Cosmology with Variable Physical Constants

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Physics

Eastern Mediterranean University
January 2020
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

The speed of light $c,$ is taken to be a constant in a vacuum. This forms the basic tool for the principle of General Covariance, which asserts that, all laws of Physics should take the same form in all frames of reference. Without putting inflation into consideration, the theory of varying speed of light (VSL) would solve basic problems of cosmology in the early universe. Furthermore, the constants, Λ and G that occurred in the Friedmann Equations may not have been real constants in the early universe but have some variation with the universe scale factor. We obtained solutions to the cosmological model of flat FRW Universe where Λ and G are taken to be variables. We used the solution to obtain deceleration parameter and state finder parameter. We also found out that the Hubble constant will be a constant if and only if Λ = 0 but conversely, it is inversely proportional to the cosmic time. For VSL, we have used the power law where we have taken $c \propto a^{-r}$ and $\Lambda \propto a^{-2r}$. For $r = 0$, we obtained the expansion rate of the universe and for $r > 0$, by numerical solution we observed that the expansion of the universe accelerating for flat space geometry where $k = 0$. Also, we assumed that $c \propto \dot{a}$, this enabled us to obtain the expression for the age of the universe for this model. We also studied the model of Linearly Varying Deceleration Parameter which reveals that the universe evolves from a big bang and ends with a big rip.

Keywords: Deceleration Parameter, Friedmann Equation, FRW Metric, CMB, Equation of State.

ÖZ

Işık c'nin hızı bir vakumda sabit olarak alınır. Bu, tüm Fizik yasalarının tüm referans çerçevelerinde aynı formu alması gerektiğini öne süren Genel Kovaryans ilkesi için temel aracı oluşturur. Enflasyon dikkate alınmadan, değişen ışık hızı teorisi (VSL) erken evrende kozmolojinin temel problemlerini çözecektir. Ayrıca, Friedmann Denklemlerinde meydana gelen Λ ve G sabitleri, erken evrende gerçek sabit olmayabilir, ancak evren ölçek faktörü ile bazı farklılıklar gösterebilir. FR ve G değişkenlerinin alındığı düz FRW Evreninin kozmolojik modeline çözümler elde ettik. Çözümü yavaşlama parametresi ve durum bulucu parametresi elde etmek için kullandık. Ayrıca Hubble sabitinin yalnızca $\Lambda = 0$ ise ve bunun tersi olarak, kozmik zamanla ters orantılı olduğu zaman sabit olacağını öğrendik. VSL için, $c \propto a$ ^ (- r) ve $\Lambda \propto a$ ^ (- 2r) aldığımız güç yasasını kullandık. R = 0 için, evrenin genişleme oranını elde ettik ve r> 0 için sayısal çözümle, k = 0 olan düz uzay geometrisi için evrenin genişlemesinin hızlandığını gözlemledik. Ayrıca, $c \propto a$'nın, bu model için evrenin yaşı için ifade almamızı sağladığını varsaydık. Ayrıca, evrenin büyük bir patlamadan evrildiğini ve büyük bir kopuşla sona erdiğini ortaya koyan Doğrusal Değişen Yavaşlama Parametresi modelini de inceledik.

Anahtar Kelimeler: Yavaşlama Parametresi, Friedmann Denklemi, FRW Metrik, CMB, Durum Denklemi

DEDICATION

This work is dedicated to Almighty God – The Omnipotent

ACKNOWLEDGEMENT

My profound gratitude goes to Asst. Prof. Dr. Ali Övgün, you did not only supervise this work; you are my anchor and my inspiration. It is a privilege to be under your tutelage. I will always remain indebted to you for the pains and sacrifice you took to ensuring that our names are written in the sand of time. May God bless you immensely Sir.

I will also express a heart of gratitude to Prof. Dr. İzzet Sakallı – the Chairman of Physics and Chemistry Department. You are not only a leader but a role model to the upcoming generation. To Prof. Dr. Mustafa Halilsoy (the father of all), studying Physics would have been meaningless without meeting you. Your reward as a teacher is indeed in heaven. To Prof. Dr. Habib Mazharimousavi, you are an interesting personality always ready to give all you best to sustain knowledge, thanks a lot. Assoc. Prof. Dr. Mustafa Riza, you do not only teach, you make friendship thereby making Quantum Physics enjoyable despite its difficulty, God bless you Sir.

Finally, a word alone is enough to express appreciation to my family, friends and members of Bethesda Chapel (EMU Chapter) for their moral and spiritual support. I shall forever remain indebted to you.

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INTRODUCTION

The study of the Cosmos can be rooted to religious and ancient philosophers in their quest for answers to the beginning of our Universe. Many myths across the world and religious philosophy tried to predict the beginning and fate of the Universe. None of such predictions have experimental backing. The Universe or Cosmos is a richly textured, with structures on a vast range of scales which includes; planets, stars collected in the galaxies and other celestial bodies. The gravitational force of attraction binds galaxies to form clusters, embedded in the cluster of galaxies are superclusters [1-3]

Amongst the theories of the beginning of the unıverse, the Big Bang theory is the scientifically accepted theory that describes how it all begins. According to [4], George Gamow and his colleagues in 1950 constructed the history of the first few minutes of the Universe's infancy; starting from a primordial hot and dense of predominantly protons, neutrons, electrons and photons. This model later became what is known as the hot Big Bang model of cosmology [5,6]. Just immediately after the birth of the universe, our cosmos experienced a period of accelerating expansion. This is what is theoretically referred to today as cosmic inflation, the period in which the observable universe expands by a factor of $\approx e^{60}$ in size [3]. At very early times, this period of inflation was followed by a period of intense reheating of the Universe at the process, matter was created. Due to high temperatures and pressures immediately after the Big Bang, atoms could not exist in the first three hundred thousand years of Universe. Instead, matter was distributed as highly ionized plasma and photons were held together in a fog at the early universe.

The Epoch of Recombination follows; the expansion of the Universe influences a drop in temperature and density of matter to a point where atomic nuclei and electrons recombined to form atoms. At this time, photons could escape the fog and travel freely. A record of the photon at the time of their escape is termed Cosmic Microwave Background radiation (CMB) (that is, left over radiation at an early state of the Universe).

One of the most important consequences of the cooling down of the Universe and its loss of symmetry would be phase transition. The inflation theory has solved some striking questions of whether there was a Big Bang: that is the homogeneity, isotropy and the spatial flatness of the observable Universe as proposed by the Big Bang model. According to [3], inflation is governed by quantum field, which brings about a small vacuum imbalance during the period of expansion by stretching microscopic to macroscopic length scales. The small fluctuations developed to all structures in the later Universe. Although, there are still many unanswered questions, quantum perturbations during the inflation period are still the best candidates to give birth to the offspring of cosmic structures [3]. At the end of inflation, the temperature falls and the energy content of the Universe became dominantly radiant energy. The energy of the Universe during phase transition, emanated mostly from pressure-free matter or dust with equation of state identically zero, this state is termed matter dominated era. Subsequently, the Universe suddenly became dominated by dark energy; it is still in this state today.

The invention of the telescope in the 20th century and merging of new gravitation with Einstein's general relativity induced physicists and astronomers to start tackling tenaciously queries arising on the origin and fate of our Universe. This gave birth to that branch of Physics known as modern cosmology which is concerned with the study of the evolution and properties of the Universe as an entity [4]. Cosmology is concerned with the universe, including its origin, nature, evolution and possibly, predictions on the fate of the universe. Cosmologists apply laws of physics to describe the universe.

Observational Cosmology has it that, everything in the Universe is moving away from us. The recession velocities of object are measured through the Redshift z, by the concept of Doppler Effect as applied to light waves [5-8]. Galaxies recedes from us by the quantity z, defined by;

$$z = \frac{\lambda_{ob} - \lambda_{em}}{\lambda_{em}} \tag{1.1}$$

where, λ_{em} and λ_{ob} are wavelength of light from the galaxy at the point of emission and wavelength as measured by an observer respectively. The, recession velocity v, of the galaxy is obtained from

$$z = \frac{v}{c} \tag{1.2}$$

According to [2], Einstein's formulation of a mathematical model of the universe via the tools of General Relativity evolves modern cosmology evolved in 1917. General Relativity is the basic theory for deriving major results in cosmology. Its major equation, the Einstein field equation given as

$$G_{\mu\nu} = -kT_{\mu\nu} \tag{1.3}$$

Where $k = \frac{8\pi G}{c^4}$, The Einstein tensor $G_{\mu\nu}$ is the Einstein tensor which represents the deformation of space-time with respect to Minkowski space-time and $T_{\mu\nu}$ is stress-energy momentum tensor that describes the matter content of the space-time. When distance, time, mass and temperature are measured in appropriate Planck units, then the speed of light c, the Boltzmann constant k, the associated Planck constant ħ, and gravitational constant are equal unity.

Studying the spectra of galaxies in 1927, Edwin Hubble observed a shift in the spectra towards the red, indicating that the galaxies were moving away from us. The theory of General Relativity asserts that the expanding universe is due to the deformation of the geometry of space as a result of matter present. The reason while the galaxies move away from us is because they are being carried by expanding geometry, a scenario analogous to a cork passively floating down a sloping river [5].

In recent times, tandem observation of type-Ia supernovae and CMB suggests that cosmic expansion is accelerating are some of the tremendous advancements in the study of the cosmos [9]. The geometry of a homogeneous and isotropic universe is characterized by a time dependent function, a curvature parameter k, and the scaling factor $a(t)$. For a closed Universe, whose geometry is like that of a sphere, $k = 1$, if it is flat, $k = 0$, or if opened like of a Saddle, $k = -1$. Basically, the scale factor is a measure of the rate at which the geometry stretches. On the assumption that an ideal gas that has an energy density $\varepsilon(t)$ and pressure $P(t)$ can be used to model matter, Einstein's Equation gives the dynamics of the scalar factor $a(t)$ by [10-17]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \qquad (1.4)$$

Where $\frac{\dot{a}}{a}$ is the Hubble parameter and G, the gravitational constant.

Equation (1.2) is the Friedmann Equation which describes how the universe expands, what causes the expansion and how the expansion rate will change over time. If we know where it began and its transitions, we can measure its expansion rate and predict its fate. The ΛCDM (Lambda-Cold Dark Matter) is the standard of cosmology model (SMC) that asserts that the universe is created from pure energy in a big bang [17]. Our Universe present composition is 5% ordinary matter, 25% dark matter and 70% dark energy [18]. The Standard Model of Cosmology conditioned assumptions on the Standard Model of Particle Physics (SMPP) and the General Theory of Relativity (GTR). However, the theory of general relativity and the Standard Model of Particle Physics are both not complete because they do not give insight into many empirical observations. For instance, the GTR do not provide knowledge to Big Bang Cosmology, cosmic inflation, nature of dark matter, etc. In the SMC, ΛCDM the big bang is parameterized such that the general theory of relativity contains cosmological constant Λ, which is related to the dark energy [17]. The extrapolation of the universe gives temperature at a finite time and infinite energy density. Hence, the Big Bang should be associated with a singularity that the breakdown of the laws of physics. Light elements were formed after the expansion and cooling hot dense mass energy with hydrogen about 75% and lithium approximately 25% [17]. The figure below shows the estimate matter content of the universe.

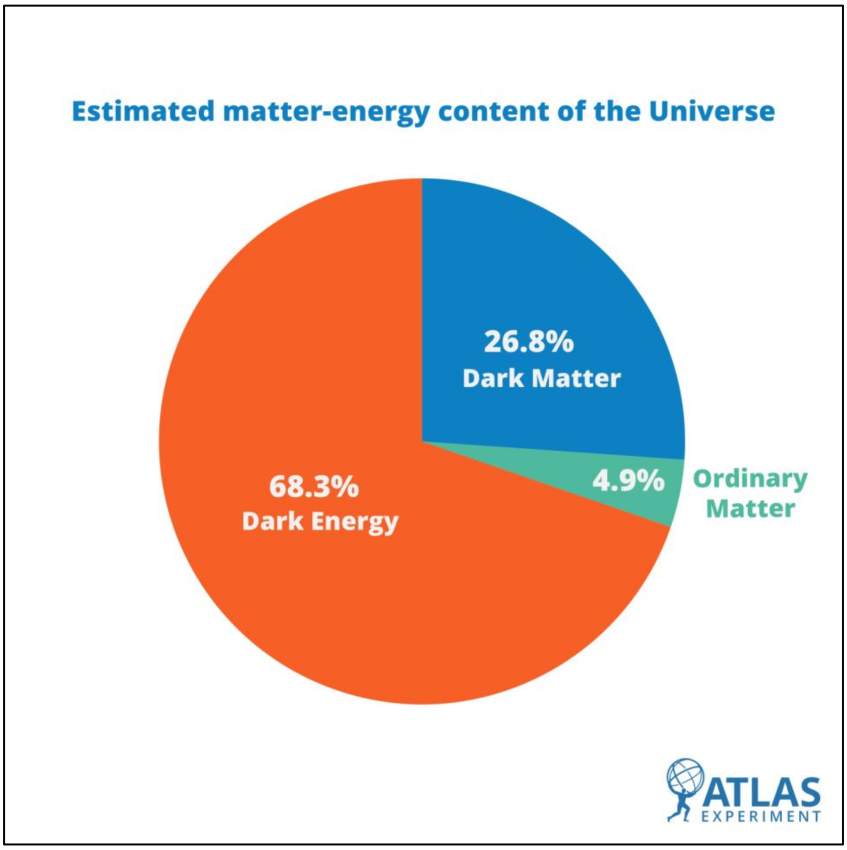

Figure 1.1: Matter content of the universe

During the expansion and cooling process, nuclei of lighter elements capture electrons to form neutral atoms. Observations revealed that if Newton's law of universal gravitation is approximately valid in the solar system, hidden more mass is required to be present in each galaxy. The invisible matter is termed dark matter whose nature remains unknown [18]. Also, observation that the expanding universe is also accelerating has also brought the notion of *dark energy*. Dark energy is hence thought of as a hypothetical form of energy that pervades the whole space and causes the expansion of the universe to accelerate at large cosmological distances . The figure below shows the accelerated expansion of the universe.

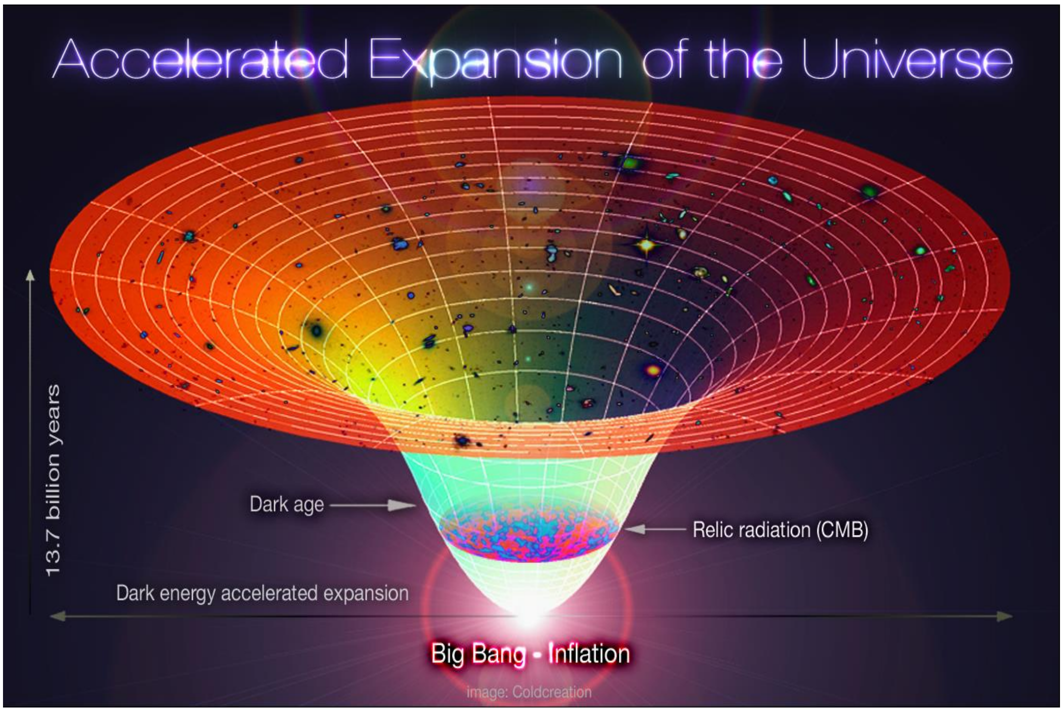

Figure 1.2: Accelerated expansion of the universe

The big bang cosmology has faced some fundamental problems such as, why are the galaxies similar, why is there isotropy in Cosmic Microwave Background (CMB)? **(Horizon Problem**). Geometry of the observable Universe is so flat (but the Friedmann Equation informs us how unnatural it is to have a flat universe). This is what is termed **Flatness Problem**. The unified theory tells us that they should be many monopoles but in reality, there is no monopole in the universe. This gives rise to **Monopole Problem**. In trying to answer the question of the origin of the galaxy and the beginning of the singularity, many scientists have applied various theories of physics to solve these problems and more is required to remove every ambiguity from the birth, evolution and fate of our universe.

Aside from the above mentioned problems, Cosmologists are now seeking answers to the following questions:

- Is dark matter a particle or superpartner? What really is it?
- What causes the accelerating expansion of the universe? Is it dark energy?
- The matter content of the universe is more than antimatter, why is this so?
- How correct is the cosmic inflation theory in the early universe? What are its phase transition and the scalar field that initiated if?
- What is the fate of the universe? Will it end in a big rip, big freeze or big crunch?

Chapter 2

NEWTONIAN COSMOLOGY

The idea of the expanding universe referred as Newtonian Cosmology was introduced in 1934 by E.A. Milne [19] and in 1955 by W.H. McCrea [42]. Newtonian dynamics and gravitation led to the formation of equations describing the time evolution of an expanding homogeneous and isotropic universe. According to [11], the large scale dynamics of the universe are governed by gravity. Therefore, the theory of gravity is the backbone of theoretical cosmology. In Newtonian cosmology, the Friedman equation is derived from the equation of motion for particles (galaxies) under the influence of gravitational force.

2.1 Hubble's Law

The relationship between distance and recession velocity of galaxies was observed by Edwin Hubble in 1929. From His observations, the concept of the expanding universe was unveiled and gave us a broader knowledge of the cosmos. The revolution in Hubble's incepts the field of observational cosmology that has shown interesting facts about the universe; its expansion and evolution and its contents such as galaxies, dark energy, and dark matter [10]. This implies that it is not up to a century that we begin to have deep knowledge about the Universe that has existed for the past 14 years [10].

According to Hubble's observations, the recession velocity of galaxies is proportionality to their distance from us. His findings also reveal that distant galaxies recede faster than closer ones. Figure (2.3) presents relationship between Hubble's observed velocities and distances from nearby galaxies showing a linear relationship between the velocity of galaxies and distance L.

If L is considered to be the proper length between a stationary observer and a distant recessing galaxy, separated by some coordinate distance l, which increases by $L(t) = a(t)l$. The distance increases at the rate given by

$$\frac{dL}{dt} = \frac{da}{dt}l = \frac{\frac{da}{dt}}{a(t)}L \qquad (2.1)$$

Or

$$v = \frac{\dot{a}}{a}L \qquad (2.2)$$

Where $\frac{\dot{a}}{a} = H_0$. This implies that the speed of recession is proportional to the separation L.

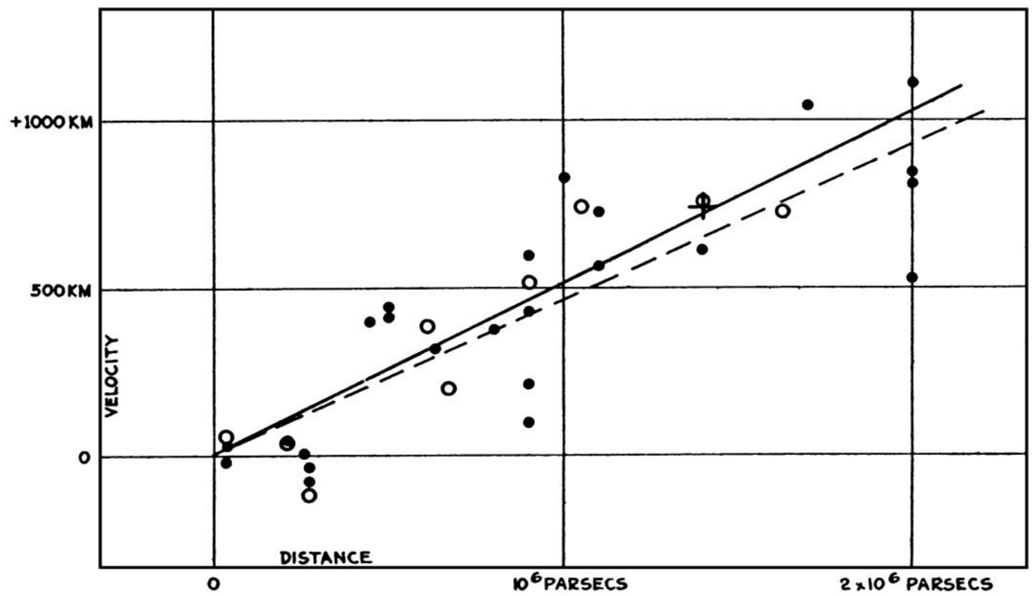

Figure 2.1: Plot of recession velocity against distance of extragalactic nebulae

The expansion rate may be constant in all directions at any given time, but the rate has been changing with time for the past 14 billion years. Where H is the Hubble constant, when expressed as a function of cosmic time $H(t)$ is known as the Hubble parameter.

At present time, the Hubble constant H_0, is obtained to be $\frac{70km}{s}/Mpc$, $1Mpc = 10^6 parsec = 3.26 \times 10^6 light\ years$ [1]. The reciprocal of the Hubble constant is the Hubble time, $t_H = \frac{1}{H_0}$ ~14 billion years, which relates to the age of the universe from big bang till present time. It presents the time from t = 0, which a linear cosmic expansion began.

2.2 The Friedmann Equations

The Friedmann's Equations are sets of equations governing the evolution of the Universe. They describe the rate at which the universe has been expanding over the past 14 billion years. They can be derived from the theory of General Relativity using the Einstein Field Equation or by Newtonian mechanics. According to Newtonian Theory, the Universe is taken to be static, but McCrea and Milne [19,42] that Friedmann Equations can be derived from the Newtonian Theory. How be it, both the two approaches give the same result. The mathematics of the Newtonian approach is simpler than the method of General Relativity. The major task for Cosmologists is solving these equations by making critical assumptions of the material content of the Universe [9]. At first, Friedmann's work describing a spatially isotropic and homogeneous expanding or contracting universe did not receive attention. It was after Hubble's results about receding galaxies were published that Einstein acknowledged the reality of the expanding Universe [1].

To derive the Friedmann equation, let us consider the universe to be homogenous of total mass M. If the sphere expands or contracts isotropically, in a manner that its radius increases or decreases with time, then it is given as R(t). If we consider a particle of mass m, at radius R(t) from a fixed point, from Newton's second law,

$$F = m\frac{d^2R}{dt^2} \tag{2.3}$$

The gravitational force F experienced by a particle of mass m, is given by

$$F = -G\frac{Mm}{R(t)^2} \tag{2.4}$$

Combining equations (2.3) and (2.4), obtain the gravitational acceleration given by

$$\frac{d^2R}{dt^2} = -G\frac{M}{R(t)^2} \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{dR}{dt}$ and integrating, we obtain

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = G\frac{M}{R(t)} + U \tag{2.6}$$

where U is a constant of integration. The quantity at the left hand side of (2.6) is the kinetic energy per unit mass E_k while, the gravitational potential energy per unit mass is given by

$$-E_P = G\frac{M}{R(t)} \tag{2.7}$$

As the sphere contracts or expands, its mass remains constant, in terms of its density, $\rho(t)$, we can write

$$M = \frac{4\pi}{3}\rho(t)R(t)^3 \tag{2.8}$$

This implies that $\rho \propto R^{-3}$ for non-relativistic particles (matter). Because the expansion of the sphere is the same at all directions about its center, its radius can be written in terms of the scale factor $a(t)$ and the comoving radius r as

$$R(t) = a(t)r \tag{2.9}$$

At current time $a(t) = 1$. Putting (2.9) and (2.8) on (2.6), we obtain

$$\frac{1}{2}r^2\dot{a}^2 = \frac{4\pi}{3}G\rho r^2 a^2 + U \quad (2.10)$$

Multiplying (2.10) by $\frac{2}{r^2a^2}$, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r^2}\frac{1}{a(t)^2} \quad (2.11)$$

Equation (2.11) is the Newtonian form of the Friedmann Equation, where $\dot{a} = \frac{da}{dt}$. If $\dot{a} < 0$ means the sphere is a contracting one with its opposite, $\dot{a} > 0$ denoting expansion. Also, considering the case where U > 0, the RHS of equation (2.11) is always positive, so $\dot{a}^2$ and the sphere expands forever. The maximum value of the scale factor is given by

$$a_{max} = -\frac{GM}{Ur} \quad (2.12)$$

after which the RHS of equation (2.11) is equal to zero, and the expansion will end and the sphere will contract. Finally, in the case U = 0, then $\dot{a} \to 0$ as t → ∞ and ρ → 0. According to [1], the correct of the Friedmann Equation with all general relativistic effects is [38]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} \quad (2.13)$$

The energy density is related to the mass density by $\varepsilon = \rho c^2$. This is a deviation from the Newtonian form of the Friedmann Equation to relativistic form. The mass density ρ, is replaced by the energy density ε, divided by the square of the speed of light and k represents the spatial curvature of the universe. The Universe is considered closed if k > 0, and has the geometry of a 3-sphere. A flat Universe is one in which k = 0, obey the laws of 3-dimensional Euclidean geometry, E^3. For k < 0, it is an open universe, and has the geometry of a 3-dimensional unbounded hyperboloid, H^3. These cases are illustrated in the figure below.

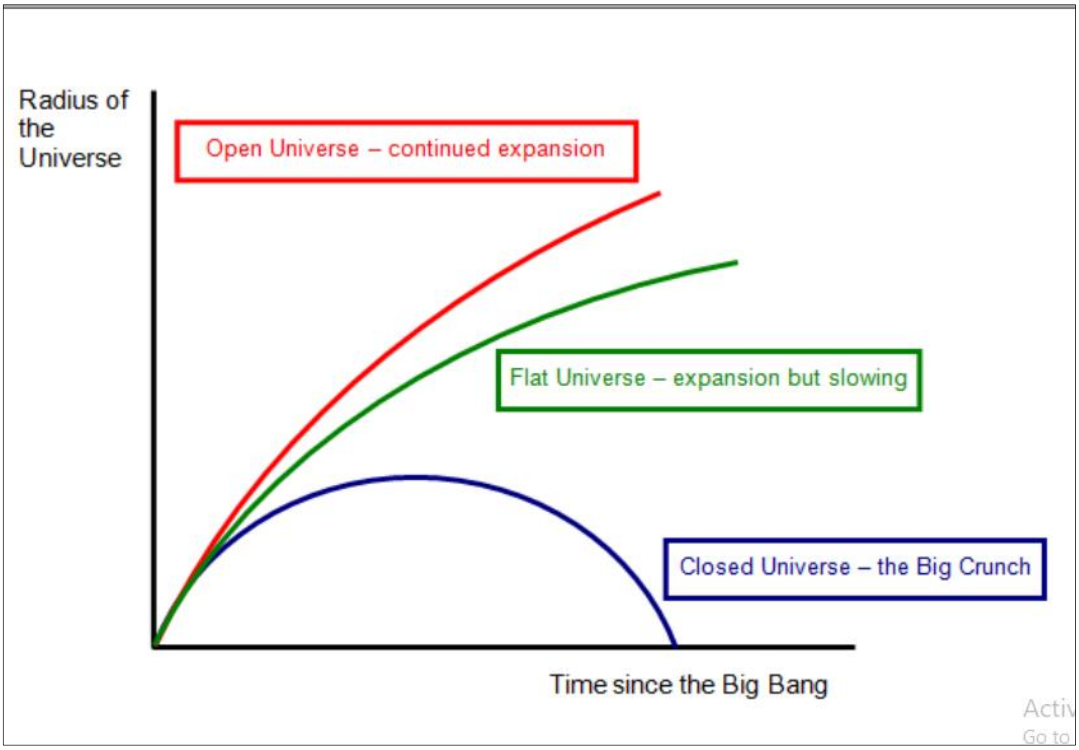

Figure 2.2: Evolution of the scale factor of the Universe with time for three possible cases of k

The possible curvature of the universe is shown in the table below:

Table 2.1: A Summary of Possible Geometries of the Universe

curvature	geometry	angles of triangle	circumference of circle	type of Universe
$k > 0$	spherical	$> 180^o$	$c < 2\pi r$	Closed
$k = 0$	flat	180^o	$c = 2\pi r$	Flat
$k < 0$	hyperbolic	$< 180^o$	$c > 2\pi r$	Open

In terms of Hubble's Parameter, the Friedmann Equation (2.13) is rewritten as

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} \tag{2.14}$$

For $k = 0$, that is flat universe, equation (2.14) becomes

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) \qquad (2.15)$$

Therefore, the critical density of the universe is defined as

$$\varepsilon_c(t) = \frac{3c^2 H(t)^2}{8\pi G} \qquad (2.16)$$

Critical density is not necessarily the true density of the Universe since the universe is not needed to be flat [9], rather, it sets a real scale for the density of the universe. In terms of energy density, Cosmologist defined a dimensionless quantity known as density parameter as

$$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)} \qquad (2.17)$$

As expressed, the density parameter Ω, is a function of time, since both ε and ε_c are time dependent, the current value of the density parameter is represented by Ω_0 [9].. Still under scrutiny [1], observations show at present day, the density parameter is in the range of $0.1 < \Omega_0 < 2$, the Fridmann equation (2.13) takes the form

$$1 - \Omega(t) = -\frac{kc^2}{R_0^2 H^2 a(t)^2} \qquad (2.18)$$

If $\Omega > 1$ at any given time, it would continue to be the same at all times. Similarly, if $\Omega < 1$, it will follow the same pattern and if it is equal to one, it will also be one at all timeAt present time, equation (2.18) is

$$1 - \Omega_0 = -\frac{kc^2}{R_0^2 H_0^2} \qquad (2.19)$$

The quantity $\frac{c}{H_0}$ is the so called Hubble distance. Equation (2.19) enables us to compute the radius of curvature R_0 of the Universe.

2.3 The Fluid Equation

The fundamental equation of cosmology, the Friedmann Equation, cannot all by itself describe the evolution of the scale factor with time. Different types of material existing in the Universe might have different pressures p, and lead to changes in its density over time. The Friedmann Equation will make more sense when combined with an equation describing how the density ρ of material in the Universe is evolving with time. We introduce the fluid equation, an equation containing p to solve for a and ε as functions of time. The fluid equation is derived from the first law of thermodynamics which states that the total energy of the universe is conserved. The actual meaning of the Friedmann Equation in the Newtonian approximation is a statement that, the sum of gravitational potential energy and kinetic energy of an expanding Universe is conserved. The first law of thermodynamics is given by

$$TdS = dE + PdV \tag{2.20}$$

where $dS = \frac{dQ}{T}$ is the change in entropy and dQ is the heat influx or efflux from a sphere of volume V with comoving radius and pressure P. A perfect homogeneous Universe is said to be adiabatic and for any given volume, $dQ = 0$, that is, there is no change in entropy. Hence, equation (2.20) takes the form

$$\dot{E} + P\dot{V} = 0 \tag{2.21}$$

If we consider a sphere of comoving radius r, expanding along with the universe expansion, its volume is given by

$$V(t) = \frac{4\pi}{3} r^3 a(t)^3 \tag{2.22}$$

The volume change at rate given by

$$\dot{V} = \frac{4\pi}{3} r^3 3a^2\dot{a} = V3\left(\frac{\dot{a}}{a}\right) \tag{2.23}$$

The internal energy $E(t)$, of the sphere is the product of its volume and energy density given by

$$E(t) = V(t)\varepsilon(t) \tag{2.24}$$

The rate of change of the internal energy is given by

$$\frac{dE}{dt} = V\frac{d\varepsilon}{dt} + \varepsilon\frac{dV}{dt} = V\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}\varepsilon\right) \tag{2.25}$$

Putting (2.24) and (2.23) into (2.21) we obtain

$$V\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}\varepsilon + 3\frac{\dot{a}}{a}P = 0\right) \tag{2.26}$$

Or

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \tag{2.27}$$

Equation (2.27) is called the Fluid Equation and the second important equation that describes the expansion of the Universe [1].

2.4 The Acceleration Equation

Combining the Friedmann Equation and the Fluid equation, we arrived at an acceleration equation which explains who the expansion of the Universe speeds up or slows down with time. The combination of these two equations is a statement of conservation of energy. Multiplying equation (2.13) by a^2, we obtain

$$\dot{a}^2 = \frac{8\pi G}{3c^2}a^2\varepsilon(t) - \frac{kc^2}{R_0^2} \tag{2.28}$$

Differentiating (2.28) with respect to t gives

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}(\dot{\varepsilon}a^2 + 2\varepsilon a\dot{a}) \tag{2.29}$$

Dividing (2.29) by $2\dot{a}a$, we obtain

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\varepsilon}\frac{a}{\dot{a}} + 2\varepsilon\right) \tag{2.30}$$

From the Fluid Equation, (2.27), we make the following simplification

$$\dot{\varepsilon}\frac{a}{\dot{a}} = -3(\varepsilon + P) \tag{2.31}$$

Substituting (2.31) into (2.30) gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) \tag{2.32}$$

Equation (2.31) is the Acceleration Equation which independent of the curvature. It contains pressure, the pressure associated with density is $P = P(\varepsilon)$. If ε is positive, the acceleration will be negative (i.e, the value of $\dot{a}$ is decreased, hence, reducing the relative velocity of any two points in the Universe).

2.5 Equation of State

To find the scale factor, energy density and pressure an equation that relates pressure and energy density. The mathematical relation is known as the equation of state given as

$$P = P(\varepsilon) \tag{2.33}$$

For cosmological substances, the equation of state is simplified as [1]

$$P = \omega\varepsilon \tag{2.34}$$

where w, is a dimensionless number, called the equation of state parameter.

For a low-density gas of non-relativistic massive particle of temperature T, and root mean square thermal velocity $\langle v^2 \rangle$ equation of state parameter is given by

$$\omega \approx \frac{\langle v^2 \rangle}{3c^2} \ll 1$$

For photons and other massless particles that are relativistic in nature, the equation of state is

$$P_{rel} = \frac{1}{3}\varepsilon_{rel} \tag{2.35}$$

Particular values of ω are of particular interest in cosmology. For a Universe with non-relativistic matter, $\omega = 0$. For a Universe filled with photons and other relativistic particles, $\omega = \frac{1}{3}$. The case where $\omega < -\frac{1}{3}$ is of great significant in cosmology because it provides a positive acceleration according to acceleration equation. The component of the Universe characterized by $\omega < -\frac{1}{3}$ indicates dark energy. For a case with $\omega = -1$, has P$= -\varepsilon$ indicates the universe contains cosmological constants.

2.6 Effect of Cosmological Constants, Λ

Einstein was one of such persons that believed in static universe but his theory of general relativity did not support a static universe. To permit a static universe he proposed a change in the equations by introducing what is termed Cosmological Constant Λ appears in the Friedmann Equation

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} + \frac{\Lambda}{3} \quad (2.36)$$

By introducing the cosmological constant, the fluid equation is not changed

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \quad (2.36b)$$

The acceleration equation takes the form

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda}{3} \quad (2.37)$$

At present time, the Friedmann Equation with cosmological constant is

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} \quad (2.38)$$

We define the critical density as

$$\varepsilon_c = \frac{3c^2 H_0^2}{8\pi G} \quad (2.39)$$

The dimensionless density parameters are defined from equation (2.38) as

$$\Omega_m = \frac{\varepsilon}{\varepsilon_c} = \frac{8\pi G\varepsilon}{3c^2 H_0^2}, \qquad \Omega_k = -\frac{kc^2}{H_0^2 R_0^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \quad (2.40)$$

The three are related conveniently by

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1 \quad (2.41)$$

2.7 A Simple Solution of the Friedmann Equation in Flat Space

From equations (2.36) and (2.34) we can write

$$\frac{\dot{\varepsilon}}{\varepsilon} = 3\frac{\dot{a}}{a}(1 + \omega) \quad (2.42)$$

We obtain the solution of the above equation to be

$$\varepsilon = \varepsilon_0 a^{-3(1+\omega)} \quad (2.43)$$

For k = 0, equation (2.28) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon_0 a^{-3(1+\omega)} \tag{2.44}$$

Or

$$\left(\frac{\dot{a}}{a}\right)^2 = {H_0}^2 a^{-3(1+\omega)}$$

Hence,

$$\dot{a} = H_0 a^{-\frac{(1+3\omega)}{2}}$$

Therefore,

$$a(t) = (H_0 t)^{\frac{2}{3(1+3\omega)}}$$

This gives the scale factor of the universe for this model.

Chapter 3

GENERAL RELATIVITY AND COSMOLOGY

The theory of General Relativity was proposed by Einstein in 1916, is a modified theory of gravity. It connects gravitational field with geometry of space-time and provides a level playground to discuss our universe. By so doing, it incorporates the gravitational field within the framework of Special Relativity to discuss the universe. The Newton's theory, which was the initial accepted theory of gravity, tends to have fatal flaws, because of its inconsistency with the theory of special relativity. According to Newton, gravitational field intensity g is given by

$$g = -\nabla\varphi, \qquad \varphi(r) = -\frac{GM}{r} \tag{3.1}$$

$\varphi(r)$ is the gravitational potential (a scalar field) and satisfied the Laplace's and Poisson's equations

$$\nabla^2\varphi = 0 \quad and \quad \nabla^2\varphi = 4\pi G\rho \tag{3.2}$$

g depends on r and not on time t. But, Special Relativity is associated with the Lorentz Covariant Field; a four-vector rather than three-vector field is both spatial and temporary dependent [14]. In this field, the equations of gravity look alike in all reference and connected by the Lorentz Transformation.

Einstein's theory of general relativity is based on two important postulates namely:

- The principle of equivalence; making use of the postulate of Galileo, the equivalence principle permits the equality of gravity and acceleration, thereby allowing a locally switching off of the gravitational effect.

- The covariance principle: this asserts that, laws of Physics should take the same form in all inertial and accelerating frames of reference.

Einstein began from the equivalence principle which is derived from the basics of Newtonian mechanics. It states that, no experiment in Physics can distinguish between a gravitational field and accelerating frame of reference. By applying the principle of the equivalence principle to a free falling body, gravity is ceased and the relativity acceleration of the body can be attributed to the curvature of space-time. The Newtonian concept of gravitation is replaced by curvature of space-time in Einstein's general relativity. Einstein summed it up as thus; mass (source of energy) causes space to curve, and curvature caused mass to move [1].

The theory of general relativity consist of the Einstein's field equation

$$G_{\mu v} = R_{\mu v} - \frac{1}{2} R g_{\mu v} = \frac{8\pi G}{C^4} T_{\mu v} \tag{3.2}$$

The description of the motion of a free falling particle is governed by space-time coordinate ξ^{α}, where $\alpha \in (0,1,2,3)$.

$$\xi^0 = ct, \quad \xi^1 = x, \ \xi^2 = y, \ \xi^3 = z \tag{3.3}$$

In Minkowski space-time, the metric is given by

$$ds^2 = \eta_{\mu v} dx^{\mu} dx^{v} = c^2 dt^2 - d\bar{x}^2 \tag{3.4}$$

Where $\mu, v = 0,1,2,3$ and the Minkowskian metric tensor is

$$\|\eta_{\mu v}\| = Diag\ (1, -1, -1, -1) \tag{3.5}$$

Under coordinate transformation, $x^{\mu} = x^{\mu}(\bar{x}^{v})$ and

$$g_{\alpha\beta} = \eta_{\alpha\beta} \tag{3.6}$$

Where $g_{\alpha\beta}$ is a symmetric metric given by $g_{\alpha\beta} = \Lambda_{\beta}^{\alpha} = \frac{\partial \xi^{\alpha}}{\partial \xi^{\beta}}$ As the particles move freely, its motion is given by four-acceleration vector whose magnitude is zero and given by

$$\frac{d^2 \xi^{\alpha}}{d\tau^2} = 0, \qquad ds^2 = c d\tau^2 \tag{3.7}$$

Infinitesimal variation $d\xi^2$ is given as

$$d\xi^{\alpha} = \frac{\partial \xi^{\alpha}}{\partial x^0} dx^0 + \frac{\partial \xi^{\alpha}}{\partial x^1} dx^1 + \cdots \tag{3.8}$$

3.1 Geodesic Equation

A geodesic, replaced the notion of a straight line in curved spacetime. A free falling particle will move along a geodesic. General Relativity envisage not to be a force but rather a consequence of a curved spacetime geometry with stress-energy tensor (matter content, for instance) as source of curvature. In particular, a path or trajectory of a particle free from external and gravitational forces is a form of geodesic. According to [14], the Geodesic equation takes the form

$$\frac{d^2 x^{\beta}}{d\tau^2} + \Gamma_{\mu\nu}^{\beta} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} = 0 \tag{3.9}$$

Where $\Gamma_{\mu\nu}^{\beta} = \frac{\partial x^{\beta}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$ denotes the Christoffel Symbol. Using Einstein's Summation, we rewrite equation (3.8) as

$$d\xi^{\alpha} = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} dx^{\mu}, \qquad d\xi^{\beta} = \frac{\partial \xi^{\beta}}{\partial x^{\nu}} dx^{\nu} \tag{3.10}$$

The invariant interval can now take the form

$$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \, dx^{\mu} dx^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{3.11}$$

Where $g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}$, is the metric tensor which has the following properties:

i. It is a symmetric tensor (i.e, $g_{\mu\nu} = g_{\nu\mu}$)

ii. Inverse matrix, defined by $g^{\mu\alpha} g_{\nu\alpha} = \delta_{\nu}^{\mu}$ (Kronecker Delta).

The metric $g_{\mu\nu}$, contains the required information concerning spacetime (curvature), it contains the needed information about gravitational field since it is equivalent to gravity.

3.2 The Christoffel Symbol (Metric Connection)

Recall that the metric tensor was given by

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}}$$

Replacing α by σ we obtain

$$g_{\mu\nu} = \eta_{\sigma\beta}\frac{\partial\xi^{\sigma}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}} \tag{3.12}$$

Also, if we replaced the index ν, with α, we have

$$g_{\alpha\mu} = \eta_{\sigma\beta}\frac{\partial\xi^{\sigma}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\alpha}} \tag{3.13}$$

Taking the partial derivative of $g_{\alpha\mu}$ with respect to x^{ν}

$$\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}}(\eta_{\sigma\beta}\frac{\partial\xi^{\sigma}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\alpha}}) = \eta_{\sigma\beta}\frac{\partial^2\xi^{\sigma}}{\partial x^{\nu}\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\alpha}} + \eta_{\sigma\beta}\frac{\partial\xi^{\sigma}}{\partial x^{\mu}}\frac{\partial^2\xi^{\beta}}{\partial x^{\nu}\partial x^{\alpha}} \tag{3.14}$$

To rewrite the Christofell symbol, given by

$$\Gamma^{\beta}_{\mu\nu} = \frac{\partial x^{\beta}}{\partial\xi^{\gamma}}\frac{\partial^2\xi^{\gamma}}{\partial x^{\mu}\partial x^{\nu}} \tag{3.15}$$

we multiply each side of the equation above by $\frac{\partial\xi^{\sigma}}{\partial x^{\beta}}$ to obtain

$$(\Gamma^{\beta}_{\mu\nu})(\frac{\partial\xi^{\sigma}}{\partial x^{\beta}}) = \frac{\partial x^{\beta}}{\partial\xi^{\gamma}}\frac{\partial^2\xi^{\gamma}}{\partial x^{\mu}\partial x^{\nu}}\left(\frac{\partial\xi^{\sigma}}{\partial x^{\beta}}\right) = (\frac{\partial x^{\beta}}{\partial\xi^{\gamma}}\frac{\partial\xi^{\sigma}}{\partial x^{\beta}})\frac{\partial^2\xi^{\gamma}}{\partial x^{\mu}\partial x^{\nu}} \tag{3.16}$$

Where $\frac{\partial x^{\beta}}{\partial\xi^{\gamma}}\frac{\partial\xi^{\sigma}}{\partial x^{\beta}} = \frac{\partial\xi^{\sigma}}{\partial\xi^{\gamma}} = \delta^{\sigma}_{\gamma} = 1$ if $\sigma = \gamma$ and 0 if $\sigma \neq \gamma$. Putting $\sigma = \gamma$

$$\left(\Gamma^{\beta}_{\mu\nu}\right)\left(\frac{\partial\xi^{\gamma}}{\partial x^{\beta}}\right) = \frac{\partial^2\xi^{\gamma}}{\partial x^{\mu}\partial x^{\nu}} \tag{3.17}$$

Substituting (3.17) into (3.14) we obtain

$$\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} = \eta_{\sigma\beta}\frac{\partial\xi^{\beta}}{\partial x^{\alpha}}\frac{\partial\xi^{\sigma}}{\partial x^{\rho}}\Gamma^{\rho}_{\mu\nu} + \eta_{\sigma\beta}\frac{\partial\xi^{\sigma}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\rho}}\Gamma^{\rho}_{\nu\alpha} \tag{3.18}$$

Equation (3.18) can be written as

$$\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} = g_{\alpha\rho}\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} \tag{3.19}$$

Swapping μ with ν in equation (3.19)

$$\frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} = g_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\Gamma^{\rho}_{\mu\alpha} \tag{3.20}$$

Again, swapping μ and α we obtain

$$\frac{\partial g_{\alpha\nu}}{\partial x^{\alpha}} = g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} + g_{\nu\rho}\Gamma^{\rho}_{\alpha\mu} \tag{3.21}$$

Adding (3.19) + (3.20) – (3.21) yield

$$\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\alpha}} = g_{\alpha\rho}\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} + g_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\Gamma^{\rho}_{\mu\alpha} - \left(g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} + g_{\nu\rho}\Gamma^{\rho}_{\alpha\mu}\right) = 2g_{\alpha\rho}\Gamma^{\rho}_{\mu\nu}$$

Or

$$g_{\alpha\rho}\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\alpha}}\right) \tag{3.22}$$

Multiplying both sides of (3.22) by $g^{\beta\alpha}$ gives

$$g^{\beta\alpha}g_{\alpha\rho}\Gamma^{\rho}_{\mu\nu} = \delta^{\beta}_{\rho}\Gamma^{\rho}_{\mu\nu} = g^{\beta\alpha}\frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\alpha}}\right) \tag{3.23}$$

Putting $\rho = \beta$ we have

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\alpha}}\right) \tag{3.24}$$

Equation (3.24) is the so called the Christoffel Symbol or Metric connection.

3.3 Covariant Derivatives

The principle of general covariance asserts that laws of physics should take the same form in all frames of reference. That is, they are expressed as balanced tensor relationships that are covariant under general coordinate transformation [2]. However, for a vector with componentV^{α}, the derivative $\frac{\partial V^{\alpha}}{\partial x^{\beta}}$ does not transform smoothly under general coordinate transformation to be a component of a rank 2 tensor. By the process of covariant differentiation, we obtain a tensor obeying the principle of general covariance. For a contravariant tensor defined as $V = V^{\alpha}e_{\alpha}$. The rate of change of the components with respect to x^{β} is

$$\frac{\partial V}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}e_{\alpha} + V^{\alpha}\frac{\partial e_{\alpha}}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}e_{\alpha} + V^{\alpha}\Gamma^{\gamma}_{\alpha\beta}\acute{e_{\gamma}} \tag{3.25}$$

Swapping γ with α in the second term of the RHS leads to

$$\frac{\partial V}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}e_{\alpha} + V^{\gamma}\Gamma^{\alpha}_{\gamma\beta}\acute{e_{\alpha}} = (\frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\gamma}\Gamma^{\alpha}_{\gamma\beta})e_{\alpha}$$

The covariant derivative of a contravariant tensor is given by

$$\nabla_{\beta}V^{\alpha} \equiv D_{\beta}V^{\alpha} \equiv V^{\alpha}{}_{;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\gamma}\Gamma^{\alpha}_{\gamma\beta} \tag{3.26}$$

Similarly, the covariant derivative of a covariant tensor is given by

$$\nabla_{\alpha}B_{\mu} \equiv D_{\alpha}B_{\mu} \equiv B_{\mu;\alpha} = \frac{\partial B_{\mu}}{\partial x^{\alpha}} - B_{\sigma}\Gamma^{\sigma}_{\alpha\mu} \tag{3.2b}$$

For other higher ranked tensors, the following equations hold

$$A^{\mu\nu}_{;\alpha} = \partial_{\alpha}A^{\mu\nu} + \Gamma^{\mu}_{\beta\alpha}A^{\beta\nu} + \Gamma^{\nu}_{\beta\alpha}A^{\beta\mu} \tag{3.27}$$

$$A_{\mu\nu;\alpha} = \partial_{\alpha}A_{\mu\nu} - \Gamma^{\beta}_{\mu\alpha}A_{\beta\nu} - \Gamma^{\beta}_{\nu\alpha}A_{\beta\mu} \tag{3.28}$$

$$A^{\mu}_{\nu;\alpha} = \partial_{\alpha}A^{\mu}_{\nu} + \Gamma^{\mu}_{\beta\alpha}A^{\beta}_{\nu} - \Gamma^{\beta}_{\nu\alpha}A^{\mu}_{\beta} \tag{3.29}$$

If we consider a vector **V**, that is parallel transported from point P around an infinitesimal flat surface of sides dx^{j} and dx^{k}. Due to the difference in curvature of the space, the victor arrived at P as $\boldsymbol{V}||P$. The difference between **V** and $\boldsymbol{V}||P$ is given by [2]:

$$\boldsymbol{V}||P^{l} - \boldsymbol{V}^{l} = \sum_{i,j,k} R^{l}_{ijk}\boldsymbol{V}^{i}dx^{j}dx^{k} \tag{3.30}$$

Where,$R\,^{l}_{ijk}$ is a measure of the curvature,in flat space, $\boldsymbol{V}||P = \mathbf{V}$ such that $R\,^{l}_{ijk} = 0$.

Under general coordinate transformation, $R\,^{l}_{ijk}$ transforms a four rank tensor with 1 contravariant index and three covariant indices [2]. $R\,^{l}_{ijk}$ is known as the **Riemann**

Curvature Tensor or simply **Riemann Tensor** which vanishes at flat space and exist at curved space. It can be shown that the Riemann tensor is given by

$$R^{\delta}_{\alpha\beta\gamma} = \frac{\partial\Gamma^{\delta}_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\delta}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\delta}_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\delta}_{\lambda\gamma} \qquad (3.31)$$

In n dimensions, the Riemann tensor has n^4components. For example, in three-dimensions, it has 81 components. Due to symmetry, in the number of components are reduced to sixteen in three-dimensional space. The contraction of the first and the last indices of the Riemann tensor gives a rank 2 tensor known as the Ricci tensor which of the form

$$R_{\alpha\beta} = g^{\delta\gamma} R_{\delta\alpha\beta\gamma} \qquad (3.32)$$

The Ricci tensor is symmetric, i.e $R_{\alpha\beta} = R_{\beta\alpha}$. A further contraction of the Ricci tensor gives

$$R = g^{\alpha\beta} R_{\alpha\beta} \qquad (3.32)$$

The quantity R is known as the *curvature scalar***.**

3.4 Stress-Energy-Momentum Tensor

The density, flux of energy and momentum in spacetime are described by a tensor quantity known as stress-energy-momentum tensor or energy-momentum tensor. In the Newtonian theory of gravity, the source of gravitation is – a conserved quantity is mass or compactly mass density. In special relativity, the mass of a particle is not a conserved quantity, but is related to its energy E, and momentum P, by [2]

$$E^2 = p^2c^2 + m^2c^4 \qquad (3.33)$$

Because there are conservation laws that relate energy and momentum, in relativistic theory the source of gravitation would not be only mass, energy and momentum are also involved. The stress-energy-momentum tensor is a source of gravitational field in Einstein field equations of General Relativity; it is the property of matter, radiation and non-gravitational fields. The energy-momentum tensor is a rank 2 tensor denoted by $T^{\mu\nu}$ specified events in spacetime by sixteen components. It is a symmetric tensor i.e, $T^{\mu\nu} = T^{\nu\mu}$, so only ten of its components are independent. It has the following general significance in the neighborhood of each event in spacetime:

- T^{00} , is the local energy density which includes all mass-energy contribution.
- $T^{oi} = T^{io}$ is the density of the i-component of momentum
- $T^{ij} = T^{ji}$ is the rate of flow the i-component of momentum per unit area perpendicular to the j-direction.

For a Universe filled with dust, the components of energy-momentum tensor is given by

$$T^{\mu\nu} = \rho U^{\mu} U^{\nu} \tag{3.34}$$

where U is a four-velocity. When described in matrix representation, the energy-momentum tensor for dust particle is given by

$$[T^{\mu\nu}] = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.35}$$

Here, the only non-zero component is T^{00} which represents the energy density. Similarly, for an ideal fluid with density ρ and pressure p that acts in every direction, the component of the energy-momentum tensor of the fluid is given by

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^{\mu} U^{\nu} - p g^{\mu\nu} \tag{3.36}$$

In matrix form, the above equation becomes

$$[T^{\mu\nu}] = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{3.37}$$

Due to conservation of relativistic energy and momentum in an inertial frame of reference, the covariant derivative of energy-momentum tensor is conserved

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.38}$$

3.5 FLRW Cosmology

To describe the Universe on large scales, the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric based on assumption of a high symmetry (Cosmological Principle) is used [2]. Since it does not have crossed terms, the FLRW metric describes an isotropic universe, and homogeneous universe due to its spherical symmetry [16]. The Friedmann-Lemaitre-Robertson-Walker is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2\right] \tag{3.39}$$

From the metric, the non-vanishing components of the Einstein tensor are

$$\left.\begin{array}{c} G_{00} = 3\left(\frac{K}{a^2} + \frac{\dot{a}}{a^2}\right) \\ G_{11} = \frac{-1}{1-Kr^2}(K + \dot{a}^2 + 2a\ddot{a}) \\ G_{22} = -r^2(K + \dot{a}^2 + 2a\ddot{a}) \\ G_{22} = -r^2 sin^2\theta(K + \dot{a}^2 + 2a\ddot{a}) \end{array}\right] \tag{3.40}$$

Assuming that the matter content of the Universe is described by ideal fluid, the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \tag{3.41}$$

Where $U^\mu = (1,\quad 0,\quad 0,\quad 0)$ and $U_\mu = g_{\mu\nu}u^\nu$

In comoving coordinates, the non-zero components of the energy-momentum tensor are

$$\left.\begin{array}{c} T_{00} = \rho \\ T_{11} = \frac{pa^2}{1-Kr^2} \\ T_{22} = pa^2r^2 \\ T_{33} = pa^2r^2 sin^2\theta \end{array}\right] \tag{3.42}$$

Using equations (3.40) and (3.42) on the Einstein Equation given by

$$G_{\mu\nu} = M_p^{-2} T_{\mu\nu} \tag{3.43}$$

We obtain two independent equations of the form

$$3\left(\frac{K}{a^2} + \frac{\dot{a}^2}{a^2}\right) = \frac{\rho}{M_p^2} \tag{3.44}$$

$$-(K + \dot{a}^2 + 2a\ddot{a}) = pa^2 \tag{3.45}$$

Solving the above two equations, we obtain the famous Friedmann equations derived earlier as

$$H^2 = \frac{8\pi\rho}{3M_p^2} - \frac{K}{a^2} \tag{3.46}$$

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{3.47}$$

Using the above two equations, we get the acceleration equation

$$\frac{\ddot{a}}{a} = \frac{-4\pi(\rho + 3p)}{6M_p^2} \tag{3.48}$$

From equation (3.46), we rewrite in terms of density parameter the Friedmann Equation as

$$\Omega - 1 = \frac{K}{a^2 H^2}$$

The homogeneity and isotropy of the metric equation (3.39) is seen in comoving coordinates (t, r, θ, φ) that define a rest frame. Stationary objects fixed values of (r, θ, φ) and physical distance between two stationary objects changes as the spacetime expands. If we consider two galaxies, A at $\overline{r_A} = (0,0,0)$ and B at $\overline{r_B} = (r,0,0)$. The physical distance measured at fixed time t $(dt = 0)$ and along a radial path $d\theta = d\varphi = 0$ is obtained by integrating the metric

$$\left.\begin{aligned} d_{AB}(t) &= \int_{X_A}^{X_B} ds = \int_0^r \sqrt{\frac{a^2(t)dr^2}{1-Kr^2}} = a(t)\int_0^r \frac{dr}{\sqrt{1-Kr^2}} \\ &= a(t)\begin{Bmatrix} \frac{1}{\sqrt{K}}\sin^{-1}\sqrt{K}r, & K > 0 \\ r, & K = 0 \\ \frac{1}{\sqrt{-K}}\sinh^{-1}\sqrt{-K}r, & K < 0 \end{Bmatrix} \end{aligned}\right| \tag{3.49}$$

Hence, the constant coordinate distance between A and B, also known as the comoving distance is

$$l_{AB}^{c} \equiv \int_0^r \frac{dr}{\sqrt{1-Kr^2}} \tag{3.50}$$

While the proper or physical distance is

$$d_{AB} = a(t) l_{AB}^{c} \tag{3.51}$$

3.6 Inflation

The Universe in its early stages of evolution involves exponential expansion in an unstable vacuum-like state. The cosmic scenario is called inflation. Inflation ended when the energy of the unstable vacuum (of a classical scalar field) transforms into the energy of hot dense matter. Subsequently, the evolution of the universe is described by hot universe theory. The standard Hot Big Bang scenario, which the very early universe dominated by radiation has three noticeable problems namely; flatness problem, horizon problem and monopole problem [1]. Alan Guth in 1987 gave a simple inflationary model in which there was supercooling during the era of phase transition [20]. Guth's model is now considered 'old model' and is problematic when considering the probability of formation of bubbles. This is because to standard problems of cosmology not being solved and the universe being anisotropic and inhomogeneous [20]. A new inflation theory asserts that inflation begins in the presence of unstable state of at peak of effective potential of false vacuum. Here inflation field rolls down slowly in effective potential that leads to homogeneous universe. The inflation field (φ) is required to have a flat plateau of its potential when $\varphi = 0$ which is another fine-tuning problem [21]. According to [20], great success was achieved in 1983 when chaotic inflation theory was introduced which solves the problems of both old and inflation theory. If we consider a scalar field φ with mass m and potential $V(\varphi)$ which has a minimum value $\varphi = 0$ define as

$$V(\varphi) = \frac{m^2}{2}\varphi^2 \quad (3.52)$$

If the universe is assumed to be homogeneous, then the Einstein's equation becomes [20].

$$H^2 + \frac{K}{a^2} = \frac{1}{6}(\varphi^2 + m^2\varphi^2) \quad (3.53)$$

3.7 Flatness Problem

From the Friedmann equation given by

$$\Omega - 1 = \frac{K}{a^2H^2} \quad (3.54)$$

The quantity a^2H^2 in the standard big bang evolution is always on the decrease. The value of Ω increases and keeps shifting away from unity. On the contrary, observations show that Ω is within the magnitude of 1, this is somewhat kind of strange and indicate that its value had to be even closer in the past. $\Omega - 1 < 0(10^{-16})$ is the required condition for during the epoch of nucleosynthesis [Liddle, 1996] and at Planck epoch the required condition is $\Omega - 1 < 0(10^{-64})$. Hence, to have the current value of Ω match with observation indicates a very fine-tuning of the conditions in the early universe. Otherwise, the universe would have collapsed after the big bang or rapidly expand and not allowing structure to form. The expansion of the universe would be thought to create a lag in its flatness, thus making it less flat. Nevertheless, the universe still maintains its flatness. This is known as flatness problem.

3.8 Horizon Problem

The evolution of the Universe over time results in wavelength of photon lesser than the Hubble radius. Particle Horizon $D_H(t)$ where photons travel from the beginning of the Universe $t = t_0$ is

$$D_H(t) = a(t)l_{AB}^c \quad (3.55)$$

Where l_{AB}^c is the comoving distance. If $t_0 = 0$, $l_{AB}^c = 3t$ and characterized an era dominated by matter. During the epoch of recombination when decoupling occurred, photons in the CMB are emitted. Causality region of photons is small, since any regions which are more than two degrees apart in CMB are causally separated at decoupling. However, due to the homogeneity of the Universe, the CMB photons are across all the sky is in thermal equilibrium with each other. This is called The Horizon Problem.

3.9 The Monopole Problem

Many modern particle theories predict the existence of magnetic monopoles, while search is still on the way for magnetic monopole which has not yet been observed. According to theoretical frameworks, these monopoles were created in the early universe during the breaking of supersymmetry. The monopoles dilutes slower than radiation, it is expected that they dominate the present universe. This completely negates the present observations. This has been termed The Monopole Problem.

Chapter 4

COSMOLOGY WITH VARIABLE PHYSICAL CONSTANTS

The principle of general covariance asserts that the laws of Physics take the same form in all frames of reference. This form one the basis in which Einstein's theory of General Relativity holds its ground. However, recent Astronomical observations from quasar linked the fine structure constant to depend on redshift – hence suggesting its variation with cosmological time [22]. According to [23], observational data relating the luminosity-redshift from type Ia supernovae now at range $z{\sim}1$. The varying speed of light theory is geared towards explaining: hard breaking the Lorentz invariance, bimetric theories (with postulates that the speeds of gravity and light are not the same), locally Lorentz invariance. By varying the speed of light, the principle of general covariance is violated and the laws of Physics are now valid only in some special frames. Varying Speed of Light (VSL) theory has successfully handled some fundamental contention in cosmology such as flatness, horizon and Lambda problem. [24], introduced the VSL scenario using the power law with $c = c_0 a(t)^n$ in which c changes from c^- to c^+ during the phase transition. Where $a(t)^n$ is the scale factor and c_0 is a constant. It is worth noting that changes in c do not influence the geometry of the Universe.

Aside c, other physical constants of nature G and Λ that occurred in both the Einstein field equation and Friedmann equation have been varied in the literatures. In the Einstein field equation for instance, the gravitational constant G serves as a coupling constant between matter and gravity [25]. At the infant stage of our universe, G appears to be time dependent [4]. In other to unify gravitation and elementary particles Physics, appendages of Einstein's Gravitation with time dependent G have been introduced [26]. By so doing, Λ which is interpreted as due to quantum mechanics and vacuum Physics is also varied. The inconsistency between observed value of the vacuum energy density and the theoretical large value of zero point-point energy as envisaged by the quantum field theory is called cosmological constant problem. [27] gives an elaborate discussion of the catastrophe and the effects of cosmology with time varying physical constants.

According to [28], the variation of the cosmological constant with time takes the form $\Lambda(t) = Bt^{-2}$ while, [29] opines that $\Lambda \propto a^{-2}$ where $a(t)$ is the scale factor in the Robertson-Walker metric. In another model, [30] proposed that $\Lambda \propto \frac{\ddot{a}}{a}$. Working on the Robertson-Walker universe with variable Λ and G, [31] proposed that $G\rho \propto \frac{\Lambda}{8\pi}$.

4.1 Cosmology with Variable G and Λ

Consider the spatially Homogeneous and isotropic Friedmann-Robertson-Walker (FRW) line element given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\varphi^2\right] \quad (4.1)$$

Where $c(t) \neq 1$ and $k = -1,\ 0,\ +1$ is the respective curvature parameter for the open, flat and closed universe. The Einstein field equation with variable c and Λ terms is given by

$$R_{\mu v} - \frac{1}{2}g_{\mu v}R + \Lambda(t) = 8\pi G T_{\mu v} \quad (4.2)$$

Assuming that the matter in the universe is represented by the energy-momentum tensor of s perfect fluid given

$$T_{\mu v} = (\rho + p)u_\mu u_v - pg_{\mu v} \tag{4.3}$$

Where ρ and p denote respective mass density and pressure of the matter and u_μ is the four velocity satisfying

$$u_\mu = (0,0,0,1) \tag{4.4}$$

Using (4.1), (4.3), (4.4) in (4.2), we obtain [31]

$$3H^2 = 8\pi G\rho + \Lambda - 3\frac{k}{a^2} \tag{4.5}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} - \Lambda = -8\pi GP \tag{4.6}$$

Taking the time derivative of equation (4.5), we obtain

$$8\pi(\dot{G}\rho + G\dot{\rho}) = -3\frac{\dot{a}}{a}\left(-2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right) - \dot{\Lambda} \tag{4.7}$$

Adding (4.5) and (4.6) yields

$$-2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} = 8\pi G(p + \rho) \tag{4.8}$$

Equation (4.7) takes the form

$$8\pi(\dot{G}\rho + G\dot{\rho}) + 3\frac{\dot{a}}{a}G\rho = -3\frac{\dot{a}}{a} - \frac{\dot{\Lambda}}{8\pi} \tag{4.9}$$

Equation (4.9) can be rewritten as

$$a^3\left(\dot{G}\rho + \frac{\dot{\Lambda}}{8\pi}\right) + G\left[\frac{d}{dt}(\rho a^3) + 3pa^2\dot{a}\right] = 0 \tag{4.10}$$

From thermodynamics perspective, the quantity $\frac{d}{dt}(\rho a^3) + 3pa^2\dot{a} = 0$ is the famous conservation equation. Hence, equation (4.10) reduces to [31]

$$\dot{G}\rho + \frac{\dot{\Lambda}}{8\pi} = 0 \tag{4.11}$$

Combining equations (4.5) and (4.6), we get

$$3\ddot{a} = -4\pi GR[3p + \rho - \frac{\Lambda}{4\pi G}] \tag{4.12}$$

Using the proposal theory that [31]

$$G\rho \propto \frac{\Lambda}{8\pi} \ or\ G\rho = \frac{\alpha\Lambda}{8\pi}, \Lambda \propto \frac{1}{a^2} \ ie\ \Lambda = \frac{\beta}{a^2} \tag{4.13}$$

Where α and β are constants, and applying the EOS of the form $\mathrm{p} = \mathrm{w}\rho$, $(0 \leq \mathrm{w} \leq 1$ to the conservation equation we get that,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1+w) \tag{4.14}$$

From equation (13), we have

$$\dot{\Lambda} = -2\beta\frac{\dot{a}}{a^3} \tag{4.15}$$

Combining equation (4.15) with (4.11), we obtain

$$\dot{G}\rho = \frac{\beta}{4\pi}a^{-3}\dot{a} \tag{4.16}$$

From equation (4.16) and (4.13) yield,

$$\frac{\dot{G}}{G} = \frac{2}{\alpha}\frac{\dot{a}}{a} \tag{4.17}$$

Using equation (4.11)

$$\dot{G}\rho = \frac{\dot{\Lambda}}{8\pi} \tag{4.18}$$

From equation (4.13) and (4.18) we obtain

$$\frac{\dot{G}}{G} = -\frac{1}{\alpha}\frac{\dot{\Lambda}}{\Lambda} \tag{4.19}$$

Using (4.17) and (4.19), we get

$$\frac{\dot{\Lambda}}{\Lambda} = -2\frac{\dot{a}}{a} \tag{4.20}$$

Taking the integral of equation (4.14), (4.19) and (4.20) with the initial conditions $\rho = \rho_{0,}\ t = t_{0,}\ G = G_{0,}\ \Lambda = \Lambda_0$ we obtain

$$\rho = \rho_0(\frac{a}{a_0})^{-(3+w)},\ G = G_0(\frac{a}{a_0})^{\frac{2}{\alpha}},\ \Lambda = \Lambda_0(\frac{a_0}{a})^2 \tag{4.21}$$

Putting equation (4.13) into (4.5) we obtain

$$\dot{a} = \sqrt{\left(\frac{1+\alpha}{3}\right)\Lambda a^2 - k} = \sqrt{\left(\frac{1+\alpha}{3}\right)\beta - k} \tag{4.22}$$

Applying the EOS and equation (4.13) into (4.12), we get

$$3\ddot{a} = [1 - \frac{\alpha(1+3w)}{2}]\frac{\alpha}{a} \quad (4.23)$$

In this model, as the density decreases from ρ to ρ_0, G increases to G_0 and R expands from R to R_0. It can be deduced from equation (4.21) that $1 + w > 0$ and $\alpha < 0$. As $G > 0$, $\rho > 0$ and $\alpha < 0$ as the universe expands. We can infer from (4.20) that $\Lambda < 0$. Also, from equation (4.11), $\rho > 0$, $\dot{G} < 0$ then $\dot{\Lambda} > 0$ such that as the universe expands, Λ must be a negative and increasing function of cosmic time. Equation (4.23) implies that $\ddot{a} < 0$ and the model is a decelerating one.

In another model, we used the equation of state of the form

$$p = (\omega - 1)\rho \quad (4.24)$$

The energy conservation equation is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad (4.25)$$

Using (4.24) in (4.25) and integrating, we get

$$\rho = \frac{B}{a^{3w}} \quad (4.26)$$

where B is a constant of integration. To obtain B, we assume that the present values of $w = w_0$, $\rho = \rho_c$ is the critical density at $t = t_0$. Equation (4.26) takes the form

$$\rho = \rho_c \frac{a_0^{3w_0}}{a^{3w}} \quad (4.27)$$

We define the deceleration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}$, for flat space, (4.5) and (4.6) respectively become

$$8\pi G\rho = 3H^2 - \Lambda \quad (4.28)$$

$$8\pi Gp = H^2(2q - 1) + \Lambda \quad (4.29)$$

Assuming that the cosmological constant varies as the square of the Hubble parameter, i.e

$$\Lambda(t) = \beta H^2 \quad (4.30)$$

Using equations (4.24), (4.28), (4.28) and (4.30) we get

$$2\dot{H} + (3-\beta)wH^2 = 0 \quad (4.31)$$

Equation (4.31) has the solution of the form

$$H(t) = \frac{-2}{(\beta-3)wt\ +\ 2B} \quad (4.32)$$

By evaluating the constant of integration B, we obtain

$$H(t) = \frac{2H_0}{H_0(3-\beta)(wt-w_0t_o)+2} \quad (4.33)$$

From (4.33) we obtain

$$t = \frac{2}{w(3-\beta)H_0}\left(\frac{H_0}{H} - 1\right) + t_0\frac{w_0}{w} \quad (4.34)$$

From equation (4.28) and (4.30), we obtain

$$\rho = \frac{1}{8\pi G}(3-\beta)H^2 \quad (4.35)$$

Integrating equation (4.33), we obtain $a(t)$ to be,

$$a(t) = C_2[(3-\beta)(tw - w_0t_0)H_0 + 2]^{\frac{2}{w(3-\beta)}} \quad (4.36)$$

Now, putting (4.36) into (4.27) we obtain

$$\rho(t) = \rho_c a_0^{3w_0} C_2^{-3w}[(3-\beta)(tw - w_0t_0)H_0 + 2]^{\frac{6}{(3-\beta)}} \quad (4.37)$$

Inserting equation (4.33) into (4.30) we find

$$\Lambda(t) = \frac{4\beta H_0^2}{[H_0(3-\beta)(wt-w_0t_o)+2]^2} \quad (4.38)$$

To find G(t), we use equation (4.11) given as $\dot{G}\rho + \frac{\dot{\Lambda}}{8\pi} = 0$ by inserting into it (4.37) and (4.38) to obtain,

$$G(t) = \frac{H_0(3-\beta)[H_0(3-\beta)(wt-w_0t_o)+2]^{\frac{2\beta}{3-\beta}}}{2\pi\rho_c a_0^{3w_0} C_2^{-3w}} \quad (4.39)$$

At this point, we can now define quantities of cosmic importance such as:

The Deceleration Parameter given as

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{w(3-\beta)}{2} \qquad (4.40)$$

The Expansion Scalar expressed as

$$\Theta = 3H = \frac{6H_0}{H_0(3-\beta)(wt - w_0 t_o) + 2} \qquad (4.41)$$

For flat universe, the density parameter is obtained as,

$$\Omega = \frac{8\pi G\rho}{3H^2} = 1 - \frac{\beta}{3} \qquad (4.42)$$

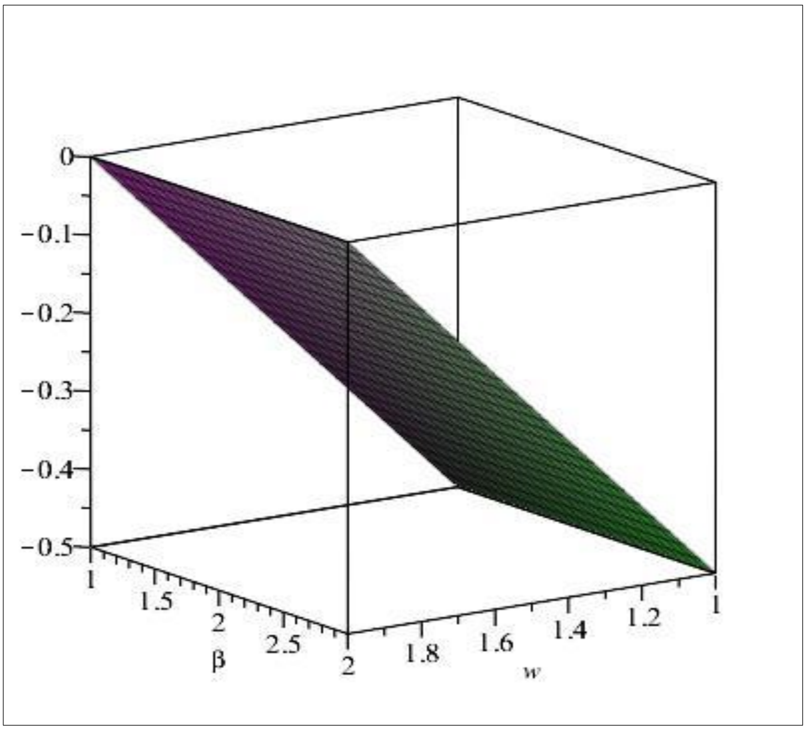

Figure 4.1: Deceleration parameter for different choices of β and w

4.2 Cosmology with Variable c and Λ

For variable c, the spatially Homogeneous and isotropic Friedmann-Robertson-Walker (FRW) line becomes

$$ds^2 = -c(t)^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2\right] \qquad (4.24)$$

Where $c(t) \neq 1$ and $k = -1,\ 0,\ +1$ remain the respective curvature parameter for the open, flat and closed universe. The Einstein field equation with variable c and Λ terms is given by

$$R_{\mu v} - \frac{1}{2} g_{\mu v} R + \Lambda(t) = \frac{8\pi G T_{\mu v}}{c^4} \quad (4.25)$$

Assuming that the matter in the universe is represented by the energy-momentum tensor of a perfect fluid given

$$T_{\mu v} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_v - p g_{\mu v} \quad (4.26)$$

Where ρ and p denote respective mass density and pressure of the matter and u_μ is the four velocity satisfying

$$u_\mu = (0,0,0,c) \quad (4.27)$$

The solution to the Einstein become [22]

$$3H^2 = 8\pi G\rho + \Lambda c^2 - 3\frac{kc^2}{a^2} \quad (4.28)$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi GP}{c^2} \quad (4.29)$$

For flat space, $k = 0$ and using the equation of state of the form

$$p = (\omega - 1)\rho c^2 \quad (4.30)$$

We obtain

$$\frac{\ddot{a}}{a} + \left(\frac{3\omega - 2}{2}\right)\left(\frac{\dot{a}}{a}\right)^2 - \omega\Lambda c^2 = 0 \quad (4.31)$$

According to [22], there is a between $c\ and\ \Lambda$ and suggest that either c to vary and Λ to be constant and vice versa. For instance, $c \propto a^{-r}$ or $c \propto H^u$ and Λ= constant corresponds to $\Lambda \propto a^{-2r}$ or $\Lambda \propto H^{2u}$ and c = constant. This follows the variation law that asserts that, one can take either c or Λ to be time depend provided on the parameters is held constant. It is obvious that equation (4.31) do not have analytical solution and [22] resolve to obtaining a numerical solution by setting Λ= constant and $\omega = 1$ and introducing the density parameters terms $\Omega_m = \frac{8\pi G}{3H^2}$ for matter and $\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$ for cosmological term. Note that the present time values for the abundance components are $\Omega_{\Lambda 0} = 0.7$ and $\Omega_{m0} = 0.3$ and the Hubble parameter is taken to be $H_o = 65 km/S/Mpc$ and the age of the universe is $T_0 = \frac{2}{3}H_0^{-1}$.

Here we have assumed that $\Lambda \propto a^{-2r}$ and $c \propto a^{-r}$ without keeping any constant and put into equation (4.31) to obtain the scale factor as a function of the cosmic time and the acceleration of the universe for the model of variable speed of light. The first instant is the case where $r = 0$, and equation (4.31) reduced to

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - 1 = 0 \qquad (4.32)$$

The solution of the above equation is given by

$$e^{\sqrt{6}t} = \frac{2a(t)^{\frac{3}{2}}\sqrt{6}e^{\frac{\sqrt{6}}{2}t}}{3} \qquad (4.33)$$

And we obtain the scale factor to be given as

$$a(t) = \left(\frac{\sqrt{6}\, e^{\sqrt{6}t}}{4e^{\frac{\sqrt{6}}{2}t}}\right)^{\frac{2}{3}} \qquad (4.34)$$

The rate of expansion of the universe is shown in figure 4.2 below. After the Big Bang scenario, the universe has been in continuous exponential expansion as seen in equation (4.34).

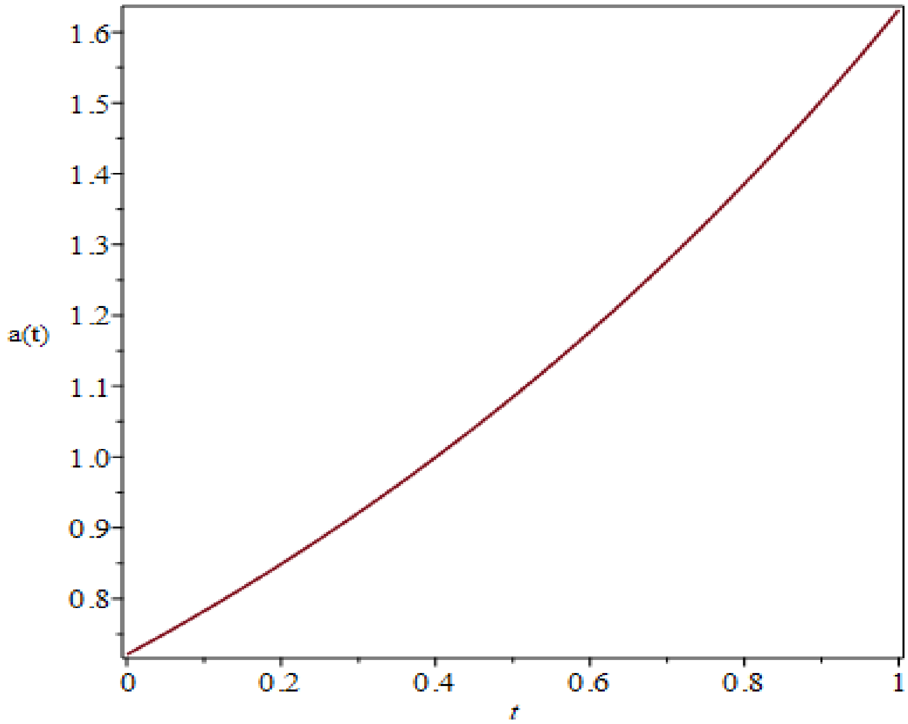

Figure 4.2: Variation of the scale factor with cosmic time

To check the acceleration of the varying speed of light model, we resolve to obtaining numerical solution of (4.31). The figure below show the cases in when $r = 0.2 \ and \ 1.0$

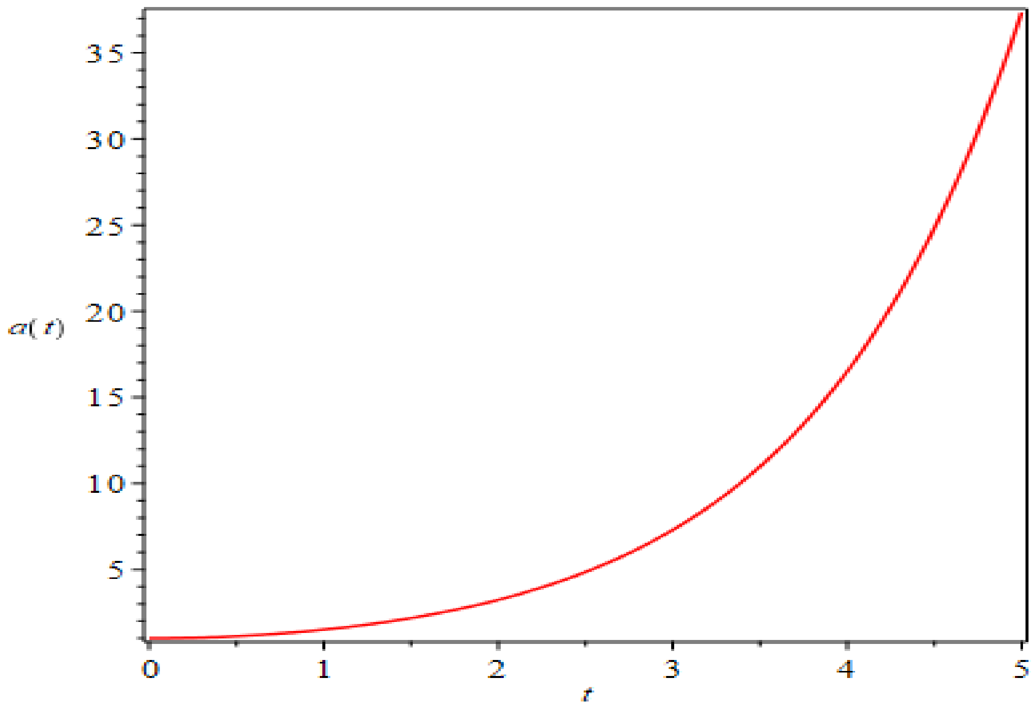

Figure 4.3: The expansion of universe for r=0.2

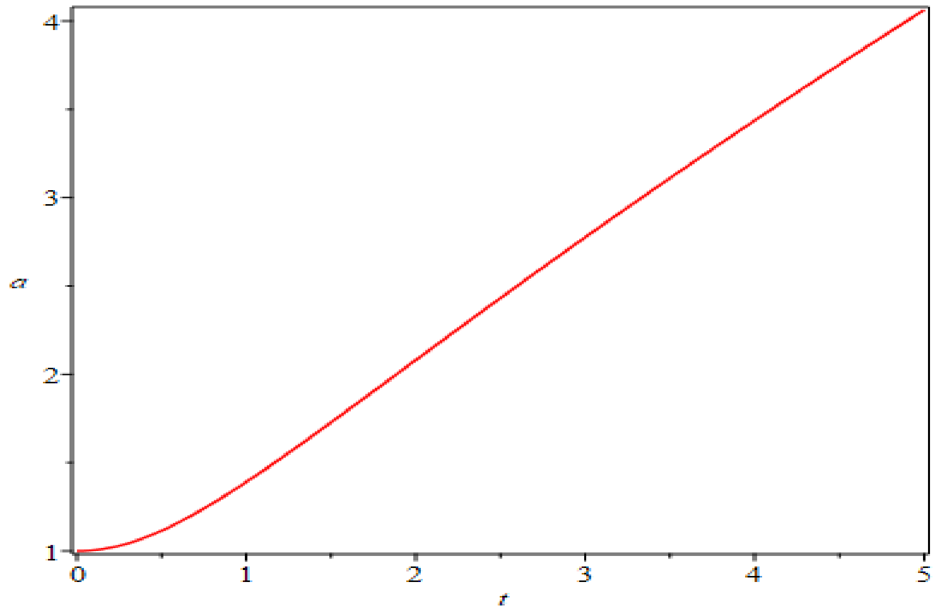

Figure 4.4: The expansion of universe for r=1.0

The solution is singular with the evolution of the universe and as $r \to 1$, the solution becomes nonsingular and inflationary model , implying $\ddot{a} \geq 0$. Hence, the model presents an accelerating one which is a good candidate for studying dark energy. The universe evolves from a singularity with $r = 0$ and undergoes rapid exponential expansion. This is characterized by what is today known as cosmic inflation. As r increases, the exponential expansion slows down as shown in figure 4.3.

$$c \propto \dot{a}(t), \ \Lambda = \frac{a_0}{a(t)^2} \tag{4.35}$$

As will be expected, $c = c_0\dot{a}(t)$ satisfies the dimensionality of physical quantities since $a(t)$ denotes length, its directional time derivative must give velocity. Therefore, equation (4.31) becomes

$$\ddot{a} + \left(\frac{3\omega-2}{2} - \frac{\omega A}{2}\right)\frac{\dot{a}^2}{a} = 0 \tag{4.36}$$

Where $A = a_0c_0^2$ is a constant.

Solving equation (9), we obtain a solution of the form

$$a(t) = C_1(A\omega t - 3\omega t + 2C_2)^{\frac{-2}{\omega(A-3)}} \tag{4.37}$$

Where C_1 and C_2 are real constants and simplifying with boundary conditions yield

$$a(t) = a_0[\left(\frac{3-A}{2}\right)\omega t H_0]^{\frac{2}{\omega(3-A)}} \tag{4.38}$$

Equation (11) gives the scale factor as a function of cosmic time for variable speed of light. The constant $A \neq 3$ but may either be negative or positive, depending on dominance of the universe. The rate of expansion of the universe is affected its content. Indeed the speed of light would not have been constant in a multiple universe, whose content changes over different cosmic epoch. Taking $a = a_0$ and $t = t_0$, we obtain the age of the universe to be

$$t_0 = \frac{2}{(3-A)\omega}{H_o}^{-1} \tag{4.39}$$

The constant A has significance in determining the current age of the universe. With adequate modification of A, we can obtain the present age of the universe with good precision than with $t_0 = \frac{2}{3(\omega+1)} {H_o}^{-1}$ model.

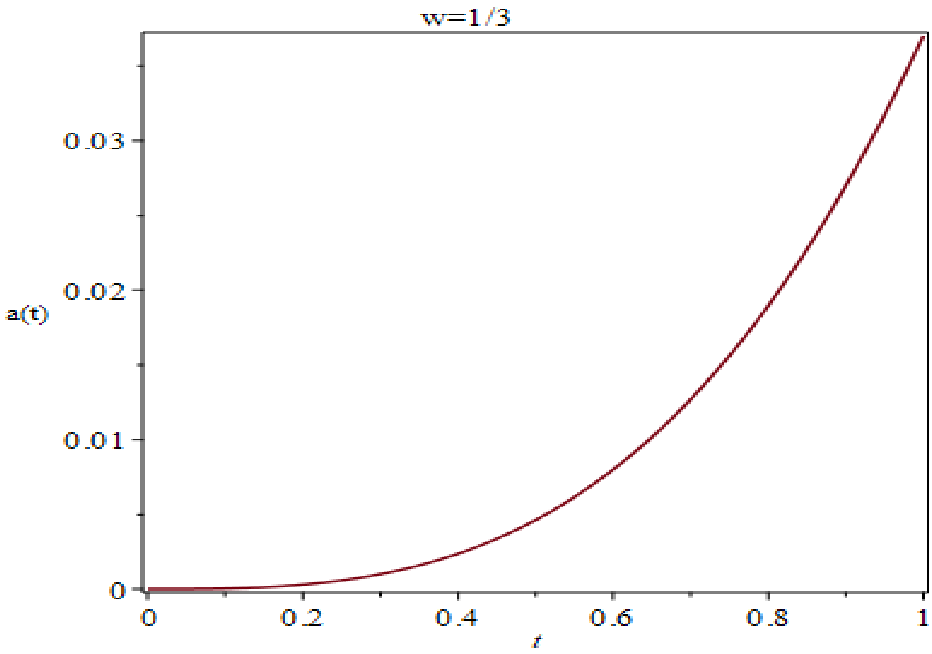

Figure 4.5: Variation of scale factor with cosmic time for c ∝ ȧ(t)

Fig shows the variation of the scale factor with cosmic as a function of cosmic time according to the power law. At $t = 0$ and $a = 0$ is characterized by a zero-sized universe with infinite density, which describes the Big Bang scenario. For a universe dominated with dust, that is $\omega = 0$ and $a = 0$ may not infer a singularity but a mathematical curiosity. If consider the case where $\omega = \frac{1}{3}$ (radiation case), we found from equation (4.35) that $\ddot{a} > 0$, which is accelerating model of the universe.

4.3 Cosmology with Linearly Varying Deceleration Parameter

The deceleration parameter q, is a factor which indicates the rate at which the universe's expansion is slowing down due to self-gravitation. In recent times, observations suggest that the rate of expansion of the universe is accelerating, thereby yielding negative values to the deceleration parameter. Berman [32],[33] proposed the theory of constant deceleration parameter (CDP) in which q is a constant. The CDP model has been used to obtain cosmological models in terms of Dark Energy (DE) and some modified theories of gravity such as $f(R)$ gravity. Berman proposed law of variation for the Hubble parameter using Robertson-Walker spacetimes and the theory of general relativity gives the deceleration parameter as

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = m - 1 \tag{4.40}$$

a is the scale factor and $m \geq 0$ is a constant. The deceleration parameter can attain the value of $q \geq 1$. The discovery of the present accelerating universe has induced the study of cosmological models in aspect of dark energy since $-1 \leq q < 0$. We view a generalized LVDP in the form [34]

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = -kt + m - 1 \tag{4.41}$$

Where $k \geq 0$ and $m \geq 0$ are constants. Equation (4.41), reduces to equation (4.40) if $k = 0$. Equation (4.41) is a generalization of (4.40). If $q > 0$, the expansion of the universe is decelerating, if $q = 0$, it is constant expansion. An accelerating power law expansion is exhibited if $-1 < q < 0$ and de Sitter expansion (exponential expansion) if $q = -1$, while $q <= -1$ corresponds to super-exponential expansion [35][36]. In the CDP model, the fastest rate of expansion is exponential expansion whereas in LVDP model, the universe without contradiction evolves into the super-exponential expansion stage except where $k = 0$ depicting the fate of the universe according to observations [36],[37]. From our metric of spacetime and the Einstein Field Equation, the Friedmann equations (4.5) and (4.6) can be written as without Λ as

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = \rho \tag{4.42}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -P \tag{4.43}$$

Now, the solutions to the solutions to the differential equation (4.41) are

$$a(t) = a_1 Exp\left[\frac{2}{\sqrt{m^2-2C_1k}}\operatorname{arctanh}\left(\frac{kt-m}{\sqrt{m^2-2C_1k}}\right)\right] \text{for } k > 0 \text{ and } m \geq 0 \tag{4.44}$$

$$a(t) = a_2(mt + C_2)^{\frac{1}{m}} \qquad \text{for } k = 0 \text{ and } m > 0 \tag{4.45}$$

$$a(t) = a_3 Exp[C_3 t] \qquad \text{for } k = 0 \text{ and } m = 0 \tag{4.46}$$

Where a_1, a_2, a_3, C_1, C_2 and C_3 are constants of integration. Equations (4.45) and (4.46) correspond to solutions to CDP model and we only dwell attention to (4.44) which corresponds to LVDP. If we put $C_1 = 0$ we obtain the scale factor to be

$$a(t) = a_1 Exp\left[\frac{2}{m}\operatorname{arctanh}\left(\frac{kt}{m} - 1\right)\right] \tag{4.46}$$

Hence, the Hubble parameter of the universe is given by

$$H = \frac{\dot{a}}{a} = \frac{2}{t(kt-2m)} \tag{4.47}$$

Substituting equation (4.46) into (4.42) and (4.43), the energy density becomes

$$\rho = \frac{12}{t^2(kt-2m)^2} + 3\frac{k}{a_1^2}Exp\left[\frac{-4}{m}\operatorname{arctanh}\left(\frac{kt}{m} - 1\right)\right] \tag{4.48}$$

And the pressure

$$p = -8\frac{kt-m+\frac{3}{2}}{t^2(kt-2m)^2} - \frac{k}{a_1^2}Exp\left[\frac{-4}{m}\operatorname{arctanh}\left(\frac{kt}{m} - 1\right)\right] \tag{4.49}$$

From the equation of state $p = \omega\rho$, the EOS parameter is obtained to be

$$\omega = \frac{-8a_1^2\left(kt-m+\frac{3}{2}\right)Exp\left[\frac{-4}{m}\operatorname{arctanh}\left(\frac{kt}{m}-1\right)\right] - \mathrm{k}t^2(kt-2m)^2}{12a_1^2 Exp\left[\frac{-4}{m}\operatorname{arctanh}\left(\frac{kt}{m}-1\right)\right] + 3\mathrm{k}t^2(kt-2m)^2} \tag{4.50}$$

The deceleration parameter as a function of redshift $z = -1 + \frac{a_{z=0}}{a}$ where $a_{z=0}$ is the present value of the scale factor is given by

$$q(z) = 2m - 1 - m\tanh\left[\frac{m}{2}\ln(z+1) - \operatorname{arctanh}\left(\frac{1+q_{z=0}}{m} - 2\right)\right] \quad (4.51)$$

Where $q_{z=0} = -m + 1$ for CDP (or $k = 0$ for the present case)

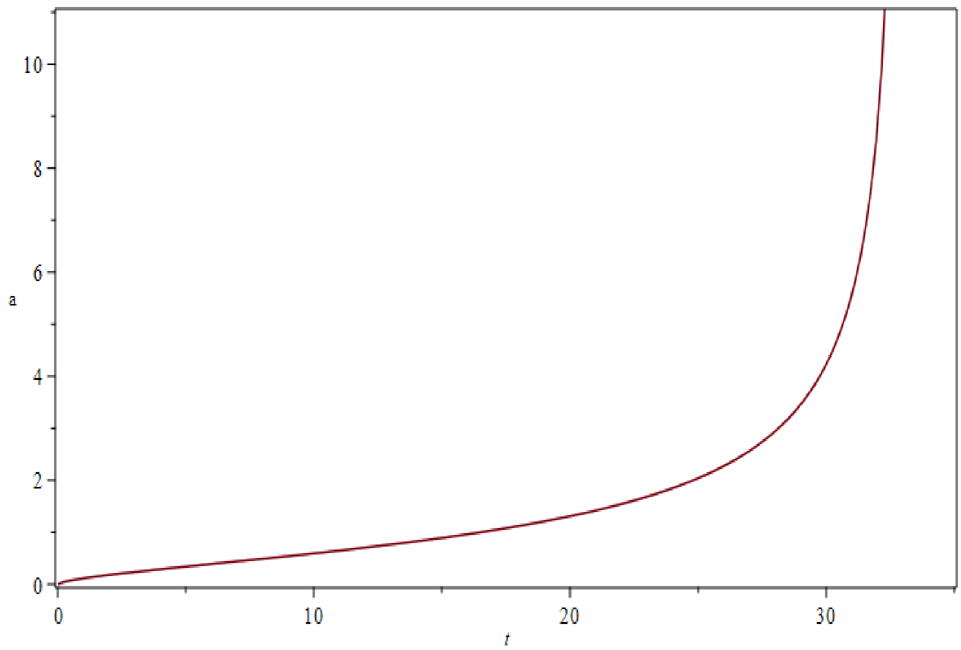

Figure 4.6: Scale factor against cosmic time, for m=1.6 and k=0.097

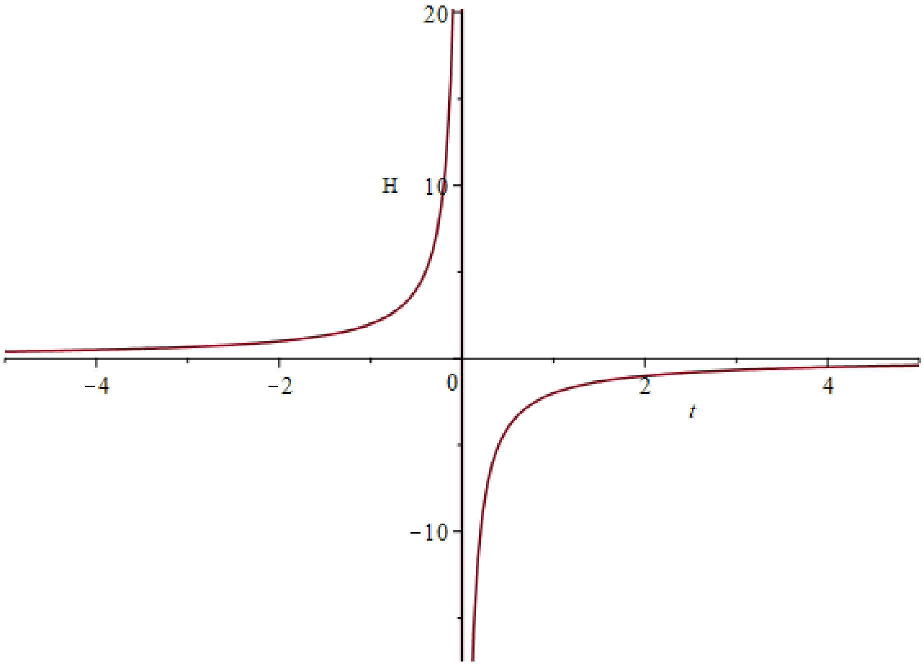

Figure 4.7: Hubble parameter against time, for m=1.6 and k=0.097

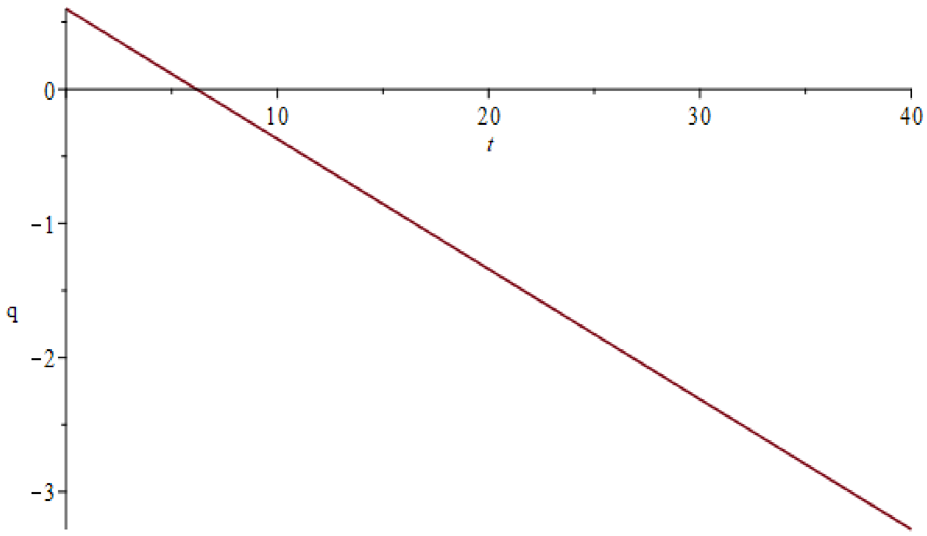

Figure 4.8: Deceleration parameter against time, for m=1.6 and k=0.097

Figure 4.5 present a plot of the scale factor against cosmic time which shows that the universe originates from a big bang at $t = 0$ and ends at a big rip at $t \cong 33$. This is closer to the lifetime of the universe given by [37] to be $t_{rip} = 35$. Figure 4.6 shows the variation of the Hubble parameter with cosmic time. It is observed that the Hubble parameter diverges at the beginning and end of the universe. A variation of the deceleration parameter with time in figure 4.7 initial $q = 0.5$ indicating that the universe started with decelerating expansion and reaches $q_{end} = -3$ which implies the phase of accelerating expansion of the universe. The universe enters this phase of accelerating expansion at $t \approx 6.1$ and $q \approx -0.2$.

Chapter 5

SUMMARY AND CONCLUSION

Indeed, the speed of light, Λ and the gravitational constant G, may not have been real physical constants at the early inhomogeneous and anisotropic universe but may be considered constants in the present day scenario [39,40,41,42]. For the time varying Λ and G case in matter dominated FWR-Universe, the Hubble constant will only be a constant if and only if Λ=0, however it has an inverse relation with cosmic time [12]. Furthermore, as proposed, for the Einstein equations with time varying G and Λ, it is observed that the conservation law asserts that $\alpha < 0$ for G > 0 and G < 0, Λ < 0 for $\dot{\Lambda} > 0$. For $\Lambda(t) = \beta H^2,$ the solution of the Friedmann equation for flat spacetime and the deceleration parameter, expansion scalar and statefinder parameter are obtained. For variable c, we proposed $\Lambda \propto a^{-2r}$ and $c \propto a^{-r}$ and obtain analytical solution of the scale for the case where $r = 0$. For $r > 0,$ we resolve to obtaining numerical solution. An interesting case is when we vary the speed of light with the rate of change of the scale factor. We found out that that the constant of variation plays an important role in determining the age of the universe under consideration. The study linearly varying deceleration parameter is a generalization of the constant deceleration parameter ansatz proposed by Berman. It is observed that the universe starts with positive deceleration parameter indicating decelerating expansion, then transits to a negative deceleration parameter which, indicates the phase of accelerating expansion.

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