Shadow Cast of the Schwarzschild-like Black Holes

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ABSTRACT

The Event Horizon Telescope has recently seen the edge of a black hole. This fruitful outcome reveals a fine structure near the black hole horizon. The photon ring and shadow have been clearly observed which provides us with a good opportunity to test general relativity in the regime of strong gravity. In this thesis first we study Schwarzschild black hole and its properties, then we calculate null geodesics of Schwarzschild metric from Euler-Lagrange to find its shadow cast which is two-dimensional dark zone in the celestial sphere caused by the strong gravity of the black hole. Last, we study the shadow of the quantum corrected Schwarzschild black hole. Hence, we discuss the effect of the quantum on the shadow of the black hole.

Keywords: Gravitation, Black holes, Shadow, Null geodesics, Quantum Corrected Black Holes.

ÖZ

Event Horizon Teleskobu yakın zamanda bir kara deliğin resmini gözlemledi. Bu önemli sonuç, kara delik ufkunun yakınında ince bir yapı ortaya koyuyor. Foton halkası ve gölge açıkça gözlemlendi, bu da bize kuvvetli yerçekimi rejiminde genel göreliliği test etmek için iyi bir fırsat sağladı. Bu tezde önce Schwarzschild kara deliği ve özelliklerini inceliyoruz, ardından kara deliğin güçlü yerçekiminin neden olduğu göksel küredeki iki boyutlu karanlık bölge olan gölge dökümünü bulmak için Euler-Lagrange'dan Schwarzschild metriğinin sıfır jeodeziklerini hesaplıyoruz. Son olarak, kuantum düzeltmeli Schwarzschild kara deliğinin gölgesini inceliyoruz. Bu nedenle kuantumun kara deliğin gölgesi üzerindeki etkisini tartışıyoruz.

Anahtar Kelimeler: Yerçekimi, Kara delikler, Gölge, Sıfır jeodezikleri, Kuantum Düzeltilmiş Kara Delikler.

DEDICATION

This work is dedicated to my family.

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INTRODUCTION

1.1 What is a black hole and why it is so important in the universe?

Black holes are scary, and the same time magnificent cosmic bodies and they seem to absorb everything in the universe. The gravity of black holes pulls so strong. From this gravity no particles and electromagnetic radiation like light can get out. When matter is compressed into a small space the gravity becomes so strong as black hole. If a star is dying this process can happen. Black holes are invisible because no light can get out from it. According to the general relativity, a black hole is formed when a sufficiently compact mass deforms spacetime [1], [2]. Event horizon is the boundary which nothing can get out from it. A black hole does not reflect light like black body so acts as an ideal black body. A black hole shadow is a two-dimensional dark zone in the celestial sphere. The strong gravitational force of the black hole caused it.

Black holes can be small, also very big. Four types of black holes have been identified: miniature, stellar, intermediate and supermassive. Scientists assume that the smallest black holes are very small like one atom. Although these black holes are very small, but their masses are too great. The mass of the “stellar” can be many times greater than the mass of the Sun. In Milky Way (Earth`s) galaxy there are maybe many stellars. There are also in universe some black holes which masses more than one million suns masses together and they are called “supermassive”. There is a supermassive black hole in the center of every large galaxy. The supermassive black hole in the center of the Milky Way galaxy is Sagittarius A*. It is very large, and its mass is about 4 million suns.

1.2 How do black holes form?

The most well-known way about formation of black holes is stellar death. By the end of stars lives, most of them will swell, lose their mass, and then cool to create white dwarfs. However, the largest of these flaming objects which at least 10 to 20 times larger than the Sun, have become either very dense neutron stars or stellar black holes. In their final stages, the giant stars cause supernova (a powerful and bright star explosion). While the star was present, nuclear fusion produced a permanent external push that balanced the star's internal attraction from its own mass. In the stellar remnants of a supernova, there are no longer forces resisting that gravity, thus the star core starts to collapse in on itself. If its mass falls to an infinitely small point, a black hole is formed. Gathering the whole mass which is many times the mass of the Sun into such a small point gives the black holes a strong gravitational force. Once a black hole has formed, this black hole can continue to grow by swallowing the surrounding masses. When black holes absorbing other stars and combining with other black holes, supermassive black holes can form. Some black holes are not of stellar origin. British astrophysicist Stephen Hawking suggested the existence of non-stellar black holes. According to this proposal, during the big bang many little primordial black holes, the mass of which is equal to or less than the mass of an asteroid, might have been formed with extremely high temperatures and density. Over time, these black holes lose mass through Hawking radiation and disappear. The smallest members of the black hole family are still theoretically. 13.7 billion years ago these tiny vortices of darkness came to life shortly after the universe formed with a big bang and then evaporated rapidly.

Astronomers doubt the existence of so-called intermediate black holes in the universe, though evidence about these black holes is still controversial.

1.3 What happens if something goes through the BH, is there a way out of the BH?

When objects like spaceship, planets, stars even light comes too close to a black hole they cannot escape from the strong gravity of the black hole. If they go through the black hole, everything will disappear even if it moves at the speed of light. When stars move too close to a supermassive black hole, they can break up into very bright sparkles. Regardless of the initial size, black holes can grow throughout objects lives, scattering gas and dust from them that passes too close. Everything that crosses the event horizon of the black hole which leads to the point where it is impossible to escape, theoretically designed for spaghettification [3] due to the sharp increase in the power of gravity. Nothing can get out from inside the event horizon. If an event occurs in the event horizon, outside observer cannot get all information from that event. As a result, it is impossible to define whether such an event has taken place. Clocks near a black hole would appear to tick more slowly than far from the black hole, for a distant observer. This effect is called gravitational time dilation [4]. An object which falling into a black hole slowly appears as it approaches the event horizon and takes an infinite amount of time to reach. From the viewpoint of a fixed external observer, all processes on this object slow down and causing any light emitted by the object to appear redder and dimmer, this is called gravitational redshift [5].

1.4 The structure of the black hole

Although the simplest static black holes have no electric charge and angular momentum, but these black holes have mass. According to Karl Schwarzschild these black holes are known as Schwarzschild black holes, who found this solution in 1916 [6]. In the following is described separetly about sructure of a black hole:

Singularity - the whole mass of the black hole is concentrated here. A gravitational singularity at the center of a black hole is a region where the spacetime curvature is infinite.

Photon sphere - the outer edge of the black hole where the light bends, but can still escape. A photon sphere is a spherical boundary of zero thickness. Photons move on tangents to that sphere are caught in a circular orbit around the black hole.

Event horizon – it is a “point of no return” around a black hole. According to the general relativity the existence of a mass deforms spacetime, the paths through which the particles pass bend towards the mass. This deformation is very strong on the event horizon of the black hole that there are no ways run away from the black hole.

Accretion disk – it is a disk of dust, gases, planets and stars which fall into the orbit of a black hole. Matter falling into a black hole can form an outward accretion disk that is heated by friction and forms some of the brightest objects (quasars) in the universe.

1.5 A brief history of the black hole from the past to the present

In 1784, John Michell suggested idea in a published letter, according to this suggestion that even light cannot escape from so massive object. Michell's calculations shows that this body might have the same density as the Sun. When the diameter of the star exceeds the diameter of the Sun by a factor of 500 such a body would form. He also showed that the speed of get out from the surface gets over the speed of light. These bodies are referred to dark star by Michell [7]. John Michell rightly pointed out that such supermassive at the same time non-radiating objects can be detected by the nearby gravitational effects of visible objects [8][9][10]. In 1915, Albert Einstein developed his theory of general relativity, showing that gravity affects the motion of light. After finding an exact solution to Einstein's approximations of general relativity a German astronomer and physicist Karl Schwarzschild proposed the modern solution of a black hole in 1916. Schwarzschild realized that it was possible to compress the mass to an infinitely small point and this would make spacetime around it bend thus nothing, not even massless photons of light, can get rid of its curvature. The part of the black hole where nothing could escape today is called as its event horizon. The distance between event horizon and the infinitely dense core (singularity) is named Schwarzschild radius. The size of the Schwarzschild radius is proportional to the mass of the collapsing star. Theoretically, the Schwarzschild radius of all masses can be calculated. For example, if the mass of the Sun was compressed into an infinitely small point, it would form a black hole with a radius of less than 3 kilometers (about 2 miles). In 1958, the Schwarzschild surface was identified as an event horizon by David Finkelstein and for the first time he published there is a region of space where nothing can run away [11]. Finkelstein's solution presented in more detailed the Schwarzschild solution. This decision supported the observers of black holes for the future investigations. In 1963, Roy Kerr solved the equations of general relativity which described rotating black holes. In 1966 the first time Synge studied black hole shadow for a Schwarzschild black hole. For the first time James Bardeen calculated how the rotation of a black hole affected the shape of the shadow. He got the result is a D-shaped shadow for a rotating black hole close to the maximum angular momentum. In 1967 the term "black hole" was invented by American astronomer John Wheeler [12]. In the same year Jocelyn Bell Burnell discovered pulsars [13], [14] and until 1969, it was shown that there were fast-rotating neutron stars [15]. It started interest in compact objects that had collapsed by gravity as a possible astrophysical reality. In 1971 some researchers independently identified Cygnus X-1, it was the first known black hole [16][17]. In 1972, Jacob Bekenstein thought so black holes should have an entropy [18]. An increase in the entropy of a black hole compensates for a decrease in the entropy carried by the absorbed object. In the same year first accretion disk model was formulated by Shakura (1972) and Sunyaev (1973). In 1974, Stephen Hawking showed that black holes emit thermal Hawking radiation [19] which corresponding to a certain temperature is called Hawking temperature. In 1992, black holes in three dimensions described by M. Banados, C. Teitelboim and J. Zanelli. In 2002, astronomers have concluded that Sagittarius A* is the Milky Way's central supermassive black hole. Sagittarius A * is very compact and the same time very bright astronomical radio source at the center of the Milky Way Galaxy. From 2009 to 2017, Event Horizon Telescope observations reveals image of the M87 * black hole. The Event Horizon Telescope is used because of to get high quality image from widely spaced observatories at different places on Earth. The Event Horizon Telescope (EHT) was designed to capture images of a black hole. The main goal of EHT to catch international collaboration from a planet-scale telescope which consists of eight ground-based radio telescopes. In 2009-2012, the M87 * black hole was observed by early Event Horizon Telescope (EHT) which telescopes located at three geographical areas and four sites in 2013. The first direct detection of gravitational waves was announced in 2016, by the LIGO Scientific Collaboration and Virgo collaboration [20]. It also reflected the first observation of a black hole combination. In 2017, the EHT with further developed telescopes located in five different geographical areas across the globe. In April 2017, the Event Horizon Telescope captured the first detailed images of the shadow of a black hole. In nearby galaxy Messier 87, new radio images of the supermassive black hole released by the Event Horizon Telescope team. The team revealed a bright emission ring surrounding a dark, circular area. Hot radiating gas surrounding the black hole. From this gas photons form the ring of light. The dark region in the center of the black hole is called the “black hole’s “shadow” [29], [30], [31], [32], [33], [34], [35]. This is captured by the black hole and consists of collection of paths of photons. On 10 April 2019, the Event Horizon Telescope (EHT) collaboration broadcast the first images of the radio emission and a dark shadow from the supermassive black hole at the center of galaxy Messier 87 [21], [22], [23]. These high-quality images are the result of development device and hardworking processing of years. This black hole is 55 million light-years from the Earth and 6.5 billion times larger than the Sun. In March 2021, for the first time a polarized-based image of the black hole was presented by the Event Horizon Telescope Collaboration. From 2021 the closest body, thought to be a black hole is about 1500 light-years away. Although only a couple dozen black holes have been found in the Milky Way so far, it is thought that there are hundreds of millions black holes in our galaxy. In July 2021, for the first time the light from behind a black hole was observed by scientists. For to observe the light from behind a black hole researchers used the NASA’s NuSTAR and European Space Agency’s XMM-Newton space telescopes. This black hole is 10 million times bigger than the Sun and lies 800 million light-years away in the spiral galaxy I Zwicky 1. The light "echo" was first predicted by Albert Einstein and published in 1916.

Chapter 2

SCHWARZSCHILD BLACK HOLE

The Schwarzschild metric have been used so far to define the spacetime outside a spherical star. In this chapter, will be explored the geometry of spacetime with the hypothesis that the Schwarzschild solution is valid everywhere. It is known that in the last stage of the life of a very massive hot star, it will undergo complete gravitational collapse in order to form a black hole. Schwarzschild geometry describes the simplest black hole solution.

The Schwarzschild solution in Schwarzschild coordinates ***(t, r, θ, ϕ)*** is

$$dS^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (2.1)$$

This is a one-parameter family of solutions. The parameter M take either sign but also it has the interpretation of a mass thus $M > 0$ here. For $0 < r < 2M$ this metric is also a solution of the vacuum Einstein equation. The Schwarzschild radius is $r = 2M$.

Theorem (Birkhoff) *- Any spherically symmetric solution of the vacuum Einstein equation is isometric to the Schwarzschild solution [24].*

Only spherical symmetry is assumed in this teorem, but there is an additional isometry in the Schwarzschild solution: ∂/∂t - hypersurface-orthogonal Killing vector field. For $r > 2M$ it is timelike, thus the $r > 2M$ Schwarzschild solution is static.

Birkhoff's theorem means that the time-independent (exterior) Schwarzschild solution described spacetime outside any spherical body. If the body itself is time-dependent this is also true. For instance, a spherical star which "uses up its nuclear fuel" collapses to form neutron star or a white dwarf. Even during the collapse, the static Schwarzschild solution describes the spacetime outside the star.

2.1 Derivation of the Schwarzschild black hole

2.1.1 Schwarzschild`s formulation of the problem

The conditions about the metric outside a spherically symmetric, static star:

1. With time the metric does not change.
2. The metric is spherically symmetric.
3. Far from the star the metric must be the same as Newton`s gravity.

In the Newtonian limit:

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}$$

$\phi = -GM/(rc^2) = -M/r$ - Newton`s gravitational potential. Space is not curved in Newton`s laws.

4. Suppose the metric is

$$dS^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (2.2)$$

5. In the space outside the star Einstein`s field equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \qquad (2.3)$$

$G_{\mu\nu}$ - Einstein`s curvature tensor.

Contract:

$$g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)= -8\pi G g^{\mu\nu}T_{\mu\nu} \tag{2.4}$$

$$g^{\mu\nu}g_{\mu\nu}=4 \tag{2.5}$$

$$R-\frac{1}{2}4R=-8\pi GT_{\alpha}^{\alpha} \tag{2.6}$$

$$R=8\pi GT_{\alpha}^{\alpha} \tag{2.7}$$

$$R_{\mu\nu}=-8\pi GT_{\mu\nu}+\frac{1}{2}g_{\mu\nu}8\pi GT_{\alpha}^{\alpha} \tag{2.8}$$

$$R_{\mu\nu}=-8\pi G\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T_{\alpha}^{\alpha}\right) \tag{2.9}$$

The stress-energy tensor in the space outside a star:

$$T^{\mu\nu}=\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & P & 0\\ 0 & 0 & 0 & P\end{pmatrix}$$

is zero, therefore

$$R_{\mu\nu}=0 \tag{2.10}$$

2.1.2 Calculation in outline

The metric is

$$g_{\mu\nu}=\begin{pmatrix}-B(r) & 0 & 0 & 0\\ 0 & A(r) & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2 sin^2\theta\end{pmatrix}$$

Calculate the Christoffel symbols

$$\Gamma_{\lambda\mu}^{\sigma}=\frac{1}{2}g^{\nu\sigma}\left(g_{\mu\nu,\lambda}+g_{\lambda\nu,\mu}-g_{\mu\lambda,\nu}\right) \tag{2.11}$$

Calculate the Ricci tensor

$$R_{\mu k}=\frac{\partial}{\partial x^k}\Gamma_{\mu\lambda}^{\lambda}-\frac{\partial}{\partial x^\lambda}\Gamma_{\mu k}^{\lambda}+\Gamma_{\mu\lambda}^{\eta}\Gamma_{k\eta}^{\lambda}-\Gamma_{\mu k}^{\eta}\Gamma_{\lambda\eta}^{\lambda} \tag{2.12}$$

The condition $R_{\mu\nu}=0$ puts restrictions on $A(r)$ and $B(r)$.

2.1.3 Christoffel symbols

Using the equation (2.11):

$$\Gamma^r_{rr} = \frac{A'}{2A} \tag{2.13}$$

$$\Gamma^r_{rr} = \frac{1}{2} g^{\nu r}\left(g_{r\nu,r} + g_{r\nu,r} - g_{\nu r,r}\right) \tag{2.14}$$

$$\Gamma^r_{rr} = \frac{1}{2}\frac{1}{A}A' + \frac{1}{2}\frac{1}{A}A' - \frac{1}{2}\frac{1}{A}A' = \frac{1}{2}\frac{1}{A}A' \tag{2.15}$$

Other terms:

$$\Gamma^r_{\theta\theta} = -\frac{r}{A} \tag{2.16}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \tag{2.17}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{2.18}$$

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B} \tag{2.19}$$

$$\Gamma^r_{tt} = \frac{1}{2}\frac{B'}{A} \tag{2.20}$$

$$\Gamma^r_{\phi\phi} = -\frac{r sin^2\theta}{A} \tag{2.21}$$

$$\Gamma^\theta_{\phi\phi} = -sin\theta cos\theta \tag{2.22}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = cot\theta \tag{2.23}$$

If we choose from Hartle book:

$$A(r) = e^{\lambda(r,t)} \tag{2.24}$$

$$B(r) = e^{\nu(r,t)} \tag{2.25}$$

Here $\frac{\partial}{\partial t}\,\nu(r,t) = 0$ and $\frac{\partial}{\partial t}\,\lambda(r,t) = 0$.

2.1.4 Ricci tensor

Using the equation (2.12), the Ricci tensor

$$R_{rr} = \frac{1}{2}\frac{B''}{B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A} \tag{2.26}$$

$$R_{tt} = -\frac{1}{2}\frac{B''}{A} + \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{A} \tag{2.27}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} \tag{2.28}$$

$$R_{\phi\phi} = sin^2\theta R_{\theta\theta} \tag{2.29}$$

The non-diagonal terms are zero. Get rid of B``:

$$\frac{R_{rr}}{A} = \frac{1}{2}\frac{B''}{BA} - \frac{1}{4}\frac{B'}{BA}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A^2} \tag{2.30}$$

$$\frac{R_{tt}}{B} = -\frac{1}{2}\frac{B''}{BA} + \frac{1}{4}\frac{B'}{BA}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{BA} \tag{2.31}$$

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{2.32}$$

Then

$$d\ log\ A + d\ log\ B = d\ log\ (AB) = 0 \tag{2.33}$$

Therefore

$$A(r)B(r) = constant \tag{2.34}$$

At $r \to \infty$, the Newtonian limit yields

$$AB \to 1 - 2M/r \to 1 \tag{2.35}$$

Consequently

$$AB = 1 \tag{2.36}$$

Consider

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = 0 \tag{2.37}$$

Since $AB = 1$

$$\frac{A'}{A} = -\frac{B'}{B} \tag{2.38}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(2\frac{B'}{B}\right) + \frac{1}{A} = -1 + rB' + B = -1 + \frac{d}{dr}(rB) = 0 \tag{2.39}$$

Integrate to get

$$rB = r + const \tag{2.40}$$

or

$$B(r) = 1 + \frac{const}{r} \tag{2.41}$$

At ∞,

$$B(r) = \frac{1}{A(r)} \to 1 + 2\phi = 1 - 2\frac{M}{r} \tag{2.42}$$

In conclusion,

$$B(r) = 1 - 2\frac{M}{r} \tag{2.43}$$

$$A(r) = (1 - 2\frac{M}{r})^{-1} \tag{2.44}$$

Spacetime is ensured with a metric tensor $g_{\mu\nu}$ thus that a line element has length

$$dS^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{2.45}$$

In flat spacetime,

$$dS^2 = -dt^2 + dx^2 \tag{2.46}$$

(x ϵ $\mathbb{R}^3$), thus $g_{\mu\nu} = \eta_{\mu\nu} = diag(-1111)$ like a matrix. The determinant of $g_{\mu\nu}$ is denoted by g. The Eistein equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \begin{cases} 0, & with\ just\ gravity\ not\ matter \\ source, & in\ the\ presence\ of\ matter \end{cases} \tag{2.47}$$

$R_{\mu\nu}$ – the Ricci tensor ($R^{\nu}_{\mu\nu\rho}$ - the contracted curvature tensor). Its trace is $R = R^{\mu}_{\mu}$ - the Ricci scalar. $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$ becomes from the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R \tag{2.48}$$

If the trace of the Einstein's equation is taken in empty space:

$$R_\mu^\mu - \frac{1}{2} R g_\mu^\mu = 0 \tag{2.49}$$

which means $R = 0$ thus $R_{\mu\nu} = 0$. Schwarzschild found the solution

$$dS^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{2.50}$$

$$d\Omega_2^2 = d\theta^2 + sin^2\theta d\varphi^2 \tag{2.51}$$

$$f(r) = 1 - \frac{2GM}{r} \tag{2.52}$$

In General Relativity

$$g_{tt} \approx -1 + 2V(r) \tag{2.53}$$

$$-1 + 2V(r) = 1 - \frac{2GM}{r} \tag{2.54}$$

$$V(r) = \frac{GM}{r} \tag{2.55}$$

$V(r)$ - the Newtonian potential. In the Schwarzschild solution if $r \rightarrow 0$ then $f(r) \rightarrow \infty$ (2.52) which is a true singularity. If $r = 2GM \equiv r_+$ it is called the horizon. Inside ($r < r_+$) even light can not run away. When $f(r_+) = 0$, proper time for an observer approaching the horizon ($dS^2 = -f(r)dt^2 \Rightarrow \sqrt{-ds^2} = \sqrt{f(r)}dt$) passes quickly while to a distant observer it appears to take an infinite time to reach the horizon.

To determine $\tau = it$ thus Minkowski space becomes Euclidean

$$ds^2 = d\tau^2 + dx^2 \tag{2.56}$$

and the Schwarzschild metric:

$$dS^2 \approx \left(1 - \frac{r_+}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)} + r^2 d\Omega_2^2 \tag{2.57}$$

if $r = r_+ + \varepsilon$ is outside but close to the horizon ($\varepsilon > 0$)

$$dS^2 \approx \frac{\varepsilon}{r_+}d\tau^2 + \frac{r_+}{\varepsilon}d\varepsilon^2 + {r_+}^2 d\Omega_2^2 \tag{2.58}$$

ε and τ do not depend on Ω thus the space near the horizon separates into a sphere S^2 of radius r_+ and a 2D manifold of metric

$$dS_2^2 = \frac{\varepsilon}{r_+}d\tau^2 + \frac{r_+}{\varepsilon}d\varepsilon^2 \tag{2.59}$$

If the coordinates are changed to $\rho = 2\sqrt{r_+\varepsilon}$ and $\chi = \frac{\tau}{2r_+}$:

$$dS_2^2 = \rho^2 d\chi^2 + d\rho^2 \tag{2.60}$$

The spacetime is a cone. If, χ is restricted to be between 0 and 2π thus τ is restricted to be between 0 and $4\pi r_+$ as a result it is to be a plane. The temperature of the black hole:

$$T = \frac{1}{4\pi r_+} \tag{2.61}$$

since $r_+ = 2GM$

$$T = \frac{1}{8\pi GM} \tag{2.62}$$

it is called Hawking temperature. In termodynamics, $dU = TdS$, it is known that U - the total energy. U must be the mass in this situation, so

$$dM = \frac{1}{8\pi GM}dS \tag{2.63}$$

$$S = \int 8\pi GM dM = 4\pi GM^2 \tag{2.64}$$

$$S = \frac{4\pi r_+^2}{4G} = \frac{A_+}{4G} \tag{2.65}$$

It is called the Bekenstein-Hawking formula and A_+ - the area of the horizon.

We know that entropy is proportional to volume and so mass. If N is count of particles, n degrees of freedom in an ordinary star and there are n^N possible states as a result entropy S is proportional to $N \ln n$ which is proportional to mass M (also volume). It leads to: the entropy is proportional to the square of the mass and to surface area. Since $T = \frac{1}{8\pi GM}$ if $T \to 0$ then $M \to \infty$ as a result entropy is increasing. The heat capacity:

$$C = \frac{dM}{dT} = \frac{d}{dT}\left(\frac{1}{8\pi GT}\right) = -\frac{1}{8\pi GT^2} \tag{2.66}$$

Since $T = \frac{1}{8\pi GM}$

$$C = -8\pi GM^2 < 0 \tag{2.67}$$

expresses an unstable thermodynamic system. Then free energy F is given by

$$F = M - TS = M - \frac{1}{8\pi GM}\frac{4\pi G^2 M^2}{G} = \frac{1}{2}M \tag{2.68}$$

Chapter 3

NULL GEODESICS AND SHADOW OF THE SCHWARZSCHILD BLACK HOLE

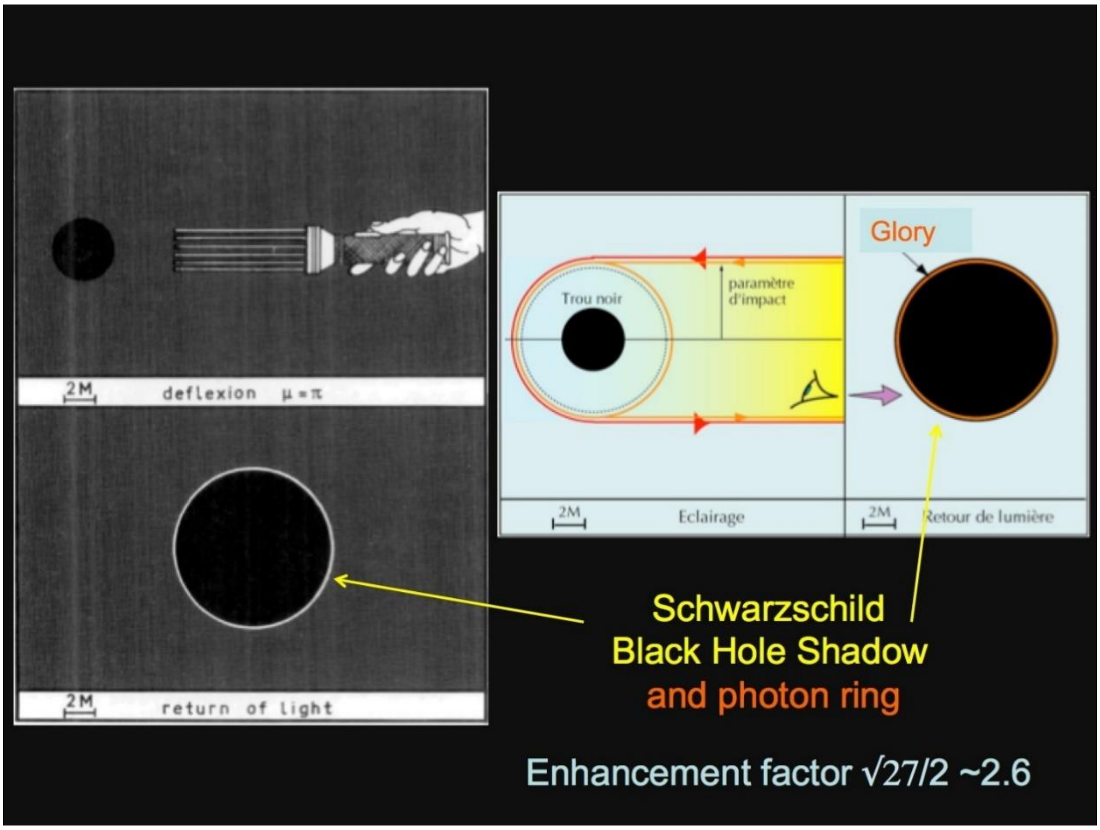

Figure 3.1: Schwarzschild black hole shadow and photon ring [27]

3.1 Geodesic equations from a variational principle

According to the Einstein's theory, particles do not fall under the action of a force. The motion of these particles corresponds to the trajectory in spacetime that extremizes the proper interval between two events. Geodesic equations or the equations of motion, can be obtained by a variational principle i.e., by extremizing the action.

$$\int d\lambda 2\mathcal{L} \equiv \int d\lambda \left(g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right) \tag{3.1}$$

$\mathcal{L}$ - Lagrangian density, it was included due to convenience. The main task is to write down the usual Euler-Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}} = \frac{\partial \mathcal{L}}{\partial x^{\alpha}} \tag{3.2}$$

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \tag{3.3}$$

$$\dot{x}^{\mu} \equiv \frac{dx^{\mu}}{d\lambda} = p^{\mu} \tag{3.4}$$

The canonical momenta related with this Lagrangian are:

$$p_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}} \tag{3.5}$$

$$p_{\mu} = g_{\mu\nu} \dot{x}^{\nu} = g_{\mu\nu} p^{\nu} \tag{3.6}$$

Introducing the notation $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}} = g_{\mu\alpha} \dot{x}^{\mu} \tag{3.7}$$

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}} = g_{\mu\alpha} \ddot{x}^{\mu} + \dot{x}^{\mu} \frac{d g_{\mu\alpha}}{d\lambda} = g_{\mu\alpha} \ddot{x}^{\mu} + \dot{x}^{\mu} \dot{x}^{\nu} \partial_{\nu} g_{\mu\alpha} \tag{3.8}$$

$$\frac{\partial \mathcal{L}}{\partial x^{\alpha}} = \frac{1}{2} \partial_{\alpha} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \tag{3.9}$$

So

$$g_{\mu\alpha} \ddot{x}^{\mu} + \dot{x}^{\mu} \dot{x}^{\nu} \partial_{\nu} g_{\mu\alpha} - \frac{1}{2} \dot{x}^{\mu} \dot{x}^{\nu} \partial_{\alpha} g_{\mu\nu} = 0 \tag{3.10}$$

μ and ν - dummy indices. By symmetrizing one gets $2\dot{x}^{\mu}\dot{x}^{\nu}\partial_{\nu}g_{\mu\alpha} = \dot{x}^{\mu}\dot{x}^{\nu}(\partial_{\nu}g_{\mu\alpha} + \partial_{\mu}g_{\nu\alpha})$ in conclusion

$$2g_{\mu\alpha}\ddot{x}^{\mu} + \dot{x}^{\mu}\dot{x}^{\nu}\left(\partial_{\nu}g_{\mu\alpha} + \partial_{\mu}g_{\nu\alpha} - \partial_{\alpha}g_{\mu\nu}\right) = 0 \tag{3.11}$$

According to the definition of the Christoffel symbols

$$\Gamma^{\mu}_{\beta\gamma} = \frac{1}{2}g^{\mu\rho}\left(\partial_{\gamma}g_{\beta\rho} + \partial_{\beta}g_{\gamma\rho} - \partial_{\rho}g_{\beta\gamma}\right) \tag{3.12}$$

it leads to

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0 \tag{3.13}$$

i.e. the usual geodesic equation.

3.2 Geodesics in static, spherically symmetric spacetimes

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.14}$$

This metric can define (for instance) a spherical star and when $f(r) = h(r) = 1 - 2M/r$ it reduces to the Schwarzschild metric. In the following the r-dependence of $f(r)$ and $h(r)$ will be omitted for convenience. So for geodesic motion the Lagrangian reduces to

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = -f\dot{t}^2 + h^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2\right) \tag{3.15}$$

The momenta related with this Lagrangian are

$$p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -f\dot{t} \tag{3.16}$$

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = h^{-1}\dot{r} \tag{3.17}$$

$$p_{\theta} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = r^2\dot{\theta} \tag{3.18}$$

$$p_{\phi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2 sin^2\theta\dot{\phi} \tag{3.19}$$

By a Legendre transform the Hamiltonian is associated with the Lagrangian

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \mathcal{L} = 2\mathcal{L} - \mathcal{L} = \mathcal{L} \tag{3.20}$$

it is equal to the Lagrangian, there is no potential energy contribution therefore the Lagrangian is "purely kinetic". As a result $\mathcal{H} = \mathcal{L}$ and both of them are constant. The Lagrangian does not depend on t and ϕ, because the metric is static and spherically symmetric. For this reason the t and ϕ give two conserved quantities E and L:

$$\frac{dp_t}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0 \Longrightarrow -p_t = f\dot{t} = E \tag{3.21}$$

$$\frac{dp_\phi}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} = 0 \Longrightarrow p_\phi = r^2 sin^2\theta \dot{\phi} = L \tag{3.22}$$

The radial equation

$$\frac{dp_r}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r} \tag{3.23}$$

The remaining equation leads to

$$\frac{dp_\theta}{d\lambda} = \frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta} = r^2 sin\theta cos\theta \dot{\phi}^2 \tag{3.24}$$

if $\theta = \pi/2 \Rightarrow \dot{\theta} = 0 \Rightarrow \ddot{\theta} = 0$ and the orbit will be limited to the equatorial plane $\theta = \pi/2$ always. Finally:

$$f\dot{t} = E \tag{3.25}$$

$$r^2\dot{\phi} = L \tag{3.26}$$

E - the energy of the particle and L - the orbital angular momentum. If there are massive particles, E and L should be explained like the particle energy and orbital angular momentum per unit rest mass: $E = E_{particle}/m_0$ and $L = L_{particle}/m_0$. The constancy of the Lagrangian now means

$$\frac{E^2}{f} - \frac{\dot{r}^2}{h} - \frac{L^2}{r^2} = -2\mathcal{L} = \delta_1 \tag{3.27}$$

for null orbits the constant $\delta_1 = 0$ and for timelike orbits $\delta_1 = 1$. The conservation of the Lagrangian is equivalent to the four-velocity of the particle`s the normalization condition: $p_\mu p^\mu = -\delta_1$, for massless particles $\delta_1 = 0$ but for massive particles $\delta_1 = 1$. This "effective potential" equation is used to study the qualitative characteristics of geodesic motion:

$$\dot{r}^2 = V_r \tag{3.28}$$

$$V_r \equiv h\left(\frac{E^2}{f} - \frac{L^2}{r^2} - \delta_1\right) \tag{3.29}$$

For null (timelike) geodesics $\delta_1 = 0$.

3.3 Geodesics of Schwarzschild metric from Euler-Lagrange

The Euler-Lagrange equations will be used to find the geodesics of the Schwarzschild metric. Note that the Schwarzschild metric is the unique metric around spherically symmetric, stationary, uncharged objects. These geodesics give information about how thing move around the Sun, the Earth and the black holes (uncharged, nonspinning objects). These are the General Relativistic extension of Newton`s and Kepler`s laws of motion. The Schwarzschild metric is

$$dS^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + \\ + r^2 sin^2\theta d\phi^2 \tag{3.30}$$

If $c = 1$ and the affine parameter $\lambda = s = \tau$ and $s = \int L d\tau$ with

$$L = 1 = \left[\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{r}} - r^2\dot{\theta}^2 - r^2 sin^2\theta\dot{\phi}^2\right]^{\frac{1}{2}} \tag{3.31}$$

For t the Euler-Lagrange equation

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0 \tag{3.32}$$

since $\frac{\partial L}{\partial t} = 0$,

$$\frac{\partial L}{\partial \dot{t}} \equiv \frac{E}{m} \tag{3.33}$$

it is a conserved quantity and known as energy per unit mass. Considering

$$\frac{\partial L}{\partial \dot{t}} = \frac{1}{2}[\dots]^{-\frac{1}{2}}\left(1 - \frac{2GM}{r}\right)2\dot{t} \tag{3.34}$$

For brevity $L = [\dots]^{1/2}$. t equation is found by using $L = 1$

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = \frac{E}{m} \tag{3.35}$$

If $r \to \infty$ the Schwarzschild metric goes to the Minkowski metric and $\dot{t} = E/m$ for the Minkowski metric. Then the ϕ equation is found:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0 \tag{3.36}$$

Since the metric is clearly independent on the angle ϕ

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \equiv -\frac{l}{m} \tag{3.37}$$

it is the angular momentum per unit mass (conserved quantity).

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}[\dots]^{-\frac{1}{2}}\left(-r^2 sin^2\theta 2\dot{\phi}\right) = -r^2 sin^2\theta\dot{\phi} \tag{3.38}$$

So the ϕ equation reads

$$\frac{l}{m} = r^2 sin^2\theta\dot{\phi} \tag{3.39}$$

To obtain θ equation

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \neq 0 \tag{3.40}$$

so there is no a conserved quantity for this equation.

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}[\dots]^{-\frac{1}{2}}\left(-r^2\dot{\phi}^2 2sin\theta cos\theta\right) = -r^2\dot{\phi}^2 sin\theta cos\theta \tag{3.41}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}[\dots]^{-\frac{1}{2}}\left(-r^2 2\dot{\theta}\right) = -r^2\dot{\theta} \tag{3.42}$$

So θ equation reads:

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) = r^2\dot{\phi}^2\ sin\theta cos\theta \tag{3.43}$$

At last, the r equation should be done, which is kind of mixed because of all the explicit r dependence in the metric. But, it is not necessarily, because the fourth equation can be obtained by the equatuions of motion from our definition of the Lagrangian $L^2 = 1$:

$$1 = \left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{\left(1 - \frac{2GM}{r}\right)} - r^2\dot{\theta}^2 - r^2sin^2\theta\dot{\phi}^2 \tag{3.44}$$

Since the object and metric are spherically symmetric the things can be simplified just considering motion in the equatorial plane, means $\theta = \pi/2 \Rightarrow \dot{\theta} = 0$. As a result, the equation $L = 1$ becomes:

$$1 = \left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{\left(1 - \frac{2GM}{r}\right)} - r^2\dot{\phi}^2 \tag{3.45}$$

By multiplying $m\dot{r}^2$ the equation becomes

$$m\dot{r}^2 = \frac{E^2}{m} - \left(m + \frac{l^2}{mr^2}\right)\left(1 - \frac{2GM}{r}\right) \tag{3.46}$$

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)\left(mc^2 + \frac{l^2}{mr^2}\right) \tag{3.47}$$

here c is a speed of photon. Consider the angular momentum $l = 0$ which can be expected for radial infall towards a spherical mass.

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)mc^2 = 0 \tag{3.48}$$

If $r \to \infty$ where the action is started at rest far from the object thus the proper time $\tau = 0$. $m(dr/d\tau)^2 = 0$. So the equation:

$$\frac{E^2}{mc^2} - \left(1 - \frac{2GM}{\infty}\right)mc^2 = 0 \tag{3.49}$$

or

$$\frac{E^2}{mc^2} = mc^2 \tag{3.50}$$

or

$$E = mc^2 \tag{3.51}$$

If $r \to \infty$ but the action is started with some velocity then $E > mc^2$ and this energy still would have been conserved. Then during this radial infall from rest, the equation becomes:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - mc^2 + \frac{2GM}{rc^2} mc^2 = \frac{2GMm}{r} \tag{3.52}$$

$$\frac{1}{2} m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0 \tag{3.53}$$

Although the equation (3.53) looks like the Newtonian case $\frac{1}{2}mv^2 - \frac{GMm}{r} = 0$, but these equations are not the same. The difference is that in our last equation the time derivative is τ (not t) and it is relativistic. By integrating these 4 equations, a complete view of motion can be obtained near the Earth, Sun or Black Hole.

3.4 Distances and times around a black hole

Consider an object flying close to a small black hole which mass is equal to three times Solar mass ($M = 3M_\odot$) [36]. The Schwarzschild radius of the Sun is $\frac{2GM}{c^2} = 2.95 \times 10^3 m$ since $M_\odot = 1.99 \times 10^{30} kg$. The Schwarzschild radius is about 3×2.95 $km = 8.85$ km for this black hole. Therefore, we pay attention to keep the object farther away from the black hole than this. Consider the object is flying all the way around the black hole and measuring a distance around (circumference) of $C = 2\pi 30\ km = 188{,}5$ km. Such a question arises, how far is the object from the black hole? For the simplest, it is expected that the object is 30 km from the black hole, but this should be checked by using the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + \tag{3.54}$$

$$+r^2 sin^2\theta d\phi^2$$

If $dt = dr = d\phi = 0$, the proper distance in the θ direction

$$dl_\theta = \sqrt{ds^2} = rd\theta \tag{3.55}$$

In flat space this is the same as the proper distance. Thus in the θ (or ϕ) direction there is no curvature. The distance around the black hole is obtained from

$$C = \int_0^{2\pi} dl_\theta = 2\int_0^{\pi} rd\theta = 2\pi r \tag{3.56}$$

exactly in flat space. So if the distance is measured around as $2\pi\ 30km$ it is known that the object is at radial coordinate $r = 30\ km$.

But does that mean the object is $30\ km$ from the center of the black hole? The metric must be used again to find out. We can set $dt = d\theta = d\phi = 0$ to find the radial direction

$$dl_r = \sqrt{ds^2} = \frac{dr}{(1 - \frac{2GM}{r})^{\frac{1}{2}}} \tag{3.57}$$

To find the distance from radial coordinate r_1 to radial coordinate r_2 it is integrated from r_1 to r_2. If $r_1 = 8.85km$ and $r_2 = 30km$, the distance from the object to the black hole horizon can be found

$$\Delta l_r = \int_{r_1}^{r_2} dr\,(1 - \frac{2GM}{r})^{-\frac{1}{2}} \tag{3.58}$$

$$A_i \equiv \sqrt{1 - \frac{2GM}{r_i c^2}} = \sqrt{1 - \frac{r_S}{r_i}} \tag{3.59}$$

the integral is evaluated as

$$\Delta l_r = \int_{r_1}^{r_2} dl_r = r_2 A_2 - r_1 A_1 + \frac{r_S}{2} \ln \left(\frac{r_2 A_2 + r_2 - \frac{r_S}{2}}{r_1 A_1 + r_1 - \frac{r_S}{2}} \right) \tag{3.60}$$

$r_S = 2GM/c^2 = 2.95326\ M/M_{\odot}$ km - the Schwarzschild radius. The distances can be found using this formula.

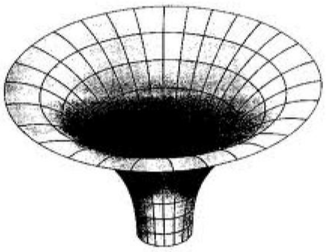

Figure 3.2: The figure describes embedding diagram of curved space close to a black hole

With the embedding diagram the spatial curvature around a black hole can be visualized. The main point in this diagram is that the radial coordinate is only the straight 3-D distance, however along the curved surface the proper distance is measured. This metric and embedding diagram work also for the Earth.

3.5 Can you fall into a black hole?

That was showed above in the θ and ϕ directions the space is flat. With proper distance $dl_r = dr/\sqrt{1 - r_S/r}$ the space is curved in the radial direction. From this equation

$$dr = dl_r \sqrt{1 - r_S/r} \tag{3.61}$$

Let`s find what happens close to the Schwarzschild radius r_S. If $r \to r_S$ then $dr \to 0$, means you are not moving at all in the radial coordinate. Does this mean that you can not get into the black hole? The answer is no, because it is a square-root singularity.

3.6 Time to fall into a black hole

Let`s return the geodesics for radial infall. If the angular momentum $l = 0$ the conserved energy was $E = mc^2$ when the object was started at rest from $r = \infty$. Then the equation of motion was (3.53):

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0 \tag{3.53}$$

For to find the proper time τ it takes for this freely falling object to go from $r_0 = 30\ km\ to\ r = r_S = 8.85\ km$ (the Schwarzschild radius of a $3M_\odot$ black hole). A falling guy`s wristwatch measured this time. In this case r is decreasing because of falling in as a result the negative square root of this equation is taken:

$$\frac{dr}{d\tau} = -\sqrt{2GM}r^{-\frac{1}{2}} \tag{3.62}$$

or

$$\int_{30}^{r} dr\, r^{\frac{1}{2}} = \int_{0}^{\tau} -\sqrt{2GM}d\tau \tag{3.63}$$

or

$$\frac{2}{3}r^{3/2}\Big|_{r_0}^{r} = -\tau\sqrt{2GM} \tag{3.64}$$

or

$$\tau = \frac{2}{3c}\left(\frac{1}{\sqrt{\frac{2GM}{c^2}}}\right)\left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}\right) \tag{3.65}$$

(For to get the correct units and to simplify the calculation c is put back). Thus to go from $r_0 = 30\ km\ to\ r = r_S = 8.85\ km$ in $3M_\odot$ black hole

$$\tau = \frac{2}{3}(1/\sqrt{(2.95)(3)km})\frac{((30km)^{\frac{3}{2}} - (8.85km)^{\frac{3}{2}})}{3 \times 10^5\frac{km}{s}} = 1.03\times 10^{-4}s \tag{3.66}$$

As a result it takes about 0.1 millisecond. From these equations can also be found how long the falling person has to live before hitting the singularity at the center of the back hole. If $r = 0$

$$\tau = \frac{2}{3}\frac{r_0^{3/2}}{\sqrt{2GM}} = 0.124\, ms \tag{3.67}$$

Thus the person gets only an extra 0,021 ms to live inside the black hole.

3.7 Shooting light into a black hole

For light the invariant interval (the metric distance) ds^2 and proper time is 0. If $d\phi = d\theta = 0$ the equation becomes

$$-\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 = 0 \tag{3.68}$$

$$\left(1 - \frac{r_S}{r}\right) dt^2 = \frac{dr^2}{\left(1 - \frac{r_S}{r}\right)} \tag{3.69}$$

$$dt = \frac{dr}{\left(1 - \frac{r_S}{r}\right)} \tag{3.70}$$

Integrating both sides from $t = 0$ at $r = r_0\ to\ t = t$ at $r = r$ can be get

$$t = r_0 - r + r_S \ln\left(\frac{r_0 - r_S}{r - r_S}\right) \tag{3.71}$$

From previous calculating it was expected that the time for light go from $r = 30\ km$ to $r = r_S$ to be less than a millisecond. However, by plugging into the equation (3.71): $ln((30 - 8.85)/(8.85 - 8.85)) \to ln(\infty) \to \infty$. Because of a logarithmic divergence it seems that never can be get into the black hole.

What would happen if someone tried to slowly lower himself into a black hole with a very strong rope? The effective g which is $GM/r^2 = 9.8\ m/s^2$ here on Earth becomes

$$g = (GM/r^2)\left(1 - \frac{r_S}{r}\right)^{-\frac{1}{2}} \tag{3.72}$$

At $r = r_S$ the force becomes infinitely strong as a result the rope will break.

3.8 Orbits in the Schwarzschild metric

For to find the orbits around a black hole or other spherical object recall the geodesic equations. Untill now only radial orbits were considered. To extend this to circular and elliptical orbits and also make calculation easier here will be used the method of effective potential.

3.8.1 Effective potential for Newtonian orbits

In Newtonian mechanics, in spherical coordinates the total energy is $E = T + V = \frac{1}{2}mv^2 - GM/r$. In 3-D non-relativistic mechanics $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$ and $\dot{z} = dz/dt$. Here $x = rsin\theta cos\phi$, $y = rsin\theta sin\phi$, $z = rcos\theta$. To simplify the method, the motion can be considered in the $x - y$ plane thus $z = 0, \theta = \pi/2 \Rightarrow \dot{\theta} = 0$. Next $\dot{x} = \dot{r}cos\phi - rsin\phi\dot{\phi}$, $\dot{y} = \dot{r}sin\phi + rcos\phi\dot{\phi}$ and $\dot{z} = 0$. Substituting this into the formula for kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) =$$

$$= \frac{1}{2}m\big((\dot{r}cos\phi - rsin\phi\dot{\phi})^2 + (\dot{r}sin\phi + rcos\phi\dot{\phi})^2\big) = \tag{3.73}$$

$$= \frac{1}{2}m(\dot{r}^2cos^2\dot{\phi} - 2r\dot{r}sin\phi cos\phi\dot{\phi} + r^2sin^2\phi\dot{\phi}^2 + \dot{r}^2sin^2\phi +$$

$$+2r\dot{r}sin\phi cos\phi\dot{\phi} + r^2cos^2\phi\dot{\phi}^2)$$

using $sin^2\phi + cos^2\phi = 1$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \tag{3.74}$$

$l = mv_\phi r = m\dot{\phi}r^2$ - the angular momentum. The total energy

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\big(\dot{r}^2 + r^2\dot{\phi}^2\big) - \frac{GMm}{r} \tag{3.75}$$

Since $r\dot{\phi} = \frac{l}{mr}$

$$E = \frac{1}{2}m\left(\dot{r}^2 + \frac{l^2}{m^2r^2}\right) - \frac{GMm}{r} \tag{3.76}$$

or

$$E = \frac{1}{2}m(\frac{dr}{dt})^2 + V_{eff} \tag{3.77}$$

with

$$V_{eff} = -\frac{GMm}{r} + \frac{l^2}{2r^2m} \tag{3.78}$$

So the equation of motion have been written in the radial coordinate r_r with energy $E = T + V_{eff}$ as a one dimensional equation. According to the total energy, the particle may run away the well and travel off to infinity or can stay in the well and oscillate back and forth in the bottom of the well. The particle might be at rest at the bottom for small enough energy. At $r = 0, V_{eff}$ drops from infinity, reaching $r = r_{min}$ and so $V_{eff} = 0\ at\ r = \infty$. Conserved quantities $(l, E\ and\ M)$ affect the exact shape. If the particle as rest at the minimum this is the easiest solution to this one dimensional problem. In this time $r = constant$ means the particle only sits there. This is the circular orbit of Kepler`s laws. By solving $dV_{eff}/dr = 0$ the minimum of the effective potential then the radius of the circular orbit is found. This leads to $0 = GMm/r^2 - l^2/(mr^3)$ or $r = GM/v^2$, where before solving for r was substituted back in $l = mvr$. This is the result was obtained from $F = ma$ as known as $GMm/r^2 = mv^2/r$. In this time the total energy is at its minimum $E_{min} = -mv^2 + \frac{1}{2}mv^2 = -\frac{1}{2}mv^2$ and it is less than zero.

If $E < 0$, thus the particle oscillates around the bottom between turning points r_1 and r_2 where $E = \frac{1}{2}m\dot{r}^2 + V_{eff}$. These orbits are the elliptical Kepler orbits. If $E > 0$, these are the hyperbolic Kepler orbits, which come in from infinity reach the nearest approach point and then return to infinity. At last, if $E = 0$ the particle will do similar things, however the orbit will be parabolic.

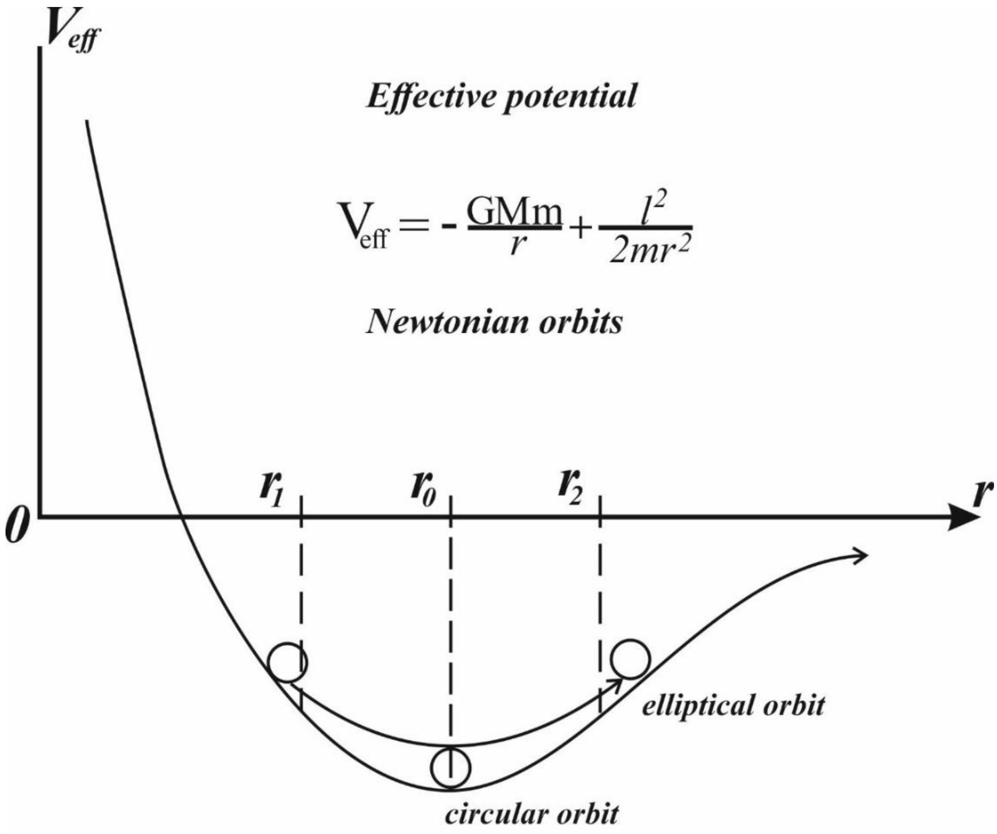

Figure 3.3: Effective potential for Newtonian orbits

3.8.2 Effective potential for Schwarzschild orbits

The same effective potential method will be applied to the full r geodesic equation

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - (1 - \frac{r_S}{r})(mc^2 + \frac{l^2}{mr^2}) \tag{3.79}$$

It is known that $r_S = 2GM/c^2$. The effective potential is

$$V_{eff} = \left(1 - \frac{r_S}{r}\right)\left(mc^2 + \frac{l^2}{mr^2}\right) = m - \frac{r_S}{r}m + \frac{l^2}{mr^2} - \frac{r_S l^2}{mr^3} \tag{3.80}$$

τ – the proper time, l - the angular momentum and E - the total energy.

There are five types of orbits:

1. There is a bound circular orbit at the minimum. The radius is determined from $dV_{eff}/dr = 0$.
2. There are also the bound orbits with turning points $r_1\ and\ r_2$ as in the Newtonian case, but not elliptical.
3. There are orbits that start at $r = \infty$ to reach a distance of nearest approach and get out again as for Newtonian hyperbolic orbits.
4. If $dV_{eff}/dr = 0$ the maximum point can be found on the top of the hill, since the slope is zero here. A particle will remain on top of the hill, if it carefully balanced there. So this is unstable circular orbit. If a particle is affected to the outward a little, the particle will run away to infinity. On the contrary, if a particle is affected to the inward a little, the particle will fall in the hole.
5. The capture orbit is another new type of orbit which not found in the Newtonian case. The particle passes through the highest point and then falls into the hole so as not to return. This can happen when the energy is high enough.

The radii of the circular orbit is found below

$$V_{eff} = m + \frac{l^2}{mr^2} - \frac{r_S m}{r} - \frac{r_S l^2}{mr^3} \tag{3.81}$$

differentiate and set to zero

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{mr^3} + \frac{r_S m}{r^2} + \frac{3r_S l^2}{mr^4} = 0 \tag{3.82}$$

Can be simplified this to a quadratic equation in r by multiplying through by mr^4

$$r_S m^2 c^2 r^2 - 2l^2 r + 3r_S l^2 = 0 \tag{3.83}$$

there are two solutions

$$r_{\pm} = \frac{l^2}{r_S m^2 c^2}\left(1 \pm \sqrt{1 - \frac{3r_S^2 m^2 c^2}{l^2}}\right) \tag{3.84}$$

There is no a real solution when the quantity in the square root is negative. So in that case there is no circular orbit. When the angular momentum is too small it will happen. There are two solutions means two circular orbits at different radii when the quantity in the square root is positive. So two circular orbit solutions were obtained if $l^2 > 3r_S^2 m^2 c^2$ and none if not. By taking the second derivative it can give information about whether the orbits are stable or unstable. The orbit is stable if the curvature of the effective potential is positive $d^2V_{eff}/dr^2 > 0$ and it is a minimum. On the contrary, the orbit is unstable if $d^2V_{eff}/dr^2 < 0$, the effective potential is a maximun. If the square root term vanishes the smallest possible stable circular orbit will happen. If $1 = 3r_S^2 m^2 c^2/l^2 = 0$ or $l^2 = 3r_S^2 m^2 c^2$ it will be obtained. In this case, the minimum stable circular radius:

$$r_{min} = \frac{l^2}{r_S m^2 c^2} = 3r_S \tag{3.85}$$

that is only three times the Schwarzschild radius. Use the equations r_{min} and l^2 to check whether the orbit is stable or unstable

$$\frac{d^2V_{eff}}{dr^2} = \frac{6l^2}{mr^4} - \frac{2r_S m}{r^3} - \frac{12r_S l^2}{mr^5} \tag{3.86}$$

$\frac{d^2V_{eff}}{dr^2} = 0$, so this orbit is neutral means that neither stable nor unstable.

The last stable orbit value is a very significant result. Because of angular momentum things have trouble getting in when fall into black holes. Accretion disks are called the tend to get ripped apart and form. The material in the disk little by little spirals inward.

There is no longer a stable circular orbit when the radius reaches $3r_S$. As a result the material all only flows into the hole.

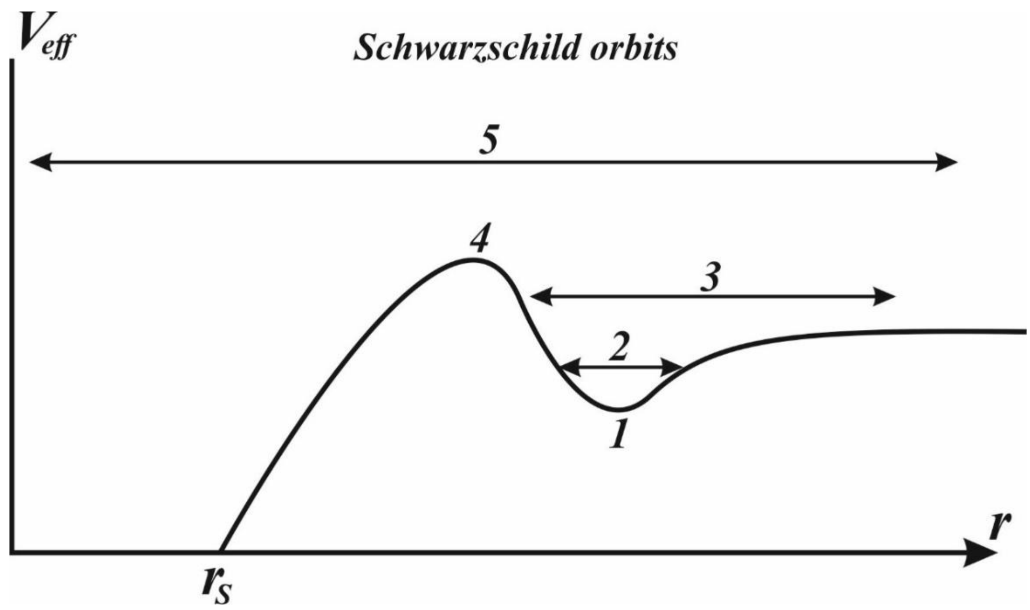

Figure 3.4: Effective potential for Schwarzschild orbits
1-Circular bound orbit (stable), 2-Bound orbit, not quite elliptical, 3-Comes is then out like hyperbolic, 4-Unstable circular orbit-not bound, 5-Capture orbit-doesn't come out

3.9 Schwarschild black holes

In Schwarschild black holes $f(r) = h(r) = 1 - 2M/r$. The radial equation reduces to

$$\dot{r}^2 = V_r^{Schw} \equiv E^2 - f\left(\frac{L^2}{r^2} + \delta_1\right) \tag{3.87}$$

This equation was rewritten by Carroll to emphasize the Newtonian limit

$$\frac{1}{2}\dot{r}^2 = \varepsilon - \nu(r) \tag{3.88}$$

where $\varepsilon \equiv E^2/2$

$$\nu_{\delta_1}(r) = \delta_1\left(\frac{1}{2} - \frac{M}{r}\right) + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \tag{3.89}$$

A more traditional option (ef. Shapiro and Teukolsky or Misner-Thorne-Wheeler) is to write

$$\dot{r}^2 = E^2 - 2\nu_{\delta_1}(r) \tag{3.90}$$

and to get

$$V_{part}(r) = 2\nu_{\delta_1=1}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \tag{3.91}$$

$$V_{phot}(r) = \frac{2\nu_{\delta_1=0}(r)}{L^2} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) \tag{3.92}$$

The motion of photons depends just on the impact parameter $b = L/E$.

$$(\frac{dr}{d\lambda})^2 = \frac{1}{b^2} - V_{phot}(r) \tag{3.93}$$

$$V_{eff} = r^4\left(\frac{1}{b^2} - \frac{f(r)}{r^2}\right) \tag{3.94}$$

The circular orbit appropriates to

$$V_{eff} = 0. V'_{eff} = 0 \tag{3.95}$$

All circular orbits form a closed surface (photon sphere) due to spherical symmetry. The radius of the photon sphere is determined by

$$(\frac{f(r_{ph})}{r_{ph}^2})' = 0 \tag{3.96}$$

The standard shadow radius is given by corresponding impact parameter (b_c)

$$b_c = \frac{r_{ph}}{\sqrt{f(r_{ph})}} \tag{3.97}$$

3.10 Circular geodesics in the Schwarzschild metric

Recall the equations (3.28) and (3.29)

$$\dot{r}^2 = V_r \tag{3.28}$$

$$V_r \equiv h\left(\frac{E^2}{f} - \frac{L^2}{r^2} - \delta_1\right) \tag{3.29}$$

it follows that $V_r = V'_r = 0$. The condition $(V_r^{Schw})' = 0$ is equivalent to

$$L^2(r - 3M) = \delta_1 M r^2 \tag{3.98}$$

Two results can be drawn from this equality:

1. Circular geodesics exist just for $r \geq 3M$.

2. Null circular geodesics ($\delta_1 = 0$) exist just at $r = 3M$.

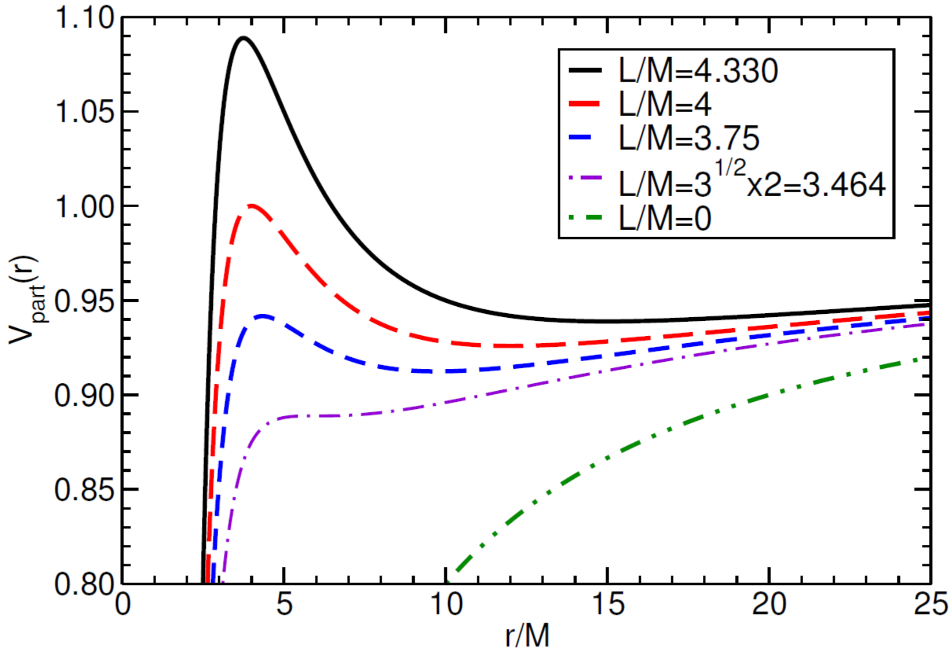

Figure 3.5: Effective potential for massive particles

The angular frequency of the circular orbit is

$$\Omega_c \equiv \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{fL}{r^2 E} \tag{3.99}$$

From the equation (3.98)

$$L^2 = \frac{\delta_1 M r^2}{r - 3M} \tag{3.100}$$

and the condition $V_r = 0$ translates to

$$E^2 = \frac{\delta_1 r f^2}{(r - 3M)} \tag{3.101}$$

As a result the relativistic analog of Kepler`s law:

$$\Omega_c^2 = \frac{f^2}{r^4}\frac{L^2}{E^2} = \frac{f^2\delta_1 M r^2}{r-3M}\frac{r-3M}{\delta_1 r f^2 r^4} = \frac{M}{r^3} \tag{3.102}$$

L^2 from (3.100) is used to get $(V_r^{Schw})''$

$$(V_r^{Schw})'' = -\frac{2\delta_1 M(r-6M)}{r^3(r-3M)} \tag{3.103}$$

So

3. With $r \geq 6M$ circular geodesics are stable (therefore $r = 6M$ is called to as the ISCO (innermost stable circular orbit)).
4. With $3M \leq r < 6M$ circular geodesics are all unstable.

3.11 The critical impact parameter

Recall the effective potential for geodesic motion (3.87)

$$\dot{r}^2 = V_r^{Schw} \equiv E^2 - f\left(\frac{L^2}{r^2} + \delta_1\right) \tag{3.87}$$

Two extreme situations should be considered: $E = 1$ and $E \to \infty$ the ultrarelativistic limit (the second situation is compatible to massless, ultrarelativistic particles so $\delta_1 = 0$). If $E = 1$ (i.e., particles dropped from rest at infinity) there is

$$\dot{r}^2 = 1 - f\left(\frac{L^2}{r^2} + 1\right) = \frac{2M^3}{r^3}\left(\frac{r^2}{M^2} + \frac{L^2}{M^2} - \frac{1}{2}\frac{L^2}{M^2}\frac{r}{M}\right) =$$

$$= \frac{2M^3}{r^3}\left(\frac{r}{M} - \frac{L}{M}\frac{\frac{L}{M} + \sqrt{\left(\frac{L}{M}\right)^2 - 16}}{4}\right)\left(\frac{r}{M} - \frac{L}{M}\frac{\frac{L}{M} - \sqrt{\left(\frac{L}{M}\right)^2 - 16}}{4}\right) \tag{3.104}$$

There is a turning point (i.e., a real value of r for $\dot{r} = 0$) and just if $(L/M)^2 - 16 > 0$. In addition, all turning points are located at $r = r_+ = 2M$, it is outside the horizon. It leads to the critical angular momentum for capture is

$$L_{crit} = 4M \tag{3.105}$$

The capture of massless particles ($\delta_1 = 0$) determining an impact parameter $b = L/E$, then

$$\dot{r}^2 = E^2 - f\left(\frac{L^2}{r^2}\right) = \frac{E^2}{r^3}(r^3 - rfb^2) \tag{3.106}$$

A cubic equation must be solved to find the turning points. According to the Chandrasekhar, for the critical b, the cubic polynomial

$$r^3 - rfb^2 = r^3 - rb^2 + 2Mb^2 \tag{3.107}$$

there is no real positive roots. It is clear that the roots (r_1, r_2, r_3) of the polynomial $ar^3 + br^2 + cr + d$ ensure $r_1 + r_2 + r_3 = -b/a$ and $r_1 r_2 r_3 = -d/a$. The polynomial tends to $-\infty$ as $r \to -\infty$ so there is always one negative root and at $r = 0$ it is positive (equal to $2Mb^2$). Therefore there are two possibilities: (1) two complex-conjugate roots and one real negative root or (2) two different real roots and one real negative root. If the two real roots of case (2) degenerate into a single real root it is a critical situation. In this case

$$(r^3 - rfb^2)' = 3r^2 - b^2 = 0 \; or \; r = \frac{b}{\sqrt{3}} \tag{3.108}$$

$$b_{crit} = 3\sqrt{3}M \tag{3.109}$$

$$r_{crit} = 3M \tag{3.110}$$

The main results of this discussion;

(1) Circular null geodesics are unstable and they are located at $r = 3M$.

(2) The critical impact parameter for capture of light rays (or gravitons or massless scalars) are compatible exactly to these circular null geodesics.

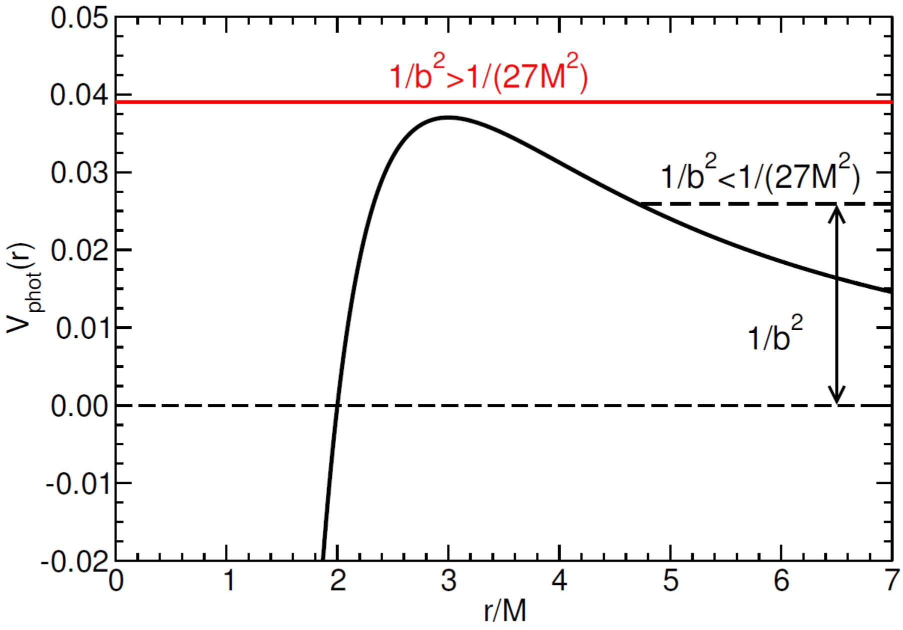

Figure 3.6: Effective potential for photons. The appropriate critical impact parameter to the light ring separates plunging orbits from scattering orbits

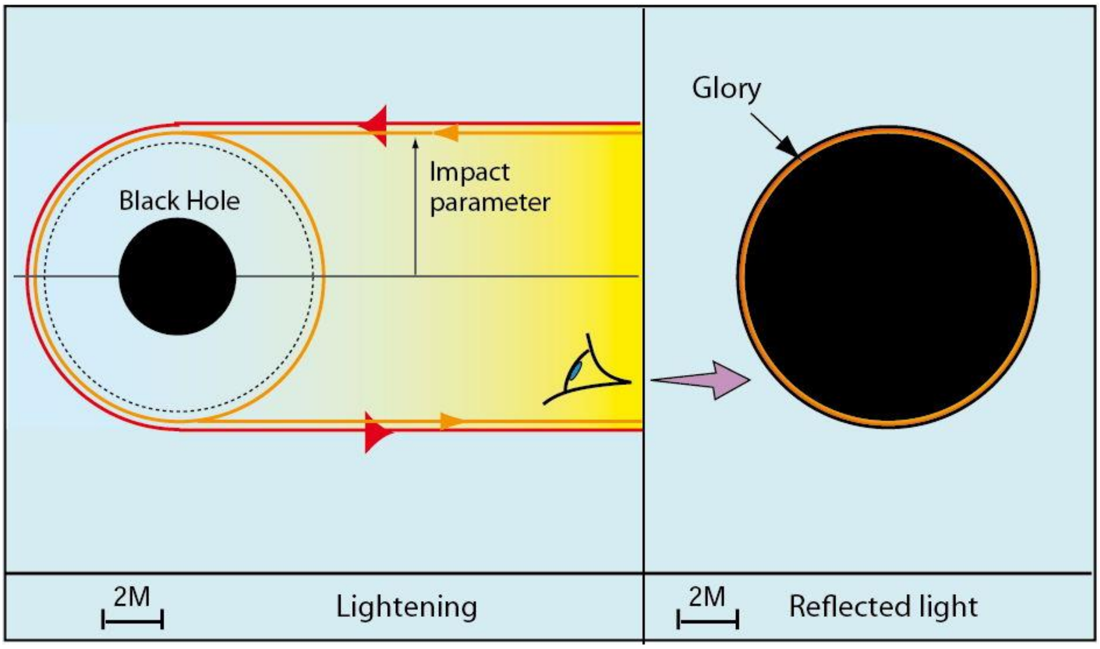

Figure 3.7: Impact parameter [27]

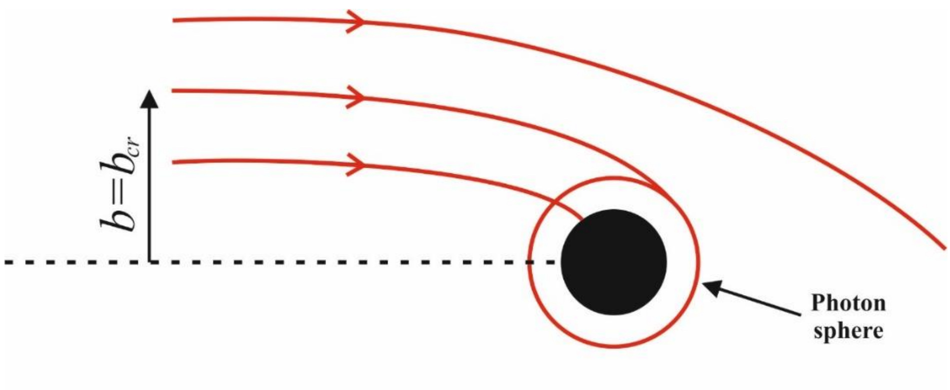

Figure 3.8: Critical impact parameter

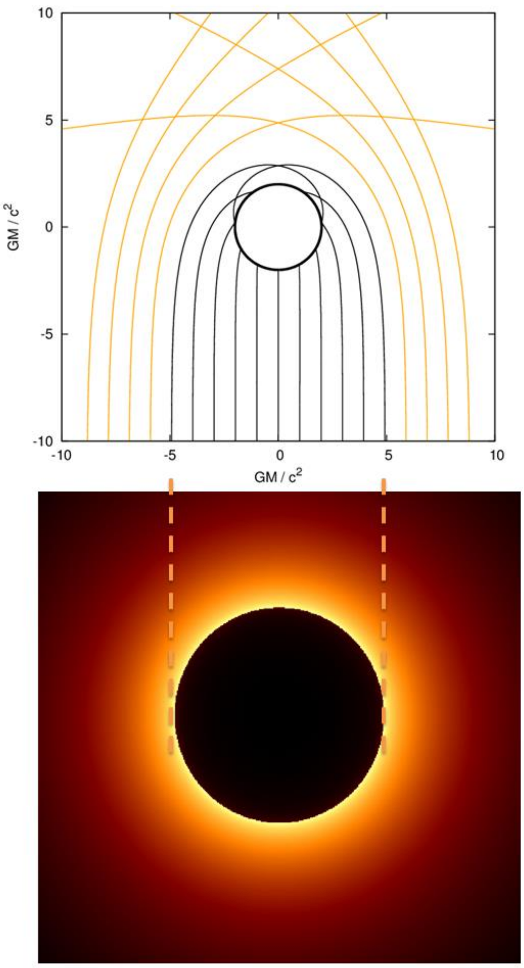

Figure 3.9: Shadow of the non-rotating black hole (Schwarzschild) [28]

Chapter 4

QUANTUM CORRECTED SCHWARZSCHILD BLACK HOLE

4.1 Null geodesic in quantum corrected Schwarzschild black hole

Solodukhin and Kazakov determined that the four-dimensional theory of gravity with Einstein action reduces to the effective two-dimensional dilaton gravity when considering spherically symmetric quantum fluctuations of a Schwarzschild black hole [25]

$$S = -\frac{1}{8}\int d^2z\sqrt{-g}\left[r^2R^{(2)} - 2(\nabla r)^2 + \frac{2}{G_N}U(r)\right] \tag{4.1}$$

Here, $U(r)$ - the dilaton potential, $R^{(2)}$ - the two-dimensional Ricci scalar and G_N - the Newtonian constant. Schwarzschild-like metric is a solution for this action

$$dS^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{4.2}$$

$$f(r) = -\frac{2M}{r} + \frac{1}{r}\int^r U(\rho)d\rho \tag{4.3}$$

$$U(r) = \frac{r}{\sqrt{r^2 - 16G_R}} \tag{4.4}$$

G_R - the renormalized Newton constant, determining a new parameter $a = 4\sqrt{G_R}$, the metric function is obtained for a quantum corrected Schwarzschild black hole

$$f(r) = -\frac{2M}{r} + \frac{\sqrt{r^2 - a^2}}{r} \tag{4.5}$$

If $U(r) = 1$ the space is empty, without quantum fluctuations and the metric is reduced to the Schwarzschild one. The central point (at $r = 0$) as singularity of the Schwarzschild black hole, is now transferred to a central 2D spherical region with radius a that is according to the Planck length. The horizon is located at

$$r_0 = \sqrt{4M^2 + a^2} \tag{4.6}$$

Though at $r = a$ the metric is finite, on this 2D sphere the curvature is different. Geodesic equations described the orbit of one particle moving in curved spacetime, which can be introduced as the Euler-Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{4.7}$$

τ - the proper time, $\mathcal{L}$ - the Lagrangian and $\dot{x}^\mu$ - the four-velocity of the particle. The Lagrangian for the line element is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(-f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\phi}^2\right) \tag{4.8}$$

The motion of the particle on the equatorial plane can always be limited for a static and spherically symmetric metric i.e., $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. Furthermore, the Lagrangian $\mathcal{L}$ does not to cover coordinates t and ϕ clearly, thus there are two conserved quantities

$$E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = f\dot{t} \tag{4.9}$$

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} \tag{4.10}$$

E - energy and L - angular momentum of one particle moving around the black hole. $\mathcal{L} = 0$ for the null geodesic, the orbit equation then can be obtained

$$\left(\frac{dr}{d\phi}\right)^2 + V_{eff} = 0 \tag{4.11}$$

$$V_{eff} = -r^4\left(\frac{1}{b^2} - \frac{f(r)}{r^2}\right) \tag{4.12}$$

$b = \frac{L}{E}$ - the impact parameter. The circular orbit appropriates to

$$V_{eff} = 0, V'_{eff} = 0 \tag{4.13}$$

Due to the spherical symmetry, a closed surface is formed by circular orbits. This is called the photon sphere and its radius is defined by

$$\left(\frac{f(r_{ph})}{r_{ph}^2}\right)' = 0 \tag{4.14}$$

the standard shadow radius is given by corresponding impact parameter b_c

$$b_C = \frac{r_{ph}}{\sqrt{f(r_{ph})}} \tag{4.15}$$

The photon sphere and shadow radius for a KS black hole are

$$r_{ph} = \sqrt{\frac{3\left(3 + x^2 + \sqrt{9 + 2x^2}\right)}{2}}M \tag{4.16}$$

$$b_c = \sqrt{\frac{\sqrt{27(3 + x^2 + \sqrt{9 + 2x^2})^3}}{2\sqrt{9 + x^2 + 3\sqrt{9 + 2x^2} - 4\sqrt{2}}}}M \tag{4.17}$$

$x = a/M$ - a dimensionless parameter. The shadow radius of a KS black hole has already been determined in [26]. For a KS black hole:

$$\frac{3}{2}r_0 \geq r_{ph} \geq \frac{b_c}{\sqrt{3}} \geq 3M \tag{4.18}$$

Table 4.1: Photon sphere for different value of x

r_{ph}	x	M
3	0	1
3.31	1	1
4.08	2	1

Table 4.2: Shadow radius for different value of x

b_c	x	M
3.3	0	1
3.69	1	1
4.66	2	1

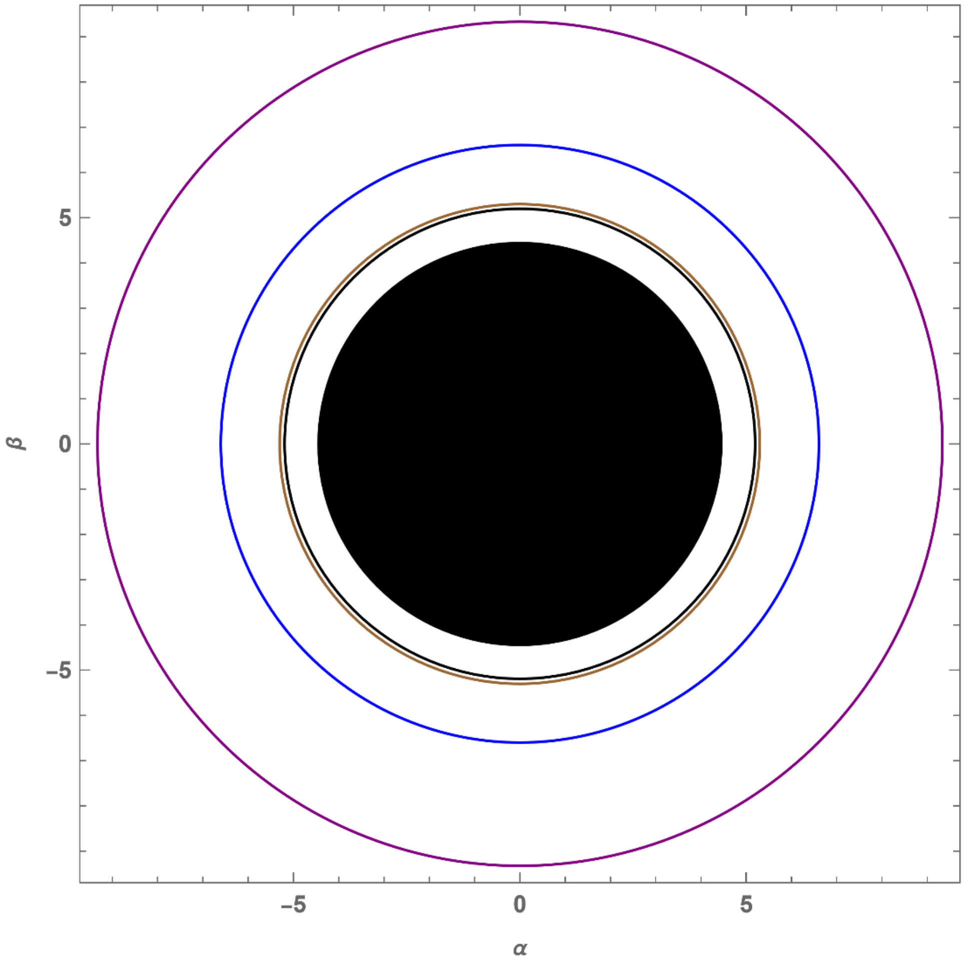

Figure 4.1: Shadow of the Schwarzschild-like black hole for various values of a=0 (black) Schwarzschild case, a=0.5 (brown), a=2 (blue), a=3 (gray), a=4 (purple) at fixed values of G=M=1.

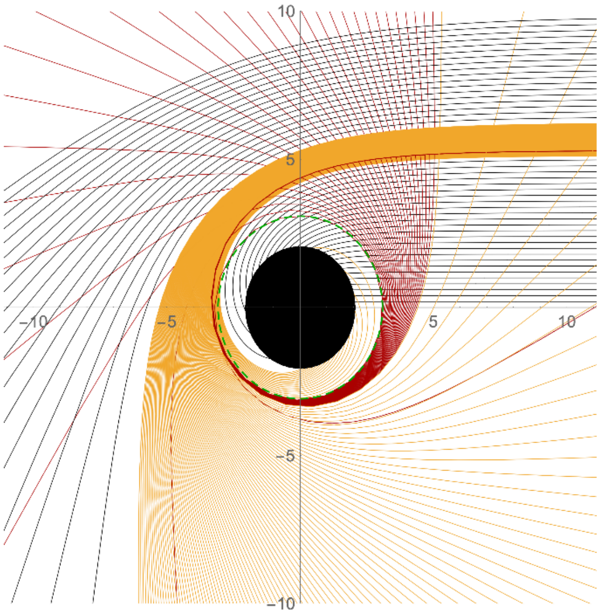

Figure 4.2: The figure shows the photon trajectories of the Schwarzschild-like black hole for values of M=G and a=0.5

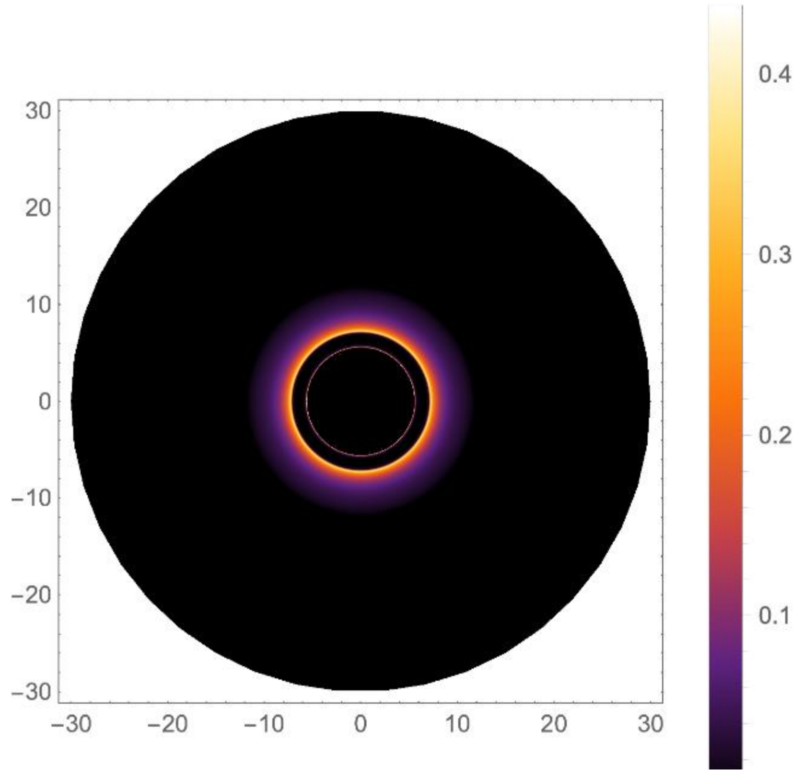

Figure 4.3: The figure shows observational appearance of the thin disk and shadow of the black hole for values of M=G and a=0.5.

Chapter 5

SUMMARY AND CONCLUSION

The main goal was here is to calculate the standard shadow radius of the Schwarzschild black hole. Black holes are regions of spacetime where gravity is too strong. Nothing can get rid of this region, hence the name 'black' hole. A Schwarzschild black hole (static black hole) is a black hole that without electric charge and angular momentum. Before calculating the standard shadow radius of the Schwarzschild black hole, the second chapter is covered the formation of the Schwarzschild black hole. In the third chapter, first geodesic equations is obtained from a variational prinsiple. Then geodesics is calculated in static, spherically symmetric spacetimes. The calculations is continued with geodesics of Schwarzschild metric from Euler-Lagrange. As a result the t, ϕ, θ and r equations are determined. Using the Schwarzschild metric, distances and times is calculated around a black hole. For calculating orbits in the Schwarzschild metric the method effective potential for Newtonian orbits is used. In the next stage different types of orbits is obtained from the method of effective potential for Schwarzschild orbits. After these steps for Schwarzschild black holes the radius of the photon sphere and the standart shadow radius are determined:

$$\left(\frac{f(r_{ph})}{r_{ph}^2}\right)' = 0\ ;\ \ b_c = \frac{r_{ph}}{\sqrt{f(r_{ph})}}$$

The last chapter is about null geodesic in quantum corrected Schwarzschild black hole. For Kazakov and Solodukhin black hole the photon sphere and shadow radius are given by

$$r_{ph} = \sqrt{\frac{3(3 + x^2 + \sqrt{9 + 2x^2})}{2}}M$$

$$b_c = \sqrt{\frac{\sqrt{27(3 + x^2 + \sqrt{9 + 2x^2})^3}}{2\sqrt{9 + x^2 + 3\sqrt{9 + 2x^2} - 4\sqrt{2}}}}M$$

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