Fixed Point Theorems and Applications

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ABSTRACT

Fixed point theory be one of the advanced topics in both pure and applied mathematics, it also has seen great interest since recent decades, because it is considered an essential tool for nonlinear analysis and many other branches of modern mathematics. In particular, when we deal with the solvability of a certain functional equation (differential equation, fractional differential equation, integral equation, matrix equation, etc), we are reformulating the problem in terms of investigating the existence and uniqueness of a fixed point of a mapping. In addition, this theory has several applications in many different fields such as biology, chemistry, economics, game theory, optimization theory, physics, etc.

The basic purpose of this thesis is to present some recent advances in this theory with some applications that is an important for our life. For example, first and second order of ordinary differential equations in Banach space and fractional differential equations involving Riemmann-Liouville and Caputo differential operators.

Keywords: Fixed points, Banach's contraction theorem, Contraction, Schauder's fixed point theorem, Brouwer's fixed point theorem, Uniqueness, Existence, Fractional differential equations, Boundary value problems.

ÖZ

Sabit nokta teorisi hem güvenli hem de uygulamalı matematiğin en ileri konularından biri olup, uzun yıllardan bu yana büyük ilgi görüyor, çünkü doğrusal olmayan analizler ve modern matematiğin diğer birçoklerinden dalında önemli bir araç olarak kabul edilir. Belirli, temizlenmemiş bir denklemin (diferansiyel denklem, kesirli diferansiyel denklem, integral denklem, matris denklemi vb), çözünebilirliği ele aldığımızda, bir haritanın sabit bir noktasının çeşitliliği araştırmak için sorun giderme düzenliyoruz. Ek olarak, bu teorinin biyoloji, kimya, ekonomi, oyun teorisi, optimizasyon teorisi, fizik vb, gibi çeşitli farklı alanda uygulamaları vardır.

Bu tezin temel hedefi, yaşamımız için önemli olan uygulamalarla, bu teorideki bazı yeni gelişmeleri sunmaktır. Örnek olarak, Banach uzayında sıradan diferansiyel denklemlerin birinci ve ikinci dereceden sıraları, Riemmann-Liouville ve Caputo diferansiyel operatörlerini içeren fraksiyonel diferansiyel denklemler.

Anahtar Kelimeler: Sabit noktalar, Banach'ın büzülme teoremi, Kasılma, Schauder'in sabit nokta teoremi, Brouwer'ın sabit nokta teoremi, Teklik, Varlık, Kesirli diferansiyel denklemler, Sınır değer problemleri.

DEDICATION

To

my parents,

my son and my daughter

(Azubair & Esraa)

ACKNOWLEDGMENT

First of all, I would like to thank Almighty and Gracious God for given me the ability and patience to complete this work.

My heartfelt thanks due to my research supervisor Prof. Dr. Sonuç Zorlu Oğurlu for her guidance and effective cooperation.

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LIST OF ABBREVIATIONS

A-AT	Arzelà-Ascoli Theorem
BCP	Banach Contraction Principle
BS	Banach Space
BVP	Boundary Value Problem
CM	Continuous Mapping (Continuous Function)
CMS	Complete Metric Space
CTLS	Convex Topological Linear Spaces
DE	Differential Equations
FP	Fixed Point
FPT	Fixed Point Theory
IVP	Initial Value Problem
LS	Linear Space (Vector Space)
MS	Metric Space
NS	Normed Space
RC	Relatively Compact
R-L	Riemann-Liouville
SFPT	Schauder's Fixed Point Theorem
TS	Topological Space
UCBS	Uniformly Convex Banach Space

Chapter 1

INTRODUCTION

The thesis displays a clear explanation of the FPT. After introducing some important preliminaries and basic theorems of FP, we focused on some applications of BCP and SFPT. The prime aim of this exposition is to offer many of the basic results and techniques of this theory. Certainly, not all aspects of involved theory could include in this work.

The thesis is divided into five chapters. In chapter 1, we give a brief introduction of some basic aspects of this thesis.

The second chapter is also devoted to provide a simple summary for some important definitions, the examples and the useful results about some of the spaces in this thesis. For examples, metric, normed, Banach, inner product and topological spaces. In addition, this chapter has illustrated some the basic concepts and several examples about the FP and contraction. The last section of this chapter studies some the relationships between FPs and convergent sequences of contraction functions.

On the other hand, the main points of this thesis are basically starting from chapter 3, which is more theoretical, develops the main abstract theorems on the existence and uniqueness of FPs of maps. We discuss the most significant theorems of FP in this chapter, starting with BCP, it deals with contraction mappings in CMS and checks the uniqueness and existence of their FPs. Moreover, we state and prove Browder-Kirk theory which is dedicated to finding FPs for non-expansive mappings of uniformly convex BS. Most of the results are discussed in of NS. For instance, Brouwer's FPT. However, it is not just confined to study contraction or non-expansive mappings. it studies problems solvability that deals with all the mapping that is defined on certain subset of IR^n. Furthermore, this the chapter also presents another results such as SFPT, which applies to solve the problems of compact and CM defined on NS. It is one of the best known classical results of FPT and it is an extension of Brouwer's FPT.

In turn, the fourth chapter focuses on several applications of this theory, and it is covering an enough variety of important results ranging from ordinary differential equations in Banach spaces to fractional differential equations. Therefore, the major interest in this chapter is to investigate existence and uniqueness of the solution of the IVP and BVP. The last type depends on the R-L and Caputo operators. In addition, it has given few explicit examples to illustrate and support our results.

The final chapter is consisted of a concise conclusion.

Chapter 2

PRELIMINARIES

2.1 Metric Space

Definition(2.1.1): A set $W \neq \emptyset$ and d is real function defined on $W \times W$ is said to be distance or metric function , if all of the following conditions are true:

(M1) $d(u, w) \geq 0$ (non-negative).

(M2) $d(u, w) = 0$ if and only if $u = w$.

(M3) $d(u, w) = d(w, u)$ (Symmetry).

(M4) $d(u, w) \leq d(u, v) + d(v, w)$(Triangle inequality),

for all $u, v, w \in W$. A nonempty set (W, d) is called metric space.

Definition(2.1.2): A sequence (w_n) in MS (W, d) is called a convergent if there exists $w \in W$ such that $\lim_{n\to\infty} d(w_n, w) = 0$. This means that

$$\forall\, \varepsilon > 0\,,\ \exists\, N \in IN \text{ such that } d(w_n, w) < \varepsilon\,, \forall\, n > N.$$

Remark(2.1.1): The sequence (w_n) in MS (W, d) is said to be a divergent, if it is not convergent.

Definition(2.1.3): A sequence (w_n) in MS (W, d) is said to be a Cauchy sequence if for all $\varepsilon > 0$ there exists $N \in IN$ such that

$$d(w_n, w_m) < \varepsilon,\ \ \forall\, n, m > N.$$

Definition(2.1.4): A MS (W, d) is said to be complete if every Cauchy sequence in W converges in W.

Definition(2.1.5): A non-empty subset D of MS W is called bounded if its diameter

$$\delta(D) = \sup\{d(u\ ,w): u, w \in D\} \text{ is a finite.}$$

Definition(2.1.6): Let W is MS, $U \subseteq W$ is called compact if every sequence in U has a convergent subsequence and also its limit in U. Also U is said to be relatively compact if its closure $\overline{U} \subseteq W$ be compact.

Definition(2.1.7): Suppose (U, d) and (W, d) are MSs and let $Y: U \longrightarrow W$ be an operator. W is said to be compact if every bounded subset of U is mapped into a RC subset of W. Equivalently, Y is compact if and only if $\{Y(w_n)\}$ contains a convergent subsequence in W for every bounded sequence $\{w_n\}$ in U.

Definition(2.1.8): The mapping $Y: (W, d) \longrightarrow (W', d)$ is said to be a continuous at $w_0 \in W$ if for $\forall\ \varepsilon > 0$ there exists $\delta > 0$ such that

$$d(\ Y(w)\ , Y(w_0)) < \varepsilon \text{ as } d(w\ , w_0) < \delta.$$

If the function Y is continuous at every point of W, it is said to be continuous on W. Also it is said to be uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\forall\ u, w \in W$, $d(u, w) < \delta$ yields

$$d(Y(u), Y(w)) < \varepsilon.$$

Definition(2.1.9): A sub-collection $\mathcal{J} \subset C(W)$ is said to be uniformly bounded if there exists $\delta > 0$ such that $|Y(w)| \leq \delta$ for every $w \in W$ and $Y \in \mathcal{J}$.

Definition(2.1.10): A sub-collection $\mathcal{J} \subset C(W, W')$ is said to be equi-continuous if for all $w_0 \in A$ and for each $\varepsilon > 0$ there exists $\delta = \delta(w_0, \varepsilon) > 0$ such that $d(w, w_0) < \delta$ yields $d(Y(w), Y(w_0)) < \varepsilon$ for all $Y \in \mathcal{J}$.

Remark(2.1.2): The collection of all CM from W into W′ be denoted by $C(W, W')$. If $W' = W$, then $C(W, W) = C(W)$.

Theorem(2.1.1)(Arzelà-Ascoli theorem): A sub-collection $\mathcal{J} \subset C(W)$ be relatively compact if and only if,

(i) $\mathcal{J}$ is equi-continuous, and

(ii) $\mathcal{J}$ is uniformly bounded.

Corollary(2.1.1): A sub-collection $\mathcal{J} \subset C(W)$ be compact if and only if it be closed, equi-continuous and uniformly bounded.

Theorem(2.1.2): A mapping $Y: W \longrightarrow U$ is continuous if for every convergent sequence (w_n) of W,

$$\lim_{n\to\infty} Y(w_n) = Y\left(\lim_{n\to\infty} w_n\right).$$

Proof

Let's assume that w_n converges to $w_0 \in W$ such that $\lim_{x\to n} w_n = w$. By the continuity of Y on W,

$$\lim_{x\to n} Y(w_n) = Y(w_0) = Y\left(\lim_{n\to\infty} w_n\right). \qquad \blacksquare$$

2.2 Normed Space

Definition(2.2.1): Assume W be a LS, the function $\|.\|: W \to IR$ is said to be a norm function on W if satisfies, (N1) $\|w\| \geq 0, \ \forall\, w \in W$.

(N2) $\|w\| = 0$ if and only if $w = 0$.

(N3) $\|\beta w\| = |\beta|\ \|w\|, \forall\, \beta \in IR$.

(N4) $\|w_1 + w_2\| \leq \|w_1\| + \|w_2\|$, $\forall\, w_1, w_2 \in W$.

The non-empty set $(W, \|.\|)$ is called a <u>normed space</u>.

Definition(2.2.2): Every complete normed space is called a <u>Banach space</u>**.**

Definition(2.2.3): Suppose W , W' are LSs over the same field , the mapping $Y: D(Y) \subseteq W \longrightarrow W'$ is said to be a <u>linear operator</u> if

(a) The domain $D(Y)$ of Y is LS and the range $R(Y) \subseteq W'$ lies in a LS over the same field .

(b) $\forall\, u, w \in D(Y)$ and $\beta \in IR$,

(i) $Y(u + w) = Y(u) + Y(w)$.

(ii) $Y(\beta w) = \beta Y(w)$.

Definition(2.2.4): Let W , W' be NSs and $Y: D(Y) \subseteq W \longrightarrow W'$ be a linear operator. The operator Y is called <u>bounded</u> if there exists a real number $\beta > 0$ such that for all $w \in D(Y)$

$$\|Y(w)\| \leq \beta\, \|w\|.$$

2.3 Inner Product Space

Definition(2.3.1): Let u, v and w be vectors in a LS W over field C, and let α, β be any scalars. The function $\langle . \ , \ . \rangle : W \times W \to C$ is said to be an <u>inner product</u> on W if satisfies the following axioms:

(IP1) $\langle w, w \rangle \geq 0$,

(IP2) $\langle w, w\rangle = 0$ if and only if $w = 0$,

(IP3) $\langle u, w\rangle = \overline{\langle w, u\rangle}$,

(IP4) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$.

The ordered pair $(W, \langle ., .\rangle)$ is called an inner product space. We call $\langle u, w\rangle$ the inner product of two elements $u, w \in W$.

Characterizations of Inner Product Spaces

1. Let $(W, \langle ., .\rangle)$ be an inner product space. Then, the function $\|.\| : W \to C$ defined by

$$\|w\| = \sqrt{\langle w, w\rangle}.$$

2. The standard inner product is

$$\langle u, w\rangle = u \,.\, w = \sum_{i=1}^{n} u_i w_i \,, \forall\, u, w \in C^n.$$

Theorem(2.3.1)(Cauchy-Schwarz inequality): Let W be an inner product space. Then,

$$|u, w| \leq \|u\|. \|w\|$$

for all $u, w \in W$.

2.4 Topological Space

Definition(2.4.1): Consider τ is a collection of subsets of non-empty set W. τ is said to be a topology on W if the following conditions are satisfied:

(T1) $\emptyset$, $W \in \tau$,

(T2) τ be closed by arbitrary unions,

(T3) τ be closed by finite intersections.

The filed (W, τ) is called topological space.

Some simple examples to illustrate this space

1) Let $Z = \{0, 5, 10\}$ and $\tau = \{\emptyset, \{0\}, \{5, 10\}, Z\}$. Then (Z, τ) is TS.

2) Consider W is any set and $\tau = \{A: A \subseteq W\}$. Then τ is called the discrete topology on W, and (W, τ) is also called the discrete space.

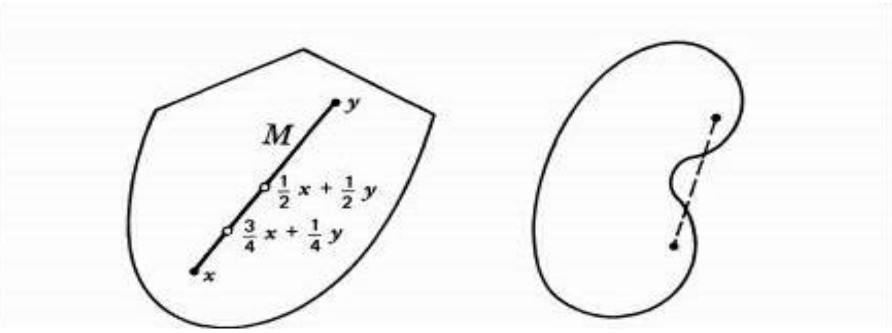
Definition(2.4.2): A subset U of TS W is said to be a neighborhood of $w \in W$ if there exists an open set $M \in \tau$ such that

$$w \in M \subset U.$$

Definition(2.4.3): A subset M of a LS W is said to be convex if for all $u, v \in M$ implies that set

$$\{ z = \alpha u + (1 - \alpha)v, 0 \leq \alpha \leq 1 \}$$

is a subset of M.



convex non-convex

Remark(2.4.1): Let C be a subset of a LS W. Then, C is convex if and only if

$$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \in C$$

for any finite set $\{w_1, \ldots, w_n\} \subset C$ and any scalars $\alpha_i \geq 0$ with $\alpha_1 + \cdots + \alpha_n = 1$.

Definition(2.4.4): Let W be a LS and C be an arbitrary subset of W. The intersection of all convex subsets of W containing C is called <u>convex hull</u> of C in W and is denoted by $\mathrm{co}(\mathrm{C})$. Symbolically, we have

$$\mathrm{co}(\mathrm{C}) = \cap \{\mathrm{K} \subset \mathrm{W} : \mathrm{C} \subseteq \mathrm{K}, \mathrm{K} \text{ is convex}\}.$$

In other words, $\mathrm{co}(\mathrm{C})$ is the set of all finite convex combination of elements of C, that is,

$$\mathrm{co}(\mathrm{C}) = \{\textstyle\sum_{i=1}^{n} \alpha_i w_i : w_i \in \mathrm{C}, 0 \leq \alpha_i \leq 1 : \sum_{i=1}^{n} \alpha_i = 1\}.$$

Example(2.4.1): Let W be a LS. The interval joining between two points $\mathrm{u}, \mathrm{w} \in \mathrm{W}$ is the set

$$[\mathrm{u}, \mathrm{w}] := \{\mathrm{tu} + (1 - \mathrm{t})\mathrm{w} : 0 \leq \mathrm{t} \leq 1\}.$$

Then $\mathrm{co}(\{\mathrm{u}, \mathrm{w}\}) = [\mathrm{u}, \mathrm{w}]$ is convex hull of $\{\mathrm{u}, \mathrm{w}\}$.

Remark(2.4.2): The closure of convex hull of C is

$$\overline{\mathrm{co}(\mathrm{C})} = \overline{\{\textstyle\sum_{i=1}^{n} \alpha_i w_i : w_i \in \mathrm{C}, 0 \leq \alpha_i \leq 1 : \sum_{i=1}^{n} \alpha_i = 1\}}$$

The closed convex hull of C in W is the intersection of all closed convex subsets of W containing C and is denoted by $\overline{\mathrm{co}}(\mathrm{C})$, as follows

$$\overline{\mathrm{co}}(\mathrm{C}) = \cap \{\mathrm{K} \subset \mathrm{W} : \mathrm{C} \subseteq \mathrm{K}, \mathrm{K} \text{ is closed and convex}\}.$$

It is easy to observe that closure of convex hull of C is closed convex hull of C such that

$$\overline{\mathrm{co}(\mathrm{C})} = \overline{\mathrm{co}}(\mathrm{C}).$$

Definition(2.4.5): A linear topology on a TS W is said to be a <u>locally convex topology</u> if every neighborhood of 0 (the zero vector of W) contains a convex neighborhood of 0. Then, W is called a <u>locally convex topological vector space</u>.

Definition(2.4.6): A NS $(W, \|\cdot\|)$ is called uniformly convex if there exists an increasing positive function $\delta: (0,2] \to (0,1]$ such that for $u, w \in W, \|u\|, \|w\| \leq r$ and $\|u - w\| \geq \varepsilon r$ imply that

$$\left\|\frac{u-w}{2}\right\| \leq (1 - \delta(\varepsilon))r.$$

2.5 Fixed Point and Contraction

Definition(2.5.1): A point $w \in W$ is called a fixed point of the mapping $Y: W \longrightarrow W$ if and only if $Y(w) = w$.

Example(2.5.1): The map $Y: IR \longrightarrow IR$ defined by $Y(w) = w^2$ has two FPs $(0 \text{ and } 1)$. On the other hand, the mapping $Y(w) = w - 1$ has no FP.

Definition(2.5.2): Suppose (W, d) is MS. A map $Y: W \longrightarrow W$ is said to be

i) A Lipschitz mapping if there exists a scalar $k \in [0, \infty)$ such that

$$d(Y(w_1), Y(w_2)) \leq k\, d(w_1, w_2), \quad \forall\, w_1, w_2 \in W.$$

ii) A contraction if there exists a scalar $k \in [0, 1)$ such that

$$d(Y(w_1), Y(w_2)) \leq k\, d(w_1, w_2), \quad \forall\, w_1, w_2 \in W.$$

iii) A non-expansive if there exists a scalar $k \in [0, 1]$ such that

$$d(Y(w_1), Y(w_2)) \leq k\, d(w_1, w_2), \forall\, w_1, w_2 \in W.$$

Remark(2.5.1): Because if $w_n \xrightarrow{n\to\infty} w \Longrightarrow Y(w_n) \xrightarrow{n\to\infty} Y(w)$, a Lipschitzian map is necessarily continuous.

Definition(2.5.3): A mapping Y of a MS W into itself and $n \in IN$, we denote by Y^n the n^{th}- iterate of Y. Namely, $Y.Y.Y. \dots \dots \dots Y$ n-times such that

$$Y^n(u) = Y(Y^{n-1}(u)), \dots \dots \dots, Y^2(u) = YY(u), Y^1(u) = Y(u),$$

$$Y^{\circ}(u) = u(\ Y^{\circ} \text{is the identity map}).$$

Remark(2.5.2): If the mapping Y is a contraction on a MS W with contraction constant k for some n, hence Y^n is also a contraction on W with contraction constant k^n for some n. But the converse does not hold in general.

2.6 Sequences of Contractions and Fixed Points

In this section, we will study two types of the convergence for the FPs, such as:

(i) Uniform convergence.

(ii) Pointwise convergence.

Definition(2.6.1): Let W is a MS and (Y_n) be a sequence of set valued functions defined on W. The sequence (Y_n) is said to

i) converge uniformly to Y if given any $\varepsilon > 0$, there exists $L = L(\varepsilon) \in IN$ such that

$$d\big(Y_n(w), Y(w)\big) \leq \varepsilon,\ \ \forall\ n \geq L \text{ and } \forall\ w \in W.$$

ii) converge pointwise to Y if given any $w \in W$ and for every $\varepsilon > 0$, there exists $L = L(w, \varepsilon) \in IN$ such that

$$d\big(Y_n(w), Y(w)\big) \leq \varepsilon,\ \ \forall\ n > L.$$

The following two main theorems will show these convergences:

Theorem(2.6.1): Let (W, d) be a MS and $Y: W \longrightarrow W$ be a contraction map with a FP u_0. Let $Y_n: W \longrightarrow W$ has at least one FP u_n. If $Y_n \longrightarrow Y$ uniformly, then $u_n \longrightarrow u_0$.

Proof

Firstly, let's consider that Y is a contraction with Lipschitz constant $k < 1$.

$$d\big(Y(w_1), Y(w_2)\big) \leq k\, d(w_1, w_2),\ \forall\ w_1, w_2 \in W.$$

Since Y_n converges uniformly to Y , then for any $\varepsilon > 0$ there exists $L = L(\varepsilon) \in IN$ such that

$$d(Y_n(w), Y(w)) \leq \varepsilon(1-k), \forall\, n \geq L, \forall\, w \in W.$$

Hence for all $n \geq L$,

$$d(w_n, w) = d(Y_n(w_n), Y(w_0)) \leq d(Y_n(w_n), Y(w_n)) + d(Y(w_n), Y(w_0))$$

$$\leq \varepsilon(1-k) + k\, d(w_n, w_0).$$

So, $d(w_n, w_0) \leq \varepsilon$, which decides that (w_n) converges to the FP w_0 . ■

Theorem(2.6.2): Let (U, d) be a locally compact MS and $Y: U \longrightarrow U$ be a contraction mapping with FP u_0. In addition, $Y_n: U \longrightarrow U$ be an equi-continuous mapping with FP u_n for each $n \geq 1$. Then convergence of the sequence (Y_n) pointwise to Y guarantees convergence (u_n) to u_0.

Proof

Set $\varepsilon > 0$ and let ε is enough small so that

$$K(u_0, \varepsilon) = \{u \in U: d(u, u_0) \leq \varepsilon\} \subset U$$

Then, by Corollary [(1)] $K(u_0, \varepsilon)$ is a compact. From the fact that (Y_n) is equi-continuous sequence of converging pointwise functions to Y , compactness of $K(u_0, \varepsilon)$ and by Theorem(2.1.1)(A-AT), the sequence $Y_n \xrightarrow{\text{uniformly}} Y$ on $K(u_0, \varepsilon)$. Indeed, since $Y_n \longrightarrow Y$ pointwise, then this implies Y_n is pointwise bounded (all convergent sequences are bounded). Define

$$a_n = d(Y_n, Y).$$

(1) Shirali, S., & Vasudeva, H. L. (2005). *Metric spaces*. Springer Science & Business Media. p180.

We wish to show that $a_n \to 0$. To see this let (a_{n_k}) be any subsequence of (a_n). The equi-continuity and pointwise boundedness of (Y_n) and (A-AT) guarantee existence a further subsequence (Y_{n_k}) that converges uniformly to some function. Since this sequence is known to converge pointwise to Y already, the uniform limit must be Y (because the uniform limit is also a pointwise limit, and pointwise limits are unique) such that , $\exists\, L = L(\varepsilon) \in \mathrm{IN}$ such that,

$$a_{n_k} = d\left(Y_{n_k}, Y\right) \xrightarrow{n_k \to \infty} 0 \text{ , for all } n_k \geq L.$$

Exactly, this means that (a_n) as well has a further subsequence (a_{n_k}), that approaches to 0. Thus, by Theorem(2), $a_n \to 0$, which implies

$$d(Y_n, Y) \xrightarrow{n \to \infty} 0 \text{ , for all } n \geq L.$$

By definition of uniform convergence, this is the same thing as saying $Y_n \longrightarrow Y$ uniformly, as desired. Choose L such that if $n > L$, then

$$d\left(Y_n(u), Y(u)\right) \leq \varepsilon(1-k) \text{ , } \forall\, n \geq L, \forall\, u \in K(u_0, \varepsilon)$$

where $k < 1$ is a Lipschitz constant for Y. Therefore, if $n \geq L$ and $u \in K(u_0, \varepsilon)$,

$$\begin{aligned} d(Y_n(u), u_0) &\leq d\left(Y_n(u), Y(u)\right) + d\left(Y(u), Y(u_0)\right) \\ &\leq \varepsilon(1-k) + kd(u, u_0) \leq \varepsilon - \varepsilon k + \varepsilon k \leq \varepsilon. \end{aligned}$$

This implies $Y_n(u) \in K(u_0, \varepsilon)$ for each $u \in K(u_0, \varepsilon)$. This proves that if $n \geq L$, hence Y_n maps $K(u_0, \varepsilon)$ into itself. Thereafter, for all $n \geq L$,

$$\begin{aligned} d(u_n, u_0) = d\left(Y_n(u_n), Y(u_0)\right) &\leq d\left(Y_n(u_n), Y(u_n)\right) + d\left(Y(u_n), Y(u_0)\right) \\ &\leq \varepsilon(1-k) + kd(u_n, u_0) \leq \varepsilon. \end{aligned}$$

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(2) Laczkovich, M., & Sós, V. T. (2015). *Real Analysis: Foundations and Functions of One Variable*. Springer. p 64.

Thus, $u_n \in K(u_0, \varepsilon)$, that means: the sequence (u_n) converges to u_0. ■

The next result refers to the general case of Theorem (2.6.1)

Theorem(2.6.3): Define $T: U \longrightarrow U$ is an uniformly CM such that T^m be a contraction for some $m \geq 1$. Suppose T_n has at least one FP $u_n = T_n(u_n)$. Then (u_n) converges to $u_0 = T(u_0)$, if (T_n) converges uniformly to T.

Proof

Firstly, since T^m is a contraction for some $m \geq 1$,

$$d\big(T^m(u_1), T^m(u_2)\big) \leq k^m d(u_1, u_2) \text{ for some } k < 1,$$

Now it is sufficiently to define a new metric P on U equivalent to d by considering

$$P(u_1, u_2) = \sum_{r=0}^{m-1} \frac{1}{k^r} d\big(T^r(u_1), T^r(u_2)\big).$$

Moreover, note that

1. T is a contraction with respect to P. To claim this, let u_1, u_2 be arbitrary elements of X.

$$P\big(T(u_1), T(u_2)\big) = \sum_{r=0}^{m-1} \frac{1}{k^r} d\big(T^{r+1}(u_1), T^{r+1}(u_2)\big) = k\sum_{r=1}^{m-1} \frac{1}{k^r} d\big(T^r(u_2), T^r(u_2)\big)$$

$$\leq k\sum_{r=1}^{m-1} \frac{1}{k^r} d\big(T^r(u_1), T^r(u_2)\big) + \frac{1}{k^{m-1}} d\big(T^m(u_1), T^m(u_2)\big)$$

$$\leq k\sum_{r=1}^{m-1} \frac{1}{k^r} d\big(T^r(u_1), T^r(u_2)\big) + k\, d(u_1, u_2)$$

$$= k\sum_{r=0}^{m-1} \frac{1}{k^r} d\big(T^r(u_1), T^r(u_2)\big) = kP(u_1, u_2).$$

2. T is a uniformly continuous with respect to P. To show this let for any $\varepsilon > 0$, there exists $\delta > 0$ $\left(\delta = \frac{\varepsilon}{k}\right)$ such that $P(x, y) \leq \delta$.

$$P\big(T(u_1), T(u_2)\big) \leq k\, P(u_1, u_2) \leq k \times \frac{\varepsilon}{k} = \varepsilon \;, \forall\, u_1, u_2 \in U.$$

3. T_n is a uniformly convergent to T respect to P. To display this let for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in IN$ such that for each $n \geq N$,

$$P(T_n(u), T(u)) = \sum_{r=0}^{m-1} \frac{1}{k^r} d\left(T^r(T_n(u)), T^r(T(u))\right)$$

$$= \sum_{r=1}^{m-1} \frac{1}{k^r} d\left(T^r(T_n(u)), T(T(u))\right) + d(T_n(u), T(u))$$

$$\leq \sum_{r=1}^{m-1} d(T_n(u), T(u)) + d(T_n(u), T(u))$$

$$= (m-1) d(T_n(u), T(u)) + d(T_n(u), T(u)) = m\, d(T_n(u), T(u)).$$

Now, we know that $T_n \xrightarrow{\text{uniform}} T$ with respect to d, hence there exists $L = L\left(\frac{\varepsilon}{m}\right) \in IN$, such that

$$d(T_n(u), T(u)) \leq \frac{\varepsilon}{m}, \ \forall\, n \geq L .$$

Now, let $N \geq L$,

$$P(T_n(u), T(u)) \leq \varepsilon \ \ , \forall\, n \geq N \text{ and } \forall\, u \in U.$$

Finally, by applying the same argument Theorem(2.6.1), we get $P(u_n, u_0) \leq \varepsilon$, for all $n \geq N$. Therefore, $u_n = T_n(u_n)$ converges to $u_0 = T(u_0)$. ■

Chapter 3

FIXED POINT THEOREMS

3.1 Banach Contraction Principle

This theory is one of the important theorems and the most used in the applications of nonlinear analysis.

Theorem(3.1.1): Assume (U, d) is a CMS, then all contraction maps $Y: U \longrightarrow U$ with contraction constant k has a unique FP $u_0 \in U$. In addition, for every $u \in U$ we have $\lim_{n\to\infty} Y^n(u) = u_0$ with

$$d(Y^n(u), u_0) \leq \frac{k^n}{1-k} d(u, Y(u)).$$

Proof

Firstly, we will claim the uniqueness. Let that Y has two FPs $u, v \in U$ with $Y(u) = u$ and $Y(v) = v$. Then

$$d(u,v) = d(Y(u), Y(v)) \leq k\, d(u,v) \Rightarrow (1-k)\, d(u,v) \leq 0$$

$$\Rightarrow d(u,v) \leq 0. \qquad (3.1.1)$$

since k is a contraction constant. Also, if d be a metric function, we conclude

$$d(u,v) \geq 0 \qquad (3.1.2)$$

From (3.1.1) and (3.1.2), we get

$$d(u,v) = 0$$

which follows $u = v$.

Secondly, to show the existence we will take any $u \in U$ and consider the sequence $(Y^n(u))$ in U. Now we need to illustrate that $Y^n(u)$ is a convergent by using the fact that (U, d) is a complete, so it is enough to show that the sequence is Cauchy. For all $n = 0, 1, 2, \ldots\ldots$ that

$$d\big(Y^n(u), Y^{n+1}(u)\big) = d\Big(Y\big(Y^{n-1}(u)\big), Y\big(Y(u)\big)\Big) \leq k\, d\big(Y^{n-1}(u), Y^n(u)\big)$$

$$= k\, d\Big(Y\big(Y^{n-2}(u)\big), Y\big(Y^{n-1}(u)\big)\Big)$$

$$\leq k^2\, d\big(Y^{n-2}(u), Y^{n-1}(u)\big) \leq \cdots\cdots\cdots \leq k^n\, d\big(u, Y(u)\big).$$

Therefore, for $m > n$

$$d\big(Y^n(u), Y^m(u)\big) \leq d\big(Y^n(u), Y^{n+1}(u)\big) + d\big(Y^{n+1}(u), Y^{n+2}(u)\big)$$

$$+ \cdots\cdots\cdots + d\big(Y^{m-1}(u), Y^m(u)\big)$$

$$\leq (k^n + k^{n+1} + \cdots\cdots\cdots + k^{m-1})\, d\big(u, Y(u)\big)$$

$$= k^n(1 + k + k^2 + \cdots\cdots + k^{m-n-1})\, d\big(u, Y(u)\big)$$

$$= k^n(1 + k + k^2 + \cdots\cdots\cdots)\, d\big(u, Y(u)\big).$$

We know that $1 + k + k^2 + \cdots$ is a geometric series, which is a convergent since $0 \leq k < 1$ and $1 + k + k^2 + \cdots\cdots\cdots = \frac{1}{1-k}$. Therefore,

$$d\big(Y^n(u), Y^m(u)\big) \leq \frac{k^n}{1-k}\, d\big(u, Y(u)\big) \tag{3.1.3}$$

Hence, $d\big(Y^n(x), Y^m(x)\big) \xrightarrow{n\to\infty} 0$, it follows $\big(Y^n(u)\big)$ is Cauchy sequence, that converges to $u_0 \in U$ since U is a CMS. That is

$$\lim_{n\to\infty} d\big(u_0, Y^n(u)\big) = 0.$$

Thus, $u_0 = \lim_{n\to\infty} Y^n(u) = \lim_{n\to\infty} Y\big(Y^{n-1}(u)\big) = Y\Big(\lim_{n\to\infty} Y^{n-1}(u)\Big) = Y(u_0)$

since Y is contraction that guarantees the continuity. Therefore, u_0 is a FP.

Finally, putting $m \to \infty$ in (3.1.3) yields

$$d(Y^n(u), u_0) \leq \frac{k^n}{1-k}\, d\big(u, Y(u)\big).$$

The proof of Theorem (3.1.1) is complete. ■

This is the best example to clear that, contractions mapping on incomplete MS might miss FPs.

Example(3.1.1): Consider $Y:(0\,,1] \to (0\,,1]$, $Y(x) = \frac{u}{2}$ such that $(0\,,1]$ is an incomplete MS. Because $\left(\frac{1}{n}\right)$ is Cauchy sequence in MS $(0\,,1]$ but $\lim_{n\to\infty}\frac{1}{n} = 0 \notin (0\,,1]$. Therefore, $(0\,,1]$ is not complete. Now, it is a clear that Y is a contraction on $(0\,,1]$. Because, for any $u, v \in (0\,,1]$

$$d\big(Y(u),Y(v)\big) = |Y(u) - Y(v)| = \left|\ \frac{u}{2} - \frac{v}{2}\ \right| = \frac{1}{2}\ |\ u - v\ | = \frac{1}{2}\ d(u - v).$$

However, for any $u \in (0\,,1]$, $Y(u) \neq u$. So, Y need no have a FP.

Theorem(3.1.2)(Local Banach's FPT): Consider $(U\,,d)$ is a CMS and let

$$B_r(u_0) = \{u \in U: d(u\,,u_0) < r\},$$

where $u_0 \in U$ and $u_0 r > 0$. Assume $Y : B_r(u_0) \to U$ be a contraction map with contraction constant $k \in [0,1)$. As well, assume that

$$d(Y(u_0)\,,u_0) < r(1 - k).$$

Then, Y has a unique FP in $B_r(u_0)$.

Proof

We have $d(Y(u_0)\,,u_0) < r(1 - k) \Longrightarrow \frac{d(Y(u_0)\,,\,u_0)}{1-k} < r$. By using archemidian property of real line IR, there exists $0 \leq r_0 < r$ such that

$$\frac{d(Y(u_0)\,,\,u_0)}{1-k} \leq r_0.$$

Now, to show that $Y : \overline{B_{r_0}(u_0)} \to \overline{B_{r_0}(u_0)}$, take any $u \in \overline{B_{r_0}(u_0)}$,

$$d(Y(u)\,,u_0) \leq d\big(Y(u),Y(u_0)\big) + d(Y(u_0),u_0)$$

$$\leq kd(u,u_0\,) + (1 - k)r_0 \leq kr_0 + r_0 - kr_0 = r_0.$$

Subsequently, $Y(u) \in \overline{B_{r_0}(u_0)}$ for all $u \in \overline{B_{r_0}(u_0)}$. Now notice that

$$Y\left(\overline{B_{r_0}(u_0)}\right) \subseteq \overline{B_{r_0}(u_0)} \subset B_r(u_0) \subset U.$$

Since $\overline{B_{r_0}(u_0)}$ is a close subset of CMS U, then by using complete subset Theorem, is a CMS. Therefore, by BCP, Y has a unique FP $\bar{u} \in \overline{B_{r_0}(u_0)}$, thus $\bar{u} \in B_r(u_0)$. So Y has a unique FP in $B_r(u_0)$. ■

Corollary(3.1.1): Suppose $Y: U \longrightarrow U$ be mapping of CMS. If $Y^N$is a contraction for some positive integer N, whereupon Y has a unique FP $u_0 \in U$ and for each $u \in U$,

$$\lim_{n\to\infty} Y^n(u) = u_0.$$

Proof

Consider that u_0 be the unique FP of Y^N, given by BCP such that $Y^N(u_0) = u_0$. Then

$$Y^N(Y(u_0)) = Y\left(Y^N(u_0)\right) = Y(u_0).$$

This implies $Y(u_0)$ is a FP of Y^N which has a unique FP, then $Y(u_0) = u_0$. So Y has a FP. Since any FP of Y is obviously a FP of Y^N, we have uniqueness as well.

Now, to show $\lim_{n\to\infty} Y^n(u) = u_0$ by using Theorem(3.1.1)(BCP), to get

$$\lim_{n\to\infty} (Y^N)^n(u) = u, \forall u \in U,\ N \geq 1.$$

Since n be any integer and $n = mN + r$ such that $0 \leq r < N$, $m \geq 0$. For any $u \in U$,

$$Y^n(u) = (Y^N)^m\left(Y^r(u)\right).$$

Therefore, $d(Y^n(u)\,,\, u_0\,) = d\left((Y^N)^m\left(Y^r(u)\right),\ u_0\right)$. By using Theorem (3.1.1) (BCP),

$$d(Y^n(u)\,,\, u_0\,) \leq \frac{k^m}{1-k} d\left(Y^r(u), Y\left(Y^r(u)\right)\right) \leq \frac{k^m}{1-k} \max_{0\leq h\leq N-1} \left\{d\left(Y^h(u), Y^{N+h}(u)\right)\right\}.$$

Now, clearly that $m \to \infty$ as$n \to \infty$.Therefore, $\lim_{n\to\infty} d(Y^n(u)\,,\, u_0\,) = 0$. So

$$\lim_{n\to\infty} Y^n(u) = u_0\,.$$ ■

3.2 Browder-Kirk's Fixed Point Theorem of Non-expansive Maps

We before present this result, will present a result known as Schauder's theorem for non-expansive maps. It is a special case of SFPT which will be presented in this chapter.

Theorem(3.2.1)(Schauder's theorem for non-expansive maps): Let $C \neq \emptyset$ be a closed, convex subset of a NS W with $\psi : C \to C$ is non-expansive and $\psi(C)$ a subset of a compact set of C. Then ψ has a FP.

Proof

Take any point $x_0 \in C$ and define

$$\psi_n = \left(1 - \frac{1}{n}\right)\psi + \frac{1}{n}x_0\,,\ \ n \geq 2.$$

It is a clear that

(i) $\psi_n : C \to C$, because for all $x \in C$,

$$\psi_n(x) = \left(1 - \frac{1}{n}\right)\psi(x) + \frac{1}{n}x_0.$$

Thus $\psi_n(x) \in C$ since C is a convex and $\psi(x),\ x_0 \in C$.

(ii) ψ_n is a contraction. To show this let's assume any $x\,, y \in C$, hence

$$\|\psi_n(x) - \psi_n(y)\| = \left(1 - \frac{1}{n}\right)\|\psi(x) - \psi(y)\| \leq \left(1 - \frac{1}{n}\right)\|x - y\|,\ \ \forall n \geq 2.$$

Theorem(3.1.1)(BCP) says that for all $n \geq 2$, ψ_n has a unique FP $x_n \in C$ such that

$$x_n = \psi_n(x_n) = \left(1 - \frac{1}{n}\right)\psi(x_n) + \frac{1}{n}x_0.$$

In addition, by our assumption $\psi(C)$ lies in a compact subset say B subset of C such that $\psi(C) \subset B \subset C$. It follows that a sequence $\left(\psi(x_n)\right) \subset \psi(C) \subset B$ has a convergent subsequence $\left(\psi\left(x_{n_k}\right)\right)$ such that

$$\psi\left(x_{n_k}\right) \xrightarrow{n_k \to \infty} \overline{x} \in C \qquad (3.2.1)$$

Therefore,

$$x_{n_k} = \psi_{n_k}(x_{n_k}) = \left(1 - \frac{1}{n_k}\right)\psi(x_{n_k}) + \frac{1}{n_k}x_0 \xrightarrow{n_k\to\infty} \bar{x}.$$

In turn, $\|\psi(x_{n_k}) - \psi(\bar{x})\| \leq \|x_{n_k} - \bar{x}\| \xrightarrow{n_k\to\infty} 0$. Then this automatically yields

$$\psi(x_{n_k}) \xrightarrow{n_k\to\infty} \psi(\bar{x}) \tag{3.2.2}$$

From $(3.2.1), (3.2.2)$ and the fact the uniqueness of the limit, $\psi(\bar{x}) = \bar{x}$. ■

The main theorem of this section is a result proved independently by Browder, Gohde and Kirk. We state it as follows:

Theorem(3.2.2)(Browder-Kirk): Let W be a UCBS and C be non-empty, closed, bounded and convex subset of W . If $\psi: C \to C$ is a non-expansive map, then ψ has a FP in C.

Proof

Let $x_* \in C$ be fixed, and consider a sequence $r_n \in (0, 1)$ converging to one. For each $n \in IN$, define the map $\psi_n: C \to C$ as

$$\psi_n(x) = r_n\psi(x) + (1 - r_n)x_*.$$

Notice that ψ_n is a contractions on C. To make sure let's take $x, y \in C$,

$$\|\psi_n(x) - \psi_n(y)\| = r_n\|\psi(x) - \psi(y)\| \leq r_n\|x - y\|.$$

Then there is a unique $x_n \in C$ such that $\psi_n(x_n) = x_n$. Since C is weakly compact, x_n has a subsequence weakly converges to some $\bar{x} \in C$. We shall prove that $\bar{x}$ is a FP of ψ. Notice initial that

$$\lim_{n\to\infty}(\|\psi(\bar{x}) - x_n\|^2 - \|\bar{x} - x_n\|^2) = \|\psi(\bar{x}) - \bar{x}\|^2.$$

Since ψ is non-expansive we have

$$\begin{aligned}\|\psi(\bar{x}) - x_n\| &\leq \|\psi(\bar{x}) - \psi(x_n)\| + \|\psi(x_n) - x_n\| \\ &\leq \|\bar{x} - x_n\| + \|\psi(x_n) - x_n\| \\ &= \|\bar{x} - x_n\| + \|\psi(x_n) - \psi_n(x_n)\|\end{aligned}$$

$$= \|\overline{x} - x_n\| + (1 - r_n)\|\psi(x_n) - x_*\|.$$

But $r_n \to 1$ as $n \to \infty$ and C is bounded, so we conclude that

$$\lim_{n\to\infty} (\|\psi(\overline{x}) - x_n\|^2 - \|\overline{x} - x_n\|^2) \leq 0.$$

Thus $\|\psi(\overline{x}) - \overline{x}\| = 0$, which yields the equality $\psi(\overline{x}) = \overline{x}$. ■

3.3 The Brouwer's Fixed Point Theorem

Brouwer's FPT is considered the bases for some FPTs. We start this section by reminding you that IR^n is endowed with its standard inner product. If $u, v \in IR^n$, hence

$$\langle u, v\rangle = u \,.\, v = \sum_{i=1}^{n} u_i v_i,$$

and norm

$$\|u\| = \sqrt{\langle u, u\rangle}.$$

Also, B^n and S^{n-1} will denote respectively, the closed unit ball and unit sphere in IR^n:

$$B^n = \{u \in IR^n : \|u\| \leq 1\}, S^{n-1} = \{u \in IR^n : \|u\| = 1\}.$$

Before we introduce the theorems, we provide the following definitions

Definition(3.3.1): A TS U has the FP property if every CM $Y: U \longrightarrow U$ has a FP.

Definition(3.3.2): A CM $Y: U \subseteq IR^n \longrightarrow IR^n$ is said to be of class C^1, if it has a continuous extension to an open neighbourhood of U on which is continuously differentiable.

Definition(3.3.3): A mapping $Y: U \subseteq IR^n \longrightarrow IR^n$ is called a

1. Non-vanishing if it satisfies for all $u \in U$, $Y(u) \neq 0$.

2. Normed if it satisfies for all $u \in U$, $\|Y(u)\| = 1$.

3. Tangent to S^{n-1} if the mapping $Y: S^{n-1} \longrightarrow IR^n$ satisfies $\langle u, Y(u)\rangle = 0$ for all $u \in S^{n-1}$.

Theorem(3.3.1): Assume $U \subseteq IR^n$ be a compact. Let $Y: U \longrightarrow IR^n$ is of class C^1 on U. Then we have the following: there exists $k \geq 0$ such that

$$\|Y(u) - Y(v)\| \leq k\|u - v\|, \forall u, v \in U.$$

With the result above, we can prove Theorem (3.3.2).

Theorem(3.3.2): Suppose that $Y: S^{n-1} \longrightarrow IR^n$ is a normed vector field of class C^1 which is tangent to S^{n-1}. Then for $t > 0$ sufficiently small,

$$Y_t(S^{n-1}) = (1 + t2)^{\frac{1}{2}} S^{n-1}, \text{ here } Y_t : u \to u + tY(u).$$

Proof

Define $Y^*: IR^n \backslash \{0\} \longrightarrow IR^n, U \subseteq IR^n$ by

$$Y^*(u) = \|u\| Y\left(\frac{u}{\|u\|}\right) \text{ and } U = \left\{u \in IR^n : \frac{1}{2} \leq \|u\| \leq \frac{3}{2}\right\}.$$

Note that $Y^*$is well defined. Furthermore, we know that $Y^*$is of class C^1 in S^{n-1} since $\|x\|$, Y and $\frac{x}{\|x\|}$ are all of class C^1 in U. U is also compact since it is bounded and closed. By Theorem(3.3.1) applied to Y^*on U,

$$\exists L \geq 0, \forall u, v \in U : \|Y^*(u) - Y^*(v)\| \leq k\|u - v\|.$$

Let $|t| \leq \min\left\{\frac{1}{3}, \frac{1}{k}\right\}$, where k is the Lipschitz constant of Y^*on U. Fix $z \in S^{n-1}$ and define $\psi: U \to IR^n$ by

$$\psi(u) = z - tY^*(u).$$

We aim to apply Theorem(3.1.1)(BCP) to G. To do so, we must show $\psi: U \longrightarrow U$ and G is a contraction. By the triangle inequality, our choice of $|t| \leq \frac{1}{3}$, and the fact that Y is normed, we have in fact that

1) $\psi : U \longrightarrow U$. To see this let that $u \in U$, hence

$$\|\psi(u)\| = \|z - tY^*(u)\| = \left\|z - t\|u\|Y\left(\frac{u}{\|u\|}\right)\right\|$$

$$\geq \|z\| - |t|\|u\| \left\| Y\left(\frac{u}{\|u\|}\right) \right\| \geq 1 - |t|\|u\| \geq 1 - \left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \frac{1}{2} \qquad (3.3.1)$$

Similarly, we also have

$$\|\psi(u)\| = \|z - tY^*(u)\| = \left\| z + (-t)\|u\| Y\left(\frac{u}{\|u\|}\right) \right\|$$

$$\leq \|z\| + |-t|\|u\| \left\| Y\left(\frac{u}{\|u\|}\right) \right\| = 1 + |t|\|u\| \leq 1 + \left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \frac{3}{2} \qquad (3.3.2)$$

Then (3.3.1) and (3.3.2) imply

$$\frac{1}{2} \leq \|\psi(u)\| \leq \frac{3}{2} \Longrightarrow \psi(u) \in U.$$

2) ψ is a contraction on U. It is easy to verify (using the Lipschitz constant k of Y^* on A and $|t| \leq \frac{1}{k}$) that $\psi: U \longrightarrow U$ is a contraction. Indeed, let $u, v \in U$

$$\|\psi(u) - \psi(v)\| = \|z - tY^*(u) - z + tY^*(v)\|$$

$$= |t|\|Y^*(v) - Y^*(u)\| \leq |t|k\|v - u\| = k|t|\|u - v\|.$$

Clearly that U is a closed subset of CMS IR^n. Therefore, U is directly complete, by Theorem (3.1.1) that ψ subsequently has a FP, say $u \in U$, such that $u = \psi(u)$. Hence $u + tY^*(u) = z$. Therefore,

$$\langle u + tY^*(u), u + tY^*(u)\rangle = \langle z, z\rangle = \|z\|^2 = 1$$

$$\Longrightarrow \langle u, u\rangle + 2t\langle u, Y^*(u)\rangle + t^2\langle Y^*(u), Y^*(u)\rangle = 1$$

$$\Longrightarrow \|u\|^2 + 2t\langle u, Y^*(u)\rangle + t^2\|Y^*(u)\|^2 = 1$$

$$\Longrightarrow \|u\|^2 + 2t\left\langle u, \|u\| Y\left(\frac{u}{\|u\|}\right)\right\rangle + t^2 \left\| \|u\| Y\left(\frac{u}{\|u\|}\right) \right\|^2 = 1$$

$$\Longrightarrow \|u\|^2 + 2t\|u\|^2 \left\langle \frac{u}{\|u\|}, Y\left(\frac{u}{\|u\|}\right)\right\rangle + t^2\|u\|^2 \left\| Y\left(\frac{u}{\|u\|}\right) \right\|^2 = 1.$$

Since Y is tangent to S^{n-1} and normed by the assumption and $\frac{u}{\|u\|}$ is unit vector, we have

$$\|u\|^2 + 2t\|u\|^2(0) + t^2\|u\|^2(1)^2 = 1 \Longrightarrow (1 + t^2)\|u\|^2 = 1 \Longrightarrow \|u\| = (1 + t^2)^{-1/2}$$

Now we can assume $v = \frac{u}{\|u\|} = (1 + t^2)^{1/2} u \in S^{n-1}$.

$Y_t(v) = v + tY(v) = (1+t^2)^{1/2}u + \frac{t}{\|u\|}Y^*(u) = (1+t^2)^{1/2}u + \frac{t}{\|u\|}\left(\frac{z-u}{t}\right)$

$= (1+t^2)^{1/2}u + (1+t^2)^{1/2}(z-u) = (1+t^2)^{1/2}(u+z-u) = (1+t^2)^{1/2}z,$

where $z \in S^{n-1}$ is an arbitrary. We have shown that for any $z \in S^{n-1}$ there exists $v \in S^{n-1}$ with

$$Y_t(v) = (1+t^2)^{1/2}z.$$

Consequently,

$$(1+t^2)^{\frac{1}{2}}S^{n-1} \subseteq Y_t(S^{n-1}). \qquad (3.3.3)$$

To show the reverse inclusion, fix $u \in S^{n-1}$. Because Y is tangent to $S^{n-1}$and normed by the assumption, we have

$\|u + tY(u)\|^2 = \langle u + tY(u), u + tY(u)\rangle = \langle u, u\rangle + t\langle u, Y(u)\rangle + t^2\langle Y(u), Y(u)\rangle$

$$= \|u\|^2 + t\langle u, Y(u)\rangle + t^2\|Y(u)\|^2 = (1)^2 + t(0) + t^2(1)^2 = 1 + t^2.$$

Fix $v = (1 + t^2)^{-1/2}(u + tY(u))$. By using above equality

$$\|v\|^2 = \left\|(1 + t^2)^{-1/2}(u + tY(u))\right\|^2 = (1 + t^2)^{-1}\|u + tY(u)\|^2 = 1$$

Since $\|v\|^2 = 1$, we have $\|v\| = 1$, that is $v \in S^{n-1}$. Furthermore, by the definition of Y_t and our choice of w, we know

$$(1 + t^2)^{-1/2}(u + tY(u)) = v \Leftrightarrow u + tY(u) = (1 + t^2)^{1/2}v$$

$$\Leftrightarrow Y_t(v) = (1 + t^2)^{1/2}y.$$

We have shown that for any $u \in S^{n-1}$, we have $v \in S^{n-1}$ with $Y_t(u) = (1 + t^2)^{1/2}v$. Therefore,

$$Y_t(S^{n-1}) \subseteq (1+t2)^{\frac{1}{2}}S^{n-1} \qquad (3.3.4)$$

From (3.3.3) and (3.3.4), we get $Y_t(S^{n-1}) = (1+t2)^{\frac{1}{2}}S^{n-1}$, as required. ■

Although we omit the proof, the following theorems are needed in order to induce a contradiction in the proof of Theorem (3.3.5).

Theorem(3.3.3): If $h \in IN$, then there are no NSs of class C^1 tangent to S^{2h}.

Theorem(3.3.4)(Weierstrass Approximation): Assume Y is a CM defined on $[b, c]$. Then $\forall\ \varepsilon > 0$, there exists a polynomial P such that for all u in $[b, c]$, we have

$$\|Y(u) - P(u)\| < \varepsilon \text{ or } \|Y - P\| < \varepsilon.$$

Theorem(3.3.5): If $h \in IN$ be fixed. Then there are no non-vanishing, continuous vector fields tangent to S^{2h}.

Proof

We aim to prove the above statement by contradiction. Suppose such a field $Y: S^{2h} \longrightarrow IR^{2h+1}$ exists and that

$$r = \min\{\|Y(u)\|: u \in S^{2h}\} > 0.$$

Since Y is CM and maps to R^{2h+1} by the assumption, we can apply Theorem (3.3.4) to each of the $2h + 1$ coordinate components of Y. Recombining the resulting polynomial components, we obtain: $\forall\ \varepsilon > 0$ there exists $P: S^{2h} \longrightarrow R^{2h+1}$ such that $\forall\ u \in S^{2h}$,

$$\| P(u) - Y(u)\| < \varepsilon.$$

Let $r = \min\{\|Y(u)\|: u \in S^{,2h}\}$. Observe that $r > 0$ since Y is non-vanishing. Thus $\frac{r}{2}$ is a valid choice for ε,

$$\exists\ P: S^{2h} \longrightarrow IR^{2h+1} \text{ such that } \forall u \in S^{2h}, \| P(u) - Y(u)\| < \tfrac{r}{2}.$$

Now P is of class C^∞since polynomials are infinitely differentiable. By the triangle inequality and the fact that r is a minimum, we get

$$\|P(u)\| \geq \|Y(u)\| - \|P(u) - Y(u)\| \geq r - \tfrac{r}{2} = \tfrac{r}{2} > 0 .$$

Thus P is non-vanishing by Definition(3.3.3). Define the vector field $\eta : S^{2k} \rightarrow IR^n$ by

$$\eta(u) = P(u) - \langle P(u), u\rangle u.$$

Because η is also a polynomial, it is also of class C^{∞} and is easily seen to be tangent to S^{2h}, as follows

$$\langle \eta(u), u\rangle = \langle P(u) - \langle P(u), u\rangle u, u\rangle = \langle P(u), u\rangle - \langle\langle P(u), u\rangle u, u\rangle$$

$$= \langle P(u), u\rangle - \langle P(u), u\rangle\langle u, u\rangle = \langle P(u), u\rangle - \langle P(u), u\rangle\|u\|^2$$

$$= \langle P(u), u\rangle - \langle P(u), u\rangle = 0.$$

By the triangle inequality, above inequalities, the tangency of Y (by the assumption) and Theorem(2.3.1)(Cauchy-Schwarz inequality), $\forall\, u \in S^{2h}$ we have

$$\|\eta(u)\| = \|P(u) - \langle P(u), u\rangle u\| \geq \|P(u)\| - |\langle P(u), u\rangle|\|u\| > \frac{r}{2} - |\langle P(u), u\rangle|$$

$$= \frac{r}{2} - |\langle P(u), u\rangle - \langle Y(u), u\rangle| = \frac{r}{2} - |\langle P(u) - Y(u), u\rangle|$$

$$\geq \frac{r}{2} - \|P(u) - Y(u)\|\|u\| > \frac{r}{2} - \frac{r}{2}(1) = 0.$$

This implies $\eta(u) \neq 0 \;\; \forall\, u \in S^{2h}$. It is well known that $\frac{\eta(u)}{\|\eta(u)\|}$ be a unit vector and $\left\|\frac{\eta(u)}{\|\eta(u)\|}\right\| = 1$. Therefore, we can consider $\frac{\eta(u)}{\|\eta(u)\|}$ is normed by definition. It is also of class C^1 and tangent to S^{2h}, since $\eta(u)$ was of class C^{∞} and tangent to S^{2h}. This contradict Theorem (3.3.3). Hence our initial assumption was false, and such that a field Y does not exist as required. ■

IR^n can be viewed as subspace of IR^{n+1} by identifying all point $x = (x_1, \cdots, x_n) \in IR^n$ with the point $(x_1, \cdots, x_n, 0) \in IR^{n+1}$. Any point of IR^{n+1} may be represented as (x, x_{n+1}), with $x \in IR^n$ and $x_{n+1} \in IR$. The unit sphere $S^n \subseteq IR^{n+1}$ may be divided into

(i) The upper hemisphere

$$S^n_+ = \{(x, x_{n+1}) \in S^n : x_{n+1} \geq 0\}$$

(ii) The lower hemisphere

$$S_-^n = \{(x, x_{n+1}) \in S^n : x_{n+1} \leq 0\}$$

The unit sphere

$$S^{n-1} = S_+^n \cap S_-^n$$

is the equator.

Let $e_{n+1} = (0, \ldots, 0, 1)$ is a north pole while $-e_{n+1} = (0, \ldots, 0, -1)$ is a south pole.

Definition(3.3.4): The stereographic projection from e_{n+1} to S^n is the mapping $S_+ : IR^n \longrightarrow S^n$ defined by

$$S_+(x) = \left(\frac{2x}{1+\|x\|^2}, \frac{\|x\|^2-1}{1+\|x\|^2}\right), \text{ for } x \in IR^n.$$

Similarly, $S_- : IR^n \longrightarrow S^n$ is the stereographic projection S_- from $-e_{n+1}$ to S^n defined by

$$S_-(x) = \left(\frac{2x}{1+\|x\|^2}, \frac{1-\|x\|^2}{1+\|x\|^2}\right), \text{ for } x \in IR^n.$$

Note that S_+ and S_- are both infinitely differentiable and thus of class C^∞. Furthermore, for any $x \in B^n$ we have

$$\frac{\|x\|^2-1}{1+\|x\|^2} \leq 0 \text{ and } \frac{1-\|x\|^2}{1+\|x\|^2} \geq 0$$

Thus $S_+ : B^n \longrightarrow S_-^n$ and $S_- : B^n \longrightarrow S_+^n$. Notice also, that for any $x \in S^{n-1}$,

$$S_+(x) = S_-(x) = \left(\frac{2x}{1+1}, \frac{1-1}{1+1}\right) = (x, 0) = x.$$

Now we will be devoted to proving Theorem (3.3.6), which will be basis for the proof of Brouwer's FPT.

Theorem(3.3.6): The closed unit ball B^n in IR^n has the FP property.

Proof

We consider two parts, n even and n odd, where n is the dimension of IR^n. Recall that we aim to prove B^n has the FP property.

Step1: We assume that $n = 2r$, where $r \in IN$. We proceed by contradiction. Assume the theorem is false. That is, there exists a CM $Y: B^{2r} \longrightarrow B^{2r}$, which has no FPs. Define the vector field η by

$$\eta(u) = u - Y(u).$$

It is immediate that η is non-vanishing on B^{2r}, and it is easy to see that at any point $u \in S^{2r-1}$ the field is directed outwards, that is,

$$0 < \langle u , \eta(u) \rangle = \langle u , u - Y(u) \rangle = \langle u , u \rangle - \langle u , Y(u) \rangle$$

$$= \| u \|^2 - \langle u, Y(u) \rangle = 1 - \langle u , Y(u) \rangle.$$

Therefore, $\langle u , Y(u) \rangle < 1$, for $u \in S^{2r-1}$.

Step 2: We can now define $\varphi : B^{2r} \longrightarrow B^{2r}$ as follows

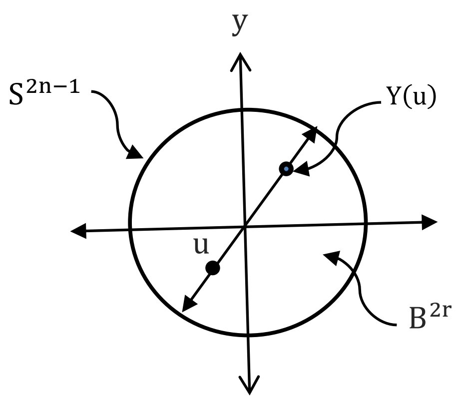
$$\varphi(u) = u - \left(\frac{1-\|u\|^2}{1-\langle u,Y(u) \rangle}\right) Y(u).$$

Note that for any $u \in S^{2r-1}$, we hold

$$\varphi(u) = u - \left(\frac{1-1}{1-\langle u,Y(u) \rangle}\right) Y(u) = u.$$

We aim to prove φ is non-vanishing by contradiction. Let that for some $u \in B^{2r}$, $\varphi(u) = 0$. Then we secure

$$u - \left(\frac{1-\|u\|^2}{1-\langle u,Y(u) \rangle}\right) Y(u) = 0 \text{ and } u = \left(\frac{1-\|u\|^2}{1-\langle u,Yu \rangle}\right) Y(u).$$



This shows that 0, u and $Y(u)$ are collinear since by above equalities u is some constant times $Y(u)$. Therefore, for some α, $Y(u) = \alpha u$. As a result,

$$\langle u, Y(u)\rangle u = \langle u, \alpha u\rangle u = \|u\|^2 \alpha u = \|u\|^2 Y(u).$$

Hence $(1 - \langle u, Y(u)\rangle)u = (1 - \|u\|^2)Y(u) \Longrightarrow u - \langle u, Y(u)\rangle u = Y(u) - \|u\|^2 Y(u)$

This in turn immediately implies $u = Y(u)$ which is contradiction, Y has no FPs. Our initial assumption was false and φ is non-vanishing.

Step 3: For any $u \in B^{2r}$, consider the set

$$\{u + t\varphi(u) : t \in [0,1]\}.$$

Since the stereographic projection S_+ is of class C^∞ and maps to S_-^{2r}. The image of this set under S_+ is a differentiable arc with initial point lying on S_-^{2r}. Therefore, define $K_- : S_-^{2r} \longrightarrow IR^{2r+1}$ by

$$K_-(v) = \left\{\frac{d}{dt} S_+\big(u + t\varphi(u)\big)\right\}\Big|_{t=0}$$

We will show $\frac{d}{dt}(\|u + t\varphi(u)\|^2)$, from known that

$$\|u + t\varphi(u)\|^2 = \langle u + t\varphi(u), u + t\varphi(u)\rangle$$

and the inner product is symmetric. Therefore,

$$\frac{d}{dt}(\|u + t\varphi(u)\|^2) = \frac{d}{dt}\langle u + t\varphi(u), u + t\varphi(u)\rangle$$

$$= \langle \frac{d}{dt}\big(u + t\varphi(u)\big), u + t\varphi(u)\rangle + \langle u + t\varphi(u), \frac{d}{dt}\big(u + t\varphi(u)\big)\rangle$$

$$= 2\,\langle u + t\varphi(u), \frac{d}{dt}\big(u + t\varphi(u)\big)\rangle = 2\langle u + t\varphi(u), \varphi(u)\rangle.$$

So $K_-(v) = \left\{\frac{d}{dt} S_+\big(u + t\varphi(u)\big)\right\}\Big|_{t=0} = \left\{\frac{d}{dt}\left(\frac{2\big(u+t\varphi(u)\big)}{1+\|u+t\varphi(u)\|^2}, \frac{\|u+t\varphi(u)\|^2-1}{1+\|u+t\varphi(u)\|^2}\right)\right\}\Big|_{t=0}$

$$= \left\{\frac{2\varphi(u)\big(1+\|u+t\varphi(u)\|^2\big)-4\big(u+t\varphi(u)\big)\langle u+t\varphi(u),\varphi(u)\rangle}{(1+\|u+t\varphi(u)\|^2)^2}, \frac{4\langle u+t\varphi(u),\varphi(u)\rangle}{(1+\|u+t\varphi(u)\|^2)^2}\right\}\Big|_{t=0}$$

$$= \left(\frac{2(1+\|u\|^2)\varphi(u)-4\langle u,\varphi(u)\rangle u}{(1+\|u\|^2)^2}, \frac{4\langle u,\varphi(u)\rangle}{(1+\|u\|^2)^2}\right)$$

$$= \frac{2}{(1+\|u\|^2)^2}\big((1 + \|u\|^2)\varphi(u) - 2\langle u, \varphi(u)\rangle u, 2\langle u, \varphi(u)\rangle\big)$$

K_- is CM since S_+ is infinitely differentiable. We also claim that K_- is non-vanishing. To see this, consider

$$\|K_-(v)\|^2 = \frac{4}{(1+\|u\|^2)^4}\left\|\left((1+\|u\|^2)\varphi(u) - 2\langle u,\varphi(u)\rangle u, 2\langle u,\varphi(u)\rangle\right)\right\|^2$$

$$= \frac{4}{(1+\|u\|^2)^4}\left(\|(1+\|u\|^2)\varphi(u) - 2\langle u,\varphi(u)\rangle u\|^2 + 4\langle u,\varphi(u)\rangle^2\right)$$

$$= \frac{4}{(1+\|x\|^2)^4}\left(\langle (1+\|u\|^2)\varphi(u) - 2\langle u,\varphi(u)\rangle u, (1+\|u\|^2)\varphi(u) - 2\langle u,\varphi(u)\rangle u\rangle + 4\langle u,\varphi(u)\rangle^2\right)$$

$$= \frac{4}{(1+\|u\|^2)^4}\left((1+\|u\|^2)^2\|\varphi(u)\|^2 - 4(1+\|u\|^2)\langle u,\varphi(u)\rangle^2 + 4\langle u,\varphi(u)\rangle^2\|u\|^2 + 4\langle u,\varphi(u)\rangle^2\right)$$

$$= \frac{4}{(1+\|u\|^2)^4}\left((1+\|u\|^2)^2\|\varphi(u)\|^2 - 4(1+\|u\|^2)\langle u,\varphi(u)\rangle^2 + 4(1+\|u\|^2)\langle u,\varphi(u)\rangle^2\right)$$

$$= \frac{4}{(1+\|u\|^2)^4}\left((1+\|u\|^2)^2\|\varphi(u)\|^2\right) = \frac{4}{(1+\|u\|^2)^2}\|\varphi(u)\|^2 = \left(\frac{2}{1+\|u\|^2}\|\varphi(u)\|\right)^2.$$

Therefore, $\|K_-(v)\| = \frac{2}{1+\|u\|^2}\|\varphi(u)\|$.

We know φ is non-vanishing ($\|\varphi(u)\| \neq 0$), thus. K_- is non-vanshing as claimed. Lastly, we claim that K_- is tangent to S_-^{2r}. Since S_+ maps to S_-^{2r}, we have

$$\langle S_+(u + tR(u)), S_+(u + tR(u))\rangle = 1$$

$$\Rightarrow \frac{d}{dt}\langle S_+(u + t\varphi(u)), S_+(u + t\varphi(u))\rangle = 0. \qquad (3.3.5)$$

We know that

$$\frac{d}{dt}\langle K_+(u + t\varphi(u)), K_+(u + t\varphi(u))\rangle = \langle \frac{d}{dt}\left(K_+(u + t\varphi(u))\right), K_+(u + t\varphi(u))\rangle$$
$$+ \langle K_+(u + t\varphi(u)), \frac{d}{dt}\left(K_+(u + t\varphi(u))\right)\rangle.$$

Because the inner product is symmetric, we also have

$$\frac{d}{dt}\langle K_+(u + t\varphi(u)), K_+(u + t\varphi(u))\rangle = 2\langle \frac{d}{dt}\left(K_+(u + t\varphi(u))\right), K_+(u + t\varphi(u))\rangle$$

$$= 2\langle K_-(v), v\rangle \text{ , } \forall\, v = S_+\big(u + t\varphi(u)\big) \in S_-^{2r} \tag{3.3.6}$$

Combining (3.3.5) and (3.3.6) we obtain

$$\langle K_-(v), v\rangle = 0 \text{ for all } v \in S_-^{2r}.$$

Thus K_- is tangent to S_-^{2r} as claimed. We define $K_+ : S_+^{2r} \longrightarrow IR^{2r+1}$ by

$$K_+(v) = \left\{\frac{d}{dt} S_-\big(u - t\varphi(u)\big)\right\}\Big|_{t=0}$$

In the same way, we also find

$$\frac{d}{dt}(\|u - t\varphi(u)\|^2) = -2\langle u - t\varphi(u), \varphi(u)\rangle.$$

Thus, $K_+(v) = \left\{\frac{d}{dt} S_-\big(u - t\varphi(u)\big)\right\}\Big|_{t=0}$

$$= \left\{\frac{d}{dt}\left(\frac{2\big(u - t\varphi(u)\big)}{1+\|u - t\varphi(u)\|^2}, \frac{1-\|u - t\varphi(u)\|^2}{1+\|u - t\varphi(u)\|^2}\right)\right\}\Big|_{t=0}$$

$$= \left\{\left(\frac{-2\varphi(u)\big(1+\|u-t\varphi(u)\|^2\big)+4\big(u-t\varphi(u)\big)\langle u-t\varphi(u),\varphi(u)\rangle}{(1+\|u-t\varphi(u)\|^2)^2}, \frac{4\langle u-t\varphi(u),\varphi(u)\rangle}{(1+\|u-t\varphi(u)\|^2)^2}\right)\right\}\Big|_{t=0}$$

$$= \left(\frac{4\langle u,\varphi(u)\rangle u - 2(1+\|u\|^2)\varphi(u)}{(1+\|u\|^2)^2}, \frac{4\langle u,\varphi(u)\rangle}{(1+\|u\|^2)^2}\right)$$

$$= \frac{2}{(1+\|u\|^2)^2}\big(2\langle u,\varphi(u)\rangle u - (1+\|u\|^2)\varphi(u), -2\langle u,\varphi(u)\rangle\big).$$

By similar arguments as those above, K_+ is also continuous, non-vanishing and tangent to S_+^{2r}. Therefore, we consider $K: S^{2r} \longrightarrow IR^{2r+1}$ defined as follows

$$K(v) = \begin{cases} K_-(v) & : v \in S_-^{2r} \\ K_+(v) & : v \in S_+^{2r} \end{cases}$$

for all $v = S_-(u) \in S_+^{2r}$. It is easy to see that $K_+(v) = K_-(v)$ for $v \in S^{2r-1}$. To prove this let $u = v \in S^{2r-1}$, then

$$K_-(v) = \frac{2}{(1+\|u\|^2)^2}\big((1+\|u\|^2)\psi(u) - 2\langle u,\psi(u)\rangle u, 2\langle u,\psi(x)\rangle\big)$$

$$= \frac{2}{(1+\|u\|^2)^2}\big((1+\|u\|^2)u - 2\langle u,u\rangle u, 2\langle u,u\rangle\big)$$

$$= \frac{2}{(1+\|u\|^2)^2}\big((1+\|u\|^2)u - 2\|u\|^2 u, 2\|u\|^2\big)$$

$$= \frac{2}{(1+(1)^2)^2}\big((1+(1)^2)u - 2(1)^2 u, 2(1)^2\big)$$

$$= \frac{2}{(1+(1)^2)^2}(2(1)^2u - (1 + (1)^2)u, 2(1)^2)$$

$$= \frac{2}{(1+\|u\|^2)^2}(2\|u\|^2u - (1 + \|u\|^2)u, 2\|u\|^2).$$

$$= \frac{2}{(1+\|u\|^2)^2}(2\langle u, u\rangle u - (1 + \|u\|^2)u, 2\langle u, u\rangle)$$

$$= \frac{2}{(1+\|u\|^2)^2}(2\langle u, \varphi(u)\rangle u - (1 + \|u\|^2)\varphi(u), 2\langle u, \varphi(u)\rangle) = K_+(v).$$

By above equality and the continuity of K_- and K_+, we conclude that K is CM on S^{2r}. Also since non-vanishing and tangency to $S^{2r}$of K_- and K_+, K is directly non-vanishing and tangent to S^{2k}. This is a contradiction, by Theorem(3.3.5) , such a vector field K should not exist. Our initial assumption was false and Y has a FP for n even as required.

Step 4: Let $n = 2r - 1$ where $r \in IN$. We proceed by contradiction. Suppose there exists a CM $Y: B^{2r-1} \longrightarrow B^{2r-1}$ with no FP. Define $\psi: B^{2r} \longrightarrow B^{2r}$ by

$$\psi(u, u_{2r}) = (Y(u), 0).$$

We know ψ is continuous since Y is CM. By our proof of the FP for n even, ψ has a FP. Therefore for some $u \in B^{2r}$ we have

$$(u, u_{2r}) = \psi(u, u_{2r}) = (Y(u), 0),$$

that is $u = Y(u)$. This contradicts the first part of the proof, Y has no FPs. Our initial assumption was false and Y has a FP for n odd as required. ■

This section of the third chapter focuses on proving Brouwer's FPT. In addition, the proof of Brouwer's FPT relies on the main result of the previous Theorem (3.3.6), as well as three additional theorems that we present below.

Definition(3.3.5): A mapping $Y: U \rightarrow W$ between two TSs is called a homeomorphism if it has the following properties:

1. Y is a bijection (one-to-one and onto).

2. Y is a continuous.

3. The inverse function Y^{-1} is a continuous (Y is an open mapping).

Also U and W are called homeomorphic.

Theorem(3.3.7): Suppose U has the FP property and U is homeomorphic to W . Then W has the FP property.

Proof

Let $h : U \longrightarrow W$ be a homeomorphism and $\mu : W \longrightarrow W$ is CM. To prove W has the FP property, it suffices to show that μ has a FP. Define $\varphi : U \longrightarrow U$ by

$$\varphi(u) = h^{-1}(\mu(h(u))).$$

We know φ is a CM since μ, h and h^{-1} are all CMs by the assumption. Because U has the FP property, φ has a FP, that is

$$\exists\, u_0 \in U \text{ such that } h^{-1}\left(\mu\big(h(u_0)\big)\right) = u_0.$$

Applying h to both sides, we obtain

$$h\left(h^{-1}\left(\mu\big(h(u_0)\big)\right)\right) = h(u_0) \Rightarrow \mu\left(h(u_0)\right) = h(u_0).$$

where $h(u_0) \in W$. Subsequently, μ has a FP $h(u_0)$ as required. ■

Definition(3.3.6): Suppose a subset $U \subset B^n$ is said to be a retract of B^n if there exists a CM $\Psi: B^n \longrightarrow D$ (called retraction) such that $\Psi(u) = u$ for all $u \in U$.

Theorem(3.3.8): Every non-empty, closed and convex subset C of IR^n is a retract.

Proof

Define $Q_C : IR^n \longrightarrow C$ by the following: for all $u \in IR^n$

$$Q_C(u) = w \in C \text{ such that } \|u - w\| = \inf\{\|u - v\|: v \in C\}.$$

We aim to show Q_C is a non-expansive, that is

$$\forall\, u, w \in IR^n, \|Q_C(u) - Q_C(w)\| \leq \|u - w\|.$$

Let $u' = Q_C(u)$ and $w' = Q_C(w)$. Because $u', w' \in C$ and C is convex, we know

$$\forall\, t \in (0,1), (1-t)u' + tw' \in C.$$

By definition $\|u' - u\|$ is the minimum distance between u and any point in C. Thus

$$\|[(1-0)u' + (0)w'] - u\|^2 = \|u' - u\|^2 \leq \|[(1-t)u' + tw'] - u\|^2.$$

Note that $\|[(1-t)u' + tw'] - u\|^2$ increases at $t = 0$, because

$$\frac{d}{dt}\|(1-t)u' + tw' - u\|^2\Big|_{t=0}$$

$$= 2\left\langle (1-t)u' + tw' - u, \frac{d}{dt}\big((1-t)u' + tw' - u\big)\right\rangle\Big|_{t=0}$$

$$= 2\langle u' - u, w' - u'\rangle \geq 0. \qquad (3.3.7)$$

Similarly, because $\|w' - w\|^2 \leq \|[tu' + (1-t)w'] - w\|^2$ by the definition, we also have

$$\frac{d}{dt}\|[tu' + (1-t)w'] - w\|^2 = 2\langle w' - w, u' - w'\rangle \geq 0 \qquad (3.3.8)$$

Consider the function $d: (0,1) \longrightarrow IR$ defined by

$$d(t) = \|u' - w' + t[u - u' - (w - w)]\|^2.$$

It is clear that $d(t)$ is a quadratic polynomial with a non-negative coefficient for t^2. Its graph is thus an upwards-opening parabola. By (3.3.7) and (3.3.8) we have

$$\frac{d}{dt}\|u' - w' + t[u - u' - (w - w')]\|^2$$

$$= \frac{d}{dt}\langle u' - w' + t[u - u' - (w - w')], u' - w' + t[u - u' - (w - w')]\rangle$$

$$= \left\langle \frac{d}{dt}(u' - w' + t[u - u' - (w - w')]), u' - w' + t[u - u' - (w - w')]\right\rangle$$

$$+ \left\langle u' - w' + t[u - u' - (w - w')], \frac{d}{dt}(u' - w' + t[u - u' - (w - w')])\right\rangle$$

$$= \langle u - u' - (w - w'), u' - w' \rangle + \langle u' - w', u - u' - (w - w')\rangle$$

$$= \langle u - u' + (w' - w), u' - w' \rangle + \langle u' - w', u - u' + (w' - w)\rangle$$

$$= \langle u - u', u' - w' \rangle + \langle w' - w, u' - w' \rangle + \langle u' - w', u - u'\rangle + \langle u' - w', w' - w\rangle.$$

From symmetric of inner product, we find

$$\frac{d}{dt}\|u' - w + t[u - u' - (w - w')]\|^2 = 2\langle u - u', u' - w' \rangle + 2\langle w' - w, u' - w' \rangle$$

$$= 2\langle u' - u, w' - u' \rangle + 2\langle w' - w, u' - w' \rangle \geq 0.$$

This means that $d(t)$ is non-decreasing at 0. Because $d(t)$ is an upwards sloping parabola $d(t)$ must also be non-decreasing on the interval $[0, \infty)$. In particular,

$$\|u' - w'\|^2 = d(0) \leq d(1) = \|u - w\|^2.$$

This leads immediately

$$\|u' - w'\| \leq \|u - w\|$$

Hence Q_C is non-expansive, because

$$\forall\, u, w \in IR^n, \|Q_C(u) - Q_C(w)\| = \|u' - w'\| \leq \|u - w\|.$$

Theorem (3.2.2) (Browder-Kirk) decides that

$$\forall\, u \in C,\; Q_C(u) = u.$$

Thus Q_C is a retraction. Hence C is a retract as required. ■

Theorem(3.3.9): Suppose W has the FP property and $U \subseteq W$ is a retract. Then U has the FP property.

Proof

Consider $\Phi: W \longrightarrow U$ be a retraction and $Y: U \longrightarrow U$ be a CM. We need to show Y has a FP. Define $\eta: W \longrightarrow U$ by

$$\eta(x) = Y(\Phi(x)).$$

We know that retraction of Φ and continuity of Y by the assumption imply that η is continuous. Moreover, we have $\eta: W \longrightarrow W$ since $U \subseteq W$. By the FP property of W, η has a FP, that is there exists $x_0 \in W$ such that

$$\eta(x_0) = x_0. \tag{3.3.9}$$

But $\eta(x_0) \in U$ since $\eta: W \longrightarrow U$ by the definition, therefore $x_0 \in U$. Because Φ is a retraction and $x_0 \in U$ such that

$$\Phi(x_0) = x_0. \tag{3.3.10}$$

Combining (3.3.9) and (3.3.10) we have

$$Y(x_0) = Y\big(\Phi(x_0)\big) = \eta(x_0) = x_0.$$

Hence Y has a FP x_0 as required. ■

Theorem(3.3.10)(Brouwer's FPT): Every non-empty, bounded, closed and convex subset C of IR^n has the FP property.

Proof

Step1: Since C is bounded, then by Definition (2.1.5), there exists $M > 0$ such that for all $c \in C$, $\|c\| \leq M$. Therefore, C is contained within the closed ball of radius M in IR^n, denoted by B^*.

Step 2: By Theorem(3.3.6), we know B^n has the FP property. By Theorem(3.3.7) and the fact B^* and B^n are homeomorphic (consider the map $Y : B^* \longrightarrow B^n$ defined by $Y(c) = \frac{1}{M}c$), B^* has the FP property.

Step 3: Since C is non-empty, closed and convex subset of IR^n by the assumption, then by using Theorem (3.3.8), C is a retract.

Step 4: Since $C \subseteq B^*$, C is a retract and B^* has the FP property. Then by applying Theorem (3.3.9), C has the FP property as required. ■

3.4 Schauder's Fixed Point Theorem

Definition(3.4.1): Consider U and W be NSs. A map $Y: U \longrightarrow W$ is called <u>compact</u> if $Y(U)$ is contained in a compact subset of W. A compact map Y is called <u>finite dimensional</u>, if $Y(U)$ is contained in a finite dimensional linear subspace of W.

Definition(3.4.2): If $w_1, w_2, \cdots, w_n$ are vectors of a LS W. An expression

$$\sum_{i=1}^{n} a_i w_i$$

is called a <u>linear combination</u> of vectors $w_1, w_2, \cdots, w_n$,where the coefficients $a_1, a_2 \ldots, a_n$ are any scalars.

Definition(3.4.3): The linear combination is called convex combination, if $w_i \geq 0$ for all $i = \{1,2, \ldots, n\}$ and

$$\sum_{i=1}^{n} w_i = 1$$

lies in C. The convex hull of a set W consists of all convex combinations of W, it is denoted by $co(W)$ .That is,

$$co(W) = \{x: \exists\, x_i \in W, w_i \geq 0(1 \leq i \leq n), \sum_{i=1}^{n} w_i = 1 \text{ and } \sum_{i=1}^{n} w_i\, x_i = x\}.$$

Remark(3.4.1): The convex hull $co(W)$ be the smallest convex set is containing W and is the intersection of all convex sets that include W.

Definition(3.4.4): Let C is a convex subset of a NS W, $U = \{u_1, \ldots, u_n\} \subseteq C$ and for fixed $\varepsilon > 0$, let

$$U_\varepsilon = \bigcup_{i=1}^{n} B(u_i, \varepsilon),$$

where $B(u_i, \varepsilon) := \{x : \|x - u_i\| < \varepsilon\}$. For each $i = 1, \ldots, n$, suppose $\mu_i: U_\varepsilon \longrightarrow IR$ be the map given by

$$\mu_i(x) = \max\{0, \varepsilon - \|x - u_i\|\}.$$

Let $co(U)$ denote the smallest convex set containing U. The map $P_\varepsilon: U_\varepsilon \longrightarrow co(U)$ given by

$$P_\varepsilon(x) = \frac{\sum_{i=1}^{n} \mu_i(x) u_i}{\sum_{i=1}^{n} \mu_i(x)} \text{ for } x \in U_\varepsilon.$$

is called the <u>Schauder projection</u>.

Remark(3.4.2):

(i) $P_\varepsilon(x)$ is well defined since $x \in U_\varepsilon$ such that $x \in B(u_i, \varepsilon)$ for some $i \in \{1,2,\dots\}$. This implies

$$\mu_i(x) = \varepsilon - \|x - u_i\|$$

for $\forall\, i \in \{1,2,\dots\}$. Therefore

$$\sum_{i=1}^{n} \mu_i(x) \neq 0.$$

(ii) $P_\varepsilon(U_\varepsilon) \subseteq co(U)$. To see this let that $\mu(x) := \sum_{i=1}^{n} \mu_i(x)$, such that

$$P_\varepsilon(x) := \sum_{i=1}^{n} \frac{\mu_i(x)}{\mu(x)} u_i .$$

Notice that

$$0 \leq \frac{\mu_i(x)}{\mu(x)} \leq 1$$

for all $i = \{1,2,\dots,n\}$. Since $P_\varepsilon(x), u_i \in C$ and the convexity of C, $\sum_{i=1}^{n} \frac{\mu_i(x)}{\mu(x)} = 1$. Therefore, $P_\varepsilon(x)$ is a convex combination of the points $u_1, \dots, u_n$. Therefore, $P_\varepsilon(x)$ lies in $co(U)$ since $co(U)$ is convex hull of U.

Theorem(3.4.1): Let C be a convex subset of a NS X, and $U = \{u_1, \dots, u_n\} \subseteq C$ be a finite. Then

(i) $P_\varepsilon : U_\varepsilon \longrightarrow co(U) \subseteq C$ is a continuous, compact mapping.

(ii) $\|x - P_\varepsilon(x)\| < \varepsilon$ for $\forall\, x \in U_\varepsilon$.

Proof

(i) The continuity of P_ε is immediate, because for all $x \in U_\varepsilon$, $\sum_{i=1}^{n} \mu_i(x) \neq 0$ and $\sum_{i=1}^{n} \mu_i(x) u_i$, $\sum_{i=1}^{n} \mu_i(x)$ are CMs since they are linear. Now to show compactness of P_ε, we know that from properties of the compact that if U is finite, then it is instantly compact. Of course, the compactness of U guarantees the compactness of $co(U)$, which contains $P_\varepsilon(U_\varepsilon)$. By Definition(3.4.1), $P_\varepsilon(U_\varepsilon)$ be compact.

(ii) Notice that for $\forall\ x \in U_\varepsilon$,

$$\|x - P_\varepsilon(x)\| = \frac{1}{\mu(x)}\|\mu(x)x - \sum_{i=1}^{n}\mu_i(x)u_i\| = \frac{1}{\mu(x)}\|\sum_{i=1}^{n}\mu_i(x)\,x - \sum_{i=1}^{n}\mu_i(x)u_i\|$$

$$= \frac{1}{\mu(x)}\|\sum_{i=1}^{n}\mu_i(x)(\,x - u_i)\| \ \le \frac{1}{\mu(x)}\sum_{i=1}^{n}\mu_i(x)\,\|x - u_i\|$$

$$< \frac{1}{\mu(x)}\sum_{i=1}^{n}\mu_i(x)\varepsilon = \varepsilon$$

since $\mu_i(x) = 0$ unless $\|x - u_i\| < \varepsilon$. ■

Definition(3.4.5): A subset V of a MS W is said to be totally bounded if for all $\varepsilon > 0$ there exists a finite subset $\{u_1, \dots, u_n\} \subset U$ such that

$$V \subseteq \bigcup_{i=1}^{n} B_\varepsilon(u_i).$$

Any MS itself is totally bounded is said to be a totally bounded metric.

Remark(3.4.3): If U is a totally bounded, then

(i) Its closure is also

(ii) Any subset of U is also totally bounded.

The next result describes the relationships between total boundedness and compactness:

Theorem(3.4.2)[(3)]: For a MS the following are equivalent:

1. The space is compact.
2. The space is complete and totally bounded.
3. The space is sequentially compact (every sequence has a convergent subsequence).

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(3) Aliprantis, C. D., & Border, K. C. (1994). Infinite-dimensional analysis, volume 4 of Studies in Economic Theory. p 84.

Our next result is known as Schauder's approximation theorem.

Theorem(3.4.3): Consider C be a convex subset of a NS W and $Y: W \longrightarrow C$ be a compact and CM. Then for $\forall\ \varepsilon > 0$, there is a finite set $U = \{u_1, \dots, u_n\}$ in $Y(W)$ and a finite dimensional CM $Y_\varepsilon: W \longrightarrow C$ with the following properties:

(i) $\|Y_\varepsilon(w) - Y(w)\| < \varepsilon,\ \forall\ w \in W.$

(ii) $Y_\varepsilon(w) \in co(U) \subseteq C,\quad \forall\ w \in W.$

Proof

$Y(W)$ is contained in a compact $M \subseteq C$, that is

$$Y(W) \subseteq M \subset C \tag{3.4.1}$$

This obviously implies M is totally bounded by Theorem(3.4.2). Therefore, since K is totally bounded, there exists a finite set $\{u_1, \dots, u_n\} \subset Y(W)$ such that

$$M \subseteq \bigcup_{i=1}^{n} B_\varepsilon(u_1) = U_\varepsilon, \tag{3.4.2}$$

Thus, we obtain from $(3.4.1)$ and $(3.4.2)$,

$$Y(W) \subseteq U_\varepsilon \tag{3.4.3}$$

Let $P_\varepsilon: U_\varepsilon \longrightarrow co(U)$ be the Schauder projection and define the map $Y_\varepsilon: W \longrightarrow C$ by

$$Y_\varepsilon(w) \coloneqq P_\varepsilon\big(Y(w)\big),\ \forall\ w \in W\,.$$

Theorem (3.4.1) now guarantees the result, as follow

$$\|Y_\varepsilon(w) - Y(w)\| = \|P_\varepsilon(Y(w)) - Y(w)\| < \varepsilon\ \forall\ w \in W.$$

Since $w \in\ W$. Then by using $(3.4.3)$, $Y(w) \in U_\varepsilon$, which leads

$$P_\varepsilon(Y(w)) \in P_\varepsilon(U_\varepsilon) \subseteq co(U).$$

This straight away is that

$$Y_\varepsilon(w) \in co(U),\ \forall\ w \in W. \qquad \blacksquare$$

Before we state and prove SFPT we first introduce the notion of an ε −FP. Let C be a subset of a NS W and $Y: C \longrightarrow W$ is a map. Given $\varepsilon > 0$, a point $c \in C$ with $\|c - Y(c)\| < \varepsilon$ is called an ε −FP for Y.

Theorem(3.4.4): Let C be a closed subset of a NS W and $Y: C \longrightarrow W$ a compact, CM. Then Y has a FP, if Y has a ε −FP.

Proof

In the first, let Y has an ε −FP. Now $\forall\, n \in \{1,2,\dots\}$, let c_n be a $\frac{1}{n}$ − FP for Y, that is,

$$\|u_n - Y(u_n)\| < \frac{1}{n} \qquad (3.4.4)$$

Since Y is compact, $Y(C)$ is contained in a compact subset $U \subseteq W$. Therefore, there exists a convergent subsequence $Y(u_{n_k})$ of $Y(u_n)$ such that

$$Y(u_{n_k}) \longrightarrow u \in U \text{ as } n_k \to \infty.$$

Now by using (3.4.4), we find

$$\|u_{n_k} - u\| \le \|u_{n_k} - Y(u_{n_k})\| + \|Y(u_{n_k}) - u\| < \frac{1}{n_k} + \|Y(u_{n_k}) - u\| \xrightarrow{n_k \to \infty} 0$$

We have that $u \in C$, since C is closed, also the continuity of Y implies that

$$Y(u_{n_k}) \longrightarrow Y(u) \text{ as } n_k \to \infty.$$

The uniqueness of the limit of $Y(c_{n_k})$ yields, $u = Y(u)$. ■

Now we ripe to state and prove SFPT.

Theorem(3.4.5)(SFPT): Let $K \neq \emptyset$ be a closed and convex subset of a NS W. Then every compact and CM $Y: K \longrightarrow K$ has at least one FP.

Proof

By Theorem(3.4.4) with $C = K$, it enough to show that Y has an ε −FP for all $\varepsilon > 0$. Fix $\varepsilon > 0$. Theorem (3.4.3) guarantees the existence of finite set

$$A = \{a_1, \ldots, a_n\}$$

in $Y(K)$ and finite dimensional, CM $Y_\varepsilon: K \longrightarrow K$ with

$$\|Y_\varepsilon(x) - Y(x)\| < \varepsilon \text{ for all } x \in K, \quad (3.4.5)$$

and $Y_\varepsilon(K) \subseteq co(A) \subseteq K$ for some finite set $A \subseteq K$. Since $co(A)$ is closed and bounded and $Y_\varepsilon(co(A)) \subseteq co(A)$, we may apply Theorem(3.3.10) (Brouwer's FPT) to deduce that there exists $x_\varepsilon \in co(A)$ with $x_\varepsilon = Y_\varepsilon(x_\varepsilon)$. Also, (3.4.5) yields

$$\|x_\varepsilon - Y(x_\varepsilon)\| = \|Y_\varepsilon(x_\varepsilon) - Y(x_\varepsilon)\| < \varepsilon.$$

This implies $x_\varepsilon = Y(x_\varepsilon)$ as required. ■

Theorem[Krasnoselskii](3.4.6): Consider C be non-empty, closed and convex subset of a BS W. Let $f, g: C \longrightarrow W$ be such that

(i) $f(x) + g(y) \in C, \ \forall\, x, y \in C$.

(ii) f is continuous and compact.

(iii) g is a contraction with k is the Lipschitz constant.

Then there is $\bar{x} \in C$ such that $f(\bar{x}) + g(\bar{x}) = \bar{x}$.

Proof

Notice first that $I - g$ maps homeomorphically C onto $(I - g)(C)$. Because it

1. A bijection, indeed

Clearly that $I - g: C \longrightarrow (I - g)(C)$ is onto and also it is one to one, because if there exist $x, y \in C$, $x \neq y$ with $(I - g)(x) = (I - g)(y)$. Then

$$0 = \|(I - g)(x) - (I - g)(y)\| \geq \|x - y\| - \|g(x) - g(y)\| \geq (1 - k)\|x - y\|$$

This implies $1 - k \leq 0 \Longrightarrow k \geq 1$, which contradicts k is the Lipschitz constant of g.

2. $I - g$ is continuous because I, g are CMs.

3. $(I - g)^{-1}$ is continuous. Indeed, first one to one of $I - g$ guarantees the existence of $(I - g)^{-1}$. Second, the continuity of $(I - g)^{-1}$ will be proved if we can show that

if (x_n) converges to x whenever (x_n) is a sequence in C and x is an element in C such that,

$$((I-g)^{-1})(x_n) \text{ converges to } (I-g)^{-1}(x).$$

Assume $y_n = ((I-g)^{-1})(x_n)$ and $y = ((I-g)^{-1})(x)$, hence $(I-g)(y_n) = x_n$ and $(I-g)(y) = x$. Suppose that $y_n \nrightarrow y$. Then there exists an $\varepsilon_0 > 0$ and a subsequence (y_{n_k}) of (y_n) such that

$$\lim_{n_k\to\infty} \|y_{n_k} - y\| > 0 \tag{3.4.6}$$

Observe,

$$\|x_{n_k} - x\| = \|(I-g)(y_{n_k}) - (I-g)(y)\| \geq \|y_{n_k} - y\| - \|g(y_{n_k}) - g(y)\|$$

$$\geq \|y_{n_k} - y\| - k\|y_{n_k} - y\| = (1-k)\|y_{n_k} - y\|$$

Take limit on both sides,

$$(1-k)\lim_{n_k\to\infty} \|y_{n_k} - y\| \leq 0.$$

Since k is a contraction constant, $\lim_{n_k\to\infty} \|y_{n_k} - y\| \leq 0$, which contradicts (3.4.6). Now for any $y \in C$, the map

$$x \longmapsto f(y) + g(x)$$

be a contraction. To see this let $x_1, x_2 \in C$,

$$\|f(y) + g(x_1) - f(y) - g(x_2)\| = \|g(x_1) - g(x_2)\| \leq k\,\|x_1 - x_2\|.$$

Hence by Theorem(3.1.1)(BCP) there is a unique $z = z(y) \in C$ such that

$$z = f(y) + g(z) \Longrightarrow z - g(z) = f(y) \Longrightarrow I(z) - g(z) = f(y) \Longrightarrow (I-g)(z) = f(y).$$

Thus

$$z = (I-g)^{-1}\big(f(y)\big) \in C.$$

On the other hand, the map $(I-g)^{-1} \circ f: C \longrightarrow C$ is CM and also by using Theorem[4] is compact, being the composition of a continuous map with a continuous and compact map. Then Theorem (3.4.5)(SFPT) entails the existence of $\overline{x} \in C$ such that

$$(I-g)^{-1}\big(f(\overline{x})\big) = \overline{x} \Longrightarrow f(\overline{x}) + g(\overline{x}) = \overline{x}. \qquad \blacksquare$$

(4) Shirali, S., & Vasudeva, H. L. (2005). *Metric spaces.* Springer Science & Business Media. p 182.

Chapter 4

APPLICATIONS OF FIXED POINT THEOREMS

4.1 The First and Second Order of Ordinary Differential Equations in BS

4.1.1 The First Order of Ordinary Differential Equations in Banach Space

It is natural to begin the applications of FP methods with existence and uniqueness of solutions of first order initial value problems as:

$$\begin{cases} z'(s) = \mu\big(s, z(s)\big), s \in [0, \alpha] \\ z(0) = z_0 \end{cases} \qquad (4.1.1.1)$$

where $\mu : [0, \alpha] \times IR \longrightarrow IR$ is CM,

$C[0, \alpha] = \left\{ q: [0, \alpha] \longrightarrow IR^n : q \text{ is CM on } [0, \alpha],\ |q|_0 = \max_{t \in [0,b]} |q(s)| \right\}$ and

$C^1[0, \alpha] = \{q \in C[0, \alpha]: q' \text{ exists and } q' \in C[0, \alpha], |q|_1 = \max\{|q|_0, |q'|_0\}\}$ are BSs.

Then, $z \in C^1[0, \alpha]$ solves $(4.1.1.1)$ if and only if $z \in C[0, b]$ solves

$$z(s) = z_0 + \int_0^s \mu\big(r, z(r)\big)dr. \qquad (4.1.1.2)$$

The operator $Y: C[0, \alpha] \longrightarrow C[0, \alpha]$ is defined by

$$Y\big(z(s)\big) = z_0 + \int_0^s \mu\big(r, z(r)\big)dr.$$

The classical solutions to $(4.1.1.1)$ are FPs of Y, that is: $Y(z) = z$.

Theorem (Picard–Lindel)(4.1.1.1): Assume that

(i) $\mu : [0, \alpha] \times IR^n \longrightarrow IR^n$ be a CM.

(ii) μ subscribes Lipschitz condition with respect to z, that is, there exists $k \geq 0$ such that

$$|\mu(s,z) - \mu(s,w)| \leq k|z - w|, \ \forall z, w \in IR^n.$$

Then $z \in IR^n$ be unique solution of (4.1.1.1).

Proof

At first glance we define a new norm in $C[0,\alpha]$ as follows:

$$\|z\|_k = \max_{s\in[0,\alpha]} e^{-ks}|z(s)|.$$

It seems natural that $\|.\|_k \approx \|.\|_\infty$. To see that to tread the following, since $0 \leq s \leq \alpha$

$\Rightarrow e^{-k\alpha} \leq e^{-ks} \leq 1 \Rightarrow e^{-k\alpha}|z(s)| \leq e^{-ks}|z(s)| \leq |z(s)|$

$\Rightarrow e^{-k\alpha} \max_{s\in[0,\alpha]}|z(s)| \leq \max_{s\in[0,\alpha]} e^{-ks}|z(s)| \leq \max_{s\in[0,\alpha]}|z(s)|$

$\Rightarrow e^{-k\alpha}\|z\|_\infty \leq \|z\|_k \leq \|z\|_\infty.$

Now define

$$Y\big(z(s)\big) = z_0 + \int_0^s \mu\big(r, z(r)\big)dr.$$

We now apply BCP to show Y has a unique FP in BS $(C[0,\alpha], \|z\|_k)$. Therefore, we will illustrate that Y is a contraction on $(C[0,\alpha], \|z\|_k)$. To see this take $u, v \in C[0,\alpha]$,

$\big|Y\big(u(s)\big) - Y\big(v(s)\big)\big| \leq \int_0^s\big|\mu\big(r,u(r)\big) - \mu\big(r,v(r)\big)\big|dr$

$$\begin{aligned}
\Rightarrow e^{-ks}\big|Y\big(u(s)\big) - Y\big(v(s)\big)\big| &\leq e^{-ks}\int_0^s\big|\mu\big(r,u(r)\big) - \mu\big(r,v(r)\big)\big|dr \\
&\leq ke^{-ks}\int_0^s|y(s) - z(s)|ds \\
&= ke^{-ks}\int_0^s e^{ks}e^{-ks}|u(s) - v(s)|ds \\
&\leq ke^{-ks}\int_0^s e^{kr}dr\ \|u - v\|_k \\
&= e^{-ks}\big(e^{ks} - 1\big)\|u - v\|_k = \big(1 - e^{-ks}\big)\|u - v\|_k.
\end{aligned}$$

Take maximum on both sides

$$\|Y(u) - Y(v)\|_k \leq \big(1 - e^{-k\alpha}\big)\|u - v\|_k.$$

Since $0 \leq 1 - e^{-k\alpha} < 1$, Y is contraction on CMS $(C[0,\alpha], \|u\|_k)$. BCP guarantees the existence and uniqueness solution $u \in C[0,\alpha]$ of Y, equivalently $u \in C^1[0,\alpha]$ is a unique solution of $(4.1.1.1)$. ■

Our first result concerns continuous and compact maps.

Lemma(Urysohn)(4.1.1.1): If V and W are disjoint $(V \cap W = \emptyset)$ closed subsets of a NS U, then there exists a CM $\eta: U \longrightarrow [0,1]$ such that $\forall\, v \in V, \eta(v) = 0$ and $\forall\, w \in W$, $\eta(w) = 1$.

Theorem(4.1.1.2)(Nonlinear Alternatives of Leray-Schauder Typy): Suppose V is a closed, convex subset of BS U, W an open subset of V and $p \in W$. Consider that $Y: \overline{W} \longrightarrow V$ is a continuous, compact (that is, $Y(\overline{U})$ is a RC subset of V) map. Then either

(i) Y has a FP in $\overline{W}$, or

(ii) $\exists\, w \in \partial W$ (the boundary of W in V) and $\beta \in [0,1]$ with $w = \beta Y(w) + (1-\beta)p$.

Proof

Let (ii) cannot be realized. Thus $w \neq \beta Y(w) + (1-\beta)p$ for $w \in \partial W$ and $\beta \in [0,1]$ and also Y has no FPs on ∂W. Define

$$H = \{h \in \overline{W}: h = tY(h) + (1-t)p \text{ for some } t \in [0,1]\}.$$

Clearly, that $H \neq \emptyset$ since $p \in W$. In addition, the continuity of Y insure the closeness of H. To check this, let $h_n \in H$ and $h_n \longrightarrow h$ and take the limit on both sides of

$$h_n = tY(h_n) + (1-t)p,$$

hence

$$h = tY(h) + (1-t)p.$$

Therefore, $h \in H$.

Notice that $H \cap \partial W = \emptyset$. Therefore, by Lemma(4.1.1.1) there is a CM $\eta: \overline{W} \longrightarrow [0,1]$ with $\eta(H) = 1$ and $\eta(\partial W) = 0$. Let

$$M(h) = \begin{cases} \eta(h)Y(h) + (1-\eta(h))p, h \in \overline{W} \\ p \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad, h \in V/W \end{cases}$$

Now it is immediate that $M: V \longrightarrow V$ is a continuous, compact map. To see compactness use Mazur's theorem together with $M(V) \subseteq \overline{co}(Y(\overline{W}) \cup \{p\})$. SFPT proves the existence of $h \in V$ with $h = M(h)$. Notice that $h \in W$ since $p \in W$. Hence

$$h = \eta(h)Y(h) + (1-\eta(h))p.$$

This means that $h \in H$. Therefore, $h = Y(h)$ since $\eta(h) = 1$. ■

Definition(4.1.1.1): Let $1 \leq \alpha \leq \infty$ and a constant β are satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Assume the following hold:

(i) $h \in C[0,1]$.

(ii) $\mu: [0,1] \times IR \longrightarrow IR$ is an L^{β}-Caratheodory function, by this we mean

(a) The map $t \longmapsto \mu(t,z)$ is measurable for all $z \in IR$, such that $\int_0^1 |\mu(t)|^{\beta}\, dt < \infty$,

$$\|\mu\|_{\beta} = \left(\int_0^1 |\mu(t)|^{\beta} dt\right)^{\frac{1}{\beta}}.$$

(b) The map $z \longmapsto \mu(s,z)$ is CM for nearly all $s \in [0,1]$.

(c) $\forall\, \lambda > 0$, there exists $\varphi_{\lambda} \in L^{\beta}[0,1]$ such that $|z| \leq \lambda$ implies that $|\mu(s,z)| \leq \varphi_{\lambda}(s)$ for nearly all $s \in [0,1]$.

(iii) $P_s(r) = P(s,r) \in L^{\alpha}[0,1]$,$\forall\, s \in [0,1]$.

(iv) The map $s \longmapsto P_s$ is CM from $[0,1]$ to $L^{\alpha}[0,1]$.

The equation

$$z(s) = h(s) + \int_0^1 P(s,r)\mu(r,z(r))ds, \forall s \in [0,1] \quad\quad (4.1.1.3)$$

is called the Fredholm integral equation.

Theorem(4.1.1.3): Suppose there is a constant $\alpha > 0$ is an independent of β, with $|z|_0 = \sup_{s\in[0,1]} |z(s)| \neq \alpha$ for any solution $z \in C[0,1]$ of

$$z(s) = \beta\left(h(s) + \int_0^1 k(s,r)\mu(r,y(r))dr\right), \qquad (4.1.1.4)$$

$s \in [0,1]$ and for each $\beta \in (0,1)$. Then the Fredholm integral equation has at least one solution $z \in C[0,1]$.

Proof

Step 1: Define the operator T by

$$T(z(s)) := h(s) + \int_0^1 P_s(r)\mu(r,z(r))dr \ \forall s \in [0,1]$$

Notice that $T: C[0,1] \longrightarrow C[0,1]$. To realize this take any $z \in C[0,1]$, then this guarantees continuity of y. There is $\lambda > 0$ such that $|z|_0 \leq \lambda$ and since μ is L^β-Caratheodory, there exists $\varphi_\lambda \in L^\beta[0,1]$ with $|\mu(r,z)| \leq \varphi_\lambda(r)$ for almost every $r \in [0,1]$. Therefore, for any $s_1,\ s_2 \in [0,1]$, we see that

$$|T(z(s_1)) - T(z(s_2))| \leq |\ h(s_1) - h(s_2)| + \int_0^1 |P_{s_1}(r) - k_{s_2}(r)||\mu(r,z(r))|dr$$

$$\leq |\ h(s_1) - h(s_2)| + \int_0^1 |P_{s_1}(r) - P_{s_2}(r)|\varphi_\lambda(r)dr$$

$$\leq |\ h(s_1) - h(s_2)| \ + \left(\int_0^1 |P_{s_1}(r) - P_{s_2}(r)|^\alpha dr\right)^{\frac{1}{\alpha}} \left(\int_0^1 \left(\varphi_{\lambda_r}(r)\right)^\beta dr\right)^{\frac{1}{\beta}}$$

$$= |\ h(s_1) - h(s_2)| + \left(\int_0^1 |(P_{s_1} - P_{s_2})(r)|^\alpha dr\right)^{\frac{1}{\alpha}} \left(\int_0^1 (\varphi_\lambda(r))^\beta dr\right)^{\frac{1}{\beta}}$$

$= |\ h(s_1) - h(s_2)| + \|P_{s_1} - P_{s_2}\|_\alpha \|\varphi_\lambda\|_\beta$. Therefore,

$$|T(z(s_1)) - T(z(s_2))| \xrightarrow{s_1 \longrightarrow s_2} 0\ . \qquad (4.1.1.5)$$

Consequently, this means that $T(z) \in C[0,1]$. Now we will apply Theorem (4.1.1.2) with $W \coloneqq \{z \in C[0,1] : |z|_0 < \alpha\}$ and $V = U = C[0,1]$.

First we show that $T: \overline{W} \longrightarrow C[0,1]$ is CM. Let $z_n \longrightarrow z$ in $C[0,1]$ with $\{z_n\}_{n=1}^\infty \subseteq \overline{W}$. We are required to show that $T(z_n) \longrightarrow T(z)$ in $C[0,1]$. There exists $\varphi_\alpha \in L^\beta[0,1]$

with $|z_n|_0 \leq \alpha$, $|z|_0 \leq \alpha$ and $|\mu(r, z_n(r))| \leq \varphi_\alpha(r), |\mu(r, z(r))| \leq \varphi_\alpha(r)$ for every $r \in [0,1]$, $n = 1,2, \ldots$

By an argument similar to the one used to obtain (4.1.1.5),

$$|T(z_n(s)) - T(z(s))| \leq \int_0^1 |P_s(r)| |\mu(r, z_n(r)) - \mu(r, z(r))| dr$$

$$= \left(\int_0^1 |P_s(r)|^\alpha dr\right)^{\frac{1}{\alpha}} \left(\int_0^1 |\mu(r, z_n(r)) - \mu(r, z(r))|^\beta dr\right)^{\frac{1}{\beta}}$$

$$= \|P_s\|_\alpha \left(\int_0^1 |\mu(r, z_n(r)) - \mu(r, z(r))|^\beta dr\right)^{\frac{1}{\beta}}$$

$$\leq \left(\sup_{s\in[0,1]} \|P_s\|_\alpha\right) \left(\int_0^1 |\mu(r, z_n(r)) - \mu(r, z(r))|^\beta dr\right)^{\frac{1}{\beta}}.$$

So $|T(z_n(s)) - T(z(s))| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus $T(z_n) \longrightarrow T(z)$ as $n \longrightarrow \infty$. Therefore, $T: \overline{W} \longrightarrow C[0,1]$ is CM.

Step 2: We will illustrate $T: \overline{W} \longrightarrow C[0,1]$ is compact. There is $\varphi_\alpha \in L^\beta[0,1]$ such that

$$|\mu(r, z(r))| \leq \varphi_\alpha(r)$$

for almost every $r \in [0,1]$ and $z \in \overline{W}$.

Since we are working in $C[0,1]$, we can use A-AT to prove compactness. Clearly $T(\overline{W})$ is a uniformly bounded since

$$T(z(s)) = h(s) + \int_0^1 P_s(r)\mu(r, z(r))\, dr, \forall\, z \in \overline{W}.$$

Subsequently,

$$|T(z(s))| \leq |h(s)| + \int_0^1 |P_s(r)| |\mu(r, z(r))| dr$$

$$\leq |h(s)| + \left(\int_0^1 |P_s(r)|^\alpha dr\right)^{\frac{1}{\alpha}} \left(\int_0^1 |\mu(r, z(r))|^\beta dr\right)^{\frac{1}{\beta}}$$

$$\leq |h(s)| + \left(\int_0^1 |P_s(r)|^\alpha dr\right)^{\frac{1}{\alpha}} \left(\int_0^1 (\varphi_\alpha(r))^\beta dr\right)^{\frac{1}{\beta}} \leq |h(s)| + \|P_s\|_p \|\varphi_\alpha\|_\beta.$$

Take sup on both sides where $0 \leq s \leq 1$.

$$|T(z)|_0 \leq |h|_0 + \left(\sup_{0\leq s\leq 1} \|P_s\|_\alpha\right) \|\varphi_\alpha\|_\beta,\ \forall\, z \in \overline{W}.$$

Using the similar argument to the one used to obtain $(4.1.1.5)$, one can see that $T(\overline{W})$ is equi-continuous since uniform continuity of T in step1. It follows from A-AT that $T(\overline{W})$ be RC. Therefore, $T: \overline{W} \longrightarrow C[0,1]$ is a compact mapping.

Step 3: We may now apply Theorem(4.1.1.2) (notice that possibility (ii) cannot occur) to deduce that T has a FP in $\overline{W}$, or equivalently, $(4.1.1.3)$ has a solution in $\overline{W}$. ■

4.1.2 The Second Order of Ordinary Differential Equations in Banach Space

To illustrate how Theorem (4.1.1.3) can be applied in practice we turn our attention to the second order homogeneous Dirichlet problem,

$$\begin{cases} z'' = f(s, z, z'), \forall s \in [a, b] \\ z(a) = z(b) = 0 \end{cases} \tag{4.1.2.1}$$

where $f: [a, b] \times IR^2 \longrightarrow IR$ is a CM. Associated with $(4.1.1.6)$, we consider the following related family of problems:

$$\begin{cases} z'' = \lambda\, f(s, z, z'),\ \forall\, s \in [a, b],\ \lambda \in (0,1) \\ z(a) = z(b) = 0 \end{cases} \tag{4.1.2.2}$$

Now the integration on both sides of $(4.1.2.1)$ respect to s on $[a, \lambda]$ implies

$$z'(\lambda) - z'(a) = \int_a^\lambda f(s)ds.$$

Since $r \in [a, \lambda]$ we can change s by r,

$$z'(\lambda) - z'(a) = \int_a^\lambda f(r)dr.$$

After that the integration respect to λ on $[a, s]$

$$z(s) - z(a) - z'(a)(s - a) = \int_a^s \int_a^\lambda f(r)\, dr\, d\lambda$$

$$\Longrightarrow z(s) = z'(a)(s - a) + \int_a^s \int_a^\lambda f(r)\, dr\, d\lambda \tag{4.1.2.3}$$

Since $s = b$,

$$z'(a) = -\frac{1}{b-a} \int_a^b \int_a^\lambda f(r)drd\lambda \tag{4.1.2.4}$$

Substitute the $(4.1.2.4)$ to $(4.1.2.3)$, to see

$$z(s) = \int_a^s \int_a^\lambda f(r)drd\lambda \ - \frac{s-a}{b-a} \int_a^b \int_a^\lambda f(r)drd\lambda$$

$$= \int_a^s \int_r^s f(r)d\lambda dr \ - \frac{s-a}{b-a} \int_a^b \int_r^b f(r)d\lambda dr$$

$$= \int_a^s (s-r)f(r)dr - \frac{s-a}{b-a} \int_a^b (b-r)f(r)dr$$

$$= \int_a^s \left(-\frac{(b-s)(r-a)}{b-a}\right) f(r)dr + \int_s^b \left(-\frac{(b-r)(s-a)}{b-a}\right) f(r)dr.$$

Consider the operator $Y: C^1[a,b] \longrightarrow C^1[a,b]$ is defined as follows

$$Y(z(s)) := \int_a^b \psi(s,r) f(r, z(r), z'(r)) dr$$

where the Green's function $\psi(s,r)$ is given by

$$\psi(s,r) =: \begin{cases} -\frac{(b-s)(r-a)}{b-a} & , a \le r \le s \le b \\ -\frac{(b-r)(s-a)}{b-a} & , a \le s \le r \le b \end{cases}$$

Lemma(4.1.2.1): Assume $Z \subseteq IR^2$ such that $f: [a,b] \times Z \longrightarrow IR$ is Lipschitz function. Let that f satisfies the following local Lipschitz condition, there exist $k_1, k_2 \in IR^+$ such that

$$|f(s, z_1, z_1') - f(t, z_2, z_2')| \le k_1|z_1 - z_2| + k_2|z_1' - z_2'|, \qquad (4.1.2.5)$$

for all $(z_1, z_1'), (z_2, z_2') \in Z$, hence

$$\|Y(z_1) - Y(z_2)\| \le \left(k_1 \ \frac{(b-a)^2}{8} + k_2 \frac{b-a}{2}\right) \|z_1 - z_2\|,$$

where $\|z\| = k_1\|z\|_\infty + k_2\|z'\|_\infty$ such that

$$\|z\|_\infty = \max_{s\in[a,b]} |z(s)| \text{ and } \|z'\|_\infty = \max_{s\in[a,b]} |z'(s)|.$$

Proof

Take $z_1, z_2 \in C^1[a,b]$, hence

$$\left|Y(z_1(s)) - Y(z_2(s))\right| \le \int_a^b |\psi(s,r)| \left|f(r, z_1(r), z_1'(r)) - f(r, z_2(r), z_2'(r))\right| dr$$

$$\le \int_a^b |\psi(s,r)| (k_1|z_1(r) - z_2(r)| + k_2|z_1'(r) - z_2'(r)|) dr$$

$$\le \int_a^b |\psi(s,r)| \left(k_1 \max_{r\in[a,b]} |z_1(r) - z_2(r)| + k_2 \max_{r\in[a,b]} |z_1'(r) - z_2'(r)|\right) dr$$

$$= \int_a^b |\psi(s,r)| dr\, (k_1 \|z_1 - z_2\|_\infty + k_2 \|z_1' - z_2'\|_\infty) = \int_a^b |\psi(s,r)| dr\, \|z_1 - z_2\|.$$

Take max on both sides where $s \in [a,b]$,

$$\|Y(z_1) - Y(z_2)\|_\infty \le \max_{s\in[a,b]} \int_a^b |\psi(s,r)| dr\, \|z_1 - z_2\|.$$

Now we find,

$$\int_a^b |\psi(s,r)| dr = \int_a^s |\psi(s,r)| dr + \int_s^b |\psi(s,r)| dr$$

$$= \frac{(b-s)}{b-a} \int_a^s (r-a) dr + \frac{(s-a)}{b-a} \int_s^b (b-r) dr.$$

$$= \frac{(b-s)}{2(b-a)} (s^2 - 2as + a^2) + \frac{(s-a)}{2(b-a)} (b^2 - 2bs + s^2)$$

$$= \frac{(b-s)(s-a)^2}{2(b-a)} + \frac{(s-a)(b-s)^2}{2(b-a)} = \frac{(b-s)(s-a)}{2(b-a)} (s - a + b - s) = \frac{(b-s)(s-a)}{2}.$$

Thus,

$$\max_{s\in[a,b]} \int_a^b |\psi(s,r)| dr = \frac{1}{2} \max_{s\in[a,b]} \big((b-s)(s-a)\big).$$

Let $h(s) = (b-s)(s-a)$. Using the second derivative test to determine the maximum value of $h(s)$, as follows:

(1) $h'(s) = b - 2s + a$.

(2) Let $h'(s) = 0 \Longrightarrow t = \frac{b+a}{2}$.

(3) $h''(s) = -2$.

(4) $h''\left(\frac{b+a}{2}\right) = -2$.

Since $h''\left(\frac{b+a}{2}\right) < 0$, $h(s)$ has maximum value at $\frac{b+a}{2}$ .Hence

$$\max_{s\in[a,b]} \int_a^b |\psi(s,r)| dr = \frac{1}{2}\left(b - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - a\right) = \frac{1}{2}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{2}\right) = \frac{(b-a)^2}{8} \qquad (4.1.2.6)$$

Thus $\|Y(z_1) - Y(z_2)\|_\infty \le \frac{(b-a)^2}{8} \|z_1 - z_2\|_\infty$. On the other hand,

$$Y'(z(s)) = \int_a^b \frac{\partial}{\partial r}\ |\psi(s,r)|\mu\big(r, z(r), z'(r)\big)dr$$

$$\Longrightarrow \left|Y'\big(z_1(s)\big) - Y\big(z_2(s)\big)\right| \le \int_a^b \left|\frac{\partial}{\partial s}\psi(s,r)\right| dr\|z_1 - z_2\|.$$

$$\int_a^b \left|\frac{\partial}{\partial s}\psi(s,r)\right| dr = \int_a^s \left|\frac{\partial}{\partial s}\psi(s,r)\right| dr + \int_s^b \left|\frac{\partial}{\partial s}\psi(s,r)\right| dr$$

$$= \frac{1}{b-a}\left(\int_a^s \left|\frac{\partial}{\partial s}(b-s)(r-a)\right| dr + \int_s^b \left|\frac{\partial}{\partial s}(b-r)(s-a)\right| dr\right)$$

$$= \frac{1}{b-a}\left(\int_a^s (r-a)dr + \int_s^b (b-r)dr\right)$$

$$= \frac{1}{2(b-a)}(s^2 - 2as + a^2 + b^2 - 2bs + s^2) = \frac{1}{2(b-a)}((s-a)^2 + (b-s)^2).$$

By an argument similar to the one used to derive $(4.1.2.6)$, we obtain

$$\max_{s\in[a,b]} \int_a^b \left|\frac{\partial}{\partial s}\psi(s,r)\right| dr = \frac{b-a}{4}.$$

Therefore,

$$\|Y'(z_1) - Y'(z_2)\|_\infty \le \frac{b-a}{4}\ \|z_1 - z_2\|_\infty.$$

Now since

$$\|Y(z_1) - Y(z_2)\| = k_1\|Y(z_1) - Y(z_2)\|_\infty + k_2\|Y'(z_1) - Y'(z_2)\|_\infty$$

$$\le \left(k_1\frac{(b-a)^2}{8} + k_2\frac{b-a}{4}\right)\|z_1 - z_2\|_\infty. \qquad (4.1.2.7) \blacksquare$$

Theorem(4.1.2.1): Consider $f: [a,b] \times Z \longrightarrow IR$ is a CM and satisfies $(4.1.2.2)$ in a set U with constants k_1 and k_2 such that

$$k_1\frac{(b-a)^2}{8} + k_2\frac{b-a}{4} < 1 \qquad (4.1.2.8)$$

is holds. There exists a bounded open set of functions $W \subseteq C^1[a,b]$ with $0 \in W$ such that $z \in W$ implies $(z(s), z'(s)) \in Z$ for all $s \in [a,b]$ and z solves $(4.1.2.2)$ for some $\lambda \in (0,1)$ leads $z \notin \partial W$. Thereafter, $(4.1.2.1)$ has a unique solution in $\overline{W}$.

Proof

Obviously, $Y: \overline{W} \longrightarrow C[a,b]$ is contraction by $(4.1.2.7)$ and $(4.1.2.8)$. Note that (ii) in Theorem$(4.1.1.2)$ cannot occur because of z solves $(4.1.2.2)$ for some $\lambda \in (0,1)$ implies $z \notin \partial W$. Hence by apply Theorems$(3.1.1)$(BCP) and $(4.1.1.2)$ Y has just one FP in $\overline{W}$, which is a unique solution of $(4.1.2.1)$ in $\overline{W}$. ■

Remark(4.1.2.1): In many important applications, the function f is independent of z', that is $f = f(s,z)$. In this case, a straightforward review of the reasoning given above shows that we can regard Y as

$$Y: C[a,b] \longrightarrow C[a,b] \ .$$

This leads to a useful variant of Theorem (4.1.2.1) in which $A \subseteq IR$, all reference to y and z is dropped in $(4.1.2.2)$ and $U \subseteq C[a,b]$.

Example(4.1.2.1): The BVP

$$\begin{cases} z''(s) = -e^{z(s)}, s \in [0,1] \\ \quad z(0) = z(1) = 0 \end{cases} \qquad (4.1.2.9)$$

possesses a unique solution with maximum norm at most 1.

To show that apply Theorem(4.1.2.1) and Remark(4.1.2.1) with $f = f(s,z) = -e^{z(s)}$. By the mean value theorem we get that $|z| \le 1$ and $|z'| \le 1$ imply there exists w, that lies between z, z' such that

$$|f(s,z) - f(s,z')| = \left|e^{z} - e^{z'}\right| = e^{w}|z - z'| \le e^{\max\{z,z'\}}|z - z'| \le e|z - z'|.$$

This means that $k_1 = e$. We take $U = [-1,1]$ and

$$W = \left\{z \in C[0,1]: |z|_0 = \max_{s\in[0,1]} |z(s)| < 1 \right\}$$

in Theorem (4.1.1.2). Then

$$k_1 \frac{(b-a)^2}{8} = \frac{e}{8} < \ 1.$$

Consider that u solves

$$\begin{cases} z''(s) = -\lambda e^{z(s)}, s \in [0,1] \\ z(0) = z(1) = 0 \end{cases} \qquad (4.1.2.10)$$

for some $\lambda \in (0,1)$.

Now integrating on both sides of (4.1.2.10) respect to s on $[0, p]$ implies that

$$z'(p) - z'(0) = -\lambda \int_0^p e^{z(s)} ds.$$

Since $t \in [0, p]$ we can change s by t,

$$z'(p) - z'(0) = -\lambda \int_0^p e^{z(t)} dt.$$

After that integrating with respect to p on $[0, r]$ yields that

$$z(r) - z(0) - rz'(0) = -\lambda \int_0^r \int_0^p e^{z(t)} dtdp$$

$$\Longrightarrow z(r) = rz'(0) - \lambda \int_0^r \int_0^p e^{z(t)} dtdp \qquad (4.1.2.11)$$

Since $r = 1$, $z'(0) = \lambda \int_0^1 \int_0^p e^{z(t)} dtdp.$ (4.1.2.12)

From (4.1.2.11) and (4.1.2.12), we see

$$\begin{aligned} Z(r) &= r\lambda \int_0^1 \int_0^p e^{z(t)} dtdp - \lambda \int_0^r \int_0^p e^{z(t)} dtdp \\ &= r\lambda \int_0^1 \int_t^1 e^{z(t)} dpdt - \lambda \int_0^r \int_t^r e^{z(t)} dpdt \\ &= r\lambda \int_0^1 (1-t) e^{z(t)} dt - \lambda \int_0^r (r-t) e^{z(t)} dt \\ &= -\lambda \int_0^r -(1-r) t e^{z(t)} dt - \lambda \int_r^1 -(1-t) r e^{z(t)} dt. \end{aligned}$$

Then

$$z(r) = -\lambda \int_0^1 \psi(t, r)\, e^{z(t)} dt, \qquad (4.1.2.13)$$

where

$$\psi(t, r) = \begin{cases} -(1-t)r & ,0 \le r \le t \le 1 \\ -(1-r)t & ,0 \le t \le r \le 1 \end{cases}.$$

Now take the norm on both sides of (4.1.2.13),

$$|z(r)| \le \lambda \int_0^1 |\psi(t, r)| e^{z(t)} dt \le e^{|z|_0} \int_0^1 |\psi(t, r)| dt$$

$$\leq e\int_0^1|\psi(t,r)|dt = e\left((1-t)\int_0^t rdr + t\int_t^1(1-r)dr\right) = \frac{e}{2}t(1-t).$$

$$\Longrightarrow |z|_0 = \max_{r\in[0,1]}|z(r)| \leq \frac{e}{2}\max_{r\in[0,1]} r(1-r).$$

Let $h(r) = r(1-r)$. Using the second derivative test to determine the maximum value of $h(r)$, as follows:

(1) $h'(r) = 1-2r$.

(2) $r = \frac{1}{2}$ as $h'(r) = 0$.

(3) $h''(r) = -2$.

(4) $h''\left(\frac{1}{2}\right) = -2$.

Since $h''\left(\frac{1}{2}\right) < 0$, $h(r)$ has maximum value at $\frac{1}{2}$. Therefore,

$$|z|_0 \leq \frac{e}{2}\max_{r\in[0,1]} r(1-r) = \frac{e}{2}\left(\frac{1}{2}\left(1-\frac{1}{2}\right)\right) = \frac{e}{8}.$$

Consequently $|z|_0 < 1$. Therefore, $z \notin \partial W$, Hence Theorem (4.1.2.1) implies that (4.1.2.9) has only one solution with norm at most 1.

4.2 Global Solution of Fractional Differential Equations

Definition(4.2.1): The function $\Gamma(\gamma)$ is defined by

$$\Gamma(\gamma) = \int_0^\infty s^{z-1}e^{-s}\,ds$$

is said to be gamma function where $z \in C$ $(\mathrm{Re}(\gamma) > 0)$.

Remark(4.2.1): If $\Gamma(\gamma)$ is gamma function then

(i) $\Gamma(\gamma+1) = \gamma\Gamma(\gamma), (\mathrm{Re}(\gamma) > 0)$.

(ii) $\Gamma(n+1) = n!$ where $n \in IN \cup \{0\}$, with $0! = 1$.

Definition(4.2.2): The fractional integral ${}^{RL}_{\alpha}I^{p}_{t}f(t)$ of order $p \in R^+$ $(n = [p]+1$, $[p]$ means the integer part of order p) defined by

$$ {}^{RL}_{\alpha}I^{p}_{t}f(t) = \frac{1}{\Gamma(p)}\int_{\alpha}^{t}(t-r)^{p-1}f(r)dr, \, t > \alpha \text{ and } p > 0. $$

is called Riemann-Liouville fractional integral.

Definition(4.2.3): The fractional derivative ${}^{RL}_{\alpha}D^{p}_{t}f(t)$ of order $p \in R^{+}$ defined by,

$$ {}^{RL}_{\alpha}D^{p}_{t}f(t) = \frac{d^n}{dt^n}\,{}^{RL}_{a}I^{n-\alpha}_{t}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_{a}^{t}(t-s)^{n-\alpha-1}f(s)ds\right), $$

is called Riemann-Liouville fractional derivative.

Definition(4.2.4): The fractional derivative ${}^{C}_{a}D^{\alpha}_{t}f(t)$ of order $\alpha \in IR^{+}$is defined by

$$ {}^{C}_{q}D^{p}_{t}f(t) = {}^{RL}_{q}D^{p}_{t}\left(f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(q)}{k!}(t-q)^{k}\right), $$

is said to be Caputo fractional derivative, where $n = [p] + 1$ for $p \notin N \cup \{0\}$, $n = p$ for $p \in N \cup \{0\}$. In particular, $p \in (0,1)$, then

$$ {}^{C}_{q}D^{p}_{t}f(t) = {}^{RL}_{q}D^{p}_{t}\big(f(t) - f(a)\big). $$

Properties(4.2.1):

(1) If $p > 0$ and $f \in L^{r}([q,w], IR^{n})$ $(1 \le r \le \infty)$, then the following equality

$$ {}^{RL}_{q}D^{p}_{t}\left({}^{RL}_{q}I^{p}_{t}f(t)\right) = f(t)\,. $$

(2) Let $p > 0$ and $n = [p] + 1$ for $p \notin N \cup \{0\}$, $n = p$ for $p \in N \cup \{0\}$. If $y \in AC^{n}([a,b], IR^{n})$ or $y \in C^{n}([a,b], IR^{n})$, then

$$ {}^{RL}_{a}I^{p}_{t}\left({}^{C}_{a}D^{p}_{t}y(t)\right) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^{k}. $$

In particular, if $0 < p \le 1$ and $y \in AC([a,b], R^{n})$ or $y \in C([a,b], IR^{n})$ then

$$ {}^{RL}_{a}I^{p}_{t}\left({}^{C}_{a}D^{p}_{t}y(t)\right) = y(t) - y(a). $$

Consider

$$ \begin{cases} {}^{C}D^{\alpha}_{t}x(t) = \beta x(t) \\ \quad x(0) = x_0 \end{cases} \tag{4.2.1} $$

such that $\alpha \in (0,1)$, W is BS and $\beta \in L(W)$ is a linear bounded operators from W to itself.

Definition(4.2.5): Assume that $\tau \in (0,\infty)$ such that

$$C^{\alpha}([0,\tau],W) = \{x \in C([0,\tau],W): {}^{C}D_t^{\alpha}x \in C([0,\tau],W)\},$$

$x \in C([0,\tau],W)$ is called a global solution of (4.2.1), if $x \in C^{\alpha}([0,\tau],W)$ for $\forall\, \tau > 0$ and satisfies (4.2.1).

Lemma(4.2.1): Assume $x: [0,\infty) \longrightarrow W$ be a CM, then x is a global solution of (4.2.1) if and only if x satisfies:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta x(s)\, ds,\ t \geq 0, \qquad (4.2.2)$$

Proof

Let's prove recessily that x satisfies (4.2.2). Therefore, let that x is a global solution of (4.2.1). By Definition (4.2.5), $x \in C^{\alpha}([0,\tau],W)$ for all $\tau > 0$ and satisfies (4.2.1). This means that $x \in C([0,\tau],W)$ and ${}^{c}D_t^{\alpha}x \in C([0,\tau],W)$. Since $0 \leq \alpha \leq 1$, property (4.2.1-2) and $x \in C([0,\tau],W)$,

$$x(t) - x(0) = {}^{RL}I_t^{\alpha}({}^{c}D_t^{\alpha}x(t))$$

$$\Rightarrow x(t) - x_0 = {}^{RL}I_t^{\alpha}\big(\beta x(t)\big)$$

$$\Rightarrow x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta x(s) ds.$$

Now let's prove sufficient that x is a global solution of (4.2.1) since x satisfies (4.2.2). For this we let $t = 0$,

$$x(0) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^0 (-s)^{\alpha-1}\beta x(s) ds \ \Rightarrow x(0) = x_0. \qquad (4.2.3)$$

Since $x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta x(s) ds$

$$\Rightarrow x(t) - x_0 = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta x(s)\,ds \Rightarrow x(t) - x(0) = {}^{RL}I_t^{\alpha}\big(\beta x(t)\big)$$

$$\Rightarrow {}^{RL}D_t^{\alpha}\big(x(t) - x(0)\big) = {}^{RL}D_t^{\alpha}\,{}^{RL}I_t^{\alpha}\big(\beta x(t)\big)$$

$$\Rightarrow {}^{c}D_t^{\alpha}x(t) = \beta x(t), \tag{4.2.4}$$

by using property (4.2.1-1) since $\beta x(t) \in L([0,\tau], X)$ and the fact that

$${}^{c}D_t^{\alpha}x(t) = {}^{RL}D_t^{\alpha}\big(x(t) - x(0)\big).$$

From (4.2.3) and (4.2.4), x satisfies (4.2.1).

Now we will show $x \in C^{\alpha}([0,\tau], W), \forall\tau > 0$. Since $x: [0,\infty) \longrightarrow W$ be CM,

$$x \in C([0,\infty], W) \Rightarrow x \in C([0,\tau], W), \forall\tau > 0. \tag{4.2.5}$$

Since $\beta \in L(W)$ linear bounded operator on W.

$\Rightarrow \beta$ is CM on $([0,\tau], W) \Rightarrow \beta \in C([0,\tau], W) \Rightarrow \beta x(t) \in C([0,\tau], W)$

$$\Rightarrow {}^{c}D_t^{\alpha}x(t) \in C([0,\tau], W), \forall\tau > 0. \tag{4.2.6}$$

From (4.2.5) and (4.2.6), $x \in C^{\alpha}([0,\tau], W)$. Therefore,

x is a global solution of (4.2.1). ■

Theorem(4.2.1): Let $0 < \alpha < 1$, $\beta \in L(W)$ and $x_0 \in W$. Then (4.2.1) has a unique global solution.

Proof

Assume $\tau > 0$ and $k_\tau = \{x \in C([0,\tau], X): x(0) = x_0\}$. Consider $K: k_\tau \longrightarrow k_\tau$ by

$$x(t) \longrightarrow K\big(x(t)\big) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta x(s)\,ds.$$

We will show that a power of K is contraction to use BCP

$$\big|K\big(x(t)\big) - K\big(y(t)\big)\big| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\beta||x(s) - y(s)|ds$$

$$\le \frac{|\beta|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\|x - y\| \le \frac{|\beta|t^{\alpha}}{\alpha\Gamma(\alpha)}\|x - y\| = \frac{|\beta|t^{\alpha}}{\Gamma(\alpha+1)}\|x - y\|.$$

Take maximum on both sides since $0 \le t \le \tau$,

$$\|K(x) - K(y)\| \le \frac{|\beta|\tau^{\alpha}}{\Gamma(\alpha+1)}\|x - y\|.$$

Notice also that

$$\|K^2(x) - K^2(y)\| \le \frac{|\beta|\tau^{\alpha}}{\Gamma(\alpha+1)}\|K(x) - K(y)\| \le \left(\frac{|\beta|\tau^{\alpha}}{\Gamma(\alpha+1)}\right)^2\|x - y\|$$

$$\le \frac{|\beta|^2\tau^{2\alpha}}{(\Gamma(\alpha+1))^2}\|x - y\| = \frac{|\beta|^2\tau^{2\alpha}}{(\alpha!)^2}\|x - y\| \le \frac{|\beta|^2\tau^{2\alpha}}{(2\alpha)!}\|x - y\| = \frac{|\beta|^2\tau^{2\alpha}}{\Gamma(2\alpha+1)}\|x - y\|.$$

By the repetition, we discover that

$$\|K^n(x) - K^n(y)\| \le \frac{|\beta|^n\tau^{n\alpha}}{\Gamma(n\alpha+1)}\|x - y\| = \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{n\alpha}}{(n\alpha)!}\|x - y\|,$$

such that $\frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{n\alpha}}{(n\alpha)!} < 1$ for n large enough. Therefore, K^n is a contraction for some $n \ge 1$.

By Corollary (3.1.1), K has a unique FP $\bar{x} \in K_{\tau}$ such that

$$\bar{x}(t) = K(\bar{x}(t)) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(\bar{x}(s))\,ds.$$

Then Lemma (4.2.1) guarantees singularity of global solution of (4.2.1). ■

Definition(4.2.6): The function $E_{\alpha}(z)$ defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)}, R(\alpha) > 0.$$

is called the basic Mittag-Leffler function. Note that when $\alpha = 1$ it is

$$\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z.$$

One generalization of $E_{\alpha}(z)$ is denoted and defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)}, R(\alpha) > 0, R(\beta) > 0.$$

where $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1), (\gamma)_0 = 1, \gamma \ne 0$,and $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, (\gamma) > 0$.

Theorem(4.2.2): Consider the same assumption of Theorem(4.2.1). In addition, assume $\{u_n(t)\}_{n=0}^{\infty}$ be a sequence of CMs $u_n: [0,\infty) \longrightarrow W$ given by $u_0(t) = x_0$ and

$$u_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta u_{n-1}(s) ds.$$

Then there exists a CM $u: [0,\infty) \longrightarrow W$ such that $u_n \longrightarrow u$ in $C([0,\tau], W)$, $\tau > 0$, u is a solution of (4.2.1) that is unique and

$$u(t) = E_\alpha(\beta t^\alpha)^k x_0.$$

Proof

Let $u_n \in C([0,\tau], W)$, $\tau > 0$ such that $u_0(t) = x_0$.

$$u_1(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta u_0(s) ds$$

$$= x_0 + \frac{\beta x_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = x_0 + \frac{\beta t^\alpha}{\Gamma(\alpha+1)} x_0.$$

$$u_2(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta u_1(s) ds$$

$$= x_0 + \frac{\beta}{\Gamma(\alpha)} x_0 \int_0^t (t-s)^{\alpha-1} ds + \frac{\beta^2}{\Gamma(\alpha)\Gamma(\alpha+1)} x_0 \int_0^t (t-s)^{\alpha-1} s^\alpha ds$$

$$= x_0 + \frac{\beta t^\alpha}{\Gamma(\alpha+1)} x_0 + \frac{\beta^2}{\Gamma(\alpha)\Gamma(\alpha+1)} x_0 \int_0^t (t-s)^{\alpha-1} s^\alpha ds.$$

Now integrating by parts to find $\int_0^t (t-s)^{\alpha-1} s^\alpha ds$,

Let $u = s^\alpha$ $\qquad du = \alpha s^{\alpha-1} ds$

$dv = (t-s)^{\alpha-1} ds$ $\qquad v = \frac{-1}{\alpha}(t-s)^\alpha$

$$\int_0^t (t-s)^{\alpha-1} s^\alpha ds = \int_0^t (t-s)^\alpha s^{\alpha-1} ds.$$

Integrating by parts again to find $\int_0^t (t-s)^\alpha s^{\alpha-1} ds$.

Let $u = s^{\alpha-1}$ $\qquad du = (\alpha-1) s^{\alpha-2} ds$

$dv = (t-s)^\alpha ds$ $\qquad v = \frac{-1}{\alpha+1}(t-s)^{\alpha+1}$

$$\int_0^t (t-s)^{\alpha-1} s^\alpha ds = \frac{\alpha-1}{\alpha+1} \int_0^t (t-s)^{\alpha+1} s^{\alpha-2} ds..$$

After use integration by parts α- times to obtain

$$\int_0^t (t-s)^{\alpha-1}s^{\alpha}ds = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-\alpha+1)}{(\alpha+1)(\alpha+2)\cdots(\alpha+\alpha-1)}\int_0^t (t-s)^{2\alpha-1}ds.$$

$$= \frac{(\alpha-1)(\alpha-2)\cdots 3.2.1}{(\alpha+1)(\alpha+2)\cdots(2\alpha-1)(2\alpha)}t^{2\alpha} = \frac{\Gamma(\alpha)t^{2\alpha}}{(\alpha+1)(\alpha+2)\cdots(2\alpha-1)(2\alpha)}.$$

Therefore,

$$u_2(t) = x_0 + \frac{\beta t^{\alpha}}{\Gamma(\alpha+1)}x_0 + \frac{\beta^2 t^{2\alpha}}{(2\alpha)(2\alpha-1)\cdots(\alpha+2)(\alpha+1)\Gamma(\alpha+1)}x_0$$

$$= x_0 + \frac{\beta t^{\alpha}}{\Gamma(\alpha+1)}x_0 + \frac{(\beta t^{\alpha})^2}{\Gamma(2\alpha+1)}x_0 = \sum_{k=0}^{2}\frac{(\beta t^{\alpha})^k}{\Gamma(k\alpha+1)}x_0.$$

$$\vdots$$

$$u_n(t) = \sum_{k=0}^{n}\frac{(\beta t^{\alpha})^k}{\Gamma(k\alpha+1)}x_0.$$

We now illustrate that $u_n(t)$ is Cauchy sequence in BS $C([0,\tau],W)$.

$$|u_{n+1}(t) - u_n(t)| = \frac{|\beta|^{n+1}t^{\alpha(n+1)}}{\Gamma((n+1)\alpha+1)}x_0.$$

Take maximum on both sides, to drive

$$\|u_{n+1} - u_n\| \le \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}x_0.$$

Let $m > n$,

$$\|u_m - u_n\| \le \|u_m - u_{m-1}\| + \cdots + \|u_{n+1} - u_n\|$$

$$\le \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{m\alpha}}{(m\alpha)!}x_0 + \cdots + \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+3)\alpha}}{((n+3)\alpha)!}x_0 + \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+2)\alpha}}{((n+2)\alpha)!}x_0 + \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}x_0$$

$$\le \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{m\alpha}}{((n+1)\alpha)!}x_0 + \cdots + \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+3)\alpha}}{((n+1)\alpha)!}x_0 + \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+2)\alpha}}{((n+1)\alpha)!}x_0 + \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}x_0$$

$$= \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}x_0\left(1 + |\beta|\tau^{\alpha} + (|\beta|\tau^{\alpha})^2 + \cdots + (|\beta|\tau^{\alpha})^{m-n-1}\right)$$

$$= \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}x_0\sum_{k=0}^{m-n-1}(|\beta|\tau^{\alpha})^k \le \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}x_0\sum_{k=0}^{\infty}(|\beta|\tau^{\alpha})^k$$

$$= \frac{\left(|\beta|^{\frac{1}{\alpha}}\tau\right)^{(n+1)\alpha}}{((n+1)\alpha)!}\left(\frac{x_0}{1-|\beta\,|\tau^{\alpha}}\right) \xrightarrow{n\to\infty} 0.$$

Therefore, $\{u_n(t)\}_{n=0}^{\infty}$ be Cauchy sequence in BS $C([0,\tau],W)$, this indicates there exists $u(t) \in C([0,\tau],W)$ such that $u_n(t) \xrightarrow{n\to\infty} u(t)$. To show that u is solution for (4.2.1), it is enough prove u satisfies (4.2.2).

$$u(t) = \lim_{n\to\infty} u_n(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta \lim_{n\to\infty} u_{n-1}(s)ds$$

$$= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta u(s)ds.$$

Hence, Lemma(4.2.1) say that u is a global solution for (4.2.1). Since $0 < \alpha < 1$, $\beta \in L(W)$ and $x_0 = u(0) \in W$, then by Theorem(4.2.1), u is a unique solution for (4.2.1). Now let $t \in [0,\tau]$ such that $\tau > 0$,

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(Bt^{\alpha})^k}{\Gamma(k\alpha+1)} x_0 = \sum_{k=0}^{\infty} \frac{(Bt^{\alpha})^k}{\Gamma(k\alpha+1)} x_0 = E_{\alpha}(t^{\alpha}B)^k x_0.$$

The proof is done. ■

4.3 Boundary Value Problems for Two-Point Fractional Differential Equations

Define the following BVP

$$\begin{cases} {}^{C}D_{a^+}^{\alpha}\eta(t) = -f\big(t,\eta(t)\big), t \in [a,b], \alpha \in (1,2] \\ \eta(a) = A, \eta(b) = B, \quad A,B \in IR \end{cases} \tag{4.3.1}$$

where $f: [a,b] \times IR \longrightarrow IR$ is a CM.

Lemma(4.3.1): Let $\alpha > 0$, then the FDE

$${}^{C}D^{\alpha}\eta(t) = 0$$

has solution

$$\eta(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, c_i \in IR, i = 0,1,2,\cdots,n, n = [\alpha] + 1.$$

Lemma(4.3.2): A function $\eta \in C^2[a,b]$ is a solution of problem (4.3.1) if and only if it satisfies the integral equation

$$\eta(t) = A + \frac{t-a}{b-a}\left(B - A + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,\eta(s))\,ds\right)$$

$$-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,\eta(s))\,ds. \tag{4.3.2}$$

Proof

$${}^{RL}I_{a^+}^{\alpha}\, {}^{C}D_{a^+}^{\alpha}\eta(t) = -{}^{RL}I_{a^+}^{\alpha} f(t,\eta(t))$$

$$\Rightarrow \eta(t) - \sum_{k=0}^{1} \frac{\eta^{(k)}(a)(t-a)^k}{k!} .= -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,\eta(s))\,ds$$

$$\Rightarrow \eta(t) - \eta(a) - \eta'(a)(t-a) = -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,\eta(s))\,ds$$

$$\Rightarrow \eta(t) = \eta(a) + \eta'(a)(t-a) - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,\eta(s))\,ds$$

$$\Rightarrow \eta(t) = \eta'(a)(t-a) - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,\eta(s))\,ds. \tag{4.3.3}$$

From (4.3.3), $t = b$ and $\eta(b) = B$, we obtain

$$\eta'(a) = \frac{1}{b-a}\left(B - A + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,\eta(s))\,ds\right) \tag{4.3.4}$$

Substitute the value of (4.3.4) to equations (4.3.3), to get (4.3.2).

The converse follows by direct computation. Indeed,

$$\eta(t) = A + \frac{t-a}{b-a}\left(B - A + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,\eta(s))\,ds\right) - {}^{RL}I_{a^+}^{\alpha} f(t,\eta(t)).$$

$$= A - \frac{a}{b-a}\left(B - A + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,\eta(s))\,ds\right)$$

$$+\frac{1}{b-a}\left(B - A + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,\eta(s))\,ds\right) t - {}^{RL}I_{a^+}^{\alpha} f(t,\eta(t)).$$

Hence by Lemma(4.3.1),

$${}^{C}D_{a^+}^{\alpha}\eta(t) = -{}^{C}D_{a^+}^{\alpha}\, {}^{RL}I_{a^+}^{\alpha} f(t,\eta(t)) = -f(t,\eta(t)).$$

Also since (4.3.2), $t = a$ and $t = b$, then $\eta(a) = A$, $\eta(b) = B$, respectively. ■

Remark(4.3.1): We can express the solution (4.3.2) in terms of Green's function as

$$\eta(t) = A + \frac{(B-A)(t-a)}{b-a} + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{(t-a)(b-s)^{\alpha-1}-(b-a)(t-s)^{\alpha-1}}{b-a} f\big(s,\eta(s)\big)\,ds.$$

$$+\frac{1}{\Gamma(\alpha)}\int_t^b \frac{(t-a)(b-s)^{\alpha-1}}{b-a} f\big(s,\eta(s)\big)\,ds$$

$$= A + \frac{(B-A)(t-a)}{b-a} + \int_a^b \psi(s,t) f\big(s,\eta(s)\big)\,ds,$$

where

$$\psi(s,t) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1}, & a \le s \le t \le b \\ \frac{(t-a)(b-s)^{\alpha-1}}{b-a} & ,a \le t \le s \le b \end{cases}$$

Let's start to define a function

$$g(t,s) = \frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1}, a \le s \le t \le b.$$

The goal is to determine the field where $g(t,s) < 0$,

$$\Longrightarrow \frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1} < 0$$

$$\Longrightarrow \frac{t-a}{b-a} < \left(\frac{t-s}{b-s}\right)^{\alpha-1} \Longrightarrow \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} < \frac{t-s}{b-s} \Longrightarrow \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}}(b-s) < t-s$$

$$\Longrightarrow \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b - t < s\left(\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} - 1\right).$$

Since $\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} - 1 < 0$, we clearly get

$$s < \frac{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b - t}{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} - 1}.$$

If we define the function h by

$$h(t) = \frac{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b - t}{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} - 1}, t \in [a,b) \text{ and } h(b) = \lim_{t\to b} \frac{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b - t}{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} - 1} = 2b - a - \alpha(b-a).$$

We now wish to show that $a < h(t) < t$ on (a,b). We have

$$a < \frac{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b - t}{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} - 1} \Leftrightarrow \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} a - a > \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b - t \;\Leftrightarrow t - a > \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}}(b-a)$$

$$\Leftrightarrow \frac{t-a}{b-a} > \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} \Leftrightarrow \left(\frac{t-a}{b-a}\right)^{\frac{2-\alpha}{\alpha-1}} < 1 \Leftrightarrow \frac{t-a}{b-a} < 1 \Leftrightarrow t < b,$$

which is true. In addition,

$$\frac{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b-t}{\left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}}-1} < t \Leftrightarrow \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} b > \left(\frac{t-a}{b-a}\right)^{\frac{1}{\alpha-1}} t \Leftrightarrow b > t, \text{ it is also true.}$$

Therefore, $a < h(t) < t$ on (a, b). Furthermore, it is easy to view that,

$$\begin{cases} g(t,s) < 0, & a \le s < h(t) \\ g(t,s) > 0, & h(t) \le s \le t \end{cases}$$

$$\int_a^b |\psi(t,s)| ds = \frac{1}{\Gamma(\alpha)} \int_a^{h(t)} \left((t-s)^{\alpha-1} - \frac{(t-a)(b-s)^{\alpha-1}}{b-a}\right) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{h(t)}^{t} \left(\frac{(t-a)(b-s)^{\alpha-1}}{b-a} - (t-s)^{\alpha-1}\right) ds + \frac{1}{\Gamma(\alpha)} \int_t^b \frac{(t-a)(b-s)^{\alpha-1}}{b-a} ds.$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{(t-a)(b-s)^{\alpha}}{b-a} - (t-s)^{\alpha}\right) \Big|_a^{h(t)}$$

$$+ \frac{1}{\Gamma(\alpha+1)} \left((t-s)^{\alpha} - \frac{(t-a)(b-s)^{\alpha}}{b-a}\right) \Big|_{h(t)}^{t} - \frac{1}{\Gamma(\alpha+1)} \left(\frac{(t-a)(b-s)^{\alpha}}{b-a}\right) \Big|_t^b$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{(t-a)\big(b-h(t)\big)^{\alpha}}{b-a} - \big(t-h(t)\big)^{\alpha} - (t-a)(b-s)^{\alpha-1} + (t-a)^{\alpha} - \right.$$

$$\left. \frac{(t-a)(b-t)^{\alpha}}{b-a} - \big(t-h(t)\big)^{\alpha} + \frac{(t-a)\big(b-h(t)\big)^{\alpha}}{b-a} + \frac{(t-a)(b-t)^{\alpha}}{b-a}\right)$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{2(t-a)\big(b-h(t)\big)^{\alpha}}{b-a} - 2\big(t-h(t)\big)^{\alpha} - (t-a)(b-a)^{\alpha-1} + (t-a)^{\alpha}\right).$$

It is clear that the right side of the previous equality has a maximum on (a, b), though we couldn't find it analytically. We define,

$M(\alpha, a, b)$

$$= \frac{1}{\Gamma(\alpha+1)} \max_{a \le t \le b} \left(\frac{2(t-a)\big(b-h(t)\big)^{\alpha}}{b-a} - 2\big(t-h(t)\big)^{\alpha} - (t-a)(b-a)^{\alpha-1} + (t-a)^{\alpha}\right).$$

Finally, we get

$$\int_a^b |\psi(t,s)| ds \le M(\alpha, a, b). \qquad (4.3.4)$$

Theorem(4.3.1): Assume that $\eta:[a,b]\times IR\longrightarrow IR$ is a CM and satisfies a uniform Lipschitz condition with respect to the second variable on $[a,b]\times IR$ with Lipschitz constant $L>0$ that is

$$|\eta(t,x)-\eta(t,y)|\leq L|x-y|,$$

for all $(t,x),(t,y)\in[a,b]\times IR$. If

$$M(\alpha,a,b)<\frac{1}{L},$$

then the BVP (4.3.1) has a unique solution.

Proof

Define an operator $K: C[a,b]\longrightarrow C[a,b]$ by

$$(Kx)(t)=A+\frac{(B-A)(t-a)}{b-a}+\int_a^b \psi(s,t)\eta\big(s,x(s)\big)\,ds.$$

Let $x,y\in C[a,b]$,

$$|(Kx)(t)-(Ky)(t)|\leq\int_a^b|\psi(s,t)|\big|\eta\big(s,x(s)\big)-\eta\big(s,y(s)\big)\big|\,ds$$

$$\leq L\int_a^b|\psi(s,t)||x(s)-y(s)|\,ds\leq L\int_a^b|\psi(s,t)|\,ds\|x-y\|.$$

From (4.3.4), we obtain

$$|(Kx)(t)-(Ky)(t)|\leq L\,M(\alpha,a,b)\|x-y\|.$$

Take maximum on both sides where $a\leq t\leq b$,

$$\|K(x)-K(y)\|\leq L\,M(\alpha,a,b)\|x-y\|$$

Notice that $L\,M(\alpha,a,b)<1$ since $M(\alpha,a,b)<\frac{1}{L}$. Therefore, K is a contraction on $C[a,b]$. It is following by an application of the BCP (3.1.1) that K has only FP

$$x(t)=A+\frac{(B-A)(t-a)}{b-a}+\int_a^b \psi(s,t)\eta\big(s,x(s)\big)\,ds.$$

Lemma(4.3.2) says that $x(t)$ is a one solution for (4.3.1). ■

Corollary(4.3.1): Assume that $\eta:[a,b]\times IR\longrightarrow IR$ is a CM and admits

$$|\eta(t,x)-\eta(t,y)| \leq L|x-y| \ ,$$

for all $(t,x),(t,y) \in [a,b] \times IR$ and $L > 0$. If

$$b-a < \frac{2\sqrt{2}}{\sqrt{L}},$$

then the BVP

$$\begin{cases} x''(t) = -\eta\big(t,x(t)\big), a < t < b, 1 < \alpha \leq 2, \\ \quad x(a) = A \ \ x(b) = B, A, B \in IR \end{cases} \tag{4.3.5}$$

has a unique solution.

Proof

Initially, we suppose that $\alpha = 2$. Then, it is not diffecult to show that

$$h(t) = \frac{\left(\frac{t-a}{b-a}\right)b-t}{\left(\frac{t-a}{b-a}\right)-1} = \frac{a(t-b)}{t-b} = a.$$

Moreover,

$$M(2,a,b) = \frac{1}{2}\max_{a\leq t\leq b}\big((t-a)(b-a)-(t-a)^2\big) = \frac{1}{2}\max_{a\leq t\leq b}\big((t-a)(b-t)\big).$$

Now we will let $K(t) = (t-a)(b-t)$, subsequently, $K'(t) = b + a - 2t$. Hence

$$t = \frac{a+b}{2},$$

since $K'(t) = 0$. Also $K''(t) = -2 < 0$. So, $K(t)$ taken its maximum at $t = \frac{a+b}{2}$. Thus

$$M(2,a,b) = \frac{1}{2}\left(\frac{a+b}{2}-a\right)\left(b-\frac{a+b}{2}\right) = \frac{1}{2}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{2}\right) = \frac{(b-a)^2}{8} < \frac{1}{8}\left(\frac{2\sqrt{2}}{\sqrt{L}}\right)^2 = \frac{1}{L}.$$

Theorem(4.3.1) and the BVP decide that $(4.3.5)$ has a unique solution. ■

4.4 Boundary Value Problems of Order $\alpha \in (0,1]$ for FDEs

Consider the given fractional BVP

$$\begin{cases} {}^{C}D^{\alpha}\vartheta(k) = \varphi\big(k,\vartheta(k)\big), t \in [0,K], \alpha \in (0,1] \\ \quad a\vartheta(0) + b\vartheta(1) = c \end{cases} \tag{4.4.1}$$

where $\varphi: [0,K] \times IR \longrightarrow IR$ is a CM, $a + b \neq 0$ and $a, b, c \in IR$.

Definition(4.4.1): A function $\vartheta \in C^1([0,K], IR)$ is said to be a solution of $(4.4.1)$ if ϑ satisfies the equation ${}^{C}D^{\alpha}\vartheta(k) = \varphi\big(k, \vartheta(k)\big)$ on $[0,K]$ and the condition

$$a\vartheta(0) + b\vartheta(1) = c.$$

For the existence of solution for $(4.4.1)$, we need the following auxiliary lemmas:

Lemma(4.4.1): Assume $0 < \alpha < 1$, $h \in C([0,K], IR)$. The solution of the fractional integral equation is given as follows,

$$\vartheta(k) = \vartheta_0 + \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1} h(s)ds. \qquad (4.4.2)$$

if and only if ϑ be a solution of the fractional IVP

$$\begin{cases} {}^{C}D_k^{\alpha}\vartheta(k) = h(k), \quad 0 \le k \le K \\ \vartheta(0) = \vartheta_0 \end{cases} \qquad (4.4.3)$$

Lemma(4.4.2): Assume $0 < \alpha < 1$, $h \in C([0,K], IR)$. The solution of the fractional integral equation is given as follows,

$$\vartheta(k) = \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1} h(s)ds - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1} h(s)ds - c\right). \qquad (4.4.4)$$

if and only if ϑ is a solution of the fractional BVP

$$\begin{cases} {}^{C}D_t^{\alpha}\vartheta(k) = h(k), \quad 0 \le k \le K \\ a\vartheta(0) + b\vartheta(K) = c \end{cases} \qquad (4.4.5)$$

Proof

Let ϑ be a solution of ${}^{C}D^{\alpha}\vartheta(k) = h(k)$

$$\Longrightarrow {}^{RL}I^{\alpha}\,{}^{C}D^{\alpha}\vartheta(k) = {}^{RL}I^{\alpha}h(k) \Longrightarrow \vartheta(k) = {}^{RL}I^{\alpha}h(k) + L, L \in IR. \qquad (4.4.6)$$

We need to find L by using the condition $a\vartheta(0) + b\vartheta(K) = c$. Let's determine $\vartheta(0), \vartheta(K)$ by $(4.4.4)$, a simple calculation gives

$$\begin{cases} \vartheta(0) = L \\ \vartheta(K) = {}^{RL}I^{\alpha}h(K) + L \end{cases}$$

Now, we have $a\vartheta(0) + b\vartheta(K) = c \Longrightarrow a\,L + b({}^{RL}I^{\alpha}h(K) + L) = c$

$$\Longrightarrow (a+b)L = c - b\,{}^{RL}I^{\alpha}h(K) \Longrightarrow L = \frac{c}{a+b} - \frac{b}{a+b}\,{}^{RL}I^{\alpha}h(K). \qquad (4.4.7)$$

Substitute the value of L to equations (4.4.6) and (4.4.7), to get

$$\vartheta(k) = \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}h(s)ds - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}h(s)ds - c\right).$$

Conversely, it is clear that if ϑ is satisfied equation (4.4.2), then equation (4.4.3) holds. Indeed,

$$\vartheta(0) = -\frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}h(s)ds - c\right).$$

From (4.4.4), we get

$$\vartheta(k) = \vartheta(0) + \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}h(s)ds.$$

Lemma (4.4.1) guarantees that ϑ is a solution for ${}^{C}D_t^{\alpha}\vartheta(k) = h(k)$. (4.4.8)

Now, to prove the condition let (4.4.4) is hold for all $k \in [0, K]$. Then

$$\begin{aligned} a\vartheta(0) + b\vartheta(K) &= \frac{a}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}h(s)ds\right) \\ &\quad + \left(1 - \frac{b}{a+b}\right)\frac{b}{\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}h(s)ds + \frac{bc}{a+b} \\ &= \frac{ac}{a+b} - \frac{ab}{(a+b)\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}h(s)ds \\ &\quad + \frac{ab}{(a+b)\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}h(s)ds + \frac{bc}{a+b} = \frac{(a+b)c}{a+b} = c. \qquad (4.4.9) \end{aligned}$$

From (4.4.8) and (4.4.9) we drive ϑ is solution for (4.4.5). ■

The BCP is a main base for first consequence.

Theorem(4.4.1): Assume that

(H_1) $\exists k > 0$ such that $|f(k,u) - f(k,\bar{u})| \le L|u - \bar{u}|$, $t \in [0, K]$ and all $u, \bar{u} \in IR$. If

$$LK^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|\Gamma(\alpha-1)}\right) < 1, \qquad (4.4.10)$$

then the BVP (4.4.1) has a unique solution on $[0, K]$.

Proof

To begin to prove the theorem we transform the problem (4.4.1) into a FP problem. To this end we introduce the following operator

$$Y: C([0,K], IR) \longrightarrow C([0,K], IR)$$

where Y is defined by

$$(Yu)(k) = {}^{RL}I^{\alpha} f\big(k, u(k)\big) - \frac{b}{a+b}\, {}^{RL}I^{\alpha} f\big(K, u(K)\big) + \frac{c}{a+b}. \qquad (4.4.11)$$

Clearly, if $u \in C[0,K]$ then $Y(u) \in C[0,K]$, this means $Y: C[0,K] \longrightarrow C[0,K]$ is CMS. Therefore, we need to show that Y is a contraction mapping. To show this suppose $u, w \in C([0,K], IR)$, then for every $k \in [0,K]$ we have

$$|Yu(k) - Yw(k)| \leq \frac{1}{\Gamma(\alpha)} \int_0^k (k-s)^{\alpha-1} \big|f\big(s, u(s)\big) - f\big(s, w(s)\big)\big| ds.$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^K (K-s)^{\alpha-1} \big|f\big(s, u(s)\big) - f\big(s, w(s)\big)\big| ds$$

$$\leq \frac{L}{\Gamma(\alpha)} \int_0^k (k-s)^{\alpha-1} |u(s) - w(s)| ds + \frac{L|b|}{|a+b|\Gamma(\alpha)} \int_0^K (K-s)^{\alpha-1} |u(s) - w(s)| ds$$

$$\leq L\left(\frac{1}{\Gamma(\alpha)} \int_0^k (k-s)^{\alpha-1} ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^K (K-s)^{\alpha-1} ds\right) \|u - w\|_{\infty}.$$

$$= L\left(\frac{k^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{|b|K^{\alpha}}{|a+b|\alpha\Gamma(\alpha)}\right) \|u - w\|_{\infty}.$$

Take maximum on both sides where $0 \leq k \leq K$,

$$\|Y(u) - Y(w)\|_{\infty} \leq \frac{LK^{\alpha}}{\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) \|u - w\|_{\infty}.$$

Consequently, by assumption (4.4.10), Y is directly contraction on CMS $C([0,K], IR)$. Application of the theorem(3.1.1)(BCP) shows the existence and uniqueness of FP of Y, which is a solution of (4.4.1). ■

The SFPT is a base for secondary consequence.

Theorem(4.4.2): Assume that

(H_1) $\exists L > 0$ such that for all $t \in [0,K]$, $u, \bar{u} \in IR$, $|f(k,u) - f(k,\bar{u})| \leq L|u - \bar{u}|$.

(H_2) The function $f: [0,K] \times IR \longrightarrow IR$ is continuous.

(H_3) There is a constant $M > 0$ such that $|f(k,u)| \le M$ for $\forall\, t \in [0,K]$ and $\forall\, u \in IR$.

Under these assumptions, the BVP (4.4.1) has at least one FP in $C[0,K]$.

Proof

The proof is created on SFPT to prove that Y has a FP. The proof will be given in several steps:

Step 1: Y is a continuous.

Let $\{u_n\}_{n=1}^{\infty} \subset C([0,K], IR)$ be a sequence such that $u_n \longrightarrow u$ in $C([0,K], IR)$. Then for each $t \in [0,K]$

$$\left|Y\big(u_n(k)\big) - Y\big(u(k)\big)\right| \le \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}\left|f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\right|ds.$$

$$+\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}\left|f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\right|ds.$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}\sup_{0\le s\le K}\left|f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\right|ds$$

$$+\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}\sup_{0\le s\le K}\left|f\big(s,u_n(s)\big) - f\big(s,u(s)\big)\right|ds$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}ds\left\|f\big(.,u_n(.)\big) - f\big(.,u(.)\big)\right\|_\infty$$

$$+\frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}ds\left\|f\big(.,u_n(.)\big) - f\big(.,u(.)\big)\right\|_\infty$$

$$= \left(\frac{k^\alpha}{\Gamma(\alpha+1)} - \frac{|b|K^\alpha}{|a+b|\Gamma(\alpha+1)}\right)\left\|f\big(.,u_n(.)\big) - f\big(.,u(.)\big)\right\|_\infty$$

Take maximum on both sides where $0 \le k \le K$,

$$\|Y(u_n) - Y(u)\|_\infty \le \frac{K^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{|b|}{|a+b|}\right)\left\|f\big(.,u_n(.)\big) - f\big(.,u(.)\big)\right\|_\infty$$

Since f is a continuous, we obtain

$$\|Y(u_n) - Y(u)\|_\infty \xrightarrow{n\to\infty} 0.$$

Hence $Y: C([0,K], IR) \longrightarrow C([0,K], IR)$, is a CM.

Step 2: Y maps the bounded sets into the bounded sets in $C([0,K], IR)$. Indeed, it is enough to show that for $\forall\, \eta > 0$ there exists a positive constant l such that for $\forall\, u$ in

$$B_\eta = \{u \in C([0,K], IR): \|u\|_\infty \le \eta\},$$

we have

$$\|Yu\|_\infty \le M.$$

By (H_3) we have for each $k \in [0,K]$

$$|(Yu)(k)| \le \frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}\left|f\big(s,u(s)\big)\right| ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}\left|f\big(s,u(s)\big)\right| ds$$

$$+\frac{|c|}{|a+b|} \le \frac{M}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}\, ds + \frac{M|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1} ds + \frac{|c|}{|a+b|}$$

$$\le \frac{M}{\Gamma(\alpha+1)}k^\alpha + \frac{M|b|}{|a+b|\Gamma(\alpha+1)}K^\alpha + \frac{|c|}{|a+b|}.$$

Take maximum on both sides where $0 \le k \le K$,

$$\|Yu\|_\infty \le MK^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|\Gamma(\alpha+1)}\right) + \frac{|c|}{|a+b|}.$$

Since $MK^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|\Gamma(\alpha+1)}\right) + \frac{|c|}{|a+b|} = l$, then $\|Y(u)\|_\infty \le l$.

Step 3: Y maps the bounded sets into the equi-continuous sets of $C([0,K], IR)$. To see this let $k_1, k_2 \in (0,K]$, $k_1 < k_2$ and $u \in B_\eta$ such that

$$B_\eta = \{u \in C([0,K], IR): \|u\|_C \le \eta\}.$$

Subsequently,

$$\left|Y\big(u(k_2)\big) - Y\big(w(k_1)\big)\right| = \frac{1}{\Gamma(\alpha)}\left(\int_0^{k_1}((k_2-s)^{\alpha-1} - (k_1-s)^{\alpha-1})\left|f\big(s,u(s)\big)\right| ds\right.$$

$$\left. + \int_{k_1}^{k_2}(k_2-s)^{\alpha-1}\left|f\big(s,y(s)\big)\right| ds\right)$$

$$= \frac{M}{\Gamma(\alpha)}\left(\int_0^{k_1}((k_2-s)^{\alpha-1} - (k_1-s)^{\alpha-1})\, ds + \int_{k_1}^{k_2}(k_2-s)^{\alpha-1}\, ds\right)$$

$$= \frac{M}{\Gamma(\alpha+1)}(k_2^\alpha - k_1^\alpha).$$

Therefore, the right hand side of the above inequality tends to zero as $k_1 \longrightarrow k_2$. Hence $Y(u)$ is equi-continuous. By steps 2 and 3 and Theorem (2.1.1)(A-AT), we conclude that $Y: C([0,K], IR) \longrightarrow C([0,K], IR)$ is a relatively compact. Definition (2.1.7) says that $Y\big(C([0,K],IR)\big)$ is compact. Therefore, $Y: C([0,T],R) \longrightarrow C([0,T],R)$ is a continuous and compact.

Step 4: A priori bounds, it remains to show that the set

$$\mathcal{M} = \{u \in C([0,K],IR): u = \lambda Y(u),\, 0 < \lambda < 1\}$$

is bounded.

Let $u \in \mathcal{M}$, then $u = \lambda Y(u)$ for some $0 < \lambda < 1$. Thus for each $k \in [0,K]$ we have

$|u(k)| = \lambda|(Yu)(k)|$

$\leq \lambda\Big(\frac{1}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}\big|f\big(s,u(s)\big)\big|\,ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}\big|f\big(s,u(s)\big)\big|\,ds$

$+\left|\frac{c}{a+b}\right|\Big) \leq \lambda\Big(\frac{M}{\Gamma(\alpha)}\int_0^k (k-s)^{\alpha-1}\,ds + \frac{M|b|}{|a+b|\Gamma(\alpha)}\int_0^K (K-s)^{\alpha-1}ds + \left|\frac{c}{a+b}\right|\Big)$

$\leq \lambda\Big(\frac{M}{\Gamma(\alpha+1)}k^{\alpha} + \frac{M|b|}{|a+b|\Gamma(\alpha+1)}K^{\alpha} + \left|\frac{c}{a+b}\right|\Big).$

Take maximum on both sides where $0 \leq k \leq K$,

$$\|u\|_{\infty} \leq \lambda\Big(MK^{\alpha}\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|\Gamma(\alpha+1)}\Big) + \left|\frac{c}{a+b}\right|\Big)$$

Since $l = \Big(MK^{\alpha}\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|\Gamma(\alpha+1)}\Big) + \left|\frac{c}{a+b}\right|\Big)$, then $\|u\|_{\infty} \leq \lambda\, l$.

This shows that the set $\mathcal{M}$ is bounded. As a consequence of Theorem(3.4.5)(SFPT), we deduce that Y has at least one FP, which is a solution of the problem (4.4.1). ■

In this section we give an example to illustrate the usefulness of our main results.

Example(4.4.1): Consider the following fractional BVP

$$\begin{cases} {}^{C}D^{\alpha}u(k) = \frac{e^{-k}|u(k)|}{(9+e^{k})(1+|u(k)|)},\, k \in [0,1]\,, 2 < \alpha \leq 3 \\ u(0) = 0, u'(0) = 1, u''(0) = 0 \end{cases} \tag{4.4.12}$$

Set $f\big(k,u(k)\big) = \frac{e^{-k}|u(k)|}{(9+e^{k})(1+|u(k)|)}$, $\big(k,u(k)\big) \in [0,1] \times [0,\infty)$, let $u, w \in [0,\infty)$ and $k \in [0,1]$. Then we have

$\big|f\big(k,u(k)\big) - f\big(k,w(k)\big)\big| = \left|\frac{e^{-k}|u(k)|}{(9+e^{k})(1+|u(k)|)} - \frac{e^{-k}|w(k)|}{(9+e^{k})(1+|w(k)|)}\right|$

$$= \frac{e^{-k}}{9+e^{k}}\left|\frac{|u(k)|}{(1+|u(k)|)} - \frac{|w(k)|}{(1+|w(k)|)}\right|$$

$$= \frac{\left||u(k)|(1+|w(k)|)-|w(k)|(1+|u(k)|)\right|}{(9e^{k}+1)(1+|u(k)|)(1+|w(k)|)}.$$

$$= \frac{\left||u(k)|-|w(k)|\right|}{(9e^{k}+1)(1+|u(k)|)(1+|w(k)|)} \leq \frac{|u(k)-w(k)|}{(9e^{k}+1)(1+|u(k)|)(1+|w(k)|)}$$

$$\leq \frac{1}{9e^{k}+1}|u(k) - w(k)| \leq \frac{1}{10}|u(k) - w(k)|.$$

Hence the condition (H_1) holds with $L = \frac{1}{10}$. We shall check that condition $(4.4.11)$ is satisfied with $K = 1$. Indeed,

$$LK^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right) < 1 \Rightarrow \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} < 10.$$

Notice that $\frac{1}{6} \leq \frac{1}{\Gamma(\alpha+1)} < \frac{1}{2}$ and $\frac{1}{2} \leq \frac{1}{2\Gamma(\alpha-1)} < c.$ $\quad (4.4.13)$

The previous inequalities decide that

$$\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} < \frac{1}{2} + c \leq 10 \Rightarrow \frac{1}{2} + c \leq 10 \Rightarrow c \leq \frac{19}{2}.$$

From $(4.4.13)$, we get

$$\frac{1}{2} \leq \frac{1}{2\Gamma(\alpha-1)} < \frac{19}{2} \Rightarrow \frac{1}{19} < \Gamma(\alpha - 1) \leq 1 \Rightarrow \frac{1}{19} < (\alpha - 2)! \leq 1, \quad (4.4.14)$$

which is satisfied for some $\alpha \in (2,3]$. Then by Theorem (4.4.1) the problem $(4.4.12)$ has a unique solution on $[0,1]$ for the values of α satisfying $(4.4.14)$.

4.5 Nonlocal BVPs for Nonlinear FDEs of Higher – Order

Consider the following nonlinear FDEs of higher q with nonlocal boundary conditions

$$\begin{cases} {}^{C}D^{q}y(t) = \eta\big(t, y(t)\big), t \in (0,1), q \in (m-1, m], m \geq 2, \\ y(0) = y'(0) = y''(0) = \cdots\cdots\cdots = y^{(m-2)}(0) = 0, \\ y(1) = \alpha y(\lambda), 0 < \lambda < 1, \alpha\lambda^{m-1} \neq 1, \alpha \in IR \end{cases} \quad (4.5.1)$$

where $\eta: [0,1] \times X \longrightarrow X$ is a CM and $(X, \|\cdot\|)$ is a BS.

Lemma(4.5.1)(Auxiliary Lemmas): Let $q > 0$, then

$$I^{q\,C}D^{q}y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in IR, i = 0,1,2,\cdots,n, n = [q]+1.$

Lemma(4.5.2): For $\sigma \in C[0,1]$, the unique solution of the BVP

$$\begin{cases} {}^{C}D^{q}y(t) = \sigma(t), t \in (0,1), q \in (m-1,m], m \geq 2, \\ y(0) = y'(0) = y''(0) = \cdots\cdots\cdots = y^{(m-2)}(0) = 0, \\ y(1) = \alpha y(\lambda), 0 < \lambda < 1, \alpha\lambda^{m-1} \neq 1, \alpha \in IR \end{cases} \tag{4.5.2}$$

is given by

$$y(t) = \int_0^t \frac{(t-r)^{q-1}}{\Gamma(q)}\sigma(r)dr. - \frac{t^{m-1}}{1-\alpha\eta^{m-1}}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\sigma(r)dr - \alpha\int_0^{\eta} \frac{(\eta-r)^{q-1}}{\Gamma(q)}\sigma(r)dr\right].$$

Proof

Let ${}^{C}D^{q}y(t) = \sigma(t) \Rightarrow {}^{RL}I^{\alpha\,C}D^{q}y(t) = {}^{RL}I^{\alpha}\sigma(t)$. The Lemma (4.5.1) says that

$$y(t) = \frac{1}{\Gamma(q)}\int_0^t (t-r)^{q-1}\sigma(r)dr - c_0 - c_1 t - c_2 t^2 - \cdots - c_{m-1}t^{m-1}. \tag{4.5.3}$$

$$y'(t) = \frac{q-1}{\Gamma(q)}\int_0^t (t-r)^{q-2}\sigma(r)dr - c_1 - 2c_2 t - \cdots - (m-1)c_{m-1}t^{m-2}.$$

$$= \frac{1}{\Gamma(q-1)}\int_0^t (t-r)^{q-2}\sigma(r)dr - c_1 - 2c_2 t - \cdots - (m-1)c_{m-1}t^{m-2}.$$

$$y''(t) = \frac{q-2}{\Gamma(q-1)}\int_0^t (t-r)^{q-3}\sigma(r)dr - 2c_2 \; -(m-1)(m-2)c_{m-1}t^{m-3}.$$

$$= \frac{1}{\Gamma(q-2)}\int_0^t (t-r)^{q-3}\sigma(r)dr - 2c_2 - \cdots . -(m-1)(m-2)c_{m-1}t^{m-3}.$$

$$\vdots$$

$$y^{(m-2)}(t) = \frac{q-2}{\Gamma(q-1)}\int_0^t (t-r)^{q-3}\sigma(r)dr - 2c_2 \; - (m-1)(m-2)c_{m-1}t$$

$$= \frac{1}{\Gamma(q-m+2)}\int_0^t (t-r)^{q-(m-1)}\sigma(r)dr.$$

$$-(m-2)(m-3)\cdots(1)c_{m-2} - (m-1)(m-2)\cdots(2)c_{m-1}t.$$

Applying the boundary conditions for assumption $c_1 = c_2 = \cdots = c_{m-3} = c_{m-2} = 0$ and thus,

$$\begin{cases} y(1) = \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_{m-1}, \\ y(\lambda) = \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)} \sigma(r)dr - c_{m-1}\lambda^{m-1} \end{cases}. \tag{4.5.4}$$

Substitute the equations of (4.5.4) to equation $y(1) = \alpha y(\lambda)$, to get

$$c_{m-1} = \frac{1}{1-\alpha\eta^{m-1}} \left(\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)} \sigma(r)dr - \alpha \int_0^\eta \frac{(\eta-r)^{q-1}}{\Gamma(q)} \sigma(r)dr \right).$$

Substitute again the values $c_1 = c_2 = \cdots = c_{m-3} = c_{m-2} = 0$ and c_{m-1} to the equation (4.5.3), to obtain

$$y(t) = \int_0^t \frac{(t-r)^{q-1}}{\Gamma(q)} \sigma(r)dr - \frac{t^{m-1}}{1-\alpha\eta^{m-1}} \left(\int_0^1 \frac{(1-r)^{q-1}}{k\Gamma(q)} \sigma(r)dr - \alpha \int_0^\eta \frac{(\eta-)^{q-1}}{\Gamma(q)} \sigma(r)dr \right),$$

as required. ■

The first basic technique is dependable on BCP.

Theorem(4.5.1): Assume $\eta: [0,1] \times X \longrightarrow X$ is jointly CM and support the condition

$$\|\eta(t,x) - \eta(t,y)\| \leq L\|x - y\|, \ \forall\, t \in [0,1], x, y \in X.$$

Then the BVP (4.5.1) has a unique solution as long as $\gamma < 1$ and ϑ is given by

$$\vartheta = \frac{L}{\Gamma(q+1)} + \gamma, \ \gamma = \frac{L(1+|\alpha|\lambda^q)}{\Gamma(q+1)|1-\alpha\lambda^{m-1}|} \tag{4.5.5}$$

Proof

$K: C \longrightarrow C$ is defined by

$$(Ky)(t) = \int_0^t \frac{(t-r)^{q-1}}{\Gamma(q)} \eta(r, y(r))dr - \frac{t^{m-1}}{1-\alpha\lambda^{m-1}} \Big[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)} \eta(r, y(r))dr$$

$$- \alpha \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)} \eta(r, y(r))dr \Big], t \in [0,1].$$

Locate $\sup_{t\in[0,1]} \|\eta(t, 0)\| = M$, and choose

$$\beta \geq \frac{M}{(1-\Lambda)\Gamma(q+1)} \left(1 + \frac{1+|\alpha|\lambda^q}{|1-\alpha\lambda^{m-1}|} \right) : \vartheta \leq \Lambda < 1. \tag{4.5.6}$$

Now we let $U_\beta = \{y \in C: \|y\| \leq \beta\}$ and show that $K(U_\beta) \subset U_\beta$. For all $y \in U_\beta$,

$|(Ky)(t)| \leq \int_0^t \frac{(1-r)^{q-1}}{\Gamma(q)} |\eta(r, y(r))| dr$

$$+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r,y(r)\right)\right|dr+|\alpha|\int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r,y(r)\right)\right|dr\right]$$

$$\le \int_0^t \frac{(1-r)^{q-1}}{\Gamma(q)}\left(\left|\eta\left(r,y(r)\right)-\eta(r,0)\right|+|\eta(r,0)|\right)dr$$

$$+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\left(\left|\eta\left(r,y(r)\right)-\eta(s,0)\right|+|\eta(r,0)|\right)dr\right.$$

$$\left.+|\alpha|\int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}\left(\left|\eta\left(r,y(r)\right)-\eta(r,0)\right|+|\eta(r,0)|\right)dr\right].$$

$$\le (L\beta+M)\left(\int_0^t \frac{(1-r)^{q-1}}{\Gamma(q)}dr+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}dr+|\alpha|\int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}dr\right]\right)$$

$$=\frac{L\beta+M}{\Gamma(q+1)}\left(t^q+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}[1+|\alpha|\lambda^q]\right)$$

Take maximum on both sides where $0\le t\le 1$.

$$\|K(y)\|\le \frac{L\beta+M}{\Gamma(q+1)}\left(1+\frac{1+|\alpha|\lambda^q}{|1-\alpha\lambda^{m-1}|}\right)$$

$$=\beta\left(\frac{L}{\Gamma(q+1)}+\frac{L(1+|\alpha|\lambda^q)}{\Gamma(q+1)|1-\alpha\lambda^{m-1}|}\right)+\frac{M}{\Gamma(q+1)}\left(1+\frac{1+|\alpha|\lambda^q}{|1-\alpha\lambda^{m-1}|}\right)$$

$$=\beta\vartheta+\frac{M}{\Gamma(q+1)}\left(1+\frac{1+|\alpha|\lambda^q}{|1-\alpha\lambda^{m-1}|}\right),\text{ (Using (4.5.5))}$$

From (4.5.6), we get

$$\|K(y)\|\le\beta\vartheta+\beta(1-\Lambda)=\beta(\vartheta+1-\Lambda)=\beta(\vartheta-\Lambda+1)\le\beta(\Lambda-\Lambda+1)=\beta.$$

Therefore, $K(y)\in U_\beta$, $\forall\, y\in U_\beta$. Now, for $x, y\in C$ and for each $t\in[0,1]$, we obtain

$$|(Kx)(t)-(Ky)(t)|\le\int_0^t \frac{(1-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r,x(r)\right)-\eta\left(r,y(r)\right)\right|dr$$

$$+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r,x(r)\right)-\eta\left(r,y(r)\right)\right|dr\right.$$

$$\left.+|\alpha|\int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r,x(r)\right)-\eta\left(r,y(r)\right)\right|dr\right]$$

$$\le L\left(\int_0^t \frac{(1-r)^{q-1}}{\Gamma(q)}dr+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}dr+|\alpha|\int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}dr\right]\right)\|x-y\|.$$

$$=\frac{L}{\Gamma(q+1)}\left(t^q+\frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}[1+|\alpha|\lambda^q]\right)\|x-y\|.$$

Take maximum on both sides where $0\le t\le 1$,

$$\|K(x)-K(y)\|\le\frac{L}{\Gamma(q+1)}\left(1+\frac{1+|\alpha|\lambda^q}{|1-\alpha\lambda^{m-1}|}\right)\|x-y\|$$

$$= \left(\frac{L}{\Gamma(q+1)} + \frac{L(1+|\alpha|\beta^q)}{\Gamma(q+1)|1-\alpha\beta^{m-1}|}\right)\|x-y\| = \vartheta\|x-y\|.$$

Since $\vartheta < 1 \Longrightarrow K$ is a contraction. Thus, the conclusion of the theorem follows by the BCP. ■

Krasnoselskii's FPT is used to prove following result.

Theorem(4.5.2): $\eta: [0,1] \times X \longrightarrow X$ be a CM maps bounded subsets of $[0,1] \times X$ into RC subsets of X. If

(A_1) $|\eta(t,x) - \eta(t,y)| \le L|x-y|$ $\forall t \in [0,1]$, $x, y \in X$,

(A_2) $|\eta(t,y)| \le \mu(t)$, $\forall (t,y) \in [0,1] \times X$, $\mu \in L^1([0,1], R^+)$,

are acceptable with $\gamma < 1$(γ is given by (4.5.5)). Then the BVP (4.5.1) has at least one solution on $[0,1]$.

Proof

Consider $\beta \ge \frac{\|\mu\|_{L^1}}{\Gamma(q)}\left(1 + \frac{1+|\alpha|\lambda^{q-1}}{|1-\alpha\lambda^{m-1}|}\right)$ and $U_\beta = \{y \in C: \|y\| \le \beta\}$. Also define two operators Φ and Ψ on U_β as $(\Phi y)(t) = \int_0^t \frac{(t-r)^{q-1}}{\Gamma(q)}\eta(r, y(r))dr$ and

$$(\Psi y)(t) = -\frac{t^{m-1}}{1-\alpha\lambda^{m-1}}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\eta(r, y(r))dr - \alpha\int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}\eta(r, y(r))dr\right].$$

For $x, y \in U_\beta$,

$$\|\Phi(x) + \Psi(y)\| \le \|\Phi(x)\| + \|\Psi(y)\| \tag{4.5.7}$$

$$|(\Phi x)(t)| \le \int_0^t \frac{(t-r)^{q-1}}{\Gamma(q)}|\eta(r, x(r))|dr \le \int_0^t \frac{(t-r)^{q-1}}{\Gamma(q)}\mu(r)dr \le \frac{t^{q-1}}{\Gamma(q)}\int_0^t \mu(r)dr.$$

Since $0 \le \beta \le t \Longrightarrow 0 \le t-\beta \le t \Longrightarrow (t-\beta)^{q-1} \le t^{q-1}$,

$$|(\Phi x)(t)| \le \frac{1}{\Gamma(q)}\int_0^1 \mu(r)dr = \frac{\|\mu\|_{L^1}}{\Gamma(q)}.$$

Take maximum on both sides where $0 \le t \le 1$,

$$\|\Phi(x)\| \le \frac{\|\mu\|_{L^1}}{\Gamma(q)}. \tag{4.5.8}$$

$$|(\Psi y)(t)| \leq \frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r, y(r)\right)\right| ds + |\alpha| \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r, y(r)\right)\right| dr\right]$$

$$\leq \frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)} \mu(r) dr + |\alpha| \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)} \mu(r) dr\right]$$

$$\leq \frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\frac{1}{\Gamma(q)} \int_0^1 \mu(r) dr + \frac{|\alpha|\lambda^{q-1}}{\Gamma(q)} \int_0^\lambda \mu(r) dr\right]$$

$$\leq \frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\frac{1}{\Gamma(q)} \int_0^1 \mu(r) dr + \frac{|\alpha|\lambda^{q-1}}{\Gamma(q)} \int_0^1 \mu(r) dr\right] = \frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\frac{\|\mu\|_{L^1}}{\Gamma(q)} + \frac{|\alpha|\lambda^{q-1}\|\mu\|_{L^1}}{\Gamma(q)}\right].$$

Take maximum on both sides where $0 \leq t \leq 1$,

$$\|\Psi(y)\| \leq \frac{1}{|1-\alpha\lambda^{m-1}|}\left[\frac{\|\mu\|_{L^1}}{\Gamma(q)} + \frac{|\alpha|\lambda^{q-1}\|\mu\|_{L^1}}{\Gamma(q)}\right]. \tag{4.5.9}$$

It follows from (4.5.7), (4.5.8) and (4.5.9),

$$\|\Phi(x) + \Psi(y)\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q)} + \frac{1}{|1-\alpha\lambda^{m-1}|}\left[\frac{\|\mu\|_{L^1}}{\Gamma(q)} + \frac{|\alpha|\lambda^{q-1}\|\mu\|_{L^1}}{\Gamma(q)}\right]$$

$$= \frac{\|\mu\|_{L^1}}{\Gamma(q)}\left(1 + \frac{1+|\alpha|\lambda^{q-1}}{|1-\alpha\lambda^{m-1}|}\right) \leq \beta.$$

Therefore, $\Phi(x) + \Psi(y) \in U_\beta$.

It follows from the assumption (A_1) that Ψ is a contraction mapping for $\gamma < 1$. To see that let's assume $x, y \in C([0,1], X)$,

$$|(\Psi x)(t) - (\Psi y)(t)| \leq \frac{t^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r, x(r)\right) - \eta\left(s, y(s)\right)\right| ds\right.$$

$$\left.+ |\alpha| \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)}\left|\eta\left(r, x(r)\right) - \eta\left(r, y(r)\right)\right| dr\right].$$

$$\leq \frac{Lt^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)} |x(r) - y(r)| dr - L\, |\alpha| \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)} |x(r) - y(r)| dr\right]$$

$$\leq \frac{Lt^{m-1}}{|1-\alpha\lambda^{m-1}|}\left[\int_0^1 \frac{(1-r)^{q-1}}{\Gamma(q)} dr - |\alpha| \int_0^\lambda \frac{(\lambda-r)^{q-1}}{\Gamma(q)} dr\right] \|x - y\|$$

$$\leq \frac{Lt^{m-1}}{|1-\alpha\lambda^{m-1}|}\left(\frac{1-|\alpha|\lambda^q}{\Gamma(q+1)}\right) \|x - y\|.$$

Take maximum on both sides where $0 \leq t \leq 1$,

$$\|\Psi(x) - \Psi(y)\| \leq \frac{L(1-|\alpha|\lambda^q)}{\Gamma(q+1)|1-\alpha\lambda^{m-1}|} \|x - y\| = \gamma\|x - y\|.$$

The continuity of η implies that the operator Φ is a CM. Also, Φ is a uniformly bounded on U_β as

$$\|\Phi(x)\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q)}, \forall x \in U_\beta. \tag{4.5.10}$$

To show that the operator Φ is compact, we use AAT. In view of (A_1) and (A_2), we define

$$\sup_{(t,x)\in[0,1]\times U_\beta} |\eta(t,x)| = f_{max},$$

Consequently, since $t_2 > t_1$, we have

$$|(\Phi x)(t_1) - (\Phi x)(t_2)|$$

$$\leq \int_0^{t_1} \frac{((t_2-r)^{q-1}-(t_1-r)^{q-1})}{\Gamma(q)} |\eta(r,x(r))| dr + \int_{t_1}^{t_2} \frac{(t_2-r)^{q-1}}{\Gamma(q)} |\eta(r,x(r))| dr$$

$$\leq f_{max}\left(\int_0^{t_1} \frac{((t_2-r)^{q-1}-(t_1-r)^{q-1})}{\Gamma(q)} dr + \int_{t_1}^{t_2} \frac{(t_2-r)^{q-1}}{\Gamma(q)} dr\right).$$

$$= f_{max}\left(\frac{t_2^q-(t_2-t_1)^q-t_1^q+(t_2-t_1)^q}{\Gamma(q+1)}\right) = \frac{f_{max}}{\Gamma(q+1)}\left(t_2^q - t_1^q\right) \xrightarrow{t_1 \to t_2} 0,$$

Thus, Φ is equi-continuous. (4.5.11)

It is following from (4.5.10),(4.5.12) and A-AT (2.1.1) that Φ is RC on U_β. This means that Φ maps BS U_β of X into a RC subset $\Phi(U_\beta)$. By definition (2.1.7), Φ is a compact on U_β. Theorem [Krasnoselskii] is satisfied and the conclusion of Theorem(3.4.6)[Krasnoselskii] implies that the BVP (4.5.1) has at least one solution on [0,1]. ■

Example(4.5.1): Consider the following BVP

$$\begin{cases} {}^{C}D^q y(t) = \frac{1}{(t+7)^2}\left(\frac{|y(t)|}{1+|y(t)|}\right), q \in (2,3], t \in [0,1] \\ y(0) = y'(0) = 0, y(1) = y\left(\frac{1}{2}\right) \end{cases} \tag{4.5.12}$$

Here, $\eta(t,y(t)) = \frac{1}{(t+7)^2}\left(\frac{|y(t)|}{1+|y(t)|}\right)$, $m = 3$, $\alpha = 1$ and $\lambda = \frac{1}{2}$. Since

$$\left|\eta(t,x(t))-\eta(t,y(t))\right| \le \frac{1}{(t+7)^2}\left|\frac{|x(t)|}{1+|x(t)|}-\frac{|y(t)|}{1+|y(t)|}\right| = \frac{1}{(t+7)^2}\frac{\left||x(t)|-|y(t)|\right|}{(1+|x(t)|)(1+|y(t)|)}$$

$$\le \frac{1}{(t+7)^2}\frac{|x(t)-y(t)|}{(1+|x(t)|)(1+|y(t)|)} \le \frac{1}{(t+7)^2}|x(t)-y(t)|$$

$$\le \frac{1}{(0+7)^2}|x(t)-y(t)| = \frac{1}{49}|x(t)-y(t)|.$$

Therefore, (A_1) is satisfied with $L=\frac{1}{49}$. Further,

$$\vartheta = \frac{L}{\Gamma(q+1)}\left(1+\frac{1+|\alpha|\lambda^q}{|1-\alpha\lambda^{m-1}|}\right) = \frac{1}{49\Gamma(q+1)}\left(1+\frac{4}{3}\left(1+\left(\frac{1}{2}\right)^q\right)\right)$$

$$< \frac{1}{49\Gamma(3)}\left(1+\frac{4}{3}\left(1+\left(\frac{1}{2}\right)^2\right)\right), \text{ since } 2<q\le 3.$$

$$\le \frac{1}{49(2!)}\left(\frac{8}{3}\right) = \frac{4}{147} < 1.$$

Thus, by Theorem(4.5.1), the BVP $(4.5.12)$ has just one solution on $[0,1]$.

Chapter 5

CONCLUSION

In this thesis, we presented some basic techniques and results of FPT with some applications.

Namely, we studied the existence and uniqueness of some ordinary and fractional differential equations by using Banach, Brouwer's and Schauder's fixed point theorems under certain conditions.

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