Nonlinear Sequential and Non Sequential Fractional Differential Equations with Integral Boundary Conditions

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ABSTRACT

This thesis relies on various fractional differential equations. Based on the classical fixed point theorem summarized by what known as the Banach contraction mapping theorem, nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorem, a three different nonlinear fractional differential equations are considered. In chapter four we study the existence and uniqueness for the solution of the nonlinear sequential fractional differential equation involving Caputo fractional derivative and associated with nonlocal integral boundary conditions. In chapter five with a little modifications on the same problem mentioned in the previous chapter lead us to define a new function space with different norm, the boundary condition for this problem can be considered as a generalization of the boundary conditions associated with the problem in chapter four. For these two chapters we illustrate our results by examples given at the end of each one.

Whereas, in chapter six which can be considered as two parts, we investigate the existence and uniqueness for the solution of the nonlinear fractional differential equations involving Hadamard and Caputo-Hadamard fractional derivative associated with three points integral boundary conditions, for the applicability of our results we give some examples at the end of this chapter as well.

Keywords: fractional differential equation, sequential, Caputo, Hadamard, nonlocal integral boundary conditions

ÖZ

Bu tez, çeşitli kesirli diferansiyel denklemlere dayanmaktadır. Banach sabit nokta teoremi, Leray-Schuader sabitnokta teoreminin doğrusal olmayan alternatifi ve Krasnoselskii sabit nokta teoremleri kullanılarak, üç farklı doğrusal olmayan kesirli diferansiyel denklemler dikkate alınmıştır.

Dördüncü bölümde, doğrusal olmayan sıralı kesirli diferansiyel denklemin çözümü için Caputo kesirli türevi içeren ve yerel olmayan integral sınır koşullarıyla ilişkili varlığı ve tekliği inceleyeceğiz. Beşinci bölümde ise, önceki bölümde bahsedilen problem üzerinde yapılan bazı değişiklikler, farklı norma sahip yeni bir fonksiyon uzayı tanımlamamıza yol açmış, ve bu bölümdeki problemle ilişkili sınır koşulları dördüncü bölümde verilen probleme ait sınır koşullarının genelleşmesi olarak düşünülmüştür. Bu iki bölümde elde edilen sonuçlar her bölümün sonunda verilen örneklerle desteklenmiştir.

Ayrıca, iki kısma ayrılan altıncı bölümde, Hadamard ve Caputo-Hadamard kesirli türevlerini içeren ve üç nokta integral sınır koşulları verilen doğrusal olmayan kesirli diferansiyel denklemlerin çözümünün varlığı ve tekliği araştırılmıştır. Bu bölümün sonunda elde edilen sonuçların uygulaması olarak da bazı örnekler verilmiştir.

Anahtar kelimeler: kesirli diferansiyel denklemler, Sıralı-Caputo- Hadamard kesirli türevleri, yerel olmayan integral sınır koşulları

DEDICATION

To My Father

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TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

BC	Boundary Condition,
RL	Riemann-Liouville,
FDE	Fractional Differential Equation
FPT	Fixed Point Theory,
SFDE	Sequential Fractional Differential Equation,
J	Any given closed interval,
$\mathbb{R}$	The set of all real numbers,
$\mathbb{N}$	The set of all natural numbers,
$\lVert\cdot\rVert$	Norm of $\cdot$,
${}_{RL}I^{(\cdot)}$	$Riemann-Liouville$ Fractional integral of order $\cdot$,
${}_{H}I^{(\cdot)}$	$Hadamard$ Fractional integral of order $\cdot$,
${}_{RL}D^{(\cdot)}$	$Riemann-Liouville$ Fractional Derivatives of order $\cdot$,
${}_{H}D^{(\cdot)}$	$Hadamard$ Fractional Derivatives of order $\cdot$,
${}_{CH}D^{(\cdot)}$	$Caputo-Hadamard$ Fractional Derivatives of order $\cdot$,

Chapter 1

INTRODUCTION

The calculus of integrals and derivatives of any arbitrary real or complex order known as what called fractional calculus, this subject has gained more importance for the duration of the past three decades because of its widely used applications in science fields, such as physics, biology, control theory,… etc. The idea of fractional calculus is generally restricted from a question raised up in 1695 by L'Hopital to Wilhelm Leibniz who proposed the nth derivative notation

$$\frac{d^n y}{dx^n}, n = 0, 1, 2, \ldots,$$

L'Hopital asked what if $n = 1/2$? . $[1]$. "Fractional derivatives were consequently mentioned in some framework, by Euler (1730), Lagrange(1772), Laplace(1812), Lacroix(1819), Fourier(1822), Liouville(1832), Riemann(1847), Greer(1859), Holmgren (1865), Griinwald (1867), Letnikov (1868), Sonin (1869), Laurent(1884), Nekrassov (1888), Krug(1890), and Weyl(1917)".

The topic of this thesis deals with the integration I^{α} and differentiation D^{α} of random order DE's. In some fractional integro-DE's a functions of more than one variable is concerned. In this work we focus on functions with one real variable. With the aim of make the thesis suitable for the readers we start the discussion from general properties and statements of fractional calculus and pass from general cases to special ones. In addition, some areas of current applications of fractional calculus involving the theories of FDE's are Probability and Statistics, Control Theory of Dynamical Systems, Optics and Signal Processing, Chemical Physics, and so on.

Recently in the theory of FDE's many new results have been obtained. The main purpose of this thesis is to study the existence and uniqueness for the solutions of the nonlinear FDE's with BC's. The theory of FDE's involving different kinds of BC's has been a field of interest in pure and applied mathematics sciences. In addition to the classical two-point BC's, great attention is paid to non-local multipoint and integral BC's. In this work we focus our study on boundary conditions for ordinary FDE's. This thesis contains a total of seven chapters. Chapter 2 (Preliminaries and Definitions) offers some basic properties and definitions from Mathematical Analysis such as special functions, functional spaces, several fractional operators. These concepts are introduced to prepare the reader for the understanding of the applications which are established in the later chapters done in this work.

Chapter 3 (FDE's Involving (Caputo/Hadamard) FD's) provides the recent researches about the nonlinear FDE and nonlinear SFDE's involving several fractional operators such as such as ${}_{RL}I^{(\cdot)}$, ${}_{H}I^{(\cdot)}$, and fractional Derivatives such as ${}_{RL}D^{(\cdot)}$, ${}_{C}D^{(\cdot)}$, ${}_{H}D^{(\cdot)}$ and ${}_{CH}D^{(\cdot)}$. With several types of BC's.

Chapter 4 (On SFDE's with nonlocal integral BC's) motivated by recent researches done on the SFDE's. This chapter provides a new type of SFDE's with nonlocal integral BC's, where Nonlocal conditions are used to describe certain features of physical, chemical or other processes occurring in the internal positions of the given region, while integral BC's provide a plausible and practical approach to modeling the problems of blood flow. For more details, check, [1], [2]. The recent research on FDE's can be found in [3]-[17]. SFDE's have also taken a notable attention, for illustration see [5]-[10]. In this chapter based on the classical FPT, the existence and uniqueness for the SFDE's given there has been discussed, provided with some examples that illustrate the obtained results. To the best of our awareness, studying of SFDE's with nonlocal integral BC's has yet to be initiated.

Chapter 5 (Nonlinear SFDE's Involving Caputo fractional derivative with Nonlocal BC's) motivated by current researches done by authors who develop the theory of FDE's and its applications by generalizing the familiar fractional operators such as RL and Caputo fractional operators. For more specification, reader can refer to [1], [2], and [18] and references mentioned therein.

FDE's with different types of BC's has been more attractive for the researchers because of their applications in applied sciences, specially what known nonlocal multipoint and integral BC's, for more illustration, generally, the nonlocal multipoint conditions describe certain structures of physical or other processes happening in the interior locations of the given region. While integral BC's help to forming the problems of blood flow. For more clarification, reader can refer to [1], [2]. For recent work done on FDE's we suggest the reader to check the articles [19]-[37] and the references cited therein. SFDE's also received extensive attention, see [4]-[10]. In this chapter we study the classical FPT for the SFDE's with nonlocal BC's. And we finalize the chapter with examples that illustrate the obtained results. To the best of our awareness, studying the SFDE's with four-point nonlocal integral BC's has yet to be initiated.

Chapter 6 (Nonlinear FDE's Involving ${}_{H}D_{a^+}^{(\cdot)}$ and ${}_{CH}D_{a^+}^{(\cdot)}$ with 3-Point Integral BC's) provides a study for FDE's which involve both of ${}_{H}D_{a^+}^{(\cdot)}$ and ${}_{CH}D_{a^+}^{(\cdot)}$ with 3-Point Integral BC's, the existence and uniqueness for solutions of both BVP's has been investigated, for the applicability of our results some examples are introduced at the end of the chapter. To the best of our knowledge, the study of FDE's with 3-point Integral BC's has yet to be initiated. It is notable that fractional operators are more proper in describing some phenomena in applied sciences and engineering, for instance R. Almeida et al [58] investigate how the best value of the order of fractional derivative can be found to minimize the error in some statistical models, they found out that for some statistical models fractional derivative gives minimum error than using the ordinal derivatives. Finally the conclusion and future work will be discussed in chapter seven.

Also in the last decades the researchers put more efforts in studying FDE's involving RL and Capoto fractional derivative more than FDE's that contain Hadamard's fractional derivative. More details are given in [45-50], moreover FDE's involving Hadamard's fractional derivative has less attention than the one contains either R-L fractional derivative or Capoto fractional derivative. The recent work done on FDE's can be found in [38-44, 51-58].

Chapter 2

PRELIMINARIES AND DEFINITIONS

In this chapter we mention the most important mathematical tools including definitions, properties, propositions, lemmas and theorems related to the most familiar fractional integrals such as ${}_{RL}I_{a^+}^{(\cdot)}$, ${}_{H}I_{a^+}^{(\cdot)}$, and the fractional Derivatives such as ${}_{RL}D_{a^+}^{(\cdot)}$, ${}_{C}D_{a^+}^{(\cdot)}$, ${}_{H}D_{a^+}^{(\cdot)}$ and ${}_{CH}D_{a^+}^{(\cdot)}$.

2.1 Special Functions

Definition 2.1.1. $\Gamma(\upsilon)=\int_0^\infty e^{-t}t^{\upsilon-1}dt, \forall \upsilon>0,$ is called a Gamma Function.

Definition 2.1.2.

$\gamma^*(v,\upsilon)=\frac{1}{v\Gamma(v)}\int_0^\upsilon x^{v-1}e^{-x}dx,$ is called the Incomplete Gamma function.

Property 2.1.3.

(i) $\Gamma(1)=1.$

(ii) $\Gamma(\upsilon+1)=\upsilon\Gamma(\upsilon), \upsilon>0.$ If $r\in\mathbb{N}$ then $\Gamma(\upsilon+1)=\upsilon!$

Definition 2.1.4. $B(y,z)=\int_0^1 \upsilon^{y-1}(1-\upsilon)^{z-1}\,d\upsilon, \forall y,z>0$, is called a Beta Function.

Definition 2.1.5.

$B_\kappa(y,z)=\int_0^\kappa \upsilon^{y-1}(1-\upsilon)^{z-1}\,d\upsilon, 0<\kappa<1,$ is called Incomplete Betta Function.

Property 2.1.6. $\forall y,z>0,$

(i) $B(y,z)=B(z,y)$

(ii) $B(y,z)=\dfrac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)}$

2.2 Function Spaces

Given the Banach space $C[a,b]$ of all continuous functions from $[a,b]\to\mathbb{R}$ with the norm $\|g\|=\sup_{a\le t\le b}|g(t)|, \forall\ t\in[a,b]$.

$\forall\,\upsilon\ge 0,$ assume $g_\upsilon(t)=(t-a)^\upsilon\,g(t)$, define the space $C_\upsilon[a,b]$ which is the space that contains g such that $g_\upsilon\in C[a,b],$ where g is any continuous function.

space $C_\upsilon[a,b]$ endowed with the norm $\|g\|_\upsilon=\sup_{t\in[a,b]}(t-a)^\upsilon|g(t)|,$ is a Banach space as well.

Define $L^1([a,b],\mathbb{R})$ the space of measurable functions, which is also a Banach space, with the norm $\|g\|_{L^1}=\int_a^b|g(t)|dt,\ \ g:[a,b]\to\mathbb{R}$ is Lebesgue integrable function.

Definition 2.2.1. Consider the Interval $J\subseteq\mathbb{R}$. A function $g:J\to\mathbb{R}$ is absolutely continuous on J if $\forall\epsilon>0$, $\exists\,\delta(\epsilon)>0$ such that for all finite set of pairwise disjoint subintervals $(u_k,v_k)\subset J$ satisfying $\sum(v_k-u_k)<\delta$ then $\sum|g(v_k)-g(u_k)|<\epsilon$.

The collection of all absolutely continuous functions on J is denoted by $AC(J)$.

Remark 2.2.2. If $J=[a,b]$, then the following are equivalent

(i) $g\in AC[a,b]$.

(ii) g has a derivative g' almost everywhere, the derivative is Lebesegue integrable and

$$g(\tau)-g(a)=\int_a^\tau g'(t)\,dt, \forall \tau\in[a,b]$$

(iii) If there exist a Lebesegue integrable function h on $[a,b]$ such that

$$g(\tau)-g(a)=\int_a^\tau h(t)\,dt, \forall \tau\in[a,b].$$

Properties 2.2.3.

(i) If $h_1, h_2\in AC[a,b]$ then $h_1+h_2, h_1h_2\in AC[a,b]$.

(ii) If $h_1\in AC[a,b]$, $h_1\neq 0$ then $\frac{1}{h_1}\in AC[a,b]$.

(iii) If h_1 is Lipschitz continuous function then h_1 is absolutely continuous.

Definition 2.2.4.

Given the function $g:J\to\mathbb{R}$ then $g\in AC^{\upsilon}(J), \upsilon=1,2,\ldots$ if $g^{(\upsilon-1)}\in AC(J)$

Particularly, $AC^1(J)=AC(J)$.

Definition 2.2.5. Let (S,d) be a metric space. $G:S\to S$ is said to be Lipschitzian if there is $l_G\geq 0$ with $d\left(G(s_1),G(s_2)\right)\leq l_G d\,(s_1,s_2), \forall s_1,s_2\in S, s_1\neq s_2$.

According to the above definition, if G is Lipschitzian then it is continuous, when when $l_G<1$ then is said to be contraction mapping.

Theorem 2.2.6. (Banach's Contraction mapping principle).

Let (S,d) be a complete metric space, if $G:S\to S$ is a contraction mapping then

$(\mathrm{i})G$ has a unique fixed point $s\in S,$ that is $G(s)=s.$

(ii) $\forall s_0\in S,$ we have $\lim_{n\to\infty}G^n(s_0)=s,$ with

$$d\left(G^n(s_0),s\right)\le\frac{l_G^n}{1-l_G}d\left(s_0,G^n(s_0)\right).$$

Theorem 2.2.7. Given a complete metric space (S,d), $G:S\to S$ satisfying

$d\left(G(s_1),G(s_2)\right)\le\omega\left(d\left(s_1,s_2\right)\right),\forall s_1,s_2\in S,$ here $\omega:[0,\infty]\to[0,\infty]$ is any monotonic

increasing function with

$\lim_{n\to\infty}\omega^n(t)=0,$ for a fixed t>0, the G has a unique fixed point with $\lim_{x\to\infty}G^n(s_0)=s,\forall s\in S.$

Theorem 2.2.8. (Nonlinear alternative of Leray-Schauder type)

Given the open subset V of a Banach space $S,0\in V$, and let $G:\overline{V}\to S$

be a contraction such that $G\left(\overline{V}\right)$ is bounded, then either

$(\mathrm{i})G$ has a fixed point in $\overline{V},$ or

$(\mathrm{ii})\exists\mu\in(0,1)$ and $v\in\partial V$ such that $v=\mu G(v)$ holds.

Theorem 2.2.9. (Arzela-Ascoli Theorem)

$G\subset C(S,\mathbb{R})$ is compact iff it is closed, bounded and equicontinuous.

Theorem 2.2.10. (Krasnoselskii's Theorem)

Given the Banach space $(E,\|\cdot\|)$, closed convex $B\subset E, A$ is open, where $A\subset B,$

and $p\in A,$ assume that $G:\overline{A}\to B$ can be written as $G=G_1+G_2,$

In addition $G\left(\overline{A}\right)$ is bounded set in B satisfying

$(\mathrm{i}) G_1 : \overline{A} \to B$ is compact

$(\mathrm{ii}) G_2 : \overline{A} \to B$ is a contraction, $\exists \omega$ a continuous nondecreasing function $\omega : [0,\infty] \to [0,\infty]$ with $\omega(a_1) > a_1, a_1 > 0,$ such that $|G_2(a_1) - G_2(a_2)| \le \omega(\|a_1 - a_2\|),$ for any $a_1, a_2 \in \overline{A}$

then either

$(\mathrm{i}) \exists a_0 \in \overline{A}$ such that $G(a_0) = a_0$,

or

$(\mathrm{ii})\ \exists\ a \in \partial A$ and $\lambda \in (0,1)$ with $a = \lambda G(a) + (1-\lambda)\, p.$

2.3 RL-Fractional Integrals and RL-Fractional Derivatives.

Definition 2.3.1. Let $g : [a,b] \to \mathbb{R}$, then

$$\left({}_{RL}I^{\theta}_{a^+} g\right)(\tau): \text{ The left sided RL-Fractional integral} = \frac{1}{\Gamma(\theta)} \int_a^{\tau} (\tau - s)^{\theta-1} g(s)\, ds,\ a < \tau, \mathrm{Re}(\theta) > 0,$$

$$\left({}_{RL}I^{\theta}_{b^-} g\right)(\tau): \text{The right sided RL-Fractional integral} = \frac{1}{\Gamma(\theta)} \int_{\tau}^{b} (s - \tau)^{\theta-1} g(s)\, ds, \tau < b, \mathrm{Re}(\theta) > 0.$$

Remark 2.3.2. If $\theta = n, n \in \mathbb{N}$, then RL-fractional integrals are equivalent to the well-known n-th integrals of the form

$$\left({}_{RL}I^{n}_{a^+} g\right)(\tau) = \int_a^{\tau}\int_a^{s_1} \dots \int_a^{s_{n-1}} g(s_n) ds_n \dots ds_1 = \int_a^{\tau} \frac{(\tau - s)^{n-1}}{(n-1)!} g(s)\, ds.$$

$$\left({}_{RL}I^{n}_{b^-} g\right)(\tau) = \int_{\tau}^{b}\int_{s_1}^{b} \dots \int_{s_{n-1}}^{b} g(s_n) ds_n \dots ds_1 = \int_{\tau}^{b} \frac{(t - \tau)^{n-1}}{(n-1)!} g(s)\, ds.$$

whereas, the RL-fractional derivatives are defined by :

$$\left({}_{RL}D^{\theta}_{a^+} g\right)(\tau) = \left(\frac{d}{d\tau}\right)^n \int_a^{\tau} \frac{(\tau - s)^{n-\theta-1}}{\Gamma(n-\theta)} g(s)\, ds, n = [\mathrm{Re}(\theta)] + 1, \theta \in \mathbb{C}, (\mathrm{Re}(\theta) \ge 0),$$

$$\left({}_{RL}D^{\theta}_{b^-} g\right)(\tau) = \left(\frac{-d}{d\tau}\right)^n \int_{\tau}^{b} \frac{(s - \tau)^{n-\theta-1}}{\Gamma(n-\theta)} g(s)\, ds, n = [\mathrm{Re}(\theta)] + 1, \theta \in \mathbb{C}, (\mathrm{Re}(\theta) \ge 0).$$

where $[\cdot]$ is the greatest integer function.

when $\theta = n, n \in \mathbb{N} \cup \{0\}$ then by definition one can easly show the following results

(i) $\left({}_{RL}D_{a^+}^{\theta} g \right)(\tau) = \left(D_{b^-}^{0} g \right)(\tau) = g(\tau).$

(ii) $\left({}_{RL}D_{a^+}^{\theta} g \right)(\tau) = g^{(n)}(\tau).$

(iii) $\left({}_{RL}D_{b^-}^{\theta} g \right)(\tau) = (-1)^n g^{(n)}(\tau), n \in \mathbb{N}.$

Now, we consider some special case

Case1. $0 < \mathrm{Re}(\theta) < 1,$

$$\left({}_{RL}D_{a^+}^{\theta} g \right)(\tau) = \frac{d}{dx} \int_a^{\tau} \frac{(\tau - s)^{n-\theta-1}}{\Gamma(n-\theta)} g(s)\, ds \ ,$$

$$\left({}_{RL}D_{b^-}^{\theta} g \right)(\tau) = -\frac{d}{d\tau} \int_{\tau}^{b} \frac{(s - \tau)^{n-\theta-1}}{\Gamma(n-\theta)} g(s)\, ds.$$

Case 2. $\theta \in \mathbb{R}^+ \ (\mathrm{Im}(\theta) = 0),$

$$\left({}_{RL}D_{a^+}^{\theta} g \right)(\tau) = \left(\frac{d}{d\tau} \right)^n \int_a^{\tau} \frac{(\tau - s)^{n-\theta-1}}{\Gamma(n-\theta)} g(s)\, ds,$$

$$\left({}_{RL}D_{b^-}^{\theta} g \right)(\tau) = \left(\frac{-d}{dx} \right)^n \int_{\tau}^{b} \frac{(s - \tau)^{n-\theta-1}}{\Gamma(n-\theta)} g(s)\, ds.$$

Properties 2.3.3. If $\mathrm{Re}(q) \geq 0$ and $\mathrm{Re}(\theta) \geq 0,$ then

$$\left({}_{RL}I_{a^+}^{q} (t-a)^{\theta-1} \right)(x) = \frac{\Gamma(\theta)(x-a)^{q+\theta-1}}{\Gamma(q+\theta)}, \mathrm{Re}(q) > 0.$$

$$\left({}_{RL}D_{a^+}^{q} (t-a)^{\theta-1} \right)(x) = \frac{\Gamma(\theta)(x-a)^{\theta-q-1}}{\Gamma(\theta-q)}, \mathrm{Re}(q) \geq 0.$$

and

$$\left({}_{RL}I_{b^-}^{q}\,(b-t)^{\theta-1}\right)(x)=\frac{\Gamma(\theta)(b-x)^{q+\theta-1}}{\Gamma(q+\theta)},\ \mathrm{Re}(q)>0.$$

$$\left({}_{RL}D_{b^-}^{q}\,(b-t)^{\theta-1}\right)(x)=\frac{\Gamma(r)(b-x)^{\theta-q-1}}{\Gamma(\theta-q)},\ \mathrm{Re}(q)\geq 0.$$

Remark 2.3.4. It is remarkable from the above properties that for a Particular case when $\theta=1$ and $\mathrm{Re}(q)\geq 0$,

$$\left({}_{RL}D_{a^+}^{q}1\right)(x)=\frac{(x-a)^{-q}}{\Gamma(1-q)},$$

$$\left({}_{RL}D_{b^-}^{q}1\right)(x)=\frac{(b-x)^{-q}}{\Gamma(1-q)}.$$

that is ${}_{RL}D^{q}c\neq 0,\ \forall c\in\mathbb{R}..$

also, for $j=1,2,\ldots,n=\left[\mathrm{Re}(q)\right]+1,$ then

$$\left({}_{RL}D_{a^+}^{q}\,(t-a)^{n-j}\right)(x)=\left({}_{RL}D_{b^-}^{q}\,(b-t)^{n-j}\right)(x)=0.$$

Properties 2.3.5. If $\mathrm{Re}(q)>0$ and $n=\left[\mathrm{Re}(q)\right]+1$

(i) The equality $\left({}_{RL}D_{a^+}^{q}g\right)(\tau)=0,$ is valid iff $\mathrm{g}(\tau)=\sum_{l=1}^{n}c_l\,(\tau-a)^{q-l},$

(ii) $\left({}_{RL}D_{b^-}^{q}g\right)(\tau)=0$ iff $\mathrm{g}(\tau)=\sum_{l=1}^{n}d_l\,(b-\tau)^{q-l}$

where, $c_l,d_l\in\mathbb{R},\ l=1,2,\ldots,n.$

Lemma 2.3.6. If $\mathrm{Re}(q)\geq 0$ and $\mathrm{Re}(r)\geq 0,$ then the following

$${}_{RL}I_{a^+}^{q}\left({}_{RL}I_{a^+}^{r}g\right)(x)=\left({}_{RL}I_{a^+}^{q+r}g\right)(x),$$

$${}_{RL}I_{b^-}^{q}\left({}_{RL}I_{b^-}^{r}g\right)(x)=\left({}_{RL}I_{b^-}^{q+r}g\right)(x).$$

hold at almost every point in $[a,b]$, $\forall g\in L_p[a,b]$, When $q+r>1.$

It is convenient to mention here that the above lemma holds true in case when Hadamard fractional integrals are also applied. Next, some properties for Caputo fractional derivatives will be discussed.

2.4. Caputo Fractional Derivative

It is turn out that ${}_{RL}D^{(\cdot)}$ has a weak points in some real models, indeed, a new definition of fractional derivatives has to be introduced. The Caputo fractional derivative of order θ proposed by an Italian mathematician is an alternative fractional derivative to the RL- fractional derivative which given by

$$\left({}_{C}D_{a^+}^{\theta} g\right)(u) = \begin{cases} \int_a^{\upsilon} \frac{(\upsilon - u)^{n-\theta-1} h^{(n)}(u)}{\Gamma(n-\theta)} du & , n-1 < \theta < n \in \mathbb{N}, \\ g^{(n)}(u) & , \theta \in \mathbb{N}, \end{cases}$$

Caputo 1967, it is important to note that the Caputo derivative is more restrictive than the RL-fractional derivative as it requires the nth derivative of the function g . Which leads to assume that it is exist whenever the ${}_{C}D^{(\cdot)}$ is used, and fortunately in the most applications the used functions have the nth derivative.

Consider the set of functions $g(t)$, continuous and integrable in any finite interval $(0, y), y \in \mathbb{R}$. For the ${}_{C}D^{(\cdot)}$ it is required that the nth derivative of the function must integrable, Next in this study all functions are already assumed to satisfy this condition.

The following results are some main properties of the ${}_{C}D^{(\cdot)}$

(i) ${}_{C}D_{a^+}^{q} g(\tau) = {}_{RL}I_{a^+}^{n-q} D^n g(\tau)$, where D^n is the standard differentiation operator $D^n = \frac{d^n}{d\tau^n}$.

(ii) $\lim_{q\to n} {}_C D_{a^+}^q g(\tau) = g^{(n)}(\tau),$

$\lim_{q\to n-1} {}_C D_{a^+}^q g(\tau) = g^{(n-1)}(\tau) - g^{(n-1)}(0).$

(iii) ${}_C D_{a^+}^q (\alpha g(\tau) + \beta h(\tau)) = \alpha\, {}_C D_{a^+}^q g(\tau) + \beta\, {}_C D_{a^+}^q h(\tau), \alpha, \beta \in \mathbb{R}.$

(iv) ${}_C D_{a^+}^q D^n g(\tau) = {}_C D_{a^+}^{q+n} g(\tau) \neq D^n\, {}^c D_{a^+}^q g(\tau)$

(v) ${}_C D_{a^+}^q (g(\tau)h(\tau)) = \sum_{l=0}^{\infty} \binom{q}{l} (D^{q-l} g(\tau)) h^{(l)}(\tau) - \sum_{l=0}^{n-1} \frac{\tau^{l-q}}{\Gamma(l+1-q)} \left((g(\tau)h(\tau))^{(l)}(0) \right).$

(vi) ${}_C D_{a^+}^q b = 0,$ b is a constant.

(vii) ${}_C D_{a^+}^q \tau^\theta = \begin{cases} \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-q)} \tau^{\theta-q}, & n-1 < q < n,\ \theta > n-1,\ \theta \in \mathbb{R}, \\ 0, & n-1 < q < n,\ \theta \le n-1,\ \theta \in \mathbb{N}. \end{cases}$

2.5. Hadamard and Caputo-Hadamard Fractional Operators

This section deals with some properties of Hadamard and Caputo-Hadamard fractional operators. One of the most difference between these operators and the others is the kernel than contains a logarithmic function.

Definition 2.5.1. Given the continuous function g, the fractional integral of order θ proposed by Hadamard is

$${}_H I^q g(\tau) = \frac{1}{\Gamma(q)} \int_a^\tau \left(\ln\frac{\tau}{r} \right)^{q-1} \frac{g(r)}{r} dr, \quad q > 0,$$

provided that the integral exist.

Definition 2.5.2. For a continuous function $g : [a, \infty) \to \mathbb{R},$ the Hadamard fractional derivative of order $\theta > 0$ is defined as

$$ {}_{H}D^{q}g(\tau)=\left(\tau\frac{d}{d\tau}\right)^{n}\int_{a}^{\tau}\frac{1}{\Gamma(n-q)}\left(\ln\frac{\tau}{r}\right)^{n-q-1}\frac{g(r)}{r}dr, $$

$q\in(n-1,n),\ \ n=[q]+1,\ \delta=\tau\left(\frac{d}{d\tau}\right),[q]$ is the integer part of $q\in\mathbb{R}$

Definition 2.5.3. For $g:[a,\infty)\to\mathbb{R}$, where g is at least a differentiable function $n-\text{times}$, the Caputo Hadamard fractional derivative (${}^{CH}D^{q}$) of order q is

$$ {}_{CH}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{n-q-1}\delta^{n}\frac{g(s)}{s}ds, $$

Lemma 2.5.4. Let

$$ u\in C_{\delta}^{n}\left([a,T],\mathbb{R}\right),\text{where } C_{\delta}^{n}[a,T]=\left\{u:[a,T]\to\mathbb{R}:\delta^{(n-1)}u\in C[a,T]\right\} $$

then

$$ {}_{H}I^{\theta}\left({}_{H}D^{\theta}u\right)(\tau)=u(\tau)-\sum_{l=1}^{n}c_{l}\left(\ln\frac{\tau}{a}\right)^{\theta-l}, $$

$$ {}_{H}I^{\theta}\left({}_{CH}D^{\theta}u\right)(\tau)=u(\tau)-\sum_{l=0}^{n-1}c_{l}\left(\ln\frac{\tau}{a}\right)^{l}. $$

Definition 2.5.5. [59], The Caputo fractional derivative is said to be of nr sequential order if the relation

$$ \left({}_{C}D^{nr}\right)u(t)={}_{C}D^{r}\left({}_{C}D^{(n-1)r}\right)u(t) $$

holds true for $n=2,3,\ldots$

Chapter 3

FDE's INVOLVING (CAPUTO/HADAMARD) FRACTIONAL DERIVATIVE

In this chapter we mention the recent research about the FDE's that include either Caputo or Hadamard fractional derivatives. We focus on the research that include a sequential type of FDE's because recently the researchers give more efforts to investigate this type of FDE's and the existence for solution for this types of FDE's have been studied.

For instance, in [60], the authors investigate a 3-point SFDE:

$$\begin{cases} {}_C D^{\gamma}\left(D+k\right)r\left(\tau\right)=h\left(\tau,r(\tau)\right), \ 1<\gamma<2, \rho<\tau<\sigma, \\ r\left(\rho\right)=r\left(\eta\right)=r\left(\sigma\right)=0 \qquad ,-\infty<\rho<\eta<\sigma<\infty. \end{cases} \tag{3.1}$$

where $\rho,\sigma,\eta,k\in\mathbb{R},k>0.$ For the SFDE (1) a new existence and uniqueness results are obtained on arbitrary interval $\left[\rho,\sigma\right],\rho,\sigma\in\mathbb{R}.$

In [61], the authors discussed the nonlinear sequential Caputo fractional Integro-DE's with nonlocal multi-point BC's. Of the form:

$$\begin{cases} {}_C D^{\alpha-1}\left(D+k\right)y\left(t\right)=h\left(t,y(t),{}_C D^{q}y(t),{}_{RL}I^{p}y(t)\right), 2<\alpha\le 3, 0<p,q<1, 0\le t\le 1, k>0, \\ y\left(0\right)=0, \quad y'\left(0\right)=0, \ \sum_{i=1}^{r}c_i y\left(\zeta_i\right)=\mu\int_0^{\eta}\frac{\left(\eta-s\right)^{\delta-1}}{\Gamma\left(\delta\right)}y\left(s\right)ds, \delta\ge 1, 0<\eta<\zeta_1<\cdots<\zeta_r<1. \end{cases}$$

(3.2)

where $h:\left[0,1\right]\times\mathbb{R}^3\to\mathbb{R}, \mu, c_i\in\mathbb{R}, i=1,2,\ldots,r$.

For SFDE (3.2) the existence uniqueness results were obtained.

Arfal et al. [62], introduced the human immunodeficiency virus (HIV) model in fractional order scene, accordingly, motivated by this model, Jiqiang et al. [63] considered the system of FDE's

$$\begin{cases} {}_C D^{\alpha-1}(D+\lambda)p(\tau) = f(\tau, p(\tau), q(\tau)), 0<\tau<1, \\ {}_C D^{\beta-1}(D+\mu)q(\tau) = g(t, p(\tau), q(\tau)), 0<\tau<1, \\ p(0) = p'(0) = 0, \qquad p(1) = \varepsilon_1 q(\varsigma), \\ q(0) = q'(0) = 0, \qquad q(1) = \varepsilon_2 p(\zeta), \end{cases} \tag{3.3}$$

where λ, μ are parameters, $1<\alpha,\beta\le 2$, $0<\varsigma,\zeta<1, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$, given the continuity of the functions $f, g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Leray-Schauder's alternative has been applied to investigate the existence for the solutions of the above system. Where the uniqueness was obtained via Banach's contraction principle.

In [64], the existence for solutions of IVP ( ) of SFDE has been studied Based on some fundamental theorems of FPT, where the SFDE there in is given by:

$$\begin{cases} \left({}_C D^q + \theta_1 {}_C D^{q-1} + \theta_2 {}_C D^{q-2}\right) q(\tau) = g(\tau, u(\tau)), q\in(2,3), 0\le a\le\tau\le T, \\ \qquad q^{(m)}(a) = c_m,\ m = 0,1,2. \end{cases} \tag{3.4}$$

where θ_1, θ_2 are parameters, $g:[a,T]\times\mathbb{R}\to\mathbb{R}$.

In [65], the following SFDE with nonlocal RL fractional BC's

$$\begin{cases} \left({}_C D^\alpha + k\, {}_C D^{\alpha-1}\right) q(\tau) = g(\tau, q(\tau)), \alpha\in(2,3], 0\le\tau\le 1, k>0, \\ q(0) = q'(0) = 0, \qquad q(\varsigma) = \dfrac{c}{\Gamma(p)} \displaystyle\int_0^\eta (\eta - x)^{p-1} q(x) dx, p>0. \end{cases} \tag{3.5}$$

where $0<\eta<\varsigma<1$, for the SFDE (3.5) based on classical FPT the existence uniqueness results were obtained.

In [66], A new set of BC's were introduced for the following SFDE, based on some fundamental theorems of FPT, existence and uniqueness property for the following SFDE has been investigated.

$$\begin{cases} {}_C D^{\alpha-1}(D+k)u(\tau) = f(\tau,u(\tau)), 2<\alpha\le 3, 0<\tau<T, k>0, \\ c_1 u(0)+\sum_{i=1}^{m} a_i u(\eta_i)+d_1 u(T)=\varepsilon_1, \\ c_2 u'(0)+\sum_{i=1}^{m} b_i u'(\eta_i)+d_2 u'(T)=\varepsilon_2, \\ c_3 u''(0)+\sum_{i=1}^{m} c_i u''(\eta_i)+d_3 u''(T)=\varepsilon_3, \end{cases} \tag{3.6}$$

where $0\le\eta_i\le T,\ c_1,c_2,c_3,d_1,d_2,d_3,\varepsilon_1,\varepsilon_2,\varepsilon_3\in\mathbb{R}, i=1,2,\ldots,m$

In [67], the existence and uniqueness for solutions of SFDE with nonlocal BC's involving lower-order fractional derivatives. The SFDE is given as follows:

$$\begin{aligned} &\left({}_C D^{\alpha}+k\,{}_C D^{\alpha-1}\right)u(\tau)=f(\tau,u(\tau)), 2<\alpha\le 3, 0\le\tau\le T, k>0, a_1 u(\eta)+b_1 u(T)=c_1, \\ &a_2\,{}^c D^{\alpha-1}u(\eta)+b_2\,{}^c D^{\alpha-1}u(T)=c_2, \\ &a_3\,{}^c D^{\alpha-2}u(\eta)+b_3\,{}^c D^{\alpha-2}u(T)=c_3, \end{aligned} \tag{3.7}$$

where $a_i,b_i,c_i\in\mathbb{R},\quad i=1,2,3,\quad 0<\eta<T.$

For the SFDE (3.7) the uniqueness results were obtained based on theorem (2.2.8), while, the existence for the solutions obtained by the theorems (2.2.10) and (2.2.12), and what so called LeraySchauder degree theory.

Next, some recent work about Hadamard type FDE's will be mentioned and as introduced before, the most difference between Caputo and Hadamrd fractional derivatives is that in the later one the kernel function in Hadamard is a logarithmic function with arbitrary exponent.

In [68], a new set of BC's were introduced. The authors applied theorems (2.2.8), (2.2.10) and (2.2.12), and what so called LeraySchauder degree theory on The FDE which given as:

$$\begin{cases} {}_H D^p x(\tau) = g(\tau, x(\tau), y(\tau)), \tau \in [1,e], p \in (1,2] \\ {}_H D^r y(\tau) = h(\tau, x(\tau), y(\tau)), \tau \in [1,e], r \in (1,2] \\ x(1) = 0, \qquad x(e) = {}^H I^\mu x(\delta_1), \mu > 0, \delta_1 \in (1,e) \\ y(1) = 0, \qquad \mathrm{y}(e) = {}^H I^\mu y(\delta_2), \mu > 0, \delta_2 \in (1,e) \end{cases} \tag{3.8}$$

Ph. Thiramanus, S. K. Ntouyasand J. Tariboon [69] studied the Hadamard type of FDE. Given as the following:

$$\begin{cases} {}_H D^p x(\tau) + q(\tau) g(x(\tau)) = 0, \qquad \tau \in (1,\infty), \quad p \in (1,2] \\ x(1) = 0, \qquad {}^H D^{p-1} x(\infty) = \sum_{i=1}^{m} \theta_i \, {}^H I^{\beta_i} x(\eta), \end{cases} \tag{3.9}$$

where $\theta_i \geq 0,\ i = 1,2,\ldots,m.$

The existence of nonnegative multiple solutions for this type of FDE's on an unbounded domain were investigated. Examples are also presented to illustrate the obtained results.

In [70] based on the classical FPT, the following FDE was investigated

$$\begin{cases} {}_H D^p r(\tau) = g(\tau, r(\tau)), \qquad 1 < \tau < e, \quad p \in (1,2] \\ r(1) = 0, \qquad \sum_{i=1}^{m} \theta_i \, {}_H I^{\alpha_i} r(\mu_i) = \sum_{k=1}^{n} \eta_k \left({}_H I^{\beta_k} r(e) - {}_H I^{\beta_k} r(\xi_k) \right). \end{cases} \tag{3.10}$$

where $\theta_i, \eta_k \in \mathbb{R}, i = 1,2,\ldots,m,\ k = 1,2,,\ldots,n.$ Also

$\mu_1 < \mu_2 < \cdots < \mu_m,\ \xi_1 < \xi_2 < \cdots < \xi_n,\ g : [1,e] \times \mathbb{R} \to \mathbb{R}.$

In chapter 6, motivated by the mentioned researches about the Hadamard fractional operators which mentioned above and the references therein, and based on the classical FPT. A new nonlinear FDE with 3-point integral BC's were investigated. Two examples are introduced to make our results more obvious.

Chapter 4

NONLINEAR SFDE's WITH NONLOCAL BC's

In this chapter we investigate the nonlinear SFDE subject to a new set of non- separated non- local integral BC's given as

$$\begin{cases} {}_C D^{p-1}(D+\lambda)u(t) = f(t,u(t)), \quad p \in (1,2], \quad t \in [0,T], \\ a_1 u(\eta) + b_1 u(T) = d_1 \int_0^{\xi} u(s)ds, \\ a_2\, {}_C D^{p-1}u(\eta) + b_2\, {}_C D^{p-1}u(T) = d_2 \int_{\zeta}^{T} u(s)ds, \end{cases} \tag{4.1}$$

where, $\eta \in (0,T)$, $0 < \xi < \zeta < T$, $\lambda > 0$, $a_1, a_2, b_1, b_2, d_1, d_2 \in \mathbb{R}$.

Applying the classical FPT on the above FDE. Results are illustrated by examples.

The following notations has been used for the rest of this chapter:

$$\theta_{11} := a_1 e^{-\lambda\eta} + b_1 e^{-\lambda T} - \frac{d_1}{\lambda}\left(1 - e^{-\lambda\xi}\right), \quad \theta_{12} := a_1 + b_1 - d_1\xi,$$

$$\theta_{21} := a_2 \frac{\lambda}{\Gamma(2-p)} \int_0^{\eta} (\eta - s)^{1-p} e^{-\lambda s} ds + b_2 \frac{\lambda}{\Gamma(2-p)} \int_0^{T} (T-s)^{1-p} e^{-\lambda s} ds + d_2 \int_{\zeta}^{T} e^{-\lambda t} dt,$$

$$\theta_{22} := d_2(T - \zeta), \ \Delta := \theta_{11}\theta_{22} - \theta_{12}\theta_{21}, \ \Delta \neq 0,$$

$$\varphi_1(t) = \frac{\theta_{21} - \theta_{22}e^{-\lambda t}}{\Delta}, \ \varphi_2(t) = \frac{\theta_{11} - \theta_{12}e^{-\lambda t}}{\Delta},$$

$$Q_1(t,s) = \frac{1}{\Gamma(p\text{-}1)} \int_s^t e^{-\lambda(t-r)} (r-s)^{p-2} dr, \ \int_0^t Q_1(t,x)h(x)dx = \int_0^t e^{-\lambda(t-s)}\, {}_{RL}I^{p-1}h(s)ds,$$

$$Q_2(t,r)=\frac{1}{\Gamma(2\text{-}p)}\int_r^t (t-s)^{1-p}\,Q_1(s,r)\,ds\,.$$

It is obvious that

$$\left|\mu_1(t)\right|\le \max\left(\frac{\left|\theta_{21}-\theta_{22}\right|}{|\Delta|},\frac{\left|\theta_{21}-\theta_{22}e^{-\lambda T}\right|}{|\Delta|}\right):=\phi_1,$$

$$\left|\mu_2(t)\right|\le \max\left(\frac{\left|\theta_{11}-\theta_{12}\right|}{|\Delta|},\frac{\left|\theta_{11}-\theta_{12}e^{-\lambda T}\right|}{|\Delta|}\right):=\phi_2,$$

and

$$\frac{1}{\Gamma(2\text{-}p)}\int_0^{\eta}(\eta-s)^{1-p}\int_0^{s} e^{-\lambda(s-x)}\,{}_{RL}I^{\,p-1}h(x)\,dx\,ds=\frac{1}{\Gamma(2\text{-}p)}\int_0^{\eta}(\eta-s)^{1-p}\int_0^{s}Q_1(s,x)\,h(x)\,dx\,ds$$

$$=\frac{1}{\Gamma(2\text{-}p)}\int_0^{\eta}\left(\int_x^{\eta}(\eta-s)^{1-p}\,Q_1(s,x)\,ds\right)h(x)\,dx=\int_0^{\eta}Q_2(\eta,x)\,h(x)\,dx.$$

Lemma 4.1. Consider $w\in C\left(\left[0,T\right],\mathbb{R}\right)$,then the solution of the following SFDE given by

$$\begin{cases} {}_C D^{p-1}(D+\lambda)u(t)=w(t)\,, & p\in(1,2],\quad 0\le t\le T\,,\\ a_1u(\eta)+b_1u(T)=d_1\int_0^{\xi}u(s)\,ds, & \\ a_2\,{}_C D^{p-1}u(\eta)+b_2\,{}_C D^{p-1}u(T)=d_2\int_{\zeta}^{T}u(s)\,ds, & \end{cases}\tag{4.2}$$

is equivalent to:

$$u(t)=\int_0^{t}Q_1(t,s)\,w(s)\,ds$$
$$+a_1\mu_1(t)\int_0^{\eta}Q_1(\eta,s)\,w(s)\,ds+b_1\mu_1(t)\int_0^{T}Q_1(T,s)\,w(s)\,ds$$

$$-\gamma_1\mu_1(t)\int_0^{\xi}\int_0^{r}Q_1(r,s)w(s)dsdr-\gamma_2\mu_2(t)\int_{\zeta}^{T}\int_0^{t}Q_1(t,s)w(s)dsdt$$

$$-\lambda a_2\mu_2(t)\int_0^{\eta}Q_2(\eta,s)w(s)ds-\lambda b_2\mu_2(t)\int_0^{T}Q_2(T,s)w(s)ds$$

$$+a_2\mu_2(t)\int_0^{\eta}w(s)ds+b_2\mu_2(t)\int_0^{T}w(s)ds. \tag{4.3}$$

Proof: consider

$$_C D^{p-1}(D+\lambda)u(t)=w(t) \tag{4.4}$$

apply I^{p-1} to both sides of (4.4)

$$_{RL}I^{p-1}\,{}_C D^{p-1}(D+\lambda)u(t)={}_{RL}I^{p-1}w(t),$$

$$(D+\lambda)u(t)+d_0={}_{RL}I^{p-1}w(t),$$

$$D\left(e^{\lambda t}u(t)\right)e^{-\lambda t}+d_0={}_{RL}I^{p-1}w(t),$$

$$D\left(e^{\lambda t}u(t)\right)+d_0e^{\lambda t}=e^{\lambda t}{}_{RL}I^{p-1}w(t) \tag{4.5}$$

integrate both sides of (4.5)

$$e^{\lambda t}u(t)-\left(u(0)+d_0\right)+d_0e^{\lambda t}=\int_0^t e^{\lambda s}{}_{RL}I^{p-1}w(s)ds,$$

$$u(t)-\left(u(0)+d_0\right)e^{-\lambda t}+d_0=\int_0^t e^{-\lambda(t-s)}{}_{RL}I^{\alpha-1}w(s)ds,$$

$$u(t)=c_1e^{-\lambda t}+c_0+\int_0^t e^{-\lambda(t-s)}{}_{RL}I^{p-1}w(s)ds \tag{4.6}$$

$$_C D^{p-1}u(t)=\frac{-\lambda c_1}{\Gamma(2-p)}\int_0^t(t-s)^{1-p}e^{-\lambda s}ds+$$

$$\frac{1}{\Gamma(2-p)}\int_0^t(t-s)^{1-p}\left(I^{p-1}w(s)-\lambda\int_0^s e^{-\lambda(s-\tau)}{}_{RL}I^{p-1}w(\tau)d\tau\right)ds$$

from (4.6)

$$u(s)=c_1e^{-\lambda s}+c_0+\int_0^s e^{-\lambda(s-\tau)}{}_{RL}I^{\alpha-1}w(\tau)\,d\tau$$

then

$$d_1\int_0^\xi u(s)\,ds=d_1\int_0^\xi c_1e^{-\lambda s}ds+d_1c_0\int_0^\xi ds+d_1\int_0^\xi\left(\int_0^s e^{-\lambda(s-\tau)}{}_{RL}I^{p-1}w(\tau)\,d\tau\right)ds$$

$$=\frac{c_1d_1}{\lambda}\left(1-e^{-\lambda\xi}\right)+d_1c_0\xi+d_1\int_0^\xi\left(\int_0^s e^{-\lambda(s-\tau)}{}_{RL}I^{p-1}w(\tau)\,d\tau\right)ds.$$

and

$$d_2\int_\zeta^T u(s)\,ds=\frac{c_1d_2}{\lambda}\left(e^{-\lambda\xi}-e^{-\lambda T}\right)+d_2c_0\left(T-\xi\right)+d_2\int_\zeta^T\left(\int_0^s e^{-\lambda(s-r)}{}_{RL}I^{p-1}w(r)\,dr\right)ds.$$

by the first boundary condition given by

$$\begin{aligned}&a_1u(\eta)+b_1u(T)\\&=a_1c_1e^{-\lambda\eta}+a_1c_0+a_1\int_0^\eta e^{-\lambda(\eta-s)}I^{p-1}w(s)\,ds+b_1c_1e^{-\lambda T}+b_1c_0+b_1\int_0^T e^{-\lambda(T-s)}I^{p-1}w(s)\,ds\\&=d_1\int_0^\xi\left(c_1e^{-\lambda r}+c_0+\int_0^r e^{-\lambda(r-s)}I^{p-1}w(s)\,ds\right)dr\\&=\frac{d_1c_1}{\lambda}\left(1-e^{-\lambda\xi}\right)+d_1c_0\xi+d_1\int_0^\xi\int_0^r e^{-\lambda(r-s)}I^{p-1}w(s)\,ds\,dr,\end{aligned}$$

$$\begin{aligned}&\left(a_1e^{-\lambda\eta}+b_1e^{-\lambda T}-\frac{d_1}{\lambda}\left(1-e^{-\lambda\xi}\right)\right)c_1+\left(a_1+b_1-d_1\xi\right)c_0\\&=d_1\int_0^\xi\int_0^r e^{-\lambda(r-s)}I^{p-1}w(s)\,ds\,dr-a_1\int_0^\eta e^{-\lambda(\eta-s)}I^{p-1}w(s)\,ds-b_1\int_0^T e^{-\lambda(T-s)}I^{p-1}w(s)\,ds.\end{aligned}$$

The second boundary condition

$$a_2\,{}_CD^{p-1}u(\eta)+b_2\,{}_CD^{p-1}u(T)=d_2\int_\zeta^T u(s)\,ds,$$

implies that

$$\left(\frac{\lambda a_2}{\Gamma(2-p)}\int_0^{\eta}(\eta-s)^{1-p}e^{-\lambda s}ds+\frac{\lambda b_2}{\Gamma(2-p)}\int_0^{T}(T-s)^{1-p}e^{-\lambda s}ds+d_2\int_{\zeta}^{T}e^{-\lambda t}dt\right)c_1+d_2(T-\zeta)c_0$$
$$=a_2\frac{1}{\Gamma(2-p)}\int_0^{\eta}(\eta-s)^{1-p}\left({}_{RL}I^{p-1}w(s)-\lambda\int_0^{s}e^{-\lambda(s-r)}{}_{RL}I^{p-1}w(r)dr\right)ds$$
$$+b_2\frac{1}{\Gamma(2-p)}\int_0^{T}(T-s)^{1-p}\left({}_{RL}I^{p-1}h(s)-\lambda\int_0^{s}e^{-\lambda(s-r)}{}_{RL}I^{p-1}w(r)dr\right)ds$$
$$-d_2\int_{\zeta}^{T}\left(\int_0^{t}e^{-\lambda(t-s)}{}_{RL}I^{p-1}w(s)ds\right)dt$$

thus

$$\theta_{11}c_1+\theta_{12}c_0=d_1\int_0^{\xi}\int_0^{r}Q_1(r,s)w(s)dsdr-a_1\int_0^{\eta}Q_1(\eta,s)w(s)ds-b_1\int_0^{T}Q_1(T,s)w(s)ds,$$
$$\theta_{21}c_1+\theta_{22}c_0=a_2\int_0^{\eta}w(s)ds+b_2\int_0^{T}w(s)ds-\lambda a_2\int_0^{\eta}Q_2(\eta,s)w(s)ds$$
$$-\lambda b_2\int_0^{T}Q_2(T,s)w(s)ds-d_2\int_{\zeta}^{T}\int_0^{t}Q_1(t,s)w(s)ds\,dt.$$

A simultaneous solution for the above equations for c_0, c_1 leads to

$$c_0=\frac{\theta_{11}a_2}{\Delta}\int_0^{\eta}w(s)ds+\frac{\theta_{11}b_2}{\Delta}\int_0^{T}w(s)ds-\frac{\theta_{11}a_2\lambda}{\Delta}\int_0^{\eta}Q_2(\eta,s)w(s)ds$$
$$-\frac{\theta_{11}b_2\lambda}{\Delta}\int_0^{T}Q_2(T,s)w(s)ds-\frac{\theta_{11}d_2}{\Delta}\int_{\zeta}^{T}\int_0^{t}Q_1(t,s)w(s)ds\,dt$$
$$-\frac{\theta_{21}d_1}{\Delta}\int_0^{\xi}\int_0^{r}Q_1(r,s)w(s)dsdr+\frac{\theta_{21}a_1}{\Delta}\int_0^{\eta}Q_1(\eta,s)w(s)ds+\frac{\theta_{21}b_1}{\Delta}\int_0^{T}Q_1(T,s)w(s)ds.$$

$$c_1=\frac{\theta_{22}d_1}{\Delta}\int_0^{\xi}\int_0^{r}Q_1(r,s)w(s)dsdr-\frac{\theta_{22}a_1}{\Delta}\int_0^{\eta}Q_1(\eta,s)w(s)ds-\frac{\theta_{22}b_1}{\Delta}\int_0^{T}Q_1(T,s)w(s)ds$$
$$-\frac{\theta_{12}a_2}{\Delta}\int_0^{\eta}w(s)ds-\frac{\theta_{12}b_2}{\Delta}\int_0^{T}w(s)ds+\frac{\theta_{12}\lambda a_2}{\Delta}\int_0^{\eta}Q_2(\eta,s)w(s)ds$$
$$+\frac{\theta_{12}\lambda b_2}{\Delta}\int_0^{T}Q_2(T,s)w(s)ds+\frac{\theta_{12}d_2}{\Delta}\int_{\zeta}^{T}\int_0^{t}Q_1(t,s)w(s)ds\,dt.$$

substituting c_0, c_1 in (4.6) we obtain

$$
\begin{aligned}
u(t) &= \int_0^t Q_1(t,s)\,w(s)\,ds \\
&+\frac{\theta_{22}d_1}{\Delta}e^{-\lambda t}\int_0^{\xi}\int_0^{r} Q_1(r,s)\,w(s)\,dsdr-\frac{\theta_{22}a_1}{\Delta}e^{-\lambda t}\int_0^{\eta} Q_1(\eta,s)\,w(s)\,ds \\
&-\frac{\theta_{22}b_1}{\Delta}e^{-\lambda t}\int_0^{T} Q_1(T,s)\,w(s)\,ds-\frac{\theta_{12}a_2}{\Delta}e^{-\lambda t}\int_0^{\eta} w(s)\,ds-\frac{\theta_{12}b_2}{\Delta}e^{-\lambda t}\int_0^{T} w(s)\,ds \\
&+\frac{\theta_{12}\lambda a_2}{\Delta}e^{-\lambda t}\int_0^{\eta} Q_2(\eta,s)\,w(s)\,ds+\frac{\theta_{12}\lambda b_2}{\Delta}e^{-\lambda t}\int_0^{T} Q_2(T,s)\,w(s)\,ds \\
&+\frac{\theta_{12}d_2}{\Delta}e^{-\lambda t}\int_{\zeta}^{T}\int_0^{t} Q_1(t,s)\,w(s)\,ds\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta_{11}a_2}{\Delta}\int_0^{\eta} w(s)\,ds+\frac{\theta_{11}b_2}{\Delta}\int_0^{T} w(s)\,ds-\frac{\theta_{11}a_2\lambda}{\Delta}\int_0^{\eta} Q_2(\eta,s)\,w(s)\,ds \\
&-\frac{\theta_{11}b_2\lambda}{\Delta}\int_0^{T} Q_2(T,s)\,w(s)\,ds-\frac{\theta_{11}d_2}{\Delta}\int_{\zeta}^{T}\int_0^{t} Q_1(t,s)\,w(s)\,ds\,dt \\
&-\frac{\theta_{21}d_1}{\Delta}\int_0^{\xi}\int_0^{r} Q_1(r,s)\,w(s)\,dsdr+\frac{\theta_{21}a_1}{\Delta}\int_0^{\eta} Q_1(\eta,s)\,w(s)\,ds+\frac{\theta_{21}b_1}{\Delta}\int_0^{T} Q_1(T,s)\,w(s)\,ds.
\end{aligned}
$$

Combining similar terms and by the help of notations the desired formula will be obtained.

By direct computation the converse of our lemma holds true.

Lemma 4.2. The following inequalities

$$
\left|\int_0^t Q_1(t,\tau)\,p_1(\tau)\,dx-\int_0^t Q_1(t,\tau)\,p_2(\tau)\,d\tau\right|\le\frac{t^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda t}+1\right)\|p_1-p_2\|_C,
$$

$$
\left|\int_0^t Q_2(t,\tau)\,p_1(\tau)\,d\tau-\int_0^t Q_2(t,\tau)\,p_2(\tau)\,d\tau\right|\le\frac{t}{\lambda}\left(-e^{-\lambda t}+1\right)\|p_1-p_2\|_C.
$$

hold true on $[0,T]$, $p_1, p_2\in C([0,T])$.

Proof. Indeed,

$$\left|\int_0^t Q_1(t,\tau)\,p_1(\tau)\,d\tau-\int_0^t Q_1(t,\tau)\,p_2(\tau)\,d\tau\right|\le\int_0^t Q_1(t,\tau)\left|p_1(\tau)-p_2(\tau)\right|d\tau$$

$$\le\int_0^t Q_1(t,\tau)\,d\tau\left\|p_1-p_2\right\|_C$$

$$\le\frac{1}{\Gamma(p-1)}\int_0^t\left(\int_\tau^t e^{-\lambda(t-r)}(r-\tau)^{p-2}\,dr\right)d\tau\left\|p_1-p_2\right\|_C$$

$$\le\frac{1}{\Gamma(p-1)}\int_0^t\int_0^r e^{-\lambda(t-r)}(r-\tau)^{p-2}\,d\tau dr\left\|p_1-p_2\right\|_C$$

$$\le\frac{t^{p-1}}{\lambda(p-1)\Gamma(p-1)}\left(-e^{-\lambda t}+1\right)\left\|p_1-p_2\right\|_C$$

$$\le\frac{T^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda T}+1\right)\left\|p_1-p_2\right\|_C.$$

and

$$\left|\int_0^t Q_2(t,\tau)\,p_1(\tau)\,ds-\int_0^t Q_2(t,\tau)\,p_2(\tau)\,d\tau\right|$$

$$=\left|\frac{1}{\Gamma(2-p)}\int_0^t(t-\tau)^{1-p}\int_0^\tau e^{-\lambda(\tau-r)}I^{p-1}\left(p_1(r)-p_2(r)\right)drd\tau\right|$$

$$=\frac{1}{\Gamma(2-p)\Gamma(p-1)}\left|\int_0^t(t-\tau)^{1-p}\int_0^\tau e^{-\lambda(\tau-r)}\int_0^r(r-m)^{p-2}\left(p_1(m)-p_2(m)\right)dmdrd\tau\right|$$

$$\le\frac{1}{(p-1)\Gamma(2-p)\Gamma(1-p)}\left|\int_0^t(t-\tau)^{1-p}\int_0^\tau e^{-\lambda(\tau-r)}r^{p-1}drd\tau\right|\left\|p_1-p_2\right\|_C$$

$$\le\frac{1}{(p-1)\Gamma(2-p)\Gamma(1-p)}\left|\int_0^t(t-\tau)^{1-p}\,\tau^{p-1}\int_0^\tau e^{-\lambda(\tau-r)}drd\tau\right|\left\|p_1-p_2\right\|_C$$

$$=\frac{\left(-e^{-\lambda t}+1\right)}{\lambda(p-1)\Gamma(2-p)\Gamma(1-p)}\left|\int_0^t(t-\tau)^{1-p}\,\tau^{p-1}d\tau\right|\left\|p_1-p_2\right\|_C$$

$$=\frac{t\left(-e^{-\lambda t}+1\right)}{\lambda(p-1)\Gamma(2-p)\Gamma(1-p)}\left|\int_0^1(t-\tau)^{1-p}\,\tau^{p-1}d\tau\right|\left\|p_1-p_2\right\|_C$$

$$=\frac{t\left(-e^{-\lambda t}+1\right)}{\lambda(p-1)\Gamma(2-p)\Gamma(1-p)}B(p,2-p)\left\|p_1-p_2\right\|_C$$

$$= \frac{t\left(-e^{-\lambda t}+1\right)}{\lambda(p-1)\Gamma(2-p)\Gamma(1-p)} \frac{\Gamma(p)\Gamma(2-p)}{\Gamma(2)} \left\| p_1 - p_2 \right\|_C$$

$$= \frac{t}{\lambda}\left(-e^{-\lambda t}+1\right)\left\| p_1 - p_2 \right\|_C .$$

Given a continuous function $f:[0,T]\to\mathbb{R},$ the Banach space $\Omega = C\left([0,T],\mathbb{R}\right)$ endowed with $\|u\|_C = \sup_{0\le t\le T}\left|u(t)\right|,$ then the operator $\mathcal{F}:\Omega\to\Omega$ can be defined as:

$$\begin{aligned}
(\mathcal{F}u)(t) &= \int_0^t Q_1(t,\tau)\, f\left(\tau,u(\tau)\right)d\tau \\
&\quad + a_1\mu_1(t)\int_0^\eta Q_1(\eta,\tau)\, f\left(\tau,u(\tau)\right)d\tau + b_1\mu_1(t)\int_0^T Q_1(T,\tau)\, f\left(\tau,u(\tau)\right)d\tau \\
&\quad - d_1\mu_1(t)\int_0^\xi\int_0^r Q_1(r,\tau)\, f\left(\tau,u(\tau)\right)d\tau dr - d_2\mu_2(t)\int_\zeta^T\int_0^t Q_1(t,\tau)\, f\left(\tau,u(\tau)\right)d\tau dt \\
&\quad - \lambda a_2\mu_2(t)\int_0^\eta Q_2(\eta,\tau)\, f\left(\tau,u(\tau)\right)d\tau - \lambda b_2\mu_2(t)\int_0^T Q_2(T,\tau)\, f\left(\tau,u(\tau)\right)d\tau \\
&\quad + a_2\mu_2(t)\int_0^\eta f\left(\tau,u(\tau)\right)d\tau + b_2\mu_2(t)\int_0^T f\left(\tau,u(\tau)\right)d\tau.
\end{aligned}$$

For computational convenience, let

$$\begin{aligned}
R &:= \frac{T^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda T}+1\right) + |a_1|\phi_1\frac{\eta^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda\eta}+1\right) \\
&+ |b_1|\phi_1\frac{T^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda T}+1\right) + |d_1|\phi_1\int_0^\xi \frac{r^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda r}+1\right)dr \\
&+ |d_2|\phi_2\int_\zeta^T \frac{t^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda t}+1\right)dt + \lambda|a_2|\phi_2\frac{\eta}{\lambda}\left(-e^{-\lambda\eta}+1\right) \\
&+ \lambda|b_2|\phi_2\frac{T}{\lambda}\left(-e^{-\lambda T}+1\right) + |a_2|\phi_2\eta + |b_2|\phi_2 T,
\end{aligned}$$

$$R^* := R - \frac{T^{p-1}}{\lambda\Gamma(p)}\left(-e^{-\lambda T}+1\right).$$

Theorem 4.3.Assume the continuity of $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, *and assume*

$(E_1)\ \exists L_f>0$ such that $\left|f(t,u_1)-f(t,u_2)\right|\le L_f\left|u_1-u_2\right|,\forall t\in[0,T],\forall u_1,u_2\in\mathbb{R}$. with

$RL_f<1$, then there exist a unique solution of the SFDE (4.1).

Proof: Define

$$B_r=\left\{u\in\Omega,\|u\|_C\le r\right\}, \text{with } r\ge\frac{MR}{1-LR} \text{ where } M=\sup_{0\le t\le T}\left|f(t,0)\right|$$

First we show that $\mathcal{F}B_r\subset B_r$ for this

$\forall u\in B_r,\forall t\in[0,T]$

$$\begin{aligned}(\mathcal{F}u)(t)=\Bigg|&\int_0^t Q_1(t,\tau)f(\tau,u(\tau))d\tau+a_1\mu_1(t)\int_0^\eta Q_1(\eta,\tau)f(\tau,u(\tau))d\tau\\&+b_1\mu_1(t)\int_0^T Q_1(T,\tau)f(\tau,u(\tau))d\tau-d_1\mu_1(t)\int_0^\xi\int_0^r Q_1(r,\tau)f(\tau,u(\tau))d\tau dr\\&-d_2\mu_2(t)\int_\zeta^T\int_0^t Q_1(t,\tau)f(\tau,u(\tau))d\tau dt-\lambda a_2\mu_2(t)\int_0^\eta Q_2(\eta,\tau)f(\tau,u(\tau))d\tau\\&-\lambda b_2\mu_2(t)\int_0^T Q_2(T,\tau)(\tau,u(\tau))d\tau+a_2\mu_2(t)\int_0^\eta f(\tau,u(\tau))d\tau+b_2\mu_2(t)\int_0^T f(\tau,u(\tau))d\tau\Bigg|.\end{aligned}$$

But generally $\left|f(t,u(t))\right|\le\left|f(t,u(t))-f(t,0)\right|+\left|f(t,0)\right|$

Implies

$$\begin{aligned}\left|f(t,u(t)\right|&\le L_f\|u\|_C+\sup_{0\le t\le T}\left|f(t,0)\right|\\&\le L_f r+M_f.\end{aligned}$$

This inequality leads to

$$\left|(\mathcal{F}u)(t)\right| \le \frac{t^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda t}\right)\left\|f(\cdot),u(\cdot)\right\|_C + |a_1||\phi_1|\frac{\eta^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda\eta}\right)\left\|f(\cdot),u(\cdot)\right\|_C$$

$$+|b_1||\phi_1|\frac{T^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda T}\right)\left\|f(\cdot),u(\cdot)\right\|_C + |d_1||\phi_1|\int_0^{\xi}\frac{r^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda r}\right)dr\left\|f(\cdot),u(\cdot)\right\|_C$$

$$+|d_2||\phi_2|\int_{\zeta}^{T}\frac{t^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda t}\right)dt\left\|f(\cdot),u(\cdot)\right\|_C + \lambda|a_2||\phi_2|\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right)\left\|f(\cdot),u(\cdot)\right\|_C$$

$$+\lambda|b_2||\phi_2|\frac{T}{\lambda}\left(1-e^{-\lambda T}\right)\left\|f(\cdot),u(\cdot)\right\|_C + |a_2||\phi_2|\eta\left\|f(\cdot),u(\cdot)\right\|_C$$

$$+|b_2||\phi_2|T\left\|f(\cdot),u(\cdot)\right\|_C,$$

$$\le R\times\left(L_f r + M_f\right)\le r.$$

Implying, $\mathcal{F}u \in B_r, \forall u \in B_r$ that is $\mathcal{F}B_r \subset B_r$.

Next, $\mathcal{F}$ is a contractionhas to be shown, for this let $u_1, u_2 \in \Omega$

$$\left|\mathcal{F}u_1(t)-\mathcal{F}u_2(t)\right| \le \left|\sup_{0\le t\le T}\int_0^t Q_1(t,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau\right|$$

$$+|a_1|\sup_{0\le t\le T}\mu_1(t)\left|\int_0^{\eta} Q_1(\eta,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau\right|$$

$$+|b_1|\sup_{0\le t\le T}\mu_1(t)\left|\int_0^{T} Q_1(T,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau\right|$$

$$+|d_1|\sup_{0\le t\le T}\mu_1(t)\left|\int_0^{\xi}\int_0^{r} Q_1(r,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau dr\right|$$

$$+|d_2|\sup_{0\le t\le T}\mu_2(t)\left|\int_{\zeta}^{T}\int_0^{t} Q_1(t,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau dt\right|$$

$$+\lambda|a_2|\sup_{0\le t\le T}\mu_2(t)\left|\int_0^{\eta} Q_2(\eta,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau\right|$$

$$+\lambda|b_2|\sup_{0\le t\le T}\mu_2(t)\left|\int_0^T Q_2(T,\tau)\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau\right|$$

$$+|a_2|\sup_{0\le t\le T}\mu_2(t)\int_0^{\eta}\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau$$

$$+|b_2|\sup_{0\le t\le T}\mu_2(t)\int_0^{T}\left|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\right|d\tau.$$

Assumption (E_1), implies

$$\|\mathcal{F}u_1-\mathcal{F}u_2\|_C\le L_f\|u_1-u_2\|_C\times$$

$$\left(\frac{T^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda T}\right)+|a_1|\phi_1\frac{\eta^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda\eta}\right)+|b_1|\phi_1\frac{T^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda T}\right)\right.$$

$$+|d_1|\phi_1\int_0^{\xi}\frac{r^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda r}\right)dr+|d_2|\phi_2\int_{\zeta}^{T}\frac{t^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda t}\right)dt$$

$$\left.+\lambda|b_2|\left(1-e^{-\lambda\eta}\right)\frac{\eta\phi_2}{\lambda}+\lambda|b_2|\left(1-e^{-\lambda T}\right)\frac{T\phi_2}{\lambda}+|a_2|\phi_2\eta+|b_2|\phi_2 T\right),$$

$$\le L_f R\|u_1-u_2\|_C\le\|u_1-u_2\|_C.$$

End up with this, we conclude that $\mathcal{F}$ is a contraction, and based on theorem (2.2.7) the uniqueness property for the solution of the SFDE (4.1) holds true on [0,T].

Theorm 4.4. Given the continuous function f,
and assume (E_2) $\exists\ \omega\in C([0,T],\mathbb{R}^+)$ and a non decreasing function $\chi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,u)|\le\omega(t)\chi(|u|),\forall(t,u)\in[0,T]\times\mathbb{R}$

(E_3) $\exists\ W>0$ such that

$$\frac{W}{\chi(W)\|\omega\|_C R}>1$$

Then there exist at least one solution of the SFDE (4.1).

Proof. Step1 show that, for this,

Consider the bounded set $B_r = \{u \in \Omega, \|u\|_C \leq r\}$ in Ω Then by (E_2)

$$\begin{aligned}
\left|(\mathcal{F}u)(t)\right| = &\left|\int_0^t Q_1(t,\tau)\, f(\tau,u(\tau))\,d\tau + a_1\mu_1(t)\int_0^\eta Q_1(\eta,\tau)\, f(\tau,u(\tau))\,d\tau\right. \\
&+ b_1\mu_1(t)\int_0^T Q_1(T,\tau)\, f(\tau,u(\tau))\,d\tau - d_1\mu_1(t)\int_0^\xi\int_0^r Q_1(r,\tau)\, f(\tau,u(\tau))\,d\tau dr \\
&- d_2\mu_2(t)\int_\zeta^T\int_0^t Q_1(t,\tau)\, f(\tau,u(\tau))\,d\tau dt - \lambda a_2\mu_2(t)\int_0^\eta Q_2(\eta,\tau)\, f(\tau,u(\tau))\,d\tau \\
&\left. - \lambda b_2\mu_2(t)\int_0^T Q_2(T,\tau)(\tau,u(\tau))\,d\tau + a_2\mu_2(t)\int_0^\eta f(\tau,u(\tau))\,d\tau + b_2\mu_2(t)\int_0^T f(\tau,u(\tau))\,d\tau\right|
\end{aligned}$$

Take $\sup\limits_{0\leq t\leq T}$, then

$$\begin{aligned}
&\leq \|\omega\|_C\, \chi(r)\times\left(\frac{t^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda t}\right) + |a_1||\phi_1|\frac{\eta^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda\eta}\right)\right. \\
&+|b_1||\phi_1|\frac{T^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda T}\right) + |d_1||\phi_1|\int_0^\xi \frac{r^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda r}\right)dr + |d_2||\phi_2|\int_\zeta^T \frac{t^{p-1}}{\lambda\Gamma(p)}\left(1-e^{-\lambda t}\right)dt \\
&\left. +\lambda|a_2||\phi_2|\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right) + \lambda|b_2||\phi_2|\frac{T}{\lambda}\left(1-e^{-\lambda T}\right) + |a_2||\phi_2|\eta + |b_2||\phi_2|T\right),
\end{aligned}$$

Implies,

$$\|\mathcal{F}u\|_C \leq \|\omega\|_C\, \chi(r) R$$

Step (2) show that $\mathcal{F}$ is equicontiuous

to do we let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, then $\forall u \in B_r$ we have

$$\begin{aligned}
\left|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\right| \leq &\left|\int_0^{t_1}\left(Q_1(t_1,\tau) - Q_1(t_2,\tau)\right) f(\tau,u(\tau))\,d\tau\right| \\
&+\left|\int_{t_1}^{t_2} Q_1(t_2,\tau)\, f(\tau,u(\tau))\,d\tau\right|
\end{aligned}$$

$$
\begin{aligned}
&+|a_1||\mu_1(t_1)-\mu_1(t_2)|\int_0^{\eta}\left|Q_1(\eta,\tau)\, f(\tau,u(\tau))\,d\tau\right| \\
&+|b_1||\mu_1(t_1)-\mu_1(t_2)|\int_0^{T}\left|Q_1(T,\tau)\, f(\tau,u(\tau))\,d\tau\right| \\
&+|d_1||\mu_1(t_1)-\mu_1(t_2)|\int_0^{\xi}\int_0^{r}\left|Q_1(r,\tau)\, f(\tau,u(\tau))\,d\tau\right|dr \\
&+|d_2||\mu_2(t_1)-\mu_2(t_2)|\int_{\zeta}^{T}\int_0^{t}\left|Q_1(t,\tau)\, f(\tau,u(\tau))\,d\tau\right|dt \\
&+\lambda|a_2||\mu_2(t_1)-\mu_2(t_2)|\int_0^{\eta}\left|Q_2(\eta,\tau)\, f(\tau,u(\tau))\,d\tau\right| \\
&+\lambda|b_2||\mu_2(t_1)-\mu_2(t_2)|\int_0^{T}\left|Q_2(T,\tau)\, f(\tau,u(\tau))\,d\tau\right| \\
&+|a_2||\mu_2(t_1)-\mu_2(t_2)|\int_0^{\eta}\left|f(\tau,u(\tau))\,d\tau\right| \\
&+|b_2||\mu_2(t_1)-\mu_2(t_2)|\int_0^{T}\left|f(\tau,u(\tau))\,d\tau\right| \\
&\to 0, \quad \text{as } t_1\to t_2,
\end{aligned}
$$

Since the above inequality is independent of u, based on theorem $(2.2.9)$
We conclude thatis completely continuous.

Finally, by showing thatuwhich is the solution for the equation $u=\delta\mathcal{F}u,\delta\in[0,1]$is bounded we complete all conditions of theorem $(2.2.10)$, to do so, and for $t\in[0,T]$, suppose that is u a solution, then

$$|u(t)|=|\delta(\mathcal{F}u)(t)|\le\delta\left(\|\omega\|_C\,\chi(r)\right)R\le\|\omega\|_C\,\chi\left(\|u\|_C\right)R.$$

implies

$$\frac{\|u\|_C}{\|\omega\|_C\,\chi\left(\|u\|_C\right)R}\le 1$$

But by (E_3) $\exists W > 0$ such that $W \neq \|u\|_C$

let $Z = \{u \in \Omega : \|u\|_C < W\}$. It is clear that $\mathcal{F} : \overline{Z} \to \Omega$

is continuous and completely continuous, by the defined Z,$\nexists u \in \partial Z$ such that

$u = \delta \mathcal{F} u, 0 < \delta < 1$, therefore, based on theorem $(2.2.10)$, we deduce that $\mathcal{F}$ has

$u \in \overline{Z}$ which solves the SFDE.

Theorem 4.5.Given the continuous function f that is satisfying the condition

$(E_1)$and $|f(t,u)| \leq y(t), \forall (t,u) \in [0,T] \times \mathbb{R}$, where, $y \in C([0,T], \mathbb{R}^+)$with $\sup_{0 \leq t \leq T} |y(t)| = \|y\|_C$

Moreover, suppose $LR^* < 1$, then the problem (4.1) has at least one solution.

Proof.

Construct $B_r = \{u \in \Omega, \|u\|_C \leq r\}$be a close set, with $r \geq R\|y\|_C$, define two operators $\mathcal{F}_1, \mathcal{F}_2$ on B_r as

$$(\mathcal{F}_1 u)(t) := \int_0^t Q_1(t,\tau) f(\tau, u(\tau)) d\tau,$$

$$\begin{aligned}(\mathcal{F}_2 u)(t) := {} & a_1\mu_1(t)\int_0^\eta Q_1(\eta,\tau) f(\tau,u(\tau))d\tau \\ & + b_1\mu_1(t)\int_0^T Q_1(T,\tau) f(\tau,u(\tau))d\tau \\ & - d_1\mu_1(t)\int_0^\xi\int_0^r Q_1(r,\tau) f(\tau,u(\tau))d\tau dr \\ & - d_2\mu_2(t)\int_\zeta^T\int_0^t Q_1(t,\tau) f(\tau,u(\tau))d\tau dt \\ & - \lambda a_2\mu_2(t)\int_0^\eta Q_2(\eta,\tau) f(\tau,u(\tau))d\tau\end{aligned}$$

$$-\lambda b_2\mu_2(t)\int_0^T Q_2(T,\tau)\, f(\tau,u(\tau))\,d\tau + a_2\mu_2(t)\int_0^\eta f(\tau,u(\tau))\,d\tau$$

$$+b_2\mu_2(t)\int_0^T f(\tau,u(\tau))\,d\tau.$$

$\forall v,u\in B_r$, then $\|\mathcal{F}_1u+\mathcal{F}_2v\|_C\le R\|y\|_C\le r.$ Thus $\mathcal{F}_1u+\mathcal{F}_2v\in B_r$.

By condition (E_1) and R^*,

then one can easy show that $\|\mathcal{F}_2u-\mathcal{F}_2v\|_C\le L_f R^*\|u-v\|_C\le\|u-v\|_C$

Implying that $\mathcal{F}_2$ is a contraction.

Moreover, the continuity of $\mathcal{F}_1$ holds because of continuity of f

and

$\|\mathcal{F}_1u\|_C\le\dfrac{T^{p-1}\|y\|_C\left(1-e^{-\lambda T}\right)}{\lambda\Gamma(p)}$, hence $\mathcal{F}_1$ is uniformly bounded.

For showing the compactness of $\mathcal{F}_1$. Fixing $\sup|f(t,u)|=f_r,\ \forall(t,u)\in[0,T]\times B_r$

and for

$t_1,t_2\in[0,T],t_1<t_2$, then ,

$$\|\mathcal{F}_1u(t_1)-\mathcal{F}_1u(t_2)\|\le$$

$$f_r\left(\left|\int_0^{t_1}\left(Q_1(t_1,\tau)-Q_1(t_2,\tau)\right) f(\tau,u(\tau))\,d\tau\right|+\left|\int_{t_1}^{t_2}Q_1(t_2,\tau)\, f(\tau,u(\tau))\,d\tau\right|\right)$$

The RHS of the above inequality approaches 0, $as\ t_1\to t_2$. It is notable that also the same side is independent of u that is $\mathcal{F}_1$ is relatively compact, theorem $(2.2.9)$ implying that $\mathcal{F}_1$ is compact.Hence, the existence property for the solution of the SFDE (4.1)holds by theorem $(2.2.10)$.

Examples.

Given the following SFDE

$$\begin{cases} \left({}_C D^{3/2} + 2\,{}_C D^{1/2}\right)u(t) = \dfrac{1}{\sqrt{t^2+49}}\left(\dfrac{t\sin u(t)}{49} + e^{-t}\cos t\right),\ 0 \le t \le 4, \\ 2u(1) + 3u(4) = -\int_0^2 u(s)ds, \\ {}_C D^{1/2}u(1) + 5\,{}_C D^{1/2}u(4) = 5\int_3^4 u(s)ds. \end{cases} \tag{4.7}$$

here

$$f(t,u(t)) = \frac{1}{\sqrt{t^2+49}}\left(\frac{t\sin u(t)}{49} + e^{-t}\cos t\right),$$

$$T = 4, \lambda = 2, p = \frac{3}{2}, a_1 = 2, a_2 = 1, b_1 = 3, b_2 = 5, d_1 = -1, d_2 = 5,$$

$$, \eta = 1, \xi = 2, \zeta = 3.$$

$$\theta_{11} = a_1 e^{-\lambda\eta} + b_1 e^{-\lambda T} - \frac{d_1}{\lambda}\left(1 - e^{-\lambda\xi}\right) \cong 0.761,$$

$$\theta_{12} := a_1 + b_1 - d_1\xi = 6,$$

$$\theta_{21} = a_2 \frac{\lambda}{\Gamma(2-p)}\int_0^\eta (\eta - s)^{1-p} e^{-\lambda s} ds + b_2 \frac{\lambda}{\Gamma(2-p)}\int_0^T (T-s)^{1-p} e^{-\lambda s} ds + d_2 \int_\zeta^T e^{-\lambda t} dt = 24.8,$$

$$\theta_{22} = \theta_2(T - \zeta) = 5,$$

$$\Delta = \theta_{11}\theta_{22} - \theta_{12}\theta_{21} = -145.$$

with

$$\mu_1(t) = \frac{\theta_{21} - \theta_{22}e^{-\lambda t}}{\Delta},\ \mu_2(t) = \frac{\theta_{11} - \theta_{12}e^{-\lambda t}}{\Delta},$$

Then

$$\phi_1 = \max\left(\frac{|\theta_{21} - \theta_{22}|}{|\Delta|}, \frac{|\theta_{21} - \theta_{22}e^{-\lambda T}|}{|\Delta|}\right) = \max\left(\frac{24.8 - 5}{145}, \frac{24.8 - 5e^{-8}}{145}\right) \cong 0.17,$$

$$\phi_2 = \max\left(\frac{|\theta_{11} - \theta_{12}|}{|\Delta|}, \frac{|\theta_{11} - \theta_{12}e^{-\lambda T}|}{|\Delta|}\right) = \max\left(\frac{0.76 - 6}{145}, \frac{0.76 - 6e^{-8}}{145}\right) \cong 0.036.$$

Implying

$R < 2.083$

To apply Theorem (4.3) we need to show that conditions

$$\begin{aligned} \left| f(t, r_1) - f(t, r_2) \right| &= \frac{t}{\sqrt{t^2 + 49}} \left| \frac{\sin r_1 - \sin r_2}{49} \right| \\ &\leq \frac{1}{49} \left| r_1 - r_2 \right|, \end{aligned}$$

and

$$L_f R < \frac{1}{49}(2.083) < 0.043 < 1.$$

Therefore, according to theorem (4.3) the uniqueness for the solution of the SFDE (4.7) holds true.

Chapter 5

NONLINEAR SFDE's INVOLVING CAPUTO FRACTIONAL DERIVATIVE WITH NONLOCAL BC's

This chapter, based on the FPT develops the existence theory for SFDE's involving ${}_C D^r$, where $r \in (1,2]$ with nonlocal integral BC's. Examples are introduced for the purpose of illustration the applications of our results.

In this chapter the uniqueness and existence for the solutions of the following SFDE associated with a new set of nonlocal integral BC's has been investigated.

$$\begin{cases} \left({}_C D^r + \lambda\, {}_C D^{r-1}\right) y(t) = g\left(t, y(t), {}^c D^{r-1} y(t)\right), \; r \in (1,2], \; 0 \le t \le T, \\ a_1 y(\mu) + b_1 y(T) = c_1 \int_0^{\tau} y(s)\,ds + d_1, \\ a_2\, {}_C D^{r-1} y(\mu) + b_2\, {}_C D^{r-1} y(T) = c_2 \int_{\omega}^{T} y(s)\,ds + d_2, \end{cases} \tag{5.1}$$

where, $0 \le \mu \le T$, $\quad a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1,2, \quad 0 < \tau < \omega < T$.

Notations:

$$Q = Q_1 + Q_2,\; Q_1 = Q_1^* + \frac{T^{r-1}\left(1 - e^{-\lambda T}\right)}{\lambda \Gamma(r)},$$

$$\eta_1(t) = \left(\frac{d_2 \rho}{q\upsilon} + \frac{d_1 \rho \sigma}{q^2 \upsilon} + \frac{d_1}{q}\right) e^{-\lambda t} - \left(\frac{d_2}{\upsilon} + \frac{d_1 \sigma}{q\upsilon}\right), \quad \eta_2(t) = \left(\frac{c_1 \rho \sigma}{q^2 \upsilon} + \frac{c_1}{q}\right) e^{-\lambda t} - \frac{c_1 \sigma}{q\upsilon},$$

$$\eta_3(t)=e^{-\lambda t}\left(\frac{c_2\rho}{q\upsilon}\right)-\left(\frac{c_2}{\upsilon}\right),\quad \eta_4(t)=\left(\frac{a_1\sigma}{q\upsilon}\right)-e^{-\lambda t}\left(\frac{a_1\rho\sigma}{q^2\upsilon}+\frac{a_1}{q}\right)e^{-\lambda t},$$

$$\eta_5(t)=\left(\frac{b_1\sigma}{q\upsilon}\right)-\left(\frac{b_1\rho\sigma}{q^2\upsilon}+\frac{b_1}{q}\right)e^{-\lambda t},\quad \eta_6(t)=\frac{a_2}{\upsilon}-\left(\frac{a_2\rho}{q\upsilon}\right)e^{-\lambda t},$$

$$\eta_7(t)=\frac{b_2}{\upsilon}-e^{-\lambda t}\left(\frac{b_2\rho}{q\upsilon}\right),\quad \eta_8(t)=e^{-\lambda t}\left(\frac{\alpha_2\lambda\rho}{q\upsilon}\right)-\frac{\alpha_2\lambda}{\upsilon},\quad \eta_9(t)=e^{-\lambda t}\left(\frac{b_2\lambda\rho}{q\upsilon}\right)-\frac{b_2\lambda}{\upsilon},$$

where

$$q=\left(a_1e^{-\lambda\mu}+b_1e^{-\lambda T}-c_1\left(1-e^{-\lambda\tau}\right)\right),\ \rho=\left(a_1+b_1-c_1\tau\right),$$

$$\sigma=\left(\frac{c_2}{\lambda}\left(e^{-\lambda\omega}-e^{-\lambda T}\right)+\frac{\lambda a_2}{\Gamma(2-r)}\int_0^{\mu}(\mu-s)^{1-r}e^{-\lambda s}ds+\frac{\lambda b_2}{\Gamma(2-r)}\int_0^{T}(T-s)^{1-r}e^{-\lambda s}ds\right),$$

$$\upsilon=c_2(T-\omega)-\frac{\sigma\rho}{q},$$

with

$$G_1(t,m)=\frac{1}{\Gamma(r\text{-}1)}\int_s^t e^{-\lambda(t-s)}(s-m)^{r-2}ds,\quad \int_0^t G_1(t,m)w(m)dm=\int_0^t e^{-\lambda(t-s)}{}_{RL}I^{r-1}h(s)ds$$

$$G_2(t,s)=\frac{1}{\Gamma(2\text{-}r)}\int_s^t(t-m)^{1-r}G_1(m,s)dm$$

$$\frac{1}{\Gamma(2\text{-}r)}\int_0^{\mu}(\mu-s)^{1-r}\int_0^s e^{-\lambda(s-m)}{}_{RL}I^{r-1}w(m)dmds=\frac{1}{\Gamma(2\text{-}r)}\int_0^{\mu}(\mu-s)^{1-r}\int_0^s G_1(s,m)w(m)dmds$$

$$=\frac{1}{\Gamma(2\text{-}r)}\int_0^{\mu}\left(\int_x^{\mu}(\mu-s)^{1-r}G_1(s,m)ds\right)w(m)dm=\int_0^{\mu}G_2(\mu,m)w(m)dm$$

Lemma 5.1. $\forall w\in C\left([0,T],\mathbb{R}\right)$.The solution of SFDE .

$$\begin{cases} \left({}_C D^r + \lambda\, {}_C D^{r-1} \right) y(t) = w(t)\ , \quad r \in (1,2]\ , \\ a_1 y(\eta) + b_1 y(T) = c_1 \int_0^{\tau} y(s)\,ds + d_1, \\ a_2\, {}_C D^{r-1} y(\eta) + b_2\, {}_C D^{r-1} y(T) = c_2 \int_{\omega}^{T} y(s)\,ds + d_2, \end{cases} \tag{5.2}$$

is

$$\begin{aligned} y(t) = {} & \eta_1(t) + \int_0^t G_1(t,m) w(m)\,dm + \eta_2(t) \int_0^{\tau}\int_0^{s} G_1(s,m) w(m)\,dmds \\ & + \eta_3(t) \int_{\omega}^{T}\int_0^{s} G_1(s,m) w(m)\,dmds + \eta_4(t) \int_0^{\mu} G_1(\mu,m) w(m)\,dm + \\ & \eta_5(t) \int_0^{T} G_1(T,m) w(m)\,dm + \eta_6(t) \int_0^{\mu} w(m)\,dm + \eta_7(t) \int_0^{T} w(m)\,dm + \\ & \eta_8(t) \int_0^{\mu} G_2(\mu,m) w(m)\,dm + \eta_9(t) \int_0^{T} G_2(T,m) w(m)\,dm \end{aligned} \tag{5.3}$$

Proof. From the previous chapter the solution of SFDE $\left({}_C D^r + \lambda\, {}_C D^{r-1} \right) y(t) = w(t)$

is

$$y(t) = M_0 e^{-\lambda t} + M_1 + \int_0^t e^{-\lambda(t-s)}\, {}_{RL}I^{r-1} w(s)\,ds\ . \tag{5.4}$$

To find M_0 and M_1 Indeed

$$\begin{aligned} {}_C D^{r-1} y(t) = {} & \frac{-\lambda M_0}{\Gamma(2-r)} \int_0^t (t-s)^{1-r}\ e^{-\lambda s} ds + \\ & \frac{1}{\Gamma(2-r)} \int_0^t (t-s)^{1-r} \left({}_{RL}I^{r-1} w(s) - \lambda \int_0^s e^{-\lambda(s-m)}\, {}_{RL}I^{r-1} w(m)\,dm \right) ds \end{aligned}$$

First BC of (5.2) implies

$$M_0\left(a_1e^{-\lambda\mu}+b_1e^{-\lambda T}-c_1\left(1-e^{-\lambda\tau}\right)\right)+M_1\left(a_1+b_1-c_1\tau\right)=d_1+c_1\int_0^\tau\int_0^s e^{-\lambda(s-x)}I^{r-1}w(m)\,dmds$$
$$-a_1\int_0^\mu e^{-\lambda(\mu-s)}I^{r-1}w(s)\,ds-b_1\int_0^T e^{-\lambda(T-s)}I^{r-1}w(s)\,ds\ .$$

(5.5)

and the second BC of (5.2), implies

$$M_0\left(\frac{c_2}{\lambda}\left(e^{-\lambda\omega}-e^{-\lambda T}\right)+\frac{\lambda a_2}{\Gamma(2-r)}\int_0^\mu(\mu-s)^{1-r}e^{-\lambda s}ds+\frac{\lambda b_2}{\Gamma(2-r)}\int_0^T(T-s)^{1-r}e^{-\lambda s}ds\right)$$
$$+M_1c_2\left(T-\omega\right) \tag{5.6}$$

A simultaneous solution of (5.5) and (5.6), leads to

$$M_0=\left(\frac{d_2\rho}{q\upsilon}+\frac{d_1\rho\sigma}{q^2\upsilon}+\frac{d_1}{q}\right)+\left(\frac{c_1\rho\sigma}{q^2\upsilon}+\frac{c_1}{q}\right)\int_0^\tau\int_0^s G_1(s,m)w(m)\,dmds$$
$$-\left(\frac{a_1\rho\sigma}{q^2\upsilon}+\frac{a_1}{q}\right)\int_0^\mu G_1(\mu,m)w(m)\,dm$$
$$-\left(\frac{b_1\rho\sigma}{q^2\upsilon}+\frac{b_1}{q}\right)\int_0^T G_1(T,m)w(m)\,dm+\left(\frac{c_2\rho}{q\upsilon}\right)\int_\omega^T\int_0^s G_1(s,m)w(m)\,dmds$$
$$-\left(\frac{a_2\rho}{q\upsilon}\right)\int_0^\mu w(m)\,dm+\left(\frac{a_2\lambda\rho}{q\upsilon}\right)\int_0^\mu G_2(\mu,m)w(m)\,dm$$
$$-\left(\frac{b_2\rho}{q\upsilon}\right)\int_0^T w(m)\,dm+\left(\frac{b_2\lambda\rho}{q\upsilon}\right)\int_0^T G_2(T,m)w(m)\,dm. \tag{5.7}$$

$$M_1=\left(-\frac{c_1\sigma}{q\upsilon}\right)\int_0^\tau\int_0^s G_1(s,m)w(m)\,dmds+\left(\frac{a_1\sigma}{q\upsilon}\right)\int_0^\mu G_1(\mu,m)w(m)\,dm$$
$$+\left(\frac{b_1\sigma}{q\upsilon}\right)\int_0^T G_1(T,m)h(m)\,dm-\frac{d_2}{\upsilon}-\frac{d_1\sigma}{q\upsilon}-\left(\frac{c_2}{\upsilon}\right)\int_\omega^T\int_0^s G_1(s,m)w(m)\,dmds+\frac{a_2}{\upsilon}\int_0^\mu w(m)\,dm$$
$$-\frac{a_2\lambda}{\upsilon}\int_0^\mu G_2(\mu,m)w(m)\,dm+\frac{b_2}{\upsilon}\int_0^T w(m)\,dm-\frac{b_2\lambda}{\upsilon}\int_0^T G_2(T,m)w(m)\,dm.$$

(5.8)

Substituting (5.7) and (5.8) in equation (5.4) implying

$$y(t)=\underbrace{\left[\left(\frac{d_2\rho}{q\upsilon}+\frac{d_1\rho\sigma}{q^2\upsilon}+\frac{d_1}{q}\right)e^{-\lambda t}-\left(\frac{d_2}{\upsilon}+\frac{d_1\sigma}{q\upsilon}\right)\right]}_{\eta_1(t)}+\int_0^t G_1(t,m)w(m)dm$$

$$+\underbrace{\left[\left(\frac{c_1\rho\sigma}{q^2\upsilon}+\frac{c_1}{q}\right)e^{-\lambda t}-\frac{c_1\sigma}{q\upsilon}\right]}_{\eta_2(t)}\int_0^\tau\int_0^s G_1(s,m)w(m)dmds+$$

$$\underbrace{\left[\left(\frac{c_2\rho}{q\upsilon}\right)e^{-\lambda t}-\left(\frac{c_2}{\upsilon}\right)\right]}_{\eta_3(t)}\int_\omega^T\int_0^s G_1(s,m)w(m)dmds$$

$$+\underbrace{\left[\left(\frac{a_1\sigma}{q\upsilon}\right)-\left(\frac{a_1\rho\sigma}{q^2\upsilon}+\frac{a_1}{q}\right)e^{-\lambda t}\right]}_{\eta_4(t)}\int_0^\mu G_1(\mu,m)w(m)dm+$$

$$\underbrace{\left[\left(\frac{b_1\sigma}{q\upsilon}\right)-\left(\frac{b_1\rho\sigma}{q^2\upsilon}+\frac{b_1}{q}\right)e^{-\lambda t}\right]}_{\eta_5(t)}\int_0^T G_1(T,m)w(m)dm$$

$$+\underbrace{\left[\frac{a_2}{\upsilon}-\left(\frac{a_2\rho}{q\upsilon}\right)e^{-\lambda t}\right]}_{\eta_6(t)}\int_0^\mu w(m)dm+\underbrace{\left[\frac{b_2}{\upsilon}-\left(\frac{b_2\rho}{q\upsilon}\right)e^{-\lambda t}\right]}_{\eta_7(t)}\int_0^T w(m)dm$$

$$+\underbrace{\left[\left(\frac{a_2\lambda\rho}{q\upsilon}\right)e^{-kt}-\frac{a_2\lambda}{\upsilon}\right]}_{\eta_8(t)}\int_0^\mu G_2(\mu,m)w(m)dm+\underbrace{\left[\left(\frac{b_2\lambda\rho}{q\upsilon}\right)e^{-\lambda t}-\frac{b_2\lambda}{\upsilon}\right]}_{\eta_9(t)}\int_0^T G_2(T,m)w(m)dm.$$

By direct computation the converse of the lemma holds true.

5.1 Existence and Uniqueness For the Solution of SFDE

Next, based on FPT we investigate the uniqueness and existence for the solution of SFDE (5.1), to do so we introduce the function space given as the following

$\Psi=\left(C_{r-1}[0,T],\|u\|_{r-1}\right)$, and the continuous $f:[0,T]\to\mathbb{R}$, it is clear that Ψ is a Banach space with the norm $\|y\|_{r-1}=\sup_{0\le t\le T}|y(t)|+\sup_{0\le t\le T}\left|{}_C D^{r-1}y(t)\right|$, an operator $\mathcal{F}:\Psi\to\Psi$ associated with the problem can be defined as

$$
\begin{aligned}
(\mathcal{F}u)(t) &= \eta_1(t) + \int_0^t G_1(t,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \eta_2(t) \int_0^\tau \int_0^s G_1(s,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm ds \\
&+ \eta_3(t) \int_\omega^T \int_0^s G_1(s,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm ds \\
&+ \eta_4(t) \int_0^\mu G_1(\mu,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \eta_5(t) \int_0^T G_1(T,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \eta_6(t) \int_0^\mu g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \eta_7(t) \int_0^T g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \eta_8(t) \int_0^\mu G_2(\mu,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \eta_9(t) \int_0^T G_2(T,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm.
\end{aligned}
\tag{5.9}
$$

One can easily show

$$
\begin{aligned}
\left({}_C D^{r-1} \mathcal{F} y\right)(t) &= \left({}^c D^{r-1} \eta_1\right)(t) + \int_0^t g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&- \lambda \int_0^t G_2(t,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \left({}_C D^{r-1} \eta_2\right)(t) \int_0^\tau \int_0^s G_1(s,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm ds \\
&+ \left({}_C D^{r-1} \eta_3\right)(t) \int_\omega^T \int_0^s G_1(s,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm ds \\
&+ \left({}_C D^{r-1} \eta_4\right)(t) \int_0^\mu G_1(\mu,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm \\
&+ \left({}_C D^{r-1} \eta_5\right)(t) \int_0^T G_1(T,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm
\end{aligned}
$$

$$+\left({}_C D^{r-1}\eta_6\right)(t)\int_0^{\mu} g\left(m, y(m), {}_C D^{r-1}y(m)\right)dm$$

$$+\left({}_C D^{r-1}\eta_7\right)(t)\int_0^{T} g\left(m, y(m), {}_C D^{r-1}y(m)\right)dm$$

$$+\left({}_C D^{r-1}\eta_8\right)(t)\int_0^{\mu} G_2(\mu, m)\, g\left(m, y(m), {}_C D^{r-1}y(m)\right)dm$$

$$+\left({}_C D^{r-1}\eta_9\right)(t)\int_0^{T} G_2(T, m)\, g\left(m, y(m), {}_C D^{r-1}y(m)\right)dm. \tag{5.10}$$

We set the following notations for computational convenience

$$Q_1 = \left(\frac{T^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda T}\right) + \|\eta_2\|\int_0^{\tau}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds + \|\eta_3\|\int_{\omega}^{T}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds\right.$$
$$+\|\eta_4\|\frac{\mu^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda\mu}\right) + \|\eta_5\|\frac{T^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda T}\right)$$
$$\left.+\mu\|\eta_6\| + T\|\eta_7\| + \|\eta_8\|\frac{\mu\left(-e^{-\lambda\mu}+1\right)}{\lambda} + \|\eta_9\|\frac{T\left(-e^{-\lambda T}+1\right)}{\lambda}\right),$$

$$Q_2 = \left(T\left(1+\left(1-e^{-\lambda T}\right)\right) + \left\|{}_C D^{r-1}\eta_2\right\|\int_0^{\tau}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds + \left\|{}_C D^{r-1}\eta_3\right\|\int_{\omega}^{T}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds\right.$$
$$+\left\|{}_C D^{r-1}\eta_4\right\|\frac{\mu^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda\mu}\right) + \left\|{}^C D^{r-1}\eta_5\right\|\frac{T^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda T}\right)$$
$$\left.+\mu\left\|{}_C D^{r-1}\eta_6\right\| + T\left\|{}_C D^{r-1}\eta_7\right\| + \left\|{}_C D^{r-1}\eta_8\right\|\frac{\mu\left(1-e^{-\lambda\mu}\right)}{\lambda} + \left\|{}_C D^{r-1}\eta_9\right\|\frac{T\left(1-e^{-\lambda T}\right)}{\lambda}\right).$$

$$Q_3 = Q_1 + Q_2 - T\left(\left(1-e^{-\lambda T}\right)\left(\frac{T^{r-2}}{\lambda\Gamma(r)}+1\right)+1\right).$$

Before introducing our main existence and uniqueness results for the SFDE (5.1) we suppose that the following assumptions holds true.

(A1) $g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is jointly continuous

(A2) $\exists L_g > 0$ such that

$$\left|g\left(t, y, \bar{y}\right) - g\left(t, z, \bar{z}\right)\right| \le L_g\left(|y-z| + \left|\bar{y}-\bar{z}\right|\right), \forall t\in[0,T], y, z, \bar{y}, \bar{z}\in\mathbb{R}$$

$(\text{A3})\,\exists$ a function $\gamma \in C\left(\left[0,T\right],\mathbb{R}^{+}\right)$ such that

$\left|g\left(t,y,z\right)\right| \leq \gamma\left(t\right),\ \forall\left(t,y,z\right)\in\left[0,T\right]\times\mathbb{R}\times\mathbb{R}$.

$(\text{A4})\,\exists h \in C\left(\left[0,T\right],\mathbb{R}^{+}\right)$ and a nondecreasing function $\chi:\mathbb{R}^{+}\to\mathbb{R}^{+}$

(A5) such that $\left|g\left(t,y,z\right)\right| \leq h\left(t\right)\chi\left(\left|y\right|+\left|z\right|\right), \forall\left(t,y,z\right)\in\left[0,T\right]\times\mathbb{R}\times\mathbb{R}$

there exist a constant $W>0$ such that

$$\frac{W}{\left\|\eta_1\right\|_{r-1}+\left\|h\right\|\chi\left(W\right)\,\mathcal{Q}}>1.$$

The uniqueness result will be introduced by next theorem.

Theorem 5.2. Assume both (A1), (A2) satisfy. If $L_g Q<1$, then there exist a unique solution for the SFDE (5.1) on $\left[0,T\right]$.

Proof. Consider $\mathcal{F}$ defined by (5.9) and construct a ball

$$B_a=\left\{y\in C_{r-1}\left[0,T\right]:\left\|y\right\|_{r-1}\leq a\right\} \text{ with, } a\geq\frac{\left\|\eta_1\right\|_{r-1}+N_g Q}{1-L_g Q} \text{ where } N_g=\sup_{0\leq t\leq T}\left|g\left(t,0,0\right)\right|.$$

First we show that $\mathcal{F}B_a\subset B_a$ for any $y\in B_a$, then

$$\left|\left(\mathcal{F}y\right)\left(t\right)\right|\leq\left\|\eta_1\right\|+\int_0^t\left|G_1\left(t,m\right)\right|\left|g\left(m,y\left(m\right),{}_C D^{r-1}y\left(m\right)\right)\right|dm$$

$$+\left\|\eta_2\right\|\int_0^\tau\int_0^s\left|G_1\left(s,m\right)\right|\left|g\left(m,y\left(m\right),{}_C D^{r-1}y\left(m\right)\right)\right|dmds+$$

$$\left\|\eta_3\right\|\int_\omega^T\int_0^s\left|G_1\left(s,m\right)\right|\left|g\left(m,y\left(m\right),{}_C D^{r-1}y\left(m\right)\right)\right|dmds+$$

$$\left\|\eta_4\right\|\int_0^\mu\left|G_1\left(\mu,m\right)\right|\left|g\left(m,y\left(m\right),{}_C D^{r-1}y\left(m\right)\right)\right|dm+$$

$$\left\|\eta_5\right\|\int_0^T\left|K_1\left(T,m\right)\right|\left|g\left(m,y\left(m\right),{}_C D^{r-1}y\left(m\right)\right)\right|dm+$$

$$\|\eta_6\|\int_0^{\mu}\left|g\left(m,y(m),{}_CD^{r-1}y(m)\right)\right|dm+$$

$$\|\eta_7\|\int_0^{T}\left|g\left(m,y(m),{}_CD^{r-1}y(m)\right)\right|dm+$$

$$\|\eta_8\|\int_0^{\mu}\left|G_2(\mu,m)\right|\left|g\left(m,y(m),{}_CD^{r-1}y(m)\right)\right|dm+$$

$$\|\eta_9\|\int_0^{T}\left|G_2(T,m)\right|\left|g\left(m,y(m),{}_CD^{r-1}y(m)\right)\right|dm.$$

but

$$\left|g\left(t\ ,y(t\ ),{}^cD^{r-1}y(t\ )\right)\right|\le L_g a+N_g.$$

then

$$\begin{aligned}\|\mathcal{F}y\|&\le\|\eta_1\|+\left(L_g a+N_g\right)\\&\times\left(\frac{T^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda T}\right)+\|\eta_2\|\int_0^{\tau}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-ks}\right)ds+\|\eta_3\|\int_{\omega}^{T}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds\right.\\&+\|\eta_4\|\frac{\mu^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda\mu}\right)+\|\eta_5\|\frac{T^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda T}\right)\\&\left.+\mu\|\eta_6\|+T\|\eta_7\|+\|\eta_8\|\frac{\mu\left(1-e^{-\lambda\mu}\right)}{\lambda}+\|\eta_9\|\frac{T\left(1-e^{-\lambda T}\right)}{\lambda}\right)\\&\le\|\eta_1\|+\left(L_g a+N_g\right)Q_1,\end{aligned}\tag{5.11}$$

and

$$\begin{aligned}\left\|{}_CD^{r-1}\mathcal{F}y\right\|&\le\left\|\left({}_CD^{r-1}\eta_1\right)\right\|+\left(L_g a+N_g\right)\\&\times\left(T\left(1+\left(1-e^{-\lambda T}\right)\right)+\left\|{}_CD^{r-1}\eta_2\right\|\int_0^{\tau}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds+\left\|{}_CD^{r-1}\eta_3\right\|\int_{\omega}^{T}\frac{s^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda s}\right)ds\right.\\&+\left\|{}_CD^{r-1}\eta_4\right\|\frac{\mu^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda\mu}\right)+\left\|{}_CD^{r-1}\eta_5\right\|\frac{T^{r-1}}{\lambda\Gamma(r)}\left(1-e^{-\lambda T}\right)\\&\left.+\mu\left\|{}_CD^{r-1}\eta_6\right\|+T\left\|{}^cD^{r-1}\eta_7\right\|+\left\|{}_CD^{r-1}\eta_8\right\|\frac{\mu\left(1-e^{-\lambda\mu}\right)}{\lambda}+\left\|{}_CD^{r-1}\eta_9\right\|\frac{T\left(1-e^{-\lambda T}\right)}{\lambda}\right).\\&\le\left\|{}_CD^{r-1}\eta_1\right\|+\left(L_g a+N_g\right)Q_2.\end{aligned}\tag{5.12}$$

Combining (5.11) and (5.12) we get

$$\|\mathcal{F}y\|_{r-1} \le \|\eta_1\|_{r-1} + \left(L_g a + N_g\right)(Q_1 + Q_2) = \|\eta_1\|_{r-1} + \left(L_g a + N_g\right)Q \le a,$$

Which implies that $\mathcal{F}B_a \subset B_a$.

Next, $\mathcal{F}$ is a contraction has to be shown, for this. $\forall y, z \in \Psi$

$$\left|(\mathcal{F}y)(t) - (\mathcal{F}z)(t)\right| \le \int_0^t \left|G_1(t,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm$$
$$+\|\eta_2\| \int_0^\tau \int_0^s \left|G_1(t,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dmds$$
$$+\|\eta_3\| \int_\omega^T \int_0^s \left|G_1(t,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dmds$$

$$+\|\eta_4\| \int_0^\mu \left|G_1(\mu,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm$$
$$+\|\eta_5\| \int_0^T \left|G_1(T,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm$$
$$+\|\eta_6\| \int_0^\mu \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm$$
$$+\|\eta_7\| \int_0^T \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm$$
$$+\|\eta_8\| \int_0^\mu \left|G_2(\mu,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm$$

$$+\|\eta_9\| \int_0^T \left|G_2(T,m)\right| \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right) - g\left(m, z(m), {}_C D^{r-1} z(m)\right)\right| dm \qquad (5.13)$$
$$\le L_g Q_1 \|y - z\|_{r-1}.$$

In a like manner

$$\left\| {}_C D^{r-1} \mathcal{F}y - {}_C D^{r-1} \mathcal{F}z \right\| \le L_g Q_2 \|y - z\|_{r-1}. \qquad (5.14)$$

From (5.13) and (5.14) it follows that

$$\|\mathcal{F}y - \mathcal{F}z\|_{r-1} \le L_g Q \|y - z\|_{r-1}.$$

From the above inequality, the contraction property for $\mathcal{F}$ has been satisfied. Implying the uniqueness for the solution of SFDE (5.1) on $[0,T]$ which guaranteed by theorem (2.2.7).

Theorem 5.3. Let (A1), (A2) and (A3) satisfy. If $L_g Q_3 < 1$, then at least there exist one solution of SFDE (5.1) on $[0,T]$.

Proof. Consider the ball $B_a = \{y \in \Psi : \|y\|_{r-1} \le a\}$ with $\|\eta_1\|_{r-1} + \|\gamma\| Q \le a$ and. We define two operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on B_a as

$$(\mathcal{F}_1 y)(t) = \int_0^t G_1(t,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm,$$

$$(\mathcal{F}_2 u)(t) = \eta_1(t) + \eta_2(t) \int_0^\tau \int_0^s G_1(s,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dmds$$

$$+\eta_3(t) \int_\omega^T \int_0^s G_1(s,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dmds$$

$$+\eta_4(t) \int_0^\mu G_1(\mu,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm$$

$$+\eta_5(t) \int_0^T G_1(T,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm$$

$$+\eta_6(t) \int_0^\mu g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm$$

$$+\eta_7(t) \int_0^T g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm$$

$$+\eta_8(t) \int_0^\mu G_2(\mu,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm$$

$$+\eta_9(t) \int_0^T G_2(T,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm.$$

For $y, z \in B_a$, it is clear that $\|\mathcal{F}_1 y + \mathcal{F}_2 z\|_{r-1} \le \|\eta_1\|_{r-1} + Q\|\gamma\| \le a$ thus $\mathcal{F}_1 y + \mathcal{F}_2 z \in B_a$.

By using the condition (A_2), then one can also easily show that

$$\|\mathcal{F}_2 y - \mathcal{F}_2 z\|_{r-1} \le L_g Q_3 \ \|y - z\|_{r-1},$$

which implies that $\mathcal{F}_2$ is a contraction.

Moreover, $\mathcal{F}_1$ is continuous because of continuity of f . And

$$\left|\left(\mathcal{F}_1 y\right)(t)\right| \leq \int_0^t G_1(t,m)\left|g\left(m, y(m), {}_C D^{r-1} y(m)\right)\right| dm$$
$$\leq \|\gamma\| \frac{T^{r-1}}{\lambda \Gamma(r)}\left(1-e^{-\lambda T}\right). \tag{5.15}$$

Note that

$$\left({}_C D^{r-1} \mathcal{F}_1 y\right)(t) = {}_C D^{r-1}\left[\int_0^t G_1(t,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm\right]$$
$$= \int_0^t g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm - \lambda \int_0^t G_2(t,m)\, g\left(m, y(m), {}_C D^{r-1} y(m)\right) dm$$
$$\left|\left({}_C D^{r-1} \mathcal{F}_1 y\right)(t)\right| \leq$$
$$\int_0^t \left|g\left(m, y(m), {}_C D^{r-1} y(m)\right)\right| dm + \lambda \int_0^t G_2(t,m)\left|g\left(m, y(m), {}_C D^{r-1} y(m)\right)\right| dm$$
$$\leq \|\gamma\| T + \frac{T^{r-1}}{\Gamma(r)}\left(1-e^{-\lambda T}\right). \tag{5.16}$$

From (5.15) and (5.16) we obtain

$$\left\|\mathcal{F}_1 y\right\|_{r-1} \leq \|\gamma\| T\left(1+\left(1-e^{-kT}\right)\left(\frac{T^{r-2}}{\lambda \Gamma(r)}+\frac{T^{r-2}}{\Gamma(r)}\right)\right).$$

For showing the compactness of $\mathcal{F}_1$. Fixing

$$\sup\left|g(t, y, z)\right| = g_a \ , \forall (t,\ y,\ z) \in [0,\ T] \times B_a \times B_a$$

For $t_1, t_2 \in [0, T]$, then

$$\left|\mathcal{F}_1 y(t_2)-\mathcal{F}_1 y(t_1)\right|\leq$$

$$g_a\left(\int_0^{t_1} G_1(t_2,m)-G_1(t_1,m)\,dm+\int_{t_1}^{t_2} G_1(t_2,m)\,dm\right)\to 0,\text{as } t_1\to t_2 \tag{5.17}$$

$$\left|{}_C D^{r-1}\mathcal{F}_1 y(t_2)-{}_C D^{r-1}\mathcal{F}_1 y(t_1)\right|\leq$$

$$g_a\left(\int_{t_1}^{t_2} dm+\lambda\int_0^{t_1} G_2(t_1,m)-G_2(t_2,m)\,dm-\lambda\int_{t_1}^{t_2} G_2(t_2,m)\,dm\right)\to 0,\text{as } t_1\to t_2 \tag{5.18}$$

(5.17) and (5.18) implies

$$\left\|\mathcal{F}_1 y(t_2)-\mathcal{F}_1 y(t_1)\right\|_{r-1}\to 0,\ \text{ as } t_1\to t_2$$

The independency of the term $\left\|\mathcal{F}_1 y(t_2)-\mathcal{F}_1 y(t_1)\right\|_{r-1}$ of u leads to deduce that $\mathcal{F}_1$ is relatively compact, based on theorem (2.2.9) we conclude that $\mathcal{F}_1$ is compact on B_a. Hence, the existence property for the solution of the SFDE holds by theorem (2.2.10).

***Theorem*5.4.** Suppose that (A1), (A4) and (A5) satisfy. Then there exist a solution for the SFDE (5.1) on [0,T].

Proof. Step 1, $\mathcal{F}$ maps bounded sets into bounded sets of Ψ has to be shown.

Consider $B_a=\left\{y\in\Psi:\|y\|_{r-1}\leq a\right\}$ be a bounded set in Ψ . Then by $(A4)$ we have

$$\begin{aligned}
\left|(\mathcal{F}y)(t)\right| &\leq \|\eta_1\|+\int_0^t \left|G_1(t,m)\right|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm\\
&+\|\eta_2\|\int_0^{\tau}\int_0^{s}\left|G_1(s,m)\right|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dmds+\\
&\|\eta_3\|\int_{\omega}^{T}\int_0^{s}\left|G_1(s,m)\right|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dmds\\
&+\|\eta_4\|\int_0^{\mu}\left|G_1(\mu,m)\right|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm
\end{aligned}$$

$$+\|\eta_5\|\int_0^T |G_1(T,m)|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm$$

$$+\|\eta_6\|\int_0^\mu \left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm$$

$$+\|\eta_7\|\int_0^T \left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm$$

$$+\|\eta_8\|\int_0^\mu |G_2(\mu,m)|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm$$

$$+\|\eta_9\|\int_0^T |G_2(T,m)|\left|g\left(m,y(m),{}_C D^{r-1}y(m)\right)\right|dm.$$

$$\leq \|\eta_1\|+\int_0^t |G_1(t,m)|h(m)\chi\left(\|y\|_{r-1}\right)dm$$

$$+\|\eta_2\|\int_0^\tau\int_0^s |G_1(s,m)|h(m)\chi\left(\|y\|_{r-1}\right)dmds+\|\eta_3\|\int_\omega^T\int_0^s |G_1(s,m)|h(m)\chi\left(\|y\|_{r-1}\right)dmds$$

$$+\|\eta_4\|\int_0^\mu |G_1(\mu,m)|h(m)\chi\left(\|y\|_{r-1}\right)dm+\|\eta_5\|\int_0^T |G_1(T,m)|h(m)\chi\left(\|y\|_{r-1}\right)dm$$

$$+\|\eta_6\|\int_0^\mu h(m)\chi\left(\|y\|_{r-1}\right)dm+\|\eta_7\|\int_0^T h(m)\chi\left(\|y\|_{r-1}\right)dm$$

$$+\|\eta_8\|\int_0^\mu |G_2(\mu,m)|h(m)\chi\left(\|y\|_{r-1}\right)dm+\|\eta_9\|\int_0^T |G_2(T,m)|h(m)\chi\left(\|y\|_{r-1}\right)dm.$$

Taking $\sup\limits_{0\leq t\leq T}$, implies that

$$\sup_{0\leq t\leq T}|\mathcal{F}y(t)|\leq \|\eta_1\|+\|h\|\chi(a)\ \mathcal{Q}_1. \tag{5.19}$$

In a like manner

$$\sup_{0\leq t\leq T}\left|{}^c D^{r-1}\mathcal{F}y(t)\right|\leq \left\|{}^c D^{r-1}\eta_1\right\|+\|h\|\chi(a)\ \mathcal{Q}_2. \tag{5.20}$$

Combining (5.19) and (5.20) we get

$$\|\mathcal{F}y\|_{r-1}\leq \|\eta_1\|_{\alpha-1}+\|h\|\chi(a)\ \mathcal{Q}\ .$$

Step (2) show that $\mathcal{F}$ is equicontiuous, to do we let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ .then $\forall u \in B_a$

we have

$$\begin{aligned}
\left|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)\right| &\le \left|\eta_1(t_2)-\eta_1(t_1)\right| \\
&+\left|\int_0^{t_1} G_1(t_1,m)h(m)\chi\left(\|y\|_{r-1}\right)dm-\int_0^{t_2} G_1(t_2,m)h(m)\chi\left(\|y\|_{r-1}\right)dm\right| \\
&+\left|\eta_2(t_2)-\eta_2(t_1)\right|\int_0^{\tau}\int_0^{s}\left|G_1(s,m)\right|h(m)\chi\left(\|y\|_{r-1}\right)dmds \\
&+\left|\eta_3(t_2)-\eta_3(t_1)\right|\int_{\omega}^{T}\int_0^{s}\left|G_1(s,m)\right|h(m)\chi\left(\|y\|_{r-1}\right)dmds \\
&+\left|\eta_4(t_2)-\eta_4(t_1)\right|\int_0^{\mu}\left|G_1(\mu,m)\right|h(m)\chi\left(\|y\|_{r-1}\right)dm \\
&+\left|\eta_5(t_2)-\eta_5(t_1)\right|\int_0^{T}\left|G_1(T,m)\right|h(m)\chi\left(\|y\|_{r-1}\right)dm \\
&+\left|\eta_6(t_2)-\eta_6(t_1)\right|\int_0^{\mu}h(m)\chi\left(\|y\|_{r-1}\right)dm \\
&+\left|\eta_7(t_2)-\eta_7(t_1)\right|\int_0^{T}h(m)\chi\left(\|y\|_{r-1}\right)dm \\
&+\left|\eta_8(t_2)-\eta_8(t_1)\right|\int_0^{\mu}\left|G_2(\mu,m)\right|h(m)\chi\left(\|y\|_{r-1}\right)dm \\
&+\left|\eta_9(t_2)-\eta_9(t_1)\right|\int_0^{T}\left|G_2(T,m)\right|h(m)\chi\left(\|y\|_{r-1}\right)dm \to 0, \quad \text{as } t_1 \to t_2
\end{aligned} \tag{5.21}$$

In a similar manner

$$\left|{}_C D^{r-1}\mathcal{F}y(t_2)-{}_C D^{r-1}\mathcal{F}y(t_1)\right| \to 0 \quad as \quad t_1 \to t_2. \tag{5.22}$$

Since the above inequalities (5.21) and (5.22) are independent of $y \in B_a$, theorem (2.2.9) leads to deduce that $\mathcal{F}$ is completely continuous.

At last, by showing that any solution y for the equation $y = \delta \mathcal{F} y$, $0 \le \delta \le 1$ is bounded, we complete all conditions of theorem (2.2.8), to do so, and for $t \in [0, T]$, suppose that y is a solution, then

$$|y(t)| \le \|\eta_1\|_{r-1} + \|h\| \chi(a)\ \mathcal{Q}\ \cdot$$

implies

$$\frac{\|y\|}{\|\eta_1\|_{r-1} + \|g\| \chi(a)\ \mathcal{Q}} \le 1.$$

But by (A_5) $\exists W > 0$ such that $W \ne \|y\|$, let $Z = \{ y \in \Psi : \|y\| < W \}$

The operator has the property that it is completely continuous by the defined Z, there is no $y \in \partial Z$ such that $y = \delta \mathcal{F} y$ for some $\delta \in (0,1)$. Therefore, theorem (2.2.8) guarantees the existence property for the solution $y \in \overline{Z}$ of $\mathcal{F}$, which solve the SFDE (5.1).

Examples. Given the following SFDE

$$\begin{cases} \left({}_C D^{3/2} + {}_C D^{1/2} \right) y(t) = K \left(\dfrac{y(t)}{5 + y(t)} + \tan^{-1}\left({}_C D^{1/2} y(t) \right) + \sqrt{\sin^5 t} \right), \ 0 \le t \le 3, \\ y(0) + y(3) = \int_0^1 y(s) ds, \\ {}_C D^{1/2} y(0) + {}_C D^{1/2} y(3) = \int_2^3 y(s) ds + 1. \end{cases} \tag{5.23}$$

here

$$g\left(t, y(t), {}_C D^{1/2} y(t)\right) = K \left(\frac{y(t)}{5 + y(t)} + \tan^{-1}\left({}_C D^{1/2} y(t) \right) + \sqrt{\sin^5 t} \right),$$
$$\lambda = 1, a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1,$$
$$\mu = 0, \tau = 1, \omega = 2, T = 3, d_1 = 0, d_2 = 1,$$

$$q=\left(a_1e^{-\lambda\mu}+b_1e^{-\lambda T}-c_1\left(1-e^{-\lambda\tau}\right)\right)=0.42,$$
$$\rho=\left(a_1+b_1-c_1\tau\right)=1\ ,$$
$$\sigma=\left(\frac{c_2}{\lambda}\left(e^{-\lambda\omega}-e^{-\lambda T}\right)+\frac{\lambda a_2}{\Gamma(2-r)}\int_0^{\mu}(\mu-s)^{1-r}\,e^{-\lambda s}ds+\frac{\lambda b_2}{\Gamma(2-r)}\int_0^{T}(T-s)^{1-r}\,e^{-\lambda s}ds\right)\le 2.04,$$
$$\upsilon=c_2\left(T-\omega\right)-\frac{\sigma\rho}{q}=-3.86,$$

$$\|\eta_2\|=\left|\left(\frac{c_1\rho\sigma}{q^2\upsilon}+\frac{c_1}{q}\right)-\frac{c_1\sigma}{q\upsilon}\right|=6.86,$$
$$\|\eta_3\|=\left|\left(\frac{c_2\rho}{q\upsilon}\right)-\left(\frac{c_2}{\upsilon}\right)\right|=0.36,$$
$$\|\mu_4\|=\left|\left(\frac{a_1\sigma}{q\upsilon}\right)-\left(\frac{a_1\rho\sigma}{q^2\upsilon}+\frac{a_1}{q}\right)e^{-\lambda T}\right|=1.23,$$

$$\|\eta_5\|=\left|\left(\frac{b_1\sigma}{q\upsilon}\right)-\left(\frac{b_1\rho\sigma}{q^2\upsilon}+\frac{b_1}{q}\right)e^{-\lambda T}\right|=1.23,$$
$$\|\eta_6\|=\left|\frac{a_2}{\upsilon}-\left(\frac{a_2\rho}{q\upsilon}\right)e^{-\lambda T}\right|=0.23,$$
$$\|\eta_7\|=\left|\frac{b_2}{\upsilon}-\left(\frac{b_2\rho}{q\upsilon}\right)e^{-\lambda T}\right|=0.23\ ,$$

$$\|\eta_8\|=\left|\left(\frac{a_2\lambda\rho}{q\upsilon}\right)-\frac{a_2\lambda}{\upsilon}\right|=0.36,$$
$$\|\eta_9\|=\left|\left(\frac{b_2\lambda\rho}{q\upsilon}\right)-\frac{b_2\lambda}{\upsilon}\right|=0.36.$$

Clearly g is continuous and $\forall t\in[0,3],\forall y,z,\bar{y},\bar{z}\in\mathbb{R}$ we have

$$\left|f(t,y,z)-f\left(t,\bar{y},\bar{z}\right)\right|\le K\left|\frac{|y|}{5+|y|}-\frac{|\bar{z}|}{5+|\bar{z}|}\right|+K\left|\tan^{-1}y-\tan^{-1}\bar{z}\right|\le L\left(\left|y-\bar{y}\right|+\left|z-\bar{z}\right|\right),$$

with $Q_1\le 11.25, Q_2\le 21.72, Q_3\le 18.17$

Choosing $K<\dfrac{1}{Q_1+Q_2}$, then theorem (5.2) holds true.

To explain Theorem (5.4), consider

$$g\left(t, y(t), {}_C D^{1/2} y(t)\right) = t^2 e^{-y^2(t)} \ln(1+2\sin^2 y(t)) + \frac{{}_C D^{3/2} y(t)}{1+{}_C D^{3/2} y(t)} + \sqrt{1+t^2},$$

$$g(t, y, z) = t^2 e^{-y^2} \ln(1+2\sin^2 y) + \frac{|z|}{1+|z|} + \sqrt{1+t^2}, t \in [0,3], \forall y, z \in \mathbb{R},$$

g is continuous and

$$|g(t, y, z)| \le t^2 \ln 3 + 1 + \sqrt{1+t^2} =: h(t), with \|h\| = 9(\ln 3) + 1 + \sqrt{10},\ h(t) \in C_{1/2}[0,3],$$

This implies that g is bounded. All conditions of theorem (5.4) satisfied which implying the existence property for the solution of the SFDE on $[0,3]$.

For example which shows that Theorem (5.3) works, we let the function

$$g\left(t, y(t), {}_C D^{1/2} y(t)\right) = L\left(\frac{y(t)}{1+y(t)} + \tan^{-1}\left({}_C D^{1/2} y(t)\right) + \sqrt{\sin^3 t}\right),$$

g is continuous and Lipschitzian.

Since

$$|g(t, y, z)| \le L\left(1 + \frac{\pi}{2} + \sqrt{\sin^3 t}\right) =: \gamma(t) \le L\left(2 + \frac{\pi}{2}\right),$$

We conclude that g is bounded, one can easily compute $Q_3 \le 11.25$, choose arbitrary L_g which satisfy the inequality $LQ_3 < 1,$, then there exist a solution for the SFDE on $[0,3]$.

Chapter 6

NONLINEAR FDE INVOLVING ${}_{H}D^{(\cdot)}$ AND ${}_{CH}D^{(\cdot)}$ WITH 3- POINT INTEGRAL BC's

In this chapter, based on FPT, the existence and uniqueness for the solution of nonlinear (Hadamard (H) / Caputo-Hadamard (CH)) FDE's of order $r \in (1,2]$ with 3-point integral BC's are obtained.

Consider the following FDE' associated with 3-point integral BC's

(a) Nonlinear Hadamard FDE:

$$\begin{cases} {}_{H}D^{r}\zeta(t) = f(t,\zeta(t)),\ 1 < r \le 2,\ 0 < a \le t \le T_0 \\ \zeta(a) = 0, \quad \zeta(T_0) = \theta\int_a^{\mu}\zeta(s)ds, \ \ a < \mu < T_0, \ \ \theta \in \mathbb{R} \end{cases} \tag{6.1}$$

and

(b) Nonlinear Caputo-Hadamard FDE:

$$\begin{cases} {}_{CH}D^{r}u(t) = h(t,u(t)), \quad r \in (1,2], \quad 0 < a \le t \le T_0, \\ u(a) = 0, \quad u(T_0) = \vartheta\int_a^{\mu}u(s)ds, \ \ \mu \in (a,T_0), \ \ \vartheta \in \mathbb{R}. \end{cases} \tag{6.2}$$

where θ, ϑ are given constants.

Lemma 6.1. Let $\omega \in C([a,T_0],\mathbb{R})$ and $\zeta \in C_{\delta}^{2}([a,T_0],\mathbb{R})$. The Hadamard FDE given by

$$\begin{cases} {}^{H}D^{r}\zeta(t) = \omega(t),\ r \in (1,2], \quad 0 < a \le t \le T, \\ \zeta(a) = 0, \quad x(T) = \theta\int_a^{\mu}\zeta(s)ds, \quad \mu \in (a,T), \quad \theta \in \mathbb{R}. \end{cases} \tag{6.3}$$

has the solution given as follows

$$\zeta(t)=\frac{1}{\Gamma(r)}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}\frac{\omega(s)}{s}ds$$
$$+\frac{\left(\ln\frac{t}{a}\right)^{r-1}\left(\theta\int_a^{\mu}\int_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}\frac{\omega(m)}{m}dmds-\int_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\frac{\omega(s)}{s}ds\right)}{\Gamma(r)\left(\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right)} \quad (6.4)$$

Proof. Applying ${}_HI^r$ to both sides of (6.3)

$${}_HI^r\left({}_HD^r\zeta\right)(t)={}_HI^r\omega(t)$$

$$\zeta(t)={}_HI^r\omega(t)+\sum_{j=1}^{2}c_j\left(\ln\frac{t}{a}\right)^{r-j}$$

$$\zeta(t)={}_HI^r\omega(t)+c_1\left(\ln\frac{t}{a}\right)^{r-1}+c_2\left(\ln\frac{t}{a}\right)^{r-2}$$

Based on $\zeta(a)=0,$ it is trivial to deduce that $c_2=0$,

then

$$\zeta(t)={}_HI^r\omega(t)+c_1\left(\ln\frac{t}{a}\right)^{r-1} \quad (6.5)$$

and

$\zeta(T_0)=\theta\int_a^{\mu}\zeta(s)ds$, implies

$${}_HI^r\omega(T_0)+c_1\left(\ln\frac{T_0}{a}\right)^{r-1}=\theta\int_a^{\mu}{}_HI^r\omega(s)ds+\theta c_1\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds$$

that is

$$c_1\left[\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right]=\theta\int_a^{\mu}{}_HI^r\omega(s)ds-{}_HI^r\omega(T_0)$$

then

$$c_1 = \frac{1}{\left(\left(\ln\frac{T_0}{a}\right)^{r-1} - \theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1} ds\right)}\left(\theta\int_a^{\eta} {}_H I^r\omega(s)ds - {}_H I^r\omega(T_0)\right) \tag{6.6}$$

Substitute (6.6) in (6.5) and expand the Hadamard fractional integrals, then equation(6.4) is obtained. By direct computation the converse of lemma (6.1) can be easily obtained.

Lemma 6.2. let $w \in C([a,T_0],\mathbb{R})$ and $u \in C_\delta^2([a,T_0],\mathbb{R})$. The linear Caputo- Hadamard F.D.E given by

$$\begin{cases} {}_{CH}D^r u(t) = w(t),\ 1 < r \le 2,\quad 0 < a \le t \le T_0 \\ u(a) = 0,\quad u(T_0) = \vartheta\int_a^{\mu} u(s)ds,\ \ a < \mu < T_0,\ \ \vartheta \in \mathbb{R}. \end{cases} \tag{6.7}$$

has a solution represented by the following integral equation

$$u(t) = \frac{1}{\Gamma(r)}\int_a^t \left(\ln\frac{t}{s}\right)^{r-1}\frac{w(s)}{s}ds$$
$$+\frac{\left(\ln\frac{t}{a}\right)\left(\vartheta\int_a^{\mu}\int_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}\frac{w(m)}{m}dmds - \int_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\frac{w(s)}{s}ds\right)}{\Gamma(r)\left(\ln\frac{T_0}{a} - \vartheta\left(\mu\left(\ln\frac{\mu}{a} - 1\right) + a\right)\right)} \tag{6.8}$$

Proof. Because of similarity to the one done in lemma 6.1, we delete the proof

6.1 Existence and Uniqueness For the Problem

Next, based on FPT we investigate the uniqueness and existence for the solution of FDE (6.1), to do so we introduce the function space given as the following

Assume $\Psi = C([a,T_0],\mathbb{R})$, $f:[0,T_0]\to\mathbb{R}$, be a continuous function, it is clear that Ψ is Banach space, we define the norm endowed with this space as $\|\zeta\| = \sup_{a\le t\le T_0}|\zeta(t)|$, and define the operator $G_H : \Psi \to \Psi$,

$$\left(G_H\zeta\right)(t)=\frac{1}{\Gamma(r)}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}\frac{f\left(s,\zeta(s)\right)}{s}ds$$

$$+\frac{\left(\ln\frac{t}{a}\right)^{r-1}\left(\theta\int_a^{\mu}\int_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}\frac{f\left(m,\zeta(m)\right)}{m}dmds-\int_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\frac{f\left(s,\zeta(s)\right)}{s}ds\right)}{\Gamma(r)\left(\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right)}.$$

Notations

$$\mathcal{Q}=\frac{1}{\Gamma(r)}\sup_{a\le t\le T_0}\int_a^t\frac{1}{s}\left(\ln\frac{t}{s}\right)^{r-1}ds+$$

$$\frac{\sup_{a\le t\le T_0}\left(\ln\frac{t}{a}\right)^{r-1}\left(|\theta|\int_a^{\mu}\int_a^{s}\frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1}dmds-\int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{r-1}ds\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}$$

$$=\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}+\frac{\left(\ln\frac{T_0}{a}\right)^{r-1}\left(\frac{|\theta|}{\Gamma(r+1)}\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r}ds-\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}\right)}{\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}$$

$$\le\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}+\frac{\left(\ln\frac{T_0}{a}\right)^{r-1}\left(\frac{|\theta|}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}(\mu-a)-\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}\right)}{\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}$$

$$=\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}+\frac{\left(\ln\frac{T_0}{a}\right)^{2r-1}\left(|\theta|(\mu-a)-1\right)}{\Gamma(r+1)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}.$$

Since, $\frac{|\theta|}{\Gamma(r+1)}\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r}ds\le\frac{|\theta|}{\Gamma(r+1)}\left(\ln\frac{\mu}{a}\right)^{r}(\mu-a)\le\frac{|\theta|}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}(\mu-a)$,
as $a\le s\le\mu\le T_0$.

.

$$\mathcal{Q}^* = \frac{\sup\limits_{a\le t\le T_0}\left(\ln\frac{t}{a}\right)^{r-1}}{\Gamma(r)\left(\left(\ln\frac{T_0}{a}\right)^{r-1} - \theta\int\limits_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1} ds\right)}\left(\theta\int\limits_a^{\mu}\int\limits_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}\frac{1}{m}dmds - \int\limits_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\frac{1}{s}ds\right)$$

$$\le \frac{\left(\ln\frac{T_0}{a}\right)^{2r-1}}{\Gamma(r+1)\left|\left(\ln\frac{T_0}{a}\right)^{r-1} - \theta\int\limits_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1} ds\right|}\left(|\theta|(\mu-a)-1\right)$$

Theorem 6.3. Consider the continuous function $f:[a,T_0]\times\mathbb{R}\to\mathbb{R}$ which satisfy the following condition $(E_1)\,\exists L_f>0$ such that

$\left|f(t,\zeta)-f(t,\xi)\right|\le L_f\left|\zeta-\xi\right|$, and $L_f\mathcal{Q}<1$, $\forall t\in[a,T_0]$, $\forall\zeta,\xi\in\mathbb{R}$. Then there exist a unique solution of the FDE (6.1) on $[a,T_0]$.

Proof. Consider the set $B_\varepsilon=\{\zeta\in\psi,\|\xi\|\le\varepsilon\}$ with $\varepsilon\ge\frac{M\mathcal{Q}}{1-L_f\mathcal{Q}}$ where $M=\sup\limits_{a\le t\le T_0}\left|f(t,0)\right|$

First we show that $G_H B_\varepsilon\subset B_\varepsilon$, for this

$\forall\zeta\in B_\varepsilon, \forall t\in[a,T_0]$

$$\left|(G_H\zeta)(t)\right|\le\frac{1}{\Gamma(r)}\int\limits_a^t\left(\ln\frac{t}{s}\right)^{r-1}\left|\frac{f(s,\zeta(s))}{s}\right|ds$$

$$+\frac{\left(\ln\frac{t}{a}\right)^{r-1}\left(|\theta|\int\limits_a^{\mu}\int\limits_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}\left|\frac{f(m,\zeta(m))}{m}\right|dmds-\int\limits_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\left|\frac{f(s,\zeta(s))}{s}\right|ds\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int\limits_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}$$

But $\forall\zeta\in B_\varepsilon, \forall t\in[0,T_0]$ we have

$$\left|f(t,\zeta(t))\right|\le L_f\|\zeta\|+\sup_{0\le t\le T_0}\left|f(t,0)\right|$$
$$\le L_f\varepsilon+M.$$

Using this inequality, implies

$$\|G_H\zeta\| \le \left(L_f\varepsilon + M\right)\times$$

$$\left[\frac{1}{\Gamma(r)}\sup_{a\le t\le T_0}\int_a^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{r-1} ds + \frac{\sup_{a\le t\le T_0}\left(\ln\frac{t}{a}\right)^{r-1}\left(|\theta|\int_a^{\mu}\int_a^{s}\frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1} dmds - \int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{r-1} ds\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1} - \theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1} ds\right|}\right]$$

$$\le \left(L_f\varepsilon + M\right)\mathcal{Q} \le \varepsilon$$

Which implies that $G_H\zeta \in B_\varepsilon, \forall \zeta \in B_\varepsilon$ that is $G_H B_\varepsilon \subset B_\varepsilon$, next we show the contraction property of G_H. $\forall \zeta, \xi \in \Psi$

$$\|G_H\zeta - G_H\xi\| \le$$

$$\frac{1}{\Gamma(r)}\sup_{a\le t\le T_0}\int_a^t \left(\ln\frac{t}{s}\right)^{r-1}\left|\frac{f(s,\zeta(s)) - f(s,\xi(s))}{s}\right| ds + \frac{\sup_{a\le t\le T_0}\left(\ln\frac{t}{a}\right)^{r-1}}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1} - \theta\int_a^{\mu}\left(\ln\frac{s}{a}\right)^{r-1} ds\right|}\times$$

$$\left(|\theta|\int_a^{\mu}\int_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}\left|\frac{f(m,\zeta(m)) - f(m,\xi(m))}{m}\right| dmds - \int_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\left|\frac{f(s,\zeta(s)) - f(s,\xi(s))}{s}\right| ds\right)$$

$$\le L_f\|\zeta - \xi\|\mathcal{Q} \le L_f\mathcal{Q}\|\zeta - \xi\|.$$

End up with this, we conclude that G_H is a contraction, which implies the uniqueness for the solution of the FDE (6.1) on $[a, T_0]$.

Theorem 6.4. Given the continuous function $f:[a,T_0]\times\mathbb{R}\to\mathbb{R}$ which satisfying the condition (E_1), and

$$|f(t,\zeta)| \le \gamma(t), \forall (t,\zeta) \in [a,T_0]\times\mathbb{R}, where, \gamma \in C\left([a,T_0],\mathbb{R}^+\right) with \sup_{a\le t\le T_0}|\gamma(t)| = \|\gamma\|$$

Assume that also $L_f\mathcal{Q}^* < 1$, then there is at least one solution for FDE (6.1) on $[a, T_0]$.

Proof.

Consider the close set $S_\lambda = \{\zeta \in \Psi, \|\zeta\| \le \lambda\}$, with $\lambda \ge \mathcal{Q}\|\gamma\|$. Define G_{H1} and G_{H2} on S_λ

as

$$(G_{H1}\zeta)(t) = \frac{1}{\Gamma(r)} \int_a^t \left(\ln\frac{t}{s}\right)^{r-1} \frac{f(s,\zeta(s))}{s} ds$$

$$(G_{H2}\zeta)(t) = \frac{\left(\ln\frac{t}{a}\right)^{r-1}\left(\theta\int_a^\mu\int_a^s\left(\ln\frac{s}{m}\right)^{r-1}\frac{f(m,\zeta(m))}{m}dmds - \int_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}\frac{f(s,\zeta(s))}{s}ds\right)}{\Gamma(r)\left(\left(\ln\frac{T_0}{a}\right)^{r-1} - \theta\int_a^\mu\left(\ln\frac{s}{a}\right)^{r-1}ds\right)}$$

For $\zeta, \xi \in S_\lambda$, then $\|G_{H1}\zeta + G_{H2}\xi\| \le \mathcal{Q}\|\gamma\| \le \lambda$ thus $G_{H1}\zeta + G_{H2}\xi \in S_\lambda$

By condition (E_1) and $\mathcal{Q}^*$,then one can easily show that:

$$\|G_{H2}\zeta - G_{H2}\xi\| \le L_f\mathcal{Q}^*\|\zeta - \xi\| \le \|\zeta - \xi\|$$

Which implies that G_{H2} is a contraction.

Moreover, G_{H1} is continuous because of continuity of f .

$$\|G_{H1}\zeta\| \le \|\gamma\| \sup_{a\le t\le T_0}\left\{\frac{1}{\Gamma(r+1)}\left(\ln\frac{t}{a}\right)^r\right\} \le \frac{\|\gamma\|}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^r,$$ hence G_{H1} is uniformly bounded.

Now it will be shown that the operator G_{H1} is compact. Fixing $\sup_{(t,\zeta)\in[a,T_0]\times S_\lambda}|f(t,\zeta)| = f_\lambda$ and

for $t_1, t_2 \in [a, T_0], (t_1 < t_2)$,

then

$$\|G_{H1}\zeta(t_2) - G_{H1}\zeta(t_1)\| \le \frac{f_\lambda}{\Gamma(r)}\left(\int_a^{t_1}\frac{1}{s}\left(\left(\ln\frac{t_2}{s}\right)^{r-1} - \left(\ln\frac{t_1}{s}\right)^{r-1}\right)ds + \int_{t_1}^{t_2}\frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{r-1}ds\right) \to 0$$

The RHS of the above inequality approaches 0, as $t_1 \to t_2$, And the same side of the same inequality is independent of ζ implies that G_{H1} is relatively compact, based on theorem (2.2.9) we deduce that G_{H1} is compact on S_λ Hence, the existence property for the solution of the SFDE holds by theorem (2.2.10).

Theorem 6.5. Given the continuous function $f:[a,T_0]\times\mathbb{R}\to\mathbb{R}$. Also assume that

(E_2) $\exists\, w \in C([a,T_0],\mathbb{R}^+)$ and a non decreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$

Such that $|f(t,\zeta)| \le w(t)\psi(\|\zeta\|), \forall (t,\zeta)\in[0,T_0]\times\mathbb{R}$

(E_3) $\exists\, M>0$ such that $\dfrac{M}{\|w\|\psi(M)\ \mathcal{Q}}>1$, then there exist a solution for FDE (6.1) on $[a,T_0]$.

Proof. Step1 we show that $G_H:\Psi\to\Psi$

Let $S_\kappa=\{\zeta\in\Psi,\|\zeta\|\le\kappa\}$ be a bounded set in Ψ. Then by (E_2)

$$|(G_H\zeta)(t)| \le \frac{1}{\Gamma(r)}\int_a^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{r-1} w(s)\psi(\|\zeta\|)ds$$
$$+\frac{\left(\ln\frac{t}{a}\right)^{r-1}\left(|\theta|\int_a^\mu\int_a^s \frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1} w(m)\psi(\|\zeta\|)dmds-\int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{r-1} w(s)\psi(\|\zeta\|)ds\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^\mu\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}$$

Take $\sup\limits_{a\le t\le T_0}$,

Implies that

$$\leq \|w\|\psi(\kappa)\ \times\left\{\frac{1}{\Gamma(r)}\sup_{a\leq t\leq T_0}\int_a^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{r-1}ds+\frac{\sup_{a\leq t\leq T}\left(\ln\frac{t}{a}\right)^{r-1}\left(|\theta|\int_a^\mu\int_a^s\frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1}dmds-\int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{-1}ds\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^\mu\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}\right\}$$

Implies,

$$\|G_H\zeta\|\leq\|w\|\psi(\kappa)\ \mathcal{Q}$$

Now, show that $G_H:\Psi\to\Psi$ is equicontinuous, to do we let $t_1,t_2\in[a,T_0]$, $(t_1<t_2)$ and $\zeta\in S_\kappa$.

$$\left|G_H\zeta(t_2)-G_H\zeta(t_1)\right|\leq\frac{1}{\Gamma(r)}\left(\begin{array}{l}\int_a^{t_1}\left(\left(\ln\frac{t_2}{s}\right)^{r-1}-\left(\ln\frac{t_1}{s}\right)^{r-1}\right)\left|\frac{f(s,\zeta(s))}{s}\right|ds\\ +\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-1}\left|\frac{f(s,\zeta(s))}{s}\right|ds\end{array}\right)$$

$$+\frac{\left(\left(\ln\frac{t_2}{a}\right)^{r-1}-\left(\ln\frac{t_1}{a}\right)^{r-1}\right)\left(|\theta|\int_a^\mu\int_a^s\frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1}\left|\frac{f(m,\zeta(m))}{m}\right|dmds-\int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{r-1}\left|\frac{f(s,\zeta(s))}{s}\right|ds\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^\mu\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}\to 0$$

As $t_1\to t_2$. Since the above inequality is independent of $\zeta\in S_\kappa$, based on theorem (2.2.10) we conclude that G_H is completely continuous.

Finally, by showing that x which is the solution for the equation $\zeta=\delta G_H\zeta,\quad 0\leq\delta\leq 1,$ is bounded we complete all conditions of theorem (2.2.9), to do so, for $t\in[0,T_0]$, then

$$|\zeta(t)|=|\delta(G_H\zeta)(t)|\leq\delta(\|w\|\psi(\kappa)\ \mathcal{Q})\leq\|w\|\psi(\kappa)\ \mathcal{Q}$$

Implies

$$\frac{\|\zeta\|}{\|w\|\psi(\kappa)\ \mathcal{Q}}\leq 1,$$

But by $(E_3)\exists\, M>0$ such that $M\neq\|\zeta\|$, construct the set $\Omega=\{\zeta\in\Psi:\|\zeta\|<M\}$.

In addition of continuity of $G_H:\overline{\Omega}\to\Psi$ it is also completely continuous, by the defined Ω, there is no $\zeta\in\partial\Omega$ such that $\zeta=\delta\mathcal{F}\zeta$ for some $\delta\in(0,1)$. Therefore, based on theorem (2.2.8), we deduce that $G_H\ \mathcal{F}$ has a fixed point $\zeta\in\overline{\Omega}$ which is a solution of the SFDE.

Example. Consider the non linear Hadamard FDE

$$\begin{cases} {}_H D^{3/2}\zeta(t)=f(t,\zeta(t)),\ \ 1\le t\le e, \\ \zeta(1)=0,\quad \zeta(e)=3\int_1^2\zeta(s)\,ds,\ \ \mu=2,\ \ \theta=3. \end{cases} \tag{6.9}$$

For the applicability of the results obtained by theorem (6.3), consider

$$f(t,\zeta(t))=\frac{\ln t^2}{e^t(t+4)^3}\frac{|\zeta(t)|}{(|\zeta(t)|+1)},\ \ t\in[1,e]$$

The function $f(t,\zeta)=\dfrac{\ln t^2}{e^t(t+4)^3}\dfrac{|\zeta|}{(|\zeta|+1)}$, is continuous on $[1,e]$

And we can show that it is Lipschitzian with

$$L_f=\frac{2}{125e}\ \text{as}\ |f(t,\zeta)-f(t,\xi)|\le\left(\frac{2}{125e}\right)|\zeta-\xi|$$

$$\text{and}\ \int_a^\mu\left(\ln\frac{s}{a}\right)^{r-1}ds=\int_1^2\left(\ln\frac{s}{1}\right)^{1/2}ds=\int_1^2(\ln s)^{1/2}\,ds\le(\ln 2)^{1/2}\int_1^2 ds=\sqrt{\ln 2}$$

One can easily compute

$$\mathcal{Q}\le\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^r+\frac{\left(\ln\frac{T_0}{a}\right)^{2r-1}}{\Gamma(r+1)\left|\left(\ln\frac{T_0}{a}\right)^{r-1}-\theta\int_a^\mu\left(\ln\frac{s}{a}\right)^{r-1}ds\right|}\left(|\theta|(\mu-a)-1\right)$$

$$\leq \frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^r + \frac{\left(\ln\frac{T_0}{a}\right)^{2r-1}}{\Gamma(r+1)}\left(|\theta|(\mu-a)-1\right)$$

$$= \frac{1}{\Gamma\left(\frac{5}{2}\right)} + \frac{2}{\Gamma\left(\frac{5}{2}\right)\left|1-3\sqrt{\ln 2}\right|} = 1.75$$

Implies that $\mathcal{Q} < 1/L_f$

All assumptions of theorem (6.3) holds true, implies the uniqueness of solution of FDE (6.9).

To illustrate theorem (6.5) consider $f\left(t,\zeta(t)\right) = \dfrac{e^{-t}\tan^{-1}\left(\zeta(t)\right)}{\pi\sqrt{t^2+144}},\ t\in[1,e]$

$f\left(t,\zeta\right) = \dfrac{e^{-t}\tan^{-1}\left(\zeta\right)}{\pi\sqrt{t^2+143}}$, is continuous on $[1,e]$,

and $\left|f\left(t,\zeta\right)\right| \leq \dfrac{e^{-t}}{\pi\sqrt{t^2+143}} = w(t)$ with $\|w\| = \dfrac{1}{12\pi e}$ and $\psi\left(\|\zeta\|\right) = 1$,

with $\|w\|\psi\left(\kappa\right), \mathcal{Q} = 0.017086506$

By taking $M > 0.017086506$ all the assumptions of theorem (6.5) hold, implies the existence of solution for the FDE (6.9).

6.2 Main Results for FDE

Based on the classical FPT we investigate the problem of existence for the solution of the FDE (6.2). Not to repeat ourselves we omit the proofs of the obtained existence results, since they are similar to those done in the previous section.

Consider the operator $G_{CH} : \Psi \to \Psi$, which given as follows

$$\left(G_{CH}u\right)(t)=\frac{1}{\Gamma(r)}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}h\left(s,u(s)\right)\frac{ds}{s}+\frac{\left(\ln\frac{t}{a}\right)}{\Gamma(r)\left(\ln\frac{T_0}{a}-\vartheta\left(\mu\left(\ln\frac{\mu}{a}-1\right)+a\right)\right)}\times$$

$$\left(\vartheta\int_a^{\mu}\int_a^{s}\left(\ln\frac{s}{m}\right)^{r-1}h\left(m,u(m)\right)\frac{dm}{m}ds-\int_a^{T_0}\left(\ln\frac{T_0}{s}\right)^{r-1}h\left(s,u(s)\right)\frac{ds}{s}\right).$$

For computational convenience, we let

$$\mathcal{D}=\frac{1}{\Gamma(r)}\sup_{a\le t\le T_0}\int_a^t\frac{1}{s}\left(\ln\frac{t}{s}\right)^{r-1}ds+\frac{\sup\limits_{a\le t\le T_0}\left(\ln\frac{t}{a}\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)-\vartheta\left(\mu\left(\ln\frac{\mu}{a}-1\right)+a\right)\right|}\times$$

$$\left(|\vartheta|\int_a^{\mu}\int_a^{s}\frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1}dmds-\int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{r-1}ds\right)$$

$$\le\frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^{r}+\frac{\left(\ln\frac{T_0}{a}\right)^{r+1}}{\Gamma(r+1)\left|\left(\ln\frac{T_0}{a}\right)-\vartheta\left(\mu\left(\ln\frac{\mu}{a}-1\right)+a\right)\right|}\left(|\vartheta|(\mu-a)-1\right).$$

$$\mathcal{D}^*=\frac{\sup\limits_{a\le t\le T_0}\left(\ln\frac{t}{a}\right)}{\Gamma(r)\left|\left(\ln\frac{T_0}{a}\right)-\vartheta\left(\mu\left(\ln\frac{\mu}{a}-1\right)+a\right)\right|}\times$$

$$\left(|\vartheta|\int_a^{\mu}\int_a^{s}\frac{1}{m}\left(\ln\frac{s}{m}\right)^{r-1}dmds-\int_a^{T_0}\frac{1}{s}\left(\ln\frac{T_0}{s}\right)^{r-1}ds\right)$$

$$\le\frac{\left(\ln\frac{T_0}{a}\right)^{r+1}}{\Gamma(r+1)\left|\left(\ln\frac{T_0}{a}\right)-\vartheta\left(\mu\left(\ln\frac{\mu}{a}-1\right)+a\right)\right|}\left(|\vartheta|(\mu-a)-1\right)$$

Theorm 6.7. Consider the continuous function $h:[a,T_0]\times\mathbb{R}\to\mathbb{R}$ and assume

$(E_1)\exists L_h > 0$ such that $\left|h(t,u_1)-h(t,u_2)\right| \le L_h \left|u_1 - u_2\right|, \quad a \le t \le T_0, \forall u_1, u_2 \in \mathbb{R}$. if

$\mathcal{D} < 1/L_h$ then there exist a unique solution for the FDE (6.2) on $[a,T_0]$.

Theorm 6.8. Consider the function $h:[a,T_0]\times\mathbb{R}\to\mathbb{R}$ which is continuous and satisfying the condition (E_1) and suppose that $\left|h(t,u)\right| \le \sigma(t), \forall(t,u)\in[a,T_0]\times\mathbb{R}$ where $\sigma \in C([a,T_0],\mathbb{R}^+)$ with $\sup_{0\le t\le T_0}\left|\sigma(t)\right| = \|\sigma\|$, moreover, it is assumed that $L_h < \frac{1}{\mathcal{D}^*}$, then $\exists$ a solution for FDE (6.2) on $[a,T_0]$.

Theorem 6.9. Consider the continuous function $h:[a,T]\times\mathbb{R}\to\mathbb{R}$. Assume that

(E_4) $\exists$ $\gamma \in C([a,T],\mathbb{R}^+)$ and a non decreasing function $\upsilon:\mathbb{R}^+\to\mathbb{R}^+$

Such that $\left|h(t,u)\right| \le \gamma(t)\upsilon(\|u\|), \forall(t,u)\in[0,T_0]\times\mathbb{R}$

(E_5) $\exists$ a constatnt $W>0$ such that $\frac{W}{\|\gamma\|\upsilon(W)\ \mathcal{D}} > 1$, where

Then the FDE (6.2) has a solution on $[a,T_0]$

Example. we can take the same examples given in example (6.6), indeed

$$\mathcal{D} \le \frac{1}{\Gamma(r+1)}\left(\ln\frac{T_0}{a}\right)^r + \frac{\left(\ln\frac{T_0}{a}\right)^{r+1}\left(|\vartheta|(\mu-a)-1\right)}{\Gamma(r+1)\left|\left(\ln\frac{T_0}{a}\right)-\vartheta\left(\mu\left(\ln\frac{\mu}{a}-1\right)+a\right)\right|} \approx 10.39849624,$$

$\mathcal{D}^* \approx 9.648496241$

Chapter 7

CONCLUSION AND FUTURE WORK

This thesis relies on various fractional differential equations. Based on the classical fixed point theorem summarized by what known as the Banach contraction mapping theorem, nonlinear alternative of Leray-Schuader type and Krasnoselskii's fixed point theorem, a three different nonlinear fractional differential equations were considered.

In chapter four we study the existence and uniqueness for the solution of the nonlinear sequential fractional differential equation involving Caputo fractional derivative and associated with nonlocal integral boundary conditions. In chapter five with a little modifications on the same problem mentioned in the previous chapter lead us to define a new function space with different norm, the boundary condition for this problem can be considered as a generalization of the boundary conditions associated with the problem in chapter four, for these two chapters we illustrate our results by examples given at the end of each one.

Whereas, in chapter six which can be considered as two parts, we investigated the existence and uniqueness for the solution of the nonlinear fractional differential equations involving Hadamard and Caputo-Hadamard fractional derivative associated with three points integral boundary conditions, for the applicability of our results we give some examples at the end of this chapter as well.

Future work concerns deeper analysis of particular fractional differential equations involving Hadamard fractional operators. Since the main subject of this thesis the SFDE, in the near future I would study the SFDE's involving Hadamard fractional derivative with nonlocal integral boundary conditions. Precisely, I would like to investigate the existence and uniqueness of the following SFDE

$$\begin{cases} \left({}_H D^r + \lambda\, {}_H D^{r-1}\right) y(t) = g\left(t, y(t), {}^c D^{r-1} y(t)\right), \ r \in (1,2], \ 0 \le t \le T, \\ a_1 y(\mu) + b_1 y(T) = c_1 \int_0^{\tau} y(s)\, ds + d_1, \\ a_2\, {}_H D^{r-1} y(\mu) + b_2\, {}_H D^{r-1} y(T) = c_2 \int_{\omega}^{T} y(s)\, ds + d_2, \end{cases}$$

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