \phi_1|x| + \phi_2|y|). \tag{15}$$

From (14), (15) we obtain

$$\|x\| + \|y\| \leq (N_1 + N_3)\delta_0 + (N_2 + N_4)\phi_0 + \big((N_1 + N_3)\delta_1 + (N_2 + N_4)\phi_1\big)\|x\|$$
$$+ \big((N_1 + N_3)\delta_2 + (N_2 + N_4)\phi_2\big)\|y\|.$$

Therefore;

$$\|(x, y)\| \leq \frac{(N_1 + N_3)\delta_0 + (N_2 + N_4)\phi_0}{N_0},$$

where $N_0 = min\{1 - (N_1 + N_3)\delta_1 - (N_2 + N_4)\phi_1, 1 - (N_1 + N_3)\delta_2 - (N_2 + N_4)\phi_2\}$ that is ε is bounded. by (Leray-Schauder theorem) the existence of solution of boundary value problems holds true on $[0, T]$.

4.3 Hyers-Ulam Stability of System (1)

This section is devoted to the investigation of Hyers-Ulam stability for our proposed system. Consider the following inequality:

$$\begin{cases} {}^cD^{\alpha-1}(D + k)x(t) - f\big(t, x(t), y(t)\big) \leq r_1, & t \in [0, T], \\ {}^cD^{\beta-1}(D + k)y(t) - g\big(t, x(t), y(t)\big) \leq r_2, & t \in [0, T], \end{cases} \qquad (16)$$

where r_1, r_2 are given two positive real numbers.

Definition 4.3.1 The boundary value problem (1) is Hyers-Ulam stable if there exist $N_i, i = 1,2,3,4$ such that for given $r_1, r_2 > 0$ and for each solution $(x, y) \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$ of inequality(16), there exists a solution $(x^*, y^*) \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$ of problem (1) with

$$\begin{cases} |x(t) - x^*(t)| \leq N_1\, r_1 + N_2\, r_2, & t \in [0, T], \\ |y(t) - y^*(t)| \leq N_3\, r_1 + N_4\, r_2, & t \in [0, T]. \end{cases}$$

Remark 4.3.1 (x, y) is a solution of inequality (16) if there exist functions $H_i \in C([0, T], \mathbb{R})$, $\mathrm{i} = 1,2$ which depend upon x, y respectively, such that

$|H_1(t)| \leq r_1, \quad |H_2(t)| \leq r_2, \quad t \in [0, T].$

$$\begin{cases} {}^cD^{\alpha-1}(D + k)x(t) = f\big(t, x(t), y(t)\big) + H_1(t), & t \in [0, T], \\ {}^cD^{\beta-1}(D + k)y(t) = g\big(t, x(t), y(t)\big) + H_2(t), & t \in [0, T]. \end{cases}$$

Remark 4.3.2 If (x, y) represent a solution of inequality (16), then (x, y) is a solution of the following inequality

$$\begin{cases} |x(t) - x^*(t)| \leq N_1\, \mathrm{r}_1 + N_2\, \mathrm{r}_2, & t \in [0, T], \\ |y(t) - y^*(t)| \leq N_3\, \mathrm{r}_1 + N_4\, \mathrm{r}_2, & t \in [0, T]. \end{cases}$$

As from Remark 4.3.1, we have

$$\begin{cases} {}^cD^{\alpha-1}(D + k)x(t) = f\big(t, x(t), y(t)\big) + H_1(t), & t \in [0, T], \\ {}^cD^{\beta-1}(D + k)y(t) = g\big(t, x(t), y(t)\big) + H_2(t), & t \in [0, T]. \end{cases}$$

With the help of Definition 4.3.1 and Remark 4.3.1 we verified Remark 4.3.2, in the following lines

$$\Bigg| x(t) - \Delta e^{-kt}\Bigg[\mu_2(I^{\alpha-1}f)(\eta_2) - k\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}f)(s)ds$$

$$- \int_0^T \big(I^{\beta-1}g\big)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,\big(I^{\beta-1}g\big)(s)dsdx\Bigg]$$

$$- \lambda e^{-kt}\Bigg[\rho_2\big(I^{\beta-1}g\big)(\zeta_2) - k\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\,\big(I^{\beta-1}g\big)(s)ds$$

$$- \int_0^T (I^{\alpha-1}f)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}f)(s)dsdx\Bigg]$$

$$- \frac{1}{T^2 - \mu_1\rho_1}\Bigg[\frac{A\mu_2}{k}(I^{\alpha-1}f)(\eta_2) - A\mu_2\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}f)(s)ds$$

$$- \frac{A}{k}\int_0^T \big(I^{\beta-1}g\big)(s)ds + (A - \rho_1)\int_0^T\int_0^x e^{-k(x-s)}\,\big(I^{\beta-1}g\big)(s)dsdx$$

$$+ \frac{B\rho_2}{k}\big(I^{\beta-1}g\big)(\zeta_2) - B\rho_2\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\,\big(I^{\beta-1}g\big)(s)ds$$

$$- \frac{B}{k}\int_0^T (I^{\alpha-1}f)(s)ds + (B - T)\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}f)(s)dsdx$$

$$+ T\rho_1\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\,\big(I^{\beta-1}g\big)(s)ds$$

$$+ \mu_1\rho_1\int_0^{\eta_1} e^{-k(\eta_1-s)}\,(I^{\alpha-1}f)(s)ds\Bigg] - \int_0^t e^{-k(t-s)}\,(I^{\alpha-1}f)(s)ds\Bigg|,$$

$$\leq |\Delta| e^{-kT}\Bigg[|\mu_2|(I^{\alpha-1}|H_1(t)|)(\eta_2) + k|\mu_2|\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}|H_1(t)|)(s)ds$$

$$+\int_0^T \left(I^{\beta-1}|H_2(t)|\right)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}|H_2(t)|\right)(s)dsdx\Bigg]$$

$$+|\lambda| e^{-kT}\Bigg[|\rho_2|\left(I^{\beta-1}|H_2(t)|\right)(\zeta_2) + k|\rho_2|\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}|H_2(t)|\right)(s)ds$$

$$+\int_0^T (I^{\alpha-1}|H_1(t)|)(s)ds + k\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}|H_1(t)|)(s)dsdx\Bigg]$$

$$+\frac{1}{|\omega|}\Bigg[\frac{|A\mu_2|}{k}(I^{\alpha-1}|H_1(t)|)\,(\eta_2) + |A\mu_2|\int_0^{\eta_2} e^{-k(\eta_2-s)}\,(I^{\alpha-1}|H_1(t)|)(s)ds$$

$$+\frac{|A|}{k}\int_0^T \left(I^{\beta-1}|H_2(t)|\right)(s)ds \; + |A-\rho_1|\int_0^T\int_0^x e^{-k(x-s)}\left(I^{\beta-1}|H_2(t)|\right)(s)dsdx$$

$$+\frac{|B\rho_2|}{k}\left(I^{\beta-1}|H_2(t)|\right)(\zeta_2) + |B\rho_2|\int_0^{\zeta_2} e^{-k(\zeta_2-s)}\left(I^{\beta-1}|H_2(t)|\right)(s)ds$$

$$+\frac{|B|}{k}\int_0^T (I^{\alpha-1}|H_1(t)|)(s)ds + |B-T|\int_0^T\int_0^x e^{-k(x-s)}\,(I^{\alpha-1}|H_1(t)|)(s)dsdx$$

$$+T|\rho_1|\int_0^{\zeta_1} e^{-k(\zeta_1-s)}\left(I^{\beta-1}|H_2(t)|\right)(s)ds + |\mu_1\rho_1|\int_0^{\eta_1} e^{-k(\eta_1-s)}\,(I^{\alpha-1}|H_1(t)|)(s)ds\Bigg]$$

$$+\int_0^T e^{-k(T-s)}\,(I^{\alpha-1}|H_1(t)|)(s)ds, \tag{17}$$

$$\leq r_1 \Bigg[|\Delta| e^{-kT} \mu_2 \left((I^{\alpha-1}1)(\eta_2) + k \int_0^{\eta_2} e^{-k(\eta_2 - s)} (I^{\alpha-1}1)(s) ds \right)$$

$$+ |\lambda| e^{-kT} \left(\int_0^T (I^{\alpha-1}1)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} (I^{\alpha-1}1)(s) ds dx \right)$$

$$+ \frac{1}{|\omega|} \Bigg(\frac{|A\mu_2|}{k} (I^{\alpha-1}1)(\eta_2) + |A\mu_2| \int_0^{\eta_2} e^{-k(\eta_2 - s)} (I^{\alpha-1}1)(s) ds$$

$$+ \frac{|B|}{k} \int_0^T (I^{\alpha-1}1)(s) ds + |B - T| \int_0^T \int_0^x e^{-k(x-s)} (I^{\alpha-1}1)(s) ds dx$$

$$+ |\mu_1 \rho_1| \int_0^{\eta_1} e^{-k(\eta_1 - s)} (I^{\alpha-1}1)(s) ds \Bigg) + \int_0^T e^{-k(T-s)} (I^{\alpha-1}1)(s) ds \Bigg]$$

$$+ r_2 \Bigg[|\Delta| e^{-kT} \left(\int_0^T (I^{\beta-1}1)(s) ds + k \int_0^T \int_0^x e^{-k(x-s)} (I^{\beta-1}1)(s) ds dx \right)$$

$$+ |\lambda| e^{-kT} \left(|\rho_2| (I^{\beta-1}1)(\zeta_2) + k |\rho_2| \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} (I^{\beta-1}1)(s) ds \right)$$

$$+ \frac{1}{|\omega|} \Bigg(\frac{|A|}{k} \int_0^T (I^{\beta-1}1)(s) ds + |A - \rho_1| \int_0^T \int_0^x e^{-k(x-s)} (I^{\beta-1}1)(s) ds dx$$

$$+ \frac{|B\rho_2|}{k} (I^{\beta-1}1)(\zeta_2) + |B\rho_2| \int_0^{\zeta_2} e^{-k(\zeta_2 - s)} (I^{\beta-1}1)(s) ds \Bigg) \Bigg]$$

$$= \mathbf{r_1} N_1 + \mathbf{r_2} N_2 .$$

By the same method we can obtain that

$$|y(t) - y^*(t)| \leq N_3 \, r_1 + N_4 \, r_2, \tag{18}$$

where $N_i, i = 1,2,3,4$ are mentioned before. Hence Remark 4.3.2 is verified, with the help of (17) and (18). Thus the nonlinear sequential coupled system of Caputo fractional differential equations is Hyers-Ulam stable and consequently, by the system (1) is Hyers-Ulam –stable.

4.4 Example

Consider the following system of fractional differential equation

$$\begin{cases} {}^{c}D^{1/2}(D+1)x(t) = \dfrac{1}{8\pi\sqrt{49+t^2}}\left(\dfrac{|x(t)|}{1+|x(t)|} + \sin y(t)\right) + \dfrac{1}{4}, & t \in [0,1] \\ {}^{c}D^{1/3}(D+1)y(t) = \dfrac{1}{2\pi(4+t)^2}\left(sin\big(x(t)\big) + \dfrac{y(t)}{1+|x(t)|}\right) + 1, & t \in [0,1] \\ \displaystyle\int_0^1 x(s)ds = y(1/2), \int_0^1 x'(s)ds = -2y'(1/2), \\ \displaystyle\int_0^1 y(s)ds = -3x(1/3), \int_0^1 y'(s)ds = x'(1). \end{cases} \tag{19}$$

Here

$$k = 1, \alpha = \frac{3}{2}, \beta = \frac{4}{3}, T = 1, \rho_1 = 1, \zeta_1 = \frac{1}{2}, \rho_2 = -2, \zeta_2 = \frac{1}{2}, \mu_1 = -3, \eta_1 = \frac{1}{3}, \mu_2 = 1,$$

$$\eta_2 = 1.$$

We found

$$N_1 = 4.5398, N_2 = 4.9766, N_3 = 2.7046, N_4 = 5.872, h_f = \frac{1}{56\pi}, h_g = \frac{1}{32\pi}.$$

It's clear that f, g are jointly continuous functions where

$$f(t,x,y) = \frac{1}{8\pi\sqrt{49+t^2}}\left(\frac{|x(t)|}{1+|x(t)|} + \sin y(t)\right) + \frac{1}{4},$$

$$g(t,x,y) = \frac{1}{2\pi(4+t)^2}\left(sin\big(x(t)\big) + \frac{y(t)}{1+|x(t)|}\right) + 1\,.$$

Now, check that $h_f(N_1+N_3) + h_g(N_2+N_4) < 1$.

Hence

$$\frac{1}{56\pi}(7.2444) + \frac{1}{32\pi}(10.8486) = 0.149 < 1.$$

Thus all the conditions of Theorem 4.2.1 are satisfied, then problem (19) has a unique solution on [0,1].

4.5 Conclusion

The current chapter studied a coupled system of Caputo type sequential fractional differential equations with integral boundary conditions. Solutions of the system were examined in details and their existence and uniqueness were established by employing

Banach’s fixed-point theorem and Schauder’s alternative. The Hyers-Ulam stability of solutions was also discussed and supporting numerical results were presented.

Chapter 5

EXISTENCE AND ULAM-HYERS STABILITY OF COUPLED SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

In this chapter we study the existence and uniqueness of solutions for sequential fractional differential equations involving Caputo derivative of order $1 < \alpha \leq 2$ with integral boundary conditions. Moreover, we discuss Hyers-Ulam stability for the problem at hand. Finally, an example is provided to illustrate the theoretical results.

We study the following nonlinear sequential fractional differential equation subject to non-separated non-local integral fractional boundary conditions

$$\begin{cases} (D^{\alpha} + \lambda D^{\alpha-1})x(t) = f_1\big(t, x(t), y(t)\big), 1 < \alpha \leq 2, 0 \leq t \leq T, \\ \big(D^{\beta} + \lambda D^{\beta-1}\big)y(t) = f_2\big(t, x(t), y(t)\big), 1 < \beta \leq 2, 0 \leq t \leq T, \\ \nu_1 x(\eta) + \mu_1 x(T) = \int_0^T h_1\big(x(s)\big)ds\,, \nu_1 y(\eta) + \mu_1 y(T) = \int_0^T h_2\big(y(s)\big)ds, \\ \nu_2 x'(\eta) + \mu_2 x'(T) = \int_0^T g_1\big(x(s)\big)ds\,, \nu_2 y'(\eta) + \mu_2 y'(T) = \int_0^T g_2\big(y(s)\big)ds, \end{cases} \quad (1)$$

where D^{α}, D^{β} denote the Caputo derivative, $0 < \eta < T, \lambda \in \mathbb{R}_{+}, \nu_1, \nu_2, \mu_1, \mu_2 \in \mathbb{R}$.

In Section 1, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 2 contains the existence results for problem (1)obtained by applying Leray-Schauder's nonlinear alternative, Banach's contraction mapping principle. In Section 3, Ulam-Hyers stability for the problem (1) is studied. Finally, in Section 4, an example is provided to illustrate the theoretical results.

5.1 Preliminaries

We begin this section with prove an auxiliary lemma, which plays a key role in defining a fixed-point problem associated with the given problem.

Lemma 5.1.1 [1] Let $\alpha > 0$. Then the differential equation $D_{0+}^{\alpha} f(t) = 0$ has solutions

$$f(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

and

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2}$$

where $c_i \in \mathbb{R}\ and\ i = 1,2,\ldots,n = [\alpha] + 1$.

Let $C([0,T];\mathbb{R})$ denote the Banach space of all continuous functions from [0,T] to $\mathbb{R}$ equipped with the sup-norm $\|x\|_\infty = sup\{|x(t)|: 0 \le t \le T\}$. For computational convenience, in what follows we use the following notations:

$$a_{11} := \nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T},\ a_{12} := \nu_1 \frac{1}{\lambda}(1 - e^{-\lambda\eta}) + \mu_1 \frac{1}{\lambda}(1 - e^{-\lambda T})$$

$$a_{21} := -\lambda\nu_2 e^{-\lambda\eta} - \lambda\mu_2 e^{-\lambda T},\ a_{22} := \nu_2 e^{-\lambda\eta} + \mu_2 e^{-\lambda T},$$

$$\Delta := a_{11}a_{22} - a_{12}a_{21},\ \Delta \neq 0,$$

$$\phi_1(t) = \left(\frac{a_{12}}{\Delta} e^{-\lambda t} - \frac{a_{11}}{\Delta}\frac{1}{\lambda}(1 - e^{-\lambda t})\right),$$

$$\phi_2(t) = \left(\frac{a_{21}}{\Delta}\frac{1}{\lambda}(1 - e^{-\lambda t}) - \frac{a_{22}}{\Delta} e^{-\lambda t}\right).$$

Lemma 5.1.2 Let $\rho, \gamma_1, \gamma_2 \in C([0,T];\mathbb{R})$. Then following boundary value problem

$$.\begin{cases} (D^\alpha + \lambda D^{\alpha-1})z(t) = \rho(t),\ 1 < \alpha \le 2,\ 0 \le t \le T, \\ \nu_1 z(\eta) + \mu_1 z(T) = \int_0^T \gamma_1(s)ds\,, \\ \nu_2 z'(\eta) + \mu_2 z'(T) = \int_0^T \gamma_2(s)ds\,, \end{cases} \tag{3}$$

is equivalent to the fractional integral equation

$$z(t) = \int_0^t (t-s)^{\alpha-1} E_{1,\alpha}(-\lambda; t-s)\rho(s)ds + \sum_{j=1}^{2} \nu_j \phi_j(t) \int_0^\eta (\eta - s)^{\alpha-j} (E_{1,\alpha+j-1}(-\lambda; \eta - s)\rho(s)ds + \sum_{j=1}^{2} \mu_j \phi_j(t) \int_0^T (T-s)^{\alpha-j} E_{1,\alpha+j-1}(-\lambda; T-s)\rho(s)ds - \sum_{j=1}^{2} \phi_j(t) \int_0^T \gamma_j(s)ds.$$

Proof. Applying $I^{\alpha-1}$ to both sides of (3)and using (2) we get

$$I^{\alpha-1}D^{\alpha-1}(D+\lambda)z(t) = I^{\alpha-1}\rho(t),$$

$$(D+\lambda)z(t) = c_0 + I^{\alpha-1}\rho(t).$$

We solve the above linear ordinary differential equation

$$z(t) = c_1 e^{-\lambda t} + c_0 \frac{1}{\lambda} - c_0 \frac{1}{\lambda} e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} r(s)ds = c_1 e^{-\lambda t} + c_0 \frac{1}{\lambda}(1 - e^{-\lambda t}) + \int_0^t (t-r)^{\alpha-1} E_{1,\alpha}(-\lambda; t-r)\rho(r)dr \quad (5)$$

It is clear that

$$z'(t) = -\lambda c_1 e^{t} + c_0 e^{-\lambda t} + \int_0^t (t-r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; t-r)\rho(r)dr \quad (6)$$

The first boundary condition implies that

$$\nu_1 z(\eta) + \mu_1 z(T) = \nu_1 c_1 e^{-\lambda \eta} + \nu_1 c_0 \frac{1}{\lambda}(1 - e^{-\lambda \eta}) + \nu_1 \int_0^\eta (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)dr + \mu_1 c_1 e^{-\lambda T} + \mu_1 c_0 \frac{1}{\lambda}(1 - e^{-\lambda T}) + \mu_1 \int_0^T (T-r)^{\alpha-1} E_{1,\alpha}(-\lambda; T-r)\rho(r)dr = \int_0^\xi \gamma_1(s)ds.$$

It follows that

$$(\nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T})c_1 + (\nu_1 \frac{1}{\lambda}(1 - e^{-\lambda\eta}) + \mu_1 \frac{1}{\lambda}(1 - e^{-\lambda T}))c_0$$

$$= \int_0^{\xi} \gamma_1(s)ds - \nu_1 \int_0^{\eta} (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)dr$$

$$- \mu_1 \int_0^{T} (T - r)^{\alpha-1} E_{1,\alpha}(-\lambda; T - r)\rho(r)dr.$$

The second boundary condition with (6) implies that

$$\nu_2 z'(\eta) + \mu_2 z'(T)$$

$$= \nu_2(-\lambda c_1 e^{-\lambda\eta} + c_0 e^{-\lambda\eta}) + \nu_2 \int_0^{\eta} (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)dr$$

$$+ \mu_2(-\lambda c_1 e^{-\lambda T} + c_0 e^{-\lambda T}) + \mu_2 \int_0^{T} (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)dr$$

$$= \int_{\zeta}^{T} \gamma_2(s)ds.$$

Thus

$$a_{11}c_1 + a_{12}c_0 = \int_0^{T} \gamma_1(s)ds - \nu_1 \int_0^{\eta} (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)dr$$

$$- \mu_1 \int_0^{T} (T - r)^{\alpha-1} E_{1,\alpha}(-\lambda; T - r)\rho(r)dr,$$

$$a_{21}c_1 + a_{22}c_0 = \int_0^{T} \gamma_2(s)ds - \nu_2 \int_0^{\eta} (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)dr$$

$$- \mu_2 \int_0^{T} (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)dr.$$

Solving the above system of equations for c_0 and c_1, we get

$c_0 = \frac{a_{11}}{\Delta}\Big(\int_0^T \gamma_2(s)ds - \nu_2 \int_0^\eta (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)dr - \mu_2 \int_0^T (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)dr\Big) - \frac{a_{21}}{\Delta}\Big(\int_0^T \gamma_1(s)ds - \nu_1 \int_0^\eta (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)dr - \mu_1 \int_0^T (T - r)^{\alpha-1} E_{1,\alpha}(-\lambda; T - r)\rho(r)dr\Big),$

$$c_1 = \frac{a_{22}}{\Delta}\left(\int_0^T \gamma_1(s)ds - \nu_1 \int_0^\eta (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)dr\right.$$

$$\left. - \mu_1 \int_0^T (T - r)^{\alpha-1} E_{1,\alpha}(-\lambda; T - r)\rho(r)dr\right)$$

$$- \frac{a_{12}}{\Delta}\left(\int_0^T \gamma_2(s)ds - \nu_2 \int_0^\eta (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)dr\right.$$

$$\left. - \mu_2 \int_0^T (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)dr\right).$$

Inserting c_0 and c_1 in (5) we obtain the desired formula (4).

Conversely, assume that u satisfies (4). By a direct computation, it follows that the solution given by (4)satisfies (3). ■

Lemma 5.1.3 For any $g, h \in C([0, T]; \mathbb{R}), \gamma > 0$, we have

$$\left|\int_0^t (t - s)^{\gamma-1} E_{1,\gamma}(-\lambda; t - s)(g(s) - h(s))ds\right| \leq t^\gamma E_{1,\gamma+1}(-\lambda; t)\|g - h\|_\infty.$$

Proof. Indeed,

$$\left|\int_0^t (t - s)^{\gamma-1} E_{1,\gamma}(-\lambda; t - s)(g(s) - h(s))ds\right|$$

$$\leq \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k + \gamma)} \int_0^t (t - s)^{k+\gamma-1} |g(s) - h(s)|ds$$

$$\leq \sum_{k=0}^{\infty} \frac{\lambda^k t^{k+\gamma}}{\Gamma(k + \gamma + 1)} = t^\gamma E_{1,\gamma+1}(|\lambda|; t)\|g - h\|_\infty.$$ ■

5.2 Main Results

By Lemma 5.1.2, we introduce a fixed point problem associated with the problem as follows:

$(x, y) = (T_1, T_2)(x, y) = T(x, y): C([0, T]; \mathbb{R}) \times C([0, T]; \mathbb{R}) \to C([0, T]; \mathbb{R}) \times C([0, T]; \mathbb{R}),$

where

$$T_1(x,y)(t) = \int_0^t (t-r)^{\alpha-1} E_{1,\alpha}(-\lambda; t-r) f_1(r, x(r), y(r)) dr$$

$$+ \sum_{j=1}^{2} \nu_j \phi_j(t) \int_0^{\eta} \eta - s)^{\alpha-j} E_{1,\alpha+j-1}(-\lambda; \eta - r) f_1(r, x(r), y(r)) dr$$

$$+ \sum_{j=1}^{2} \mu_j \phi_j(t) \int_0^{T} (T-s)^{\alpha-j} E_{1,\alpha+j-1}(-\lambda; T-r) f_1(r, x(r), y(r)) dr$$

$$+ \sum_{j=1}^{2} \phi_j(t) \int_0^{T} h_j\big(r, x(r), y(r)\big) dr. \qquad (6)$$

$$T_2(x,y)(t) = \int_0^t (t-r)^{\beta-1} E_{1,\beta}(-\lambda; t-r) f_2(r, x(r), y(r)) dr$$

$$+ \sum_{j=1}^{2} \nu_j \phi_j(t) \int_0^{\eta} \eta - s)^{\beta-j} E_{1,\beta+j-1}(-\lambda; \eta - r) f_2(r, x(r), y(r)) dr$$

$$+ \sum_{j=1}^{2} \mu_j \phi_j(t) \int_0^{T} (T-s)^{\beta-j} E_{1,\beta+j-1}(-\lambda; T-r) f_2(r, x(r), y(r)) dr$$

$$+ \sum_{j=1}^{2} \phi_j(t) \int_0^{T} g_j\big(r, x(r), y(r)\big) dr. \qquad (7)$$

Evidently, the existence of fixed points of the operator T is equivalent to the existence of solutions for problem (1) for $\gamma = \alpha, \beta$, let

$$R^{\gamma} := max \left\{ \begin{array}{c} (T^{\gamma} E_{1,\gamma+1}(|\lambda|; T) + \sum_{j=1}^{2} |\nu_j| \left|\phi_j(t)\right| \eta^{\gamma-j} E_{1,\gamma+j-1}(|\lambda|; \eta) \\ + \sum_{j=1}^{2} \left|\mu_j\right| \left|\phi_j(t)\right| T^{\gamma-j} E_{1,\gamma+j-1}(|\lambda|; T)), (\|\phi_1\| + \|\phi_2\|)T \end{array} \right\}.$$

Here, we prove the existence and uniqueness of solutions for problem(1). We apply a fixed-point theorem due to Banach.

Theorem 5.2.1 Let $f_i, h_i, g_i: [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that the following conditions hold:

(A_1) There exists $L_f, L_h, L_g > 0$ such that

$$|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)| \leq L_f(|x_1 - x_2| + |y_1 - y_2|),$$

$$|h_i(t,x_1,y_1)-h_i(t,x_2,y_2)|\le L_h(|x_1-x_2|+|y_1-y_2|),$$

$$|g_i(t,x_1,y_1)-g_i(t,x_2,y_2)|\le L_g(|x_1-x_2|+|y_1-y_2|),$$

$$\forall (t,x_1,y_1),(t,x_2,y_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}.$$

(A_2) $$1-2(L_f+L_h)R^{\alpha}>0, 1-2(L_f+L_g)R^{\beta}>0.$$

Then problem (1) has a unique solution in $C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$.

Proof. Consider a ball

$$B_r:=u\in C([0,T],\mathbb{R}):\|u\|_\infty\le r$$

with radius

$$r\ge max\left\{\frac{(M_f+M_h)R^{\alpha}}{1-2(L_f+L_h)R^{\alpha}},\frac{((M_f+M_g)R^{\beta}}{1-2(L_f+L_g)R^{\beta}}\right\},$$

where

$$M_f:=max(\|f_1(t,0,0)\|_\infty,\|f_2(t,0,0)\|_\infty),$$

$$M_h:=max(\|h_1(t,0,0)\|_\infty,\|h_2(t,0,0)\|_\infty),$$

$$M_g:=max(\|g_1(t,0,0)\|_\infty,\|g_2(t,0,0)\|_\infty).$$

It is clear that for all $x,y\in\mathbb{R}$

$$|f_i(t,x,y)|\le L_f(|x|+|y|)+M_f,$$

$$|h_i(t,x,y)|\le L_h(|x|+|y|)+M_h,$$

$$|g_i(t,x,y)|\le L_g(|x|+|y|)+M_g.$$

Using this inequality and Lemma 5.1.3, from (6) it follows that

$$|T_1(x,y)(t)|\le t^{\alpha}E_{1,\alpha+1}(|\lambda|;t)\|f_i(\cdot,x(\cdot),y(\cdot))\|+\sum_{j=1}^{2}|\nu_j||\phi_j(t)|\eta^{\alpha-j}E_{1,\alpha+j-1}(|\lambda|;\eta)\,\|f_i(\cdot,x(\cdot),y(\cdot))\|+\sum_{j=1}^{2}|\mu_j||\phi_j(t)|\,T^{\alpha-j}E_{1,\alpha+j-1}(|\lambda|;T)\|f_i(\cdot,x(\cdot),y(\cdot))\|_\infty$$

$$+\sum_{j=1}^{2}|\phi_j(t)|\int_0^T\left|h_i\big(r,x(r),y(r)\big)\right|dr.$$

$$\leq \left(t^{\alpha} E_{1,\alpha+1}(-\lambda; t) + \sum_{j=1}^{2} |\nu_j| |\phi_j(t)| \eta^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|; \eta) \right.$$

$$\left. + \sum_{j=1}^{2} |\mu_j| |\phi_j(t)| \, T^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|; T) \right) \times (L_f(|x| + |y|) + M_f) + (\|\phi_1\|$$

$$+ \|\phi_2\|) T (L_h(|x| + |y|) + M_h) \leq ((L_f + L_h) r + M_f + M_h) R^{\alpha}$$

$$\leq \frac{r}{2}. \tag{8}$$

In the like manner we have

$$|T_2(x, y)(t)| \leq ((L_f + L_h) r + M_f + M_h) R^{\beta} \leq \frac{r}{2}. \tag{9}$$

From (8) and (9) it follows that $TB_r \subset B_r$. Next, using condition (A_1), we obtain

$$|T_1(x_1, y_1)(t) - T_1(x_2, y_2)(t)| \leq R^{\alpha} \| f_1(\cdot, x_1(\cdot), y_1(\cdot)) - f_1(\cdot, x_2(\cdot), y_2(\cdot)) \|_{\infty}$$

$$\leq (L_f + L_h) R^{\alpha} \| (x_1, y_1) - (x_2, y_2) \|_{\infty}. \tag{10}$$

Similarly,

$$|T_2(x_1, y_1)(t) - T_2(x_2, y_2)(t)| \leq R^{\beta} \| f_2(\cdot, x_1(\cdot), y_1(\cdot)) - f_2(\cdot, x_2(\cdot), y_2(\cdot)) \|_{\infty}$$

$$\leq (L_f + L_g) R^{\beta} \| (x_1, y_1) - (x_2, y_2) \|_{\infty}. \tag{11}$$

It follows from (10) and (11) that

$$\| T(x_1, y_1) - T(x_2, y_2) \| \leq [(L_f + L_h) R^{\alpha} + (L_f + L_g) R^{\beta}] \| (x_1, y_1) - (x_2, y_2) \|_{\infty}.$$

By (A_2) the operator T is a contraction. Thus by the Banach fixed point theorem T has a unique fixed point in $C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$. This completes the proof. ■

In the next result, we prove the existence of solutions for problem(1) by applying the Leray-Schauder alternative.

Theorem 5.2.2 Let $f. [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following condition holds:

(A_3) There exists $\gamma_f, \gamma_h, \gamma_g \in C([0,T], \mathbb{R}_+)$ and a nondecreasing function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f_i(t,x,y)| \le \gamma_f(t)\psi_f(|x| + |y|), \forall (t,u) \in [0,T] \times R.$$

$$|h_i(t,x,y)| \le \gamma_h(t)\psi_h(|x| + |y|),$$

$$|g_i(t,x,y)| \le \gamma_g(t)\psi_g(|x| + |y|).$$

(A_4) There exists $M > 0$ such that

$$\frac{\|(x,y)\|_\infty}{(\|\gamma_f\|_\infty\psi_f(\|(x,y)\|_\infty) + \|\gamma_h\|_\infty\psi_h(\|(x,y)\|_\infty))R^\alpha + (\|\gamma_f\|_\infty\psi_f(\|(x,y)\|_\infty) + \|\gamma_g\|_\infty\psi_g(\|(x,y)\|_\infty))R^\beta)}$$

$\le 1.$

Then BVP (1) has at least one solution.

Proof. Step 1: Show that $T: C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R})$ maps bounded sets into bounded sets and is continuous.

Let B_r be a bounded set in $C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R})$.Then

$$|f_1(t,x(t),y(t))| \le \|\gamma_f\|\psi_f(r), |f_2(t,x(t),y(t))| \le \|\gamma_f\|\psi_f(r),$$

and by Lemma 5.1.3,

$$|T_1(x,y)(t)| \le t^\alpha E_{1,\alpha+1}(|\lambda|;t)\|f_1(\cdot,x(\cdot),y(\cdot))\|$$

$$+ \sum_{j=1}^{2} |\nu_j| \left|\phi_j(t)\right| \eta^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|;\eta) \, \|f_i(\cdot,x(\cdot),y(\cdot))\|$$

$$+ \sum_{j=1}^{2} \left|\mu_j\right| \left|\phi_j(t)\right| T^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|;T) \|f_i(\cdot,x(\cdot),y(\cdot))\|_\infty$$

$$+ \sum_{j=1}^{2} \left|\phi_j(t)\right| \int_0^T \left|h_i\big(r,x(r),y(r)\big)\right| dr$$

$$\leq \left(t^{\alpha} E_{1,\alpha+1}(|\lambda|;t) + \sum_{j=1}^{2} |\nu_j| \left|\phi_j(t)\right| \eta^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|;\eta) \right.$$

$$\left. + \sum_{j=1}^{2} \left|\mu_j\right| \left|\phi_j(t)\right| T^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|;T) \right) \times (\gamma_f(t)\psi_f(|x|+|y|) + M_f)$$

$$+ (\|\phi_1\| + \|\phi_2\|) T (L_h(|x|+|y|) + M_h)$$

$$\leq (\|\gamma_f\|\psi_f(\|(x,y)\|) + \|\gamma_h\|\psi_h(\|(x,y)\|)) R^{\alpha}.$$

Similarly

$$|T_2(x,y)(t)| \leq (\|\gamma_f\|\psi_f(\|(x,y)\|) + \|\gamma_g\|\psi_g(\|(x,y)\|)) R^{\beta}.$$

It follows that $T(B_r)$ is bounded.

Step 2: Next we show that T maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $(x,y) \in B_r$. Then we obtain

$|T_1(x,y)(t_1) - T_1(x,y)(t_2)|$

$$\leq \left| \int_0^{t_1} (t_1 - r)^{\alpha-1} E_{1,\alpha}(-\lambda; t_1 - r) f_1(r, x(r), y(r)) dr \right.$$

$$\left. - \int_0^{t_2} (t_2 - r)^{\alpha-1} E_{1,\alpha}(-\lambda; t_2 - r) f_1(r, x(r), y(r)) dr \right|$$

$$+ \sum_{j=1}^{2} |\nu_j| \left|\phi_j(t_1) - \phi_j(t_2)\right| \eta^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|;\eta) \, \|f_i(\cdot, x(\cdot), y(\cdot))\|$$

$$+ \sum_{j=1}^{2} \left|\mu_j\right| \left|\phi_j(t_1) - \phi_j(t_2)\right| T^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|;T) \|f_i(\cdot, x(\cdot), y(\cdot))\|_\infty$$

$$+ \sum_{j=1}^{2} |\phi_j(t_1) - \phi_j(t_2)| \int_0^T \left|h_i\big(r, x(r), y(r)\big)\right| dr. \qquad (12)$$

It is clear that the last three terms approach to zero independently of $(x,y) \in B_r$ as $t_1 \to t_2$. Now, we estimate the first term of (12)

$$\left|\int_0^{t_1}(t_1-r)^{\alpha-1}E_{1,\alpha}(-\lambda;t_1-r)f_1(r,x(r),y(r))dr-\int_0^{t_2}(t_2-r)^{\alpha-1}E_{1,\alpha}(-\lambda;t_2-r)f_1(r,x(r),y(r))dr\right|$$

$$=\left|\int_0^{t_1}e^{-\lambda(t_1-s)}\left(I^{\alpha-1}f_1(\cdot,x(\cdot),y(\cdot))\right)(s)ds-\int_0^{t_2}e^{-\lambda(t_2-s)}\left(I^{\alpha-1}f_1(\cdot,x(\cdot),y(\cdot))\right)(s)ds\right|$$

$$\leq\left|\int_{t_1}^{t_2}e^{-\lambda(t_1-s)}\left(I^{\alpha-1}f_1(\cdot,x(\cdot),y(\cdot))\right)(s)ds\right|$$

$$+\left|\int_0^{t_1}[e^{-\lambda(t_2-s)}-e^{-\lambda(t_1-s)}]\left(I^{\alpha-1}f_1(\cdot,x(\cdot),y(\cdot))\right)(s)ds\right|.$$

Obviously, the right-hand side of the above inequality tends to zero independently of $(x,y)\in B_r$ as $t_1\to t_2$. A Similar result is true for $T_1(x,y)$. As T is uniformly bounded and equicontinuous, therefore it follows by the Arzelá-Ascoli theorem that $T: C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have proved the boundedness of the set of all solutions to equations $u=\theta Tu$ for $0\leq\theta\leq 1$.

Let (x,y) be a solution. Then, using the computations employed in proving that T is bounded, we have

$$|(x,y)(t)|=\theta|T(x,y)(t)|$$

$$\leq(\|\gamma_f\|\psi_f(\|(x,y)\|)+\|\gamma_h\|\psi_h(\|(x,y)\|))R^\alpha+(\|\gamma_f\|\psi_f(\|(x,y)\|)+\|\gamma_g\|\psi_g(\|(x,y)\|))R^\beta.$$

Consequently, we have

$$\frac{\|(x,y)\|_\infty}{(\|\gamma_f\|_\infty\psi_f(\|(x,y)\|_\infty)+\|\gamma_h\|_\infty\psi_h(\|(x,y)\|_\infty))R^\alpha+(\|\gamma_f\|_\infty\psi_f(\|(x,y)\|_\infty)+\|\gamma_g\|_\infty\psi_g(\|(x,y)\|_\infty))R^\beta)}\leq 1.$$

In view of (A_4), there exists M such that $\|u\|_C\neq M$. Let us set

$$U=\{u\in C([0,T],\mathbb{R}):\|u\|_C<M\}.$$

Note that the operator $T: U \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \theta T u$ for some $0 < \theta < 1$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that T has a fixed point $u \in U$ which is a solution of problem (1). This completes the proof. ∎

5.3 Ulam -Stability

In this section, we discuss the Ulam stability for problem (1) by means of integral representation of its solution given by

$$x(t) = T_1(x,y)(t), \qquad y(t) = T_2(x,y)(t),$$

where T_1 and T_2 are defined by (6) and (7).

Define the following nonlinear operators $Q_1, Q_2: C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$:

$$Q_1(x,y)(t) := (D^{\alpha} + \lambda D^{\alpha-1})x(t) - f_1(t, x(t), y(t)),$$

$$Q_2(x,y)(t) := (D^{\beta} + \lambda D^{\beta-1})x(t) - f_2(t, x(t), y(t))$$

For some $\varepsilon_1, \varepsilon_2 > 0$, we consider the following inequality:

$$\{\|Q_1(x,y)\| \leq \varepsilon_1, \|Q_2(x,y)\| \leq \varepsilon_2. \tag{13}$$

Definition 5.3.1 The coupled system (1) is said to be Ulam-Hyers stable, if there exist $V_1, V_2 > 0$ such that for every solution $(x^*, y^*) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of the inequality (13), there exists a unique solution $(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of problem (1) with

$$\|(x,y) - (x^*, y^*)\|_{\infty} \leq V_1\varepsilon_1 + V_2\varepsilon_2. \tag{14}$$

Theorem 5.3.1 Let the assumptions of Theorem 5.2.1 hold. Then problem (1) is Ulam-Hyers stable.

Proof. Let $(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ be the solution of problem (1) satisfying $(6)$and (7). Let (x^*, y^*) be any solution satisfying(13):

$$(D^{\alpha} + \lambda D^{\alpha-1})x^*(t) = f_1(t, x^*(t), y^*(t)) + Q_1(x^*, y^*)(t),$$

$$(D^{\beta} + \lambda D^{\beta-1})x^*(t) = f_2(t, x^*(t), y^*(t)) + Q_2(x^*, y^*)(t).$$

So

$$x^*(t) = T_1(x^*, y^*)(t) + \int_0^t (t-r)^{\alpha-1} E_{1,\alpha}(-\lambda; t-r) Q_1(x^*, y^*)(r) dr$$

$$+ \sum_{j=1}^{2} \nu_j \phi_j(t) \int_0^\eta \eta - s)^{\alpha-j} E_{1,\alpha+j-1}(-\lambda; \eta - r) Q_1(x^*, y^*)(r) dr$$

$$+ \sum_{j=1}^{2} \mu_j \phi_j(t) \int_0^T (T-s)^{\alpha-j} E_{1,\alpha+j-1} E_{1,\alpha+j-1}(-\lambda; T-r) Q_1(x^*, y^*)(r) dr.$$

It follows that

$$|T_1(x^*, y^*)(t) - x^*(t)|$$

$$\leq \int_0^t (t-r)^{\alpha-1} E_{1,\alpha}(|\lambda|; t-r) dr \varepsilon_1$$

$$+ \sum_{j=1}^{2} |\nu_j| \|\phi_j(t)\| \int_0^\eta \eta - s)^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|; \eta - r) dr \varepsilon_1$$

$$+ \sum_{j=1}^{2} |\mu_j| \|\phi_j(t)\| \int_0^T (T-s)^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|; T-r) dr \varepsilon_1$$

$$\leq \left(T^\alpha E_{1,\alpha+1}(|\lambda|; T) + \sum_{j=1}^{2} |\nu_j| \|\phi_j(t)\| \eta^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|; \eta) \right.$$

$$\left. + \sum_{j=1}^{2} |\mu_j| \|\phi_j(t)\| T^{\alpha-j} E_{1,\alpha+j-1}(|\lambda|; T) \right) \varepsilon_1 =: U^\alpha \varepsilon_1.$$

Similarly,

$$|T_2(x^*, y^*)(t) - y^*(t)|$$

$$\leq \left(T^\beta E_{1,\beta+1}(|\lambda|; T) + \sum_{j=1}^{2} |\nu_j| \|\phi_j(t)\| \eta^{\beta-j} E_{1,\beta+j-1}(|\lambda|; \eta) \right.$$

$$\left. + \sum_{j=1}^{2} |\mu_j| \|\phi_j(t)\| T^{\beta-j} E_{1,\beta+j-1}(|\lambda|; T) \right) \varepsilon_2 =: U^\beta \varepsilon_1.$$

Therefore, we deduce by the fixed-point property of the operator T,given by (6)and (7), that

$$|x(t)-x^*(t)| = |x(t)-T_1(x^*,y^*)(t)+T_1(x^*,y^*)(t)-x^*(t)|$$
$$\leq |T_1(x,y)(t)-T_1(x^*,y^*)(t)| + |T_1(x^*,y^*)(t)-x^*(t)|$$
$$\leq (L_f+L_h)R^\alpha \|(x,y)-(x^*,y^*)\|_\infty + U^\alpha \varepsilon_1, \tag{15}$$

and similarly

$$|y(t)-y^*(t)| = |y(t)-T_2(x^*,y^*)(t)+T_2(x^*,y^*)(t)-y^*(t)|$$
$$\leq (L_f+L_g)R^\beta \|(x,y)-(x^*,y^*)\|_\infty + U^\beta \varepsilon_2. \tag{16}$$

From (15) and (16) it follows that

$$\|(x,y)-(x^*,y^*)\|_\infty$$
$$\leq ((L_f+L_h)R^\alpha + (L_f+L_g)R^\beta)\|(x,y)-(x^*,y^*)\|_\infty + U^\alpha\varepsilon_1 + U^\beta\varepsilon_2,$$

and

$$\|(x,y)-(x^*,y^*)\|_\infty \leq \frac{U^\alpha\varepsilon_1 + U^\beta\varepsilon_2}{1-((L_f+L_h)R^\alpha + (L_f+L_g)R^\beta} = V_1\varepsilon_1 + V_2\varepsilon_2,$$

with

$$V_1 = \frac{U^\alpha}{1-((L_f+L_h)R^\alpha + (L_f+L_g)R^\beta},$$

$$V_2 = \frac{U^\beta}{1-((L_f+L_h)R^\alpha + (L_f+L_g)R^\beta}.$$

Thus, we obtain the Ulam-Hyers stability condition. ■

5.4 Application

We consider the following fractional order coupled system:

$$\begin{cases} (D^\alpha + \lambda D^{\alpha-1})x(t) = L_f \dfrac{|y(t)|}{1+|y(t)|}, \quad 1<\alpha\leq 2, \quad 0\leq t\leq T, \\ \left(D^\beta + \lambda D^{\beta-1}\right)y(t) = L_f\left(sinx(t) + (cost)x(t)\right), \quad 1<\beta\leq 2, \quad 0\leq t\leq T, \\ \nu_1 x(\eta) + \mu_1 x(T) = L_h \displaystyle\int_0^T \frac{|x(t)|}{11+|x(t)|}ds, \nu_1 y(\eta) + \mu_1 y(T) = L_h \int_0^T (siny(t) + cosy(t))ds, \\ \nu_2 x'(\eta) + \mu_2 x'(T) = L_g \displaystyle\int_0^T \frac{|x(t)|}{21+|x(t)|}ds, \nu_2 y'(\eta) + \mu_2 y'(T) = L_g \int_0^T (siny(t) + cosy(t))ds, \end{cases}$$

Here

$$f_1(t,x,y) = L_f\frac{|y|}{1+|y|}, f_2(t,x,y) = L_f(sinx + (cost)x), h_1(x) = L_h\frac{|x|}{11+|x|},$$

$$h_2(y) = L_h(siny + cosy), g_1(x) = L_g \frac{|x|}{21 + |x|}, g_2(y) = L_g(siny + cosy)$$

As

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \leq L_f|y_1 - y_2|, |f_2(t, x_1, y_1) - f_2(t, x_2, y_2)| \leq L_f|x_1 - x_2|,$$

$$|h_1(t, x_1) - h_1(t, x_2)| \leq L_h|x_1 - x_2|, |h_2(t, y_1) - h_2(t, y_2)| \leq L_h|y_1 - y_2|,$$

$$|g_1(t, x_1) - g_1(t, x_2)| \leq L_g|x_1 - x_2|, |g_2(t, y_1) - g_2(t, y_2)| \leq L_g|y_1 - y_2|,$$

therefore (A_1) is satisfied. It is obvious that $L_f, L_h, L_g > 0$ can be chosen so that condition (A_2) is satisfied. Therefore, coupled system (1) has a unique solution and Ulam-Hyers stable.

5.5 Conclusion

Here we have studied the existence and uniqueness of the solutions as well as the Ulam-Hyers stability for a coupled sequential fractional system with integral boundary conditions. As a future work, one can generalize a different concepts of stability and existence results to an impulsive fractional system, a neutral time-delay system/inclusion, and a time-delay system/inclusion with finite delay. neutral time-delay system/inclusion, and a time-delay with finite delay.

REFERENCES

[1] Kilbas, A.A., Marichev, O.I., Samko, S.G.(1993) : Fractional Integrals and Derivatives (Theory and Applications). *Gordon and Breach*, Switzerland.

[2] Miller, K.S., Ross, B (1993) .: An Introduction to the Fractional Calculus and Fractional Differential Equations. *Wiley*, New York .

[3] Podlubny, I (1999) .: Fractional Differential Equations, Mathematics in Science and Engineering. *Academic Press*, New York.

[4] Hilfer, R.:(2000) Applications of Fractional Calculus in Physics. *World Scientific*, Singapore.

[5] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.:(2006) Theory and Applications of Fractional Differential Equations. *Elsevier*, Amsterdam

[6] Benchohra, M., Graef, J.R., Hamani, S.:(2008) Existence results for boundary value problems with nonlinear fractional differential equations. *Appl. Anal*. 87, 851–863 .

[7] Ahmad, B., Sivasaundaram, S.(2010) : On four-point non-local boundary value problems of non-linear integro-differential equations of fractional order. *Appl. Math. Comput.* 217, 480–487

[8] Agarwal, R.P., Benchohra, M., Hamani, S(2010) .: A survey on existence equations and inclusions. *Acta Appl. Math*. 109, 973–1033

[9] Wang, J., Li, X.(2016) : A uniformed method to Ulam–Hyers stability for some linear fractional equations. *Mediterr. J. Math*. 13, 625–635

[10] Wang, J., Ibrahim, A.G., Fečkan, M.(2015) : Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces. *Appl. Math. Comput*. 257, 103–118

[11] Wang, J., Fečkan, M., Zhou, Y.(2016) : A survey on impulsive fractional differential equations. Fract. Calc. *Appl. Anal*. 19, 806–831

[12] Wang, J., Fečkan, M., Zhou, Y.(2017) : Fractional order differential switched systems with coupled nonlocal initial and impulsive conditions. *Bull. Sci. Math.* 141, 727–746

[13] Wang, J., Zhou, Y., Fečkan, M.(2012) : Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl*. 64, 3389–3405

[14] Yang, D., Wang, J., O'Regan, D.(2018) : On the orbital Hausdorff dependence of differential equations with non-instantaneous impulses. *C. R. Acad. Sci*. Paris, Ser. I. 356, 150–171

[15] Bai, Z., Dong, X., Yin, C.(2016) : Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions. *Bound. Value Probl.* 2016, 63

[16] Shah, K., Khan, R.A.(2015) : Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti-periodic boundary conditions. *Differ. Equ. Appl.* 7, 245–262

[17] Shah, K., Khalil, H., Khan, R.A.(2015) : Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations. *Chaos Solitons Fractals* 77, 240–246

[18] Wang, Y., Liu, L., Wu, Y.(2011) : Positive solutions for a nonlocal fractional differential equation. *Nonlinear Anal*. 74, 3599–3605

[19] Zhang, X., Liu, L., Wu, Y.(2012): Existence results for multiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives. *Appl. Math. Comput*. 219, 1420–1433

[20] Zhang, X., Mao, C., Liu, L., Wu, Y.(2017) : Exact iterative solution for an abstract fractional dynamic system model for bioprocess. Qual. *Theory Dyn. Syst*. 16, 205–222

[21] Zhou, Y., Vijayakumar, V., Murugesu, R.(2015) : Controllability for fractional evolution inclusions without compactness. *Evol. Equ. Control Theory* 4, 507–524

[22] Sudsutad, W., Tariboon, J.(2012) : Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions. *Adv. Differ. Equ*. 2012 10 pp.

[23] Gafiychuk, V., Datsko, B., Meleshko, V., Blackmore, D.(2009) : Analysis of the solutions of coupled nonlinear fractional reaction-diffusion equations. *Chaos Solitons Fractals* 41, 1095–1104

[24] Shah, K., Khan, R.A.(2016) : Iterative scheme for a coupled system of fractional-order differential equations with three-point boundary conditions. *Math. Methods Appl. Sci*. 2016 7 pp.

[25] Ahmad, B., Nieto, J.J.(2009) : Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Comput. Math. Appl*. 58, 1838–1843

[26] Dong, X., Bai, Z., Zhang, S.(2017) :Positive solutions to boundary value problems of p-Laplacian with fractional derivative. *Bound. Value Probl*. 2017,5

[27] Bai, Z., Zhang, S., Sun, S., Yin, C.(2016) : Monotone iterative method for fractional differential equations. *Electron. J. Differ. Equ.* 2016, 6

[28] Yang, W.(2012) : Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations. *Comput. Math. Appl*. 63, 288–297

[29] Wang, G.(2010) : Boundary value problems for systems of nonlinear integro-differential equations with deviating arguments. J. *Comput. Appl. Math.* 234, 1356–1363

[30] Jiang, J.Q., Liu, L.S., Wu, Y.H.(2009) : Second-order nonlinear singular Sturm–Liouville problems with integral boundary conditions. *Appl. Math. Comput.* 215, 1573–1582

[31] Sudsutad, W., Tariboon, J.(2012) : Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions. *Adv. Differ. Equ*. 2012, 93

[32] Wang, Y., Yang, Y.(2014) : Positive solutions for a high-order semipositone fractional differential equation with integral boundary conditions. J. *Appl. Math. Comput*. 45, 99–109

[33] Wang, X., Wang, L., Zeng, Q.(2015) : Fractional differential equations with integral boundary conditions. J. *Nonlinear Sci. Appl*. 8, 309–314

[34] Cabada, A., Hamdi, Z.(2014) : Nonlinear fractional differential equations with integral boundary value conditions. *Appl. Math. Comput*. 228, 251–257

[35] Li, Y., Shah, K., Khan, R.A.(2017) : Iterative technique for coupled integral boundary value problem of non-integer order differential equations. *Adv. Differ. Equ.* 2017, 251

[36] Li, Y., Sang, Y., Zhang, H.(2016) : Solvability of a coupled system of nonlinear fractional differential equations with fractional integral conditions. J. *Appl. Math. Comput.* 50, 73–91

[37] Ulam, S.M.(1940): Problems in Modern Mathematics, Chapter 6. *Wiley*, New York

[38] Hyers, D.H.(1941) : On the stability of the linear functional equation. Proc. *Natl. Acad. Sci*. 27, 222–224

[39] Su, X.(2009) Boundary value problem for a coupled system of nonlinear fractional differential equations. *Appl. Math.* Lett. , 22, 64–69.

[40] S. K. Ntouyas and M. Obaid,2012, "A coupled system of fractional differential equations with nonlocal integral boundary conditions,*" Advances in Difference Equations*, vol. no. 1, pp. 130-139, 2012.

[41] B. Ahmad and S. K. Ntouyas,(2017) A Coupled System of Nonlocal Fractional Differential Equations with Coupled and Uncoupled Slit-Strips-Type Integral Boundary Conditions, *Journal of Mathematical Sciences*, vol. 226, no. 3, pp. 175-196, .

[42] H. H. Alsulami, S. K. Ntouyas, R. P. Agarwal, B. Ahmad and A. Alsaedi(2017) A study of fractional-order coupled systems with a new concept of coupled non-separated boundary conditions,*Boundary Value Problems*, vol. no. 1, pp. 68-74, 2017.

[43] Granas, A, Dugundji, J(2005): Fixed Point Theory. *Springer*, New York.

[44] Rus, Ioan A. (2010)"Ulam stabilities of ordinary differential equations in a Banach space." *Carpathian journal of Mathematics*: 103-107.

[45] Ahmad, Bashir, and Juan J. Nieto.(2013) "Boundary value problems for a class of sequential integrodifferential equations of fractional order." *Journal of Function Spaces and Applications*.

[46] Mahmudov, Nazim I., and Areen Al-Khateeb. (2019):Stability, Existence and Uniqueness of Boundary Value Problems for a Coupled System of Fractional Differential Equations. *Mathematics* 7.4 354.

[47] Mahmudov, Nazim I., Sameer Bawaneh, and Areen Al-Khateeb.(2019): On a Coupled System of Fractional Differential Equations with Four Point Integral Boundary Conditions.*Mathematics* 7.3 (2019): 279.

[48] Mahmudov, Nazim I., and Areen Al-Khateeb.(2019)Existence and Ulam–Hyers stability of coupled sequential fractional differential equations with integral boundary conditions.*Journal of Inequalities and Applications* 2019.1: 165.