Geodesics of Black Holes in Bumblebee Gravity Theory

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ABSTRACT

In this thesis, our main purpose is to study the geodesics of bumblebee black hole (BBH) in 4-dimension (4D), which is an exact solutions to the bumblebee gravity theory. The motions of photon and massive particles are going to be studied via the standart Lagrangian method in asymptotically flat geometry. Due to the physical constraints, instead of spacelike geodesics, we concantrate on null and timelike solutions.

After finding the Euler Lagrange equations, next step is invastigating radial motions of geodesics. Moreover, exact analytical solutions are also planned to be studied for the radial and angular geodesic equations. According to our purpose, we use some numerical simulations in order to plot graphs for visualizing the geodesics. We also investigate the limit of Lorentz symmetry breaking term (LSB) by using classicsal tests which are the advance of the perihelion and bending of light.

Keywords: General Relativity, Standart Model Extensions, Bumblebee Gravity Theory, Black Hole, Geodesics, Perihelion, Lorentz Symmetry Breaking, Deflective Angle.

ÖZ

Bu tezde, Bumblebee yerçekimi teorisi'nin kesin çözümleri olan 4 boyutlu (4D) kara deliklerin (BH) jeodeziklerini araştırıyoruz. Kütlesiz ve kütleli parçacıkların hareketleri asimptotik olarak düz geometride, standart Lagrangian yöntemi ile incelenecektir. Fiziksel kısıtlamalar nedeniyle, uzay benzeri jeodezikler yerine, ışık ve zaman benzeri jeodezikler üzerinde çalışılacaktır.

Euler-Lagrange denklemlerini elde ettikten sonra, jeodeziklerin radyal hareketlerini analiz edeceğiz. Ayrıca, hem radyal hemde açısal jeodezik denklemler için kesin analitik çözümlerin de çalışılması planlanmaktadır. Aynı düşüncede, jeodeziklerin görüntülenmesini sağlayacak, birçok grafiği çizmek için bazı sayısal simülasyonlar yapacağız. Günberi ilerlemesi ve ışığın bükülmesi gibi bazı klasik testleri de araştırdık.

Anahtar Kelimeler: Genel Görelilik, Standart Model Uzantıları, Bumblebee Yerçekimi Modeli, Kara Delik, Jeodezikler, Günberi, Lorentz Simetri Kırınımı, Sapma Açısı.

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Chapter 1

INTRODUCTION

The mysteries in the universe have always attracted the attention of human beings. Nicolaus Copernicus was a mathematician and astronomer who made a pioneering contribution to the scientific revolution [1]. The motion of stars and planets were discovered by Copernicus in the earlys of 1500's and he formulated a model of the universe that the planets orbit around the Sun. Laterly, discoveries of science continues with Kepler who correctly defined orbits. The development of the modern scientific approach started with Johannes Kepler (1571-1630), then continue with Isaac Newton (1642-1726) and Albert Einstein (1879-1955) [2].

In 1609, Kepler hypothesized the first two of his three laws in his work Astronomia Nova [3] the existence of a force radiated by the Sun, which decreases with distance and which causes the planets to move faster in the event that closer to the Sun and he asserted that planets, while orbiting around the sun, follow elliptical geometry [4], not circles (as claimed up to that time), then, he explained his last law, which used mathematic principles to relate the time a planet takes to orbit the sun to the average distance of the planet from the sun. With these three laws, Kepler became one of the pioneers of modern astronomy.

Nonetheless, the understanding of gravitational force is due to Isaac Newton. The impressive discoveries of Newton sprang quickly from those of Kepler, and completed the incredible chain of truths which constitude laws of the planetary system [5]. Kepler believed that some force from the sun pushed the planets around in their orbits, but he was unable to identify the force. Laterly, Newton's work on gravity revealed why the planets orbit the way they do. When applied to the planets and the Sun, Newton's law of universal gravitation accurately predicts the motion of the planets. Newton put forth his laws in Philosophiae Naturalis Principia Mathematica, published in 1687 [6].

1.1 General Relativity

General relativity is a gravitation theory that was evolved by Albert Einstein between 1907 and 1915 [7]. In General Relativity, the gravitational effect is causing by the distortion of space-time between masses. In the early 20th century, Newton's universal law of attraction was adopted as a valid definition of gravitational force among the masses for more than two hundred years. In Newton's principle, gravity is causing of an appealing force between large objects. Even Newton, though disturbed by the unknown nature of this force, was extremely successful in explaining fundamental frame motion. As time progressed it is showed that Einstein's definition of gravity explains several effects that cannot reach accurate results from Newton's law, such as orbits of planets and an effect of gravity on light. In Newton's definition of gravity, matter causes the force of gravity. More precisely, it results from a certain property of material objects: their mass. In Einstein's theory and related theories of gravity, the curvature at every point in space-time originates from everything that exists. Here, mass is also a key property in determining the gravitational effect of matter. However, in a theory of general relativity, mass cannot be the only source of gravity. Relativity combines mass with energy and energy with momentum. Today Einstein’s General Relativity remains scientists’ best understanding of gravity and a key to our understanding of the cosmos on the grandest scale.

1.2 Geodesics

In general, the shortest distance between two points on a plane is represented by a straight line. The shortest distance between two points on a sphere is expressed by the arc segment whose center is the center of the sphere and passes through these two points. We call this curve that indicates the shortest distance between two points on a surface, the geodesic curve [8]. More importantly, when a particle undergoes a free falling, the gravitational force dissapears and this path is a particular geodesic. In other words, a freely moving or falling particle always moves along a geodesic [9]. In general relativity, gravity can be considered not as a force, but as a result of a curved space-time geometry, where the source of curvature is the stress energy momentum tensor (i.e. matter). On the authority of Einstein, the gravitational field is nothing more than a deviation of the properties of real spacetime moves along a geodesic line, which is independent of its mass and composition. This geodesic motion in curved spacetime is percieved by us as curved motion with variable velocity. Einstein’s theory postulates from the very beginning that the curvature of the trajectory and the variation of speed are spacetime properties, properties of the geodesics; and, hence, that accelerations of all bodies must be equal [10].

1.3 Black Holes

Black holes are a number of the strangest and maximum captivating items in outer space. They're extraordinarily dense,with such robust gravitational enchantment that even light cannot escape out theirhold close if it comes close to enough [11]. The principle of general relativity predicts that compact galaxies in time can deform space time to create a black hole [12]. There is a line where light cannot escape from the BH, we call it the event horizon. Until now, we could not make observation of event horizon although it has great effect to the objects. In many ways, a BH acts as an ideal blackbody, as it does not reflect light [13]. The most important property of BHs are being very dense. Density is a measure of how tightly mass is packed into a space.

1.4 Bumblebee Gravity

There are currently two theories of physics that explain the universe: first, Einstein's theory of relativity, which can explain very large objects like galaxies, and second, quantum mechanics, which can explain very small-sized matter like atoms. These two theories explain the same universe, when the two theories are combined in one theory. However, this has not been achieved so far. Some possibilities were found in several experiments, but these possibilities were found to be infinite. However, probability should not be less than 0 and greater than 1. By using string theory, reasonable results were obtained in solutions by getting rid of these infinities. This combination is already considered the greatest step in the history of science. V. Alan Kostelecký is a smart theoretical physicist that he verified presence of an anisotropy in string theory models, and described a modified version of the Standard Model of particle physics, called the Standard-Model Extension [14] that space-time symmetry can be violated.

Bumblebee models are modified gravity models which describing a vector field in space with a non vanishing vacuum expectation value that spontaneously breaks Lorentz symmetry. A bumblebee model is the basic model of a theory with Lorentz symmetry breaking. The improvement of bumblebee models was first motivated by the discovery of unisotropy in string theory (and other modified quantum theories of gravity) can play an important role in non vanishing vacuum expectation values.

In my master thesis, my main motivation is to analyse the geodesics of the 4-dimensional non rotating black hole by Euler Lagrange method, which are exact solutions to the bumblebee gravity theory in asymptotically flat geometry. We study the radial motion of a particle both in null [15] and timelike geodesics via the standart Lagrangian method. We also inspect the V_{eff} to observe the motion of test particles, both in null (photon) and timelike geodesics. In addition, we find the exact analytical solutions of the geodesics equations in bumblebee graviy model. Finally, we observed the limit of Lorentz violation term by getting help from some experimental tests which are the advance of the perihelion and bending of light.

As i will mention here, the order of my master thesis; in chapter 2, we discussed the bumblebee spacetime and giving properties of Lorentz symmetry breaking [16] under the bumblebee model. In chapter 3, we derived the geodesics equations via the standart Lagrangian method and we also study the radial motion of a particle both in null and timelike geodesics without angular momentum. In chapter 4, we find the exact analytical solutions of the geodesics equations in bumblebee graviy model. Chapter 5 represents some classical tests about advance of the perihelion and bending of light. Finally, in chapter 6, we wrote our conclusion.

Chapter 2

BUMBLEBEE SPACETIME

In Riemann spacetime, the bumblebee term and coupling term cause changes in the Lagrangian density [17]. Therefore, our modified Lagrangian density of the bumblebee gravity model [18] gives the following extended vacuum Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} \tag{2.1}$$

where $G_{\mu\nu}$ is Einstein tensor, $R_{\mu\nu}$ is Ricci tensor, R is Ricci scalar, $g_{\mu\nu}$ is the metric tensor, k is a gravitational coupling and $T_{\mu\nu}$ is total energy momentum tensor respectively. In addition, $T_{\mu\nu}$ is the total source of combining of matter and Bumblebee field. In mathematically,

$T_{\mu\nu} = T^{M}_{\mu\mu} + T^{B}_{\mu\nu}$ and $T^{B}_{\mu\nu}$ is given by [19]

$$\begin{aligned} T^{B}_{\mu\nu} = -B_{\mu\nu}B^{\alpha}_{\nu} - \frac{1}{4} B_{\alpha\beta}B^{\beta\alpha}g_{\mu\nu} - Vg_{\mu\nu} + 2V'B_{\mu}B_{\nu} \\ + \frac{\xi}{k}\Big[\frac{1}{2} B^{\alpha}B^{\beta}R_{\alpha\beta}g_{\mu\nu} - B_{\mu}B^{\alpha}R_{\alpha\nu} - B_{\nu}B^{\alpha}R_{\alpha\mu} \\ + \frac{1}{2}\nabla_{\alpha}\nabla_{\mu}(B^{\alpha}B_{\nu}) + \frac{1}{2}\nabla_{\alpha}\nabla_{\nu}(B^{\alpha}B_{\mu}) - \frac{1}{2}\nabla^{2}(B_{\mu}B_{\nu}) \\ - \frac{1}{2} g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}(B^{\alpha}B^{\beta})\Big] \end{aligned} \tag{2.2}$$

where ξ is coupling constant relation with gravity-bumblebee interaction. In Eqn. (2.2), potential term (V) satisfied non vanishing vacuum expectation value for bumblebee vector.

On the other hand, vacuum solutions of bumblebee field determined when $V = V' = 0$. Vacuum in space means, there is no matter or pressure close to any particles in the space and do not affect any processes being carried on there. In other words, it means that there is not any source and not any time dependence. For this reason, the bumblebee vector should be a simple as stated below;

$$\mathrm{B\mu} = (0, \mathrm{br(r)}, 0, 0) \tag{2.3}$$

When the bumblebee field B_μ vanishes, the equation (2.1) reduces to the Einstein equations. Recently, the vacuum solution in the bumblebee gravity model induced by the Lorentz symmetry breaking has been derived by Casana [20].

In arrange to explore the Lorentz symmetry breaking [21] in standart model extension, there should be theories which gives the Lorentz violation, occurs in the vector Bμ that gives a nonzero vacuum expectation value. These speculations are called bumblebee models and are among the only illustrations of field hypotheses with unconstrained Lorentz and diffeomorphism infringement.

In addition, (-,+,+,+) is applying to the metric signature. In a bumblebee gravity model, a spherically symmetric vacuum solution is obtained as follows [22],

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + (1 + l)\left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\emptyset^2) \tag{2.4}$$

or

$$ds^2 = -f dt^2 + (1 + l) f^{-1} dr^2 + r^2 d\Omega^2 \tag{2.5}$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\emptyset^2 \tag{2.6}$$

and l is the non-zero Lorentz symmetry breaking parameter. If l takes zero value in the metric tensor, then it reduces to the Schwarzschild solution model. The differences between Schwarzschild and bumblebee model is non zero vaalue of l.

$$f = 1 - \frac{2M}{r}, \quad r_H = 2M, \quad r_H: Event\ Horizon \tag{2.7}$$

$$T_H = \frac{1}{8\pi M\sqrt{1+l}} \tag{2.8}$$

The non-zero LSB parameter has the effect of reducing the Hawking temperature [23] of a Schwarzschild BH.

In addition, Kretschmann scalar is given by;

$$K = \frac{4(12M^2 + 4lMr + l^2r^2)}{r^6(1+l)^2} \tag{2.9}$$

and r = 0 is the real singularity.

Chapter 3

RADIAL SOLUTION OF THE BUMBLEBEE BLACKHOLE

3.1 Analysing of Complete Geodesics Equations of the Bumblebee Blackhole from Euler Lagrange Equations

In this section, geodesics of the test particles in the bumblebee BH is the main motivation. Standart Lagrangian method is applied to the metric to find the equations of motion. The suitable Lagrange (L) equation of a photon and massive particle in the BBH geometry is shown below;

$$2L = -f\dot{t}^2 + (1+l)f^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + sin^2\theta\dot{\emptyset}^2) \tag{3.10}$$

where dot over proportion represents the derivative with respect to the affine parameter. The left side of Eqn. (3.10) is the metric condition in general and stated by;

$$L = \frac{\varepsilon}{2} \tag{3.11}$$

in which ε = 0 represents null and ε = -1 represents timelike geodesics. In this metric, (t, ϕ) are cyclic coordinates, therefore, their conjugate momenta $\mathfrak{p}_t, \mathfrak{p}_\emptyset$ are defined as;

$$\frac{d}{d\sigma}(\mathfrak{p}_t) = \frac{\partial L}{\partial t} = 0 \tag{3.12}$$

$$\frac{d}{d\sigma}(\mathfrak{p}_\emptyset) = \frac{\partial L}{\partial \emptyset} = 0 \tag{3.13}$$

where $\mathfrak{p}_t = \frac{\partial L}{\partial \dot{t}}$ and $\mathfrak{p}_\emptyset = \frac{\partial L}{\partial \dot{\emptyset}}$.

As can be understood from the Eqns. (3.12) and (3.13), the conjugate momenta of t and ϕ are independent variable. For this reason, they are equal to constant number.

$$\mathfrak{p}_t = \frac{\partial L}{\partial \dot{t}} = -f\dot{t} = const \tag{3.14}$$

$$\mathfrak{p}_\emptyset = \frac{\partial L}{\partial \dot{\emptyset}} = r^2 sin^2\theta\dot{\emptyset} = const \tag{3.15}$$

where, σ is affine parameter. Time variable t is associated with energy and ϕ variable is associated with angular momentum respectively.

$$\mathfrak{p}_t = -E \implies f\dot{t} = E \implies E: Energy\ of\ the\ test\ prticles$$

$$\dot{t} = \frac{E}{f} = \frac{E}{1-\frac{r_H}{r}} = \frac{Er}{r-r_H}, \qquad (r_H = 2M) \tag{3.16}$$

Moreover, E represent the total energy of the massive and massless particles investigated by an external observer located at

$$r \to \infty\ (spatial\ infinity) : r \to \infty \implies \dot{t} \to E$$

On the other hand; ℓ is an integration constant which include angular momentum relation and stated as below;

$$r^2 sin^2\theta\dot{\emptyset} = \ell \implies \dot{\emptyset} = \frac{\ell}{r^2 sin^2\theta} \tag{3.17}$$

Without loss of generality, to make it simple, we can project the problem into an equilateral plane: $\theta = \frac{\pi}{2}$.

Thus the Lagrangian become

$$L = -\frac{1}{2}f\dot{t}^2 + \frac{1}{2}(1+l)f^{-1}\dot{r}^2 + \frac{r^2}{2}(\dot{\emptyset}^2) \tag{3.18}$$

and also can be written as

$$L = -\frac{1}{2}\frac{E^2}{f} + \frac{1}{2}(1+l)f^{-1}\dot{r}^2 + \frac{\ell^2}{2r^2} \tag{3.19}$$

EL-equation of the radial coordinates then reads as follow

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{3.20}$$

where $\left(L = \frac{-E^2}{2f} + \frac{\dot{r}^2}{2f}(1+l) + \frac{\ell^2}{2r^2}\right)$

$$\frac{d}{d\sigma}\left(\frac{1+l}{2f}\dot{r}\right) = \frac{E^2}{2f^2}f' - \frac{\dot{r}^2(1+l)}{2f^2}f' - \frac{\ell^2}{r^3} \tag{3.21}$$

and from the metric condition; we introduce $\frac{\epsilon}{2}$ instead of L in Eqn. (3.19), then, we obtain Eqn. (3.22);

$$L = \frac{\epsilon}{2} \Longrightarrow \frac{\epsilon}{2} = \frac{-E^2}{2f} + \frac{\dot{r}^2}{2f}(1+l) + \frac{\ell^2}{2r^2}f$$

$$f\epsilon = -E^2 + \dot{r}^2(1+l) + \frac{\ell^2}{r^2}f$$

$$\dot{r}^2 = \frac{1}{1+l}\left[E^2 + f\epsilon - \frac{\ell^2 f}{r^2}\right] \tag{3.22}$$

making some additions, corrections and editions to the Eqn. (3.22), we find as

$$\frac{1}{2}\dot{r}^2 = \frac{1}{2(1+l)}\left[E^2 + f\left(\epsilon - \frac{\ell^2}{r^2}\right)\right] \tag{3.23}$$

where dot means derivative with respect to σ. Moreover, we reformat our Eqn. (3.23) where the effective potential and effective energy can be seen easily as shown below.

$$\frac{1}{2}\dot{r}^2 + V_{eff} = E_{eff} \tag{3.24}$$

$$\frac{1}{2}\dot{r}^2 = \frac{f}{2(1+l)}\left(\frac{\ell^2}{r^2} - \epsilon\right) = \frac{E^2}{2(1+l)} \tag{3.25}$$

Therefore; we can separate E_{eff} and V_{eff} equations respectivley.

$$E_{eff} = \frac{E^2}{2(1+l)} \tag{3.26}$$

$$V_{eff} = \frac{f}{2(1+l)}\left(\frac{\ell^2}{r^2} - \epsilon\right) \tag{3.27}$$

In addition, with the help of chain rule, we change the variables,

$$\frac{d}{d\sigma}=\frac{\ell}{r^2}\frac{d}{d\emptyset} \Longrightarrow \dot{r}=\frac{\ell}{r^2}r',\ \ r'=\frac{\partial r}{\partial \emptyset}=\frac{dr}{d\emptyset}\Longrightarrow\ \dot{r}^2=\frac{\ell^2}{r^4}r'^2$$

$$r'^2=\frac{2r^4}{\ell^2}\left(E_{eff}-V_{eff}\right) \tag{3.28}$$

Setting $r=\frac{1}{u}=u^{-1}$ and $dr=-u^{-2}du$. Then, introducing this equation into a standart Kepler problem which is very important for analysing the circular motion [24]. ($Hint$: $\frac{dr}{d\emptyset}=r'=-u^{-2}u'$)

$$u'^2=\frac{2}{\ell^2}\left[E_{eff}(u)-V_{eff}(u)\right] \tag{3.29}$$

Or in open form is shown as below;

$$u'^2=\frac{1}{\ell^2(1+l)}\left[E^2+(ur_H-1)(\ell^2u^2-\epsilon)\right] \tag{3.30}$$

3.2 Radial Geodesics without Angular Momentum

In this part, we are focusing on zero angular momentum case in which ℓ=0. Thus, we reduced our 4D into a 3D and the motion of particle is only in radial direction. Therefore, the Eqn. (3.23) reduces to

$$\dot{r}^2=\frac{1}{1+l}\left[E^2+f\epsilon\right] \tag{3.31}$$

Moreover, when the geodesics refer to a null case, it means that massless particle (photon) is taken to the consideration and the above equation becomes as;

$$\dot{r}^2=\frac{E^2}{1+l}=2E_{eff} \tag{3.32}$$

where $\dot{r}=\frac{dr}{d\sigma}$.

Again, using the properties of Eqn. (3.16), and changing the variable of Eqn. (3.32), from affine parameter σ to the time t, we obtain ; $\dot{t}=\frac{E}{f}\Longrightarrow\frac{dt}{d\sigma}=\frac{E}{f}\Longrightarrow d\sigma=\frac{f}{E}dt$.

Therefore; it becomes;

$$\frac{dr}{dt} = \pm \frac{f}{\sqrt{1+l}} \tag{3.33}$$

where $f = 1 - \frac{r_H}{r}$. Therefore it becomes as

$$\frac{r\sqrt{1+l}}{r - r_H} dr = \pm dt \tag{3.34}$$

and we take integral of both sides with respect to t and r. Then, we obtain,

$$\sqrt{1+l} \int_{r_0}^{r} \frac{rdr}{r - r_H} = \pm \int_{t_0}^{t} dt \tag{3.35}$$

$$\Rightarrow \Delta r + r_H \, ln \left(\frac{r - r_H}{r_0 - r_H} \right) = \pm \frac{\Delta t}{\sqrt{1+l}}$$

In which $\Delta r = r - r_0 \;\; and \;\; r_0 \geq r_H$ respectively.In addition, letting $c = r_0 - r_H$, we obtain,

$$\Delta r + r_H \, ln \left(\frac{r - r_H}{c} \right) = \pm \frac{\Delta t}{\sqrt{1+l}} \tag{3.36}$$

Finally, we find the radial solution of $r(t)$ as follows;

$$r = r_H + c \; exp \left[\frac{1}{r_H(1+l)} \left[Lambertw \left(\frac{e^y c}{r_H} \right) r_H (1+l) - \left(s\sqrt{1+l}\Delta t + c(1+l) \right) \right] \right] \tag{3.37}$$

where t is nothing but the time measured by an external observer as higher initial time and c is chosen to be a constant parameter. We can reorganize our Eqn. (3.37) into this form as stated below;

$$r = r_H + c \; exp \left[-Y + Lambertw \left(\frac{ce^Y}{r_H} \right) \right] \tag{3.38}$$

where $H_0 = \; s\sqrt{1+l}\Delta t + c(1+l)$ and $Y = \frac{H_0}{r_H(1+l)}$.

Now, instead of null geodesics, we concentrate on timelike geodesics ($\epsilon = -1$) or massive particle without angular momentum. Therefore, Eqn. (3.31) leads to become as follow,

$$\dot{r}^2 = \frac{1}{1+l}[E^2 - f] \tag{3.39}$$

Substituting $f = 1 - \frac{r_H}{r}$ into the above equation and get the derivative of Eqn. (3.39) with respect to σ parameter, then we reach the 2nd order radial equation.

$$\ddot{r} = -\frac{1}{2(1+l)}\frac{r_H}{r^2} \tag{3.40}$$

Moreover, just reorganizing τ (proper time) instead of the affine parameter σ, we reach the radial force per unit mass (i.e. the acceleration).

$$\frac{d^2r}{d\tau^2} = a_r = \frac{-r_H}{2(1+l)r^2} \tag{3.41}$$

In which a_r gives us the centrifugal acceleration due to its NEGATIVE sign. This is not a surprised result cause a_r is directed toward the center of the black hole. Now, if we assume that a particle indicates its motion from rest at an initial point r_i , using the following equation:

$$\dot{r}^2 = \frac{1}{1+l}[E^2 + f\epsilon] \tag{3.42}$$

When we take the timelike particle into the consideration ($\epsilon = -1$). Our equation reduces to;

$$\dot{r}^2 = \frac{1}{1+l}[E^2 - f] \tag{3.43}$$

and, considering a particle that its initial radial point $r = r_i$ with $\sigma = \tau$, we get

$$E^2 = 1 - \frac{r_H}{r_i} = \frac{r_i - r_H}{r_i} \tag{3.44}$$

then, substituting Eqn. (3.42) into an Eqn. (3.44), we obtain,

$$\dot{r}^2 = \frac{1}{1+l}\left[\frac{r_H(r_i - r)}{r r_i}\right] \tag{3.45}$$

Moreover, from the previous equation which is Eqn. (3.27), we obtain the effective potential. For photons, introducing the $\ell = 0$ condition to this potential, our equation leads to;

$$V_{eff} = \frac{f}{2(1+l)} \tag{3.46}$$

One can get the above result from the definition of conservative force for a test particle having m=1 as shown below,

$$F = m\frac{d^2r}{d\tau^2} = \frac{d^2r}{d\tau^2} = \frac{-1}{2(1+l)}\frac{r_H}{r^2} = -\Delta V_{eff} \tag{3.47}$$

$$= \frac{-dV_{eff}}{dr}$$

$$V_{eff} = \int \frac{1}{2(1+l)}\frac{r_H}{r^2}\,dr \tag{3.48}$$

$$\Rightarrow V_{eff} = \frac{-1}{2(1+l)}\frac{r_H}{\mathrm{r}} + c$$

Where c is an integration constant. For the sake of conformity of Eqn. (3.27), we can get $c = \frac{1}{2(1+l)}$. Therefore we proved that our effective potential is same as shown below; (Hint: 2M=Mass of BH)

$$V_{eff} = \frac{1}{2(1+l)}\left[1 - \frac{2M}{r}\right] \tag{3.49}$$

Finally, we impose timelike geodesics to Eqn. (3.31) which means $\varepsilon = -1$, and using the previous condition which was $\partial_\sigma = \partial_\tau = \frac{E}{f}\partial_t$. Thus, we obtain,

$$\frac{E^2}{f^2}\left(\frac{dr}{dt}\right)^2 = \frac{1}{1+l}\left[E^2 - f\right] \tag{3.50}$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{1}{1+l}\left(f^2 - \frac{f^3}{E^2}\right)} = \sqrt{x} = x^{1/2}$$

In addition, setting $x = \frac{1}{1+l}\left(f^2 - \frac{f^3}{E^2}\right)$ and differenriating the above expression with respect to t, one can easily find that

$$\frac{d}{dt}\left[\frac{dr}{dt} = \sqrt{x}\right] \Longrightarrow \frac{d^2r}{dt^2} = \frac{1}{2}(x)^{-\frac{1}{2}}\frac{dx}{dt} = \frac{1}{2}(x)^{-\frac{1}{2}}\frac{dx}{dr}\frac{dr}{dt} \tag{3.51}$$

$$= \frac{1}{2}(x)^{-1/2}(x)^{1/2}\frac{dx}{dr}$$

$$\frac{d^2r}{dt^2} = \frac{1}{2}\frac{dx}{dr}$$

$$\frac{dx}{dr} = \frac{f}{1+l}\left(2 - \frac{3f}{E^2}\right)\frac{r_H}{r^2}$$

$$\frac{d^2r}{dt^2} = \frac{1}{2}\frac{f r_H}{r^2(1+l)}\left(2 - \frac{3f}{E^2}\right)$$

On the other hand, for a massive particle starting its motion from rest ($V_0 = 0$)

$$V_0^2 = \frac{f_i^2}{(1+l)E^2} \tag{3.52}$$

where $E^2 = f_i = 1 - \frac{r_H}{r_i} = \frac{r_i - r_H}{r_i}$.

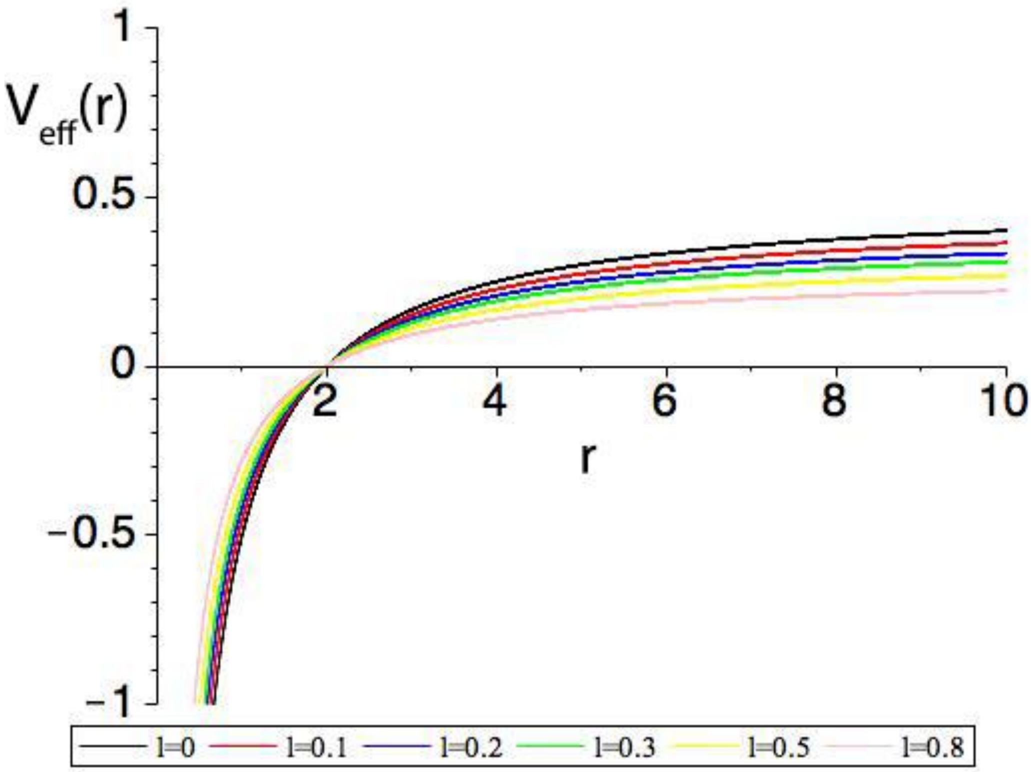

Figure 1: Showing the $V_{eff}(r)$ potential versus r. The plot is governed by [Eq. (3.49)]

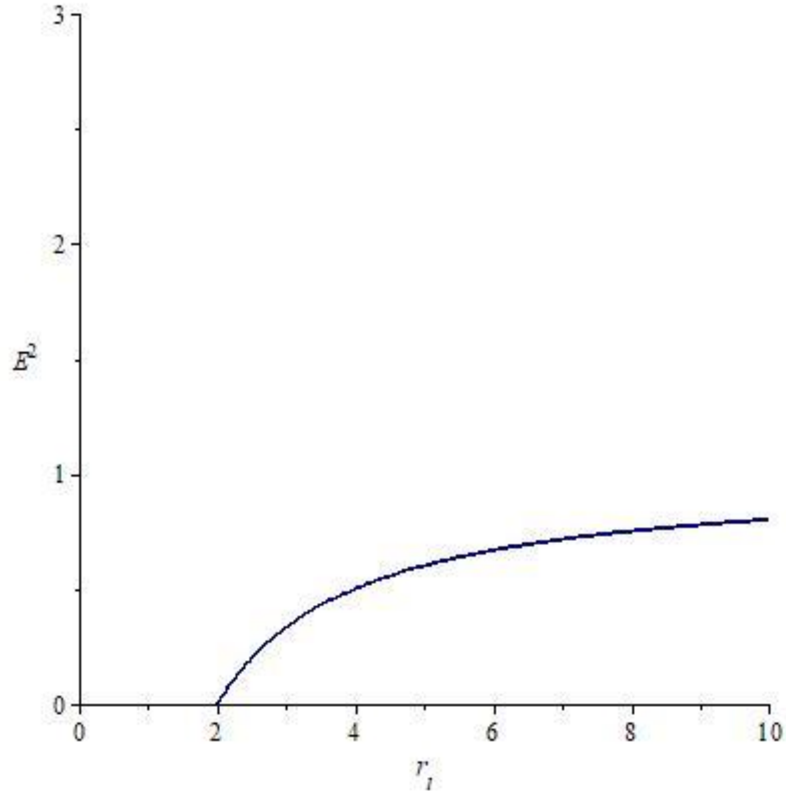

Figure 2: The plot of the E^2 vs r_i. The plot is governed by [Eq. (3.52)]

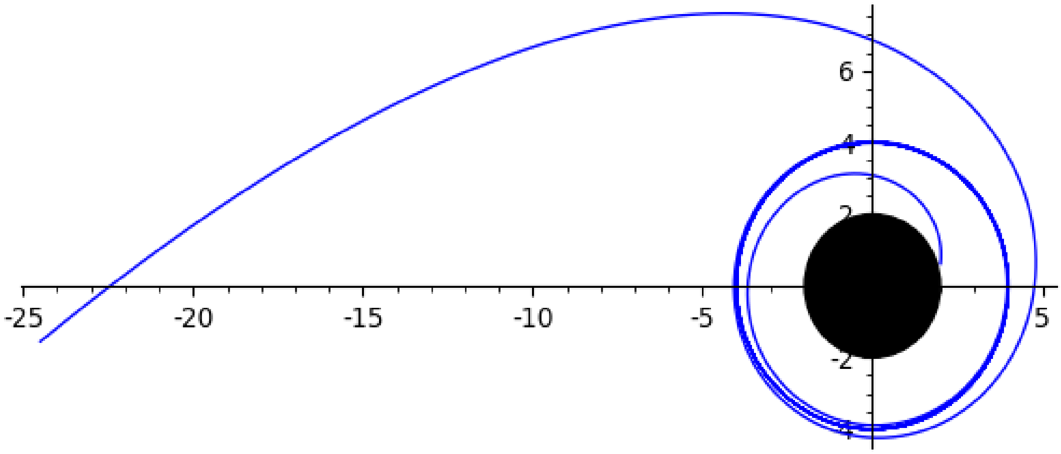

Figure 3: Showing the orbit of a photon around a Schwarzschild black hole *(l=0)*

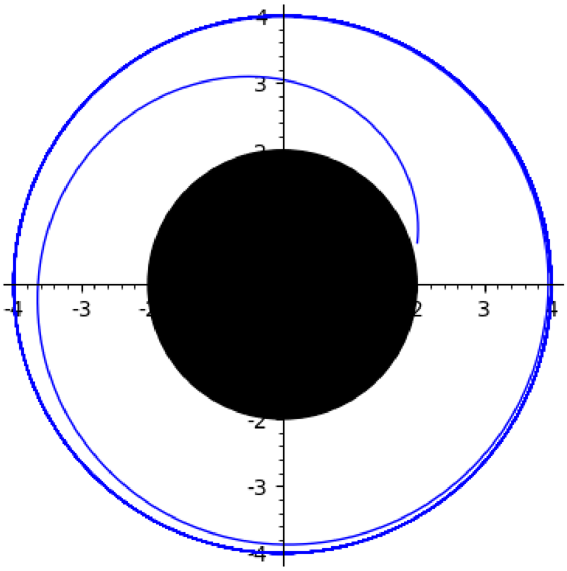

Figure 4: Showing the orbit of a photon around a Bumblebee black hole *(l=0.1)*

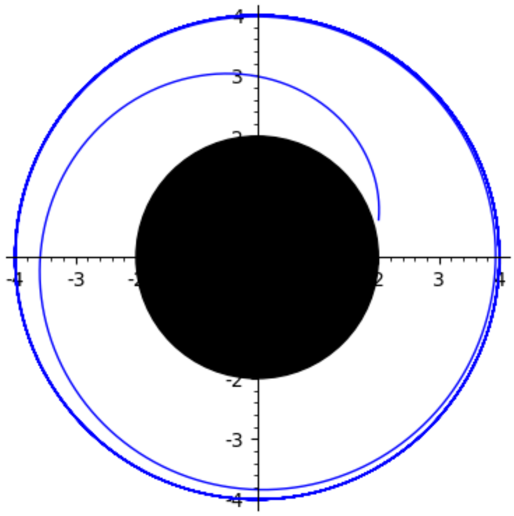

Figure 5: Showing the orbit of a photon around a Bumblebee black hole *(l=0.3)*

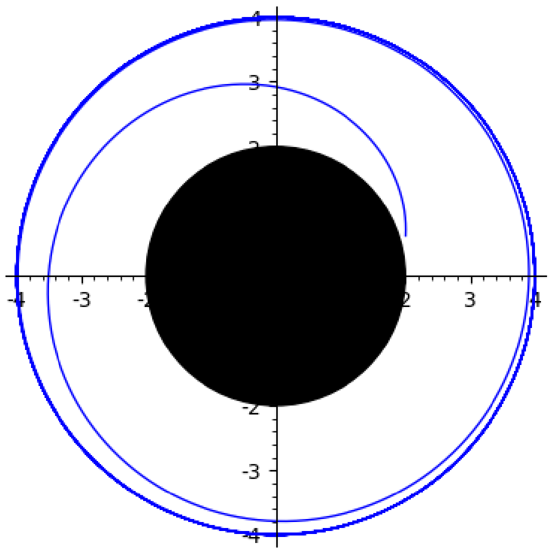

Figure 6: Showing the orbit of a photon around a Bumblebee black hole *(l=0.6)*

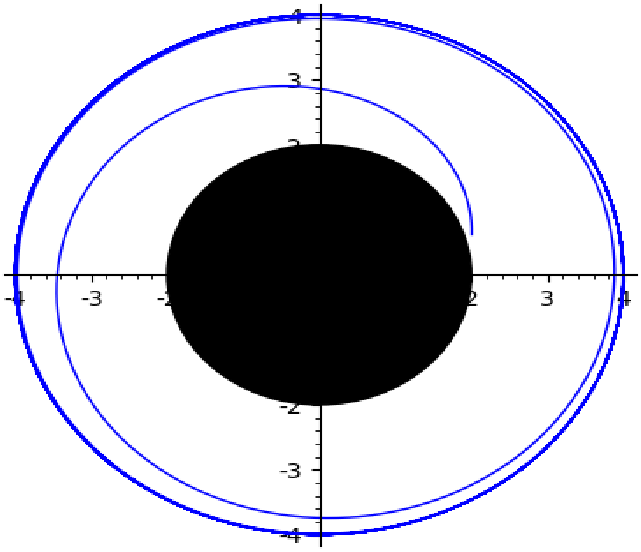

Figure 7: Showing the orbit of a photon around a Bumblebee black hole *(l=0.9)*

Chapter 4

ANALYTICAL SOLUTION OF GEODESICS EQUATIONS OF BUMBLEBEE BLACK HOLE IN MODIFIED GRAVITY

4.1 Euler Lagrange Equations with Mino Proper Time

In this section, the Lagrangian (3.11) is reordering by using the mino proper time (γ) which is ruled by the following differential expression in bumblebee gravity model.

$$d\sigma = r d\gamma \tag{4.1}$$

By using the chain rule $\frac{\partial}{\partial\sigma} = \frac{1}{r}\frac{\partial}{\partial\gamma}$ and describe the $' = \frac{\partial}{\partial\gamma}$, Thus, we obtain reforming of Lagrangian which is stated as below;

$$L = \frac{-f}{2r^2}(t')^2 + \frac{(1+l)f^{-1}}{r^2}(r')^2 + \frac{(\theta')^2}{2} + \frac{sin^2\theta}{2}(\emptyset')^2 \tag{4.2}$$

where $f = 1 - \frac{2M}{r}, r_H = 2M \; and \; 2L = \epsilon.$

And its corresponding metric condition is in the same form with Eq. (3.11). After applying the EL method, we get

$$\frac{d}{d\gamma}\left(\frac{-ft'}{r^2}\right) = 0 \;\Rightarrow\; t' = \frac{r^2\alpha}{f} \tag{4.3}$$

$$\frac{d}{d\gamma}(sin^2\theta\emptyset') = 0 \;\Rightarrow\; \emptyset' = \frac{\beta}{sin^2\theta} \tag{4.4}$$

In which α and β are integration constants respectively. Moreover, from Eqn. (4.2), we get the equation as follows;

$$\frac{d}{d\gamma}(\theta') = sin\theta cos\theta(\emptyset')^2 \tag{4.5}$$

$$\Rightarrow \theta'' = cos\theta \frac{\beta^2}{sin^3\theta}$$

$$\Rightarrow 2\theta'\theta'' = \frac{cos\theta}{sin^3\theta}\beta^2 2\theta' \Rightarrow \; 2\theta' d\theta' = 2\beta^2 \frac{cos\theta}{sin^3\theta} d\theta$$

$$\Rightarrow \theta'^2 = 2\beta^2 \int \frac{cos\theta d\theta}{sin^3\theta} + k$$

where $\theta' = \frac{d\theta}{d\gamma}$ and k is the integration constant. In addition, by integrating new variable to the above equation such that u=sinθ and du=cosθdθ, then, applying the integral, our Eqn. (4.5), becomes as follows;

$$\theta'^2 = k - \left(\frac{\beta}{sin\theta}\right)^2 \tag{4.6}$$

With the help of metric condition, the radial equation can be direved as stated below;

$$L = \frac{-f}{r^2}\left(\frac{r^2\alpha}{f}\right)^2 + \frac{(1+l)}{fr^2}(r')^2 + \left[k - \left(\frac{\beta}{sin\theta}\right)^2\right] \tag{4.7}$$

$$+ sin^2\theta \frac{\beta^2}{sin^4\theta} = \epsilon$$

$$\Rightarrow \; -\frac{r^2\alpha^2}{f} + \frac{(1+l)(r')^2}{fr^2} = \epsilon - k$$

$$(r')^2 = \frac{\epsilon - k}{1+l} fr^2 + \frac{\alpha^2}{1+l} r^4$$

where $\rho^2 = \frac{\alpha^2}{1+l}$ and $\alpha = \frac{ft\prime}{r^2}$ respectively. Finally we get;

$$(r')^2 = \frac{\epsilon - k}{1+l} fr^2 + \rho^2 r^4 \tag{4.8}$$

4.2 Exact Analytical Solution of the Radial Geodesics in Bumblebee Gravity Model

Under the leadership of transformations given in [25, 26], we make a change in the radial-coordinate as follows:

$$r(y) = \frac{s}{x(y)} + z \tag{4.9}$$

where, $s = \pm 1$, $s^2 = 1$, $y \equiv \gamma$, respectively.

From Eqn. (4.8), when r is equal to z, the radial equation become zero. Therefore, z satisfied the zero condition when its equal to r.

$$(r')^2 = \frac{\epsilon - k}{1 + l} f r^2 + \rho^2 r^4, \quad at\, z = r \tag{4.10}$$

$$\Rightarrow (r')^2 = 0$$

where $f = \frac{r(y) - r_h}{r(y)}$, $r_h = 2M$. Moreover, when we put the all parameters into an Eqn. (4.8), we obtain the new form of radial equation as stated below;

$$(x')^2 = a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 \tag{4.11}$$

and we find

$$a_1 = \frac{\epsilon - k}{1 + l} f r^2 - \rho^2 r^4 \tag{4.12}$$

$$a_2 = \frac{(3r_H - 2z)}{r_H - z} \rho^2 s z^3 \tag{4.13}$$

$$a_3 = \frac{6r_H - 5z}{r_H - z} \rho^2 z^2 \tag{4.14}$$

$$a_4 = \rho^2 4sz \tag{4.15}$$

$$a_5 = \rho^2 \tag{4.16}$$

then, letting $a_2 = b_3$; $a_3 = b_2$; $a_4 = b_1$; $a_5 = b_0$, we obtain,

$$(x')^2 = b_3 x^3 + b_2 x^2 + b_1 x + b_0 \tag{4.17}$$

setting

$$x(y) = \frac{1}{b^3}\left(4u(y) - \frac{b_2}{3}\right) \tag{4.18}$$

One can get;

$$(u')^2 - 4u^3 + g_{22}u + g_{33} = 0 \tag{4.19}$$

where $g_{22} = \frac{b_2^2}{12} - \frac{b_1 b_3}{4}$ and $g_{33} = \frac{-b_2^3}{216} + \frac{b_1 b_2 b_3}{48} - \frac{b_0 b_3^2}{16}$.

Eqn. (4.19) is nothing but the equation of Weierstrass P function (an elliptic function). Solution of this equation is stated below,

$$u(y) = WP(c_1 + y, g_{22}, g_{33}) \tag{4.20}$$

finally, we reach the full solution in radial direction.

$$x = \frac{1}{b^3}\left(4u(y) - \frac{b_2}{b_3}\right) \tag{4.21}$$

4.3 Exact Analytical Solution of the Angular Geodesics in Bumblebee Gravity Model

In this section, analytical solution of the angular geodesics are investigated.

From our $\frac{d\theta}{d\gamma}$ Eqn. (4.6), which showed below, should be integrated to find the analytical solution.

$$\frac{d\theta}{d\gamma} = \sqrt{k - \frac{\beta^2}{sin^2\theta}} = \frac{\sqrt{ksin^2\theta - \beta^2}}{sin\theta} \tag{4.22}$$

and integration calculations are showed below step by step.

$$\frac{sin\theta d\theta}{\sqrt{ksin^2\theta - \beta^2}} = d\gamma \tag{4.23}$$

Let cosθ= u , du= -sinθdθ and from basic knowledge $sin^2\theta = 1 - u^2$, one can write this Eqn. in this form;

$$\frac{-du}{\sqrt{k(1-u^2) - \beta^2}} = d\gamma \tag{4.24}$$

$$-\int \frac{du}{\sqrt{k(1-u^2)-\beta^2}} = \gamma - \gamma_0 \tag{4.25}$$

then, our integration become;

$$-\sqrt{k}(\gamma - \gamma_0) = tan^{-1}\left(\frac{\sqrt{k}u}{k - ku^2 - \beta^2}\right) \tag{4.26}$$

Hint : ($1 + tan^w v = \frac{1}{cos^2 v}$) and plus one for both sides of Eqn. (4.26)

$$1 + tan^2\left(\sqrt{k}(\gamma - \gamma_0)\right) = \frac{ku^2}{k - ku^2 - \beta^2} + 1 \tag{4.27}$$

$$\Rightarrow \frac{1}{cos^2\left(\sqrt{k}(\gamma - \gamma_0)\right)} = \frac{k - \beta^2}{k - ku^2 - \beta^2}$$

$$(k - \beta^2)\left(1 - cos^2\left(\sqrt{k}(\gamma - \gamma_0)\right)\right) = ku^2 \tag{4.28}$$

$$\Rightarrow (k - \beta^2)sin^2\left(\sqrt{k}(\gamma - \gamma_0)\right) = ku^2$$

finally, we reached the angular solution with new proper time parameter (γ) as stated below;

$$u = \sqrt{\frac{k - \beta^2}{k} sin\left(\sqrt{k}(\gamma - \gamma_0)\right)} \tag{4.29}$$

According to the mathematical computer programming Mapple, we can obtain θ as;

$$\theta = \pi \pm cos^{-1}(u) \tag{4.30}$$

then, putting Eqn. (4.29) into the Eqn. (4.30) and substituting into Eqn. (4.22), Finally we obtained $\emptyset$ - equation's solution.

$$\gamma - \gamma_0 = -\frac{1}{\sqrt{k}} tan^{-1}\left(\frac{\beta}{\sqrt{k}} tan(c_i - \emptyset)\right) \tag{4.31}$$

where c_i is also an integration constant.

Chapter 5

PERIHELION AND BENDING OF LIGHT

Light bending and perihelion precession are the two most vital impacts on orbits caused by the general relativity redresses to the Newtonian gravitational field of the sun [27]. The main concept of his section is to find the upper limit of Lorentz violation term (*l*) in spherically symmetric geodesic equation. Thus, we handle the Solar system to study the effects of LV term on the bending of light around the Sun and perihelion precession of inner planets. In other words, we use these technniques to compare the result with GR. Geodesics of the particles describe as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{5.1}$$

In which λ is an affine parameter. However, due to the metric compatibility, it is continuously conceivable to utilize a constant motion, χ , characterized as;

$$\chi = -g_{\mu\nu}U^\mu U^\nu \tag{5.2}$$

in which the vector is defined as

$$U^\mu = \frac{dx^\mu}{d\lambda} \equiv \dot{\chi}^\mu \tag{5.3}$$

where dot simply explains the derivative with respect to λ. For gigantic particles, the relative parameter is ordinarily chosen to be the proper time τ. In addition, massive and massless particles get the value of χ = +1 (timelike geodesics), χ = 0 (null geodesics) respectively.

5.1 Advance of the Perihelion

From the geodesic Eqn. (5.1), the equations of motion is detected for a massive particle.

$$\frac{d}{d\tau}[(1-\frac{2M}{r})\dot{t}]=0 \tag{5.4}$$

$$\ddot{r}+\frac{M(r-2M)}{r^3(l+1)}\dot{t}^2-\frac{M}{r(r-2M)}\dot{r}^2 \tag{5.5}$$

$$-\frac{r-2M}{l+1}(\dot{\theta}^2+sin^2\,\theta\dot{\emptyset}^2)=0 \tag{5.5}$$

$$\frac{d}{d\tau}(r^2sin^2\,\theta\dot{\emptyset})=0 \tag{5.6}$$

For simplicity, we reduces our 4D to 3D by taking the $\theta=\frac{\pi}{2}$. Therefore, we understood that any differential orders of θ in Eqn. (5.6), equal to zero and the motion of the particle continue on equilateral plane. From the cyclic coordinates we had two vectors that represent the energy and the angular momentum. The time-like geodesic is related to the energy given as follow;

$$E=-g_{\mu\nu}K^{\mu}U^{\nu}=(1-\frac{2M}{r})\dot{t} \tag{5.7}$$

and the angular momentum of the particle,

$$\ell=g_{\mu\nu}\psi^{\mu}U^{\nu}=r^2\dot{\phi} \tag{5.8}$$

thus, both Eqn. (5.8) and Eqn. (5.9) are the conserved quantities and when we put this equations into the time-like geodesics which is Eq. (5.2), we obtain

$$(1+l)\dot{r}^2+(1-\frac{2M}{r})(\frac{\ell^2}{r^2}+1)=E^2 \tag{5.9}$$

The Eqn. (5.10) is nothing but just explains how we can use the property of Eqn. (5.2) in our time-like geodesic to find conserved equation. When we introducing the new variable which is $u = r^{-1}$ and substitude into above equation which is Eqn. (5.10), we obtain; (Hint: $\dot{r} = \frac{dr}{d\phi}\dot{\phi} = -\ell\frac{du}{d\phi}$)

$$(1+l)(\frac{du}{d\phi})^2 + u^2 = \frac{E^2-1}{\ell^2} + \frac{2M}{\ell^2}u + 2Mu^3 \tag{5.10}$$

As can easily be seen above, it is preferred to solve the second order differential equation obtained by differentiating equation with respect to ϕ. Thus, we obtain;

$$(1+l)\frac{d^2u}{d\phi^2} + u - \frac{M}{\ell^2} - 3Mu^2 = 0 \tag{5.11}$$

in Eqn. (5.12), only the first term contain LV property if we compare with the standart GR. For the purpose of solving Eqn. (5.12) perturbatively, the Lorentz violating term should be smaller than one ($l \ll 1$). it is still significant to consider the last term as a relativistic redress when compared with the Newtonian case. The perturbative solution is including in terms of a small parameter, $\in = \frac{3M^2}{\ell^2}$.

$$u \simeq u^{(0)} + \in u^{(1)} \tag{5.12}$$

The zeroth order of differential equation in $\in$ yields

$$(1+l)\frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} - \frac{M}{\ell^2} = 0 \tag{5.13}$$

where the solution is given by

$$\mathrm{u}^{(0)} = \frac{\mathrm{M}}{\ell^2}[1 + \mathrm{e}\cos(\frac{\phi}{\sqrt{1+l}})] \tag{5.14}$$

It is similar to Newtonian result. In addition, the integration constants we have considered are the orbital eccentricity e and the initial value $\emptyset_0 = 0$. The first order differential equation in $\in$ is

$$(1+l)\frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} - \frac{L^2}{M}(u^{(0)})^2 = 0 \tag{5.15}$$

which shows approximated solution of the form

$$u^{(1)} \simeq \frac{M}{\ell^2} e \frac{\phi}{\sqrt{1+l}} sin(\frac{\phi}{\sqrt{1+l}}) + \frac{M}{\ell^2}[(1+\frac{e^2}{2}) - \frac{e^2}{6} cos(\frac{2\phi}{\sqrt{1+l}})] \tag{5.16}$$

The second term can be ignored because of giving non-effective result, Therefore, the perturbative solution of Eqn. (5.13) reads,

$$u \simeq \frac{M}{\ell^2}[1 + ecos(\frac{\phi}{\sqrt{1+l}}) + \epsilon e \frac{\phi}{\sqrt{1+l}} sin(\frac{\phi}{\sqrt{1+l}})] \tag{5.17}$$

Because of $\in \ll 1$, the perturbative arrangement can be revised within the shape of an ellipse condition

$$u \simeq \frac{M}{\ell^2}\left[1 + ecos\left(\frac{\phi(1-\epsilon)}{\sqrt{1+l}}\right)\right] \tag{5.18}$$

although the priority of Lorentz violation, with period Φ, the orbit remains periodic.

$$\Phi = \frac{2\pi\sqrt{1+l}}{1-\epsilon} \approx 2\pi + \Delta\Phi \tag{5.19}$$

In general, the minimum order of ∈ and *l* expansion gives the advance of perihelion which is (ΔΦ) and it is stated below;

$$\Delta\Phi = 2\pi\epsilon + \pi l = \Delta\Phi_{GR} + \delta\Phi_{LV} \tag{5.20}$$

where $\Delta\emptyset_{GR}$ is the prediction of GR

$$\Delta\Phi_{GR} = 2\pi\epsilon = \frac{6\pi G_N m}{c^2(1-e^2)a} \tag{5.21}$$

İn which c represents the speed of light, m is the mass, e is the half major axis of ellipse. Therefore, from the above Eqn. (5.21), we can easily understood that the contribution to the GR is coming from the Lorentz Symmetry Breaking term and it showed below;

$$\delta\emptyset_{LV} = \pi l \tag{5.22}$$

The expression (5.21) shows the effects of Lorentz violation term to the GR result.

5.2 Bending of Light

In this section, we are going to use null geodesics instead of timelike geodesics because of massless test particles motion. For this reason, in our Eqn. (5.2) yields χ=0. substituting our conserved quantities in null geodesic, we obtain;

$$(1+l)\dot{r}^2 + (1-\frac{2M}{r})\frac{l^2}{r^2} = E^2 \tag{5.23}$$

where dot defines the differentiation with respect to affine parameter. Similar to the previous part of this chapter, we again altering the variable by using $u = r^{-1}$. Moreover, substituting this new parameter into the Eqn. (5.24) and the differentiation with respect to $\emptyset$, we created a new form of null geodesics, as stated below,

$$(1+l)\frac{d^2u}{d\phi^2} + u - 3Mu^2 = 0 \tag{5.24}$$

Just a simple observation, the deflection of light rays, in Eqn. (5.25), reduces to the normal GR result when $\ell \to 0$. Thus, by using the perturbation, we can write the solution

$$u \simeq u^{(0)} + 3Mu^{(1)} \tag{5.25}$$

when we substitute the above equation in Eqn. (5.25), it gives the following differential equation for u^0,

$$(1+l)\frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} = 0 \tag{5.26}$$

in which the solution is

$$u^{(0)} = \frac{1}{D}sin(\frac{\phi}{\sqrt{1+l}}) \tag{5.27}$$

For simplicity, we have considered the initial angle of $\emptyset_0 = 0$, in addition, D is an integration constant. This result corresponds to a straight line equation similar to the Newton estimate. The differential equation for u^1, then, becomes;

$$(1+l)\frac{d^2u^{(1)}}{d\emptyset^2}+u^{(1)}-\frac{1}{D^2}sin^2(\frac{\emptyset}{\sqrt{1+l}})=0 \tag{5.28}$$

and its arrangement is depicting as

$$u^{(1)}=\frac{1}{3D^2}[1+Acos(\frac{\emptyset}{\sqrt{1+l}})+cos^2(\frac{\emptyset}{\sqrt{1+l}})] \tag{5.29}$$

On account of this, the solution for u(∅) is representing in this form

$$u \cong \frac{1}{D}sin(\frac{\emptyset}{\sqrt{1+l}}) + \frac{M}{D^2}[1+Acos(\frac{\emptyset}{\sqrt{1+l}})+cos^2(\frac{\emptyset}{\sqrt{1+l}})] \tag{5.30}$$

As we know, A is a constant parameter and the main purpose is detecting the angle of deflection rate for a light, thus, he boundary conditions are presenting as follow:

(i) because of source $r \to \infty$ it means $u(r \to \infty) \to 0$ and $\varphi = -\delta 1$, and (ii) because of observer $r \to \infty$ it means $u(r \to \infty) \to 0$ and $\varphi = +\delta 2$, therefore, $\delta = \delta 1 + \delta 2$ is the total angle of deflection. By using these conditions in Eqn. (5.31), taking in consideration $\ell \ll 1$ and $\delta 1, \delta 2 \ll 1$, the first-order equation provides

$$\delta_1 = \frac{\mathrm{M}}{\mathrm{D}}(2+\mathrm{A}) \tag{5.31}$$

$$\delta_2 = \frac{M}{D}(2-A)+\frac{\pi l}{2} \tag{5.32}$$

Therefore, the deflection angle of light in our metric tensor (2.4) becomes,

$$\delta = \delta_{GR}+\delta_{LV} = \frac{4G_N m}{c^2 D}+\frac{\pi l}{2} \tag{5.33}$$

Here m is the mass of the deflecting object and D is the parameter that indicates the path closest to the center of the deflecting object. The first term presents as follow;

$$\delta_{\mathrm{GR}} = \frac{4\mathrm{G_N m}}{\mathrm{c^2 D}} \tag{5.34}$$

which gives the deflection of light in standart GR model and the second term in Eqn. (5.34) is the Lorentz symmetry breaking term.

$$\delta_{LV} = \frac{\pi l}{2} \tag{5.35}$$

Moreover, taking the limit $l \to 0$ in Eqn. (5.34), it means that we cancel the second term and automatically we reach standart GR model for the bending of light.

Chapter 6

CONCLUSION

In this thesis, main purpose was to investigate the geodesics of Bumblebee BH. The only differences between Scharwzchild BH and BBH is the Lorentz violation term that breaks the symmetry. These BHs are said to be the solutions of modified gravity since the bumblebee field is coupled with the standart GR field equations. Moreover, the BBHs reduce to the Schwarzchild BH in the limit of vanishing Lorentz symmetry breaking. To have the geodesics equations, we first employed the Lagrangian method in the geometry of the bumblebee black hole. Then we derived full solutions of radial equations of both massive and massless particles in bumblebee gravity model. We also plotted the effective potential versus radius of BH in Figure 1 by changing the magnitude of lorentz symmetry breaking term. In figure 2, we draw E^2 vs r_i and the rest of the figures are showing the orbit of a photon around a bumblebee black hole with different *l* values. Furthermore, we obtained the exact analytical solutions of BBH both in null and timelike geodesics. We then investigated the upper limit of lorentz violation term by using some experimental methods. I believed that the work presented in this thesis may shed light on the observational studies in the future about the signs of the existence of the Lorentz symmetry breaking in the cosmos. I also plan to extend my studies to the rotating BBHs for revealling the effect of angular momentum geodesics. This will be my near future project.

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