Logical Puzzles

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ABSTRACT

In this thesis, we use commonsense reasoning and graph representation to study two different new types of logical puzzles with three types of people. In the first type of puzzles, any person in the puzzle can be either Strong Truth-teller, Strong Liar or Strong Crazy. While in the second type, any person in the puzzle can be either Strong Truth-teller, Strong Liar or Weak Crazy. Strong Truth-tellers say only true atomic statements, Strong Liars say only false atomic statement and Strong Crazy people say only self-contradicting statements, while Weak Crazy person must say at least one self-contradictory statement if he/she say anything. Self-contradicting statements are connected to the Liar paradox, i.e., no Truth-teller or a Liar could say “I am a Liar”. A puzzle is clear if only the statements of its people are given to solve it and a puzzle is good if it has exactly one solution. It is known that there is no clear and good Strong Truth-teller–Strong Liar (also called SS-) puzzle. However, in this thesis, we show that there are good and clear Strong Truth-teller, Strong Liar and Strong Crazy puzzles (SSS-puzzles) and Strong Truth-teller, Strong Liar and Weak Crazy puzzles (SSW-puzzles). The newly investigated types Weak and Strong ‘Crazy’ changes drastically the scenario of SS-puzzles. Some properties of the new types of puzzles are analyzed and some statistics are also given. Also we provide a comparison between the three different types of puzzles along with characterization of graph representation of good puzzles of the new types of puzzles.

Keywords: SS-puzzles; SSS-puzzles; SSW-puzzles; Strong Crazy persons; Weak Crazy persons; Self-contradictory statements; Graph representation of the puzzles.

ÖZ

Bu tezde üç tür insanla iki farklı yeni mantıksal bulmaca türü üzerinde çalışmak için sağduyulu akıl yürütme ve grafik gösterimi kullanıyoruz. İlk bulmaca türünde, bulmacadaki herhangi bir kişi Güçlü Doğrucu, Güçlü Yalancı ve ya Güçlü Çılgın olabilir. İkinci tipteyken, bulmacadaki herhangi bir kişi Güçlü Doğrucu, Güçlü Yalancı ve ya Zayıf Çılgın olabilir. Güçlü doğrucular sadece gerçek atomik ifadeler söyler, Güçlü Yalancılar sadece yanlış atomik ifadeler söyler ve Güçlü Çılgın insanlar sadece kendiyle çelişen ifadeler söylerken, Zayıf Çılgın kişi bir şey söylerse en az bir kendiyle çelişkili ifade söylemelidir. Kendine çelişen ifadeler Yalancı paradoksuyla bağlantılıdır, yani hiç bir doğru söyleyen ve Yalancı "Ben bir Yalancıyım" diyemez. Bir bulmaca sadece insanlarının ifadeleri onu çözmek için verilirse açıktır ve bir bulmaca tam olarak bir çözümü varsa, iyidir. Yapboz Net ve iyi bir Güçlü Doğrucu, Güçlü Yalancı (SS olarak da bilinir) olmadığı bilinmektedir. Ancak, bu tezde, iyi ve net Güçlü Doğrucu, Güçlü Yalancı ve Güçlü Çılgın bulmacalar (SSS-bulmacalar) ve Güçlü Doğrucu, Güçlü Yalancı ve Zayıf Çılgın bulmacalar (SSV-bulmacalar) olduğunu gösteriyoruz. Yeni araştırılan Zayıf ve Güçlü 'Çılgın' türleri SS-bulmaca senaryosunu önemli ölçüde değiştiriyor. Yeni bulmaca türlerinin bazı özellikleri analiz edilir ve bazı istatistikler de verilmekdedir.Ayrıca yeni bulmaca türlerinin bulmacalarını iyice çözmek için grafik gösteriminin karakterizasyonu ile birlikte üç farklı bulmaca türü arasında bir karşılaştırma sağlıyoruz.

Anahtar Kelimeler: SS-bulmaca; SSS-bulmaca; GGB-bulmaca; Güçlü Çılgın kişiler; Zayıf Çılgın kişiler; kendiliğinden çelişen ifadeler; bulmaca grafik gösterimi.

DEDICATION

I am dedicating this thesis to Almighty Allah, my creator; to my master, my great teacher and messenger, Mohammed (May Allah bless and grant him), who taught us the purpose of life. Wholeheartedly to my beloved parents, who have been my source of inspiration and give me strength when I thought of giving up, who continually provide moral, spiritual, emotional support. To brothers, sisters, and relatives for their sincere love, and ever-present support of my personal endeavors towards learning, and achieving my goal.

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Chapter 1

INTRODUCTION

Logical puzzles have various roles in our life. They are helpful for training our brains, to learn logical thinking and also for recreation. In terms of problem solving processes, without human intervention, some techniques and methods have been proposed in order to solve logical puzzles (see, e.g. [1] for recent results). The formal study of various types of puzzles can be done by commonsense reasoning or by formal logic. Knight and Knave puzzles are logical puzzles that were introduced by Smullyan [2; 3, 4]. Knight-knave puzzles are wide spread and popular logical puzzles which are close connection to recreational mathematics, see e.g., [5, 6]. Formal solution methods for solving these puzzles using classical propositional calculus have been discussed in [7]. Similarly, in [8], based on logical representations, automated reasoning is used to solve puzzles. In a nutshell, Knights are Truth-tellers and Knaves are Liars. Various types of Truth-teller–Liar puzzles have been further popularised by various books and papers [5, 6, 9, 10, 11, 12] under various names. In the simplest type of them, the participants say only statements about their types. Strong and Weak Truth-tellers and Liars were introduced in [13, 14]. Strong Truth-tellers can say only true atomic statements, whilst Strong Liars can tell only false atomic statements. (In contrast, Weak Liars must say at least one false atomic statement, whilst Weak Truth-tellers must say at least one true atomic statement, if they say anything.) In [13, 15], it was proven that there is no SS-puzzle, i.e., puzzle with Strong Truth-tellers and Strong Liars such that it has a unique solution without any additional information, e.g., without knowing the number of

Truth-tellers. In various puzzles, there are not only Truth-tellers and Liars: to complete the picture, Normals are persons who can say false and true atomic statements, see, e.g., [10]. In [16], people who cannot say true and cannot say false statements are defined as Mutes (see Table 1.1). In puzzles studied in [16], Mutes cannot say anything, as their name infers. In puzzles where the allowed statements are of the form “X can/cannot say the truth”, “X can/cannot say false statement”, in the solution, it could be that some people neither can say true, nor false statement, and thus they are Mutes in [16]. However, there is another way to work with this issue based on the liar paradox [17, 18]. In this context, the statement “I am a Liar” is neither true nor false if True-teller or Liar, respectively, would say it. Accordingly, a person who could say such statement is not saying the truth and does not lie, thus the term Mute does not fit for them. In order to fix this issue, we define a new type of persons called Crazy persons.

We also note that puzzles have various connections to graphs, to Boolean programming [19, 20] and other scientific disciplines. In [10, 21], diagrammatical logic has been used to represent various Knight-Knave and Knight-Normal-Knave puzzles and their solutions. Various degrees of truthfulness of people in the puzzle and some paradoxical statements were also presented in [22]. Neutrals (a type somehow between Knights and Knaves) were used in [12].

As it is proven in [13, 15], there are no SS-puzzles with one solution, our central aim is to reconsider these simplest puzzles by extending the possible types of people. On the other hand, there could be such scenario in some puzzles in which some statements cannot be true and cannot be false, thus people who cannot say true and cannot say false, still can say some (paradoxical) statement. In this thesis, we present two new types of puzzles. We investigate two new types of persons, the (Strong and Weak) Crazy people and use them along with strong Truth-tellers and strong Liars in our SSS-puzzles and SSW-puzzles. In contrast to the SS-puzzles, there are SSS-puzzles with a unique solution without any additional information [23]. Similarly, we prove in this thesis that there are SSW-puzzles with a unique solution without any additional information. In our puzzles, we use atomic statements of the form "X is a Liar" or "X is a Truth-teller". However, we are still able to show an interesting extension of the binary world and see how to manage if there are more types of people in the puzzles, especially, if the set of allowed truth-values of the statements is extended. Now, we use a novel approach to deal with paradoxical statements. In [23], as we have already mentioned, we defined the so-called Strong Crazy type of people, as people who cannot say true and cannot say false, but they may say self-contradictory statements. In this terminology, self-contradictory statements are those statements which cannot be true or false whenever a Truth-teller or a Liar says them. They are closely connected to the ancient Greek paradox known as the Liar paradox [17, 18]. We may distinguish two variants of each type of people, Strong and Weak. Each statement said by a Strong variant reflects the type of the person. In contrast, the Weak variant must say at least one statement according to his or her type if he or she says anything. Thus, a Weak Crazy person must say at least one self-contradicting statement if he or she says anything. We investigate the new type, the Strong Crazy persons, instead of Mutes (see Table 1.2). This is a new approach in puzzles to deal with sentences that have a third truth-value; Crazy people (Strong or Weak) can say self-contradictory statements, those are not true and not false if a Truth-teller or a Liar, respectively, would say them. We will prove that these statements contain self-reference. In the present thesis, we are working with either two or three types of people in puzzles.

Table 1.1: Type of people and their possible statements by their truth value in the earlier literature [16]

	Can say false statements	**Cannot say false statements**
Can say true statements	Normal	Knight/Truth-teller
Cannot say true statements	Knave/Liar	Mute

Table 1.2: Type of people and their possible statements by their truth value in our new approach (SSS-puzzles, see also [23])

	Can say false statements	Cannot say false statements
Can say true statements	Normal	Truth-teller
Cannot say true statements	Liar	Crazy

Note that in Table 1.1 and Table 1.2, Truth-teller, Liar and Crazy are shortened forms of Strong Truth-teller, Strong Liar and Strong Crazy people, respectively. In the next sections, we recall SS-puzzles and define our SSS- and SSW-puzzles. We also show that many, but not all unsolvable SS-puzzles become solvable if we shift the type of the puzzle to SSS-puzzle or SSW-puzzle. Also, we show some statistical data about number of good, solvable and unsolvable SS-, SSS- and SSW-puzzles and other statistical data about good SSS-puzzles and good SSW-puzzles. Some characteristics of maximal and minimal graph representation of the puzzles will be also discussed and explained.

Chapter 2

PRELIMINARIES

In this part of the thesis, we introduce some definitions and notations that are necessary in our investigation of the three different foresaid puzzle types. Some of these definitions are common in the three puzzle types and some of them analogues to each other, that is depending on the type of the puzzle.

Definition 2.1 Let A be a person. *Atomic statements* are the statements which cannot be divided into smaller statements, and in our puzzles, they could have two forms: "person A is a Liar" and "person A is a Truth-teller". In this thesis, for all puzzle types, all puzzle statements are atomic statements.

Definition 2.2 Assume that we have a set of persons and the atomic statements about them, then the *puzzle* is a function which assigns a set of atomic statements to each person, such that these statements said by that person, about other persons or about himself/herself. Intuitively, a *solution* of a puzzle is a function assigning the type of the people such that their statements match to their types (formal definition will be provided later on depending on the type of the puzzle).

Definition 2.3 Any puzzle that has at least one solution is called *solvable puzzle*. Further, a puzzle is said to be *good* if it has exactly one solution.

Definition 2.4 A puzzle that does not need any additional information to solve it, is called *clear puzzle*, i.e. only the persons and their statements are given.

As an example, the number of true and false statements is an additional information in the puzzle. In this thesis, we assume that all puzzles of the three puzzle types (SS-, SSS- and SSW-puzzles) are clear.

2.1 Graph representation of the puzzle

In this thesis, any puzzle of the three type of puzzles (SS, SSS and SSW) can be represented by its *Graph representation* G as follows:

- Nodes represent the persons of the puzzle.
- Directed edges represent the atomic statements:
  - Solid edge from X to Y: X said that Y is a Truth-teller ($\overline{XY}$).
  - Broken edge from X to Y: X said that Y is a Liar ($\ddot{X}\ddot{Y}$).
- $\vec{X}$: The set of all directed edges directed away from X, and these edges are in the form $\overline{XY}$ or $\ddot{X}\ddot{Y}$, where Y is any node in G.

In graph representation G, if two persons in the puzzle say the same type of statements about each other, then the representation of these statements, the two edges pointing in opposite directions can be substituted by a bidirectional same type (i.e., broken or solid) edge between the two nodes.

2.2 Notations and Definition

- N: The set of all people in the puzzle.
- G is the graph representation of the puzzle.

Let N be a set of people in the puzzle, and $A_i \in N$, then $\Gamma_{i,0}$ is a set of persons named by A_i as Liars, and $\Gamma_{i,1}$ is a set of persons named by A_i as Truth-tellers. Further in this thesis, we refer to a statement as $A_j \in \Gamma_{i,m}$, where A_j, $A_i \in N$ and $m \in \{0,1\}$.

Chapter 3

STRONG TRUTH-TELLER AND STRONG LIAR PUZZLES (SS-PUZZLES)

In this chapter, we study and recall one of the simplest types of Truth-teller–Liar puzzles. Firstly, for better understanding, we will give an example about these puzzles, then we discuss the formal definitions (based, e.g., on [9, 15]).

Example 3.1 Let us have a puzzle with three persons: Alice, Beth, and Chris. Alice said that Chris is a Liar and Beth is a Truth-teller, Beth claimed that Chris is a Liar. Determine who of the three persons is Liar or Truth-teller (see Figure 3.1).

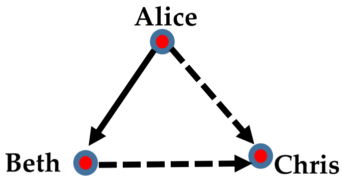


Figure 3.1: The graph representation of the SS-puzzle in Example 3.1

Now, let us study some definitions about SS-puzzles. Later on, we will solve the previous puzzle.

Definition 3.1 A person is a *Strong Liar* if he/she makes only false atomic statements (if he/she makes any statement), while a person is a *Strong Truth-teller* if he/ she makes only true atomic statements (if he/she makes any statement).

Definition 3.2 An SS-puzzle with n persons is a puzzle, where each person in the puzzle is Strong Truth-teller or Strong Liar. (The double S in "SS-puzzle", is to show that both types of persons, the Truth-tellers and the Liars are of Strong type).

In Example 3.1 we have three atomic statements.

As already mentioned, there are various ways to define or understand the concept of Truth-tellers and Liars, e.g., putting conjunction or disjunction between the atomic statements of a person, and evaluating the obtained sentence. Depending on these concepts various puzzles are investigated in, e.g., [13, 14]. In this paper, we use one of the simplest and yet very usual approach which leads to the concepts of Strong Truth-tellers and Strong Liars.

The concept of solutions is attached to the concept of puzzles as similar as models and logic are connected. The next definition of puzzle solution will give a clear picture about Strong Liars and Strong Truth-tellers in SS-puzzles.

Definition 3.3 Assume an SS-puzzle with set N of people is given. The *solution* of the puzzle is a function $N \to$ {Truth-teller, Liar} such that, for each person A_k who claimed any statement in the puzzle the following holds:

- Person A_k is Truth-teller, if the following condition holds for every $A_i \in N$:

  if A_k claimed atomic statement about A_i, and this atomic statement is in the form:

  - "A_i is a Liar", then A_i is a Liar (the atomic statement is true).
  - "A_i is a Truth-teller", then A_i is a Truth-teller (the atomic statement is true).

- Person A_k is Liar, if the following condition holds for every $A_i \in N$: if A_k claimed atomic statement about A_i, and this atomic statement is in the form:
    - "A_i is a Liar", then A_i is a Truth-teller (the atomic statement is false).
    - "A_i is a Truth-teller", then A_i is a Liar (the atomic statement is false).

Remark 3.1 In any SS-puzzle, if there is a statement "I am a Liar", then the puzzle is unsolvable.

Example 3.1 (continued) There are two solutions for this puzzle. The first solution is if we assume that Alice is Liar, then Beth is Lair and Chris is Truth-teller. In the second solution if we assume that Alice is Truth-teller, then Beth is Truth-teller and Chris is Liar. Thus, we have more than one solution.

The solutions can be seen in a different way, as it is given by the following lemma.

Lemma 3.1 A solution of SS-puzzle with set N of persons can also be expressed such that the following satisfies for each person A_k about whom any person said any atomic statement in the puzzle:

- If person A_k is Truth-teller, then for all A_i who said atomic statements about A_k, if these atomic statements are in the form:
    - "A_k is a Liar", then A_i is a Liar.
    - "A_k is a Truth-teller", then A_i is a Truth-teller.
- If person A_k is Liar, then for all A_i who said atomic statements about A_k, if these atomic statements are in the form:
    - "A_k is a Liar", then A_i is a Truth-teller.
    - "A_k is a Truth-teller", then A_i is a Liar.

Proof. It is obvious by Definition 3.3.

Definition 3.4 A *solution* of the graph representation G of an SS-puzzle is a function which assigns either L or T to each node, such that all statements represented by the edges of the graph are satisfied.

Proposition 3.1 The solution(s) of any SS-puzzle are the same as the solution(s) of the graph representation G of that puzzle.

Proof. In the graph representation G, the nodes and the edges of G represent the persons and the statements in the puzzle represented by G, respectively. Hence, based on Definition 3.3 and Definition 3.4, the solution set of the puzzle is the same as the solution set of G. ■

Theorem 3.1 There is no clear and good SS-puzzle.

Proof. See [13].

Example 3.2 Assume that we have an SS-puzzle with three persons: Alice, Beth and Chris. Alice claimed that Beth is Liar, Beth said “I am a Liar”, Chris claimed that Alice and Beth are Truth-tellers. Determine who is Truth-teller and who is Liar. Figure 3.2 shows the graph representation of the puzzle.

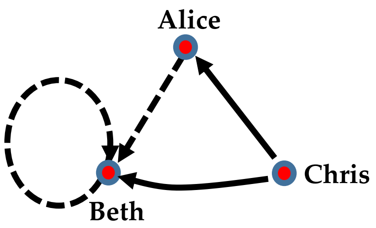


Figure 3.2: The graph representation of the SS-puzzle in Example 3.2

The previous SS-puzzle is unsolvable since in the solution, Beth is neither Truth-teller nor Liar, since she claimed the statement “I am a Liar”.

Chapter 4

STRONG TRUTH-TELLER, STRONG LIAR AND STRONG CRAZY PUZZLES (SSS-PUZZLES)

In this chapter, we introduce a new type of puzzles. As Table 1.2 shows, we distinguish the four types of people not in the previously shown (Table 1.1) usual way. In binary logic every statement is either true or false, which led to the previous Mute type of people in the lower right corner of Table 1.1. However, there is another option, if we allow statements which are not true and not false (e.g., typical paradoxical self-reference statements), people who cannot say true and cannot say false, may still say some statements. In our first newly investigated puzzles, those people who cannot say any other type statements but only paradoxical statements are called Strong Crazy people. In this section, puzzles are considered in which three types of people may appear, namely: Strong Truth-teller, Strong Liar and Strong Crazy. The following results have been published in [23].

Definition 4.1 A self-contradictory statement is an atomic statement which has the following property independently of other statements: it is neither false nor true if a Truth-teller or Liar says it.

Definition 4.2 An SSS-puzzle is a puzzle with a set of persons $N = \{A_1, A_2, \ldots, A_n\}$ and their atomic statements, in which A_i (where $0 < i \leq n$) can be Strong Liar, Strong Truth-teller, or a Strong Crazy such that:

- Strong Truth-teller (T) and Strong Liar (L) previously have been defined (see Definition 3.1).
- A person is a Strong Crazy person (C) if all of his/her statements are self-contradictory statements (if he/she says any statement).

Definition 4.3 Let an SSS-puzzle with set N of persons be given. A function $N \rightarrow$ {Truth-teller, Liar, Crazy} is a solution of the puzzle if the following holds for each person A_k who said anything in the puzzle:

- A_k is Truth-teller exactly as in Definition 3.3.
- Person A_k is Liar if the following holds for every $A_i \in N$: if A_k said atomic statement about A_i, and this atomic statement is in the form:
  - "A_i is a Liar", then A_i is a Truth-teller or a Crazy (not a Liar).
  - "A_i is a Truth-teller", then A_i is a Liar or a Crazy (not a Truth-teller).
- Person A_k is Crazy person if A_k said only self-contradictory statements.

Clear, solvable and good SSS-puzzles can be defined analogously based on Definitions 2.3 and 2.4.

Let us fix a/the solution of a solvable puzzle. Let

- T: The set of Strong Truth-tellers, L: The set of Strong Liars, and C: The set of Strong Crazy people in the puzzle. T, L and C are disjoint sets, moreover, $N = T \cup L \cup C$.
- True atomic statements (α): The set of all statements that are logically true.
- False atomic statements (β): The set of all statements that are logically false.

Paradoxical statements (γ): The set of all self-contradictory statements in the puzzle.

4.1 Preliminary Results

The first result about our new type of puzzles is already connected to the new type of people.

Proposition 4.1 Let an SSS-puzzle be given with set N of persons, and let $A_k, A_i \in N$ with $i \neq k$. Let a/the solution be given such that $A_i \in C$. If $A_k \in T$ or $A_k \in C$ in the solution, then A_k cannot say any statement about A_i. Therefore, if $i \neq k$, then A_k can say a statement about A_i only if $A_k \in L$.

Proof. In contrary, let us assume, first, that person $A_k \in T$ says a statement about $A_i \in C$. If $A_i \in \Gamma_{k,0}$, then it is a false statement said by a Truth-teller. Analogously, $A_i \in \Gamma_{k,1}$ leads also to a contradiction. Second, a Crazy person cannot say any statement about another Crazy person, since this statement would be a false statement and it contradicts to the definition of Crazy person. Thus, based on these arguments, only Liars (e.g., $A_k \in L$) can say statements about a Crazy person A_i (with the condition $i \neq k$). ■

Definition 4.4 A *solution* of the graph G is a function which assigns either C, L or T to each node, such that all statements represented by the edges of the graph are satisfied.

Proposition 4.2 The solutions of the SSS-puzzle are the same as the solutions of the graph representation G of the puzzle.

Proof. In graph G, the nodes and the edges of G represent the persons and the statements in the puzzle represented by G, respectively. Therefore, based on Definition 4.3 and Definition 4.4, the solution set of the puzzle is the same as the solution set of G. ■

4.2 Analyzing SSS-puzzles

In contrary to Theorem 3.1, if we consider the puzzle of Example 3.2 as an SSS-puzzle, then this puzzle has exactly one solution (see also Example 4.1 below). This fact is not only an important property of SSS-puzzles, but also highlights the different nature of SS- and SSS-puzzles. On the other hand, Theorem 3.1 has a straightforward implication for our SSS-puzzles.

Corollary 4.1 In the solution of good and clear SSS-puzzle, $C \neq \{\}$.

Example 4.1 Let an SSS-puzzle with three persons be given: Alice, Beth and Chris. Alice said that Beth is Liar, Beth said "I am a Liar", Chris claimed that Alice and Beth are Truth-tellers. Determine who is Truth-teller or Liar (See Figure 3.2)!

This puzzle has one solution such that Alice and Chris are Liars, and Beth is a Crazy person. This is because Beth said a self-contradictory statement and both Alice and Chris said about Beth false statements.

As Crazy people play importance in solutions, let us identify them. The first step is about Crazy people who are not silent in the puzzle.

Lemma 4.1 In a solvable SSS-puzzle with set N of persons and their atomic statements, if $A_i \in \Gamma_{i,0}$, then $A_i \in C$.

Proof. Let us fix a solution, if $(A_i \in \Gamma_{i,0}) \in \alpha$, then $A_i \in T$, since he/she said a true atomic statement, but A_i said about himself/herself that he/she is a Liar, then $A_i \in L$. That implies that $(A_i \in \Gamma_{i,0}) \in \beta$, which contradicts to our assumption $(A_i \in \Gamma_{i,0}) \in \alpha$. If we assume $(A_i \in \Gamma_{i,0}) \in \beta$, then $A_i \in L$ since he/she said a false atomic statement, but A_i said about himself/herself that he/she is a Liar, then $A_i \in T$, which implies that $(A_i \in \Gamma_{i,0}) \in \alpha$ which contradicts to our assumption $(A_i \in \Gamma_{i,0}) \in \beta$. Consequently, the only possibility is $(A_i \in \Gamma_{i,0}) \in \gamma$, which means that $A_i \in C$. ■

By Lemma 4.1, in the solution of Example 4.1, Beth is a Crazy person, since she said the statement "I am a Liar".

Lemma 4.2 Let a solvable SSS-puzzle be given with a/the solution. If the type of A_j is the same as the type of A_k $(A_j, A_k \in N, j \neq k)$ in the solution, then $A_j \notin \Gamma_{k,0}$ and $A_k \notin \Gamma_{j,0}$.

Proof. Let us fix a solution, let A_j, $A_k \in N$ $(j \neq k)$ such that at least one of A_j and A_k says some statement. Let us assume that $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$). Then, if A_j, $A_k \in C$, this means that A_j, A_k said only self-contradictory statements, but $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$) is a false atomic statement, which means that this statement is not self-contradictory which contradicts to assumption that A_j, $A_k \in C$. If we assume that A_j, $A_k \in T$, then $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$) means that $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$) is a false atomic statement said by a Truth-teller which is also a contradiction. Finally, if A_j, $A_k \in L$, then $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$) implies that $(A_j \in \Gamma_{k,0}) \in \alpha$ (or $(A_k \in \Gamma_{j,0}) \in \alpha$), which means that $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$) is a true atomic statement said by a Liar which contradicts to Definition 4.3. Finally, if none of A_j and A_k says any statement in the puzzle, then, clearly, both of $A_j \notin \Gamma_{k,0}$ and $A_k \notin \Gamma_{j,0}$ are satisfied. Therefore, there is no solution for an SSS-puzzle in which two persons A_j, A_k have the same type and $A_j \in \Gamma_{k,0}$ (or $A_k \in \Gamma_{j,0}$). ■

Consequently, by Lemma 4.2, if the puzzle is solvable, then in the graph representation *G*, there is no broken edge between any two persons who have the same type (i.e. both of them are Truth-tellers, or both of them are Liars, or both of them are Crazy) in the solution of the puzzle.

Corollary 4.2 In an SSS-puzzle, if there are two persons A_i and A_j, such that $A_i \in \Gamma_{j,0}$ (or $A_j \in \Gamma_{i,0}$), and both A_i and A_j said the same statement about a third person A_k, then the puzzle is unsolvable.

As the corollary states, there are SSS-puzzles without solution. We have seen already that some of the SS-puzzles which have no solutions become solvable if we think about them as SSS-puzzles, i.e., if we allow not only Truth-tellers and Liars in the target of the solution function, but also the type Crazy can be used. Now, it is turned out that not every puzzle becomes solvable in this way.

Now, we continue to find out who are the Crazy people in the solutions.

Lemma 4.3 In the solution of an SSS-puzzle with set *N* of persons, for a person $A_i \in N$ if

- $\Gamma_{i,0} = \emptyset$, $\Gamma_{i,1} = \emptyset$ ($\vec{A}_i = \{\}$ in G) and

- $\exists\ A_j,\ A_k \in N/\{A_i\}$ such that the type of A_j is the same as the type of A_k, and $A_i \in \Gamma_{j,0}$, $A_i \in \Gamma_{k,1}$ (or $A_i \in \Gamma_{k,0}$ and $A_i \in \Gamma_{j,1}$),

then $A_i \in C$ and $A_j,\ A_k \in L$.

Proof. In case $j \neq k$ and the type of A_j is the same as the type of A_k, but they say two different statements about A_i ("A_i is a Liar" and "A_i is a Truth-teller"), then in order to have a solution, $A_j,\ A_k \in L$ and $A_i \in C$, otherwise the puzzle is unsolvable.

If $j = k$, then this means that there is one person A_j who said two different statements about A_i. Therefore, in order to have a solution, $A_j \in L$ and $A_i \in C$, otherwise the puzzle is unsolvable. ■

Now, we are ready to give a characterization of the Crazy people.

Theorem 4.1 In a solvable SSS-puzzle with set *N* of person, if $A_i \in C$, then A_i can say at most one atomic statement, which is $A_i \in \Gamma_{i,0}$.

Proof. In an SSS-puzzle, any person A_i can say only two forms of atomic statements: either $A_j \in \Gamma_{i,0}$ or $A_j \in \Gamma_{i,1}$, where $A_j \in N$. Suppose that $A_i \in C$ and $A_j \in \Gamma_{i,0}$, where $i \neq j$. If $A_j \in L$, then $(A_j \in \Gamma_{i,0}) \in \alpha$, which contradicts to $A_i \in C$. If $A_j \in T$, then $(A_j \in \Gamma_{i,0}) \in \beta$ which contradicts to $A_i \in C$. If $A_j \in C$, then $(A_j \in \Gamma_{i,0}) \in \beta$, thus $(A_j \in \Gamma_{i,0}) \notin \gamma$ and $A_i \notin C$.

Assume now that $A_i \in C$ and $A_j \in \Gamma_{i,1}$ with $i \neq j$. If $A_j \in L$, then $(A_j \in \Gamma_{i,1}) \in \beta$, which contradicts to $A_i \in C$. If $A_j \in T$, then $(A_j \in \Gamma_{i,1}) \in \alpha$, thus $A_i \notin C$. If $A_j \in C$, then ($A_j \in$

$\Gamma_{i,1}) \in \beta$, thus $(A_j \in \Gamma_{i,1}) \notin \gamma$ and $A_i \notin C$. If $A_i \in C$, then $(A_i \in \Gamma_{i,1}) \in \beta$, which also contradicts to $A_i \in C$.

Consequently, the only statement a Crazy person A_i can tell is $A_i \in \Gamma_{i,0}$, which is a self-contradictory statement. ■

Based on Theorem 4.1 and Lemma 4.1, if $A_i \in C$, A_i can say only $A_i \in \Gamma_{i,0}$, or by Lemma 4.3, A_i does not say any atomic statement about other persons.

For further analysis, we use the graph representations of the puzzles. Let us recall the notion of complete graphs from graph theory. K_n is a graph of *n* nodes such that each pair of nodes is connected by an edge. Particularly, K_3, the triangle graph, is a graph containing three nodes such that there is an edge between any two vertices. In the next result we show how triangle graphs are connected to solvable puzzles.

Proposition 4.3 If the graph representation G of an SSS-puzzle P contains a subgraph whose underlying undirected graph is K_3 with broken edges, then P has no solution.

Proof. Let us assume that the graph representation G of the puzzle contains a subgraph with broken edges, such that the underlying undirected graph of this subgraph is exactly K_3. Considering this subgraph, there are two possible forms of non-isomorphic subgraphs in G with broken edges. Figure 4.1 shows these two subgraphs. Let us start with the graph shown in Figure 4.1(a). In the solution of the puzzle, A and D have same type, since they say the same statement about B. But, according to Lemma 4.2, in order to have a solution $A \notin \Gamma_{D,0}$ and $D \notin \Gamma_{A,0}$, that is a contradiction to formation of puzzle having edges represented by subgraph shown in Figure 4.1(a). Considering Figure 4.1(b), in the solution of the puzzle, $A \notin C$, $B \notin C$ and $D \notin C$, because by Theorem 4.1, a Crazy person cannot say any statement about other persons. Let us assume that $A \in T$, therefore $D \in L$ and $B \in L$, but $B \in \Gamma_{D,0}$, which contradicts Lemma 4.2. On the other hand, if we assume that $A \in L$, then $D \in T$ and $B \in T$, but $B \in \Gamma_{D,0}$, which also contradicts Lemma 4.2. ∎

Figure 4.1: Form (a) and form (b) non-isomorphic graphs of SSS-puzzles that are represented by K_3 in the underlying undirected graph representation.

The last proposition is helpful if one wants to provide an unsolvable puzzle using its graph representation.

Example 4.2 Assume that we have an SSS-puzzle with three persons: Alice, Beth and Chris. Alice claimed that Beth is Liar, Beth claimed that Chris is a Liar and "I am a Liar", Chris claimed that Alice and Beth are Truth-tellers. Determine who is Truth-teller, Liar and/or Crazy (see Figure 4.2)!

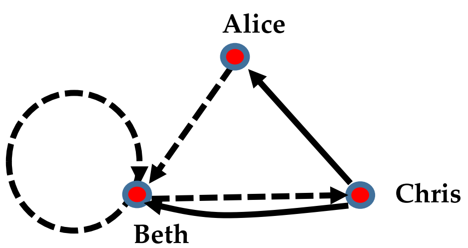


Figure 4.2: Graph representation of the SSS-puzzle in Example 4.2

Note that the SSS-puzzle in the previous example is unsolvable, since Beth said a non-self-contradictory statement and a self-contradictory statement ("I am a Liar"). Hence, Beth cannot be Truth-teller, Liar or Crazy person. As we can see there are still a lot of unsolvable SSS-puzzles. Therefore, in order to reduce the number of unsolvable SS-puzzles and SSS-puzzles (e.g. puzzles similar the puzzle presented the previous example) a new type of puzzles is introduced in Chapter 5.

Chapter 5

STRONG TRUTH-TELLER, STRONG LIAR AND WEAK CRAZY PUZZLES (SSW-PUZZLES)

In this chapter, as the crucial part of the thesis, another new type of puzzles, namely SSW-puzzles are defined and studied. In SS-puzzles there are two distinct types of people: Truth-tellers and Liars, whilst in SSW-puzzles, there are three distinct types of persons: Truth-tellers, Liars and Weak Crazy. Unlike SSS-puzzles, instead of having a Strong Crazy type, in SSW-puzzles we have Weak Crazy persons. In the new type of persons (Weak Crazy), we are merging the properties of the statements said by Strong Crazy and statements said by Normal persons (see Table 1.2). Weak Crazy type is a kind of usual way of extension, like Weak Truth-teller and Weak Liar for the Strong Truth-tellers and Strong Liars, respectively. Our aim here is to increase number of solvable and good puzzles, by shifting some of unsolvable puzzles from SS-puzzles and SSS-puzzle to be solvable and good SSW-puzzles.

Firstly, we will have some fundamental formal definitions of our new type puzzles (SSW-puzzle) and some theorems and lemmas about new type of persons (Weak Crazy).

Definition 5.1 An SSW-puzzle is a puzzle with a set N of persons and their atomic statements, where $\forall A_i \in N, A_i$ can be Strong Truth-teller, Strong Liar or Weak Crazy such that:

- Strong Truth-teller (T) and Strong Liar (L) have been defined previously (see Definition 3.1).
- A person is a Weak Crazy person (C) if at least one of his/her statements is self-contradictory statement (if he/she says any statement).

Table 5.1 presents the types of people used in SSW-puzzles with the types of their possible statements. Of course, to talk about the truth value of a statement we may need to assign to each person its type, which leads to the following definition of solution.

Table 5.1: Three different types of persons with all possible types of statements in SSW-puzzles.

	Truth-teller	Liar	Weak Crazy
True atomic statements	Yes	No	Yes*
False atomic statements	No	Yes	Yes*
Self-contradictory statement	No	No	Yes

*If he/she speaks, he/she needs to say at least one self-contradictory statement

Definition 5.2 Assume that an SSW-puzzle with set N of persons is given. The *solution* of the puzzle is a function $N \rightarrow$ {Truth-teller, Liar, Crazy} such that, for each person A_k who said any statement in the puzzle the following holds:

- Person A_k is Truth-teller exactly as in Definition 3.3.
- Person A_k is Liar, such that, the following condition holds for every $A_i \in N$: if A_k said atomic statement about A_i, and this atomic statement is in the form:
  - "A_i is a Liar", then A_i is a Truth-teller or Crazy (A_i not a Liar).
  - "A_i is a Truth-teller", then A_i is a Liar or a Crazy (A_i not a Truth-teller).
- Person A_k is Crazy if A_k says at least one self-contradictory statement (if he/she says any statement).

The main difference between the Strong Crazy person and Weak Crazy, that the Strong Crazy persons are restricted to say only self-contradictory statements (if they say any statement), while Weak Crazy persons must say at least one self-contradictory statement beside other statements that are non-self-contradictory statements.

Definition 5.3 A solution of graph representation G of any SSW-puzzle is a function which assigns either C, L or T to each node, such that all statements represented by the edges of the graph are satisfied.

Proposition 5.1 The solution(s) any SSW-puzzle are the same as the solutions of the graph representation G of that puzzle.

Proof. In graph representation G, the nodes and the edges of G represent the persons and the statements in the puzzle represented by G, respectively. Therefore, based on Definition 5.2 and Definition 5.3, the solution set of the puzzle is the same as the solution set of G. ■

Clear, solvable and good SSW-puzzles can be defined analogously based on Definitions 2.3 and 2.4.

5.1 Notations

Let us fix a/the solution of a solvable puzzle. Then let

- *T*: The set of Strong Truth-tellers,
- *L*: The set of Strong Liars, and
- *C*: The set of Weak Crazy people in the puzzle.

  T, L and C are disjoint sets, moreover, $N = T \cup L \cup C$.

- True atomic statements of the puzzle, α: The set of all statements that are logically true in the solution of the puzzle.
- False atomic statements of the puzzle, β: The set of all statements that are logically false in the solution.
- Paradoxical statements of the puzzle, γ: The set of all self-contradictory statements in the puzzle.

5.2 Analyzing SSW-puzzles

In the next proposition, we are studying the relation between the new type of persons (Weak Crazy) and other types in the puzzle (Truth-tellers and Liars).

Proposition 5.2 Let an SSW-puzzle be given with set N of persons and its graph representation G, let $A_k, A_i \in N$ with $i \neq k$. Let a/the solution be given such that $A_i \in C$. If $A_k \in T$ in the solution, then A_k cannot say any statement about A_i. Therefore, if $i \neq k$, then A_k can say a statement about A_i only if $A_k \in L$ or $A_k \in C$.

Proof. In contrary, firstly, suppose that person $A_k \in T$ says a statement about $A_i \in C$. If $\overset{\cdots}{A_k A_i} \in G$, then $\overset{\cdots}{A_k A_i} \in \beta$ which represents a false statement said by a Truth-teller. Analogously, $\overline{A_k A_i} \in G$ leads also to a contradiction. Secondly, if $A_k \in L$ or $A_k \in C$, then in both cases, any statement made by A_k about A_i is a false atomic statement, which is acceptable by Definition 5.2 in the solution of the puzzle. Thus, based on these arguments, only Liars and Crazy persons (e.g., $A_k \in L$ or $A_k \in C$) can make statements about a Crazy person A_i (with the condition that $i \neq k$). ■

Corollary 5.1 There is no good and clear SSW-puzzle such that $C = \{\}$ in its solution.

Let us consider the puzzle in Example 4.2 as an SSW-puzzle; in contrast to Theorem 3.1, this puzzle becomes solvable, moreover it has a unique solution, as we will see in Example 5.1. This helps to characterize the main feature of SSW-puzzles, also it highlights the difference of the solution sets of SSW-puzzles from other puzzle types.

Example 5.1 Assume that we have an SSW-puzzle with three persons: Alice, Beth and Chris. Alice said that Beth is Liar, Beth said that Chris is Liar and "I am a Liar", Chris said that Alice and Beth are Truth-tellers. Determine who is Truth-teller, Liar and/or Crazy (see Figure 4.2)!

By shifting the type of the puzzle from SSS to SSW, there is exactly one solution for this puzzle as follows: Beth is Crazy, since she said self-contradictory statement, Alice and Chris are not Crazy person, since they didn't say any self-contradictory statements, but both made false atomic statements about Beth. Therefore, the solution of the previous puzzle is: (Alice, Chris $\in L$ and Beth $\in C$).

Note that this puzzle is unsolvable if it is considered as SS-puzzle or SSS-puzzle, since Beth cannot be either Truth-teller, Liar or Strong Crazy person, because only one of her statements is a self-contradicting statement. As in the previous example, Crazy persons play significant role in the solution of SSW-puzzle. Hence, the next lemma is used to identify Weak Crazy person in the puzzle.

Lemma 5.1 In the graph representation G of a solvable SSW-puzzle with set N of persons, for $A_k \in N$, if $A_k \overset{\dots}{} A_k \in G$, then $A_k \in C$.

Proof. The proof of this lemma is very similar to the proof of Lemma 4.1 ■

Note that by Lemma 5.1, in the solution of Example 5.1, Beth is a Crazy person, since she stated that “I am a Liar”.

Now we show a lemma which is analogue to Lemma 4.2.

Lemma 5.2 Let a solvable SSW-puzzle and its graph representation G be given with a/the solution. For any $A_m, A_k \in N$, if $A_m, A_k \in L$ or $A_m, A_k \in T$ in the solution of the puzzle, then $A_m\overset{\cdots}{\,}A_k, A_k\overset{\cdots}{\,}A_m \notin G$.

Proof. If none of A_m and A_k says any statement in the puzzle, then, obviously, both $A_k\overset{\cdots}{\,}A_m \notin G$ and $A_m\overset{\cdots}{\,}A_k \notin G$ are automatically satisfied. The rest of the proof is by contradiction. Let $A_m, A_k \in N$, such that at least one of A_m and A_k says some statement. Let us suppose that $A_m\overset{\cdots}{\,}A_k \in G$ (or $A_k\overset{\cdots}{\,}A_m \in G$). First, if we assume that $A_m, A_k \in T$, then the atomic statement represented by $A_m\overset{\cdots}{\,}A_k$ in G (or $A_k\overset{\cdots}{\,}A_m$) is a false atomic statement said by a Truth-teller which is a contradiction. Second, if $A_m, A_k \in L$, then the atomic statement represented by $A_m\overset{\cdots}{\,}A_k$ in G (or $A_k\overset{\cdots}{\,}A_m$) implies that $A_m\overset{\cdots}{\,}A_k \in \alpha$ (or ($A_k\overset{\cdots}{\,}A_m \in \alpha$)), which means that $A_m\overset{\cdots}{\,}A_k \in G$ (or $A_k\overset{\cdots}{\,}A_m \in G$) represents a true atomic statement made by a Liar which contradicts to Definition 5.2. All the cases are proven, hence the lemma. ∎

Consequently, by Lemma 5.2, if the SSW-puzzle is solvable, then in the graph representation of the puzzle, there is no broken edge between any two persons who both are Liar or both are Truth-teller in any of the solutions of the puzzle.

Corollary 5.2 In an SSW-puzzle represented by G, if there are two persons A_k and A_m, such that $\overset{\dots}{A_k A_m} \in G$ (or $\overset{\dots}{A_m A_k} \in G$), $\overset{\dots}{A_k A_k} \notin G$, $\overset{\dots}{A_m A_m} \notin G$ and both A_k and A_m said the same statement about a third person A_i, then the puzzle is unsolvable.

As the previous corollary shows, there are some unsolvable SSW-puzzles. In Example 5.1, we found a solution of an SSW-puzzle that was unsolvable if we consider it as SS-puzzle or SSS-puzzle. In other words, if we permit not only Truth-tellers and Liars in the target of the solution function as in SS-puzzles (or Truth-tellers, Liars and Strong Crazy in the target of the solution function as in SSS-puzzles), but also add (replace in SSS-puzzles) the Weak Crazy type then we will have more solvable than SS-puzzles and SSS-puzzles. It is turned out that not every puzzle becomes solvable in this way.

Now, let us give further properties of Crazy people in SSW-puzzles.

Lemma 5.3 In the solution of an SSW-puzzle with set N of persons and its graph representation G, for a person $A_k \in N$ if

- $\overrightarrow{A_k} = \{\,\}$ and
- $\exists\, A_i, A_j \in N/\{A_k\}$ such that the type of A_i is the same as the type of A_j, but $A_i, A_j \notin C$, and $\overset{\dots}{A_j A_k}$, $\overline{A_\imath A_k} \in G$,

then $A_k \in C$ and $A_i, A_j \in L$.

Proof. In case $i \neq j$, $A_i, A_j \notin C$ and the type of A_i is the same as the type of A_j (i.e., both $A_i, A_j \in L$ or both $A_i, A_j \in T$), but they make two different statements about A_k ("A_k is a Liar" and "A_k is a Truth-teller"), then in order to have a solution, $A_i, A_j \in L$ and to make both above statements false, $A_k \in C$. The case when both $A_i, A_j \in T$ does not lead to solution, since these two statements about A_k cannot the true at the same time.

If $i = j$, then this means that there is one person A_i who made two different statements about A_k and $A_i \notin C$. Moreover, $A_i \notin T$, since the mentioned two statements cannot be both true at the same time. Hence, to have a solution, $A_i \in L$ and $A_k \in C$. ■

Example 5.2 Assume that we have an SSW-puzzle with four persons: Alice, Beth, Chris and Dani. Alice said that Dani is Liar, Beth didn't say anything, Chris claimed that Dani and Beth are Truth-tellers, Dani said that Beth is Liar. Determine who is Truth-teller, Liar and/or Crazy (see Figure 5.1)!

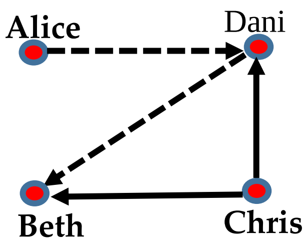


Figure 5.1: Graph representation of the SSW-puzzle in Example 5.2

Let us find the solution(s) of the puzzle. Beth is a Crazy person by Lemma 5.3, since Chris and Dani have the same type (otherwise the puzzle is unsolvable) and they said different statements about Beth who didn't say anything. Neither Chris nor Dani has a self-contradicting statement, therefore, Chris and Dani are Liars: they say false atomic statements about Beth's type. Alice is, then, a Truth-teller, since she said a true atomic statement about Dani's type. Therefore, the puzzle is a good puzzle: it has only one solution.

The following theorem characterize one of the important properties of Weak Crazy persons and number of self-contradictory statements they can make in the puzzle. We recall that in Lemma 5.1 we have already seen self-contradictory statements.

Theorem 5.1 In any solvable SSW-puzzle with set N of persons and its graph representation G, if $A_k \in C$, then A_k can say at most one self-contradictory statement, which is represented by $A_k\dddot{A}_k$ in G.

Proof. In an SSW-puzzle, any person A_k can say only two forms of atomic statements which are represented in G by: either $A_k\dddot{A}_m$ or $\overline{A_kA_m}$ where $A_m \in N$. Suppose that $A_k \in C$ and $A_k\dddot{A}_m \in G$, where $k \neq m$. If $A_m \in L$, then $A_k\dddot{A}_m \in \alpha$, which means that $A_k\dddot{A}_m \notin \gamma$. If $A_m \in T$, then $A_k\dddot{A}_m \in \beta$ which implies that $A_k\dddot{A}_m \notin \gamma$. If $A_m \in C$, then $A_k\dddot{A}_m \in \beta$, thus again $A_k\dddot{A}_m \notin \gamma$.

Suppose now that $A_k \in C$ and $\overline{A_kA_m} \in G$ with $k \neq m$. If $A_m \in L$, then $\overline{A_kA_m} \in \beta$, which means that $\overline{A_kA_m} \notin \gamma$. If $A_m \in T$, then $\overline{A_kA_m} \in \alpha$, thus $\overline{A_kA_m} \notin \gamma$. If $A_m \in C$, then $\overline{A_kA_m} \in \beta$, hence $\overline{A_kA_m} \notin \gamma$. If $k = m$, then $\overline{A_kA_k} \in \beta$, which also means that $\overline{A_kA_k} \notin \gamma$. Consequently, the only self-contradictory statement a Crazy person A_k can say, is represented by $A_k\dddot{A}_k$ in G. ∎

Thus, a Crazy person either says "I am a Liar" statement and maybe some other statements, or does not say anything.

For the next theorem, again as in Proposition 4.3, we recall the definition of complete undirected graph with *n* nodes (denoted by K_n). Specifically, K_3 is a triangle graph. In the next theorem we will show how the complete graph K_3 will play an important role in defining unsolvable SSW-puzzles.

Theorem 5.2 In the graph representation G of an SSW-puzzle, if G contains subgraph G' with three vertices (let us say $A, B, D \in N$), such that the underlying undirected subgraph of G' form K_3 graph with broken edges, and $\ddot{A}A, \ddot{B}B, \ddot{D}D \notin G$, then the SSW-puzzle represented by G is unsolvable.

Proof. Suppose that graph representation G contains subgraph G', whose underlying undirected graph form K_3 graph with broken edges. Considering such subgraph, there are two possible non-isomorphic subgraphs in G, as they are shown in Figure 4.1.

Let us start with the subgraph representation in Figure 4.1(a). In the puzzle, A and D say the same type of statements about B, but there is broken edge also between them. Hence, by Corollary 5.2, the puzzle is unsolvable.

If we consider subgraph represented in Figure 4.1(b), by Theorem 5.1, $A, B, D \notin C$, because none of them say a self-contradictory statement, but each of them says other statement(s). Now, if we assume in the solution of the puzzle that $A \in T$, then $B, D \in L$, which means that B and D have the same type in the solution of G, but $\ddot{D}B \in G$ which is contradicting to Lemma 5.2. On the other hand, if we assume that $A \in L$, then $B, D \in T$ in the solution of the puzzle, but $\ddot{D}B \in G$, which contradicts again to Lemma 5.2.

Consequently, the puzzle is unsolvable for any graph representation which contains either the subgraph in Figure 4.1(a) or the subgraph in Figure 4.2(b). ∎

Chapter 6

STATISTICAL COMPARISON BETWEEN SS-, SSS- AND SSW-PUZZLES

In this chapter, many statistical results are presented in order to show the significance of puzzle type in finding the solution set of the puzzle, since we saw some puzzles in Chapters 4 and 5 that were unsolvable if we consider them as an SS-puzzles.

One may ask what is the chance for a “random” puzzle to be solvable or good. Some statistics about the number of good, solvable and unsolvable SS-puzzles and SSS-puzzles generated by computer were presented in [9, 23]. In this section, we present also some statistics about the puzzles studied here, e.g., about the number of puzzles depending on the number of persons. Then, we give some details on the number of edges in the puzzle graphs.

In the following statistical data, firstly we have considered puzzles in which each person can say only one statement about any person in the puzzle. In second part of the comparison between the three types of puzzles, we will consider the case were any person may say both types of statements about any person in the puzzle (including himself/herself). We consider two forms of comparison to highlight the importance of puzzle specification, and show how slight change may affect the number of good, solvable, and unsolvable puzzles.

Total number of possible puzzles depends on the number of persons in the puzzle and on the number of the possible statements that the person can say about any person in the puzzle. Suppose that we have a puzzle with two persons A_i and A_j, then in the puzzle, if we consider the first type (where each person can say one statement about any person), then A_i will not say anything about A_j, or A_i says that A_j is a Liar, or A_i says that A_j is a Truth-teller. Hence any person has three choices to talk about any person in the puzzle (including himself/herself). Therefore, the total number of possible puzzles with two persons is 3^4. In the second type of puzzles, each person has four possible ways to talk about any person in the puzzle, therefore, the total number of possible puzzles with two persons is 4^4.

6.1 Comparison Type I Between SS-, SSS-, SSW-puzzles

In this type of comparison between the three predefined puzzle types, each person can say only one statement about any person in the puzzle. Since in SS-puzzles, if any person says both types of statements about any person in the puzzle, then the puzzle is unsolvable, this restriction is applied in this comparison type. Table 6.1 shows the number of good puzzles, solvable puzzles (including the good ones), and unsolvable puzzles with two, three and four persons of SS-, SSS- and SSW-puzzles. Table 6.2 shows the number of solvable SS-, SSS-and SSW-puzzles with two, three and four persons with respect to the number of solutions for the puzzles. In fact, the SS-, SSS- and SSW-puzzles correspond to each other by the statements, but their solvability differ as it is shown. Table 6.3 shows the number of good SSS- and SSW-puzzles with one, two, three or four Crazy persons in their solutions if there are two, three or four persons in the puzzle.

Table 6.1: The number of good, solvable (including good ones) and unsolvable SS-, SSS- and SSW-puzzles with two, three and four persons, in case if each person can say at most one statement about any other person.

	Two-person			Three-person			Four-person		
Puzzle type	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**
Good puzzles	0	9	33	0	553	8577	0	136017	13157025
Solvable	28	41	73	1880	2619	12563	506896	668849	15113617
Unsolvable	53	40	8	17803	17064	7120	42539825	42377872	27933104
Total	81	81	81	19683	19683	19683	43046721	43046721	43046721

Table 6.2: The number of SS-, SSS- and SSW-puzzles with one, two, three, four, five or more solutions for puzzles with two, three or four persons, in case if each person can say at most one statement about any other person.

	Two-person			Three-person			Four-person		
Puzzle type	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**
Good puzzles	0	9	33	0	553	8577	0	136017	13157025
Two-solution puzzles	24	18	22	1728	1467	2667	490752	443876	1466092
Three-solution puzzles	0	10	14	0	411	1035	0	66836	412636
Four-solution puzzles	4	1	1	144	51	75	15552	7350	27942
Five-solution puzzles	0	0	0	0	24	24	0	3963	6432
Puzzles with more than five solutions	0	3	3	8	113	185	592	10807	43490

Table 6.3: The number of good SSS-an SSW-puzzles with one, two, three or four Crazy persons in the solution in case if each person can say at most one statement about any other person.

	Two-person puzzle		Three-person puzzle		Four-person puzzle	
Number of Crazy persons in the solution	**SSS-puzzle**	**SSW-puzzle**	**SSS-puzzle**	**SSW-puzzle**	**SSS-puzzle**	**SSW-puzzle**
One	8	24	504	3960	124480	2572224
Two	1	9	48	3888	11328	5959296
Three	-	-	1	729	208	4094064
Four	-	-	-	-	1	531441

Observe that, the number of unsolvable puzzles increases dramatically as number of persons in the puzzle increases (see Table 6.1). In SS-puzzles, there are no puzzles with odd number of solutions, since a solution and its dual (assigning Truth-teller to the former Liars and Liar to the former Truth-tellers) are both solutions for the SS-puzzle. In contrast, as Table 6.2 shows there are various SSS- and SSW-puzzles with odd number of solutions. Table 6.3 shows that the majority of the good SSS-puzzles have exactly one Crazy in their solution.

6.2 Comparison Type II Between SS-, SSS-, SSW-puzzles

In this section, we introduce some appealing statistics about SSS-puzzles and SSW-puzzles in a comparison to SS-puzzles depending on the number of persons. In these statistics we assumed that it is allowed to each person to make two different statements about any other person in the puzzle. Note that it is shown in [9], that if any person says more than one type of statements about any person in an SS-puzzle, then the puzzle will be unsolvable.

In SSS- and SSW-puzzles we show that there are some good puzzles, hence it will be interesting if it is possible to predict what is the chance to have a good, solvable or unsolvable puzzle. Table 6.4 shows the number of good, solvable (including the good ones) and unsolvable puzzles of SS-, SSS- and SSW-puzzles with two, three and four persons. Note that in SSW-puzzles, the number of unsolvable puzzles decreases dramatically as the number of persons increases in a comparison to SS- and SSS-puzzles. On the other side, the number of solvable puzzles increases. Table 6.5 shows the number of solvable puzzles with one (good puzzles), two, three, four, five or more solutions for two, three and four persons in SS-, SSS- and SSW-puzzles. As it is observed in Table 6.5, the majority of solvable SSW-puzzles are good ones which is noticeable in comparison with SSS-puzzles. Table 6.6 presents the number of good SSS- and SSW-puzzles with one, two, three or four Crazy persons in the solution(s) of puzzles with two, three and four persons. As it is shown in Table 6.6, the majority of good SSS-puzzles has one crazy person, but in SSW-puzzles this property doesn't hold. Indeed, SS-, SSS- and SSW-puzzles have the same structure in term of statements and persons, but they are different in their solvability.

Table 6.4: The number of good, solvable (including good ones) and unsolvable SS-, SSS- and SSW-puzzles with two, three and four persons if each person can say two statements about any person

	Two-person			Three-person			Four-person		
Puzzle type	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**
Good puzzles	0	17	164	0	1369	152246	0	403361	1774218584
Solvable	28	49	224	1880	3531	167614	506896	961921	1822388936
Unsolvable	228	207	32	260264	258613	94530	4294460400	4294005375	2472578360
Total	256	256	256	262144	262144	262144	4294967296	4294967296	4294967296

Table 6.5: The number of SS-, SSS- and SSW-puzzles with one, two, three, four, five or more solutions for puzzles with two, three or four persons if each person can say two statements about any person.

	Two-person			Three-person			Four-person		
Puzzle type	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**	**SS**	**SSS**	**SSW**
Good puzzles	0	17	164	0	1369	152246	0	403361	1774218584
Two-solution puzzles	24	18	32	1728	1539	9444	490752	465188	31692456
Three-solution puzzles	0	10	24	0	435	5364	0	70484	15826104
Four-solution puzzles	4	1	1	144	51	144	15552	7782	212760
Five-solution puzzles	0	0	0	0	24	24	0	3963	16128
Puzzles with more than five solutions	0	3	3	8	113	392	592	11170	422904

Table 6.6: The number of good SSS-an SSW-puzzles with one, two, three or four Crazy persons in the solution in case if each person can say two statements about any person.

	Two-person puzzle		Three-person puzzle		Four-person puzzle	
Number of Crazy persons in the solution	**SSS-puzzle**	**SSW-puzzle**	**SSS-puzzle**	**SSW-puzzle**	**SSS-puzzle**	**SSW-puzzle**
One	16	100	1224	25776	342272	26128352
Two	1	64	144	93702	60096	416386416
Three	-	-	1	32768	992	1063268360
Four	-	-	-	-	1	268435456

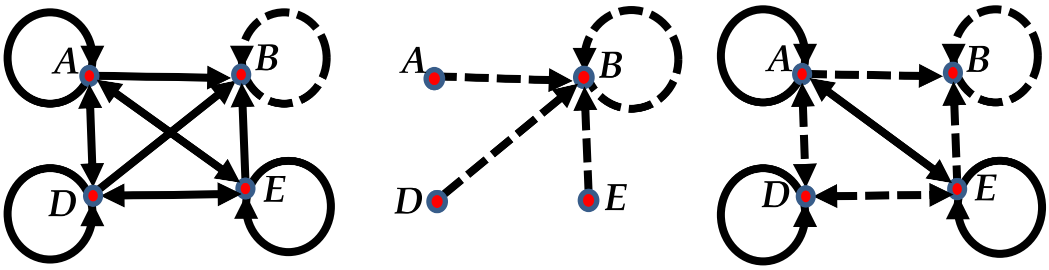
6.3 Maximal and Minimal Graphs of Good Puzzles

In this part of the thesis, we turn to analyze the number of edges in the graphs of SS-, SSS- and SSW-puzzles. That will help to figure out how the good puzzles, solvable and unsolvable puzzles can be recognized through their graphs. Proposition 4.3 and Theorem 5.2 give examples of how the graph of the puzzle plays a significant role on defining unsolvable SSS- and SSW-puzzles.

In SS-puzzles, as it is already mentioned, there is no good SS-puzzles. Considering solvable puzzles, each person is allowed to say maximum 1 statement about each person in the puzzle (including himself/herself), then the maximum number of edges in any solvable SS-puzzle with *n* persons is n^2. On the other hand, the minimum number of edges in any solvable SS-puzzle is 0, since if nobody says anything in the puzzle, then each person can be either Truth-teller or Liar meaning that the number of the solution for such puzzle with *n* persons is 2^n.

In the first type of puzzles, where each person is allowed to say only one statement about any person (including himself/herself). In SSS-puzzles, the maximum number of edges in a good SSS-puzzle with four persons is 13 (see, e.g., Figure 6.1(a), observe that some of the edges are bidirectional), with three persons is 7, and with two persons is 3. The minimum number of edges in good SSS-puzzles with four persons is 4 (as it is shown, e.g., in Figure 6.1(b)), with three persons is 3, and with two persons is 2, while maximum number of broken edges in good SSS-puzzles with four persons is 7 (see Figure 6.1(c)), with three persons is 4, and with two persons is 2. Also, minimum number of broken edges is in good SSS-puzzles with four persons is 1 (see Figure 6.1(a) for a graph with this property), and maximum number of solid edges is 12

(Figure 6.1(a) has also this feature) and minimum number of solid edges is 0 (see Figure 6.1(b) for an example).



(a) (b) (c)

Figure 6.1: (a) An example of maximal graph of a good SSS-puzzle with 13 edges which contains maximum number of solid edges: 12. (b) Graph of a good puzzle with minimum number of edges which is 4 and minimum number of solid edges which is 0. (c) An example of graph representation of a good puzzle with maximum number of broken edges which is 7.

On the other side, in SSW-puzzles, the maximum number of edges in a good SSW-puzzle with four persons is 16 (see, e.g., Figure 6.2(a), observe that some of the edges are bidirectional), with three persons is 9, and with two persons is 4. The minimum number of edges in good SSW-puzzles with four persons is 4 (same as it is shown, e.g., in Figure 6.1(b) if the puzzle is considered as an SSW-puzzle), with three persons is 3 and with two persons is 2. While the maximum number of broken edges in good SSW-puzzles with four persons is 16 (see Figure 6.2(b)), with three persons is 9, and with two persons is 4. Also, the minimum number of broken edges in good SSW-puzzles with four persons is 1 (see Figure 6.2(a) for a graph of an SSW-puzzle with this property), and the maximum number of solid edges is 15 (Figure 6.2(a) has also this feature) and the minimum number of solid edges is 0 (see Figure 6.2(b) for an example).

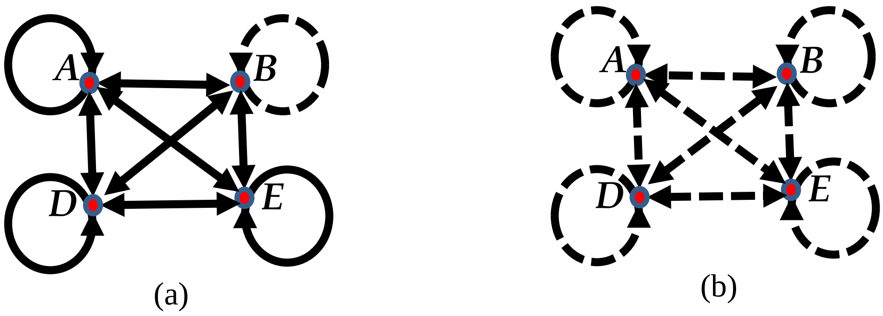


Figure 6.2: (a) An example of maximal graph of a good SSW-puzzle with 16 edges which contains maximum number of solid edges: 15. (b) An example of graph representation of a good puzzle with maximum number of broken edges which is 16.

Now, we will figure out the maximal and minimal graphs of the second type of puzzles, where each person is allowed to say two statements about any person (including himself/herself). In the SSS-puzzles, the maximum number of edges in any good SSS-puzzle with four persons is 16, with three persons is 9 (see e.g. Figure 6.3(a), note that we have used bidirectional edges between any two different nodes) and with two persons is 4. The minimum number of edges in good SSS-puzzles with four persons is 4, with three persons is 3 (Figure 6.3(b) shows such graph) and with two persons is 2. Whilst, the maximum number of broken edges in good SSS-puzzles with four persons is 7, the maximum number of broken edges in good puzzles with three persons is 4 (Figure 6.3(b) presents an example with such property). In addition, the maximum number of broken edges in good puzzles with two persons is 2. On the other hand, the minimum of broken edges in good SSS-puzzles with four, three or two persons is 1 (see Figure 6.3(c)), and the maximum number of solid edges in SSS-puzzles with four persons is 12 (same as the graph presented in Figure 6.1(a)), with three persons is 6, and with two persons is 2. While the minimum number of solid edges in good SSW-puzzles with four, three and two persons is 0 (as is it shown in Figure 6.1(b)).

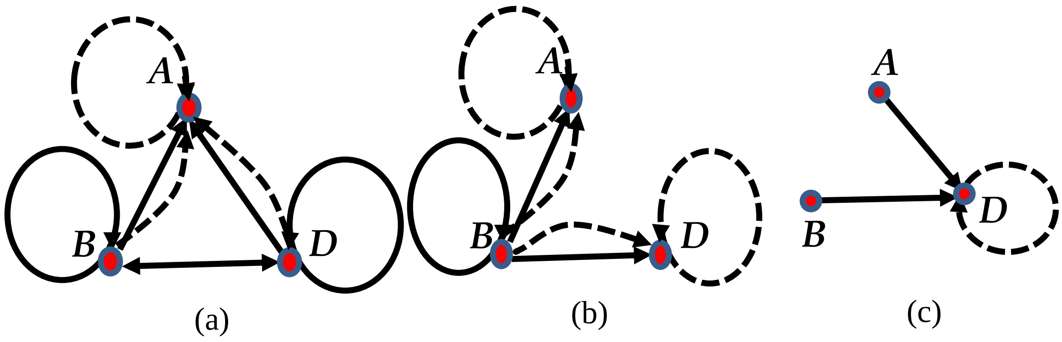


Figure 6.3: (a) An example of graph of a good SSS-puzzle with three persons that has maximum number of edges which is 9. (b) An example of graph representation of good SSS-puzzle with maximum number of broken edges that equals 4. (c) A graph of an SSS-puzzle with minimum number of edges, where this graph contains minimum number of broken edges which is 1.

In the SSW-puzzles, the maximum number of edges in any good SSW-puzzles with four persons is 32, with three persons is 18 (see e.g. Figure 6.4) and with two persons is 8. Observe that in the graph representation of puzzle with maximum number of edges, each person says both types of statements about any person in the puzzle (including himself/herself). The minimum number of edges in good SSW-puzzles with four persons is 4. The minimum number of edges in good SSW-puzzles with three persons is 3 and the minimum number of edges in good puzzles with two persons is 2. Whilst, the maximum number of broken edges (or solid edges, respectively) in good SSW-puzzles with four persons is 16, the maximum number of broken edges (or solid edges, respectively) in good puzzles with three persons is 9 (Figure 6.4 presents an example with such property). Further, the maximum number of broken edges (or solid edges, respectively) in good puzzles with two persons is 4. On the other hand, the minimum of broken edges in good SSW-puzzles with four, three and 2 persons is 1, and the minimum number of solid edges in good SSW-puzzles with 4 persons is 0.

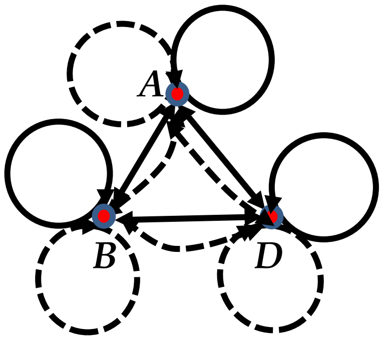


Figure 6.4: An example of graph of a good SSW-puzzle with maximum number of edges which is 18, where we have 9 broken edges and 9 solid edges.

In the next lemmas, we show some interesting facts about the possible number of edges in the graphs of good SSS- and SSW-puzzles.

Lemma 6.1 For any good SSS- and SSW-puzzle with *n* persons, the graph representation *G* of the puzzle has at least *n* edges.

Proof. Let us consider minimal puzzles in term of the number of edges in their graph representation *G*. We may have m different connected components, where $m \leq n$. From Corollary 4.1 and 5.1, each one of these components must have a Crazy person, otherwise, the puzzle is not a good puzzle. Let us assume that in each component (let k_i denote the nodes of the *i*-th component) of these *m* components, the number of the nodes equals to n_i, such that $\sum_{i=1}^{m} n_i = n$ and $\exists A_{k_i} \in C,\ A_{k_i} \in k_i$. Therefore, in any connected component in graph *G*, there are at least $n_i - 1$ edges that make that component connected and one more edge that makes A_{k_i} Crazy person. Thus, the total number of edges in each component is at least n_i. Consequently, the total number of edges in graph representation *G* equals to:

$$\sum_{i=1}^{m} n_i = n \qquad \blacksquare$$

Example 6.1 Figure 6.5 shows the graph representation of a minimal 4-person good SSS-puzzle, which has 4 edges. The solution of this puzzle is: $A \in L, B \in C, D \in L$ and $E \in L$.

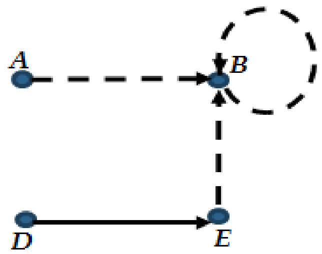


Figure 6.5: A 4-persons good SSS- puzzle with minimum number of edges

Since in good SSS-puzzles and SSW-puzzles, one person may say two different statements about another, it is interesting to count the maximal number of edges in such puzzles. Although, the case of SSS-puzzles and SS-puzzles are completely different from this point view, the result is very similar to the maximal number of edges in solvable SS-puzzles and good SSS-puzzles. Formally, we have:

Lemma 6.2 For any good SSS-puzzle with *n* persons, the graph representation *G* of the puzzle has at most n^2 edges.

Proof. Let us consider maximal puzzles in term of the number of edges in their graph representation *G*. First, we show that between any two nodes *A* and *B*, there are at most two edges. If in the solution of the puzzle, $A \notin C$ and $B \notin C$, then between *A* and *B*, there will be at most two same type edges in opposite directions. If *A* or *B* have two different types of outgoing edges toward the other, then the puzzle will be unsolvable. If $A \in C$ and $B \in C$, then by Theorem 4.1, both of them can have self-broken edge and there are no edges between them. Finally, if $A \in C$ and $B \notin C$ ($B \in L$ or $B \in T$), then in the maximal graph G, $\ddot{BA} \in G$ and $\overline{BA} \in G$ (in case of $B \in L$).

In the solution of the puzzle with maximal graph G, let us assume that there are m Crazy persons and $n - m$ non-crazy persons. Thus, the maximum number of edges between non-Crazy persons equals to $(n - m)^2$, since each one of these non-crazy persons have $n - m - 1$ outgoing edges toward others and one self-solid edge. And the maximum number of edges between non-Crazy and Crazy persons is $2(n - m)m$, since each non-Crazy person can have two types of outgoing edges toward every Crazy person (if the non-crazy persons are liars). By Theorem 4.1, graph G has m self-broken edges, since each crazy person can have one self-broken edge. Therefore, the total maximum number of edges is given by:

$$\begin{aligned}(n-m)^2 + 2(n-m)m + m &= n^2 - 2nm + m^2 + 2nm - 2m^2 + m \\ &= n^2 - m^2 + m\end{aligned}$$

That is maximal when $m = 1$, since $m = 0$ is not possible (the puzzle cannot be good if there is no Crazy person in the solution). Hence, in the maximal graph G, the maximum number of edges equals to n^2 and the number of Crazy persons equals 1. ■

Example 6.2 Figure 6.6 shows the graph representation of a maximal 4-person good SSS-puzzle, which has 16 edges, where the solution for such puzzle is: $A \in L, B \in L, D \in L$ and $E \in C$.

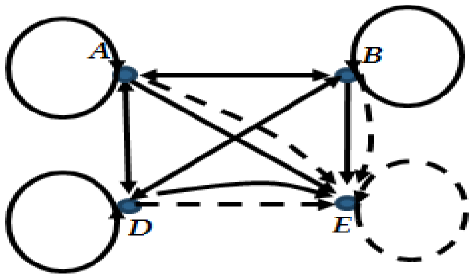


Figure 6.6: A 4-person good SSS- puzzle with maximum number of edges

Now, we will study maximal graphs of good SSW-puzzles.

Lemma 6.3 For any good SSW-puzzle with n persons, the graph representation G of the puzzle has at most $2n^2$ edges (n^2 solid and n^2 broken edges).

Proof. Let us consider maximal puzzles in terms of the number of edges in their graph representation G. By definition any person can tell at most two sentences about a person, this makes at most $2n^2$ statements in the puzzle (equivalent to edges in G). Let us prove that this puzzle is a good puzzle. Each person has the self-contradictory statement “I am a Liar”, thus everybody could be only Crazy in the solution. Then, by the definition of Weak Crazy people, it is allowed to say any other statements independently if they are true or false. In fact, all other statements will be false. Thus, for any two people, E and F, in the puzzle, in the graph, both of them have self-broken edges and self-solid edges, moreover, there are all the four edges: $\ddot{FE}, \ddot{EF}, \overline{FE}$ and $\overline{EF}$. Consequently, the maximum number of edges in a good SSW-puzzles appear in the case when all people are (non-silent) Crazy in the solution. Furthermore, exactly half of the edges are solid and half of them are broken. ■

Example 6.3 Figure 6.4 shows the graph representation of a maximal 3-person good SSW-puzzle, which has 18 edges, where the solution for such puzzle is: $A \in C, B \in C$, and $D \in C$.

Chapter 7

CONCLUSION

Truth-tellers and Liars appear in various puzzles. In this thesis, we have discussed three distinct types of puzzles: SS-puzzles, SSS-puzzles and SSW-puzzles. Two new types of people are investigated: Strong Crazy people can say only statements that are self-contradictory, i.e., no Truth-teller or Liar could say them; and Weak Crazy people who say at least self-contradictory statement (if they say anything in the puzzle). Thus, SS-puzzles are similar to SSS-puzzles and SSW-puzzles, but there can be three types of people in the solution instead of the original two (in SS-puzzles). By investigating the Strong and Weak Crazy people, several unsolvable puzzles become solvable and also there are clear and good SSS-puzzles and SSW-puzzles (which was not the case with SS-puzzles), just to recall our main results. On the other hand, many unsolvable SSS-puzzles become also solvable and there are clear and good SSW-puzzles. It is also shown that there are unsolvable SSS-puzzles and SSW-puzzles (based on Proposition 4.3 and Theorem 5.2 one can easily create such puzzles). Some statistical data are used to compare SS-puzzles, SSS-puzzles and SSW-puzzles with two, three and four persons. Two types of comparison were presented and discussed, where in type I, each person can say only one statement about any person in the puzzle. While in the comparison of type 2, each person can say both types of statements about any person in the puzzle.

Considering the newly introduced types of persons: Strong Crazy and Weak Crazy, as an application of SSS-puzzles and SSW puzzles, wireless sensor networks might benefit from such improved models. In such networks, the messages between the neighbour sensors plays significant role in studying the field and the environment that they are placed in. A generalized scenario of satellite model that has been discussed in [6].

In studying the previous three puzzle types, some questions are left open. In the future, we would like to find some easily checkable properties that characterize the good, solvable and unsolvable SSS-puzzles and SSW-puzzles. Another type of puzzles can also be considered in the future, where the person can say a new type of atomic statement which is "A_i is a Crazy person".

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