Controllability of Deterministic Systems

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ABSTRACT

This thesis is focused on the controllability of deterministic systems in Hilbert spaces. We basically consider linear systems in finite and infinite dimensional spaces and then mostly, we examine the existing controllability concepts of linear deterministic systems in both finite and infinite dimensional spaces. In Chapter 2, various concepts and their properties are given such as Kalman Rank condition with its proof, definitions of exact and approximate controllability, resolvent conditions, and partial controllability with its conditions. Moreover, controllability of semilinear systems are examined by using contraction mapping theorem and its generalization.

Keywords: Exact controllability; Approximate controllability; Partial controllability; Deterministic systems; Kalman Rank Condition; Resolvent Conditions; Contraction mapping

ÖZ

Bu tezin konusu, Hilbert uzaylarında deterministik sistemlerin kontrol edilebilirliğine odaklanmıştır. Temel olarak doğrusal sistemleri sonlu ve sonsuz boyutlu uzaylarda ele alıyoruz, daha sonra çoğunlukla, doğrusal deterministik sistemlerin hem sonlu hem de sonsuz boyutlu uzaylarda mevcut kontrol edilebilirlik kavramlarını inceliyoruz. Doğrusal deterministik sistemlerin Kontrol Edilebilirliği bölümünde, ispatıyla birlikte Kalman Sırası koşulu, kesin ve yaklaşık kontrol edilebilirlik tanımları, çözücü koşulları ve koşullarıyla kısmi kontrol edilebilirlik gibi çeşitli kavramlar ve özellikleri verilmiştir. Ayrıca, yarı doğrusal sistemlerin kontrol edilebilirliği, büzülme haritalama teoremi ve genellemesi kullanılarak incelenmiştir.

Anahtar Kelimeler: Tam kontrol edilebilirlik; yaklaşık kontrol edilebilirlik; kısmi kontrol edilebilirlik; Deterministic sistemler; Kalman sıra koşulu; çözücü koşullar; büzülme haritası

To my family

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Chapter 1

INTRODUCTION

There are large number of processes or systems to be controlled in the field of control engineering. The control engineers have a duty to build and receive the desired response from controller so that the constructed controller can have an interactive communication with such systems. However, it may not be possible to control these systems easily. Thus, controllability plays a key role in many issues of control including the reliability of dysfunctional systems through feedback or optimal control who are essential features of a control system. Rudolf E. Kalman is the first researcher who has the reputation for publishing a work about deterministic linear systems for the controllability concept and as a definition, a control system is controllable if any initial point can be transferred to any final destination point within a finite duration considering admissible controls.

Before passing through the controllability concepts, derivation of the Variation of Constant formula is introduced by supplying suitable linear systems in finite dimensional space that this formula is also a unique solution for a linear system. After that, the important equations are defined which are heat equation, wave equation and delay equation for infinite dimensional space. Derivation of heat and wave equation are described with related examples. Considering the controllability concepts, Kalman's rank condition is a significant method to check the controllability of a control system which is perfectly useful for finite dimensional linear deterministic control systems, but if we switch the finite dimension to infinite dimension, then this method cannot be as beneficial as it is used to be. According to this fact, in order to have a better control on linear deterministic control systems in infinite dimensions, researchers have developed the controllability concept into two main parts which are exact controllability and approximate controllability. The difference between these two concepts is the concept of exact controllability is similar with the defined controllability method by Kalman so that it is a stronger version in comparison with approximate controllability.

The controllability concept have been improved by further studies of Bashirov and Mahmudov (1999) related with resolvent conditions. Then, the concept of partial controllability is initiated and by making enlargement onto ordinary controllability conditions, partial controllability concepts are obtained [2]. Partial controllability concepts are more helpful for the control systems including wave and delay equations and higher-order differential equations instead of using the normal controllability concepts, because basic concepts are too strong for these equations so, by expanding the state space's dimension, it will be possible to rewrite and show the related equations in a more preferred form (first-ordered linear differential equation). The next and final concept is the controllability of semilinear systems. Description of these systems are divided into two parts which are exact controllability and approximate controllability concepts. Here only sufficient conditions are considered and in general, fixed point theorems and contraction mapping theorem and its generalization are introduced and used.

This dissertation is organized as follows: In Chapter 2, some general information is given that is mostly used and needed while considering the theorems and proofs. Then, a brief review is made for finite dimensional space. Also, for infinite dimensional space, three important equations are introduced and related examples are supplied for them. At the end of Chapter 2, a short introduction is made for semilinear systems. In Chapter 3, the concept of controllability for linear deterministic control systems are defined in both finite and infinite dimensions. Partial controllability concept is briefly introduced as well. Chapter 4 provides information about semilinear controllability concept.

Chapter 2

DETERMINISTIC SYSTEMS

In this chapter, our aim is to provide some fundamental information for the following chapters where the proofs, lemmas, propositions, etc. will be more understandable and clear. The provided concepts which are definitions, corollaries, propositions in this section can also be used in the upcoming topics in this research therefore the reader will have a chance of struggling less to notice where the origin of the definitions or properties of the theorems came from. These beneficial facts will be expressed without proofs since there are a lot of books that the facts can be found inside them and simply give the required clarifications.

2.1 Linear Systems

2.1.1 General Definition

Definition 2.1: A vector space (linear space) V is defined over a field $\mathbb{R}$ that has two binary conditions as addition and scalar multiplication. The following properties must be hold for all $u, v, w \in V$ and $t, r \in \mathbb{R}$.

i) For every $u, v \in V, u + v = v + u$ (Commutativity with addition)

ii) For every $u, v, w \in V, (u + v) + w = u + (v + w)$ (Associativity with addition)

iii) $\exists 0 \in V$ such that $0 + u = u + 0 = u, \forall u \in V$ (Additive Identity)

iv) For every $u \in V, \exists -u \in V$ such that $u + (-u) = 0$ (Additive inverse)

v) For every $u \in V$, $1u = u$ (Identity with multiplication)

vi) For every $t, r \in \mathbb{R}$ and for every $u \in V$, $(tr)u = t(ru)$ (Associativity with multiplication)

vii) For every $t \in \mathbb{R}$ and for every $u,v \in V$, $t(u+v) = tu+tv$ (Left distributivity)

viii) For every $t,r \in \mathbb{R}$ and for every $u \in V$, $(t+r)u = tu+ru$ (Right distributivity)

Definition 2.2: Let the linear vector space K be a normed space. The expression $\|x\|$ (norm of x) satisfies the following three properties;

i) For all $x \in K$, $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

ii) For all $c \in \mathbb{R}$ and $x \in K$, $\|cx\| = |c| \, \|x\|$

iii) For all $x,y \in K$, $\|x+y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

Given x and y be vectors with a norm $\|.\|$. An expression can be given for the length between the vectors x and y as

$$d(x,y) = \|x-y\|$$

and $(K, \|.\|)$ together is called a normed space K.

Definition 2.3: Let K be a normed space. To be able to stress that K is a Banach space, a condition needs to be hold which is every Cauchy sequence has to be convergent in K.

Definition 2.4: Let K be a vector space over $\mathbb{R}$. An inner product on K is a function that assigns to each pair of vectors u,v in K, a scalar in $\mathbb{R}$, denoted by $\langle u,v \rangle$ if the followings hold;

i) $\forall u,v,z \in K$, $\langle u+z,v \rangle = \langle u,v \rangle + \langle z,v \rangle$

ii) $\forall u.v \in K$, $\langle u,v \rangle = \langle v,u \rangle$

iii) $\forall u \in K$, $\langle u,u \rangle \geq 0$

iv) $\forall u \in K$, $\langle u,u \rangle = 0 \iff x = 0$

v) $\forall u,v \in K$ and $\forall c \in \mathbb{R}$, $\langle cu,v \rangle = c \langle u,v \rangle$

Definition 2.5: Let A and B be two vector spaces. The operator $N : A \to B$ is said to be linear, if

$$N(ay_1 + by_2) = aN(y_1) + bN(y_2)$$

for every $y_1, y_2 \in A$ and for every $a, b \in \mathbb{R}$.

Definition 2.6: Let Y and Z be two normed vector spaces. The linear operator $N : Y \to Z$ is said to be a bounded linear operator, if

$$\|Ny\|_Z \leq m \|y\|_Y$$

for $m > 0$ and for all $y \in Y$.

Here, m becomes an operator norm for N when m has the smallest possible number that holds the condition above and represented by

$$\|N\| = \sup_{\|y\|=1} \|Ny\|_Z \ .$$

Definition 2.7: Assume that collection of all linear operators from Y to Z is denoted by $\mathbf{L}$. Define Y and Z be two Banach spaces, where $N \in \mathbf{L}(Y, Z)$. N^* is the adjoint of the operator N that $N^* \in \mathbf{L}(Y^*, Z^*)$ therefore $\forall y \in Y$ and $\forall z^* \in Z^*$,

$$(N^* z^*) y = z^*(Ny).$$

If the previous definition is considered in Hilbert space, the expression of the adjoint operator N^* for $N : Y \longrightarrow Z$ is

$$\langle Ny, z \rangle_Z = \langle y, N^* z \rangle_Y \qquad \forall y \in Y, \forall z \in Z.$$

Definition 2.8: Let Y be Hilbert space with $N \in \mathbf{L}(Y, Y) = \mathbf{L}(Y)$. If $N = N^*$, then N is self-joint such that the following properties hold:

i) Nonnegative if $\forall w \in Y$, $\langle Nw, w\rangle \geq 0$.

ii) Positive if $\forall w \in Y$ except $w = 0$, $\langle Nw, w\rangle > 0$.

iii) Coercive if there exists a positive c value such that $\forall w \in Y$, $\langle Nw, w\rangle \geq c\,\|w\|^2$.

Definition 2.9: Let Y and Z be two Banach spaces. If a sequence $\{N_n\} \in \mathbf{L}(Y,Z)$ is said to converge to the operator $N \in \mathbf{L}(Y,Z)$, then

i) N_n converges uniformly if $\|N_n - N\|$ goes to 0 as $n \longrightarrow \infty$.

ii) N_n converges strongly for all $y \in Y$ if $\|N_n y - Ny\|$ goes to 0 as $n \longrightarrow \infty$.

iii) N_n converges weakly for all $y \in Y$ and $z^* \in Z^*$ if $z^*((N_n - N)y) \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 2.10: Suppose Z is a Banach space and the operator G maps Z into itself. G is called a contraction mapping if there is $0 \leq c < 1$ so that $\forall x, y \in Z$,

$$\|G(x) - G(y)\| \leq c\,\|x - y\|.$$

Theorem 2.1: Let the operator $G : Z \to Z$ be a contraction mapping and Z be a Banach space. Then, there exists only one fixed point $z_0 \in Z$ so that $G(z_0) = z_0$.

Theorem 2.2: Assume Z is a Banach space and the nonlinear operator G maps Z into itself. Let $G^1 = G$, $G^2 = G \circ G, \cdots, G^n = G^{n-1} \circ G$ for every $n \in \mathbb{N}$. If G^n is a contraction mapping for some $n \in \mathbb{N}$, then G has a unique fixed point in Z.

Theorem 2.3: (Fubini's Theorem) Let $g : K = [x,y] \times [m,n] \to \mathbb{R}$ is integrable concerning total variable $(a,b) \in [x,y] \times [m,n]$. If $\forall b \in [m,n]$, $g(a,b)$ is integrable concerning $a \in [x,y]$ and the integral of function b, $\int_x^y g(a,b)\, da$ is integrable on $[m,n]$. Then

$$\int_K g(a,b)dK = \int_m^n \left(\int_x^y g(a,b)da\right) db = \int_x^y \left(\int_m^n g(a,b)db\right) da.$$

In addition, if $g(a,b) = f(a)h(b)$, then

$$\int_K g(a,b)dK = \left(\int_x^y f(a)da\right)\left(\int_m^n h(b)db\right).$$

Theorem 2.4: (Gronwall's Inequality) Suppose the function $h \geq 0$ exists on $[x,y] \in \mathbb{R}$ that holds

$$h(t) \leq d_1 + d_2 \int_x^t h(s)\, ds, \quad x \leq t \leq y,$$

where d_1 and d_2 are both positive constants. Then

$$h(t) \leq d_1 e^{d_2(t-x)}, \quad x \leq t \leq y.$$

Theorem 2.5: (Lebesgue's dominated convergence theorem) Assume Z is a Hilbert space and also $\Lambda \subseteq \mathbb{R}$. Let $\{g_n\}$ be sequence of Lebesgue integrable functions $L^1(\Lambda; Z)$ so that $f_n(r) \to f(r)$ for $n \to \infty$ on Z. Assume there exists an integrable function $h \in L^1(\Lambda)$ so that $\forall n \in \mathbb{N}$, $|f_n(r)| \leq h(r)$ on Z. Then, f is Lebesgue integrable on Z and $\lim_{n\to\infty} \int_Z g_n(r)\, dr = \int_Z g(r)\, dr$.

2.1.2 Finite Dimensional Linear Systems

This section aims to give information about ordinary differential equations by investigating them with vector space form and also to show the derivation of Variation of Constants formula. The controllability concepts for finite dimensional systems will not be included in this section; however, for the upcoming chapters, their properties and proofs related to controllability will be stressed and explained.

First, we will use the following facts for vector spaces.

i) Recall Definition 2.1 that the axioms are in $\mathbb{R}$.

ii) Let $D^n[d_0, d_1]$ define $n-tuples$ that are continuous at time $t \in [d_0, d_1]$. The elements can be expressed as vectors:

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and } a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{pmatrix}.$$

iii) Assume $\mathbb{R}^{n\times m}$ define all the set of n by m real numbers as follows:

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix}.$$

Now, let us consider a system with first order equations

$$\begin{aligned} z_1'(t) &= f_1\left[z_1(t),...,z_n(t),t\right] \\ z_2'(t) &= f_2\left[z_1(t),...,z_n(t),t\right] \\ &\vdots \\ z_n'(t) &= f_n\left[z_1(t),...,z_n(t),t\right] \end{aligned} \tag{2.1}$$

It is possible to set the following higher order scalar equations in this form

$$x^n(t) = f\left[x(t),x^1(t),...,x^{(n-1)}(t),t\right],$$

by assuming $z_j = x^{(j-1)}$ for $j = 1,...,n$ to obtain

$$\begin{aligned} z_1'(t) &= z_2(t) \\ z_2'(t) &= z_3(t) \\ &\vdots \\ z_n'(t) &= f\left[z_1(t),...,z_n(t),t\right]. \end{aligned}$$

It is also possible to enlarge this form to simultaneous higher order equations in a form of a single vector differential equation

$$z'(t) = f\left[z(t), t\right],$$

where z is a column vector.

Borrowing (2.1) and assuming they are homogeneous, we can rewrite (2.1) as follows,

$$\begin{aligned}
z_1'(t) &= a_{11}(t)z_1(t) + a_{12}(t)z_2(t) + \cdots + a_{1n}(t)z_n(t) \\
z_2'(t) &= a_{21}(t)z_1(t) + a_{22}(t)z_2(t) + \cdots + a_{2n}(t)z_n(t) \\
&\vdots \\
z_n'(t) &= a_{n1}(t)z_1(t) + a_{n2}(t)z_2(t) + \cdots + a_{nn}(t)z_n(t)
\end{aligned} \tag{2.2}$$

where

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

therefore, one can express (2.2) as

$$z'(t) = A(t)z(t). \tag{2.3}$$

Now, consider two common initial-valued first order linear differential equation,

$$z'(t) = A(t)z(t) + f(t) \tag{2.4}$$

$$z'(t) = A(t)z(t), \quad z(t_0) = z_0. \tag{2.5}$$

Here the aim is to show derivation of Variation of Constants formula by using (2.4) and (2.5). The solution of the homogeneous equation (2.5) is known as

$$z(t) = z(t_0)\exp\left(\int_{t_0}^{t} A(r)\, dr\right) = z_0 \exp(A(t - t_0)). \tag{2.6}$$

Since

$$\frac{z'(t)}{z(t)} = A(t) = \frac{d}{dt} \log|z(t)|,$$

then

$$\begin{aligned}
\log|z(t)| &= c + \int_{t_0}^{t} A(r)\, dr \\
|z(t)| &= \exp(c) \exp\left(\int_{t_0}^{t} A(r)\, dr\right) \\
z(t) &= \pm \exp(c) \exp\left(\int_{t_0}^{t} A(r)\, dr\right).
\end{aligned}$$

Now, let us consider the inhomogeneous equation (2.4) by rearranging it as:

$$z'(t) - A(t)z(t) = f(t). \tag{2.7}$$

By introducing an integrating factor $\mu(t)$ and applying it for the equation (2.7), we have

$$\mu(t)z'(t) - A(t)\mu(t)z(t) = \mu(t)f(t). \tag{2.8}$$

It is assumed that in some sense, the left-hand sides of both (2.7) and (2.8) are formed as the result of an application of the product rule, so according to that idea, left-hand side of the equation (2.8) will change as,

$$\mu'(t)z(t) + \mu(t)z'(t) = \mu(t)f(t), \tag{2.9}$$

$$\frac{d}{dt}(\mu(t)z(t)) = \mu(t)f(t). \tag{2.10}$$

In order to figure out if the previous assumption can be applicable, the integrating factor $\mu(t)$ needs to satisfy (2.6). Therefore, by considering (2.8) and (2.9), we get

$$\begin{aligned}
\mu'(t) &= -A(t)\mu(t), \\
\mu(t) &= z(t_0) \exp\left(-\int_{t_0}^{t} A(r)\, dr\right).
\end{aligned} \tag{2.11}$$

From (2.11), it is approved that it is possible to understand the left-hand sides as the result of an application of the product rule if the integrating factor $\mu(t)$ is chosen as

(2.11). Moreover, by applying integration on both sides of (2.10), we obtain

$$\mu(t)z(t) = \int_{t_0}^{t} \mu(r)f(r)\,dr + c,$$

$$z(t) = \mu(t)^{-1}c + \mu(t)^{-1}\int_{t_0}^{t} \mu(r)f(r)\,dr. \tag{2.12}$$

By substituting (2.12) into (2.6),

$$z(t) = \exp\left(\int_{t_0}^{t} A(r)\,dr\right)c + \exp\left(\int_{t_0}^{t} A(r)\,dr\right)\int_{t_0}^{t} \exp\left(-\int_{t_0}^{r} A(s)\,ds\right)f(r)\,dr$$

So,

$$z(t) = z_0\exp\left(\int_{t_0}^{t} A(r)\,dr\right) + \int_{t_0}^{t} \exp\left(\int_{r}^{t} A(s)\,ds\right)f(r)\,dr. \tag{2.13}$$

The equation (2.13) is also known as the Variation of Constants formula that is a unique solution of (2.4). Its look can vary according to the given initial value so that if the initial value changes to $z(0) = z_{0,}$ then we get

$$z(t) = \exp(At)z_0 + \int_{0}^{t} \exp(A(t-r))f(r)\,dr. \tag{2.14}$$

which is a well-known general version of the Variation of Constants formula.

2.1.3 Heat Equation

In this section, we will examine infinite dimensional systems that are quite beneficial when the systems are defined by partial differential equations because finite dimensional systems are just perfectly suitable for ordinary differential equations, not for partially. There are 3 important equation concepts that play a vital role for these systems which are heat, wave and delay equations. Firstly, a brief information will be given about heat equation and then derivation of heat equation with a one-dimensional solution concept will be introduced by Fourier method. Heat is a representative physical energy that can be transferred depending on the variation of temperature in that body such as moving from a certain area of higher temperature to a region with lower temperature.

(Derivation of Heat Equation) Firstly, an equation is defined as $G = cmT$ that shows heat energy on a homogenous body which is proportional to its temperature T and mass m. There is also a representation for the heat capacity of the body as a constant c. The defined equation above will have its general form for any changement for temperature within time t and space variable y as follows:

$$G(t) = cp \int\int\int_A T(t,y)\, dy$$

Here A is the area in three dimensional space used by an object. By considering constant mass density which is $p = \frac{\mu}{\gamma(A)}$, we get

$$G'(t) = cp \int\int\int_A T'(t,y)\, dy. \tag{2.15}$$

Let the derivative and triple integral be interchangeable (have the same meaning). Apart from that, as an assumption, there is no heat flow within the area A, then there is a connection proportionally between the rate of heat transfer across the partial derivative of A and the total of outer normal components of ∇T over the partial derivative of A which implies

$$G'(t) = \phi \int\int_{\partial A} \nabla T(t,y) \cdot R(y)\, ds,$$

where constant ϕ refers to heat conductivity. By the derivation of generic conservation equation, we obtain

$$G'(t) = \phi \int\int\int_A \nabla \cdot \nabla T(t,y)\, dy. \tag{2.16}$$

Equating the right sides of (2.15) and (2.16),

$$\int\int\int_A (cpT'(t,y) - \phi \nabla^2 T(t,y))\, dy = 0.$$

Choosing A arbitrarily,

$$T'(t,y) = \frac{\phi}{cp} \nabla^2 T'(t,y).$$

Assuming positive constant $\alpha^2 = \frac{\phi}{cp}$, we have the heat equation as

$$T'(t,y) = \alpha^2 \nabla^2 T(t,y). \tag{2.17}$$

Solution of Heat Equation by Fourier Method:

Consider one-dimensional heat equation

$$T_t'(t,y) = T_{yy}''(t,y), \qquad \alpha = 1 \tag{2.18}$$

with the boundary condition

$$T(t,0) = T(t,1) = 0 \tag{2.19}$$

where $t \geq 0$ and $y \in [0,1]$.

and initial condition as

$$T(0,y) = f(y), \quad 0 \leq y \leq 1. \tag{2.20}$$

The assumption of $\alpha = 1$ represents the heat on a homogenous rod and as it is insulated, the heat cannot penetrate into the rod so that it only moves along the body. By separation of variables, we get

$$T(t,y) = w(t)z(y). \tag{2.21}$$

When either w or z is 0, $T = 0$ which is a trivial solution but this contradicts with the initial condition that is not 0. According to this fact, let w and z be non-zero functions by considering (2.19),

$$z(0) = z(1) = 0. \tag{2.22}$$

Substituting (2.21) into (2.18),

$$w'(t)z(y) = w(t)z''(y)$$

$$\frac{w'(t)}{w(t)} = \frac{z''(y)}{z(y)}.$$

In the previous equality, left and right-hand sides are independent on t and y so if we rewrite by defining them with k, we have

$$w'(t) = kw(t) \text{ and } z''(y) = kz(y).$$

Let $k = 0$, then assuming a and b as constants, $z(y) = ay + b$. Considering (2.22), a and b equals to 0, so $z = 0$. On the other hand, accepting $k > 0$ will lead to $z(y) = a\sinh\sqrt{ky} + b\cosh\sqrt{ky}$ and again $z = 0$ according to (2.21). Therefore $k \geq 0$ causes a trivial solution meaning $T = 0$. In addition, when $k < 0$ with an assumption of $k = -\mu^2$ for some $\mu \neq 0$, the previous equations of w and z will be

$$w'(t) = -\mu^2 w(t) \text{ and } z''(y) = -\mu^2 z(y)$$

with a solution

$$w(t) = A\exp(-\mu^2 t) \text{ and } z(y) = B\sin\mu y + C\cos\mu y$$

where A, B and C are constants. By (2.22), we have $C = 0$ and $\mu = mT$ that m is the set of all integers. From the fact that $B\sin(-m\pi y) = -B\sin(m\pi y)$, one can accept m as a set of all positive integers so every one of

$$T_m(t,y) = H_m \exp(-m^2\pi^2 t)\sin(m\pi y) \tag{2.23}$$

can be a problem solver for obtaining a nontrivial function of two variables that (2.18)–(2.20) holds. In general, linear combination of two solutions in (2.18) satisfying (2.19) is acceptable as a solution again. Therefore it is vital to search a possible solution in (2.18)–(2.20) in the form

$$T(t,y) = \sum_{m=1}^{\infty} T_m(t,y) = \sum_{m=1}^{\infty} H_m \exp(-m^2\pi^2 t)\sin(m\pi y).$$

Here H_m can be denoted from (2.20) as:

$$f(y) = \sum_{m=1}^{\infty} H_m \sin(m\pi y)$$

where $H_m = 2\int_0^1 f(r)\sin m\pi r \, dr$ and the function $f(y)$ is the half range fourier sine expansion. By assuming $f(y)$ converges, one can have a unique solution for (2.18)–(2.20) as:

$$T(t,y) = 2\sum_{m=1}^{\infty} \exp(-m^2\pi^2 t)\sin(m\pi y)\int_0^1 f(r)\sin(m\pi r)\,dr.$$

Let T_1 and T_2 be two solutions then as the function above is zero, $D = T_1 - T_2$ that (2.18)–(2.20) holds. To prove $D = 0$, suppose

$$S(t) = \int_0^1 D(t,y)^2\,dy$$

then,

$$\begin{aligned} S'(t) &= 2\int_0^1 DD'_t\,dy = 2\int_0^1 DD''_{yy}\,dy = 2\int_0^1 ((DD'_y)'_y - (D'_y)^2)\,dy \\ &= [2DD'_y|_0^1 - 2\int_0^1 (D'_y)^2\,dy = -2\int_0^1 (D'_y)^2\,dy \leq 0 \end{aligned}$$

Thus, S is a decreasing function that leads to

$$0 \leq S(t) \leq S(0) = 0 \qquad \text{showing that } D = 0.$$

2.1.4 Wave Equation

Another significant partial differential equation (PDE) is the wave equation that contributes to define the oscillations of a material's waves. The following demonstration (**Derivation of Wave Equation**) has an aim to extract an equation for z that has the following properties:

i) A homogeneous elastic string is set on a horizontal plane(u-axis).

ii) It is pulled tightly in an interval $[0,L]$ where L refers to length.

iii) Initially when time is zero, it is released to form a vibration.

iv) Vertical displacement of the string is defined by $z(t,u)$ at the point (t,u).

Assume a part of the string is stretched and released between the point $[u, u+\Delta u]$ on the horizontal axis then that point after the release changes its position of movement vertically creating the function's graph on $[u, u+\Delta u]$. Considering the tension that occurs in the chosen part of the string with the forces $\mathbf{F}(t,u)$ and $\mathbf{F}(t,u+\Delta u)$, if both $\mathbf{F}(t,u)$ and u-axis have an angle $\beta(t,u)$, then one can notice the absence of horizontal movement since $\mathbf{F}(t,u)\cdot \exp_u = \mathbf{F}(t,u+\Delta u)\cdot \exp_u$ meaning

$$\|\mathbf{F}(t,u)\|\cos\beta(t,u) = \|\mathbf{F}(t,u+\Delta u)\|\cos\beta(u+\Delta u) = F \tag{2.24}$$

Besides horizontal movement, vertical movement also exists. By using Newton's Second Law,

$$\mathbf{F}(t,u+\Delta u)\cdot \exp_z - \mathbf{F}(t,u)\cdot \exp_z = ma$$

where acceleration $a = z''_{tt}(t,u)$ and mass $m = p\Delta u$ that p is the mass density per unit length. By considering these in (2.24),

$$\|\mathbf{F}(t,u+\Delta u)\|\sin\beta(u+\Delta u) - \|\mathbf{F}(t,u)\|\sin\beta(t,u) = pz''_{tt}(t,u)\Delta u \tag{2.25}$$

From (2.24) & (2.25),

$$\frac{\|\mathbf{F}(t,u+\Delta u)\|\sin\beta(t,u+\Delta u)}{\|\mathbf{F}(t,u+\Delta u)\|\cos\beta(t,u+\Delta u)} = \frac{\|\mathbf{F}(t,u)\|\sin\beta(t,u)}{\|\mathbf{F}(t,u)\|\cos\beta(t,u)} = \frac{p}{F}z''_{tt}(t,u)\Delta u.$$

Moreover,

$$\frac{\tan\beta(t,u+\Delta u) - \tan\beta(t,u)}{\Delta u} = \frac{p}{F}z''_{tt}(t,u).$$

Since $\tan\beta(t,u) = z'_u(t,u)$,

$$\frac{z'_u(t,u+\Delta u) - z'_u(t,u)}{\Delta u} = \frac{p}{F}z''_{tt}(t,u).$$

As Δu approaches to 0 through the limit, we have

$$z''_{tt}(t,u) = c^2 z''_{uu}(t,u). \tag{2.26}$$

Here the constant $c = (\frac{F}{p})^{\frac{1}{2}}$, $c > 0$. In order to express (2.26) in a higher order form when $n > 1$, $z''_{uu}(t,u)$ is altered with $\nabla^2 z$ to have the following wave equation

$$z''_{tt}(t,u) = c^2\nabla^2 z(t,u).$$

Solution of Wave Equation by Fourier Method:

Consider one-dimensional wave equation

$$z''_{tt}(t,u) = z''_{uu}(t,u), \quad c = 1 \tag{2.27}$$

with the boundary condition as

$$z(t,0) = z(t,\pi) = 0 \tag{2.28}$$

and initial condition

$$z(0,u) = f(u),\ z'_t(0,u) = g(u),\ 0 \leq u \leq \pi \tag{2.29}$$

First, by separation of variables, let $z(t,u) = w(t)\phi(u)$. By substituting the equality into (2.27), we have

$$w''(t)\phi(u) = w(t)\phi''(u)$$
$$\frac{w''(t)}{w(t)} = \frac{\phi''(u)}{\phi(u)}.$$

In the previous equality, left and right-hand sides are independent on t and u, so if we rewrite by defining them with k, we obtain

$$w''(t) = kw(t) \text{ and } \phi''(u) = k\phi(u),$$

by using the solution method for the heat equation, one can see that we have a trivial solution $z(t,u) = 0$ occured when $k \geq 0$ so when $k < 0$, we get

$$w''(t) = -\mu^2 w(t) \text{ and } \phi''(u) = -\mu^2\phi(u) \qquad \text{where } \mu \neq 0$$

with a solution

$$w(t) = A\cos\mu t + B\sin\mu t \text{ and } \phi(u) = C\sin\mu u + D\cos\mu u$$

where A, B, C and D are constants.

Respectively,

$$z(t,u) = (A\cos\mu t + B\sin\mu t)(C\sin\mu u + D\cos\mu u).$$

Verification of (2.28) causes $D = 0$ and μ becomes the set of all integer numbers but due to the fact of sin(-μt) $= -\sin(\mu t)$. It is possible to accept just $\mu = 1, 2, 3, \ldots$ Then every one of

$$z_m(t,u) = (a_m \cos mt + b_m \sin mt) \sin mu \tag{2.30}$$

can be a problem solver for obtaining a nontrivial function of two variables that (2.27)–(2.29) holds. In general, linear combination of two solutions in (2.27) that (2.28) holds as well is again acceptable as a solution. Therefore, it is vital to search a possible solution in (2.27)–(2.29) as follows:

$$z(t,u) = \sum_{m=1}^{\infty} z_m(t,u) = \sum_{m=1}^{\infty} (a_m \cos mt + b_m \sin mt) \sin mu \tag{2.31}$$

Here both a_m and b_m can be denoted from (2.29) as:

$$f(u) = z(0,u) = \sum_{m=1}^{\infty} a_m \sin mu$$
$$g(u) = z_t'(0,u) = \sum_{m=1}^{\infty} m b_m \sin mu.$$

The above equations are the half range fourier sine expansions belonging to the functions $f(u)$ and $g(u)$. Furthermore,

$$a_m = \frac{2}{\pi} \int_0^{\pi} f(r) \sin mr \, dr \ \text{ and } \ b_m = \frac{2}{m\pi} \int_0^{\pi} g(r) \sin mr \, dr \tag{2.32}$$

with an assumption of keeping the fourier series convergent that

$$\sum_{m=1}^{\infty} a_m^2 < \infty \ \text{ and } \ \sum_{m=1}^{\infty} m^2 b_m^2 < \infty.$$

Therefore, (2.31) and (2.32) solve the problem for (2.27)–(2.29).

For checking the uniqueness of this solution, assume z_1 and z_2 are two solutions. Equations (2.27)–(2.29) are satisfied by $S = z_1 - z_2$ with $f = g = 0$. Consider

$$R(t) = \frac{1}{2} \int_0^{\pi} (S_t'(t,u)^2 + S_u'(t,u)^2) \, du.$$

Then,

$$
\begin{aligned}
R'_t &= \int_0^{\pi} (S'_t S''_{tt} + S'_u S''_{ut})\, du \\
&= \int_0^{\pi} S'_t S''_{tt}\, du + [S'_u S'_t|_0^{\pi} - \int_0^{\pi} S'_t S''_{uu}\, du \\
&= \int_0^{\pi} S'_t (S''_{tt} - S''_{uu})\, du = 0.
\end{aligned}
$$

Thus, $R(t)$ is constant. Since $R(0) = 0$, it leads to $R(t) = 0$ meaning $S'_t = S'_u = 0$ that shows S is constant. Moreover $S = 0$ as it satisfies the condition (2.27)–(2.29), so $z_1 = z_2$.

What's more, after describing both heat and wave equations, a discussion can be made between initial and boundary conditions. The initial condition from the heat equation declares the initial temperature distribution inside the rod so it can be called as initial state instead, but when boundary condition is the case and gathers with heat equation, then the formation of them likely shares the initial state to other non-initial states. In comparison with wave equation of this fact, not only will there be one initial state $z(0,u)$ but also $z'_t(0,u)$ will be included. Therefore, the expression of wave equation will be

$$
z(t,u) = \begin{bmatrix} z(t,u) \\ z'_t(t,u) \end{bmatrix}
$$

and as a matrix, (2.27) is stated as:

$$
\begin{bmatrix} z \\ z'_t \end{bmatrix}'_t (t,u) = \begin{bmatrix} 0 & 1 \\ \frac{\partial^2}{\partial u^2} & 0 \end{bmatrix} \begin{bmatrix} z \\ z'_t \end{bmatrix} (t,u).
$$

Furthermore, $z(t,u)$ from (2.31) is came together with

$$
z'_t(t,u) = \sum_{m=1}^{\infty} (-ma_m \sin mt + mb_m \cos mt) \sin mu
$$

concluding,

$$\begin{bmatrix} z \\ z'_t \end{bmatrix}(t,u) = \sum_{m=1}^{\infty} \sin mu \begin{bmatrix} \cos mt & m^{-1}\sin mt \\ -m\sin mt & \cos mt \end{bmatrix} \begin{bmatrix} f_m \\ g_m \end{bmatrix}$$

where $f_m = a_m$ and $g_m = mb_m$ are n^{th} coefficients of the Fourier sine expansions of f and g respectively.

2.1.5 Delay Equations

Delay is a well-known used feature in differential equations especially for the engineers that has an interest in control systems which likely contributes to the developing models to act close to a real system and will also make the related processes more precisely predicted. As a definition, time-delayed systems are also used for Delay Differential Equations (DDEs) that leads to do actual delays and time lags because by doing that, solving high order models will have a simplified approach which are a good contributor for the simplification of higher order models. A typical form of a delay equation can be represented as

$$x'(t) = f(t, x(t), x(t-\tau)), \;\; x(t) \in \mathbb{R},$$

and positive delay τ is constant. The following examples will show the semigroups of delay equation.

Let $X \in H$, let $[x,y]$ be a finite interval in $\mathbb{R}$ and let $W^{1,\rho}(x,y;X)$ be the class of all functions $f : [x,y] \to X$ that can be expressed in the form

$$f_t = f_x + \int_x^t g_r \, dr = f_b - \int_t^y g_r \, dr, \; x \leq t \leq y,$$

for some $g \in L_\rho(x,y;X)$, $1 \leq \rho \leq \infty$. The notation $W^{n,\rho}(x,y;X)$, where $n = 2,3,...$ and $1 \leq \rho \leq \infty$, will denote the class of functions $f : [x,y] \to X$ which have $(n-1)st$ derivative in $W^{1,\rho}(x,y;X)$. Under the corresponding norm, $W^{n,\rho}(x,y;X)$ is a Banach space. Particularly, $W^{1,2}(x,y;X)$ is a Hilbert space in which a scalar product can be defined by

$$\langle f,g\rangle_{W^{1,2}} = \langle f_y, g_y\rangle + \int_x^y \langle f'_t, g'_t\rangle \, dt.$$

Proposition 2.1: Consider $X \in H$ and assume the interval $[x,y]$ is finite in $\mathbb{R}$.

1. The differential operator $\frac{d}{dr} \in \tilde{L}(L_2(x,y;X))$ where the operator is defined as

$$D(\frac{d}{dr}) = \{b \in W^{1,2}(x,y;X) : b_y = 0\},$$

and the adjoint of the operator $(\frac{d}{dr})^*$ equals to $-\frac{d}{dr}$ with

$$D(-\frac{d}{dr}) = \{b \in W^{1,2}(x,y;X) : b_x = 0\}.$$

2. The differential operator $\frac{d^2}{dr^2} \in \tilde{L}(L_2(x,y;X))$ where the operator is defined as

$$D(\frac{d^2}{dr^2}) = \{b \in W^{2,2}(x,y;X) : b_x = b_y = 0\},$$

and the adjoint of the operator $(\frac{d^2}{dr^2})^*$ equals to $\frac{d^2}{dr^2}$.

Example 2.1: Let us consider a partial differential equation as

$$\frac{\partial}{\partial t} x_{t,\omega} = \frac{\partial}{\partial \omega} x_{t,\omega} \ , t > 0$$

with its initial and boundary conditions respectively

$$x_{0,\omega} = f_\omega, \ -\varepsilon \le \omega \le 0, \ f \in W^{1,2}(-\varepsilon,0;X)$$

$$x_{t,0} = 0, \ t \ge 0,$$

that has a solution as

$$x_{t,\omega} = \begin{Bmatrix} f_{\omega+t}, & \omega+t \le 0 \\ 0, & \omega+t > 0 \end{Bmatrix}, \ [-\varepsilon,0], \ t \ge 0.$$

Assume $\tilde{X}$ be the space $L_2(-\varepsilon,0;X)$ and also accept $\frac{d}{d\omega}$ as a differential operator with

$$D(\frac{d}{d\omega}) = \left\{b \in W^{1,2}(-\varepsilon,0;X) : b_0 = 0\right\}.$$

By using Proposition 2.1, since $\frac{d}{d\omega}$ is densely defined closed linear operator, it can be expressed as $\frac{d}{d\omega} \in \tilde{L}(\tilde{X})$. By considering $y_t = [x_t]$ and

$$[T_t^* b]_\omega = \left\{ \begin{array}{ll} b_{\omega+t}, & \omega+t \leq 0 \\ 0, & \omega+t > 0 \end{array} \right\}, \ [-\varepsilon, 0], \ t \geq 0, \ b \in \tilde{X}, \tag{2.33}$$

the mentioned problem above and solution of it can be expressed as

$$y_t' = \frac{d}{d\omega} y_t, \ t > 0, \ y_0 = f \in D(\frac{d}{d\omega}) \quad \text{where} \ \ y_t = T_t^* f, \ t \geq 0.$$

The notation indicates the differential operator $\frac{d}{d\omega}$ generates the notation T_t^* in (2.33) so that it becomes continuous semigroup in a strong topology and also the notation T^* is called a semigroup of left translation.

Example 2.2: Suppose from Example 2.1, the assumptions are also valid here. Let $\varepsilon > 0$, and $\tilde{X} = L_2(-\varepsilon, 0; X)$ and also the differential operator $\frac{d}{d\omega} \in \tilde{L}(\tilde{X})$. Again by using Proposition 2.1, it is known that $(\frac{d}{d\omega})^* = -\frac{d}{d\omega}$ with

$$D(-\frac{d}{d\omega}) = \{b \in W^{1,2}(-\varepsilon, 0; X) : b_{-\varepsilon} = 0\}.$$

From (2.33), one can say that T is generated by the operator $-\frac{d}{d\omega}$ which is a continuous semigroup in a strong topology and has an expression as

$$[T_t b]_\omega = \left\{ \begin{array}{ll} b_{\omega-t}, & \omega - t \geq -\varepsilon \\ 0, & \omega - t < -\varepsilon \end{array} \right\}, \ [-\varepsilon, 0], \ t \geq 0, \ b \in \tilde{X}. \tag{2.34}$$

The definition above shows the semigroup of right translation.

Example 2.3: Consider the following one-dimensional semilinear system

$$x'(t) = ax(t) + bx(t-\varepsilon) + u(t) + f(t, x(t), u(t)), \tag{2.35}$$

since a delay exists above, the system (2.35) is in infinite dimensional space and controlled by a first-order partial differential equation. Here $\varepsilon > 0$ and b must be non-zero in order to keep the existence of the delay in (2.35) that is shown as

$bx(t-\varepsilon)$. Then, according to the following assumed initial conditions;

$$x(0)=\lambda \quad \text{and} \quad x(\phi)=\kappa(\phi), \quad \text{where } -\varepsilon\le\phi\le 0,$$

it is possible for the state space to be defined as $X=\mathbb{R}\times L_2(-\varepsilon,0)$ that means $\mathbb{R}$ and square integrable functions in a border of $[-\varepsilon,0]$. It is known that

$$\tilde{x}(t)=\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}, \; so \; \frac{d}{dt}\tilde{x}(t)=\begin{bmatrix} \frac{d}{dt}x(t) \\ \frac{d}{dt}\bar{x}(t) \end{bmatrix}=\begin{bmatrix} x'(t) \\ x'(t+\phi) \end{bmatrix},$$

where $\bar{x}(t)$ is the function between the values of t and $t-\varepsilon$.

Suppose a linear operator is defined as A, then

$$A\begin{bmatrix} \lambda \\ \kappa \end{bmatrix}=\begin{bmatrix} a\lambda+b\kappa(-\varepsilon) \\ \frac{d}{d\phi}(\kappa-\lambda) \end{bmatrix}, \tag{2.36}$$

with an expression for its domain,

$$D(A)=\{\begin{bmatrix} \lambda \\ \kappa \end{bmatrix}\in X \;\; such\ that\ \frac{d}{d\phi}\kappa\in L_2(-\varepsilon,0)\ and\ \kappa(0)=\lambda\}. \tag{2.37}$$

Moreover

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}=\begin{bmatrix} ax(t)+bx(t-\varepsilon) \\ \frac{d}{d\phi}x(t+\phi) \end{bmatrix}+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)+\begin{bmatrix} f(t,x(t),u(t)) \\ 0 \end{bmatrix} \tag{2.38}$$

where

$$\tilde{x}(t)=\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}, \; \tilde{x}(0)=\begin{bmatrix} \lambda \\ \kappa \end{bmatrix}, \; B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; F(t,\tilde{x},u)=\begin{bmatrix} f(t,x,u) \\ 0 \end{bmatrix}, \tag{2.39}$$

with

$$[\bar{x}(t)](\phi)=\begin{cases} x(t+\phi) \quad if \quad t+\phi>0 \\ \kappa(t+\phi) \quad if \quad t+\phi\le 0. \end{cases}. \tag{2.40}$$

In (2.38), we know that $\bar{x}(t)=x(t+\phi)$ and if separately the derivative of $\bar{x}(t)$ with respect to t and $x(t+\phi)$ with respect to ϕ are taken, then it is clear that an identity solution of $x'(t+\phi)$ occurs. By considering this fact, it is not possible to be in one dimensional space anymore as the combination of the two spaces $\mathbb{R}$ and L_2 will create infinite dimensional space and this result aids the system to be able to get rid of the delay in (2.35). Therefore, the system (2.35) can be rewritten as

$$\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + Bu(t) + F(t, \tilde{x}(t), u(t)). \tag{2.41}$$

2.2 Semilinear Systems

Consider the following linear system

$$z'(t) = Az(t) + f(t), \qquad 0 < t \leq T \tag{2.42}$$

$$z(0) = z_0 \in Z.$$

Here the following assumptions are made:

i) Z is a separable Hilbert space

ii) $f \in L_1(0, T; Z)$

iii) $A : D(A) \subseteq Z \to Z$ is densely defined linear closed operator on Z generating a strongly continuous semigroup e^{At}.

If Z is a finite dimensional Euclidean space, then according to variation of constant formula,

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-r)} f(r)\, dr \tag{2.43}$$

is a unique solution of (2.42), but in infinite dimensional spaces, $z(t)$ defined by (2.43) may not belong to $D(A)$, so (2.43) is not a solution of (2.42) in ordinary(strong) sense however as the function in (2.43) is still considered as a mild solution of (2.42). By developing this idea, one can consider f in (2.42) depending on z and obtain semilinear differential equation which is

$$z'(t) = Az(t) + f(t, z(t)), \qquad 0 < t \leq T \tag{2.44}$$

$$z(0) = z_0 \in Z.$$

Under mild solution of equation (2.44), it is possible to understand the solution of integral equation

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-r)} f(r, z(r))\, dr. \tag{2.45}$$

It is known that (2.45) has unique solution (which stands for mild solution for (2.44)) under these conditions that $f : [0,T] \times Z \to Z$ holds:

i) $\forall z \in Z$, $f(\cdot, z)$ is strongly measurable

ii) there exists $H \in L_1(0,T)$ such that

$$\left\{ \begin{array}{c} |f(t,z) - f(t,x)| \leq H(t)\, |z - x|, \\ |f(t,0) \leq H(t)|, \end{array} \right\} \tag{2.46}$$

iii) $\forall z, x \in Z$ and $\forall t \in [0,T]$.

Theorem 2.6: Let the previous assumptions for A are hold and f be continuous on $[0,T]$ and Lipschitz continuous on Z. Then $\forall z \in Z$, the equation (2.42) has only one mild equation $z \in \mathbb{C}(0,T;Z)$. Furthermore, from Z into $z \in \mathbb{C}(0,T;Z)$, mapping of $z_0 \to z$ is Lipschitz continuous.

Proof. let $z_0 \in Z$, then by the equation

$$(Fz)(t) = e^{At} z_0 + \int_0^t e^{A(t-r)} f(r, z(r))\, dr, \quad [0,T] \tag{2.47}$$

it is possible to create a mapping

$$F : z \in \mathbb{C}(0,T;Z) \to z \in \mathbb{C}(0,T;Z).$$

Since $\|z\|_\infty \in \mathbb{C}(0,T;Z)$, by the definition of F,

$$\|(Fz)(t) - (Fw)(t)\| \leq MKt\, \|z - w\|_\infty \tag{2.48}$$

where M represents the boundedness of $\|T(t)\|$ on $[0,T]$. By (2.47) & (2.48) and applying induction method on μ,

$$\|(F^\mu z)(t) - (F^\mu w)(t)\| \leq \frac{(MKt)^\mu}{\mu!} \|z - w\|_\infty$$

thus,

$$\left\| \left\| F^{\mu} z - F^{\mu} w \right\| \leq \frac{(MKT)^{\mu}}{\mu!} \left\| z - w \right\|_{\infty} \right\|. \tag{2.49}$$

In the case of having μ large enough $\frac{(MKT)^{\mu}}{\mu!} < 1$ and considering the contraction principle will lead to a unique fixed point z that F has in $\mathbb{C}(0,T;Z)$. This result shows the existence of the mild solution (2.45) .

Assume that w is a solution of (2.42) with an initial value w_0 on $[0,T]$. Then,

$$\begin{aligned} \|z(t) - w(t)\| &\leq \left\| e^{At} z_0 - e^{At} w_0 \right\| + \int_0^t \left\| e^{A(t-r)} [f(r, z(r)) - f(r, w(r))] \right\| \, dr \\ &\leq M \|z_0 - w_0\| + MK \int_0^t \|z(r) - w(r)\| \, dr \end{aligned} \tag{2.50}$$

which indicates by Gronwall's Inequality,

$$\|z(t) - w(t)\| \leq M e^{MKT} \|z_0 - w_0\|$$

so,

$$\|z - w\|_{\infty} \leq M e^{MKT} \|z_0 - w_0\|.$$

This shows the uniqueness of z and the Lipschitz continuity of the map $z_0 \to z$. $\square$

Chapter 3

CONTROLLABILITY OF LINEAR DETERMINISTIC SYSTEMS

In this chapter, some controllability concepts and their conditions will be introduced and explained which are related to controllability of Linear Deterministic Systems. Firstly, we will consider finite dimensional system that Kalman's rank condition takes place. Originally, controllability was introduced by Rudolf E. Kalman in 1960 that is a significant property of control systems and as a definition, a control system is controllable if it can transfer any initial state to any final state within a finite time using admissible controls. The rank condition is the most valuable in finite dimensional systems; however, in infinite dimensional systems, it does not work. Therefore, two major concepts of controllability are developed which are exact and approximate controllability. These two concepts are defined for infinite dimensional systems and will be examined in upcoming sections.

3.1 Controllability in Finite Dimensions for Linear Deterministic Systems

The aim of this part is to study the Kalman's rank condition that is only used in finite dimensions and also to discuss the controllability concept of linear systems for finite dimensions. Therefore, the basic form of the initial valued linear control system is shown below:

$$z'(t) = Az(t) + Bu(t), \quad 0 < t \leq T \tag{3.1}$$

$$z_0 = \beta \in Z$$

In the system (3.1), $A \in M(n \times n)$ and $B \in M(n \times m)$ are defined as matrices of respective dimensions. $M(n \times m)$ notation gathers all the matrices together in a set where n and m are rows and columns respectively. By considering the presence of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, let the state and control spaces be $Z = \mathbb{R}^n$ and $U = \mathbb{R}^m$ respectively. The system (3.1) has only one solution as follows:

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-r)} Bu(r)\, dr, \;\; 0 < t \leq T. \tag{3.2}$$

The control u has a task of moving the initial state a to the final state b within a finite time $T > 0$ and this process can be represented with $z^{a,u}(T) = b$. The notation can also be explained as b is reachable or attainable from a. For the system (3.1), Q_T matrix also exists that is called the controllability Gramian and is shown as follows:

$$Q_T = \int_0^T e^{Ar} BB^* e^{A^* r}\, dr, \tag{3.3}$$

where Q_T is a matrix and A^* and B^* are defined as the transpose of the A and B matrices respectively.

Proposition 3.1: If Q_T is non-singular matrix, then

i)
$$u(r) = -B^* e^{A^*(T-r)} Q_T^{-1} (e^{AT} c - d) \tag{3.4}$$

is the control that moves state c to final state d at $T > 0$ for every $c, d \in \mathbb{R}^n$ where $r \in [0, T]$.

ii) The functional $\int_0^T |u(r)|^2 dr$ considered over all controls transferring c to d takes minimal value at $u(r)$ defined in (3.4). Moreover for u from (3.4),

$$\int_0^T |u(r)|^2 dr = \left\langle Q_T^{-1}(e^{AT} c - d), e^{AT} c - d \right\rangle \tag{3.5}$$

Proof. Combining (3.4) and solution of control system (3.2), we have

$$
\begin{aligned}
z^{c,u}(T) &= e^{AT}c - \left(\int_0^T e^{A(T-r)}BB^* e^{A^*(T-r)}dr\right)(Q_T^{-1}(e^{AT}c-d)) \\
&= e^{AT}c - Q_T Q_T^{-1}(e^{AT}c-d) \\
&= e^{AT}c - e^{AT}c + d = d.
\end{aligned}
$$

By showing this, (i) is achieved.

For (ii), let us observe the expansion of the integral (3.5):

$$
\begin{aligned}
\int_0^T |u(r)|^2 dr &= \int_0^T \left| B^* e^{A^*(T-r)} Q_T^{-1}(e^{AT}c-d)\right|^2 dr \\
&= \left\langle \int_0^T e^{A(T-r)}BB^* e^{A^*(T-r)}(Q_T^{-1}(e^{AT}c-d))\, dr, Q_T^{-1}(e^{AT}c-d)\right\rangle \\
&= \left\langle Q_T Q_T^{-1}(e^{AT}c-d), Q_T^{-1}(e^{AT}c-d)\right\rangle \\
&= \left\langle Q_T^{-1}(e^{AT}c-d), e^{AT}c-d\right\rangle.
\end{aligned}
$$

Moreover, for any arbitrary control $u(\cdot)$ that transfers c to d at time T, can be assumed that $u(\cdot)$ is square integrable on $0 \le r \le T$. Then,

$$
\begin{aligned}
\int_0^T \langle u(r), \widehat{u}(r)\rangle \; dr &= -\int_0^T \left\langle u(r), B^* e^{A^*(T-r)} Q_T^{-1}(e^{AT}c-d)\right\rangle \; dr \\
&= -\left\langle \int_0^T e^{A(T-r)}Bu(r)\; dr, Q_T^{-1}(e^{AT}c-d)\right\rangle \\
&= \left\langle e^{AT}c-d, Q_T^{-1}(e^{AT}c-d)\right\rangle.
\end{aligned}
$$

Thus,

$$
\int_0^T \langle u(r), \widehat{u}(r)\rangle \; dr = \int_0^T \langle \widehat{u}(r), \widehat{u}(r)\rangle \; dr.
$$

By considering the previous equivalence, we get

$$
\int_0^T |u(r)|^2 dr = \int_0^T |u(r)|^2 dr + \int_0^T |u(r) - \widehat{u}(r)|^2 dr.
$$

$\square$

Proposition 3.2: If for any state $d \in \mathbb{R}^n$ is reachable from 0, then for any $T > 0$, the matrix Q_T is non-singular.

Proof. Assume the following integral is an expression of a linear operator that is taken from $U_T = L_1[0, T : \mathbb{R}^m]$ into $\mathbb{R}^n$,

$$G_T u = \int_0^T e^{Ay} B u(T-y)\, dy. \tag{3.6}$$

Note that

$$G_T u = z^{0,u}(T). \tag{3.7}$$

Considering (3.7), one can have a result D_T is non-decreasing on $T > 0$ where $D_T = G_T(U_T)$. Since all the elements of D_T are in $\mathbb{R}^n$, one can define $D_{\tilde{T}} = \mathbb{R}^n$ for some $\tilde{T}$ according to the dimensions of D_T. For any $T > 0$, $w \in \mathbb{R}^n$ and $u \in U_T$,

$$\begin{aligned} \langle Q_T w, w \rangle &= \left\langle \left(\int_0^T e^{Ay} B B^* e^{A^* y}\, dy \right) w, w \right\rangle \\ &= \int_0^T \left| B^* e^{A^* y} w \right|^2 dy. \end{aligned} \tag{3.8}$$

$$\langle G_T u, w \rangle = \int_0^T \left\langle u(y), B^* e^{A^*(T-y)} w \right\rangle\, dy. \tag{3.9}$$

According to (3.8) and (3.9), one can get $Q_T w = 0$ if one of the two conditions are satisfied which are the linear space Q_T is orthogonal to v or $B^* e^{A^*(\cdot)} w$ has an equality with zero on $0 \leq y \leq T$. Therefore the function equals to 0 everywhere leading to $Q_T w = 0\ \forall T > 0$ and in addition $Q_{\tilde{T}} w = 0$. It is given that $D_{\tilde{T}} = \mathbb{R}^n$, so $w = 0$ and proof is achieved. □

Theorem 3.1: Consider the polynomial $h(\mu) = \mu^n + c_1 \mu^{n-1} + \ldots + c_n$ and any $A \in M(n \times n)$, then

$$A^n + c_1 A^{n-1} + \ldots + c_n I = 0.$$

The following theorem will introduce Kalman's Rank Condition that is beneficial for finite dimensional control systems and it can be taken into account as necessary and sufficient condition of controllability of linear systems in finite dimensions. For any assigned matrices such as $A \in M(n \times n)$ and $B \in M(n \times m)$, the linear system (3.1) is controllable if $rank[B, AB, A^2B, ..., A^{n-1}B] = n$. This means that the number of $n \times n$ (row) is less than the number of $n \times m$ (column) so the rank is n. In addition, the matrix $[A : B]$ has a representation as $[B, AB, ..., A^{n-1}B] \in M(n \times nm)$.

Theorem 3.2: [15] The following statements are equivalent:

i) An arbitrary state $b \in \mathbb{R}^n$ is reachable from 0.

ii) The control system (3.1) is controllable which means from the initial state μ, every state in $\mathbb{R}^n$ is reachable.

iii) Q_T is non-singular for some $T > 0$.

iv) Q_T is non-singular for an arbitrary $T > 0$.

v) $rank[A : B] = n$.

Propositions 3.1 and 3.2 give the proof for the implications $(i) - (v)$, but to prove the implications for (iv), the next lemma should be considered.

Lemma 3.1: Suppose d_n has an identical image with the transformation G_T and particularly, G_T is onto if and only if d_n is onto.

Proof. First, let d_n be a linear mapping with an expansion as follows:

$$dn(u_0, ..., u_{m-1}) = \sum_{i=0}^{m-1} A^i B u_i, \quad \text{where } u_i \in \mathbb{R}^m, \;\; i = 0, 1, ..., m-1$$

then arbitrarily choosing $w \in \mathbb{R}^n, u \in L_1[0, T; \mathbb{R}^m], u_i \in \mathbb{R}^m$, $i = 0, 1, ..., m-1$:

$$\langle G_T u, w \rangle = \int_0^T \left\langle u(r), B^* e^{A^*(T-r)} w \right\rangle \, dr,$$

$$\langle d_n(u_0, u_1, ..., u_{m-1}), w \rangle = \langle u_0 B^* w \rangle + ... + \left\langle u_{m-1}, B^*(A^*)^{m-1} w \right\rangle .$$

Let $\langle d_n(u_0, u_1, ..., u_{m-1}), w \rangle = 0$, then $B^* w = 0, ..., B^*(A^*)^{m-1} w = 0$. Considering

Theorem 3.1, and using matrix A^*, one can obtain

$$(A^*)^m = \sum_{z=0}^{m-1} k_z (A^*)^z, \quad \text{where constant } k = k_0, k_1, ..., k_{m-1}.$$

Application of induction will result for any $j = 0, 1, ...$ $\exists k_{j,0}, ..., k_{j,m-1}$ such that

$$(A^*)^{m+1} = \sum_{z=0}^{m-1} k_{j,z} (A^*)^z.$$

Thus,

$$B^*(A^*)^z w = 0 \text{ for } z = 0, 1, ...$$

Furthermore,

$$B^* e^{A^*(s)} w = \sum_{z=0}^{\infty} B^*(A^*)^z \frac{ws^z}{z!}, \quad s \geq 0,$$

we conclude that

$$B^* e^{A^*(s)} w = 0 \text{ for any } T > 0 \text{ with an interval } 0 \leq s \leq T,$$

therefore,

$$\langle G_T u, w \rangle = 0 \quad \text{where } u \in L_1[0, T; \mathbb{R}^m].$$

Now, let us consider u reversely that for any $u \in L_1[0, T; \mathbb{R}^n]$, $\langle G_T u, w \rangle = 0$. After that, $B^* e^{A^*(s)} w = 0$ and differentiation of the next identity

$$\sum_{z=0}^{\infty} B^*(A^*)^z \frac{ws^z}{z!} = 0, \quad 0 \leq s \leq T,$$

$0, 1, ..., (m-1)$ times and assigning $\forall s = 0$, we get $B^*(A^*)^z w = 0$ where $z = 0, 1, 2, ..., m-1$. According to that,

$$\langle d_n(u_0, u_1, ..., u_{m-1}), w \rangle = 0 \text{ for any } u_0, u_1, ..., u_{m-1} \in \mathbb{R}^m.$$

□

By considering the system(3.1) as controllable, system will conclude the transformation G_T is onto $\mathbb{R}^n$ for all $T > 0$ and with previous lemma that has been proved, the matrix $[A:B]$ equals to rank of n. Reversely, considering $rank[A:B]=n$ will lead to d_n to be onto $\mathbb{R}^n$, thus, the transformation G_T is onto $\mathbb{R}^n$,so $(iv)-(i)$ equivalences are also satisfied.

Example 3.1: Assume that matrix A and vector B exists in $\mathbb{R}^2$ with a control system (3.1)

$$A=\begin{bmatrix}1 & 2\\ 0 & -4\end{bmatrix}, \quad B=\begin{bmatrix}-1\\ 2\end{bmatrix}.$$

It can be calculated:

$$rank[A:B]=rank\begin{bmatrix}-1 & 3\\ 2 & -8\end{bmatrix}=2.$$

Here, by using Theorem 3.2, the system (3.1) is controllable.

Example 3.2: Assume that matrix A and vector B exists in $\mathbb{R}^2$ with a control system (3.1)

$$A=\begin{bmatrix}3 & 0\\ -2 & -1\end{bmatrix}, \quad B=\begin{bmatrix}2\\ -1\end{bmatrix}.$$

Obviously after some calculations,

$$rank[A:B]=rank\begin{bmatrix}2 & 6\\ -1 & -3\end{bmatrix}=1\neq \dim\mathbb{R}^2=2.$$

Therefore, considering Theorem 3.2 and since the $\dim\mathbb{R}^2$ doesn't match with the rank number 1, then the system (3.1) is not controllable.

3.2 Controllability in Infinite Dimensions for Linear Deterministic Systems

This section will contribute to understand the two concepts which are exact and approximate controllability for infinite dimensional spaces. The infinite case is similar to the finite case that the control system (3.1) can be used here as well but the properties of that will alter in this case. Let z and u be the state and control processes, therefore consider the following linear system

$$z'(t) = Az(t) + Bu(t), \quad 0 < t \leq T, \quad u \in U_{ad} = L_2(0,T;U)$$
$$z_0 = \beta \in Z. \tag{3.10}$$

Here, Z and U are separable Hilbert spaces and A is the generator of a strongly continuous semigroup e^{At} on Z and B is bounded operator from U to Z.

3.2.1 Exact Controllability For Linear Systems

Now, before passing through the definition of exact controllability, it is required to define an attainable set with time $t \in [0,T]$ which is

$$D_t^{\beta} = \{z_t^{\beta,u} such\ that\ u \in U_{ad}\}, \ \ \beta \in Z. \tag{3.11}$$

Definition 3.1: Let Z be the state space then the system (3.10) is said to be exactly controllable if $D_T^{\beta} = Z$ for time $T > 0$ for all $\beta \in Z$.

$R(\mu, -Q_T)$ is defined as the resolvent of the operator $-Q_T$ that

$$R(\mu, -Q_T) = (\mu I + Q_T)^{-1}, \quad where\ Q_T \geq 0 \text{ and } T > 0. \tag{3.12}$$

The linear operator $(\mu I + Q_T)^{-1}$ is coercive and therefore well-defined for $\mu > 0$.

Theorem 3.3: The following properties are equivalent:

i) The system (3.10) is exactly controllable,

ii) Q_T is coercive,

iii) $R(\mu, -Q_T)$ converges in a uniform topology as $\mu \to 0$,

iv) $R(\mu, -Q_T)$ converges in a strong topology as $\mu \to 0$,

v) $R(\mu, -Q_T)$ converges in a weak topology as $\mu \to 0$,

vi) $\mu R(\mu, -Q_T)$ converges in a uniform topology to the zero operator as $\mu \to 0$.

Proof. The equivalences of $(i) \leftrightarrow (ii)$ is clarified in plenty of books such as [1, 6]. As it is mentioned in the previous brief description for the resolvent operator, (ii) is also well-defined linear operator, therefore for the beginning of the proof, these two properties $(ii) \to (iii)$ will be considered. Assume Q_T is coercive then there exists $n > 0$ such that $\forall z \in Z$ and $\forall \mu \geq 0$,

$$\langle z, (\mu I + Q_T)z \rangle \geq (\mu + n) \|z\|^2.$$

The following equality shows the boundedness of $\|R(\mu, -Q_T)\|$ for $\mu \geq 0$,

$$\|R(\mu, -Q_T)\| = \left\|(\mu I + Q_T)^{-1}\right\| \leq \frac{1}{\mu + n} \leq \frac{1}{n}.$$

Then we obtain,

$$\begin{aligned}
\left\|R(\mu, -Q_T) - Q_T^{-1}\right\| &= \left\|(\mu I + Q_T)^{-1} - Q_T^{-1}\right\| \\
&= \left\|Q_T^{-1}(Q_T - \mu I - Q_T)(\mu I + Q_T)^{-1}\right\| \\
&\leq \mu \left\|Q_T^{-1}\right\| \left\|(\mu I + Q_T)^{-1}\right\| \\
&\leq \frac{\mu}{n^2}.
\end{aligned}$$

As a conclusion, $R(\mu, -Q_T)$ converges to Q_T^{-1} uniformly as $\mu \to 0$.

The proof of the implications $(iii) \Rightarrow (iv) \Rightarrow (v)$, by checking Definition 2.9 which has a content of the properties of convergent sequences of operators are trivial results. The proof of the implications of $(v) \Rightarrow (vi)$ will be straightforward consequence due to the boundedness of a weakly convergent sequence of operators.

Eventually, for $(vi) \Rightarrow (i)$, assume that

$$\mu \|R(\mu, -Q_T)\| = \mu \left\|(\mu I + Q_T)^{-1}\right\| \to 0, \quad as\ \mu \to 0. \tag{3.13}$$

Then, the following equation will form below after the application of square root on both sides to (3.13),

$$\mu^{\frac{1}{2}} \left\| (\mu I + Q_T)^{-\frac{1}{2}} \right\| \to 0, \quad as\ \mu \to 0.$$

One can obtain for a sufficiently small $\mu_0 > 0$,

$$\mu^{\frac{1}{2}} \left\| (\mu I + Q_T)^{-\frac{1}{2}} \right\| \leq \frac{1}{\sqrt{2}}.$$

Therefore, $\forall z \in Z$,

$$\begin{aligned} \|z\|^2 &= \left\| (\mu_0^{\frac{1}{2}} (\mu_0 I + Q_T)^{-\frac{1}{2}}) (\mu_0^{-\frac{1}{2}} (\mu_0 I + Q_T)^{\frac{1}{2}}) z \right\|^2 \\ &\leq \frac{1}{2} \left\| \mu_0^{-\frac{1}{2}} (\mu_0 I + Q_T)^{\frac{1}{2}} z \right\|^2 \\ &= \frac{1}{2} \left\langle \mu_0^{-1} (\mu_0 I + Q_T) z, z \right\rangle, \end{aligned}$$

implying

$$\left\langle \mu_0^{-1} (\mu_0 I + Q_T) z, z \right\rangle \geq 2 \|z\|^2$$

and finally,

$$\langle Q_T z, z \rangle \geq \mu_0 \|z\|^2 .$$

This tells that Q_T is coercive and as (i) is satisfied, the proof is completed. □

3.2.2 Approximate Controllability for Linear Systems

Exact controllability allows the system to move from any initial point into any final point but unfortunately, application of this concept may not be possible for some of the control systems that are in infinite dimensional spaces. According to that fact, a new concept of controllability has to be revealed so that is where approximate controllability takes place. It enables the system to move from any point to the set which is dense in that state space and because of that, the concept of approximate controllability is a weaker version in comparison with exact controllability. In this section, approximate controllability for deterministic systems will be initiated.

Definition 3.2: Suppose the control system (3.10) is given. It is said to be an approximately controllable if $\forall \beta \in Z$, $\overline{D_T^{\beta}} = Z$ for $T > 0$.

Lemma 3.2: For $\mu > 0$ and $f \in Z$, there exists only one optimal control $u^{\mu} \in U_{ad}$ where the following functional obtains its minimum value according to (3.10):

$$J(u) = \|z_T^u - f\|^2 + \mu \int_0^T \|u_t\|^2 \, dt. \tag{3.14}$$

For every $0 \leq t \leq T$,

$$u_t^{\mu} = -B^* e^{A^*(T-t)} R(\mu, -Q_T)(e^{AT}\mu - f), \tag{3.15}$$

and

$$z_T^{u^{\mu}} - f = \mu R(\mu, -Q_T)(e^{AT}\mu - f), \tag{3.16}$$

where $R(\mu, -Q_T)$ is defined as the resolvent of the operator $-Q_T$.

Proof. There is only one optimal control $u^{\mu} \in U_{ad}$ for the functional (3.14) that by computing the variation of the functional J, an optimal solution u^{μ} will be obtained as

$$u_t^{\mu} = -\frac{1}{\mu} B^* e^{A^*(T-t)} (x_t^{u^{\mu}} - f), \;\; a.e. \;\; 0 \leq t \leq T. \tag{3.17}$$

Substituting (3.17) into the equation (3.10),

$$\begin{aligned} z_T^{u^{\mu}} &= e^{AT}\beta + \frac{1}{\mu} \int_0^T e^{A(T-r)} B B^* e^{A^*(T-r)} (z_T^{u^{\mu}} - f) \, dr \\ &= e^{AT}\beta - \frac{1}{\mu} Q_T (z_T^{u^{\mu}} - f). \end{aligned}$$

Then,

$$\mu z_T^{u^{\mu}} = \mu e^{AT}\beta - Q_T (z_T^{u^{\mu}} - f), \tag{3.18}$$

which implies

$$(\mu I + Q_T) z_T^{u^{\mu}} = \mu e^{AT}\beta + Q_T f. \tag{3.19}$$

Since $(\mu I + Q_T)^{-1}$ exists, we get

$$
\begin{aligned}
z_T^{u^\mu} &= (\mu I + Q_T)^{-1}\mu e^{AT}\beta + (\mu I + Q_T)^{-1}(\mu I + Q_T - \mu I)f \\
&= \mu(\mu I + Q_T)^{-1}(e^{AT}\beta - f) + f,
\end{aligned}
$$

and eventually,

$$
z_T^{u^\mu} - f = \mu R(\mu, -Q_T)(e^{AT}\beta - f). \tag{3.20}
$$

This proves (3.16). In order to obtain (3.15), just substitute (3.16) into (3.17). □

Theorem 3.4: The following properties are equivalent

i) The system (3.10) is approximately controllable,

ii) $Q_T > 0$,

iii) For every $t \in [0,T]$, the equality $B^* e^{A^* t} z = 0$ points out that $z = 0$,

iv) $\mu R(\mu, -Q_T)$ converges in a strong topology to the zero operator as $\mu \to 0$,

v) $\mu R(\mu, -Q_T)$ converges in a weak topology to the zero operator as $\mu \to 0$.

Proof. The equivalences of $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$ are all clarified and shown in various books for instance [1,6]. To show the proof of $(i) \Leftrightarrow (iv)$, let the control system (3.10) be approximately controllable. Then by considering Lemma 3.2, for arbitrary $k \in Z$, the sequence $\{s^r\}$ that is in U_{ad} exists such that

$$
\left\| z_T^{s^r} - f \right\| \to 0 \ \text{ as } r \text{ approaches to } \infty. \tag{3.21}
$$

After that, for u^μ as a control and since μ is positive, the given functional in Lemma 3.2 takes on its minimum value so,

$$
\begin{aligned}
\left\| z_T^{u^\mu} - f \right\|^2 &\le \left\| z_T^{u^\mu} - f \right\|^2 + \mu \int_0^T \|u_t^\mu\|^2 \, dt \\
&\le \left\| z_T^{s^r} - f \right\|^2 + \mu \int_0^T \|s^r\|^2 \, dt.
\end{aligned} \tag{3.22}
$$

Accepting $\varepsilon > 0$ and choosing sufficiently large r will lead to

$$
\left\| z_T^{s^r} - f \right\| < \frac{\varepsilon}{\sqrt{2}}, \tag{3.23}
$$

and then one can accept $\alpha > 0$ as a sufficiently small value so for $0 < \mu < \alpha$,

$$\mu \int_0^T \|s_t^r\|^2 \, dt \le \frac{\varepsilon^2}{2}. \tag{3.24}$$

In addition, substituting (3.23) and (3.24) into (3.22), one can get $\left\| z_T^{u^\mu} - f \right\| \le \varepsilon^2$ for all $0 < \mu < \alpha$ that concludes the convergence of $z_T^{u^\mu}$ to f as $\mu \to 0$. By borrowing (3.16) from Lemma 3.2,

$$\left\| z_T^{u^\mu} - f \right\| = \left\| \mu R(\mu, -Q_T)(e^{AT}\mu - f) \right\| \le \varepsilon,$$

for arbitrary $f \in Z$, the strong convergence of $\mu R(\mu, -Q_T) \to 0$ as $\mu \to 0$ which is the statement (iv) holds.

Reversely from $(iv) \Rightarrow (i)$, let (iv) be satisfied then, considering Lemma 3.2, μ is chosen as a sufficiently small value such that

$$\left\| z_T^{u^\mu} - f \right\| = \left\| \mu R(\mu, -Q_T)(e^{AT}\mu - f) \right\|. \tag{3.25}$$

With respect to the statement (iv) and (3.25), $\left\| z_T^{u^\mu} - f \right\| \to 0$ and eventually, $z_T^{u^\mu} \to f$ as $\mu \to 0$. This tells that the system (3.10) is approximately controllable.

For $(iv) \Leftrightarrow (v)$, it is known that $(iv) \Rightarrow (v)$ exists directly as this is a true fact in functional analysis so to prove its reverse $(v) \Rightarrow (iv)$, assume (v) holds meaning for all $a, b \in Z$,

$$\langle \mu R(\mu, -Q_T)a, b \rangle \to 0 \text{ as } \mu \to 0.$$

To imply strong convergence, the fact $R(\mu, -Q_T) \ge 0$ is required to be used, so that

$$\begin{aligned} \|\mu R(\mu, -Q_T)a\|^2 &= \langle \mu R(\mu, -Q_T)a, \mu R(\mu, -Q_T)a \rangle \\ &\le (\|\mu R(\mu, -Q_T)\|^2)^{\frac{1}{2}} \mu \langle R(\mu, -Q_T)a, a \rangle \\ &\le \langle \mu R(\mu, -Q_T)a, a \rangle \to 0 \quad \text{as} \quad \mu \to 0. \end{aligned}$$

Here, a is an arbitrary value that $a \in Z$, therefore $\mu R(\mu, -Q_T)$ is strongly convergent which also proves (iv) and satisfies the theorem. $\square$

Example 3.3: Assume that there is a control system as follows:

$$v'(t) = Av(t) + Bu(t), \quad 0 < t \leq T, \;\; v_0 \in Z. \tag{3.26}$$

Here $Z = l_2$ is considered as a Hilbert space that is formed by numerical sequences $\{z_n\}$ which satisfy the condition $\sum\limits_{m=1}^{\infty} z_m^2 < \infty$. For Hilbert space, the inner product is shown below:

$$\langle (z_m, v_m) \rangle = \sum_{m=1}^{\infty} z_m v_m. \tag{3.27}$$

In addition, there is a fundamental basis set for this space as,

$$R = \{e_1 = (1,0,0,...),\; e_2 = (0,1,0,...),\; e_3 = (0,0,1,0,...),...\}.$$

Choosing $A = 0$ will lead to $e^{At} = e^{A^*t} = I$ and consider a matrix B,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The third implication in Theorem 3.4 will be used to check if the control system (3.26) is approximately controllable, so clearly,

$$\sum_{m=1}^{\infty} \langle Be_m, Be_m \rangle = B^2 \sum_{m=1}^{\infty} \langle e_m, e_m \rangle = \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.$$

Here B is represented as a Hilbert-Schmidt operator on l_2 so that $B \in L(l_2)$ and since $B = B^*$,

$$B^* e^{A^*t} z = 0 \;\text{ points }\; Bz = 0,$$

which clearly concludes that $z = 0$. Therefore, the system (3.26) is approximately controllable. Besides checking its approximate controllability, let us consider the other case that if it can be exactly controllable. As $B = B^*$,

$$Q_T = \int_0^T e^{Ar} BB^* e^{A^* r}\, dr = TB^2.$$

Thus,

$$\langle Q_T e_m, e_m \rangle = T \left\langle B^2 e_m, e_m \right\rangle = \frac{T}{m^2} \to 0 \text{ as } m \to \infty.$$

Since no value is found for $d > 0$ to show the inequality $\langle Q_T e_m, e_m \rangle \geq d \|e_m\|^2$ holds, the operator Q_T is not coercive and this fact implies that the system (3.26) is not exactly controllable.

3.2.3 Partial Controllability Concepts for Linear Systems

Until this section, concept of controllability of linear deterministic systems is studied in both finite and infinite dimensional spaces. According to Bashirov (2003), some control systems with higher order differential equations can be rewritten as a first-order differential equations (standard-formed systems) basically by expanding the dimension of the state space. This new concept is called partial controllability that is a weaker case in comparison with the conditions of ordinary deterministic control systems as the expanded state space is already involved in those conditions. Now, let us consider the following control system,

$$\begin{aligned} z(t) &= Az(t) + Bu(t), \quad 0 < t \leq T, \quad u \in U_{ad} = L_2(0,T;U) \\ z_0 &= \beta \in Z. \end{aligned} \tag{3.28}$$

As it is mentioned before, z and u are state and control processes respectively. Let Z, U, A and B has the identical properties with the system (3.10). There is only one mild solution for the system (3.28) as

$$z(t) = e^{At} z_0 + \int_0^t e^{A(t-r)} Bu(r)\, dr, \;\; 0 < t \leq T. \tag{3.29}$$

It is known that the controllability gramian Q_T has a vital role in controllability theory that its partial version $\tilde{Q}_T$ has a similar role in partial controllability theory. Therefore, by enlarging the controllability gramian into its partial version, it is possible to have

$$\tilde{Q}_T = LQ_TL^*, \quad [0,T]. \tag{3.30}$$

As the partial version $\tilde{Q}_T$ has an identical function with Q_T, it has also similar properties meaning, for all $[0,T]$, $R(\mu,-\tilde{Q}_T)$ is expressed as the resolvent of the operator $-\tilde{Q}_T$ then $R(\mu,-\tilde{Q}_T) = (\mu I + \tilde{Q}_T)^{-1}$ is well-defined for $\mu > 0$ where $\tilde{Q}_T \geq 0$.

Definition 3.3: Suppose Z is a separable Hilbert space and $\mathbf{H} \subset \mathbf{Z}$ is a closed subspace of $\mathbf{Z}$. Let the operator L project Z onto $\mathbf{H}$ as well**.** Then,

i) If $D_T^{\beta} = \mathbf{H}$ for all $\beta \in Z$, the control system (3.28) is L-partially exactly controllable.

ii) If $\bar{D}_T^{\beta} = \mathbf{H}$ for all $\beta \in Z$, the control system (3.28) is L-partially approximately controllable.

Theorem 3.5: The following properties are equivalent considering the conditions and notation in this section.

i) The control system (3.28) is L-partially exactly controllable,

ii) $\tilde{Q}_T$ is coercive,

iii) $R(\mu,-\tilde{Q}_T)$ converges in a uniform topology as $\mu \to 0$,

iv) $R(\mu,-\tilde{Q}_T)$ converges in a strong topology as $\mu \to 0$,

v) $R(\mu,-\tilde{Q}_T)$ converges in a weak topology as $\mu \to 0$,

vi) $\mu R(\mu,-\tilde{Q}_T)$ converges in a uniform topology to the zero operator as $\mu \to 0$.

Proof. When $Z = \mathbf{H}$, controllability concept of L-partially deterministic systems are similar to the concept of controllability for ordinary deterministic systems so the proof of this theorem can coincide with the proof of Theorem 3.3. Therefore, by changing the notation of controllability gramian Q_T with its partial version which is $\tilde{Q}_T$, it is possible to prove it again so no need for repetition of this proof. □

Theorem 3.6: The following properties are equivalent considering the conditions and notation in this section.

i) The control system (3.28) is L-partially approximately controllable,

ii) $\tilde{Q}_T > 0$,

iii) For every $t \in [0,T]$, the equality $\tilde{B}^* e^{\tilde{A}^* t} z = 0$ points out that $z = 0$,

iv) $\mu R(\mu, -\tilde{Q}_T)$ converges in a strong topology to zero as $\mu \to 0$,

v) $\mu R(\mu, -\tilde{Q}_T)$ converges in a weak topology to zero as $\mu \to 0$.

Example 3.4: Consider the following non-linear system

$$y^{(n)}(t) = f(t, y(t), y'(t), ..., y^{(n-1)}(t), u(t)). \tag{3.31}$$

$\mathbb{R}$ is considered as the state space of the system (3.31) where $y \in \mathbb{R}$. One can express the system (3.31) as a first-order differential equation

$$z'(t) = Az(t) + F(t, z(t), u(t)) \tag{3.32}$$

where

$$z(t) = \begin{pmatrix} z(t) \\ z'(t) \\ \vdots \\ z^{(n-2)}(t) \\ z^{(n-1)}(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and

$$F(t,z,u)=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t,y,y',\dots,y^{(n-1)},u) \end{pmatrix}.$$

For the system (3.32), the n-dimensional Euclidean space $\mathbb{R}^n$ is considered as the state space and equivalently, its reachable set becomes a subset of $\mathbb{R}^n$. Thus, controllability concepts for the system (3.32) are stronger than the ones in the system (3.31). On the other hand, by considering the following projection operator L

$$L=[1\ \ 0\ \ \cdots\ \ 0\ \ 0]:\mathbb{R}^n\to\mathbb{R},$$

it will be possible to keep the L-partial controllability concepts for the systems (3.31) and (3.32) similar.

Example 3.5: Consider the following semilinear wave equation

$$\frac{\partial^2 z_{t,\eta}}{\partial t^2}=\frac{\partial z_{t,\eta}}{\partial\eta^2}+f(t,z_{t,\eta},\partial z_{t,\eta}/\partial t,u_t), \tag{3.33}$$

where z is a real-valued binary function with $t\geq 0$ and $0\leq\eta\leq 1$. $L_2(0,1)$ is assigned as the state space of (3.33) that represents the space of all Lebesque measurable and square integrable functions. It is likely to express (3.33) in a form of first-order differential equation as follows:

$$x_t'=Ax_t+F(t,x_t,u_t) \tag{3.34}$$

where

$$x_t=\begin{pmatrix} z_{t,\eta} \\ \partial z_{t,\eta}/\partial t \end{pmatrix},\ \ A=\begin{pmatrix} 0 & I \\ d^2/d\eta^2 & 0 \end{pmatrix},$$

$$F(t,x,u) = \begin{pmatrix} 0 \\ \\ f(t,x_1,x_2,u) \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ \\ b \end{pmatrix}.$$

Here $x \in L_2(0,1) \times L_2(0,1)$. By enlarging the state space of the system (3.33), $L_2(0,1) \times L_2(0,1)$ is obtained which is the state space of the system (3.34). The concepts of controllability for (3.33) is strong in comparison with (3.34), therefore by constructing the operator L

$$L = [I \quad 0] : L_2(0,1) \times L_2(0,1) \to L_2(0,1),$$

the L-partial controllability concepts for the system (3.34) can be similar with the same concepts for the system (3.33).

Chapter 4

CONTROLLABILITY OF SEMILINEAR SYSTEMS

The sufficient conditions for the concept of controllability of semilinear control systems will be observed and studied in this chapter, since researches in necessary conditions are not well-known for the controllability of semilinear control systems. For the sufficient conditions, mostly fixed-point theorems are considered so that the given problems in controllability can be converted into a fixed-point problem. Not only fixed-point theorems are considered, but also principle of contraction mapping will be used in this chapter. It is likely to divide this chapter into two sections as exact controllability and approximate controllability of semilinear systems that both of them will contain results belonging to contraction mapping principles.

Let us consider a semilinear system in its general form

$$\begin{aligned} z(t) &= Az(t)+Bu(t)+f(t,z(t),u(t)), \quad 0<t\leq T \\ z(0) &= \beta \in Z. \end{aligned} \tag{4.1}$$

As usual, the state and control are respectively $z \in Z$ and $u \in U_{ad} = \mathbb{C}(0,T;U)$. The following assumptions are made according to this system :

i) Z and U are separable Hilbert spaces,

ii) A is the generator of a strongly continuous semigroup e^{At} on Z and B is a bounded operator from U to Z,

iii) The bounded function f is continuous on $[0,T]\times Z\times U$,

iv) The function f is Lipschitz continuous with respect to z and u, that is $\forall u,w \in U$

and $\forall z, x \in Z$, $\quad \|f(t,z,u) - f(t,x,w)\| \leq D(\|z-x\| + \|u-w\|)$

where $\forall t \in [0,T]$ and for some $D \geq 0$.

Considering the previous assumptions, for all $u \in U_{ad}$ and $\beta \in Z$, there exists a continuous function $z \in \mathbb{C}(0,T;Z)$ that consists of a unique mild solution satisfying the system (4.1) which is

$$z_t = e^{At}\beta + \int_0^t e^{A(t-r)}(Bu(r) + f(r,x(r),u(r)))\,dr. \tag{4.2}$$

4.1 Exact Controllability for Semilinear Systems

Contraction mapping principle will be considered in this section to observe the sufficient conditions of exact controllability in semilinear systems.

Let $\tilde{Z} = \mathbb{C}(0,T;Z)$. Then, $\left(\tilde{Z} \times U_{ad}, \|(\cdot,\cdot)\|\right)$ is a Banach space with

$$\|(\cdot,\cdot)\| = \|(\cdot,\cdot)\|_{\tilde{Z}\times U_{ad}} = \|\cdot\|_{\tilde{Z}} + \|\cdot\|_{U_{ad}}.$$

Lemma 4.1: Let Z and U be separable Hilbert spaces. Assume that A is a closed operator generating C_0-semigroup e^{At} and B is a bounded operator. Then, the following inequality holds

$$\|Q_t\| \leq \|Q_T\|, \quad t \in [0,T].$$

Proof. It is obvious that $Q_t = Q_t^*$ and $\langle Q_t z, z\rangle \geq 0$ for all $z \in Z$. Thus,

$$\|Q_t\| = \sup_{\|z\|=1} \langle Q_t z, z\rangle.$$

Therefore,

$$
\begin{aligned}
\langle Q_T z, z \rangle &= \int_0^T \left\langle e^{Ar} B B^* e^{A^* r} z, z \right\rangle \, dr \\
&= \langle Q_t z, z \rangle + \int_t^T \left\langle e^{Ar} B B^* e^{A^* r} z, z \right\rangle \, dr \\
&= \langle Q_t z, z \rangle + \int_t^T \left\langle B^* e^{A^* r} z, B^* e^{A^* r} z \right\rangle \, dr \\
&= \langle Q_t z, z \rangle + \int_t^T \left\| B^* e^{A^* r} z \right\|^2 \, dr \\
&\geq \langle Q_t z, z \rangle .
\end{aligned}
$$

So, this points out $\|Q_t\| \leq \|Q_T\|$. □

Lemma 4.2: Let Z and U be separable Hilbert spaces. Suppose the bounded function f is continuous on $[0,T] \times Z \times U$ and also $\exists d > 0$ such that $\langle Q_T z, z \rangle \geq d \|z\|^2 \ \forall z \in Z$. By considering these properties, for any arbitrary $c \in Z$, the non-linear operator $D : \tilde{Z} \times U_{ad} \to \tilde{Z} \times U_{ad}$, that is defined by

$$
D(x,w)(t) = (X(t), W(t)), \text{ for all } 0 \leq t \leq T, \tag{4.3}
$$

where

$$
\begin{aligned}
X(t) = {} & Q_t e^{A^*(T-t)} Q_T^{-1} \left(c - e^{AT}\beta - \int_0^T e^{A(T-t)} f(r, x(r), w(r)) \, dr \right) \\
& + e^{At}\beta + \int_0^t e^{A(t-r)} f(r, x(r), w(r)) \, dr,
\end{aligned} \tag{4.4}
$$

$$
W(t) = B^* e^{A^*(T-t)} Q_T^{-1} \left(c - e^{AT}\beta \right) - B^* e^{A^*(T-t)} Q_T^{-1} \int_0^T e^{A(T-r)} f(r, x(r), w(r)) \, dr, \tag{4.5}
$$

holds the inequality

$$
\|D(x,w)(t) - D(y,s)(t)\| \leq \left(\frac{1 + \|Q_T\| N + \|B\| N}{\mu} \right) MKT \left(\|x - y\| + \|w - s\| \right), \tag{4.6}
$$

where

$$
M = \sup_{0 \leq t \leq T} \left\| e^{At} \right\| .
$$

Proof. Assume the space $\tilde{Z} \times U_{ad}$ contains two functions (x,w) and (y,s) where $D(x,w) = (X,W)$ and $D(y,s) = (Y,S)$. Then,

$$\|D(x,w) - D(y,s)\|_{\tilde{Z}\times U_{ad}} = \|X-Y\|_{\tilde{Z}} + \|W-S\|_{U_{ad}}. \tag{4.7}$$

As a first step, let us make an estimation for $\|X-Y\|_{\tilde{Z}}$:

$$\begin{aligned}
\|X-Y\| = \max_{0\le\, t\, \le T} \Bigg\| \int_0^t e^{A(t-s)} \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\, dr \\
- \int_0^t e^{A(t-\rho)} BB^* e^{A^*(t-\rho)} e^{A^*(T-t)} Q_T^{-1} \\
\times \int_0^T e^{A(T-r)} \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\, dr\, d\rho \Bigg\| \\
= \max_{0\le\, t\, \le T} \Bigg\| \int_0^t e^{A(t-r)} \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\, dr \\
- \int_0^T \int_0^t e^{A(t-\rho)} BB^* e^{A^*(t-\rho)} e^{A^*(T-t)} Q_T^{-1} e^{A(T-r)} \times \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\, d\rho\, dr \Bigg\| \\
= \max_{0\le\, t\, \le T} \Bigg\| \int_0^t e^{A(t-r)} \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\, dr \\
- \int_0^T Q_t e^{A^*(T-t)} Q_T^{-1} e^{A(T-r)} \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\, d\rho\, dr \Bigg\| \\
\le \max_{0\le\, t\, \le T} \left(M + \|Q_t\| M^2\right) \int_0^T \left\| Q_T^{-1} \left(f(r,x(r),w(r)) - f(r,y(r),s(r))\right)\right\|\, dr \\
\le \frac{1+\|Q_T\|M}{\mu} M \int_0^T \|(f(r,x(r),w(r)) - f(r,y(r),s(r)))\|\, dr \\
\le \frac{1+\|Q_T\|M}{\mu} MK \int_0^T \left(\|x(r)-y(r)\| + \|w(r)-s(r)\|\right)\, dr \\
\le \frac{1+\|Q_T\|M}{\mu} MKT \left(\|x-y\| + \|w-s\|\right).
\end{aligned} \tag{4.8}$$

Now, for the next step consider $\|W-S\|_{U_{ad}}$,

$$
\begin{aligned}
\|W-S\| &= \max_{0\leq\, t\,\leq T}\left\| -B^{*}e^{A^{*}(T-t)}\int_0^t Q_T^{-1}e^{A(T-r)}\left(f(r,x(r),w(r))-f(r,y(r),s(r))\right)\,dr\right\| \\
&\leq \frac{\|B\|M^2}{\mu}\int_0^T \|(f(r,x(r),w(r))-f(r,y(r),s(r)))\|\;dr \\
&\leq \frac{\|B\|M}{\mu}MK\int_0^T \left(\|x(r)-y(r)\|+\|w(r)-s(r)\|\right)\,dr \\
&\leq \frac{\|B\|M}{\mu}MKT\left(\|x-y\|+\|w-s\|\right). && (4.9)
\end{aligned}
$$

By putting (4.8) and (4.9) into together, we obtain

$$
\begin{aligned}
\|D(x,w)(t)-D(y-s)(t)\| &\leq \left(\frac{1+\|Q_T\|M}{\mu}MKT+\frac{\|B\|M}{\mu}MKT\right)\left(\|x-y\|+\|w-s\|\right) \\
&= \left(\frac{1+\|Q_T\|M+\|B\|M}{\mu}\right)MKT\left(\|x-y\|+\|w-s\|\right).
\end{aligned}
\tag{4.10}
$$

This shows the proof. □

In order to get rid of the complex form of large coefficient in (4.10), let

$$
R=\left(\frac{1+\|Q_T\|M+\|B\|M}{\mu}\right)MKT. \tag{4.11}
$$

Lemma 4.3: Let Z and U be separable Hilbert spaces and assume the assumption (iv) is hold. If

$$
R<1, \tag{4.12}
$$

then operator D which transforms $\tilde{Z}\times U_{ad}$ into $\tilde{Z}\times U_{ad}$, has only one fixed point $(z,u)\in\tilde{Z}\times U_{ad}$.

Proof. Firstly as it is stressed above, there is a transformation from $\tilde{Z}\times U_{ad}$ into $\tilde{Z}\times U_{ad}$ by the operator D. By means of Lemma 4.2, since D is a contraction mapping on the Banach space $\tilde{Z}\times U_{ad}$, then D has a unique fixed point $(z,u)\in\tilde{Z}\times U_{ad}$. □

Theorem 4.1: Let Z and U be separable Hilbert spaces and the assumption (iv) satisfy. Suppose there exists $d > 0$ so that $\langle Q_T z, z\rangle \geq d\,\|z\|^2\ \forall z \in Z$. If

$$R < 1 \tag{4.13}$$

satisfies, then the system (4.1) is exactly controllable.

Proof. Take any $\beta \in Z$ and $c \in Z$. The aim is to show $\exists u \in U_{ad}$ such that $c = z_T$, so now consider the following equality,

$$u_t = B^* e^{A^*(T-t)} Q_T^{-1}\left(c - e^{AT}\beta\right) - \int_0^T B^* e^{A^*(T-t)} Q_T^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr. \tag{4.14}$$

By substituting (4.14) into (4.2) and applying Theorem 2.3,

$$\begin{aligned}
z_t &= e^{At}\beta + \int_0^t e^{A(t-r)} BB^* e^{A^*(t-r)} e^{A^*(T-t)} Q_T^{-1}\left(c - e^{AT}\beta\right)\, dr \\
&- \int_0^t e^{A(t-\rho)} BB^* e^{A^*(t-\rho)} e^{A^*(T-t)} \int_0^T Q_T^{-1} e^{A(T-r)}\, dr\, d\rho \\
&+ \int_0^t e^{A(t-r)} f(r, z(r), u(r))\, dr \\
&= e^{At}\beta + Q_t e^{A^*(T-t)} Q_T^{-1}(c - e^{AT}\beta) + \int_0^t e^{A(t-r)} f(r, z(r), u(r))\, dr \\
&- \int_0^T Q_t e^{A^*(T-t)} Q_T^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr.
\end{aligned} \tag{4.15}$$

Considering Lemma 4.3, fixed point $(z, u) \in \tilde{Z} \times U_{ad}$ exists that satisfies both (4.14) and (4.15). Therefore, $u \in U_{ad}$. Furthermore, when $t = T$, we get

$$\begin{aligned}
z_T &= Q_T Q_T^{-1}\left(c - e^{AT}\beta - \int_0^T e^{A(T-r)} f(r, z(r), u(r))\, dr\right) \\
& e^{AT}\beta + \int_0^T e^{A(T-r)} f(r, z(r), u(r))\, dr \\
&= c.
\end{aligned}$$

Thus, the control system (4.1) is exactly controllable. □

4.2 Approximate Controllability for Semilinear Systems

Theorem 4.1 indicates that strongly improving the conditions imposed on the Lipschitz coefficient is a necessity to use contraction mapping theorem. For the upcoming section, since approximate controllability concept is weaker than the complete one, generalized contraction mapping theorem will be applied to clarify the proof of sufficient conditions of approximate controllability of semilinear control systems. Suppose the previous assumptions and notation are taken into consideration similarly in this section as well.

Lemma 4.4: Let Z and U be separable Hilbert spaces. Assume that the assumption (iv) holds. Then, considering $c \in Z$ arbitrarily, and $\mu > 0$, the operator $D_\mu : \tilde{Z} \times U_{ad} \to \tilde{Z} \times U_{ad}$, that is described by

$$D_\mu(x,w)(t) = (X_\mu(t), W_\mu(t)),\ \forall t \in [0,T], \tag{4.16}$$

where

$$\begin{aligned} X_\mu(t) &= e^{At}\beta + Q_t e^{A^*(T-t)}(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \\ &\quad - \int_0^t Q_{t-r} e^{A^*(T-t)}(\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r,x(r),w(r))\, dr \\ &\quad + \int_0^t e^{A(t-r)} f(r,x(r),w(r))\, dr, \end{aligned} \tag{4.17}$$

$$\begin{aligned} W_\mu(t) &= B^* e^{A^*(T-t)}(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \\ &\quad - \int_0^t B^* e^{A^*(T-t)}(\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r,x(r),w(r))\, dr, \end{aligned} \tag{4.18}$$

has only one fixed point in $\tilde{Z} \times U_{ad}$.

Proof. Assume there exists two functions (x,w) and (y,s) in $\tilde{Z} \times U_{ad}$ so that $D_\mu(x,w) = (X_\mu, W_\mu)$ and $D_\mu(y,s) = (Y_\mu, S_\mu)$. Borrowing the same procedure that is followed by Lemma 4.2, it is likely to get,

$$\begin{aligned}\left\|D_\mu(x,w)(t)-D_\mu(y,s)(t)\right\| &\leq \left(\frac{1+\|Q_T\|M+\|B\|M}{\mu}\right)MK \\ &\int_0^t \left(\|x(r)-y(r)\|+\|w(r)-s(r)\|\right)\,dr \\ &= \left(\frac{1+\|Q_T\|M+\|B\|M}{\mu}\right)MKt\left(\|x-y\|+\|w-s\|\right) \\ &= R_\mu t\left(\|x-y\|+\|w-s\|\right). \end{aligned} \tag{4.19}$$

Then, one can get the following by repeating the argument on D_μ^2 in a similar way,

$$\begin{aligned}\left\|D_\mu^2(x,w)(t)-D_\mu^2(y,s)(t)\right\| &\leq R_\mu \int_0^t \left\|D_\mu(x,w)(r)-D_\mu(y,s)(r)\right\|\,dr \\ &\leq R_\mu^2\left(\|x-y\|+\|w-s\|\right)\int_0^t r\,dr \\ &= R_\mu^2\frac{t^2}{2!}\left(\|x-y\|+\|w-s\|\right). \end{aligned} \tag{4.20}$$

After that,

$$\left\|D_\mu^2(x,w)(t)-D_\mu^2(y,s)(t)\right\| \leq R_\mu^2\frac{T^2}{2!}\left(\|x-y\|+\|w-s\|\right). \tag{4.21}$$

Eventually, by the induction method for $n\geq 1$,

$$\left\|D_\mu^n(x,w)(t)-D_\mu^n(y,s)(t)\right\| \leq R_\mu^m\frac{T^m}{m!}\left(\|x-y\|+\|w-s\|\right). \tag{4.22}$$

Since

$$\lim_{m\to\infty}(R_\mu)^m\frac{T^m}{m!}=0, \tag{4.23}$$

the inequality below satisfies for sufficiently large m,

$$0\leq (R_\mu)^m\frac{T^m}{m!}<1. \tag{4.24}$$

For large enough m, D_μ^m is a contraction mapping on $\tilde{Z}\times U_{ad}$, and thus D_μ also does. By considering this fact, it can be concluded that D_μ has only one fixed point $(z,u)\in \tilde{Z}\times U_{ad}$ where z related to u is a solution of the system (4.1). □

Theorem 4.2: Let Z and U be separable Hilbert spaces and the assumption (iv) hold. Suppose $\mu R(\mu,-Q_T)\to 0$ uniformly as $\mu\to 0$ for all $0<t\leq T$. By considering these

properties, the system (4.1) is approximately controllable.

Proof. Suppose $\beta \in Z$ and $c \in Z$. Here the aim is to show that $\exists u \in U_{ad}$ such that $\|c - z_T\| \to 0$ as $\mu \to 0$ where z_T is a solution of system (4.1) at time T. In order to achieve this, let u has such an expression:

$$\begin{aligned} u_t = {} & B^* e^{A^*(T-t)}(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \\ & - \int_0^t B^* e^{A^*(T-t)}(\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, x(r), u(r))\, dr \end{aligned} \tag{4.25}$$

Applying substitution from (4.25) into (4.2) and also Theorem 2.3, we have

$$\begin{aligned} z_t = {} & e^{At}\beta + \int_0^t e^{A(t-r)} B B^* e^{A^*(t-r)} e^{A^*(T-t)} (\mu I + Q_T)^{-1}(c - e^{AT}\beta)\, dr \\ & - \int_0^t e^{A(t-\rho)} B B^* e^{A^*(t-\rho)} e^{A^*(T-t)} \int_0^\rho (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr\, d\rho \\ & + \int_0^t e^{A(t-r)} f(r, z(r), u(r))\, dr \\ = {} & e^{At}\beta + Q_t e^{A^*(T-t)}(\mu I + Q_T)^{-1}(c - e^{AT}\beta) + \int_0^t e^{A(t-r)} f(r, z(r), u(r))\, dr \\ & - \int_0^t \int_r^t e^{A(t-\rho)} B B^* e^{A^*(t-\rho)} e^{A^*(T-t)} (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, d\rho\, dr \\ = {} & e^{At}\beta + Q_t e^{A^*(T-t)}(\mu I + Q_T)^{-1}(c - e^{AT}\beta) + \int_0^t e^{A(t-r)} f(r, z(r), u(r))\, dr \\ & - \int_0^t Q_{t-r} e^{A^*(T-t)} (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr. \end{aligned} \tag{4.26}$$

By means of Lemma 4.4, only one point $(x, u) \in \tilde{Z} \times U_{ad}$ satisfies both (4.25) and (4.26). Thus, $u \in U_{ad}$. Moreover, we have

$$
\begin{aligned}
z_T &= e^{AT}\beta + Q_T(\mu I + Q_T)^{-1}(c - e^{AT}\beta) + \int_0^T e^{A(T-r)} f(r, z(r), u(r))\, dr \\
&- \int_0^T Q_{T-r}(\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr \\
&= e^{AT}\beta + Q_T(\mu I + Q_T)^{-1}(c - e^{AT}\beta) + \int_0^T e^{A(T-r)} f(r, z(r), u(r))\, dr \\
&+ \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) - \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \\
&+ \mu \int_0^T (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr \\
&- \mu \int_0^T (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr \\
&= c - \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) - \mu \int_0^T (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|z_T - c\| &= \left\| \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) - \mu \int_0^T (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr \right\| \\
&\leq \mu \left\| \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \right\| + \left\| \mu \int_0^T (\mu I + Q_{T-r})^{-1} e^{A(T-r)} f(r, z(r), u(r))\, dr \right\| \\
&\leq \left\| \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \right\| + N \int_0^T \left\| \mu(\mu I + Q_{T-r})^{-1} f(r, z(r), u(r))\, dr \right\|.
\end{aligned}
$$

Recalling Theorem 2.3 and applying it on the integral term, $N \int_0^T \left\| \mu(\mu I + Q_{T-r})^{-1} \right\| \cdot \| f(\rho, z(\rho), u(\rho)) \| \, d\rho \to 0$ as $\mu \to 0$, $\forall 0 \leq r < T$, since $\left\| \mu(\mu I + Q_{T-r})^{-1} \right\| \to 0$ as $\mu \to 0$ and $\left\| \mu(\mu I + Q_T)^{-1}(c - e^{AT}\beta) \right\| \to 0$ as $\mu \to 0$. Hence, $\|z_T - c\| \to 0$ as $\mu \to 0$. Therefore, the system (4.1) is approximately controllable. □

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