Compact Finite Difference Scheme for One Dimensional Parabolic Inverse Problem

Vincent Chukwuebuka Onwuegbuna

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
February 2021
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazım Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Derviş Subaşı
Supervisor

Examining Committee

1. Prof. Dr. Derviş Subaşı
2. Assoc. Prof. Dr. Suzan C. Buranay
3. Asst. Prof. Dr. Buğçe Eminağa Tatlıcıoğlu

ABSTRACT

Inverse Problems is a research area dealing with inversion of models or data. An inverse problem is a mathematical model that is used to obtain information about a physical object or system from observed measurements. The solution to this problem is useful because it generally provides information about a physical parameter that we cannot directly observe or measure. These problems play a very important role in many areas of science and engineering. In this thesis, we formulated a compact finite difference scheme for a one dimensional parabolic inverse problem to determine the solution $\phi(x,t)$ and control parameter $c(t)$. The global existence and uniqueness of the solution $\phi(x,t)$ was proved using the method of retardation of the time variable along with an a priori estimation. Also, the existence of solution or solvability of the formulated compact scheme was proved by employing the homogeneous system of the tridiagonal system resulted from the formulated scheme. Some numerical results are presented to show that the accuracy of the space and time directions are improved, and computation time is shortened largely.

Keywords: Parabolic Inverse Problems, Control Parameter, Existence and Uniqueness, Compact Difference Scheme.

ÖZ

Ters Problemler, modellerin veya verilerin ters çevrilmesi ile ilgilenen bir araştırma alanıdır. Ters problem, gözlemlenen ölçümlerden fiziksel bir nesne veya sistem hakkında bilgi elde etmek için kullanılan matematiksel bir modeldir. Bu sorunun çözümü yararlıdır çünkü genellikle doğrudan gözlemleyemediğimiz veya ölçemediğimiz bir fiziksel parametre hakkında bilgi sağlar. Bu sorunlar bilim ve mühendisliğin birçok alanında çok önemli bir rol oynamaktadır. Bu tezde, $\phi(x,t)$ çözümünü ve $c(t)$ kontrol parametresini belirlemek için tek boyutlu bir parabolik ters problem için kompakt sonlu bir fark şeması formüle edilmiştir. $\phi(x,t)$ çözümünün küresel varlığı ve benzersizliği, bir ön tahmin ile birlikte zaman değişkeninin geciktirme yöntemi kullanılarak kanıtlanmıştır. Ayrıca, formüle edilmiş kompakt şemanın çözümünün veya çözülebilirliğinin varlığı, formüle edilen şemadan elde edilen homojen tridiyagonal sistemi kullanılarak kanıtlanmıştır. Uzay ve zaman yönlerinin doğruluğunun geliştirildiğini ve hesaplama süresinin büyük ölçüde kısaltıldığını göstermek için bazı sayısal sonuçlar sunulmuştur.

Anahtar Kelimeler: Parabolik Ters Problemler, Kontrol Parametresi, Varlık ve Teklik, Kompakt Fark Şeması.

To My Mom

ACKNOWLEDGMENT

Firstly, I would like to give thanks and praise to God Almighty, for His unconditional love and faithfulness towards me throughout the period of this program, particularly during my research work.

My profound gratitude goes to Prof. Dr. Derviş Subaşı for his supervision, advice, and guidance from the very early stage of this thesis as well as his enthusiasm throughout the work. I am greatly overwhelmed by his patience, motivation, immense knowledge, and experience. I could not have wished for a better advisor and mentor for my research work.

My sincere thanks go to the members of my examining committee for their insightful comments and encouragement. I want to also thank the entire staff of Mathematics Department, most especially the Head of Department Prof. Dr. Nazım Mahmudov, Asst. Prof. Dr. Müge Saadetoğlu, Asst. Prof. Dr. Arran Fernandez and a host other who inspired me in one way or another.

I am extremely grateful to my family for their love, prayers, and support in making sure my dream become a reality. I am very much thankful to my mom, Mrs. Janet Onwuegbuna who sacrificed so much to see that I complete this program. Finally, I want to thank St. Mark's Anglican Church Famagusta and my friends who helped and encouraged me during the period of my studies and this thesis, few of whom I must mention are, Mr. and Mrs. Ojo Gbenga, Prince Ngene, Obinna Ezugwu and Muhammed Suleman.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

Inverse problems constitute the most essential mathematical problems in science and engineering, for reasons that they give us information about parameters that cannot be measured directly.

In this work, we are interested in the problem of finding the solution $\phi(x,t)$ and control parameter $c(t)$ in the one-dimensional parabolic inverse problem below:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + c(t)\phi + \psi(x,t), \quad 0 < x < 1, \; 0 < t \leq T, \tag{1.1a}$$

together with initial condition

$$\phi(x,0) = \gamma(x) \tag{1.1b}$$

and boundary conditions

$$\phi(0,t) = z_0(t), \quad \phi(1,t) = z_1(t), \quad 0 \leq t \leq T \tag{1.1c}$$

Subject to the overspecification at a point in the space coordinate

$$\phi(x^*,t) = H(t), \qquad\qquad 0 \leq t \leq T \tag{1.1d}$$

where $\gamma(x)$, $z_0(t)$, $z_1(t)$, $\psi(x,t)$ and $H(t)$ are known functions, $|H(t)| \geq H_0 > 0$, $x^* \in (0,1)$, while $\phi(x,t)$ and $c(t)$ are unknown functions. Equation (1.1) could be regarded as a control problem of determining the control parameter $c(t)$ in which the inner restraint Equation (1.1d) is fulfilled, provided $\phi(x,t)$ stands for the temperature of the system. The objective of finding the solution to the above inverse problem is to determine the control parameter $c(t)$ which will yield a desirable temperature at a specified location x^*in the space direction, at each time level t.

In applied mathematics, physics and medical sciences, the determination of the parameter (i.e., control parameter or conductivity) in partial differential parabolic equations with respect to overspecified data is of the essence. For example, parabolic inverse problems are encountered in the study of vibration problems, chemical diffusion, heat transfer processes, nuclear reactor dynamics, thermo-elasticity, control theory, inverse problems, population dynamics, biochemistry, and certain biological processes. To a great extent, the unknown properties of an extended spatial location of the space coordinate can be ascertained using parameter determination method, by measuring only data on the bounds of the region or a given point in the domain. An example of such unknown properties; is conductivity, which oftentimes we cannot measure directly, or it is too costly to be measured even though it plays a crucial role to the physical process. There is rapidly increasing research interest in finding the solution of parabolic PDE’s with a standardized boundary conditions [1-3].

Lately, there has been increasing interest towards the investigation of parabolic inverse problems. In the past three decades or more, there has been increasing interest given in research towards developing, analyzing, and implementing accurate techniques for numerically finding the solutions of parabolic inverse problems. To obtain such approximate solutions, a lot of different methods such as the finite difference, finite volume, finite element and boundary element methods have been suggested, but little investigation has been made towards the determination of approximate numerical solution of parabolic PDE’s subject to over-specified boundary data. Apparently, in recent years, we have seen that parabolic PDE's with source control parameters can be used to describe numerous natural processes [2,3].

In [3], inverse problems which involves identifying parameters were discussed as well as some applications of inverse problems. Cannon and Lin in [1], proved that the solution to these inverse problems exist and are unique, by making a transformation of problems (1.1a) – (1.1d). Cannon, Lin and Xu in [6], proved the convergence of a backward implicit finite difference method by first transforming the original problem. The solution ϕ of the scheme was shown to have order of convergence of $O(\tau + h^2)$, while the parameter c with order of convergence of $O\left(\tau^{1/2}\right)$ when $\tau = O(h^2)$. In [6], Dehghan gave four different difference schemes for problems (1.1a) – (1.1d). These includes 3-point FTCS method, 5-point explicit formula, 3-point BTCS method and implicit (3,3) Crandall's method. Two of these finite difference schemes; 3-point FTCS method and 3-point BTCS method are second-order accurate, while the implicit (3,3) Crandall's method and the 5-point explicit method have fourth-order precision. In [7], the authors presented a high-level order compact finite difference scheme for solving a one dimensional parabolic equation of a coefficient inverse problem. The proposed method is an efficient fourth order numerical method, founded on the Padé approximation, the functional alteration, and the Richardson extrapolation. The algorithm is used to find the solution $\phi(x,t)$ and the unidentified diffusion coefficient $a(t)$ which depends on time t, in the parabolic partial differential equation. Initially, the new algorithm was second order accurate in the time trend and fourth order accurate in the spatial domain, because it was developed from the Crank–Nicolson algorithm. Thus, the algorithm was improved to fourth order accuracy in the time trend using the Richardson's extrapolation method. The newly formulated high order method can as well be applied to a more general response diffusion equation and is proved to be unconditionally stable. In [7], Dehghan and Saadatmandi suggested for problems (1.1a) – (1.1d) an estimation method known as tau method, that is founded on Shifted Legendre tau theories. First, the method involves carrying out an expansion of the estimate solution $\phi(x,t)$ and $c(t)$ as a shifted Legendre function with unidentified coefficients, which reduces the problem to a system of algebraic equations. So, to determine the unidentified coefficients of the Shifted Legendre functions, the tau method in combination with matrices are used. Dehghan and Shakeri in [8] presented a method of line technique for solving problems (1.1a) – (1.1d). The idea was centered on making use of traditional finite difference approximations and the Runge-Kutta technique. The process requires discretizing the spatial domain such that an initial boundary value partial differential equation (PDEs) problem is reduced to a set of ordinary differential equations (ODEs) in time. Therefore, the subsequent set of ordinary differential equations may then be solved with the use of a basic ODE solver such as the Runge-Kutta method. In [9], a meshless method known as the moving least square (MLS) method was applied in solving problems (1.1a) – (1.1d). The method involves making use of the moving least square estimation, for the discretization of both time and spatial variables. The solution $\phi(x,t)$ is considered as a basis, and with the application of the collocation method, the control parameter $c(t)$ is recovered. Also, the approximation of moving least-square (MLS) has been used in [9] for finding the solution of problems (1.1a) – (1.1d). The method is a meshless method, which involves using the moving least-square approximation for discretizing both the time and spatial variables. The solution $\phi(x,t)$ is considered as a basis, and the control parameter $c(t)$ is obtained by the collocation method. In [10], a one dimensional parabolic inverse problem together with source and Neumann boundary conditions was solved using finite volume technique. The method involves dividing problems (1.1a) – (1.1d) into two separate problems, and the emerging problems are solved, respectively. The first part is solved as a direct problem, by integrating Eq. (1.1a) using Green's formula which results to a difference scheme that is unconditionally stable. But a different difference scheme was developed for the second part by means of right rectangle formula which results to a difference scheme that is stable for $|r| > 1$. In [11], Wang et al. presented a general form of problems (1.1a) – (1.1d). The inverse heat problem was solved by applying the technique of reproducing kernel space. The method involved; firstly, to redefine the inner products so that the reproducing kernel space can be obtained. As a result, the cumulative inaccuracies are decreased thereby improving the accuracy and decreasing the period of time for the system to run; secondly, problems (1.1a) – (1.1d) is reduced to a set of linear equation, thereby avoiding Gram Schmidt orthogonalization process. In [12], Kerimova and Ismailov presented a parabolic inverse problem together with nonlocal boundary and integral overspecification stipulations, with an investigation on how a time dependent coefficient in the inverse problem can be recovered. The method involved reducing problems (1.1a) – (1.1d) to an operator equation of the first kind, using Green's function method. The authors established that the solution of the inverse problem exists and are unique, likewise the continuous dependence upon the information of the solution, by means of the general Fourier method. In [13], Limin and Zongmin presented a category of parabolic inverse problems with two or more dimensions, which was solved by applying the Radial Basis Functions (RBFs) technique. The Radial Basis Functions is among the widely used meshless techniques in contemporary approximation concept. The technique of Radial Basis Functions approximation involves, to determine an estimated solution of the inverse problem by applying Radial

Basis estimation in the spatial domain for each time t, while the Finite Difference technique is applied in the time direction. The technique gives a global interpolation formula for both the derivatives of the solution and the solution itself, and processes high-level of accuracy. Therefore, for a category of inverse problem having a control parameter, the RBFs technique provides a quick and accurate meshless method of solution. Mohebbi and Dehghan in [14], presented a scheme with high-level accuracy for finding the unidentified solution $\phi(x,t)$ and unidentified control parameter $c(t)$ of parabolic inverse problems (1.1a) – (1.1d) together with overspecification at a precise location in the space coordinate and integral overspecification. In the method, the spatial domain was first approximated using fourth-order Compact Difference technique, which reduced problems (1.1a) – (1.1d) to a set of Ordinary Differential Equations (ODEs). The resulting set of ODEs is subsequently solved using Boundary Value Method (BVM) of fourth-order. Thus, the suggested method possesses high precision, i.e. it is fourth-order accurate in time coordinate as well as the spatial domain. Ye and Sun in [15] proposed a numerical method which improves the result obtained by Cannon and Lin in [4]. The authors following the idea in [4], formulated a difference scheme for problems (1.1a) – (1.1d) by making a transformation of the problem and as well proved that the proposed method is unique, completely stable and convergent. The approximations of both the control parameter $c(t)$ and the solution $\phi(x,t)$ have convergence orders of $O(\tau + h^2)$. In [16] Daoud and Subasi presented a different approach for determining the control parameter $c(t)$ and solution $\phi(x,y,t)$ of inverse control problems of two of more dimensions. The new procedure proposed is a predictor corrector parallel type of technique where the parallel splitting up technique is used to define both the solution and the predictor and corrector methods. The solution $\phi(x,y,t)$ is updated by a non-repetitive method in a two or more dimensions inverse control problem. The multiple dimensioned inverse problem is split up into several one-dimensional problems, which are then discretized using finite difference approximation. The algorithm is highly flexible in the sense that it allows one to choose a unique grid spacing for different variables in the space domain. Also, the authors proved that the method is unconditionally stable with high order of accuracy. In addition, the local and global convergency of the method was established to be second order and first order, respectively. In [2], Dehghan presented various schemes that can be used to determine the unknown function $c(t)$ in the problems (1.1a) – (1.1c), dependent on the integral overdetermination along the space coordinate

$$\int_0^1 k(x)\phi(x,t)dx = H(t), \qquad 0 \leq t \leq T,$$

Or the overdetermination condition on the space coordinate at a given spot

$$\phi(x^*,t) = H(t), \quad 0 \leq t \leq T,$$

respectively. The numerical methods proposed for determining the solution of problem (1.1a) – (1.1d) at the internal mesh points are; the 3-point second order forward time centered space (Explicit) scheme, the 3-point second order backward time centered space (Implicit) method, the Crank Nicolson (3,3) method, the Saulyev's method of first and second kind. The various methods are used to find the approximate result to the problem dependent on the over-specified data. The numerical approximations proposed are employed in finding the unidentified function $c(t)$, that is capable of producing the required temperature dissemination at any instant along the space coordinate, or the required energy dissemination along the space coordinate, at any point in time. Different techniques were suggested for finding the approximate value of $c(t)$ by applying the heat overdetermination constraint, or the energy overdetermination constraint.

Mohebbi and Abbasi in [3], proposed a scheme that has high accuracy for finding the unknown solution $\phi(x,t)$ along with unknown function $c(t)$ of parabolic inverse problems (1.1a) – (1.1d), subject to the overdetermination at a given position in the space coordinate. A Compact difference scheme of high-level order of accuracy was used to approximate the spatial domain in the new method formulated by the authors. The new method was proved to be stable and convergent, with a convergence order of $O(\tau^2 + h^4)$.

In this literature review, a number of the difference schemes proposed gives a method with low-level order of precision [5,7,15] or have high order of accuracy with no mention of how stable or convergent the scheme is [8,14].

Consider the nonnegative integers M and N. Let $h = \frac{1}{M},\ \tau = \frac{T}{N}$ and $r = \frac{\tau}{h^2}$. Define $\Omega_h = \{x_i = ih,\ 0 \le i \le M\}$, $\Omega_\tau = \{\mathrm{t}_n | \mathrm{t}_n = n\tau,\ 0 \le n \le N\}$, $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Assuming we have an integer k_0 which implies $x^* = \mathrm{x}_{k_0}$. This is feasible in real applications as indicated in [9].

Given a mesh function $\{\phi_i^n | 0 \le i \le M,\ 0 \le n \le N\}$ with respect to $\Omega_{h\tau,}$ we present the notations below:

$$\phi_i^{n+\frac{1}{2}} = \frac{\phi_i^{n+1} + \phi_i^n}{2}, \qquad \delta_t \phi_i^{n+\frac{1}{2}} = \frac{\phi_i^{n+1} - \phi_i^n}{\tau}, \qquad D_{\hat{t}} \phi_i^n = \frac{\phi_i^{n+1} - \phi_i^{n-1}}{2\tau},$$

$$\phi_i^{\overline{n}} = \frac{\phi_i^{n+1} + \phi_i^{n-1}}{2}, \qquad \delta_x^2 \phi_i^n = \frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{h^2},$$

$$\|\phi^n\| = \left[h \sum_{i=1}^{M-1} (\phi_i^n)^2 \right]^{1/2}, \qquad \|\phi^n\|_\infty = \max_{0 \le i \le M} |\phi_i^n|.$$

Denote the mesh functions

$\psi_i^n = \psi(x_i, t_n), \qquad H^n = H(t_n), \qquad (H')^n = H'(t_n), \qquad 0 \le i \le M,$

$0 \le n \le N.$

Dehghan in [5] suggested for problems (1.1a) – (1.1d) the subsequent implicit Crandall's scheme:

$$\frac{1}{12}\left(\delta_t \phi_{i-1}^{n+\frac{1}{2}} + 10\delta_t \phi_i^{n+\frac{1}{2}} + \delta_t \phi_{i+1}^{n+\frac{1}{2}}\right) = \delta_x^2 \phi_i^{n+\frac{1}{2}} + c^{n+1}\phi_i^n + \lambda_i^{n+1}\ ,$$

$$1 \le i \le M-1, \quad 0 \le n \le N-1, \tag{1.2a}$$

$$c^{n+1} = \frac{1}{H^{n+1}}\Big[(H')^{n+1} - \frac{1}{12h^2}\big(-\phi_{k_0-2}^{n+1} + 16\phi_{k_0-1}^{n+1} - 30\phi_{k_0}^{n+1} + 16\phi_{k_0+1}^{n+1} - \phi_{k_0+2}^{n+1}\big) - \lambda_{k_0}^{n+1}\Big], \qquad 0 \le n \le N-1, \tag{1.2b}$$

$$c^0 = \frac{1}{\gamma(x^*)}\left[(H')^0 - \gamma''(x^*) - \lambda_{k_0}^0\right], \tag{1.2c}$$

$$\phi_i^0 = \gamma(x_i), \qquad 1 \le i \le M-1, \tag{1.2d}$$

$$\phi_0^n = g_0(t_n), \qquad \phi_M^n = g_1(t_n),\ 0 \le n \le N. \tag{1.2e}$$

Dehghan provided a reiterative approach for solving the scheme. Suppose $(\phi_i^{n+1}, c^{n+1})^T$ is regarded as unidentified solution pair, then a linear algebraic system can be developed from equations (1.2a) – (1.2e). The linearized scheme is responsible for the accuracy of the time domain with an order of $O(\tau)$ only.

This thesis is organized as follows: In chapter 2, we established the existence and uniqueness of the solution of problems (1.1a) – (1.1d). This was achieved by firstly, making a transformation of the original problem to obtain a parabolic equation that is nonlocal and comparable to the original problem. Next, we established some a priori estimations together with some assumptions and lemmas to establish the proof that the solution of the problem exists and is unique. In chapter 3, we constructed a compact finite different scheme for the one dimensional parabolic inverse problems (1.1a) – (1.1d). The existence and uniqueness of the constructed scheme is as well presented. In chapter 4, two numerical examples are presented and solved using the constructed compact scheme to investigate or determine the accuracy and efficiency of the scheme, this was done by presenting the errors for $\phi(x,t)$ and $c(t)$. Finally, conclusion and observation are made for the experiment.

Chapter 2

EXISTENCE AND UNIQUENESS OF SOLUTION

2.1 Introduction

In this chapter we intend to establish the existence and uniqueness of the solution of problems (1.1a) – (1.1d). First, we shall begin by utilizing the following transformations:

$$\phi(x,t) = g(x,t)e^{\int_0^t c(s)ds} \tag{2.1}$$

$$r(t) = e^{-\int_0^t c(s)ds} \tag{2.2}$$

$$g = g(x,t) = \phi(x,t)e^{-\int_0^t c(s)ds} \tag{2.3}$$

$$c(t) = \frac{-r'(t)}{r(t)} \quad , \qquad r'(t) = \frac{dr(t)}{dt} \tag{2.4}$$

The transformation above, enables us to get rid of the term $c(t)\phi$ from problem (1.1a), so as to achieve a parabolic equation that is nonlocal and comparable to the original problem given that a certain number of conditions that enables it to be compatible are met. Next, the a *priori* estimations regarding the solution of the corresponding nonlocal parabolic problem will be derived. Then employing the strong maximum principle in conjunction with the compactness arguments, the solution of the problem is therefore proved to exist uniquely. Lastly, making use of the inverse transformation (2.4), we are able to establish that problems (1.1a) – (1.1d) has a global solution subject to appropriate assumptions on the data.

2.2 A Priori Estimate for Problems (1.1a) – (1.1d)

In this section we shall take into consideration the problem below, which arises from

$$\phi_t = L\phi + c(t)\phi + \psi\big(x,t,\phi,\phi_x,c(t)\big), \qquad 0 < x < 1, \qquad 0 < t < T \tag{2.5a}$$

$$\phi(x,0) = \gamma(x), \qquad x \in \Omega \tag{2.5b}$$

$$\phi(0,t) = z_0(t), \qquad \phi(1,t) = z_1(t), \qquad \text{on } \partial\Omega \times (0,T], \tag{2.5c}$$

$$\phi(x^*,t) = H(t), \qquad x^* \in \Omega, \qquad 0 \le t \le T \tag{2.5d}$$

Where the linear elliptic operator L is given by

$$L\phi = \sum_{i,j=1}^{n} (a_{i,j}(x,t)\,\phi_{x_i})\phi_{x_j}$$

Equations (2.5a) – (2.5d) becomes

$$\phi_t = \sum_{i,j=1}^{n} a_{i,j}\phi_{x_i x_j} + \sum_{i=1}^{n} a_i x_i + c(t)\phi + \psi\big(x,t,\phi,\phi_x,c(t)\big)$$

$$0 < x < 1, \qquad 0 < t < T \tag{2.6a}$$

$$\phi(x,0) = \gamma(x), \qquad x \in \Omega \tag{2.6b}$$

$$\phi(0,t) = z_0(t), \qquad \phi(1,t) = z_1(t), \qquad \text{on } \partial\Omega \times (0,T], \tag{2.6c}$$

$$\phi(x^*,t) = H(t), \qquad x^* \in \Omega, \qquad 0 \le t \le T \tag{2.6d}$$

Where $Q_T = \Omega \times (0,T], T > 0$ and Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega \in C^{2+\alpha}, (0 < \alpha < 1)$. We also assume that $G(t) \subset \Omega$ with smooth boundary $\partial G(t)$ for all $0 \le t \le T$ [17]. Now, applying transformation (2.1) – (2.4) to Equations (2.6a) – (2.6d), it's easy to see that $(\phi, c) \to (g, r)$, that we have:

$$g_t = \sum_{i,j=1}^{n} a_{i,j} g_{x_i x_j} + \sum_{i=1}^{n} a_i g_{x_i} + r(t)\psi\left(x,t,\frac{g}{r}\right) \quad \text{in} \quad \mathrm{Q_T} \tag{2.7a}$$

$$g(x,0) = \gamma(x), \qquad x \in \Omega \tag{2.7b}$$

$$g(0,t) = r(t)z_0(t), \quad g(1,t) = r(t)z_1(t), \quad \text{on } \partial\Omega \times (0,T], \tag{2.7c}$$

and

$$r(t) = \frac{g(x^*,t)}{H(t)}, \qquad 0 \le t \le T \tag{2.7d}$$

Now, to develop a priori estimation for the solution pair (g, r), we shall assume that $r \neq 0$ together with the assumptions on the data named below. Because the operator L is linear, we have the liberty to utilize $1 + \delta$ type estimates to achieve the bounds on (g, r).

Assumption H1: ([17]) $\gamma \ge 0$, $z_i (i = 0,1) \ge 0$, $\psi \ge 0$, $H > 0$, and for some $0 < \alpha < 1$,

$\gamma \in C^{2+\alpha}(\overline{\Omega})$, $\qquad z_i \in C^{(2+\alpha)/2}(\partial\Omega \times [0, T])$, $\qquad H \in C^{1+\alpha/2}([0, T])$.

Then meas $\{G(t)\}$ and meas $\{\partial G(t)\}$ are smooth functions on $[0, T]$.

Assumption H2: ([17]) $\psi = \psi(x, t, \phi, \phi_x, q)$ is a smooth function in regard to every one of its variables, $\psi \ge 0$ and $|\psi(x, t, \phi, c, q)| \le \delta|q| + C(|\phi| + |c| + 1)$, where $\delta > 0$ is such that

$$0 \le \delta^* = \delta \max_{0 \le t \le T} \left\{ H^{-1}(t) \int_{G(t)} \phi(x, t) dx \right\} < 1$$

Assumption H3: ([17]) $\qquad a_{ij}, a_{ijx_k} \in C^{\alpha}(\overline{\mathrm{Q_T}})$, $\quad y \in \Re^n$, $\qquad a_0, A_0 > 0$

$$a_0|y|^2 \le \sum_{i,j=1}^{n} a_{ij} y_i y_j \le A_0|y|^2$$

With the assumption that the data satisfies the basic compatibility condition, i.e., Eq. (2.6c) is satisfied on $\partial\Omega \times \{0\}$ by the data.

Assumption H4: ([17]) Assume the data a_{ij}, a_i, γ, z_i and H satisfies the equivalent statements in H1 – H3.

Assumption H5: ([17]) $\psi = \psi(x, t, \phi) \geq 0$ is a smooth function $|\psi(x, t, \phi)| \leq C(1 + |\phi|)$, and the condition that makes it compatible is satisfied on $\partial\Omega \times \{0\}$ by the data and $\gamma(x_0) = H(0) > 0$.

Lemma 2.1: ([17]) Suppose that equations (2.7a) – (2.7d) has a classified solution g with $g(x^*, t) \neq 0$. Then there exists a positive constant $M > 0$ that depends just on the data such that

$$\max_{(x,t)\in Q_T} |g(x, t)| + \max_{[0,T]} |r(t)| \leq M \tag{2.8}$$

Proof: Let

$$g(x, t) = w(x. t) \exp\left\{\delta \sum_{i=1}^{n} (x_i - x_i^*)^2 + \beta t\right\}$$

We realize by computations that w satisfies

$$w_t = \sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} + \sum_{i=1}^{n} \bar{a}_i w_{x_i} + (\bar{a} - \beta)w + r(t)e^{-\xi}\psi\left(x, t, \frac{e^{\xi} w}{r(t)}\right) \qquad \text{in } Q_T$$

$$w(0, t) = \exp\{-\delta \textstyle\sum_{i=1}^{n} (x_i - x_i^*)^2\} \frac{z_0(t)}{H(t)} g(x^*, t) \qquad \text{On } \partial\Omega\times(0,T]$$

$$w(1, t) = \exp\left\{-\delta \sum_{i=1}^{n} (x_i - x_i^*)^2\right\} \frac{z_1(t)}{H(t)} g(x^*, t)$$

$$w(x, 0) = \exp\left\{-\delta \sum_{i=1}^{n} (x_i - x_i^*)^2\right\} \gamma(x), \quad x \in \Omega$$

Where:

$$\xi = \delta \sum_{i=1}^{n} (x_i - x_i^*)^2 + \beta t$$

and

$$\bar{a}_i = a_i + 2\delta \textstyle\sum_{j=1}^{n} (a_{ij} + a_{ji}), \qquad \delta_{ij} = \begin{cases} 1, & if\ i = j, \\ 0, & if\ i \neq j, \end{cases} \tag{2.9}$$

$$\bar{a} = 2\delta\left(\sum_{i=1}^{n} a_i(x_i - x_i^*) + \sum_{i,j=1}^{n} a_{ij}\left(\delta_{ij} + 2\delta(x_i - x_i^*)(x_j - x_j^*)\right)\right) \tag{2.10}$$

Since $x^* \in \Omega$ we see that $d = dist(x^*, \partial\Omega) > 0$. It follows that if we take $\delta > 0$ large enough such that

$$exp\{-\delta n d^2\} \max_{\partial\Omega\times[0,T]} \frac{|z_i(x,t)|}{|H(t)|} \leq \frac{1}{2}$$

We will find that $|w(x,t)|$ cannot reach Its maximum on the boundary $\partial\Omega \times [0,T]$. Assuming at this point that w reaches its positive maximum at (x', t') in the interior. Thus, it follows that:

$$\begin{aligned}\left(\beta - \bar{a}(x',t')\right)w(x',t') &\leq r(t')e^{-\xi}\psi\left(x', t', \frac{e^{\xi}w(x',t')}{r(t')}\right)\\ &\leq K\left(w(x',t') + r(t')e^{-\xi(x',t')}\right)\\ &\leq K\left(w(x',t') + e^{\beta t'}exp\{\delta \textstyle\sum_{i=1}^{n}(x_i' - x_i^*)^2 -\right.\\ &\quad \left.\beta t'\}w(x',t')H^{-1}(t')\right)\\ &\leq (K + exp\{\delta n\, diam(\Omega)^2\}H_*^{-1})w(x',t') \end{aligned} \tag{2.11}$$

Here we have used $r(t) = \phi(x^*, t)H^{-1}(t)$ and $H_* = inf H(t)$. If we take $\beta > 0$ large enough such that:

$$\beta - (K + exp\{\delta n\, diam(\Omega)^2\}H_*^{-1}) - \max_{Q_T}|\bar{a}| \geq 0 \tag{2.12}$$

Clearly w can not attain its maximum positive point in the interior. Likewise, if we consider β to be sufficiently large, then we can prove that w can not reach its minimum negative point in the interior. Hence, w is bounded and, therefore, g and r are also bounded. ■

Lemma 2.2: **(**[17]) There exists $C = C(M) > 0$ and $0 < \alpha < 1$ such that in the Hölder space we have,

$$\|g\|C^{1+\alpha}(\overline{Q_T}) + \|r\|C^{1+\alpha/2}([0,T]) \leq C(M) \tag{2.13}$$

Proof: Take $g = w^1 + w^2$, such that w^1 satisfy

$$w_t^1 = A(x,t)w^1 + r(0)\psi\left(x, 0, \frac{\gamma(x)}{r(0)}\right) \text{ in } Q_T$$

$$w^1(0,t) = r(t)\frac{z_0(t)}{H(t)}, \quad w^1(1,t) = r(t)\frac{z_1(t)}{H(t)}, \quad \text{on } \partial\Omega \times [0,T]$$

$$w^1(x,0) = u_0(x), \quad x \in \Omega \tag{2.14}$$

and w^2 satisfies

$$w_t^2 = A(x,t)w^2 + r(t)\psi\left(x, t, \frac{g}{r}\right) - r(0)\psi\left(x, 0, \frac{\gamma(x)}{r(0)}\right) \quad \text{in } Q_T$$

$$w^2(0,t) = 0, \; w^2(1,t) = 0, \; \text{on } \partial\Omega \times [0,T]$$

$$w^2(x,0) = 0. \; x \in \Omega. \tag{2.15}$$

Where:

$$A(x,t)w = \sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} + \sum_{i=1}^{n} a_i w_{x_i}$$

Therefore, from Lemma 2.1 and Schauder's interior estimations [17] we realize that there exists $0 < \alpha < 1$ so that

$$\|w^2\|C^{2+\alpha}(Q_T) \leq C(M) \tag{2.16}$$

and

$$\|w^2\|C^{1+\alpha}(\overline{Q_T}) \leq Cmax\left|r\psi\left(x, t, \frac{g}{r}\right) - r(0)\psi\left(x, 0, \frac{\gamma(x)}{r(0)}\right)\right| \leq C(M) \tag{2.17}$$

Thus from Eq. (2.16) and Assumption H4 we have

$$\|g(x_*,\cdot)\|C^{1+\alpha/2}([0,T]) \leq C(M) \tag{2.18}$$

Therefore, according to the general theory [17] and Eq. (2.14) we have

$$\|w^1\|C^{1+\alpha/2}(\overline{Q_T}) \leq C(M)$$

Therefore,

$$\|g\|C^{1+\alpha}(\overline{Q_T}) \leq C(M) \blacksquare$$

Then from Equations (2.7a) – (2.7d), (2.13) and [17] we realize that there exists

$C = C(M, r_*) > 0$ such that:

$$\|g\|C^{2+\alpha}(\overline{Q_T}) \leq C(M, r_*), \quad r_* = inf\{|r(t)||t \in [0, T]\} \tag{2.19}$$

2.3 Existence and Uniqueness

To establish the existence and uniqueness of the solution, we intend to employ the technique of retardation of the time variable along with the a priori estimation gotten in the preceding section to show the existence and uniqueness of the solution for the problems (1.1a) – (1.1d).

Theorem 2.1: ([17]) Regarding assumptions H4 and H5, there exists a unique solution pair (g, r) for the problem (2.7a) – (2.7d) which is continuously dependent upon the data.

Proof: Take $\theta > 0$ to be a small parameter, we define g^θ by

$$g_t^\theta = Ag^\theta + \frac{r^\theta}{H}\psi\left(x, t, \frac{g^\theta H(t)}{r^\theta}\right), \text{ in } Q_T$$

$$g^\theta(x, 0) = \gamma(x), \qquad x \in \Omega$$

$$g^\theta(0, t) = \frac{z_0(t)}{H(t)} r^\theta(t),$$

$$g^\theta(1, t) = \frac{z_1(t)}{H(t)} r^\theta(t), \qquad \text{on } \partial\Omega \times [0, T]. \tag{2.20}$$

Where $r^\theta(0) = H(0) > 0$, and

$$r^{\theta}(t) = \begin{cases} \gamma(x_*), & 0 \le t \le \theta \\ g^{\beta}(x_*,\ t-\theta), & 0 \le t \le T \end{cases} \tag{2.21}$$

With respect to the classical theory of parabolic equations, it results that g^{θ} exists in $0 \le t \le \theta$ and $r^{\theta}(t) = g^{\theta}(x_*, t) > 0$ by the strong maximum principle [17]. From (2.21) it can be seen that g^{θ} exists for all $0 \le t \le T$ and $r^{\theta} > 0$. We find that there exists $0 \le \alpha \le 1$ when the a priori estimations (2.13) and (2.19) are applied, such that

$$\|g^{\theta}\| C^{1+\alpha}(\overline{Q_T}) + \|r^{\theta}\| C^{1+\alpha}([0,T]) \le C \tag{2.22}$$

and

$$\|g^{\theta}\| C^{1+\alpha}(\overline{Q_T}) \le C\left(r_*^{\theta}\right), \quad r_*^{\theta} = inf\left\{r^{\theta}(t)\middle| t \in [0,T]\right\} \tag{2.23}$$

From the compactness we have that there exists a $g = g(x,t) \in C^{1+\alpha}(\overline{Q_T})$ and a subsequence of g^{θ}, as well designated by itself, such that

$$g^{\theta} \to g, \qquad r^{\theta} \to g(x_*, t) \qquad \text{as } \theta \to 0 \tag{2.24}$$

Besides, g converges uniformly in $C^{1+\beta}(\overline{Q_T}) \times C^{\beta/2}([0,T])$ for any $0 \le \beta \le \alpha$. Firstly, we have to establish that there exists a $r_* > 0$ such that $r^{\theta}(t) \ge r_* > 0$ for all θ small, so that we can take the limit as $\theta \to 0$ for all $0 \le t \le T$. Since the solution g converges uniformly, its therefore enough to prove that $r(t) > 0$ for all $0 \le t \le T$. Take $t_0 \in (0,T)$ to be expressed by

$$t_0 = inf\{r(t) = 0 | t \in (0,T)\} > 0 \tag{2.25}$$

Hence, we apparently see that the limit $g\big(\text{and } r(t) = g(x_*, t)\big)$ becomes the local solution in $\Omega \times [0, t_0)$ for the problems (2.7a) – (2.7d) if we let $\theta \to 0$. However, with the strong maximum principle [17] along with the assumptions, we get

$$r(t_0) = g(x_*, t_0) > 0 \tag{2.26}$$

Which is a contradiction to (2.25). Therefore, we have been able to prove that the limit function $r(t) > 0$ for all $0 \le t \le T$, is an implication that the lower bound $r_*^{\theta} > 0$ is not dependent on θ. Now taking the limit as $\theta \to 0$ in (2.20) - (2.21) we find that g is a global solution of (2.7a) – (2.7d) in $[0, \mathrm{T}]$.

To show that the solution is unique and continuously depend upon the data, assume g^k to be the solutions with the data γ^k, H^k, z_i^k, ψ^k, $k = 1,2$, $i = 1,2$, it therefore results from Lemma 2.1 and 2.2 that we have $C(M) > 0$ which implies

$$\|g^k\|C^{2,1}(\overline{Q_T}) + \|r^k\|C^{1+\alpha}([0,T]) \leq C(M), \qquad k = 1,2 \tag{2.27}$$

Where $M > 0$ is only dependent on the data. With related arguments to the proof of the a priori estimations of Lemma 2.1 and 2.2, we have that for some positive constant $C = C(M) > 0$, we get

$$\|g^1 - g^2\|C^{1+\alpha}(\overline{Q_T}) + \|r^1 - r^2\|C^{1+\alpha}([0,T])$$
$$\leq C\big(\|\gamma^1 - \gamma^2\|C^{1+\alpha}(\overline{\Omega}) + \|H^1 - H^2\|C^{1+\alpha}([0,T])\big)$$
$$+ \big(\|{z_i}^1 - {z_i}^2\|C^{2+\alpha}(\partial\Omega \times [0,T]) + \|\psi^1 - \psi^2\|L^\infty(\overline{Q_T} \times [-N,N])\big) \tag{2.28}$$

for some $0 < \alpha < 1$, where $N > 0$ is such that $\left|\frac{g}{r}\right| \leq N$ for all $(x,t) \in \overline{Q_T}$ ■

Chapter 3

A LINEARIZED COMPACT FINITE DIFFERENCE SCHEME

3.1 Introduction

In this chapter, we shall construct a compact finite difference scheme proposed by Ye and Sun in [18] for the one dimensional parabolic inverse problems (1.1a) – (1.1d). The compact difference scheme constructed is a linearized high ordered scheme, having a truncation error of order $O(\tau^2 + h^4)$. Hence, the constructed scheme retains a high-level order of precision. In the last section, we shall establish the existence of solution of the constructed scheme.

Given the nonnegative numbers M and N. Let $h = \frac{L}{M}$ designate the step size of the space domain, x, and $\tau = \frac{T}{N}$ designate step size for the time direction, t, whereas $L = 1$ and T is the terminal point in time.

Let's define the mesh points (x_i, t_n) by

$$x_i = ih, \quad i = 0,1,2, \dots, M$$

$$t_n = n\tau, \quad n = 0,1,2, \dots, N$$

Where solutions of the exact and estimate solution at the point (x_i, t_n) is designated by ϕ_i^n and φ_i^n respectively.

3.2 Construction of the Compact Scheme

Constructing the compact finite difference scheme below, basically revolves around the application of Taylors expansion to problems (1.1). The process involves some computation using the Taylors expansion and a few assumptions on the data, as well as applying the notations in Chapter 1.

Consider the equation below

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + c(t)\phi + \psi(x,t) \tag{3.1}$$

Let's take

$$w = \frac{\partial^2 \phi}{\partial x^2} \tag{3.2}$$

Then Eq. (3.1) becomes

$$w = \frac{\partial \phi}{\partial t} - c(t)\phi + \psi(x,t), \tag{3.3}$$

Describe the mesh functions as follows

$$\varphi_i^n = \phi(x_i, t_n), \qquad W_i^n = w(x_i, t_n), \qquad C^n = c(t_n),$$

$$0 \le i \le M, \;\; 0 \le n \le N \tag{3.4}$$

Eq. (3.3) at the point (x_i, t_n), becomes

$$w(x_i, t_n) = \frac{\partial \phi}{\partial t}(x_i, t_n) - c(t_n)\phi(x_i, t_n) - \psi(x_i, t_n),$$

$$0 \le i \le M, \;\; 0 \le n \le N \tag{3.5}$$

Putting Eq. (3.4) into Eq. (3.5) we have

$$W_i^n = \frac{\partial \phi}{\partial t}(x_i, t_n) - C^n \varphi_i^n - \psi_i^n, \qquad 0 \le i \le M, \quad 0 \le i \le N \tag{3.6}$$

Applying Taylors expansion, that is:

$$\phi^{i+1} = \phi^i + h\phi_x^i + \frac{h^2}{2!}\phi_{xx}^i + \frac{h^3}{3!}\phi_{xxx}^i + \frac{h^4}{4!}\phi_{iv}^i + O(h^5) \tag{3.7}$$

$$\phi^{i-1} = \phi^i - h\phi_x^i + \frac{h^2}{2!}\phi_{xx}^i - \frac{h^3}{3!}\phi_{xxx}^i + \frac{h^4}{4!}\phi_{iv}^i + O(h^5) \tag{3.8}$$

$$\phi_x^{i+1} = \phi_x^i + h\phi_{xx}^i + \frac{h^2}{2!}\phi_{xxx}^i + \frac{h^3}{3!}\phi_{iv}^i + \frac{h^4}{4!}\phi_v^i + O(h^5) \quad (3.9)$$

$$\phi_x^{i-1} = \phi_x^i - h\phi_{xx}^i + \frac{h^2}{2!}\phi_{xxx}^i - \frac{h^3}{3!}\phi_{iv}^i + \frac{h^4}{4!}\phi_v^i + O(h^5) \quad (3.10)$$

$$\phi_{xx}^{i+1} = \phi_{xx}^i + h\phi_{xxx}^i + \frac{h^2}{2!}\phi_{iv}^i + \frac{h^3}{3!}\phi_v^i + \frac{h^4}{4!}\phi_{vi}^i + O(h^5) \quad (3.11)$$

$$\phi_{xx}^{i-1} = \phi_{xx}^i - h\phi_{xxx}^i + \frac{h^2}{2!}\phi_{iv}^i - \frac{h^3}{3!}\phi_v^i + \frac{h^4}{4!}\phi_{vi}^i + O(h^5) \quad (3.12)$$

Multiplying Equations (3.7) with r, (3.8) with s, (3.9) with ch and (3.10) with dh

$$r\phi^{i+1} = r\phi^i + rh\phi_x^i + \frac{rh^2}{2!}\phi_{xx}^i + \frac{rh^3}{3!}\phi_{xxx}^i + \frac{rh^4}{4!}\phi_{iv}^i + O(h^5) \quad (3.13)$$

$$s\phi^{i-1} = s\phi^i - sh\phi_x^i + \frac{sh^2}{2!}\phi_{xx}^i - \frac{sh^3}{3!}\phi_{xxx}^i + \frac{sh^4}{4!}\phi_{iv}^i + O(h^5) \quad (3.14)$$

$$ch\phi_x^{i+1} = ch\phi_x^i + ch^2\phi_{xx}^i + \frac{ch^3}{2!}\phi_{xxx}^i + \frac{ch^4}{3!}\phi_{iv}^i + \frac{ch^5}{4!}\phi_v^i + O(h^5) \quad (3.15)$$

$$dh\phi_x^{i-1} = dh\phi_x^i - dh^2\phi_{xx}^i + \frac{dh^3}{2!}\phi_{xxx}^i - \frac{dh^4}{3!}\phi_{iv}^i + \frac{dh^5}{4!}\phi_v^i + O(h^5) \quad (3.16)$$

Add Equations (3.13), (3.14), (3.15) and (3.16)

$$r\phi^{i+1} + s\phi^{i+1} + ch\phi_x^{i+1} + dh\phi_x^{i-1} = (r+s)\phi^i + (r-s+c+d)h\phi_x^i$$

$$+(r+s+2c-2d)\frac{h^2}{2!}\phi_{xx}^i + (r-s+3c+3d)\frac{h^3}{3!}\phi_{xxx}^i$$

$$+(r+s+4c-4d)\frac{h^4}{4!}\phi_{iv}^i + (c+d)\frac{h^5}{5!}\phi_v^i \quad (3.17)$$

Collecting terms, we have

$$r\phi^{i+1} + s\phi^{i+1} = (r+s)\phi^i = 0$$

$$ch\phi_x^{i+1} + dh\phi_x^{i-1} = (r-s+c+d)h\phi_x^i$$

$$c\phi_x^{i+1} - (r-s+c+d)\phi_x^i + d\phi_x^{i-1} = K \quad (3.18)$$

Assume that.

$$(r-s+c+d) = -\frac{4}{6}$$

Then Eq. (3.18) becomes

$$\frac{1}{6}\phi_x^{i+1} + \frac{4}{6}\phi_x^i + \frac{1}{6}\phi_x^{i-1} = \frac{f(x+ih)-f(x-ih)}{2h} \quad (3.19)$$

Similarly, multiplying Eq. (3.11) by eh^2 and Eq. (3.12) by fh^2, we have

$$eh^2\phi_{xx}^{i+1} = eh^2\phi_{xx}^i + eh^3\phi_{xxx}^i + \frac{eh^4}{2!}\phi_{iv}^i + \frac{eh^5}{3!}\phi_v^i + \frac{eh^6}{4!}\phi_{vi}^i + O(h^5) \quad (3.20)$$

$$fh^2\phi_{xx}^{i-1} = fh^2\phi_{xx}^i - fh^3\phi_{xxx}^i + \frac{fh^4}{2!}\phi_{iv}^i - \frac{fh^5}{3!}\phi_v^i + \frac{fh^6}{4!}\phi_{vi}^i + O(h^5) \quad (3.21)$$

Adding Equations (3.17), (3.20) and (3.21)

$$r\phi^{i+1} + s\phi^{i+1} + ch\phi_x^{i+1} + dh\phi_x^{i-1} + eh^2\phi_{xx}^{i+1} + fh^2\phi_{xx}^{i-1} = (r+s)\phi^i$$

$$+(r-s+c+d)h\phi_x^i + (r+s+2c-2d+2e+2f)\frac{h^2}{2!}\phi_{xx}^i + \cdots \quad (3.22)$$

Collecting terms

$$r\phi^{i+1} + s\phi^{i+1} = (r+s)\phi^i = 0 \quad (3.23)$$

$$ch\phi_x^{i+1} + dh\phi_x^{i-1} = (r-s+c+d)h\phi_x^i \quad (3.24)$$

Equations (3.23) and (3.24) has been considered before

$$eh^2\phi_{xx}^{i+1} + fh^2\phi_{xx}^{i-1} = \frac{(r+s+2c-2d+2e+2f)}{2}h^2\phi_{xx}^i$$

$$e\phi_{xx}^{i+1} - \frac{(r+s+2c-2d+2e+2f)}{2}\phi_{xx}^i + f\phi_{xx}^{i-1} = L \quad (3.25)$$

Assume that.

$$\frac{(r+s+2c-2d+2e+2f)}{2} = -\frac{10}{12}$$

Equation (3.25) becomes

$$\frac{1}{12}\phi_{xx}^{i+1} + \frac{10}{12}\phi_{xx}^i + \frac{1}{12}\phi_{xx}^{i-1} = L$$

$$\frac{1}{12}\left(\phi_{xx}^{i+1} + 10\phi_{xx}^i + \phi_{xx}^{i-1}\right) = \frac{f(x+ih) - 2f'(x) + f(x-ih)}{h^2} = \delta_x^2\phi_i^n$$

Therefore, the above equation can be written as

$$\delta_x^2\varphi_i^n = \frac{1}{12}\left(W_{xx}^{i-1} + 10W_{xx}^i + W_{xx}^{i+1}\right) + O(h^4) \quad (3.26)$$

Thus, applying the result of the Taylors expansion Eq. (3.26) to Eq. (3.6) we have

$$\delta_x^2\varphi_i^n = \frac{1}{12}\left[\left(\frac{\partial\phi}{\partial t}(x_{i-1},t_n) - C^n\varphi_{i-1}^n - \psi_{i-1}^n\right) + 10\left(\frac{\partial\phi}{\partial t}(x_i,t_n) - C^n\varphi_i^n - \psi_i^n\right) + \left(\frac{\partial\phi}{\partial t}(x_{i+1},t_n) - C^n\varphi_{i+1}^n - \psi_{i+1}^n\right)\right] \quad (3.27)$$

$$\delta_x^2\varphi_i^n = \frac{1}{12}\left[\frac{\partial\phi}{\partial t}(x_{i-1},t_n) + 10\frac{\partial\phi}{\partial t}(x_i,t_n) + \frac{\partial\phi}{\partial t}(x_{i+1},t_n) - (C^n\varphi_{i-1}^n + 10C^n\varphi_i^n + C^n\varphi_{i+1}^n) - (\psi_{i-1}^n + 10\psi_i^n + \psi_{i+1}^n)\right] \quad (3.28)$$

Thus, this becomes

$$\frac{1}{12}\left(\frac{\partial\phi}{\partial t}(x_{i-1},t_n) + 10\frac{\partial\phi}{\partial t}(x_i,t_n) + \frac{\partial\phi}{\partial t}(x_{i+1},t_n)\right) = \delta_x^2\varphi_i^n + \frac{1}{12}C^n(\varphi_{i-1}^n + 10\varphi_i^n + \varphi_{i+1}^n) + \frac{1}{12}(\psi_{i-1}^n + 10\psi_i^n + \psi_{i+1}^n), \quad 1 \le i \le M-1,\ 1 \le n \le N-1 \quad (3.29)$$

Using Taylors expansion again

$$\frac{1}{12}(D_{\hat{t}}\varphi_{i-1}^n + 10D_{\hat{t}}\varphi_i^n + D_{\hat{t}}\varphi_{i+1}^n) = \delta_x^2\varphi_i^{\bar{n}} + \frac{1}{12}C^n\left(\varphi_{i-1}^{\bar{n}} + 10\varphi_i^{\bar{n}} + \varphi_{i+1}^{\bar{n}}\right) + \frac{1}{12}(\psi_{i-1}^n + 10\psi_i^n + \psi_{i+1}^n) + (e_1)_i^n, \quad 1 \le i \le M-1, \quad 1 \le n \le N-1, \quad (3.30)$$

Where c_1 is such that

$$|(e_1)_i^n| \le c_1(\tau^2 + h^4), \qquad 1 \le i \le M-1, \quad 1 \le n \le N-1$$

From Eq. (3.1) we have that

$$c(t)\phi = \frac{\partial\phi}{\partial t} - \frac{\partial^2\phi}{\partial x^2} - \psi(x,t)$$

$$c(t) = \frac{1}{\phi}\left[\frac{\partial\phi}{\partial t} - \frac{\partial^2\phi}{\partial x^2} - \psi(x,t)\right] \quad (3.31)$$

Where the overspecification is given as $\phi(x^*,t) = H(t) = H^n$, $\frac{\partial\phi}{\partial t} = (H')^n$.

Suppose that there is an integer k_0 such that $x^* = x_{k_0}$, where $C^n = c(t_n)$. Eq. (3.31) becomes

$$\mathrm{c}(t_n) = \frac{1}{H^n}\left[(H')^n - \frac{\partial^2\phi}{\partial x^2}\left(x_{k_0},t_n\right) - \psi_{k_0}^n\right], \qquad 1 \le n \le N, \quad (3.32)$$

Now we derive the differential formula for, $\frac{\partial^2 \phi}{\partial x^2}\left(x_{k_0}, t_n\right)$:

Consider the Taylors expansions below:

$$\phi(i+h) = \phi_i + h\phi_i' + \frac{h^2}{2!}\phi_i'' + \frac{h^3}{3!}\phi_i''' + \frac{h^4}{4!}\phi_i'^{v} + \frac{h^5}{5!}\phi_i^{v} + \frac{h^6}{6!}\phi_i^{v'} + \cdots \quad (3.33)$$

$$\phi(i-h) = \phi_i - h\phi_i' + \frac{h^2}{2!}\phi_i'' - \frac{h^3}{3!}\phi_i''' + \frac{h^4}{4!}\phi_i'^{v} - \frac{h^5}{5!}\phi_i^{v} + \frac{h^6}{6!}\phi_i^{v'} + \cdots \quad (3.34)$$

Adding Equations (3.33) and (3.34)

$$\phi(i+h) + \phi(i-h) = 2\phi_i + h^2\phi_i'' + \frac{h^4}{12}\phi_i'^{v} + \frac{h^6}{360}\phi_i^{v'} + \cdots \quad (3.35)$$

Also consider

$$\phi(i+2h) = \phi_i + 2h\phi_i' + \frac{4h^2}{2!}\phi_i'' + \frac{8h^3}{3!}\phi_i''' + \frac{16h^4}{4!}\phi_i'^{v} + \frac{32h^5}{5!}\phi_i^{v} + \frac{64h^6}{6!}\phi_i^{v'} + \cdots \quad (3.36)$$

$$\phi(i-2h) = \phi_i - 2h\phi_i' + \frac{4h^2}{2!}\phi_i'' - \frac{8h^3}{3!}\phi_i''' + \frac{16h^4}{4!}\phi_i'^{v} - \frac{32h^5}{5!}\phi_i^{v} + \frac{64h^6}{6!}\phi_i^{v'} + \cdots \quad (3.37)$$

Adding Equations (3.36) and (3.37)

$$\phi(i+2h) + \phi(i-2h) = 2\phi_i + 4h^2\phi_i'' + \frac{4h^4}{3}\phi_i'^{v} + \frac{8h^6}{45}\phi_i^{v'} + \cdots \quad (3.38)$$

Multiply Eq. (3.35) by 16

$$16\phi(i+h) + 16\phi(i-h) = 32\phi_i + 16h^2\phi_i'' + \frac{16h^4}{12}\phi_i'^{v} + \frac{16h^6}{360}\phi_i^{v'} + \cdots \quad (3.39)$$

Subtract Eq. (3.38) from (3.39)

$$-\phi(i-2h) + 16\phi(i-h) + 16\phi(i+h) - \phi(i+2h) = 30\phi_i + 12h^2\phi_i'' + O(h^6) \quad (3.40)$$

$$12h^2\phi_i'' = -\phi(i-2h) + 16\phi(i-h) - 30\phi_i + 16\phi(i+h) - \phi(i+2h) + O(h^6)$$

$$\phi_i'' = \frac{1}{12h^2}\big(-\phi(i-2h) + 16\phi(i-h) - 30\phi_i + 16\phi(i+h) - \phi(i+2h)\big) + O(h^4)$$

Therefore, we can write

$$\frac{\partial^2\phi}{\partial x^2}\left(x_{k_0},t_n\right)=\frac{1}{12h^2}\left(-\phi_{k_0-2}^n+16\phi_{k_0-1}^n-30\phi_{k_0}^n+16\phi_{k_0+1}^n-\phi_{k_0+2}^n\right)+O(h^4) \tag{3.41}$$

Putting Eq. (3.41) in (3.32), we obtain

$$C^n=\frac{1}{H^n}\left[(H')^n-\frac{1}{12h^2}\left(-\phi_{k_0-2}^n+16\phi_{k_0-1}^n-30\phi_{k_0}^n+16\phi_{k_0+1}^n-\phi_{k_0+2}^n\right)-\psi_{k_0}^n\right]+(e_2)^n,\ 1\le n\le N \tag{3.42}$$

Where c_2 is such that

$|(e_2)^n|\le c_2h^4, \qquad 1\le n\le N$

Once more we consider the Taylors expansion

$$\phi_i^1=\phi_i^0+\tau\frac{\partial\phi}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2\phi}{\partial t^2}+\cdots$$

Recall that;

$$\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial x^2}+c(t)\phi+\psi(x,t)$$

$$\phi_i^1=\phi_i^0+\tau\left(\frac{\partial^2\phi}{\partial x^2}+c(t)\phi+\psi(x,t)\right)+O(\tau^2)$$

From the initial condition, we have that $\phi_i^0=\gamma(x_i)$ and $t=0$.

$$\phi_i^1=\gamma(x_i)+\tau[\gamma''(x_i)+C^0\gamma(x_i)+\psi_i^0]+(e_3),\quad 1\le i\le M-1 \tag{3.43}$$

Where c_3 is such that

$|(e_3)^n|\le c_3\tau^2, \qquad 1\le i\le M-1$

From Eq. (3.32), for $n=0$, we can write

$$C^0=\frac{1}{\gamma(x^*)}\left[(H')^0-\gamma''(x^*)-\psi_{k_0}^0\right], \tag{3.44}$$

Neglecting error terms in Equations (3.30), (3.42) and (3.43), we formulate for problems (1.1a) – (1.1d), the compact difference scheme below:

$$\frac{1}{12}\left(D_{\hat{t}}\phi_{i-1}^{n} + 10D_{\hat{t}}\phi_{i}^{n} + D_{\hat{t}}\phi_{i+1}^{n}\right) = \delta_x^2\phi_i^{\bar{n}} + \frac{1}{12}C^n\left(\phi_{i-1}^{\bar{n}} + 10\phi_i^{\bar{n}} + \phi_{i+1}^{\bar{n}}\right) + \frac{1}{12}\left(\psi_{i-1}^{n} + 10\psi_i^{n} + \psi_{i+1}^{n}\right), \quad 1 \le i \le M-1, \quad 1 \le n \le N-1 \qquad (3.45)$$

$$C^n = \frac{1}{H^n}\left[(H')^n - \frac{1}{12h^2}\left(-\phi_{k_0-2}^{n} + 16\phi_{k_0-1}^{n} - 30\phi_{k_0}^{n} + 16\phi_{k_0+1}^{n} - \phi_{k_0+2}^{n}\right) - \psi_{k_0}^{n}\right] \quad 1 \le n \le N \qquad (3.46)$$

$$C^0 = \frac{1}{\gamma(x^*)}\left[(H')^0 - \gamma''(x^*) - \psi_{k_0}^{0}\right], \qquad (3.47)$$

$$\phi_i^1 = \gamma(x_i) + \tau[\gamma''(x_i) + C^0\gamma(x_i) + \psi_i^0] \quad 1 \le i \le M-1 \qquad (3.48)$$

$$\phi_i^0 = \gamma(x_i), \quad 1 \le i \le M-1, \qquad (3.49)$$

$$\phi_0^n = z_0(t_n),\ \phi_M^n = z_1(t_n),\ 0 \le n \le N \qquad (3.50)$$

Above difference scheme, Equations (3.45) – (3.50) is a three-level linearized scheme. $\{\phi_i^0,\ \phi_i^1 | 0 \le i \le M\} \cup \{c^0, c^1\}$ can be easily obtained from (3.46) – (3.50).

If $\{\phi_i^{n-1},\ \phi_i^n | 0 \le i \le M\} \cup \{c^n\}$ has been obtained, then from (3.45) and (3.50) we get:

$$\frac{1}{12}\left(D_{\hat{t}}\phi_{i-1}^{n} + 10D_{\hat{t}}\phi_{i}^{n} + D_{\hat{t}}\phi_{i+1}^{n}\right) = \delta_x^2\phi_i^{\bar{n}} + \frac{1}{12}C^n\left(\phi_{i-1}^{\bar{n}} + 10\phi_i^{\bar{n}} + \phi_{i+1}^{\bar{n}}\right) + \frac{1}{12}\left(\psi_{i-1}^{n} + 10\psi_i^{n} + \psi_{i+1}^{n}\right), \quad 1 \le i \le M-1, \qquad (3.51)$$

$$\phi_0^{n+1} = z_0(t_{n+1}), \qquad \phi_M^{n+1} = z_1(t_{n+1}), \qquad (3.52)$$

So, we need to solve the tridiagonal system above to obtain $\{\phi_i^{n+1} | 0 \le i \le M\}$. Afterwards c^{n+1} is obtained from Eq. (3.46).

3.3 Local Truncation Error and Consistency

From equation (3.45), we have the equation below

$$\frac{1}{12}\left(D_{\hat{t}}\phi_{i-1}^{n} + 10D_{\hat{t}}\phi_{i}^{n} + D_{\hat{t}}\phi_{i+1}^{n}\right) = \delta_x^2\phi_i^{\bar{n}} + \frac{1}{12}C^n\left(\phi_{i-1}^{\bar{n}} + 10\phi_i^{\bar{n}} + \phi_{i+1}^{\bar{n}}\right) + \frac{1}{12}\left(\psi_{i-1}^{n} + 10\psi_i^{n} + \psi_{i+1}^{n}\right), \quad 1 \le i \le M-1, \quad 1 \le n \le N-1 \qquad (3.53)$$

We have the following notations below

$$D_{\hat{t}}\phi_i^n = \frac{\phi_i^{n+1}-\phi_i^{n-1}}{2\tau}, \qquad \phi_i^{\overline{n}} = \frac{\phi_i^{n+1}+\phi_i^{n-1}}{2}, \qquad \delta_x^2\phi_i^n = \frac{\phi_{i-1}^n-2\phi_i^n+\phi_{i+1}^n}{h^2}$$

Applying the above notations to Eq. (3.53) above, we get

$$\phi_{i-1}^{n+1}\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right)+\phi_i^{n+1}\left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right)+\phi_{i+1}^{n+1}\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right)=$$
$$\phi_{i-1}^{n-1}\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)+\phi_i^{n-1}\left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right)+\phi_{i+1}^{n-1}\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)+$$
$$\frac{1}{12}(\psi_{i-1}^n+10\psi_i^n+\psi_{i+1}^n), \quad 1\le i\le M-1 \tag{3.54}$$

Let's take

$$A=\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right), \qquad B=\left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right), \qquad AA=\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right),$$

$$BB=\left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right), \qquad D=\frac{1}{12}(\psi_{i-1}^n+10\psi_i^n+\psi_{i+1}^n),$$

Factorizing, Eq. (3.54) becomes

$$A(\phi_{i-1}^{n+1}+\phi_{i+1}^{n+1})+B\phi_i^{n+1}=AA(\phi_{i-1}^{n-1}+\phi_{i+1}^{n-1})+BB\phi_i^{n-1}+D \tag{3.55}$$

We have the following Taylors expansion

$$\phi_{i-1}^{n+1}=\phi_{i,n}-h\phi_x+\tau\phi_t+\frac{h^2}{2}\phi_{xx}-hk\phi_{xt}+\frac{\tau^2}{2}\phi_{tt}-\frac{h^3}{6}\phi_{xxx}+\frac{h^2\tau}{2}\phi_{xxt}$$
$$-\frac{h\tau^2}{2}\phi_{xtt}+\frac{\tau^3}{6}\phi_{ttt}+\frac{h^4}{24}\phi_{xxxx}-\frac{h^3\tau}{6}\phi_{xxxt}+\frac{h^2\tau^2}{4}\phi_{xxtt}$$
$$-\frac{h\tau^3}{6}\phi_{xttt}+\frac{\tau^4}{24}\phi_{tttt}+\cdots$$

$$\phi_{i+1}^{n+1}=\phi_{i,n}+h\phi_x+\tau\phi_t+\frac{h^2}{2}\phi_{xx}+h\tau\phi_{xt}+\frac{\tau^2}{2}\phi_{tt}+\frac{h^3}{6}\phi_{xxx}+\frac{h^2\tau}{2}\phi_{xxt}$$
$$+\frac{h\tau^2}{2}\phi_{xtt}+\frac{\tau^3}{6}\phi_{ttt}+\frac{h^4}{24}\phi_{xxxx}+\frac{h^3\tau}{6}\phi_{xxxt}+\frac{h^2\tau^2}{4}\phi_{xxtt}$$
$$+\frac{h\tau^3}{6}\phi_{xttt}+\frac{\tau^4}{24}\phi_{tttt}+\cdots$$

$$\phi_{i-1}^{n-1} = \phi_{i,n} - h\phi_x - \tau\phi_t + \frac{h^2}{2}\phi_{xx} + h\tau\phi_{xt} + \frac{\tau^2}{2}\phi_{tt} - \frac{h^3}{6}\phi_{xxx} - \frac{h^2\tau}{2}\phi_{xxt}$$

$$- \frac{h\tau^2}{2}\phi_{xtt} - \frac{\tau^3}{6}\phi_{ttt} + \frac{h^4}{24}\phi_{xxxx} + \frac{h^3\tau}{6}\phi_{xxxt} + \frac{h^2\tau^2}{4}\phi_{xxtt}$$

$$+ \frac{h\tau^3}{6}\phi_{xttt} + \frac{\tau^4}{24}\phi_{tttt} + \cdots$$

$$\phi_{i+1}^{n-1} = \phi_{i,n} + h\phi_x - \tau\phi_t + \frac{h^2}{2}\phi_{xx} - h\tau\phi_{xt} + \frac{\tau^2}{2}\phi_{tt} + \frac{h^3}{6}\phi_{xxx} - \frac{h^2\tau}{2}\phi_{xxt}$$

$$+ \frac{h\tau^2}{2}\phi_{xtt} - \frac{\tau^3}{6}\phi_{ttt} + \frac{h^4}{24}\phi_{xxxx} - \frac{h^3\tau}{6}\phi_{xxxt} + \frac{h^2\tau^2}{4}\phi_{xxtt}$$

$$- \frac{h\tau^3}{6}\phi_{xttt} + \frac{\tau^4}{24}\phi_{tttt} + \cdots$$

$$\phi_i^{n+1} = \phi_{i,n} + \tau\phi_t + \frac{\tau^2}{2}\phi_{tt} + \frac{\tau^3}{6}\phi_{ttt} + \frac{\tau^4}{24}\phi_{tttt} + \cdots$$

$$\phi_i^{n-1} = \phi_{i,n} - \tau\phi_t + \frac{\tau^2}{2}\phi_{tt} - \frac{\tau^3}{6}\phi_{ttt} + \frac{\tau^4}{24}\phi_{tttt} + \cdots$$

Equation (3.55) becomes

$$T_{i,n} = A(\phi_{i-1}^{n+1} + \phi_{i+1}^{n+1}) + B\phi_i^{n+1} - AA(\phi_{i-1}^{n-1} + \phi_{i+1}^{n-1}) - BB\phi_i^{n-1} - D \quad (3.56)$$

Substituting the Taylors expansion above into Eq. (3.56) we get:

$$T_{i,n} = A\Big[\phi_{i,n} - h\phi_x + \tau\phi_t + \tfrac{h^2}{2}\phi_{xx} - h\tau\phi_{xt} + \tfrac{\tau^2}{2}\phi_{tt} - \tfrac{h^3}{6}\phi_{xxx} + \tfrac{h^2\tau}{2}\phi_{xxt} -$$

$$\tfrac{h\tau^2}{2}\phi_{xtt} + \tfrac{\tau^3}{6}\phi_{ttt} + \tfrac{h^4}{24}\phi_{xxxx} - \tfrac{h^3\tau}{6}\phi_{xxxt} + \tfrac{h^2\tau^2}{4}\phi_{xxtt} - \tfrac{h\tau^3}{6}\phi_{xttt} + \tfrac{\tau^4}{24}\phi_{tttt} + \phi_{i,n} +$$

$$h\phi_x + \tau\phi_t + \tfrac{h^2}{2}\phi_{xx} + h\tau\phi_{xt} + \tfrac{\tau^2}{2}\phi_{tt} + \tfrac{h^3}{6}\phi_{xxx} + \tfrac{h^2\tau}{2}\phi_{xxt} + \tfrac{h\tau^2}{2}\phi_{xtt} + \tfrac{\tau^3}{6}\phi_{ttt} +$$

$$\tfrac{h^4}{24}\phi_{xxxx} + \tfrac{h^3\tau}{6}\phi_{xxxt} + \tfrac{h^2\tau^2}{4}\phi_{xxtt} + \tfrac{h\tau^3}{6}\phi_{xttt} + \tfrac{\tau^4}{24}\phi_{tttt}\Big] + B\Big(\phi_{i,n} + \tau\phi_t + \tfrac{\tau^2}{2}\phi_{tt} +$$

$$\tfrac{\tau^3}{6}\phi_{ttt} + \tfrac{\tau^4}{24}\phi_{tttt} + \cdots\Big) - AA\Big[\phi_{i,n} - h\phi_x - \tau\phi_t + \tfrac{h^2}{2}\phi_{xx} + h\tau\phi_{xt} + \tfrac{\tau^2}{2}\phi_{tt} -$$

$$\tfrac{h^3}{6}\phi_{xxx} - \tfrac{h^2\tau}{2}\phi_{xxt} - \tfrac{h\tau^2}{2}\phi_{xtt} - \tfrac{\tau^3}{6}\phi_{ttt} + \tfrac{h^4}{24}\phi_{xxxx} + \tfrac{h^3\tau}{6}\phi_{xxxt} + \tfrac{h^2\tau^2}{4}\phi_{xxtt} +$$

$$\tfrac{h\tau^3}{6}\phi_{xttt} + \tfrac{\tau^4}{24}\phi_{tttt} + \phi_{i,n} + h\phi_x - \tau\phi_t + \tfrac{h^2}{2}\phi_{xx} - h\tau\phi_{xt} + \tfrac{\tau^2}{2}\phi_{tt} + \tfrac{h^3}{6}\phi_{xxx} -$$

$\frac{h^2\tau}{2}\phi_{xxt} + \frac{h\tau^2}{2}\phi_{xtt} - \frac{\tau^3}{6}\phi_{ttt} + \frac{h^4}{24}\phi_{xxxx} - \frac{h^3\tau}{6}\phi_{xxxt} + \frac{h^2\tau^2}{4}\phi_{xxtt} - \frac{h\tau^3}{6}\phi_{xttt} +$

$$\frac{\tau^4}{24}\phi_{tttt}\Big] - BB\left(\phi_{i,n} - \tau\phi_t + \frac{\tau^2}{2}\phi_{tt} - \frac{\tau^3}{6}\phi_{ttt} + \frac{\tau^4}{24}\phi_{tttt} + \cdots\right) - D \qquad (3.57)$$

Cancelling out like terms we have:

$$T_{i,n} = (2A + B - 2AA - BB)\phi_{i,n} + (2A + B + 2AA + BB)\tau\phi_t + (A - AA)h^2\phi_{xx} + (2A + B - 2AA - BB)\frac{\tau^2}{2}\phi_{tt} + (A + AA)h^2\tau\phi_{xxt} + (2A + B + 2AA + BB)\frac{\tau^3}{6}\phi_{ttt} + (A - AA)\frac{h^4}{12}\phi_{xxxx} + (A - AA)\frac{h^2\tau^2}{12}\phi_{xxtt} + (2A + B - 2AA - BB)\frac{\tau^4}{24}\phi_{tttt} - D \qquad (3.58)$$

But we have that

$$(2A + B - 2AA - BB) = \left[2\left(\frac{1}{24\tau} - \frac{1}{2h^2} - \frac{C^n}{24}\right) + \left(\frac{5}{12\tau} + \frac{1}{h^2} - \frac{5C^n}{12}\right) - 2\left(\frac{1}{24\tau} + \frac{1}{2h^2} + \frac{C^n}{24}\right) - \left(\frac{5}{12\tau} + \frac{1}{h^2} - \frac{5C^n}{12}\right)\right] = -C^n$$

$$(2A + B + 2AA + BB) = \left[2\left(\frac{1}{24\tau} - \frac{1}{2h^2} - \frac{C^n}{24}\right) + \left(\frac{5}{12\tau} + \frac{1}{h^2} - \frac{5C^n}{12}\right) + 2\left(\frac{1}{24\tau} + \frac{1}{2h^2} + \frac{C^n}{24}\right) + \left(\frac{5}{12\tau} + \frac{1}{h^2} - \frac{5C^n}{12}\right)\right] = \frac{1}{\tau}$$

$$(A - AA) = \left[\left(\frac{1}{24\tau} - \frac{1}{2h^2} - \frac{C^n}{24}\right) - \left(\frac{1}{24\tau} + \frac{1}{2h^2} + \frac{C^n}{24}\right)\right] = \left(-\frac{1}{h^2} - \frac{C^n}{12}\right)$$

$$(A + AA) = \left[\left(\frac{1}{24\tau} - \frac{1}{2h^2} - \frac{C^n}{24}\right) + \left(\frac{1}{24\tau} + \frac{1}{2h^2} + \frac{C^n}{24}\right)\right] = \frac{1}{12\tau}$$

Putting the above results into Eq. (3.60) we get

$$T_{i,n} = (-C^n)\phi_{i,n} + \left(\frac{1}{\tau}\right)\tau\phi_t + \left(-\frac{1}{h^2} - \frac{C^n}{12}\right)h^2\phi_{xx} + (-C^n)\frac{\tau^2}{2}\phi_{tt} + \left(\frac{1}{12\tau}\right)h^2\tau\phi_{xxt} + \left(\frac{1}{\tau}\right)\frac{\tau^3}{6}\phi_{ttt} + \left(-\frac{1}{h^2} - \frac{p^n}{12}\right)\frac{h^4}{12}\phi_{xxxx} + \left(-\frac{1}{h^2} - \frac{p^n}{12}\right)\frac{h^2\tau^2}{12}\phi_{xxtt} + (-p^n)\frac{\tau^4}{24}\phi_{tttt} - D$$

$$T_{i,n} = -C^n\phi_{i,n} + \phi_t - \phi_{xx} - \frac{C^nh^2}{12}\phi_{xx} - \frac{C^n\tau^2}{2}\phi_{tt} + \frac{h^2}{12}\phi_{xxt} + \frac{\tau^2}{6}\phi_{ttt} - \frac{h^2}{12}\phi_{xxxx} - \frac{C^nh^4}{144}\phi_{xxxx} - \frac{\tau^2}{2}\phi_{xxtt} - \frac{C^nh^2\tau^2}{24}\phi_{xxtt} - \frac{C^n\tau^4}{24}\phi_{tttt} - D$$

$$T_{i,n} = \left(\phi_t - \phi_{xx} - C^n\phi_{i,n} - D\right) - \frac{C^n h^2}{12}\phi_{xx} - \frac{C^n\tau^2}{2}\phi_{tt} + \frac{h^2}{12}\phi_{xxt} + \frac{\tau^2}{6}\phi_{ttt} - \frac{h^2}{12}\phi_{xxxx} - \frac{C^n h^4}{144}\phi_{xxxx} - \frac{\tau^2}{2}\phi_{xxtt} - \frac{C^n h^2\tau^2}{24}\phi_{xxtt} - \frac{C^n\tau^4}{24}\phi_{tttt} \tag{3.59}$$

Where from Eq. (3.59), we have that

$$\left(\phi_t - \phi_{xx} - C^n\phi_{i,n} - D\right) = 0$$

$$T_{i,n} = -\frac{C^n h^2}{12}\phi_{xx} - \frac{C^n\tau^2}{2}\phi_{tt} + \frac{h^2}{12}\phi_{xxt} + \frac{\tau^2}{6}\phi_{ttt} - \frac{h^2}{12}\phi_{xxxx} - \frac{C^n h^4}{144}\phi_{xxxx} - \frac{\tau^2}{2}\phi_{xxtt} - \frac{C^n h^2\tau^2}{24}\phi_{xxtt} - \frac{C^n\tau^4}{24}\phi_{tttt} \tag{3.60}$$

From the main problem we have that

$$\phi_{xx} = \phi_t - C^n\phi - \psi \tag{3.61}$$

Taking the fourth derivative of Eq. (3.61) w.r.t x, we have

$$\phi_{xxxx} = \phi_{xxt} - C^n\phi_{xx} - \psi_{xx} \tag{3.62}$$

Putting Eq. (3.62) into Eq. (3.60) we have

$$T_{i,n} = -\frac{C^n h^2}{12}\phi_{xx} - \frac{C^n\tau^2}{2}\phi_{tt} + \frac{h^2}{12}\phi_{xxt} + \frac{\tau^2}{6}\phi_{ttt} - \frac{h^2}{12}(\phi_{xxt} - C^n\phi_{xx} - \psi_{xx}) - \frac{C^n h^4}{144}\phi_{xxxx} - \frac{\tau^2}{2}\phi_{xxtt} - \frac{C^n h^2\tau^2}{24}\phi_{xxtt} - \frac{C^n\tau^4}{24}\phi_{tttt}$$

$$T_{i,n} = -\frac{C^n h^2}{12}\phi_{xx} - \frac{C^n\tau^2}{2}\phi_{tt} + \frac{h^2}{12}\phi_{xxt} + \frac{\tau^2}{6}\phi_{ttt} - \frac{h^2}{12}\phi_{xxt} + \frac{C^n h^2}{12}\phi_{xx} + \frac{h^2}{12}\psi_{xx} - \frac{C^n h^4}{144}\phi_{xxxx} - \frac{\tau^2}{2}\phi_{xxtt} - \frac{C^n h^2\tau^2}{24}\phi_{xxtt} - \frac{C^n\tau^4}{24}\phi_{tttt}$$

Cancelling out like terms we have

$$T_{i,n} = -\frac{C^n\tau^2}{2}\phi_{tt} + \frac{\tau^2}{6}\phi_{ttt} - \frac{C^n h^4}{144}\phi_{xxxx} - \frac{\tau^2}{2}\phi_{xxtt} - \frac{C^n h^2\tau^2}{24}\phi_{xxtt} - \frac{C^n\tau^4}{24}\phi_{tttt} + \frac{h^2}{12}\psi_{xx} \tag{3.63}$$

Therefore, the principal part of the local truncation error is

$$\left(-\frac{C^n\tau^2}{2}\phi_{tt} - \frac{C^n h^4}{144}\phi_{xxxx}\right)$$

Hence,

$T_{i,n} = O(\tau^2) + O(h^4)$

From Eq. (3.63), we see that

As $\quad \tau, h \longrightarrow 0 \quad$ then $\quad T_{i,n} \longrightarrow 0$

Therefore, the scheme is CONSISTENT.

3.4 Existence of Solution of the Compact Scheme

In this section, we shall employ the homogeneous system of the tridiagonal system above, together with the notations in Chapter 1, to establish the existence of solution of the compact difference scheme constructed.

Lemma 1: ([18]) Let $z = \{z_i | 0 \leq i \leq M\}$ be a mesh function on $\Omega_h = \{x_i | x_i = ih, \ 0 \leq i \leq M, \ Mh = 1\}$ and satisfy $z_0 = 0, \ \ z_M = 0,$ then

$$h \sum_{i=1}^{M-1} z_i^2 \leq \frac{1}{6} h \sum_{i=1}^{M} \left(\frac{z_i - z_{i-1}}{h} \right)^2$$

Theorem 1: ([18]) Supposing $\{\phi_i^{n-1}, \ \phi_i^n | 0 \leq i \leq M\} \cup \{c^n\}$ has been obtained. Then, $\{\phi_i^{n+1} | 0 \leq i \leq M\} \cup \{c^{n+1}\}$ can be uniquely determined from the difference scheme (3.45) – (3.50) when $c^n \leq 6$, or $c^n > 6$ and $\tau < {}^{1}\!/_{(c^n - 6)}$

Proof: Given the homogeneous system of (3.51) and (3.52)

$$\frac{1}{12\tau}(\phi_{i-1}^{n+1} + 10\phi_i^{n+1} + \phi_{i+1}^{n+1}) = \frac{1}{2}\delta_x^2 \phi_i^{n+1} + \frac{1}{24} c^n (\phi_{i-1}^{n+1} + 10\phi_i^{n+1} + \phi_{i+1}^{n+1}),$$

$$1 \leq i \leq M - 1, \qquad (3.64)$$

$$\phi_0^{n+1} = 0, \qquad \phi_M^{n+1} = 0, \qquad (3.65)$$

If we Multiply both sides of Eq. (3.64) by $2h\phi_i^{n+1}$ and sum up for i from 1 to $M - 1$, we obtain

$$\frac{1}{12\tau}h\sum_{i=1}^{M-1}(\phi_{i-1}^{n+1}+10\phi_i^{n+1}+\phi_{i+1}^{n+1})\,\phi_i^{n+1}=h\sum_{i=1}^{M-1}(\delta_x^2\phi_i^{n+1})\phi_i^{n+1}$$

$$+\frac{1}{12}c^n h\sum_{i=1}^{M-1}(\phi_{i-1}^{n+1}+10\phi_i^{n+1}+\phi_{i+1}^{n+1})\phi_i^{n+1}$$

Applying the boundary condition of Eq. (3.65), we get

$$\frac{1}{12\tau}h\sum_{i=1}^{M-1}(\phi_{i-1}^{n+1}+10\phi_i^{n+1}+\phi_{i+1}^{n+1})\,\phi_i^{n+1}=-h\sum_{i=1}^{M-1}\left(\delta_x\phi_{i-\frac{1}{2}}^{n+1}\right)^2$$

$$+\frac{1}{12}c^n h\sum_{i=1}^{M-1}(\phi_{i-1}^{n+1}+10\phi_i^{n+1}+\phi_{i+1}^{n+1})\phi_i^{n+1}$$

Now applying lemma1 to the above equation and simplifying the terms we get

$$\frac{1}{12}\left(\frac{1}{\tau}-c^n\right)h\sum_{i=1}^{M-1}(\phi_{i-1}^{n+1}+10\phi_i^{n+1}+\phi_{i+1}^{n+1})\,\phi_i^{n+1}+6\|\phi^{n+1}\|^2\le 0$$

Where:

$$\|\phi^2\|=\left[h\sum_{i=1}^{M-1}(\phi_i^n)\right]^{1/2}$$

So, we can write:

$$\frac{2}{3}\|\phi^{n+1}\|^2\le\frac{1}{12}h\sum_{i=1}^{M-1}(\phi_{i-1}^{n+1}+10\phi_i^{n+1}+\phi_{i+1}^{n+1})\,\phi_i^{n+1}\le\|\phi^{n+1}\|^2$$

When $\frac{1}{\tau}-p^n\ge 0$, we have

$$\frac{2}{3}\left(\frac{1}{\tau}-c^n\right)\|\phi^{n+1}\|^2+6\|\phi^{n+1}\|^2\le 0$$

Thus, $\|\phi^{n+1}\|=0$

When $\frac{1}{\tau}-c^n<0$, we have:

$$6\|\phi^{n+1}\|^2\le\left(\frac{1}{\tau}-c^n\right)\|\phi^{n+1}\|^2$$

$$\left(6-c^n+\frac{1}{\tau}\right)\|\phi^{n+1}\|^2\le 0$$

If $c^n \leq 6$, or $c^n > 6$ and $\tau < {}^{1}\!/_{(c^n-6)}$, we have:

$\|\phi^{n+1}\| = 0$

So, in any case we have:

$\|\phi^{n+1}\| = 0$

That is:

$\phi^{n+1} = 0, \qquad 1 \leq i \leq M-1$

Thus, the homogeneous system (3.64) and (3.65) has only a trivial solution.

Chapter 4

NUMERICAL RESULTS AND DISCUSSION

In this chapter, we shall present two examples of numerical solutions of problems (1.1a) – (1.1d). The two model problems to be considered will be solved using the compact scheme of equations (3.45) – (3.50) proposed by Ye and Sun in [18]. For convenience of discussion, we shall present our results in tables and figures in other to give a clear overview of the method, and how it improves the accuracy of the time and space directions.

Firstly, we shall employ the notations presented in Chapter 1, to rewrite the tridiagonal system (3.51) and (3.52) obtained in Chapter 3, in algebraic form and thereafter represent them in matrix form.

From equations (3.51) and (3.52) we have.

$$\frac{1}{12}(D_{\hat{t}}\phi_{i-1}^{n} + 10D_{\hat{t}}\phi_{i}^{n} + D_{\hat{t}}\phi_{i+1}^{n}) = \delta_x^2\phi_i^{\bar{n}} + \frac{1}{12}C^n\left(\phi_{i-1}^{\bar{n}} + 10\phi_i^{\bar{n}} + \phi_{i+1}^{\bar{n}}\right) + \frac{1}{12}(\psi_{i-1}^{n} + 10\psi_i^{n} + \psi_{i+1}^{n}), \quad 1 \le i \le M-1, \tag{4.1}$$

$$\phi_0^{n+1} = z_0(t_{n+1}), \qquad \phi_M^{n+1} = z_1(t_{n+1}), \tag{4.2}$$

Applying the following notations to Eq. (4.1)

$$D_{\hat{t}}\phi_i^n = \frac{\phi_i^{n+1} - \phi_i^{n-1}}{2\tau}, \quad \phi_i^{\bar{n}} = \frac{\phi_i^{n+1} + \phi_i^{n-1}}{2}, \qquad \delta_x^2\phi_i^n = \frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{h^2},$$

We get:

$$\frac{1}{12}\left[\left(\frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n-1}}{2\tau}\right)+10\left(\frac{\phi_{i}^{n+1}-\phi_{i}^{n-1}}{2\tau}\right)+\left(\frac{\phi_{i+1}^{n+1}-\phi_{i+1}^{n-1}}{2\tau}\right)\right]=\left(\frac{\phi_{i-1}^{\overline{n}}-2\phi_{i}^{\overline{n}}+\phi_{i+1}^{\overline{n}}}{h^2}\right)+$$

$$\frac{1}{12}C^n\left[\left(\frac{\phi_{i-1}^{n+1}+\phi_{i-1}^{n-1}}{2}\right)+10\left(\frac{\phi_{i}^{n+1}+\phi_{i}^{n-1}}{2}\right)+\left(\frac{\phi_{i+1}^{n+1}+\phi_{i+1}^{n-1}}{2}\right)\right]+\frac{1}{12}(\psi_{i-1}^{n}+10\psi_{i}^{n}+\psi_{i+1}^{n})$$

$$\frac{\phi_{i-1}^{n+1}}{24\tau}-\frac{\phi_{i-1}^{n-1}}{24\tau}+\frac{5\phi_{i}^{n+1}}{12\tau}-\frac{5\phi_{i}^{n-1}}{12\tau}+\frac{\phi_{i+1}^{n+1}}{24\tau}-\frac{\phi_{i+1}^{n-1}}{24\tau}=\frac{1}{h^2}\left[\left(\frac{\phi_{i-1}^{n+1}+\phi_{i-1}^{n-1}}{2}\right)-2\left(\frac{\phi_{i}^{n+1}+\phi_{i}^{n-1}}{2}\right)+\right.$$

$$\left.\left(\frac{\phi_{i+1}^{n+1}+\phi_{i+1}^{n-1}}{2}\right)\right]+C^n\left(\frac{\phi_{i-1}^{n+1}}{24}+\frac{\phi_{i-1}^{n-1}}{24}+\frac{5\phi_{i}^{n+1}}{12}+\frac{5\phi_{i}^{n-1}}{12}+\frac{\phi_{i+1}^{n+1}}{24}+\frac{\phi_{i+1}^{n-1}}{24}\right)+\frac{1}{12}(\psi_{i-1}^{n}+$$

$$10\psi_{i}^{n}+\psi_{i+1}^{n}),$$

$$\frac{\phi_{i-1}^{n+1}}{24\tau}+\frac{5\phi_{i}^{n+1}}{12\tau}+\frac{\phi_{i+1}^{n+1}}{24\tau}-\frac{\phi_{i-1}^{n-1}}{24\tau}-\frac{5\phi_{i}^{n-1}}{12\tau}-\frac{\phi_{i+1}^{n-1}}{24\tau}=-\frac{\phi_{i-1}^{n+1}}{2h^2}-\frac{2\phi_{i}^{n+1}}{2h^2}+\frac{\phi_{i+1}^{n+1}}{2h^2}+\frac{\phi_{i-1}^{n-1}}{2h^2}-$$

$$\frac{2\phi_{i}^{n-1}}{2h^2}+\frac{\phi_{i+1}^{n-1}}{2h^2}+\frac{C^n\phi_{i-1}^{n+1}}{24}+\frac{5C^n\phi_{i}^{n+1}}{12}+\frac{C^n\phi_{i+1}^{n+1}}{24}+\frac{C^n\phi_{i-1}^{n-1}}{24}+\frac{5C^n\phi_{i}^{n+1}}{12}+\frac{C^n\phi_{i+1}^{n-1}}{24}+$$

$$\frac{1}{12}(\psi_{i-1}^{n}+10\psi_{i}^{n}+\psi_{i+1}^{n}),$$

Then we have:

$$\phi_{i-1}^{n+1}\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right)+\phi_{i}^{n+1}\left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right)+\phi_{i+1}^{n+1}\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right)=$$

$$\phi_{i-1}^{n-1}\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)+\phi_{i}^{n-1}\left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right)+\phi_{i+1}^{n-1}\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)+$$

$$\frac{1}{12}(\psi_{i-1}^{n}+10\psi_{i}^{n}+\psi_{i+1}^{n}),\quad 1\le i\le M-1$$

$$\begin{bmatrix} \left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right) & \left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right)\cdots & 0 & 0 \\ \left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right) & \left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right) & \left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right)\cdots & 0 \\ \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 \\ \vdots & \left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right) & \left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right) & \left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right) \\ 0 & 0\cdots & \left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right) & \left(\frac{5}{12\tau}+\frac{1}{h^2}-\frac{5C^n}{12}\right) \end{bmatrix} \begin{bmatrix} \phi_1^{n+1} \\ \phi_2^{n+1} \\ 0 \\ \vdots \\ 0 \\ \phi_{M-2}^{n+1} \\ \phi_{M-1}^{n+1} \end{bmatrix}$$

$$+\begin{bmatrix} \phi_0^{n+1}\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right) \\ 0 \\ \vdots \\ 0 \\ \phi_M^{n+1}\left(\frac{1}{24\tau}-\frac{1}{2h^2}-\frac{C^n}{24}\right) \end{bmatrix} =$$

$$
\begin{bmatrix}
\left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right) & \left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)\cdots & 0 & 0 \\
\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right) & \left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right) & \left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)\cdots & 0 \\
\ddots & \ddots & \ddots & \vdots \\
0 & & & 0 \\
\vdots & \left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right) & \left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right) & \left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right) \\
0 & 0\cdots & \left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right) & \left(\frac{5}{12\tau}-\frac{1}{h^2}+\frac{5C^n}{12}\right)
\end{bmatrix}
\begin{bmatrix}
\phi_1^{n-1} \\ \phi_2^{n-1} \\ 0 \\ \vdots \\ 0 \\ \phi_{M-2}^{n-1} \\ \phi_{M-1}^{n-1}
\end{bmatrix}
$$

$$
+\begin{bmatrix}
\phi_0^{n-1}\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right) \\ 0 \\ \vdots \\ 0 \\ \phi_M^{n-1}\left(\frac{1}{24\tau}+\frac{1}{2h^2}+\frac{C^n}{24}\right)
\end{bmatrix}
+\frac{1}{12}(\psi_{i-1}^n+10\psi_i^n+\psi_{i+1}^n),
$$

4.1 Numerical Example 1

Consider the parabolic problem with control parameter in [18]:

$$
\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial x^2}+c(t)\phi+[\pi^2-(t+1)^2]e^{-t^2}[\cos(\pi x)+\sin(\pi x)],
$$

$$
0<x<1, \quad 0<t\le 1 \tag{4.3}
$$

$$
\phi(x,0)=\cos(\pi x)+\sin(\pi x), \qquad 0<x<1, \tag{4.4}
$$

$$
\phi(0,t)=e^{-t^2}, \qquad \phi(1,t)=-e^{-t^2}, \qquad 0\le t\le 1, \tag{4.5}
$$

$$
\phi(0.25,t)=\sqrt{2}e^{-t^2}, \qquad 0\le t\le 1, \tag{4.6}
$$

The exact solution of the above problem is:

$$
\phi(x,t)=e^{-t^2}[\cos(\pi x)+\sin(\pi x)], \qquad c(t)=1+t^2.
$$

Define:

$$E_\infty(h,\tau) = \max_{0\le n\le N}\left\{\max_{0\le i\le M}|\phi_i^n - \varphi_i^n|\right\}, \qquad F_\infty(h,\tau) = \max_{0\le n\le N}|C^n - c^n|.$$

We apply the difference scheme of equations (3.45) – (3.50) to solve equations (4.3) – (4.6). Table 1 gives the comparison of the maximal errors of the numerical solution $\phi(x,t)$ for the implicit Crandall's scheme in Equations (1.2a) – (1.2e) and the present scheme (the Compact finite difference scheme). While Table 2 gives the comparison of the maximal errors of the numerical solution $c(t)$ for the implicit Crandall's scheme in Eq. (1.2) and the present scheme. The accuracy of the present method is tested by solving the above problem with several values of h and τ at the final time $T = 1$.

Table 1: Comparison of absolute error in $\phi(x,1)$ for different methods

M	N	Crandall	Present Method
		$\boldsymbol{E_\infty(h,\tau)}$	$\boldsymbol{E_\infty(h,\tau)}$
40	80	7.2100×10^{-2}	2.4000×10^{-3}
80	320	2.6400×10^{-2}	3.1350×10^{-4}
160	1280	7.6000×10^{-3}	7.7472×10^{-5}
320	5120	2.0000×10^{-3}	1.9428×10^{-5}

Table 2: Comparison of absolute error in $c(t)$ for different methods

M	N	Crandall	Present Method
		$\boldsymbol{F_\infty(h,\tau)}$	$\boldsymbol{F_\infty(h,\tau)}$
40	80	1.8139	8.3100×10^{-2}
80	320	6.7380×10^{-1}	1.8000×10^{-2}
160	1280	1.9410×10^{-1}	3.7000×10^{-3}
320	5120	5.0500×10^{-2}	9.0301×10^{-4}

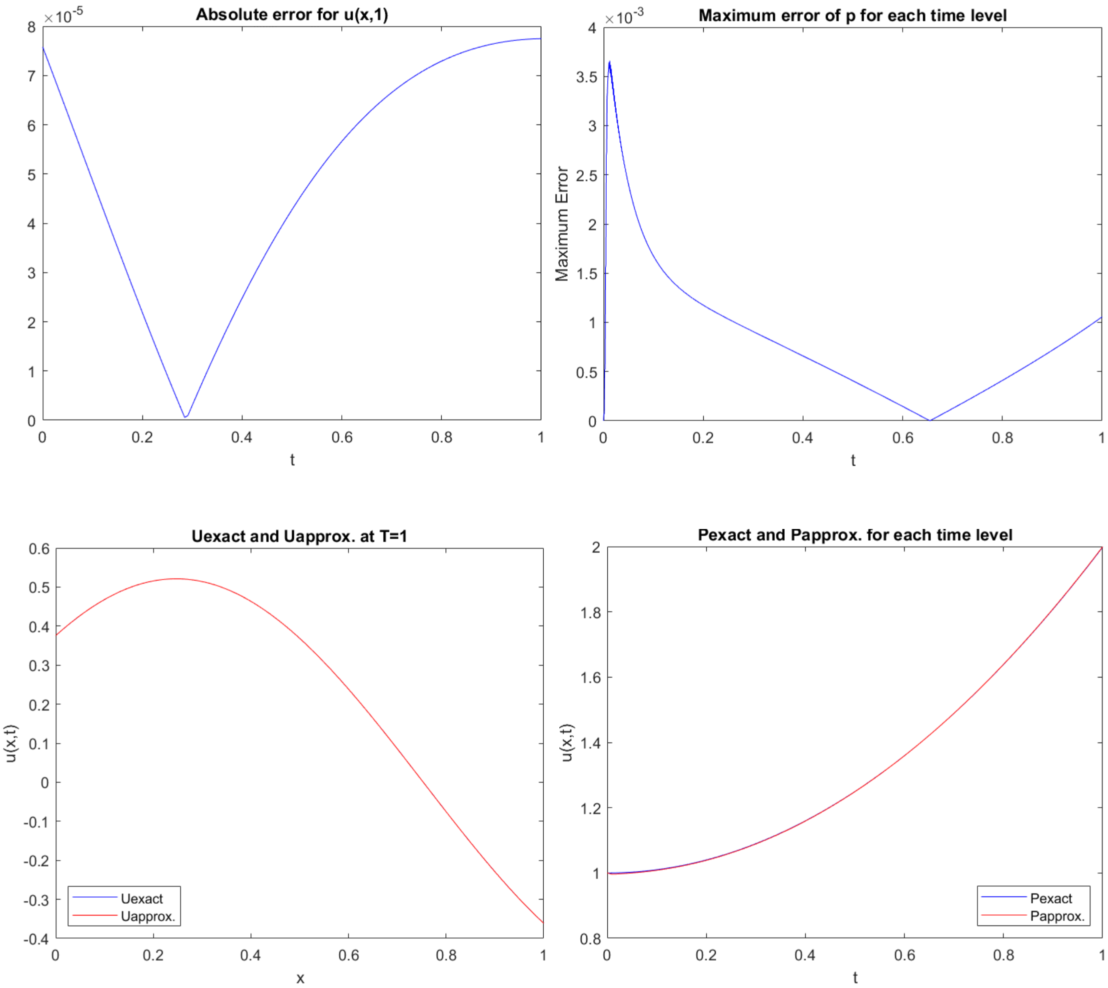

Figure 1: Graphs of Problem 1 with $h = {}^1/_{160}$ and $\tau = {}^1/_{1280}$
Using the Compact Difference Scheme

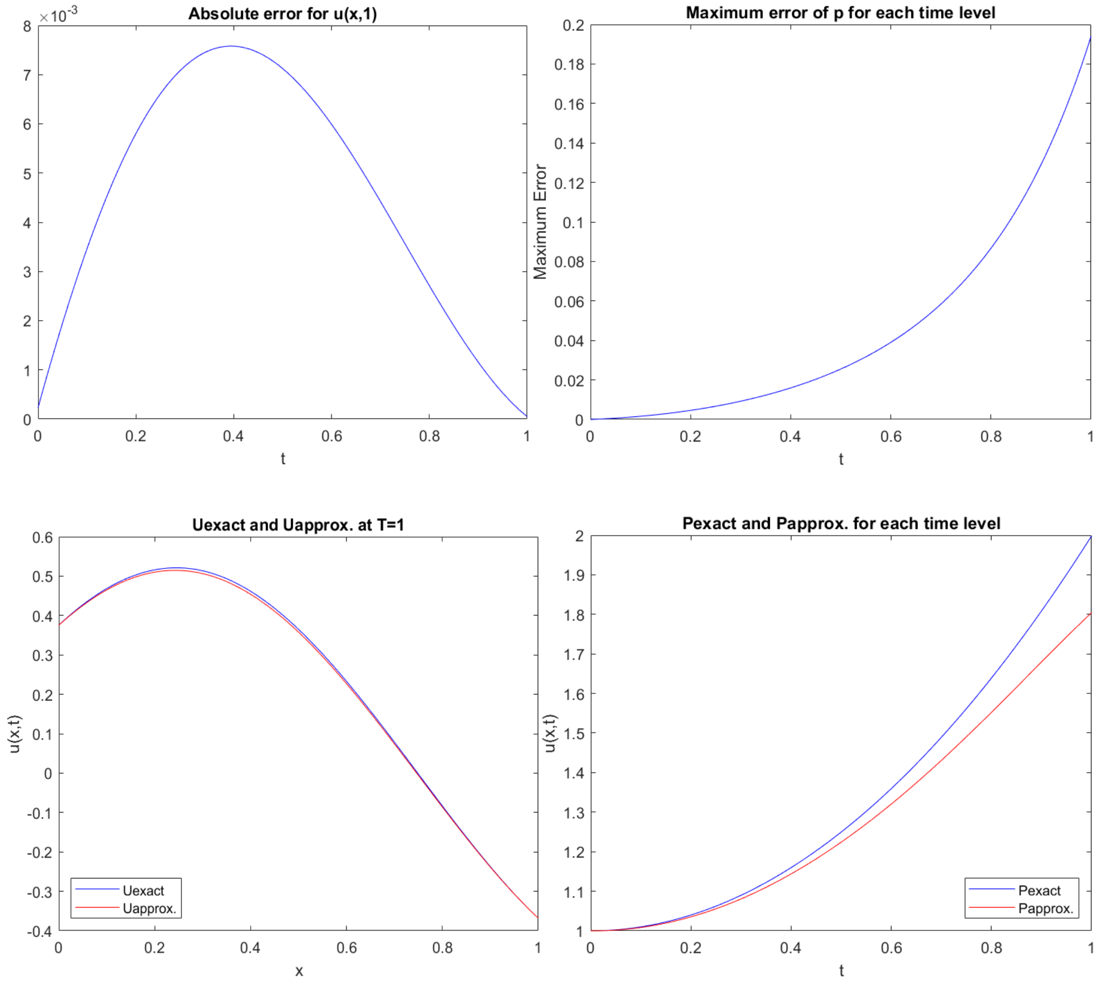

Figure 2: Graphs of Problem 1 with $h = {}^{1}/_{160}$ and $\tau = {}^{1}/_{1280}$. Using the Implicit Crandall's scheme

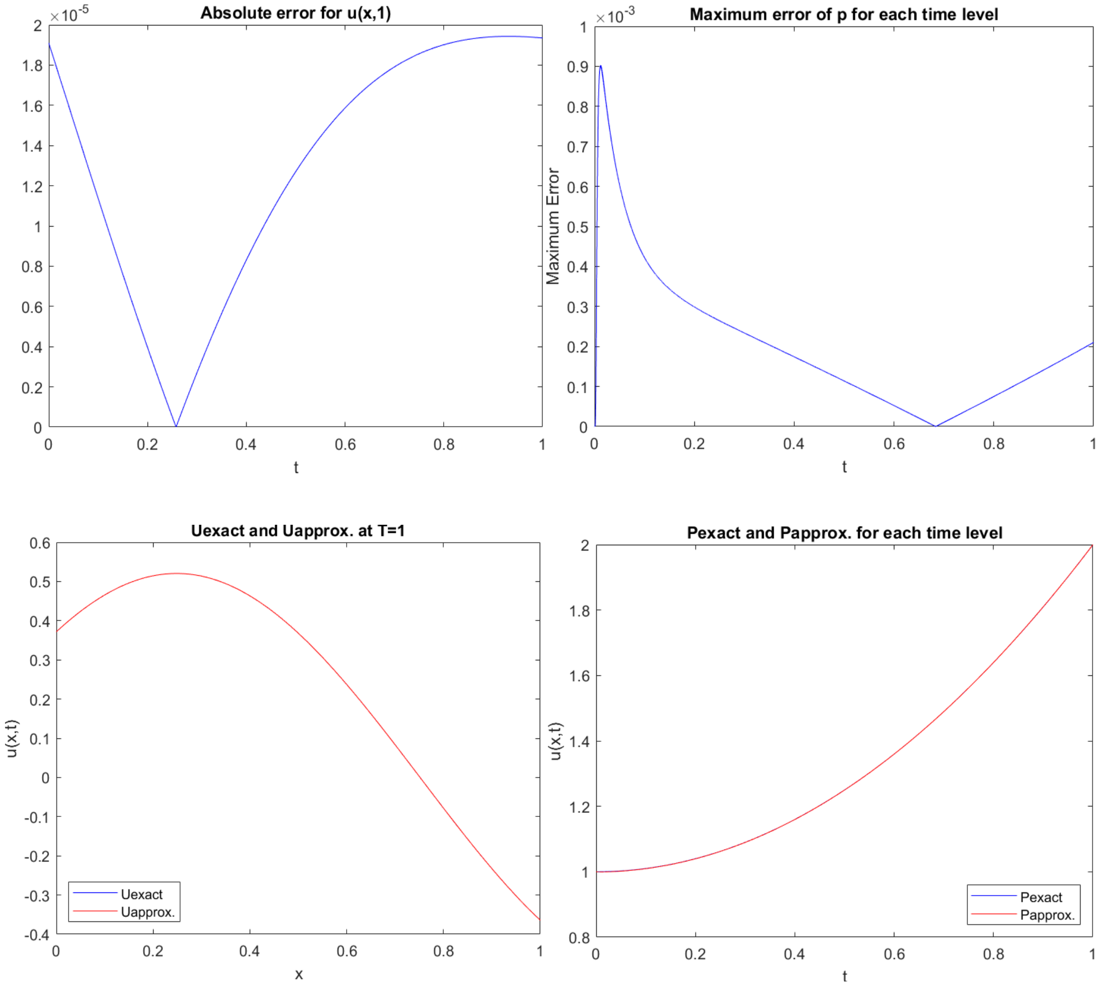

Figure 3: Graphs of Problem 1 with $h = {}^{1}/_{320}$ and $\tau = {}^{1}/_{5120}$ Using the Compact Difference Scheme

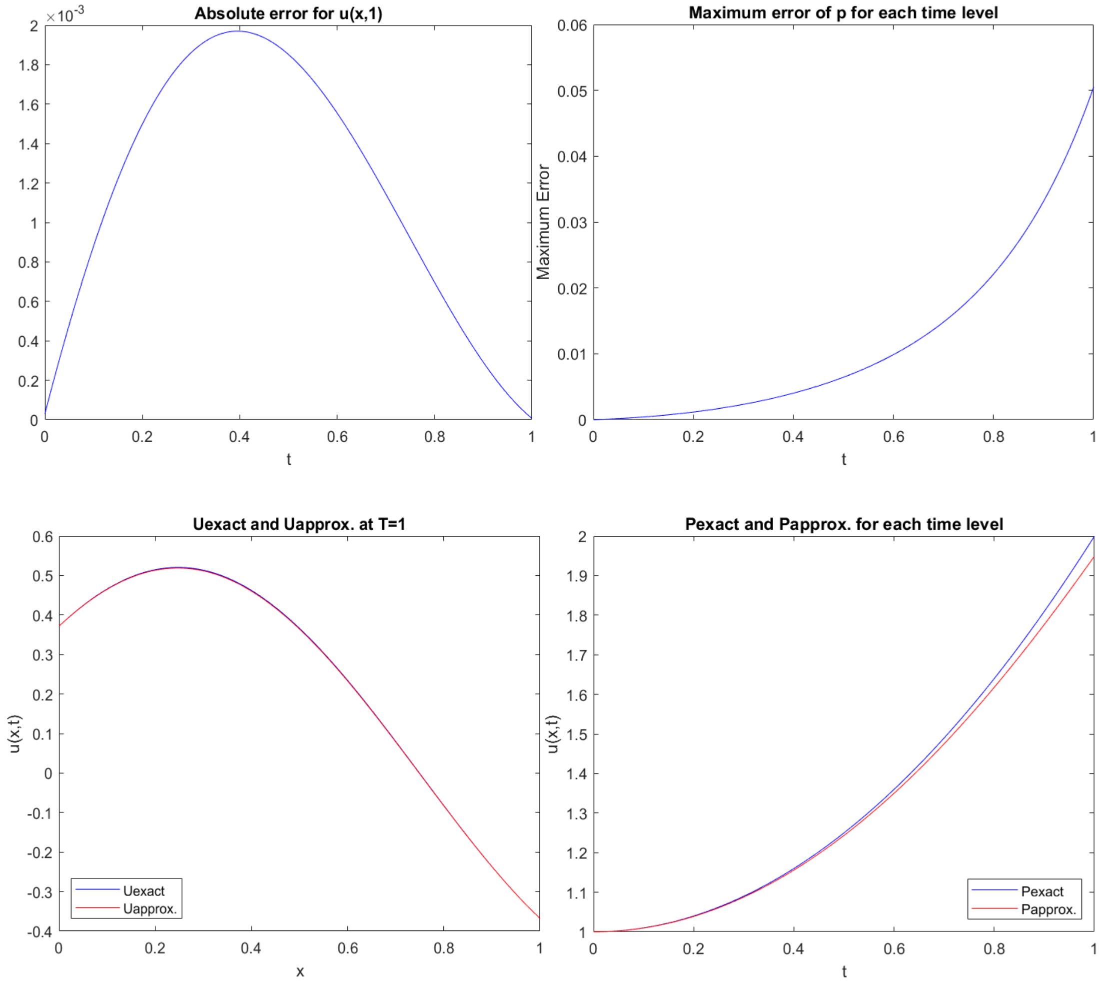

Figure 4: Graphs of Problem 1 with $h = {}^{1}/_{160}$ and $\tau = {}^{1}/_{1280}$. Using the Implicit Crandall's scheme

As we see from Tables 1 and 2, and the Figures above, the results of the present method improve the accuracy of the space and time directions. Figures 1 and 3 shows the maximum error obtained for $\phi(x,t)$ and $c(t)$ for different values of h and τ using the present method. While Figures 3 and 4 shows the maximum errors for $\phi(x,t)$ and $c(t)$ obtained using the implicit Crandall's method.

4.2 Numerical Example 2

Consider the second inverse problem below:

$$\phi(x,t) = e^t[x + \cos(\pi x) + \pi^2 \cos(\pi x)] - e^t(1+t^2)[x + \cos(\pi x)]$$

$$0 < x < 1, \quad 0 < t \le 1 \qquad (4.7)$$

$$\phi(x,0) = \cos(\pi x) + x, \qquad 0 < x < 1, \qquad (4.8)$$

$$\phi(0,t) = e^t, \qquad \phi(1,t) = 0, \qquad 0 \le t \le 1, \qquad (4.9)$$

$$E(t) = e^t[\cos(\pi x^*) + x^*], \qquad 0 \le t \le 1 \qquad (4.10)$$

With the exact solution of the problem given as:

$$\phi(x,t) = e^t[\cos(\pi x) + x], \quad c(t) = 1 + t^2.$$

Again, we apply the difference scheme of equations (3.45) – (3.50) to solve equations (4.7) – (4.10). Table 3 gives the comparison of the maximal errors of the numerical solution $\phi(x,t)$ for the implicit Crandall's scheme in Equations (1.2a) – (1.2e) and the present method. While Table 4 gives the comparison of the maximal errors of the numerical solution $c(t)$ for the implicit Crandall's scheme in Eq. (1.2) and the present scheme.

Table 3: Comparison of absolute error in $\phi(x,1)$ for different methods

M	N	Crandall $\boldsymbol{E_\infty(h,\tau)}$	Present Method $\boldsymbol{E_\infty(h,\tau)}$
40	80	9.9400×10^{-2}	1.7100×10^{-2}
80	320	2.5500×10^{-2}	4.2000×10^{-3}
160	1280	6.4000×10^{-3}	1.1000×10^{-3}
320	5120	1.6000×10^{-3}	2.6601×10^{-4}

Table 4: Comparison of absolute error in $c(t)$ for different methods

M	N	Crandall $\boldsymbol{F_\infty(h,\tau)}$	Present Method $\boldsymbol{F_\infty(h,\tau)}$
40	80	5.9110×10^{-1}	1.3600×10^{-1}
80	320	1.5030×10^{-1}	2.9800×10^{-2}
160	1280	3.7500×10^{-2}	6.0000×10^{-2}
320	5120	9.4000×10^{-3}	1.5000×10^{-3}

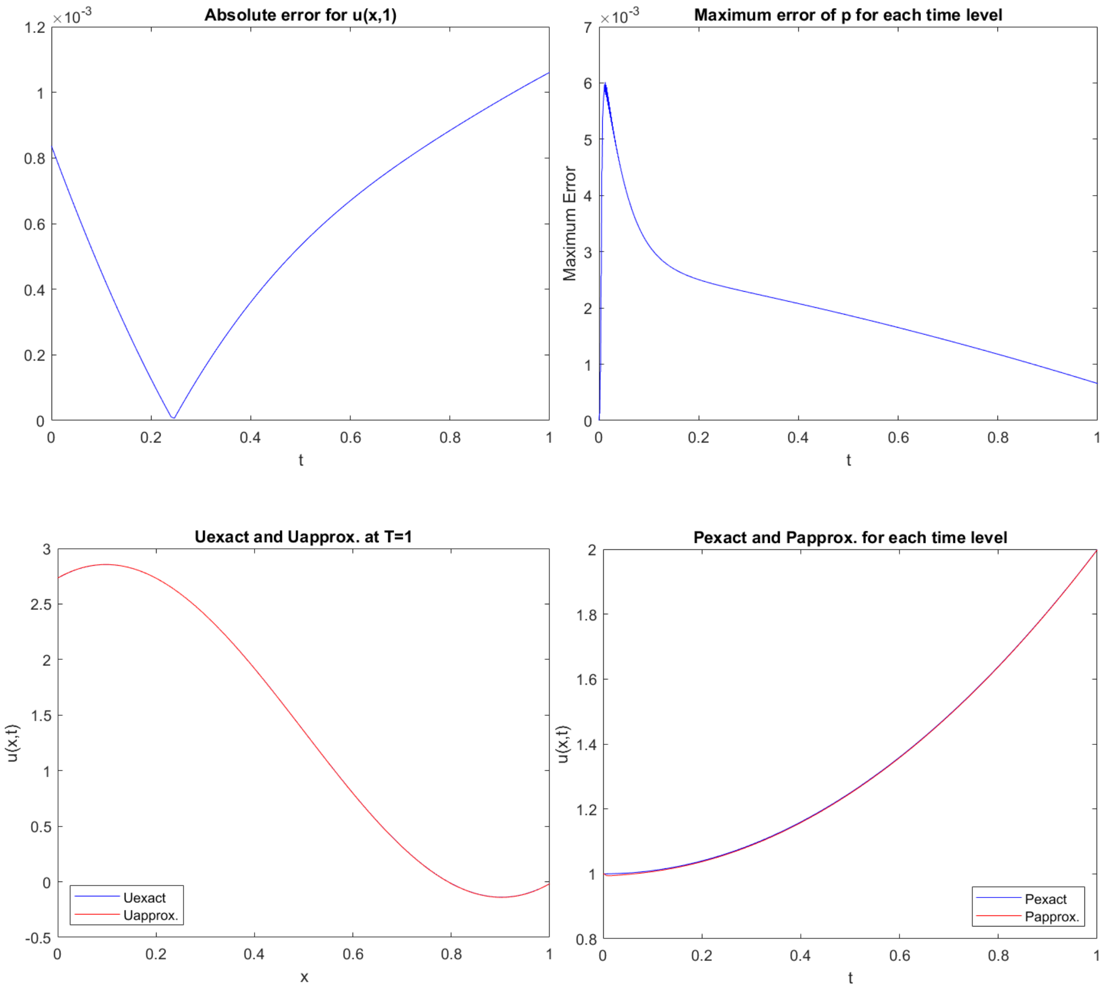

Figure 5: Graphs of Problem 2 with $h = {}^{1}/_{160}$ and $\tau = {}^{1}/_{1280}$ Using the Compact Difference Scheme

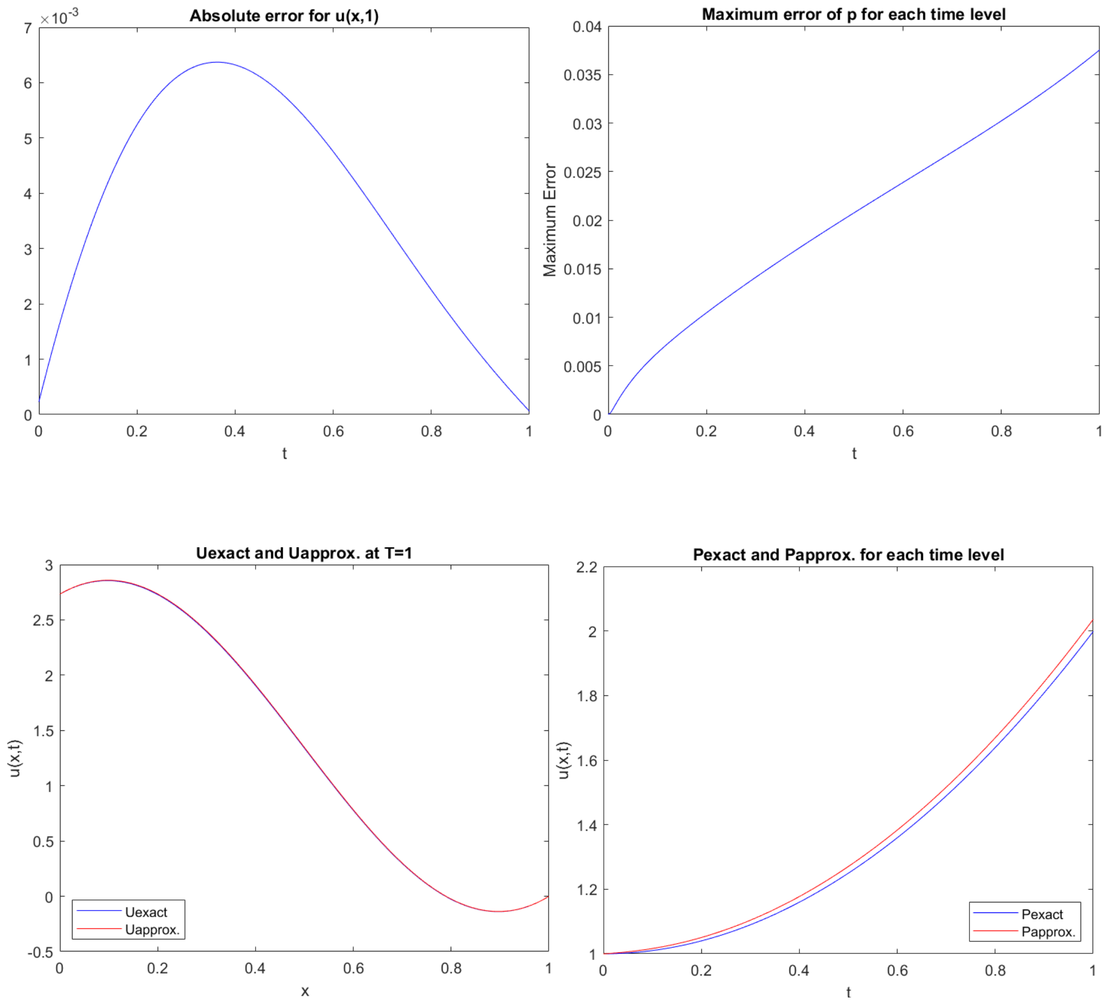

Figure 6: Graphs of Problem 1 with $h = {}^{1}/_{160}$ and $\tau = {}^{1}/_{1280}$. Using the Implicit Crandall's Scheme

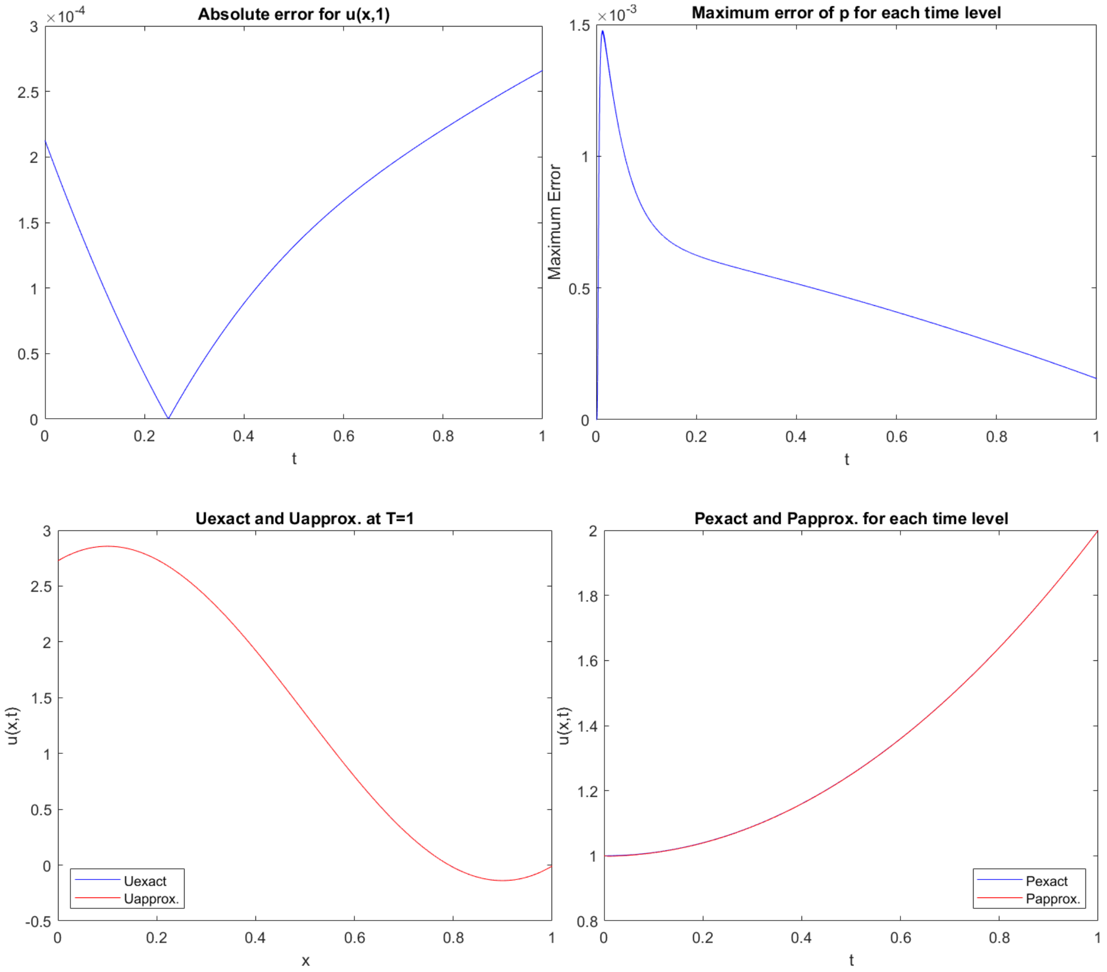

Figure 7: Graphs of Problem 1 with $h = {}^{1}/_{320}$ and $\tau = {}^{1}/_{5120}$ Using the Compact Difference Scheme

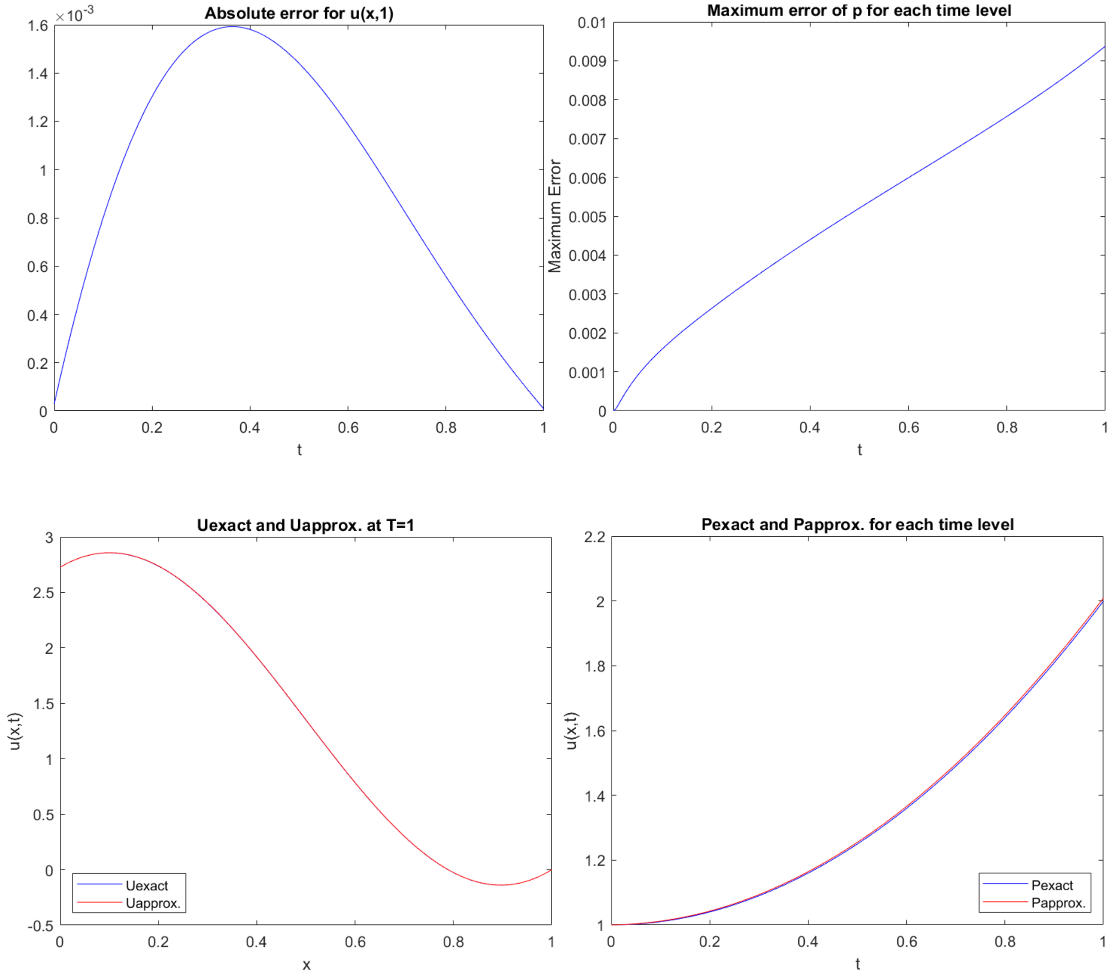

Figure 8: Graphs of Problem 1 with $h = {}^{1}/_{160}$ and $\tau = {}^{1}/_{1280}$. Using the Implicit Crandall's Scheme

Again, we see from Tables 3 and 4, and the Figures of problem 2 above, that the results of the present method improve the accuracy of the space and time directions. Figures 5 and 7 shows the maximum error obtained for $\phi(x, t)$ and $c(t)$ for different values of h and τ using the present method. While Figures 6 and 8 shows the maximum errors for $\phi(x, t)$ and $c(t)$ obtained using the implicit Crandall's method.

Chapter 5

CONCLUSION AND FUTURE WORK

In this thesis, we constructed a linearized compact difference scheme for determining unknown control parameter and unknown solution of a one dimensional parabolic inverse problem with overspecification at a point in the spatial domain. In the constructed scheme, we approximated the time and space directions to two and four order accuracy respectively. The existence and consistence of the constructed scheme was proved as well. We stated two test problems and presented some numerical results which were compared with the implicit Crandall's scheme given by Dehghen to confirm the efficiency of the method in this thesis. The numerical results show that the linearized compact difference scheme improve the accuracy of the time and space directions. Therefore, the linearized compact difference scheme is reasonably satisfactory. For further work, the method in this thesis can also be applied to two-dimensional inverse problem, whereby a splitting up algorithm is applied to split the two-dimensional problem into one-dimensional problems which can then be solved using the linearized compact difference scheme formulated in this thesis.

REFERENCE

[1] Cannon, J.R., & Lin, Y. (1990). An inverse problem of finding a parameter in a semi-linear heat equation. *J. Math. Anal. Appl*, 145 (2), 470–484.

[2] Dehghan, M. (2005). Parameter determination in a partial differential equation from the overspecification data. *Math. And Computer Modelling*, 41, 197-213.

[3] Mohebbi, A., & Abbasi, M. (2015). A fourth-order compact difference scheme for the parabolic inverse problem with an overspecification at a point. *Inverse Problems in Science and Engineering*, 457-478.

[4] Cannon, J.R., Lin, Y., & Xu, S. (1994). Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic differential equations. *Inverse Problems,* 10, 227–243.

[5] Dehghan, M. (2003). Finding a control parameter in one-dimensional parabolic equations. *Appl. Math. Comput,* 135, 491–503.

[6] Liao, W., Dehghan, M., & Mohebbi, A. (2009). Direct numerical method for an inverse problem of a parabolic partial differential equation. *J. Comput. Appl. Math.,* 232, 351–360.

[7] Dehghan, M., & Saadatmandi, A. (2006). A tau method for the one-dimensional parabolic inverse problem subject to temperature overspecification. *Comput. Math. Appl.,* 52, 933–940.

[8] Dehghan, M., & Shakeri, F. (2009). Method of lines solutions of the parabolic inverse problem with an overspecification at a point. *Numer. Algor.*, 50, 417–437.

[9] Cheng, R. (2008). Determination of a control parameter in a one-dimensional parabolic equation using the moving least-square approximation. *Int. J. Compu. Math.*, 85, 1363–1373.

[10] Wang, B., Zou, G., Zhao, P., & Wang, Q. (2011). Finite volume method for solving a one-dimensional parabolic inverse problem. *Appl. Math. Comput.*, 217, 5227–5235.

[11] Wang, W., Hana, B., & Yamamoto, Ma. (2013). Inverse heat problem of determining time-dependent source parameter in reproducing kernel space. *Nonlinear Anal. Real World Appl.*, 14, 875–887.

[12] Kerimova, N. B., & Ismailov, M. I. (2011). An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions. *Math. Comput. Simul.*, 81, 1656–1672.

[13] Maa, L., & Wua, Z. (2011). Radial basis functions method for parabolic inverse problem. *Int. J. Comput. Math.,* 88, 384–395.

[14] Mohebbi A, & Dehghan M. (2010). High-order scheme for determination of a control parameter in an inverse problem from the over-specified data. *Comput. Phys. Commun.*, 181, 1947–1954.

[15] Ye, C. R., & Sun, Z. Z. (2007). On the stability and convergence of a difference scheme for an one-dimensional parabolic inverse problem. *Appl. Math. Comput*., 188, 214–225.

[16] Daoud, D. S., & Subasi, D. (2005). A splitting up algorithm for the determination of the control parameter in multidimensional parabolic problem. *Appl. Math. Comput.*, 166, 584–595.

[17] Cannon, J.R., Lin, Y., & Wang, S. (1992). Determination of source parameter in parabolic equations. *Meccanica*, 27, 85–94.

[18] Ye, C. R, & Sun, Z. Z. (2009). A linearized compact difference scheme for an one-dimensional parabolic inverse problem. *Appl. Math. Modell*., 33, 1521–1528.

APPENDICES

Appendix A: Matlab Code for Numerical Example 1

```
%-----------------------------------------------------------------------------------------------
% LINEARIZED COMPACT FINITE DIFFERENCE FOR PARABOLIC INVERSE
PROBLEM
%-----------------------------------------------------------------------------------------------
%                  PROBLEM 1
%   U_t(x,t) = U_xx(x,t)+p(t)U(x,t)+Q(x,t)
%  Initial condition        :  %U(x,o)=cos(pi*x)+sin(pi*x)
%    Boundary  conditions           :   U(0,t)  =  e^((-)(t)^2)   ,  U(1,t)  =  -e^((-)(t)^2)%
Overspecification condition :
% E(t)=U(x^*,t) = e^((-)(t)^2)*(cos(pi*x^*)+sin(pi*x^*))
% Exact Solution :U(x,t) = e^((-)(t)^2)*(cos(pi*x)+sin(pi*x))
% ------------------------------------------------
%                  INPUT  X AND T VALUES
%  -----------------------------------------------
clc, clear all;
M = 40; %NUMBER OF SPACE STEPS
N = 80; %NUMBER OF TIME STEPS
T = 1;
t = T/N; % CHANGE IN TIME T
h = 1/M; % CHANGE IN X VARIABLE
x0 = 0.25;
k0 = (x0/h);
h2 = h*h;
```

```
%---------------------------------------------------------------------------
%    BOUNDARY CONDITION CALCULATION (LBC, RBC)
%---------------------------------------------------------------------------
for i = 1:N
    tn = (i-1)*t;
    u(1,i) = exp(-tn^2);
    u(M-1,i) = -exp(-tn^2);
end
%---------------------------------------------------------------
%    INITIAL CONDITION CALCULATION (U(I,1))
%---------------------------------------------------------------
for i = 1:M-1
    xx = i*h;
    u(i,1) = cos(pi*xx)+sin(pi*xx);
end
%---------------------------------------------
%    P0 CALCULATION
%---------------------------------------------
F = cos(pi*x0)+sin(pi*x0);
FF = 1/F;
p0 = FF*(0+((pi^2)*F)-((pi^2)-1)*F);
P(1) = p0;
%-------------------------------------------------------
%    2ND TIME LEVEL U(I,2) CALCULATION
%-------------------------------------------------------
```

```
for i = 1:M-1
    ii = i*h;
    t0 = 0;
    fx = cos(pi*ii)+sin(pi*ii);
    f2x = (-1*(pi^2))*fx;
    fp = ((pi^2)-(t0+1)^2)*(exp(-(t0)^2));
    qx = fp*fx;
    u(i,2) = fx+(t*(f2x+p0*fx+qx));
end
%------------------------------------------------------------------------
% CREATING A TRIDIAGONAL MATRIX FOR PHI ϕ(X,T)
%------------------------------------------------------------------------
n = M-1;
m = 1;
q = 10;
r = 1;
TM = full(gallery('tridiag',n,m,q,r));
% ---------------------------------------------------------------
%   CALCULATION FOR U AND P ALL TIME LEVEL
% ---------------------------------------------------------------
for k = 2:N % TIME LOOP. THIS IS THE MAIN LOOP
    % LOOP FOR RHS TRIDIAGONAL MATRIX 'B'
    for i = 1:M-1
        for j= 1: M-1
            if i == j+1
```

```
            B(i,j)= ((1/(24*t))+(1/(2*h2))+(P(k-1)/24)); % LOWER DIAGONAL
        elseif i == j
            B(i,j) = ((5/(12*t))-(1/(h2))+(5*P(k-1)/12)); % MAIN DIAGONAL
        elseif i == j-1
            B(i,j) = ((1/(24*t))+(1/(2*h2))+(P(k-1)/24)); % UPPER DIAGONAL
        end
    end
end
%------------------------------------------------------------
% COLUMN VECTOR FOR MARIX 'A' AND 'B'
%------------------------------------------------------------
z = (k-2)*t;
r = k*t;
C(1,k-1) = (1/(24*t)+1/(2*h2)+P(k-1)/24)*exp(-(z^2))-(1/(24*t)-1/(2*h2)-P(k-1)/24)*(exp(-r^2));
C(M-1,k-1) = (1/(24*t)+1/(2*h2)+P(k-1)/24)*(-exp(-(z^2)))-(1/(24*t)-1/(2*h2)-P(k-1)/24)*(-exp(-r^2));
C(2:M-2,k-1) = 0;
CC = C(1:M-1,k-1);
%------------------------------------------------------------
% SOLVING FOR Phi TRIDIAGONAL MATRIX
%------------------------------------------------------------
tv = (k-1)*t;
for i = 1:M-1
    x = i*h;
```

```
    D(i,k-1) = (pi^2-(tv+1)^2)*(exp(-(tv)^2)*(cos(pi*x)+sin(pi*x)));
end
EE(1,k-1) = (pi^2-(tv+1)^2)*(exp(-(tv)^2));     %EE IS A COLUMN MATRIX
EE(2:M-2,k-1) = 0;
EE(M-1,k-1) = (pi^2-(tv+1)^2)*(exp(-(tv)^2))*(cos(pi)+sin(pi));
GG(1:M-1,k-1) = ((1/12)*TM)*(D(1:M-1,k-1))+EE(1:M-1,k-1);
%---------------------------------------------------------------------------
        %  SOLUTION OF U WITH THOMAS ALGORITHM.
%----------------------------------------------------------------------------
ld1 = ((1/(24*t))-(1/(2*h2))-(P(k-1)/24));
md1 = ((5/(12*t))+(1/(h2))-(5*P(k-1)/12));
ud1 = ((1/(24*t))-(1/(2*h2))-(P(k-1)/24));
for i =1:M-1
    ld(i) = ld1;
end
for i = 1:M-1
    md(i) = md1;
end
for i = 1:M-1
    ud(i) = ud1;
end
UU = u(1:M-1,k-1);
RHS = B*UU+CC+GG(1:M-1,k-1);
V = M-1;
ud(1) = ud(1)/md(1) ; RHS(1) = RHS(1)/md(1) ;
```

```
    for i = 2:V-1
        temp = md(i)-ld(i)*ud(i-1);
        ud(i) = ud(i)/temp;
        RHS(i) = (RHS(i)-ld(i)*RHS(i-1))/temp;
    end
    RHS(V) = (RHS(V)-ld(V)*RHS(V-1))/(md(V)-ld(V)*ud(V-1));
    AA(V) = RHS(V);
    % NOW BACK SUBSTITUTION
    for i = V-1:-1:1
        AA(i) = RHS(i)-ud(i)*AA(i+1);
    end
    u(1:M-1,k+1) = AA;     % NEXT TIME COLUMN OF U CALCULATED
    SSUM = (-u(k0-2,k+1)+16*u(k0-1,k+1)-30*u(k0,k+1)+16*u(k0+1,k+1)-
u(k0+2,k+1));   %THIS IS THE SUM IN THE BRACKET OF P^(N+1) FORMULA
    r = k*t;
    E = sqrt(2)*exp(-(r)^2);      %OVERSPECIFICATION
    EE = 1/E;
    E_p = -2*r*sqrt(2)*exp(-(r)^2);     %DERIVATIVE OF OVERSPECIFICATION
    Phi = (pi^2-(r+1)^2)*exp(-(r)^2)*(cos(pi*x0)+sin(pi*x0)); %PHI AS GIVEN IN
THE NUMERICAL EXAMPLE
    P(k) = EE*(E_p-(1/(12*h2))*(SSUM)-Phi); %THE VALUES OF P2, P3,...P79
end
%-------------------------------------
% THE EXACT SOLUTION U
%-------------------------------------
```

```
for j = 1:N
    for i = 1:M-1
        t_e = (j-1)*t;
        x_e = i*h;
        u_exact(i,j) = (exp(-(t_e)^2)*(cos(pi*x_e)+sin(pi*x_e)));
    end
end
%---------------------------------------
% P EXACT
%---------------------------------------
for i = 1:N
    t_e = (i-1)*t;
    p_exact(i) = 1+(t_e)^2;
end
%---------------------------------------------
% MAXIMUM ERROR FOR U
%---------------------------------------------
for j = 1:N
    for i = 1:M-1
        Maxerr(i,j) = (abs( u(i,j)-u_exact(i,j)));
        Abs_error(i,N) = (abs( u(i,N)-u_exact(i,N)));
    end
    maxuerr = max(Maxerr);
    Max_error_U = max(maxuerr);
    Abso_error = max(Abs_error);
```

```
    Absolute_error = max(Abso_error);
end
%---------------------------------------------
% MAXIMUM ERROR FOR P
%---------------------------------------------
for i=1:N
    Maxerrp = (abs(P(i)-p_exact(i)));
    erp(i) = Maxerrp;
end
Max_error_P = max(erp);
%-----------------------------------------------------------------------------
% GRAPH OF MAXIMUM ERROR OF U AT EACH TIME LEVEL
%------------------------------------------------------------------------------
%FIGURE 1;
figure;
x = linspace(0,1,N);
y1 = maxuerr(:);
plot( x,y1,'b-');
title('Maximum error of u for each time level');
xlabel('t'); ylabel('Maximum Error');
disp(' Press any key to continue...')
pause
%--------------------------------------------------------
% GRAPH OF MAXIMUM ERROR OF U AT T=1
%--------------------------------------------------------
```

```
%FIGURE 2;
figure;
x = linspace(0,1,M-1);
y1 = Abs_error(:,N);
plot( x,y1,'b-');
title('Absolute error for u(x,1)');
xlabel('t');
disp(' Press any key to continue...')
pause
%--------------------------------------------------------------------------------
% GRAPH OF MAXIMUM ERROR OF P AT EACH TIME LEVEL
%--------------------------------------------------------------------------------
%FIGURE 3;
figure;
x = linspace(0,1,N);
y1 = erp(:);
plot(x,y1,'b-');
title('Maximum error of p for each time level');
xlabel('t'); ylabel('Maximum Error');
disp(' Press any key to continue...')
pause
%----------------------------------------------------------------
% GRAPH OF U_EXACT AND U_APPROX. AT T=1
%----------------------------------------------------------------
```

```
%FIGURE 4;

figure;

xv = linspace(0,1,M-1);

yy1 = u_exact(:,N);

yy2 = u(1:M-1,N);

plot(xv,yy1,'b-',xv,yy2,'r-');

title('Uexact and Uapprox. at T=1');

legend('Uexact','Uapprox.','location','southwest');

xlabel('x'); ylabel('u(x,t)');

disp(' Press any key to continue...')

pause

%--------------------------------------------------------------------------------

% GRAPH OF PEXACT AND PAPPROX. FOR EACH TIME LEVEL

%--------------------------------------------------------------------------------

%FIGURE 5;

figure;

x = linspace(0,1,N);

y1 = p_exact(:);

y2 = P(:);

plot(x,y1,'b-',x,y2,'r-');

title('Pexact and Papprox. for each time level');

legend('Pexact','Papprox.','location','southeast');

xlabel('t'); ylabel('u(x,t)');

disp(' Press any key to continue')

pause
```

```
%--------------------------------------------------------
% PLOTTING SOLUTION FOR ALL X AND T
%--------------------------------------------------------
%FIGURE 6;
figure;
t = linspace(0,1,N+1);    x=linspace(0,1,M-1);
[T,X] = meshgrid(t,x);    mesh(X,T,u(1:M-1,:));
title('Approximate Solution');
xlabel('x');    ylabel('t');    zlabel('u(x,t)');
disp(' Press any key to continue')
pause
%---------------------------------------------------------
% PLOTTING SOLUTION FOR ALL X AND T
%---------------------------------------------------------
%FIGURE 7;
figure;
t = linspace(0,1,N);    x = linspace(0,1,M-1);
[T,X] = meshgrid(t,x);    mesh(X,T,u_exact);
title('Exact Solution');
xlabel('x');    ylabel('t');    zlabel('u(x,t)').
```

Appendix B: Matlab Code for Numerical Example 2

```
%-----------------------------------------------------------------------------------------------
% LINEARIZED COMPACT FINITE DIFFERENCE FOR PARABOLIC INVERSE
PROBLEM
%-----------------------------------------------------------------------------------------------
%                   PROBLEM 2
%   U_t(x,t) = U_xx(x,t)+p(t)U(x,t)+Q(x,t)
%  Initial condition         :  U(x,o) = cos(pi*x)+x
%  Boundary conditions       :  U(0,t) = e^(t)  , U(1,t)=0
% Overspecification condition :
% E(t) = U(x^*,t) = e^(t)*(cos(pi*x^*)+x*)
% Exact Solution :U(x,t) = e^(t)*(cos(pi*x)+x)
% ------------------------------------------------
%                    INPUT  X AND T VALUES
%  ----------------------------------------------
clc, clear all;
M = 40; %NUMBER OF SPACE STEPS
N = 80; %NUMBER OF TIME STEPS
T = 1;
t = T/N; % CHANGE IN TIME T
h = 1/M; % CHANGE IN X VARIABLE
x0 = 0.25;
k0 = (x0/h);
h2 = h*h;
```

```
%--------------------------------------------------------------------------
%    BOUNDARY CONDITION CALCULATION (LBC, RBC)
%--------------------------------------------------------------------------
for i = 1:N
    kk = (i-1)*t;
    u(1,i) = exp(kk);
    u(M-1,i) = 0;
end
%---------------------------------------------------------------
%    INITIAL CONDITION CALCULATION (U(I,1))
%---------------------------------------------------------------
for i = 1:M-1
    xx = i*h;
    u(i,1) = cos(pi*xx)+xx;
end
%---------------------------------------------
%    P0 CALCULATION
%---------------------------------------------
t0 = 0;
fs = cos(pi*x0)+x0;
fs1 = 1/fs;
sb = 1/sqrt(2);
ED = exp(t0)*(x0+sb);
f2 = -1*(pi^2)*(cos(pi*x0));
sc = (x0+cos(pi*x0)+((pi^2)*cos(pi*x0)));
```

```
sd = (1+(t0^2))*(x0+cos(pi*x0));
qf = exp(t0)*(sc-sd);
p0 = fs1*(ED-f2-qf);
P(1) = p0;
%-----------------------------------------------------------
%    2ND TIME LEVEL U(I,2) CALCULATION
%-----------------------------------------------------------
for i = 1:M-1
   ii = i*h;
   t0 = 0;
   f = cos(pi*ii)+ii;
   f2 = (-1*(pi^2))*cos(pi*ii);
   sc = (ii+cos(pi*ii)+((pi^2)*cos(pi*ii)));
   sd = 1+(t0^2);
   sd1 = ii+cos(pi*ii);
   sa = sd*sd1;
   qf = exp(t0)*(sc-sa);
   gf1 = t*(f2+(p0*f)+qf);
   u(i,2) = f+gf1;
end
%--------------------------------------------------------------------------
% CREATING A TRIDIAGONAL MATRIX FOR PHI Q(X,T)
%--------------------------------------------------------------------------
n = M-1;
m = 1;
```

```
q = 10;
r = 1;
TM = full(gallery('tridiag',n,m,q,r));
% -----------------------------------------------------------------
%   CALCULATION FOR U AND P ALL TIME LEVEL
% -----------------------------------------------------------------
for k=2:N % TIME LOOP. THIS IS THE MAIN LOOP
   % LOOP FOR RHS TRIDIAGONAL MATRIX 'B'
   for i = 1:M-1
      for j = 1: M-1
         if i == j+1
            B(i,j) = ((1/(24*t))+(1/(2*h2))+(P(k-1)/24)); % LOWER DIAGONAL
         elseif i == j
            B(i,j) = ((5/(12*t))-(1/(h2))+(5*P(k-1)/12)); % MAIN DIAGONAL
         elseif i == j-1
            B(i,j) = ((1/(24*t))+(1/(2*h2))+(P(k-1)/24)); % UPPER DIAGONAL
         end
      end
   end
   %------------------------------------------------------------
   % COLUMN VECTOR FOR MARIX 'A' AND 'B'
   %-------------------------------------------------------------
   z = (k-2)*t;
   r = k*t;
```

```
C(1,k-1) = (1/(24*t)+1/(2*h2)+P(k-1)/24)*(exp(z))-(1/(24*t)-1/(2*h2)-P(k-1)/24)*(exp(r));
C(M-1,k-1) = ((1/(24*t)+1/(2*h2)+P(k-1)/24)*0)-(1/(24*t)-1/(2*h2)-P(k-1)/24)*0;
C(2:M-2,k-1) = 0;
CC = C(1:M-1,k-1);
%------------------------------------------------------------
% SOLVING FOR PHI TRIDIAGONAL MATRIX
%------------------------------------------------------------
tv = (k-1)*t;
for i = 1:M-1
    xx = i*h;
    f = exp(tv);
    fp = cos(pi*xx);
    D(i,k-1) = f*(xx+fp+(pi^2)*fp)-f*(1+(tv^2))*(xx+fp);
end
EE(1,k-1) = exp(tv)*(0+1+(pi^2)*1)-(exp(tv)*(1+(tv^2)*(0+1)));     %EE IS A COLUMN MATRIX
EE(2:M-2,k-1) = 0;
EE(M-1,k-1) = exp(tv)*(1+(-1)+(pi^2)*(-1))-(exp(tv)*(1+(tv^2)*(1+(-1))));
GG(1:M-1,k-1) = ((1/12)*TM)*(D(1:M-1,k-1))+EE(1:M-1,k-1);
%---------------------------------------------------------------------
%       SOLUTION OF U WITH THOMAS ALGORITHM.
%---------------------------------------------------------------------
ld1 = ((1/(24*t))-(1/(2*h2))-(P(k-1)/24));
md1 = ((5/(12*t))+(1/(h2))-(5*P(k-1)/12));
```

```
ud1 = ((1/(24*t))-(1/(2*h2))-(P(k-1)/24));
for i = 1:M-1
    ld(i) = ld1;
end
for i = 1:M-1
    md(i) = md1;
end
for i = 1:M-1
    ud(i) = ud1;
end
UU = u(1:M-1,k-1);
RHS = B*UU+CC+GG(1:M-1,k-1);
V = M-1;
ud(1) = ud(1)/md(1) ; RHS(1) = RHS(1)/md(1) ;
for i = 2:V-1
    temp = md(i)-ld(i)*ud(i-1);
    ud(i) = ud(i)/temp;
    RHS(i) = (RHS(i)-ld(i)*RHS(i-1))/temp;
end
RHS(V) = (RHS(V)-ld(V)*RHS(V-1))/(md(V)-ld(V)*ud(V-1));
AA(V) = RHS(V);
% NOW BACK SUBSTITUTION
for i = V-1:-1:1
    AA(i) = RHS(i)-ud(i)*AA(i+1);
end
```

```
    u(1:M-1,k+1) = AA;     % NEXT TIME COLUMN OF U CALCULATED
    SSUM = (-u(k0-2,k+1)+16*u(k0-1,k+1)-30*u(k0,k+1)+16*u(k0+1,k+1)-u(k0+2,k+1)); %THIS IS THE SUM IN THE BRACKET OF P^(N+1) FORMULA
    r = k*t;
    sb = 1/sqrt(2);
    E = exp(r)*(x0+sb); %OVERSPECIFICATION
    EE = 1/E;
    E_p = exp(r)*(x0+sb); %DERIVATIVE OF OVERSPECIFICATION
    f = exp(r);
    fp = cos(pi*x0);
    Phi = f*(x0+fp+(pi^2)*fp)-f*(1+(r^2))*(x0+fp);     %PHI AS GIVEN IN THE NUMERICAL EXAMPLE
    P(k) = EE*(E_p-(1/(12*h2))*(SSUM)-Phi); %THE VALUES OF P2, P3,...P80
end
%-------------------------------------
% THE EXACT SOLUTION U
%-------------------------------------
for j = 1:N
    for i = 1:M-1
        t_e = (j-1)*t;
        x_e = i*h;
        u_exact(i,j) = exp(t_e)*((cos(pi*x_e)+x_e));
    end
end
```

```
%------------------------
% P EXACT
%------------------------
for i = 1:N
    t_e = (i-1)*t;
    p_exact(i) = 1+(t_e)^2;
end
%--------------------------------------
% MAXIMUM ERROR FOR U
%---------------------------------------
for j = 1:N
    for i = 1:M-1
        Maxerr(i,j) = (abs( u(i,j)-u_exact(i,j)));
    end
    maxuerr = max(Maxerr);
    Max_error_U = max(maxuerr);
end
%--------------------------------------------
% MAXIMUM ERROR FOR U
%--------------------------------------------
for j = 1:N
    for i = 1:M-1
        Maxerr(i,j) = (abs( u(i,j)-u_exact(i,j)));
        Abs_error(i,N) = (abs( u(i,N)-u_exact(i,N)));
    end
```

```
    maxuerr = max(Maxerr);
    Max_error_U = max(maxuerr);
    Abso_error = max(Abs_error);
    Absolute_eeror = max(Abso_error);
end
%---------------------------------------------
% MAXIMUM ERROR FOR P
%---------------------------------------------
for i =1:N
    Maxerrp = max(abs(P(i)-p_exact(i)));
    erp(i) = Maxerrp;
end
Max_error_P = max(erp);
%--------------------------------------------------------------------------------
% GRAPH OF MAXIMUM ERROR OF U AT EACH TIME LEVEL
%--------------------------------------------------------------------------------
%FIGURE1;
figure;
x = linspace(0,1,N);
y1 = maxuerr(:);
plot( x,y1,'b-');
title('Maximum error of u for each time level');
xlabel('t'); ylabel('Maximum Error');
disp(' Press any key to continue...')
pause
```

```
%-----------------------------------------------------------

% GRAPH OF MAXIMUM ERROR OF U AT T=1

%-----------------------------------------------------------

%FIGURE2;

figure;

x = linspace(0,1,M-1);

y1 = Abs_error(:,N);

plot( x,y1,'b-');

title('Absolute error for u(x,1)');

xlabel('t');

disp(' Press any key to continue...')

pause

%-------------------------------------------------------------------------------

% GRAPH OF MAXIMUM ERROR OF P AT EACH TIME LEVEL

%-------------------------------------------------------------------------------

%FIGURE3;

figure;

x = linspace(0,1,N);

y1 = erp(:);

plot(x,y1,'b-');

title('Maximum error of p for each time level');

xlabel('t'); ylabel('Maximum Error');

disp(' Press any key to continue...')

pause
```

```
%-----------------------------------------------------------------
% GRAPH OF U_EXACT AND U_APPROX. AT T=1
%-----------------------------------------------------------------
%FIGURE 4;
figure;
xv = linspace(0,1,M-1);
yy1 = u_exact(:,N);
yy2 = u(1:M-1,N);
plot(xv,yy1,'b-',xv,yy2,'r-');
title('Uexact and Uapprox. at T=1');
legend('Uexact','Uapprox.','location','southwest');
xlabel('x'); ylabel('u(x,t)');
disp(' Press any key to continue...')
pause
%----------------------------------------------------------------------------------
% GRAPH OF P_EXACT AND P_APPROX. FOR EACH TIME LEVEL
%----------------------------------------------------------------------------------
%FIGURE 5;
figure;
x = linspace(0,1,N);
y 1= p_exact(:);
y2 = P(:);
plot(x,y1,'b-',x,y2,'r-');
title('Pexact and Papprox. for each time level');
legend('Pexact','Papprox.','location','southeast');
```

```
xlabel('t'); ylabel('u(x,t)');
disp(' Press any key to continue')
pause
%--------------------------------------------------------
% PLOTTING SOLUTION FOR ALL X AND T
%--------------------------------------------------------
%FIGURE 6;
figure;
t = linspace(0,1,N+1);    x = linspace(0,1,M-1);
[T,X] = meshgrid(t,x);    mesh(X,T,u(1:M-1,:));
title('Approximate Solution');
xlabel('x');    ylabel('t');    zlabel('u(x,t)');
disp(' Press any key to continue')
pause
%------------------------------------------------------
% PLOTTING SOLUTION FOR ALL X AND T
%------------------------------------------------------
%FIGURE 7;
figure;
t = linspace(0,1,N);    x = linspace(0,1,M-1);
[T,X] = meshgrid(t,x);    mesh(X,T,u_exact);
title('Exact Solution');
xlabel('x');    ylabel('t');    zlabel('u(x,t)');
```